MATH 3110 HOMEWORK #7

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Problem 11.1.4

Proof. We are given that e^x is continuous at $x = 0$, that is, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|e^y - 1| < \epsilon$ if $|y| < \delta$. Since $|e^x - e^y| = e^y|e^{x-y} - 1|$, pick $\epsilon' = \frac{\epsilon}{e^y} > 0$. By continuity at 0, if $|x - y| < \delta$ then $|e^{x-y} - 1| < \epsilon'$ for some δ. Therefore, $|e^x - e^y| < e^y\epsilon' = e^y \frac{\epsilon}{e^y} = \epsilon$, and so e^x is continuous everywhere. □

Problem 11.2.2

Proof. $\lim_{x\to 0^+} f(x) = L$ means that for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $0 < x < \delta$. This implies that $|f(-x) - L| < \epsilon$ if $0 < -x < \delta$. Also, since $f(x)$ is even, $|f(x) - L| = |f(-x) - L|$, and so $|f(x) - L| < \epsilon$ if $0 < -x < \delta$, that is, if $-\delta < x < 0$. So, $\lim_{x\to 0^-} f(x) = L$, and therefore $\lim_{x\to 0} f(x) = L$. □

Problem 11.4.1

Proof. Pick $\epsilon > 0$, and consider the value of $|f(x)-f(0)|$. By algebraic manipulation, $|f(x)-f(0)| = |f(x)| = \left|\sqrt{x}\cos\left(\frac{1}{x}\right)\right| \leq |\sqrt{x}| = \sqrt{x}$. Set $\delta = \epsilon^2$: if $x < \delta$ then $\sqrt{x} < \epsilon$ (negative value of x need not be considered, as the function is assumed to have a real value). □

Problem 11.4.4

Proof. By 2.4.1, we can write these two functions as follows:

$$\max(f, g) = \frac{f + g + |f - g|}{2} = \frac{f + g}{2} + \frac{|f - g|}{2}$$
$$\min(f, g) = \frac{f + g - |f - g|}{2} = \frac{f + g}{2} - \frac{|f - g|}{2}$$

Since both f and g are continuous, $f + g$ and $\frac{f+g}{2}$ are also continuous. Also, by Q11.4.3, since $f - g$ is also continuous (due similarly to the continuity of f and g), so is $|f - g|$ and in turn $\frac{|f-g|}{2}$. Thus, both $\max(f, g)$ and $\min(f, g)$, being the sum and difference of the functions $\frac{f+g}{2}$ and $\frac{|f-g|}{2}$, respectively, are continuous. □

Problem 11-1

(a) Lemma: $f\left(\sum_{i=1}^{k} a_i\right) = \sum_{i=1}^{k} f(a_i)$.

Proof. By definition, this is true if $k = 2$. Now, suppose this is the case for $k = K$, that is, $f\left(\sum_{i=1}^{K} a_i\right) = \sum_{i=1}^{K} f(a_i)$. Proceeding to add another term, we get:

$$\begin{aligned} f\left(\sum_{i=1}^{K+1} a_i\right) &= f\left(a_{K+1} + \sum_{i=1}^{K} a_i\right) \\ &= f(a_{K+1}) + f\left(\sum_{i=1}^{K} a_i\right) \\ &= f(a_{K+1}) + \sum_{i=1}^{K} f(a_i) \\ &= \sum_{i=1}^{K+1} f(a_i) \end{aligned}$$

Thus, by induction, the statement is true for any value of k. □

Suppose $f(1) = C$. The first case, when $x = n \in \mathbb{Z}, n \neq 0$:

Proof. Since $n \in \mathbb{Z}$, $n = \sum_{i=1}^{n} 1$. Therefore, $f(n) = f\left(\sum_{i=1}^{n} 1\right) = \sum_{i=1}^{n} f(1) = \sum_{i=1}^{n} C = Cn = Cx$. □

The second case, when $x = \frac{1}{n}$:

Proof.

$$\begin{aligned} C = f(1) &= f(n \cdot \frac{1}{n}) \\ &= f\left(\sum_{k=1}^{n} \frac{1}{n}\right) \\ &= \sum_{k=1}^{n} f\left(\frac{1}{n}\right) \\ &= nf\left(\frac{1}{n}\right) \end{aligned}$$

Therefore, $Cx = \frac{C}{n} = f\left(\frac{1}{n}\right) = f(x)$. □

The third case, when $x = \frac{m}{n}, m \in \mathbb{Z}$:

Proof. Since $m \in \mathbb{Z}$, $m = \sum_{i=1}^{m} 1$ and so $\frac{m}{n} = \frac{\sum_{i=1}^{m} 1}{n} = \sum_{i=1}^{m} \frac{1}{n}$. Therefore, $f(x) = f\left(\sum_{i=1}^{m} \frac{1}{n}\right) = \sum_{i=1}^{m} f\left(\frac{1}{n}\right) = \sum_{i=1}^{m} \frac{C}{n} = C\frac{m}{n} = Cx$. □

(b) We now know that $f(x) = Cx$ if $x \in \mathbb{Q}$. Now, we can extend this to $\mathbb{R}$.

Proof. By the completeness property, for any $x \in \mathbb{R}$, there exists a sequence $\{a_n\}$ such that $a_n \to x$ and $a_n \in \mathbb{Q}\ \forall\, n$. Using this sequence, we can write $f(x) = f(a_n)$. We are given that f is continuous everywhere, and so by the sequential continuity theorem, $\lim_{n\to\infty} f(a_n) = f(x)$. But, using the previous proofs, $f(a_n) = Ca_n$, so $f(x) = \lim_{n\to\infty} Ca_n = C \lim_{n\to\infty} a_n = Cx$. □