STSCI 3080 HW #5

KIRILL CHERNYSHOV

Problem 4.6.12

By the rules of variance, $\mathrm{Var}(2X-3Y+8) = 4\mathrm{Var}(X) + 9\mathrm{Var}(Y)$. The means of X and Y are

$$\begin{aligned}
\mathrm{E}(X) = \int_0^1 x \int_0^2 f(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \frac{1}{3}\int_0^1 x \int_0^2 (x+y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{1}{3}\int_0^1 x(2x+2)\,\mathrm{d}x = \frac{2}{3}\int_0^1 (x^2+x)\,\mathrm{d}x \\
&= \frac{2}{3}(\frac{1}{3}+\frac{1}{2}) = \frac{5}{9}
\end{aligned}$$

and similarly

$$\mathrm{E}(Y) = \int_0^2 y \int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{11}{9}$$

We also need $\mathrm{E}(X^2)$ and $\mathrm{E}(Y^2)$:

$$\begin{aligned}
\mathrm{E}(X^2) &= \int_0^1 x^2 \int_0^2 f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \frac{7}{18} \\
\mathrm{E}(Y^2) &= \int_0^2 y^2 \int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{16}{9}
\end{aligned}$$

So, we have $\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{13}{162}$ and similarly $\mathrm{Var}(Y) = \frac{23}{81}$. So, $\mathrm{Var}(2X-3Y+8) = 2.877$.

Problem 4.6.18 By definition, $\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y)$. This distribution is symmetric, so we know that $\mathrm{E}(X) = \mathrm{E}(Y)$, which is

$$\begin{aligned}
\mathrm{E}(X) = \int_0^1 x \int_0^1 (x+y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 x(x+\frac{1}{2})\,\mathrm{d}x \\
&= \int_0^1 (x^2 + \frac{x}{2})\,\mathrm{d}x = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{E}(XY) = \int_0^1 \int_0^1 (xy)(x+y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \int_0^1 (x^2y + xy^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3}\right)\mathrm{d}x \\
&= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\end{aligned}$$

Therefore, the covariance of X and Y is $\mathrm{Cov}(X,Y) = \frac{1}{3} - \frac{7}{12}\frac{7}{12} = -\frac{1}{144}$.

Problem 4.7.7

First we need to find the conditional probability density function of y $g_2(y|x)$, which is defined as $\frac{f(x,y)}{f_1(x)}$, where f_1 is the marginal p.d.f. of X:

$$\begin{aligned} f_1(x) &= \int_0^1 f(x,y)\,\mathrm{d}y = \int_0^1 (x+y)\,\mathrm{d}y \\ &= x + \frac{1}{2} \end{aligned}$$

Therefore, $g_2(y|x) = \frac{x+y}{x+\frac{1}{2}}$. By definition,

$$\begin{aligned} \mathrm{E}(Y|x) &= \int_0^1 y g_2(y|x)\,\mathrm{d}y = \int_0^1 \frac{y(x+y)}{x+\frac{1}{2}}\,\mathrm{d}y \\ &= \frac{1}{x+\frac{1}{2}}\frac{3x+2}{6} \\ &= \frac{3x+2}{6x+3} \end{aligned}$$

and therefore, $\mathrm{E}(Y|X) = \frac{3X+2}{6X+3}$. Also by definition, $\mathrm{Var}(Y|X) = \mathrm{E}(Y^2|X) - \mathrm{E}(Y|X)^2$.

$$\begin{aligned} \mathrm{E}(Y^2|X) &= \int_0^1 y^2 g_2(y|x)\,\mathrm{d}y = \int_0^1 \frac{y^2(x+y)}{x+\frac{1}{2}}\,\mathrm{d}y \\ &= \frac{4x+3}{12x+6} \end{aligned}$$

So, $\mathrm{Var}(Y|X) = \mathrm{E}(Y^2|X) - \mathrm{E}(Y|X)^2 = \frac{4x+3}{12x+6} - \left(\frac{3x+2}{6x+3}\right)^2 = \frac{6x^2+6x+1}{18(2x+1)^2}$.

Problem 5.2.6

Let A be the number of times the target is hit by person A, and ditto for B and C. Since these are all sums of Bernoulli trials, they all have binomial distributions $B(3, 0.125)$, $B(5, 0.25)$ and $B(2, 0.5)$ respectively. We want to find $\mathrm{E}(A+B+C) = \mathrm{E}(A)+\mathrm{E}(B)+\mathrm{E}(C)$, so we can use the fact that the expected value of $B(n,p)$ is simply np, therefore $\mathrm{E}(A+B+C) = \mathrm{E}(A)+\mathrm{E}(B)+\mathrm{E}(C) = \frac{3}{8} + \frac{5}{4} + \frac{2}{2} = 2.625$.

Problem 5.2.8

Let X be the number of components that has failed; then, $X \sim B(10, 0.2)$. We want to work out $\mathbb{P}(X \geq 2 | X \geq 1)$, which by definition is equal to $\frac{\mathbb{P}(X \geq 2 \cap X \geq 1)}{\mathbb{P}(X \geq 1)}$. Since $(X \geq 2) \subseteq (X \geq 1)$, $\mathbb{P}(X \geq 2 \cap X \geq 1) = \mathbb{P}(X \geq 2)$, and so $\mathbb{P}(X \geq 2 | X \geq 1) = \frac{\mathbb{P}(X \geq 2)}{\mathbb{P}(X \geq 1)}$. $\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X < 2) = 1 - \mathbb{P}(X = 1) + \mathbb{P}(X = 0) = 1 - (10 \cdot 0.2^1 0.8^9 + 0.8^{10}) = 0.6242$, and $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - 0.8^{10} = 0.8926$, so $\mathbb{P}(X \geq 2 | X \geq 1) = \frac{0.6242}{0.8926} = 0.699$.

Problem 5.3.2

Let R be the amount of red balls drawn; then, $\mathbb{P}(R = r) = \frac{\binom{5}{r}\binom{10}{7-r}}{\binom{15}{7}}$ for $0 \leq r \leq 5$. $\mathbb{P}(R \geq 3) = 1 - \mathbb{P}(R < 3) = 1 - (\mathbb{P}(R = 0) + \mathbb{P}(R = 1) + \mathbb{P}(R = 2)) = 1 - (0.01865 + 0.1632 + 0.3916) = 0.4266$.

Problem 5.4.3

Suppose X represents the amount of defects found in five bolts of cloth. Then, $X \sim Po(2)$. $\mathbb{P}(X \geq 6) = 1 - \mathbb{P}(X \leq 5) = 1 - \sum_{i=0}^{5} \mathbb{P}(X = 0) = 1 - e^{-2} \sum_{i=0}^{5} \frac{2^i}{i!} = 1 - 0.9834 = 0.0166$.

Problem 5.6.10

Suppose $X \sim N(\mu, 2^2)$. If $\bar{X}$ represents the average value of X from 25 samples, then $\bar{X} \sim N(\mu, \frac{4}{25})$. In this case, $\sigma = \frac{2}{5}$ and so $1 = \frac{5}{2}\sigma$. Using a lookup table, the probability of the sample mean lying within 2.5 standard deviations of μ is $\Phi(2.5) - \Phi(-2.5) = 0.9938 - 0.0062 = 0.9876$.

Problem 5.7.8

We are given that X_i has p.d.f. $f_i(x) = \beta_i e^{-\beta_i x}$, and c.d.f. $F_i(x) = \int_0^x f(t)\,\mathrm{d}t = 1 - e^{-\beta_i x}$. Suppose $Y = \min(X_1, X_2 \cdots X_k)$; then the c.d.f. of Y is

$$\begin{aligned} G(y) = \mathbb{P}(Y \leq y) &= 1 - \mathbb{P}(Y \geq y) \\ &= 1 - \mathbb{P}(\bigcup_{i=0}^{k} X_i \geq y) = 1 - \prod_{i=0}^{k} \mathbb{P}(X_i \geq y) \\ &= 1 - \prod_{i=0}^{k} (1 - \mathbb{P}(X_i \leq y)) = 1 - \prod_{i=0}^{k} (1 - F_i(y)) \\ &= 1 - \prod_{i=0}^{k} (e^{-\beta_i y}) = 1 - e^{-y \sum_{i=0}^{k} \beta_i} \end{aligned}$$

Let $\lambda = \sum_{i=0}^{k} \beta_i$. By definition, $f(y) = \frac{\mathrm{d}}{\mathrm{d}y} G(y) = \frac{\mathrm{d}}{\mathrm{d}y}(1 - e^{-\lambda y}) = \lambda e^{-\lambda y}$.