STSCI 3080 HW #3

KIRILL CHERNYSHOV

Problem 3.4.2 This is simply a matter of adding up entries in the table.

(a) Using the law of total probability, we need to add all the entries in the table for which $X = 2$, that is, $\sum_{i=0}^{4} \mathbb{P}(X = 2|Y = i)$. This is equal to $0.05 + 0.06 + 0.09 + 0.04 + 0.03 = 0.27$.

(b) $\mathbb{P}(Y \geq 2) = \mathbb{P}(Y = 2) + \mathbb{P}(Y = 3) + \mathbb{P}(Y = 4)$. Calculating these values as before, this is equal to $(0.06+0.12+0.09+0.03)+(0.01+0.05+0.04+0.03)+(0.01+0.02+0.03+0.04) = 0.53$.

(c) Once again, summing the entries for which this is true, we get $(0.08 + 0.06 + 0.05) + (0.07 + 0.10 + 0.06) + (0.06 + 0.12 + 0.09) = 0.69$.

(d) For this we need to sum the entries on the main diagonal: $0.08 + 0.10 + 0.09 + 0.03 = 0.3$.

(e) For this we need to sum the entries strictly below the main diagonal: $(0.06 + 0.05 + 0.02) + (0.06 + 0.03) + (0.03) = 0.25$.

Problem 3.4.4

(a) From the laws of probability we know that $\int_0^1 \int_0^2 f(x, y)\,\mathrm{d}x\,\mathrm{d}y = 1$. $\int_0^1 \int_0^2 f(x, y)\,\mathrm{d}x\,\mathrm{d}y = c \int_0^1 \int_0^2 y^2\,\mathrm{d}x\,\mathrm{d}y = c \int_0^1 xy^2\Big|_0^2 \mathrm{d}y = c \int_0^1 2y^2\,\mathrm{d}y = c\frac{2y^3}{3}\Big|_0^1 = \frac{2}{3}c$. Since $\frac{2}{3}c = 1$, we have $c = \frac{3}{2}$.

(b) $\mathbb{P}(X + Y > 2) = \mathbb{P}(X > 2 - Y)$. This can be calculated by adjusting the integration limits: $\int_0^1 \int_{2-y}^2 \frac{2y^2}{3}\,\mathrm{d}x\,\mathrm{d}y$. Since $\int_{2-y}^2 \frac{2y^2}{3}\,\mathrm{d}x = \frac{2xy^2}{3}\Big|_{2-y}^2 = \frac{4y^2}{3} - \frac{2(2-y)y^2}{3} = \frac{4y^2-4y^2+2y^3}{3} = \frac{2y^3}{3}$, then we have $\int_0^1 \int_{2-y}^2 \frac{2y^2}{3}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{2y^3}{3}\,\mathrm{d}y = \frac{y^4}{6}\Big|_0^1 = \frac{1}{6}$.

(c) Using the law of total probability, this is equivalent to calculating the total probability that $\mathbb{P}\left(Y < \frac{1}{2}\right)$ for all values of X. This means calculating $\int_0^{\frac{1}{2}} \int_0^2 \frac{2y^2}{3}\,\mathrm{d}x\,\mathrm{d}y$. We know from before that $\int_0^2 \frac{2y^2}{3}\,\mathrm{d}x = \frac{4y^2}{3}$, so $\int_0^{\frac{1}{2}} \int_0^2 \frac{2y^2}{3}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{1}{2}} \frac{4y^2}{3}\,\mathrm{d}y = \frac{4y^3}{9}\Big|_0^{\frac{1}{2}} = \frac{1}{18}$.

(d) Similarly, for this we need to evaluate $\int_0^1 \int_0^1 \frac{2y^2}{3}\,\mathrm{d}x\,\mathrm{d}y$. Since $\int_0^1 \frac{2y^2}{3}\,\mathrm{d}x = \frac{2y^2}{3}$, we have $\int_0^1 \int_0^1 \frac{2y^2}{3}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{2y^2}{3}\,\mathrm{d}y = \frac{2}{9}$.

(e)

Problem 3.5.2

(a) The marginal p.f. of X is $f_1(x) = \frac{1}{30} \sum_{i=0}^{3} f(x, i) = \frac{1}{30}(x+0+x+1+x+2+x+3) = \frac{4x+6}{30} = \frac{2x+3}{15}$. The marginal p.f. of Y is $f_2(y) = \frac{1}{30} \sum_{i=0}^{2} f(i, y) = \frac{1}{30}(0 + y + 1 + y + 2 + y) = \frac{3y+3}{30} = \frac{y+1}{10}$.

(b) X and Y are independent if and only if $f_1(x)f_2(y) = f(x, y)$. $f_1(x)f_2(y) = \frac{2x+3}{15}\frac{y+1}{10} = \frac{2xy+2x+3y+3}{150} \neq f(x, y)$. Therefore, X and Y are not independent.

Problem 3.5.4

(a) The marginal p.d.f. of X is $f_1(x) = \frac{15}{4}\int_0^{1-x^2} x^2\,\mathrm{d}y = \frac{15}{4}x^2(1-x^2)$. Since $0 \le y \le 1-x^2$, we have $0 \le x^2 \le 1-y \implies x \le \sqrt{1-y}$ The marginal p.d.f. of Y is $\frac{15}{4}\int_0^{\sqrt{1-y}} x^2\,\mathrm{d}x = \frac{15}{4}\frac{(\sqrt{1-y})^3}{3}$.

(b) Since the set $\{(x,y) : f(x,y) > 0\}$ has a curved boundary (since y depends on $1-x^2$), X and Y cannot be independent.

Problem 3.5.6

(a) Since X and Y are independent, $f(x,y) = g(x)g(y) = \frac{9}{64}x^2y^2$.

(b)

(c) $\mathbb{P}(X > Y) = \int_0^2\int_0^x f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \frac{9}{64}\int_0^2\int_0^x x^2y^2\,\mathrm{d}y\,\mathrm{d}x = \frac{9}{64}\int_0^2 \frac{x^5}{3}\,\mathrm{d}x = \frac{9}{64}\frac{32}{9} = \frac{1}{2}$.

(d) $\mathbb{P}(X+Y \le 1) = \mathbb{P}(X \le 1-Y) = \int_0^2\int_0^{1-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{9}{64}\int_0^2\int_0^{1-y} x^2y^2\,\mathrm{d}x\,\mathrm{d}y = \frac{9}{64}\int_0^2 \frac{(1-y)^3y^2}{3}\,\mathrm{d}y$. This evaluates to

Problem 3.5.13 $x^2+y^2 \le 1 \implies y^2 \le 1-x^2 \implies -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$. Therefore, $f_1(x) = k\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2\,\mathrm{d}y = k\frac{x^2y^3}{3}\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = 2k\frac{x^2y^3}{3}\Big|_0^{\sqrt{1-x^2}} = \frac{2kx^2\sqrt{1-x^2}^3}{3}$. The range for x is $-1 \le x \le 1$, since $x^2+y^2 \le 1 \implies -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$, and the largest possible value of $\sqrt{1-y^2}$ is 1 (when $y=0$). Since x and y are interchangeable in both the range $x^2+y^2=1$ and the function $f(x,y) = kx^2y^2$, this means they have the same marginal distributions.

Problem 3.6.4

(a) By definition, the conditional p.d.f. is $g_1(x|y) = \frac{f(x,y)}{f_2 y}$. $f_2(y) = \int_0^1 f(x,y)\,\mathrm{d}x = c\int_0^1 (x+y^2)\,\mathrm{d}x = c\left(\frac{x^2}{2}+xy^2\right)\Big|_0^1 = c\left(y^2+\frac{1}{2}\right)$. ~~To calculate c we can use the fact that $c\int_0^1\int_0^1 x+y^2\,\mathrm{d}x\,\mathrm{d}y = 1 = c\frac{5}{6}$, so $c = \frac{6}{5}$. Therefore, $f_2(y) = \frac{6}{5}\left(y^2+\frac{1}{2}\right)$.~~

Therefore, $g_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{c(x+y^2)}{c\left(y^2+\frac{1}{2}\right)} = \frac{x+y^2}{y^2+\frac{1}{2}}$

(b) Using the function from before, $\mathbb{P}\left(X > \frac{1}{2}|Y = \frac{3}{4}\right) = \int_{\frac{1}{2}}^1 \frac{x+\frac{9}{16}}{\frac{9}{16}+\frac{1}{2}}\,\mathrm{d}x = 0.602$.

Problem 3.6.8

(a) $\mathbb{P}(X > 0.8) = \frac{2}{5}\int_0^1\int_0^{0.8}(2x+3y)\,\mathrm{d}x\,\mathrm{d}y = \frac{2}{5}\int_0^1 2.4y+0.64\,\mathrm{d}y = 1.84\frac{2}{5} = 0.736$.

(b) $f_2(y) = \frac{2}{5}\int_0^1(2x+3y)\,\mathrm{d}x = \frac{2}{5}(3y+1)$. Therefore, $g_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{2x+3y}{3y+1}$. By definition, $\mathbb{P}(X > 0.8|Y = 0.3) = \int_0^{0.8} g_2(x|0.3)\,\mathrm{d}x = \int_0^{0.8}\frac{2x+0.9}{0.9+1}\,\mathrm{d}x = 0.716$.

(c) $f_1(x) = \frac{2}{5}\int_0^1(2x+3y)\,\mathrm{d}y = \frac{2}{5}\left(2x+\frac{3}{2}\right)$. Therefore, $g_2(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{2x+3y}{2x+\frac{3}{2}}$. By definition, $\mathbb{P}(Y > 0.8|X = 0.3) = \int_0^{0.8} g_2(y|0.3)\,\mathrm{d}y = \int_0^{0.8}\frac{0.6+3y}{0.6+\frac{3}{2}}\,\mathrm{d}y = 0.686$.