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Assignment #2

Section 5.1

74. Prove that if p is a prime number and r is an integer with 0 < r < p, then $\binom{p}{r}$ is divisible by p.

<u>Proof:</u> Suppose p is a prime number and r is an integer with 0 < r < p.
Then $\binom{p}{r} = \frac{p!}{r!(p-r)!} = \frac{p\cdot(p-1)\cdot(p-2)\cdot...\cdot(p-r+1)\cdot(p-r)!}{r!(p-r)!} = \frac{p\cdot(p-1)\cdot(p-2)\cdot...\cdot(p-r+1)}{r!}$
The prime factorization of the numerator will contain a p and the primes in the prime factorization of the denominator will be less than p. Since $\binom{p}{r}$ is an integer, all the primes in the denominator will cancel out with the primes from the numerator. This will result in p times an integer. Therefore, $\binom{p}{r}$ is divisible by p.

Section 5.2

16. Prove each of the statements in 10–17 by mathematical induction.
$(1 - \frac{1}{2^2})\ (1 - \frac{1}{3^2})\ ...\ (1 - \frac{1}{n^2}) = \frac{n+1}{2n}$, for all integers n ≥ 2.
<u>Proof:</u> P(n): $(1 - \frac{1}{2^2})\ (1 - \frac{1}{3^2})\ ...\ (1 - \frac{1}{n^2}) = \frac{n+1}{2n}$
Basis Step: Show that P(2) is true.
$(1 - \frac{1}{2^2}) = \frac{2+1}{2\cdot 2}$
$(1 - \frac{1}{4}) = \frac{3}{4}$
$\frac{3}{4} = \frac{3}{4}$ ✓
Inductive Hypothesis - P(k): $(1 - \frac{1}{2^2})\ (1 - \frac{1}{3^2})\ ...\ (1 - \frac{1}{k^2}) = \frac{k+1}{2k}$, for all integers k ≥ 2.
Suppose P(k) is true. Show that if P(k) is true, then P(k + 1) is true:
P(k+1): $(1 - \frac{1}{2^2})\ (1 - \frac{1}{3^2})\ ...\ (1 - \frac{1}{(k+1)^2}) = \frac{(k+1)+1}{2(k+1)}$
P(k+1): $(1 - \frac{1}{2^2})\ (1 - \frac{1}{3^2})\ ...\ (1 - \frac{1}{(k+1)^2}) = \frac{k+2}{2k+2}$
P(k+1) $= (1 - \frac{1}{2^2})\ (1 - \frac{1}{3^2})\ ...\ (1 - \frac{1}{k^2})\ (1 - \frac{1}{(k+1)^2})$
P(k+1) $= \frac{k+1}{2k}\ (1 - \frac{1}{(k+1)^2})$
P(k+1) $= \frac{k+1}{2k}\ (1 - \frac{1}{k^2+2k+1})$
P(k+1) $= \frac{k+1}{2k}\ (\frac{k^2+2k+1}{k^2+2k+1} - \frac{1}{k^2+2k+1})$
P(k+1) $= \frac{k+1}{2k}\ (\frac{k^2+2k}{k^2+2k+1})$
P(k+1) $= \frac{k+1}{2k}\ (\frac{k^2+2k}{(k+1)^2})$
P(k+1) $= \frac{k^2+2k}{2k\cdot(k+1)}$
P(k+1) $= \frac{k\cdot(k+2)}{2k\cdot(k+1)}$
P(k+1) $= \frac{(k+2)}{2\cdot(k+1)}$

P(k+1) = $\frac{k+2}{2k+2}$ ✓

27. Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sums in 20–29 or to write them in closed form.

$5^3 + 5^4 + 5^5 + ... + 5^k$, where k is any integer with k $\geq$ 3.

$5^3 + 5^4 + 5^5 + ... + 5^k = 1 + 5 + 5^2 + 5^3 + 5^4 + 5^5 + ... + 5^k$ - $(1 + 5 + 5^2)$

$1 + 5 + 5^2 + 5^3 + 5^4 + 5^5 + ... + 5^k$ - $(1 + 5 + 5^2) = \frac{5^{k+1}-1}{4}$ - $\frac{5^3-1}{4} = \frac{5^{k+1}-1}{4}$ - $31 = \frac{5^{k+1}-1}{4}$ - $\frac{124}{4} = \frac{5^{k+1}-125}{4}$

Section 5.3

21. Prove each statement in 8–23 by mathematical induction.

$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$, for all integers n $\geq$ 2.

Proof: P(n): $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$

Basis Step: Show that P(2) is true.

$\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$

$\sqrt{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} < 1 + \frac{1}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} < \frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} < \frac{\sqrt{2}+1}{\sqrt{2}}$ ✓

Inductive Hypothesis - P(k): $\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}}$, for all integers k $\geq$ 2.

Suppose P(k) is true. Show that if P(k) is true, then P(k + 1) is true:

P(k + 1): $\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$\sqrt{k} + \frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$\frac{\sqrt{k}\cdot\sqrt{k+1}+1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$\frac{\sqrt{k\cdot(k+1)}+1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$\frac{\sqrt{k^2+k}+1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$\sqrt{k+1} = \sqrt{k+1} \cdot \frac{\sqrt{k+1}}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}}$

$\sqrt{k^2+k} + 1 >$ k + 1

$\sqrt{k^2+k} >$ k

$k^2 + k > k^2$

k > 0 This is true because it is assumed k $\geq$ 2.

$\therefore \sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$ ✓

Section 5.5

29. In 24–34, F_0, F_1, F_2, ... is the Fibonacci sequence.

Prove that F^2_{k+1} - $F^2_k = F_{k-1}F_{k+2}$, for all integers k $\geq$ 1.

For all integers k $\geq$ 1,

F^2_{k+1} - F^2_k = $(F_{k+1} + F_k) \cdot (F_{k+1} - F_k)$ - (difference of two squares)

= $(F_{k+2}) \cdot (F_{k+1} - F_k)$ - (definition of the Fibonacci sequence)

= $F_{k+2}\ F_{k+1} - F_{k+2}\ F_k$

= $F_{k+2}\ (F_k + F_{k-1}) - F_{k+2}\ F_k$ - (definition of the Fibonacci sequence)

= $F_{k+2}\ F_k + F_{k-1}\ F_{k+2} - F_{k+2}\ F_k$

= $F_{k-1}\ F_{k+2}$ ✓

Section 5.6

15. In each of 3–15 a sequence is defined recursively. Use iteration to guess an explicit formula for the sequence. Use the formulas from Section 5.2 to simplify your answers whenever possible.

$y_k = y_{k-1} + k^2$, for all integers $k \geq 2$

$y_1 = 1$

$y_2 = y_1 + 2^2 = 1 + 2^2 = 5$

$y_3 = y_2 + 3^2 = 1 + 2^2 + 3^2 = 14$

$y_4 = y_3 + 4^2 = 1 + 2^2 + 3^2 + 4^2 = 30$

$y_5 = y_4 + 5^2 = 1 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

$y_k = 1 + 2^2 + 3^2 + ... + (k - 1)^2 + (k)^2 = \frac{(k)\cdot(k+1)\cdot(2k+1)}{6}$

This formula is the summation of the first k Squares.

46. In each of 43–49 a sequence is defined recursively. (a) Use iteration to guess an explicit formula for the sequence. (b) Use strong mathematical induction to verify that the formula of part (a) is correct.

$s_k = 2s_{k-2}$, for all integers $k \geq 2$,

$s_0 = 1$, $s_1 = 2$.

(a) $s_0 = 1$

$s_1 = 2$

$s_2 = 2s_0 = 2 \cdot 1 = 2$

$s_3 = 2s_1 = 2 \cdot 2 = 4$

$s_4 = 2s_2 = 2 \cdot 2 \cdot 1 = 4$

$s_5 = 2s_3 = 2 \cdot 2 \cdot 2 = 8$

$s_6 = 2s_4 = 2 \cdot 2 \cdot 2 \cdot 1 = 8$

$s_7 = 2s_5 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$

$s_8 = 2s_6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 = 16$

$$s_k = \begin{cases} 2^{(k+1)/2} \text{ if k is odd} \\ 2^{k/2} \text{ if k is even} \end{cases}$$

$= 2^{\lceil k/2 \rceil}$ for all integers $k \geq 0$.

(b) <u>Proof:</u>

Let $s_0, s_1, s_2, ...$ be the recursive sequence defined by $s_0 = 1$, $s_1 = 2$ and $s_k = 2s_{k-2}$, for all integers $k \geq 2$.

P(n): $s_n = 2^{\lceil n/2 \rceil}$ for all integers $n \geq 0$.

Basis Step: Show that P(0) is true.

$P(0) = 2^{\lceil 0/2 \rceil} = 1$ ✓

This equals s_0 in the sequence s_0, s_1, s_2, ...

Inductive Hypothesis -

P(k): $s_k = 2^{\lceil k/2 \rceil}$ for all integers $k \geq 0$.

Suppose P(i) is true for all integers i with $0 \leq i \leq k$. Show that P(k + 1) is true:

P(k + 1): $s_{k+1} = 2^{\lceil (k+1)/2 \rceil}$

$s_{k+1} = 2s_{k-1}$

$s_{k+1} = 2 \cdot 2^{\lceil (k-1)/2 \rceil}$

$s_{k+1} = 2^{\lceil ((k-1)/2)+1 \rceil}$

$s_{k+1} = 2^{\lceil ((k-1)/2)+(2/2) \rceil}$

$s_{k+1} = 2^{\lceil (k+1)/2 \rceil}$ ✓

Section 6.1

23. Let $V_i = \{x \varepsilon \mathbb{R} \mid -\frac{1}{i} \leq x \leq \frac{1}{i}\} = [-\frac{1}{i}, \frac{1}{i}]$ for all positive integers i.

$V_1 = [-1, 1]$, $V_2 = [-\frac{1}{2}, \frac{1}{2}]$, $V_3 = [-\frac{1}{3}, \frac{1}{3}]$, $V_4 = [-\frac{1}{4}, \frac{1}{4}]$

a. $\bigcup_{i=1}^{4} V_i = ?$

$\bigcup_{i=1}^{4} V_i = [-1, 1] \cup [-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{1}{3}, \frac{1}{3}] \cup [-\frac{1}{4}, \frac{1}{4}] = [-1, 1]$

b. $\bigcap_{i=1}^{4} V_i = ?$

$\bigcap_{i=1}^{4} V_i = [-1, 1] \cap [-\frac{1}{2}, \frac{1}{2}] \cap [-\frac{1}{3}, \frac{1}{3}] \cap [-\frac{1}{4}, \frac{1}{4}] = [-\frac{1}{4}, \frac{1}{4}]$

c. Are V_1, V_2, V_3, ... mutually disjoint? Explain.

V_1, V_2, V_3, ... are not mutually disjoint because they intersect each other. $V_{k+1} \subseteq V_k$ for all integers $k \geq 1$.

d. $\bigcup_{i=1}^{n} V_i = ?$

$\bigcup_{i=1}^{n} V_i = [-1, 1] \cup [-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{1}{3}, \frac{1}{3}] \cup [-\frac{1}{4}, \frac{1}{4}] \ldots \cup [-\frac{1}{n}, \frac{1}{n}] = [-1, 1]$

e. $\bigcap_{i=1}^{n} V_i = ?$

$\bigcap_{i=1}^{n} V_i = [-1, 1] \cap [-\frac{1}{2}, \frac{1}{2}] \cap [-\frac{1}{3}, \frac{1}{3}] \cap [-\frac{1}{4}, \frac{1}{4}] \ldots \cap [-\frac{1}{n}, \frac{1}{n}] = [-\frac{1}{n}, \frac{1}{n}]$

f. $\bigcup_{i=1}^{\infty} V_i = ?$

$\bigcup_{i=1}^{\infty} V_i = [-1, 1] \cup [-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{1}{3}, \frac{1}{3}] \cup [-\frac{1}{4}, \frac{1}{4}] \ldots \cup [-\frac{1}{\infty}, \frac{1}{\infty}] = [-1, 1]$

g. $\bigcap_{i=1}^{\infty} V_i = ?$

$\bigcap_{i=1}^{\infty} V_i = [-1, 1] \cap [-\frac{1}{2}, \frac{1}{2}] \cap [-\frac{1}{3}, \frac{1}{3}] \cap [-\frac{1}{4}, \frac{1}{4}] \ldots \cap [-\frac{1}{\infty}, \frac{1}{\infty}] = [0]$

Section 6.2

19. Use an element argument to prove each statement in 7–19. Assume that all sets are subsets of a universal set U.

For all sets A, B, and C,

A $\times$ (B $\bigcap$ C) = (A $\times$ B) $\bigcap$ (A $\times$ C).

Case 1: A $\times$ (B $\bigcap$ C) $\subseteq$ (A $\times$ B) $\bigcap$ (A $\times$ C)

Suppose (x , y) ε A $\times$ (B $\bigcap$ C).

1. x ε A - definition of cartesian product

2. y ε (B $\bigcap$ C) - definition of cartesian product

3. y ε B - definition of set intersection

4. y ε C - definition of set intersection

5. (x, y) ε (A $\times$ B) - (1), (3) cartesian product

6. (x, y) ε (A $\times$ C) - (1), (4) cartesian product

7. (x, y) ε (A $\times$ B) $\bigcap$ (A $\times$ C) - (5), (6) set intersection

Case 2: (A $\times$ B) $\bigcap$ (A $\times$ C) $\subseteq$ A $\times$ (B $\bigcap$ C)

Suppose (x, y) ε (A $\times$ B) $\bigcap$ (A $\times$ C)

1. (x, y) ε (A $\times$ B) - definition of set intersection

2. (x, y) ε (A $\times$ C) - definition of set intersection

3. x ε A - (1), definition of cartesian product

4. y ε B - (1), definition of cartesian product

5. y ε C - (2), definition of cartesian product

6. y ε (B $\bigcap$ C) - (4), (5), set intersection

7. (x, y) ε A $\times$ (B $\bigcap$ C) - (3), (6), cartesian product

Section 6.3

35. In 30–40, construct an algebraic proof for the given statement. Cite a property from Theorem 6.2.2 for every step.

For all sets A and B, A - (A - B) = A $\bigcap$ B.

A - (A - B)

= A $\bigcap$ (A - B)^c - Set Difference Law

= A $\bigcap$ (A $\bigcap$ B^c)^c - Set Difference Law

= A $\bigcap$ (A^c $\bigcup$ (B^c)^c) - De Morgan's Law for $\bigcap$

= A $\bigcap$ (A^c $\bigcup$ B) - Double Complement Law

= (A $\bigcap$ A^c) $\bigcup$ (A $\bigcap$ B) - Distributive Law for $\bigcap$

= $\emptyset$ $\bigcup$ (A $\bigcap$ B) - Complement Law for $\bigcap$

= (A $\bigcap$ B) $\bigcup$ $\emptyset$ - Commutative Law for $\bigcup$

= (A $\bigcap$ B) - Identity Law for $\bigcup$