MATH 2220 SECTION 203
HW #7

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Problem 1

(a) $f(1,2) = \left(1 + \frac{1}{2}, \frac{1}{1} + 2\right) = (1.5, 3)$

(b) By definition, we know that

$$\mathbf{Df} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & -y^{-2} \\ -x^{-2} & 1 \end{bmatrix}$$

$\det \mathbf{Df} = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{x^2 y^2}}$. This matrix is invertible (the determinant is nonzero) at all points (x, y) where $x^2 \neq \frac{1}{y^2}$.

(c) By the inverse function theorem, $\mathbf{Df^{-1}(f(a,b)) = (Df(a,b))^{-1}}$. The linear approximation for $f^{-1}(x,y)$ near a point (a,b) is $L(x,y) = f^{-1}(a,b) + \mathbf{Df^{-1}(a,b)} \begin{bmatrix} (x-a) \\ (y-b) \end{bmatrix} = \mathbf{f^{-1}(a,b)} + \mathbf{(Df(f^{-1}(a,b)))} \begin{bmatrix} (x-a) \\ (y-b) \end{bmatrix}$. In this case, $(a,b) = (1.5, 3)$ and $f^{-1}(a,b) = (1,2)$, so:

$$\mathbf{Df^{-1}(1.5,3) = (D(1,2))^{-1}} = \left(\begin{bmatrix} 1 & -0.25 \\ -1 & 1 \end{bmatrix} \right)^{\mathbf{-1}} = \frac{\mathbf{1}}{\mathbf{0.75}} \begin{bmatrix} 1 & 0.25 \\ 1 & 1 \end{bmatrix}$$

So $L(1.49, 2.9) = f^{-1}(1.5,3) + \mathbf{Df^{-1}(f(1.5,3))} \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix} = \mathbf{(1,2) + (0.04667, 0.1467) = (1.04667, 2.1467)}$.

Problem 2

(a) No. This would be the case if the determinant of the derivative matrix of the function with respect to the vector consisting solely of its third variable, at $(1,2,-3)$. However, since $\nabla f(1,2,-3) = (0,0,0)$, this derivative matrix is just (0), so its determinant is zero.

(b) $\nabla f(1,0,3) = (0,1,0)$, so the derivative matrix of f with respect to the vector consisting solely of its second variable if (1). This matrix has a nonzero determinant, so it is possible to find a function $g(x,z)$ such that around $(1,0,3)$, $f(x,y,z) = f(x, g(x,z), z)$.

(c) By similar reasoning, at $(1,2,3)$ the derivative matrix of f with regards to the vector consisting solely of y is just (1), so a similar $g(x,z)$ also exists in this case.

(d)

(e) In this case, we cannot guarantee the existence of such function $g(y,z)$ such that $f(x,y,z) = f(g(y,z), y, z)$, since at $(1,0,3)$ the derivative of f with respect to x is 0. This means that the relevant matrix is not invertible.

Problem 3

(a) $f(2,-2,2) = (2-2+2, -4+4-4) = (2,-4)$.

(b)

$$\mathbf{Df} = \begin{bmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \end{bmatrix}$$

$$\mathbf{Df(2,-2,2)} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

(c) We want to check the invertibility of the derivative matrix of f with respect to the vector (v,w). This matrix is $\begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix}$; its determinant is $-4 \neq 0$, so we know it is invertible. Therefore, the implicit function theorem guarantees the existence of a function $g(v,w)$ such that near $(2,-2,2)$ we have $f(u,v,w) = f(g(v,w),v,w)$. Therefore, we can solve for v and w in terms of u.

(d) By similar reasoning, this time we exclude w from the vector, so the relevant matrix is $\begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$. This matrix has determinant 4 so once again we have g. Therefore we can solve for u and v in terms of w near $(2,-2,2)$.

Problem 4 (a) The vector (x,y) must be on the unit circle, since $x^2+y^2 = 1$. (u,v) is tangent to the unit circle, since (u,v) is orthogonal to (x,y) as their dot product is 0. Therefore, T is the set of vectors tangent to the unit circle.

(b) $f(x,y,u,v) = (x^2+y^2, ux+vy)$. By definition,

$$\mathbf{Df} = \begin{bmatrix} 2x & 2y & 0 & 0 \\ u & v & x & y \end{bmatrix}$$

(c) The set of these points is the set of points where the matrix of derivatives of f with respect to x and u is invertible. This matrix is:

$$\mathbf{M} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ u & x \end{bmatrix}$$

This (and any) matrix is invertible iff its derivative is nonzero. $\det M = 2x^2$, so wherever $x \neq 0$, this matrix is invertible, and therefore by the implicit function theorem there exists a function g such that $g(x,u) = (y,v)$.

(d) We have $x^2+y^2 = 1 \implies y = \pm\sqrt{1-x^2}$, and $ux+vy = 0 \implies v = -\frac{ux}{y} = \mp\frac{ux}{\sqrt{1-x^2}}$.

Problem 5

(a) It is given that $f(c(x)) = k$ for all $x \in \mathbb{R}$, where $k \in \mathbb{R}$ is a constant. This means that for all values of $\mathbf{v} = \mathbf{c}(\mathbf{x})$, $f(\mathbf{v})$ is constant, which means by definition that the image of c is contained in a level set of f.

(b) By the chain rule, $\nabla(f \circ c)(t) = \nabla f(c(t)) \cdot \nabla c(t)$. Since $f \circ c$ is constant, $\nabla(f \circ c)(t) = 0$, so $\nabla f(c(t)) \cdot \nabla c(t) = 0$.

(c) In my proof of part (b), I assumed nothing about the dimension of the vectors. The proof holds for any such $c : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ regardless of the value of n.

(d) Since the image of c is contained in a level set of f, and $\nabla f(c(t))$ is the gradient of f at $c(t)$, which is in a level set of f, geometrically this means that the gradient of a function is always perpendicular to the direction of its level set (since their dot product is 0).

(e)