MATH 3110 HOMEWORK #3

KIRILL CHERNYSHOV

Problem 3.1.1 (e)

Proof. We begin by examining the difference $|\sqrt{n^2+1}-n-0| = |\sqrt{n^2+1}-n|$. Since for $n>0$, $\sqrt{n^2}=n$ and $n^2+1>n^2$, and since $\sqrt{n}$ increases as n increases, we have $\sqrt{n^2+1}>n$ for $n>0$. Therefore, $|\sqrt{n^2+1}-n| = \sqrt{n^2+1}-n$, since the quantity is always positive. Now, pick $\epsilon>0$ and solve the inequality:

$$\begin{aligned}\sqrt{n^1+1}-n &< \epsilon \\ \sqrt{n^2+1} &< \epsilon+n \\ n^2+1 &< \epsilon^2+n^2+\epsilon n \\ \epsilon n &> 1-\epsilon^2 \\ n &> \frac{1}{\epsilon}-\epsilon\end{aligned}$$

Therefore, given $N>\frac{1}{\epsilon}-\epsilon$, for any ϵ, for all $n\geq N$, $|\sqrt{n^2+1}-n|<\epsilon$. $\square$

Problem 3.1.2 Prove that if $\{a_n\}$ is a non-negative sequence, then $\lim_{n\to\infty} a_n = 0 \implies \lim_{n\to\infty}\sqrt{a_n} = 0$.

Proof. Pick any $\epsilon>0$. By definition, $\lim_{n\to\infty} a_n = 0$ means that, for any $\epsilon'>0$ there exists N such that for all $n\geq N$, $|a_n|<\epsilon'$, equivalent to $a_n<\epsilon'$ since we are also given that $a_n\geq 0\ \forall\ n$. $\sqrt{n}$ increases as n increases, so this means that $\sqrt{a_n}<\sqrt{\epsilon'}$. Therefore, we can pick $\epsilon'=\epsilon^2$, and using the same N, we know that $\sqrt{a_n}<\sqrt{\epsilon'}=\epsilon$. Therefore, $\lim_{n\to\infty}\sqrt{a_n}=0$. $\square$

Problem 3.2.3

(a) Show that the sequence

$$a_n = \sum_{i=1}^{n}\frac{1}{n+i}$$

has a limit.

Proof. To show that this sequence has a limit, it suffices to show that it is both increasing and has an upper bound. First, examine the difference between two consecutive terms:

$$\begin{aligned}
a_{n+1} - a_n &= \sum_{i=1}^{n+1} \frac{1}{n+1+i} - \sum_{i=1}^{n} \frac{1}{n+i} \\
&= \sum_{i=2}^{n+2} \frac{1}{n+i} - \sum_{i=1}^{n} \frac{1}{n+i} \\
&= \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} \\
&> \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{n+1} - \frac{1}{n+1} = 0
\end{aligned}$$

Since the difference is strictly greater than 0, the sequence is strictly increasing. Also, note that $\sum_{i=1}^{n} \frac{1}{n+i} < \sum_{i=1}^{n} \frac{1}{n} = \frac{n}{n} = 1$, that is, the sequence is bounded above by 1. Thus, the sequence must have a limit. $\square$

The limit is actually $\ln 2$, since the sequence is actually the riemann sum for $\int_0^1 \frac{1}{1+x}\, \mathrm{d}x$

(b) In the last step, the $K - \epsilon$ principle cannot be applied, since it works if and only if K is a constant, and n is not a constant.

Problem 4.3.1 The generic recursive formula for Newton's method is

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(x_n)}$$

For $f(x) = x^2 - 3$ we have $f'(x) = 2x$, and therefore the recursive formula is $a_{n+1} = a_n - \frac{a_n^2 - 3}{2a_n}$, where $a_n \neq 0$.

Problem 3-1 (a) Suppose that $b_n = \frac{\sum_{i=1}^{n} a_i}{n}$ for some sequence $\{a_n\}$. Show that if $\lim_{n\to\infty} a_n = 0$

then $\lim_{n\to\infty} b_n = 0$.

Proof. By definition, given $\epsilon > 0$, there exists N such that for all $n \geq N$, $|a_n| < \epsilon$. Now examine the value of $|b_n|$:

$$\begin{aligned}
|b_n| &= \left|\frac{\sum_{i=1}^{n} a_i}{n}\right| \\
&= \left|\frac{\sum_{i=1}^{\lfloor N \rfloor} a_i}{n} + \frac{\sum_{i=\lfloor N \rfloor+1}^{n} a_i}{n}\right| \\
&< \frac{\sum_{i=1}^{\lfloor N \rfloor} |a_i|}{n} + \frac{\sum_{i=\lfloor N \rfloor+1}^{n} \epsilon}{n} \\
&= \frac{\sum_{i=1}^{\lfloor N \rfloor} |a_i|}{n} + \frac{(n - \lfloor N \rfloor)\epsilon}{n} \\
&< \frac{\sum_{i=1}^{\lfloor N \rfloor} a_i}{n} + \frac{n\epsilon}{n} \\
&= \frac{\sum_{i=1}^{\lfloor N \rfloor} a_i}{n} + \epsilon
\end{aligned}$$

Since a_n must be bounded, we know that there exists k such that for all n, $|a_n| \leq k$. This means that $\frac{\sum_{i=1}^{\lfloor N \rfloor} a_i}{n} \leq \frac{\sum_{i=1}^{\lfloor N \rfloor} k}{n} = \frac{k \cdot \lfloor N \rfloor}{n}$. If we pick n such that $n \geq \max(N, \frac{k \cdot \lfloor N \rfloor}{\epsilon})$, then $|b_n| < \frac{\sum_{i=1}^{\lfloor N \rfloor} a_i}{n} + \epsilon \leq \epsilon + \epsilon = 2\epsilon$. Therefore, by the $K - \epsilon$ principle, $\lim_{n \to \infty} b_n = 0$. $\square$

Problem 3-4 Prove that a convergent sequence $\{a_n\}$ must be bounded.

Proof. Suppose $\lim_{n \to \infty} a_n = L$. Then, by definition, for any $\epsilon > 0$, there exists N such that for all $n \geq N$, $|a_n - L| < \epsilon$. For any such ϵ, we know that the sequence $\{a_n\}$ where $n \geq N$ must be bounded, since $|a_n - L| < \epsilon \implies L - \epsilon < a_n < L + \epsilon$. The rest of the sequence (the terms $a_0, a_1 \cdots a_{\lfloor N \rfloor}$) is a finite set of finite terms, therefore it must also be bounded by some constants P and M, such that for all $i : 0 < i \leq \lfloor N \rfloor$, $P \leq a_i \leq M$. Therefore, we can construct a lower and an upper bound on the entire sequence:

$$\min(P, L - \epsilon) \leq a_n \leq \max(M, L + \epsilon) \; \forall \; n$$

$\square$

For any $\epsilon > 0$, there exists N such that for all $n \geq N$, $|a_n - L| < \epsilon$.