Exercise 0.1

(a) 2 black aces and 2 red queens: $\frac{2}{52} + \frac{2}{52} = \frac{4}{52} = \frac{1}{13}$.
(b) 12 face cards, of which 6 are black, and 26 black cards overall: $\frac{12}{52} + \frac{26}{52} - \frac{6}{52} = \frac{32}{52} = \frac{8}{13}$.
(c) 13 hearts, 4 queens of which 1 queen of hearts: probability of finding a heart or a queen is $\frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$. The probability of not finding this is $1 - \frac{4}{13} = \frac{9}{13}$.
(d) Remove one card: the probability that it is a spade is $P(S) = \frac{13}{52}$. For two removed cards, the probability is $P(S1) \cdot P(S2) = \frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652}$ that they are both spades, $P(S1) \cdot P(\neg S2) + P(\neg S1) \cdot P(S2) = \frac{13}{52} \cdot \frac{39}{51} + \frac{39}{52} \cdot \frac{13}{51} = \frac{1014}{2652}$ that one is a spade and $P(\neg S1) \cdot P(\neg S2) = \frac{39}{52} \cdot \frac{38}{51} = \frac{1482}{2652}$ that neither is a spade. If two spades have been removed, the probability of drawing a spade is $P(S|S1, S2) = \frac{11}{50}$; if one has been removed, it's $P(S|S1, \neg S2 \wedge \neg S1, S2) = \frac{12}{50}$ and if none has been removed, it's $P(S|\neg S1, \neg S2) = \frac{13}{50}$. The total probability thus becomes $P(S) = P(S|S1, S2)P(S1, S2) + P(S|\neg S1, S2 \wedge S1, \neg S2)P(\neg S1, S2 \wedge S1, \neg S2) + P(S|\neg S1, \neg S2)P(\neg S1, \neg S2) = \frac{11}{50} \cdot \frac{156}{2652} + \frac{12}{50} \cdot \frac{1014}{2652} + \frac{13}{50} \cdot \frac{1482}{2652} = \frac{1716+12168+19266}{132600} = \frac{33150}{132600} = \frac{1}{4}$.
This answer can also easily be argued: if we remove a number of cards at random from a deck, we don't expect to favour one suit or the other. The probabilities should therefore not change.

Exercise 0.2

(a) $E(X) = \int_{-\infty}^{\infty} xf(x)dx$, so here $E(X) = \int_1^2 x6(x-1)(2-x)dx = 1.5$.
(b) First, we determine the variance $\text{Var}(X) = E((X-E(X))(X-E(X))) = E(X^2) - E(X)^2$. $E(X^2) = \int_1^2 x^2 f(x)dx = 2.3$. Therefore, $\text{Var}(X) = 2.3 - (1.5)^2 = 0.05$. The standard deviation is the square root of that, $\sqrt{0.05} \approx 0.223$.
(c) Denoting the 7 times given as x_i, $\mu = \frac{1}{7}\sum_{i=1}^{7} x_i = \frac{1}{7}(1.5+1.5+1.5+1.67+1.67+1.2+1.9) \approx 1.563$ hours. The unbiased estimate of the sample variance is $\frac{1}{7-1}\sum_{i=1}^{7}(x_i - \mu)^2 \approx 0.0467$ and so the standard deviation $\approx \sqrt{0.0467} \approx 0.216$.

Exercise 0.3

Let X be a normally distributed random variable with mean 125 and standard deviation 10. Then $Z = \frac{X-125}{10}$ is a random variable with standard normal distribution. The probability of one student having an IQ larger than 135 is $P(X \geq 135) = P(\frac{X-125}{10} \geq \frac{135-125}{10}) = P(\frac{X-125}{10} \geq 1) = P(Z \geq 1) = 1 - \Phi(1) \approx 1 - 0.8413 = 0.1587$. Therefore, the probability that out of 20 randomly selected Delft students, five have an IQ of at least 135 is $\binom{20}{5}(0.1587)^5(0.8413)^{15} \approx 0.117$.

Exercise 0.4

(a) $F_{X,Y}(x, y) = \int_0^x \int_x^y kxydydx = \frac{1}{8}kx^2(2y^2 - x^2)$.
(b) The cumulative distribution function should be 1 for $(x, y) = (\infty, \infty)$, but here does not change for (x, y) larger than $(1, 1)$. So setting $F_{X,Y}(1, 1) = 1$ gives $k = 8$.
(c) A: $F_{X,Y}(x, y) = 0$
C: $F_{X,Y}(x, y) = F_{X,Y}(x, y)$ for $y = 1$: $x^2(2 - x^2)$
D: $F_{X,Y}(x, y) = 1$
E: $F_{X,Y}(x, y) = F_{X,Y}(x, y)$ for $x = y$: y^4

(d) $f_X(x) = \int_x^1 8xydy = 4x(1-x^2)$ for $0 \le x \le 1$, 0 otherwise;
$f_Y(y) = \int_0^y 8xydx = 4y^3$ for $0 \le y \le 1$, 0 otherwise.
(e) $E(Y) = 4\int_0^1 y \cdot y^3 dy = \frac{4}{5}$
$E(Y^2) = 4\int_0^1 y^2 \cdot y^3 dy = \frac{4}{6}$
$\text{var}(Y) = E(Y^2) - E(Y)^2 = \frac{4}{6} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$
(f) No: for independent random variables $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. This is not the case here.

Exercise 0.5

(a) Call X the length of the first piece, then the second piece has length $Y = 1 - X$. X is uniformly distributed, with PDF $f(x) = 1$ for $0 \le x < 1$ and 0 otherwise.
The covariance of X and Y, Cov(X,Y) is defined as $E((X - E(X))(Y - E(Y)))$. $E(X) = \int_0^1 x1dx = \frac{1}{2}$; likewise, $E(Y) = \frac{1}{2}$.
$E((X - E(X))(Y - E(Y))) = E((X - \frac{1}{2})(Y - \frac{1}{2})) = \int_0^1 (x - \frac{1}{2})((1-x) - \frac{1}{2})dx = \int_0^1 x - x^2 - \frac{1}{2}x - \frac{1}{2} + \frac{1}{2}x + \frac{1}{4}dx = \int_0^1 -x^2 + x - \frac{1}{4}dx = -\frac{1}{12}$.
(b) The correlation coefficient $\rho = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$. $\text{Var}(X) = E(X^2) - E(X)^2$ with $E(X^2) = \int_0^1 x^2 1dx = \frac{1}{3}$; $E(X) = \int_0^1 x1dx = \frac{1}{2}$, so $\text{Var}(X) = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{12}$ and $\sigma_X = \sqrt{\text{Var}(X)} = \sqrt{\frac{1}{12}}$.
Likewise, $\sigma_Y = \sqrt{\frac{1}{12}}$ and therefore $\rho = \frac{-\frac{1}{12}}{\sqrt{\frac{1}{12}}\sqrt{\frac{1}{12}}} = -1$.
(c) No, since they are correlated.

Exercise 0.6

If X_1 and X_2 are not correlated, $R_{12} = R_{21} = 0$ and so $\boldsymbol{\Sigma}$ becomes a diagonal matrix. This means we can write $f(\boldsymbol{x})$ as $f(\boldsymbol{x}) = \frac{1}{((2\pi)^2\sigma_1^2\sigma_2^2)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\frac{(x_1-\mu_1)^T(x_1-\mu_1)}{\sigma_1^2} + \frac{(x_2-\mu_2)^T(x_2-\mu_2)}{\sigma_2^2}\right)\right) = \frac{1}{(2\pi\sigma_1^2)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\frac{(x_1-\mu_1)^T(x_1-\mu_1)}{\sigma_1^2}\right) \frac{1}{(2\pi\sigma_2^2)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\frac{(x_2-\mu_2)^T(x_2-\mu_2)}{\sigma_2^2}\right) = f(x_1)f(x_2)$, i.e. X_1 and X_2 are independent.

Exercise 0.7

(a) $P(-|H) = 1 - P(+|H) = 1 - 0.95 = 0.05$.
(b) Bayes' rule: $P(H|+) = \frac{P(+|H)P(H)}{P(+)}$, with $P(+) = P(+|H)P(H) + P(+|\neg H)P(\neg H)$.
$P(+|H) = 0.95$ and $P(-|\neg H) = 0.95$; so $P(+|\neg H) = 1 - P(-|\neg H) = 1 - 0.95 = 0.05$.
$P(\neg H) = 1 - P(H) = 1 - \frac{1}{1000} = \frac{999}{1000}$ and so
$P(+) = 0.95 \cdot \frac{1}{1000} + 0.05 \cdot \frac{999}{1000} = 0.0509$, giving
$P(H|+) = \frac{0.95 \cdot \frac{1}{1000}}{0.0509} = 0.0187$.
(c) The answer is modelled by a binomial random variable with $n = 100$ and $p = P(+) = 0.0509$.
$P(\ge 1+) = 1 - P(0\,+) = 1 - \binom{n}{0}(1-p)^n = 1 - (1 - 0.0509)^{100} = 1 - 0.0054 = 0.9946$.