CS 2800 HW #10

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Key: as. = assumption, abs. = reductio ad absurdum, e. m. = excluded middle.

Problem 1

(a) Suppose $\neg P$. Q can either evaluate to T or F:

$\neg P$	P	Q	$P \land Q$	$\neg(P \land Q)$
T	F	T	F	T
T	F	F	F	T

Therefore, $\neg P \models \neg(P \land Q)$.

(b)

$$\dfrac{\dfrac{}{\vdash (P \land Q) \lor \neg(P \land Q)}\ \mathit{e.\ m.} \qquad \dfrac{\dfrac{\dfrac{}{P \land Q \vdash P \land Q}\ \mathit{as.}}{P \land Q \vdash P}\ \land\ \mathit{elim} \qquad \dfrac{}{\neg P \vdash \neg P}\ \mathit{as.}}{\neg P, P \land Q \vdash \neg(P \land Q)}\ \mathit{abs.} \qquad \dfrac{}{\neg(P \land Q) \vdash \neg(P \land Q)}\ \mathit{as.}}{\neg P \vdash \neg(P \land Q)}\ \lor\ \mathit{elim}$$

Problem 2

Inductively assume $P(\phi)$ and $P(\psi)$. $I \models \phi \to \psi$ means that eval$(\phi \to \psi, I) = T$, that is, $I \models \neg\phi$ or $I \models \psi$. If $I \models \neg\phi$ then using our assumption, $I \vdash \neg\phi$, and so $I \vdash \phi \to \psi$. This can be seen from the following proof tree:

$$\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi}\ \mathit{as.} \qquad \dfrac{}{\neg\phi \vdash \neg phi}\ \mathit{as.}}{\neg\phi, \phi \vdash \psi}\ \mathit{abs.}}{\neg\phi \vdash \phi \to \psi}\ \to\ \mathit{intro}$$

If $I \models \psi$, then using our assumption, $I \vdash \psi$, and once again we have $I \vdash \phi \to \psi$:

$$\dfrac{\dfrac{}{\phi, \psi \vdash \psi}\ \mathit{as.}}{\psi \vdash \phi \to \psi}\ \to\ \mathit{intro}$$

Conversely, if $I \not\models \phi \to \psi$, then $I \models \phi \land \neg\psi$. Using the inductive assumption, we have $I \vdash \phi$ and $I \vdash \neg\psi$. Therefore:

$$\dfrac{\dfrac{}{\vdash (\phi \to \psi) \lor \neg(\phi \to \psi)}\ \mathit{e.\ m.} \qquad \dfrac{\dfrac{\dfrac{}{\phi \vdash \phi}\ \mathit{as.} \qquad \dfrac{}{\phi \to \psi \vdash \phi \to \psi}\ \mathit{as.}}{\phi, \phi \to \psi \vdash \psi}\ \to\ \mathit{elim} \qquad \dfrac{}{\neg\psi \vdash \neg\psi}\ \mathit{as.}}{\phi, \neg\psi, \phi \to \psi \vdash \neg(\phi \to \psi)}\ \mathit{abs.} \qquad \dfrac{}{\neg(\phi \to \psi) \vdash \neg(\phi \to \psi)}\ \mathit{as.}}{\phi, \neg\psi \vdash \neg(\phi \to \psi)}\ \lor\ \mathit{el}$$