Inverses

Any square matrix $\vec{A}_{n\times n}$ is **invertible** if $\exists \vec{C}_{n\times n} \ni \vec{C}\vec{A} = \vec{A}\vec{C} = \vec{I}_n$. If such a matrix $\vec{C}$ does exist, it is called the **inverse** of $\vec{A}$, and can be expressed as $\vec{A}^{-1}$.

> Theorem: Given a square matrix $\vec{A}_{n\times n}$, if $\vec{A}^{-1}$ exists it is unique.

Proof. Assume $\vec{B}$ and $\vec{C}$ are both inverses of $\vec{A}$. Then, $\vec{B} = \vec{B}\vec{I} = \vec{B}(\vec{A}\vec{C}) = (\vec{B}\vec{A})\vec{C} = \vec{I}\vec{C} = \vec{C}$. Therefore, $\vec{A}$ cannot have two distinct inverses. $\square$

It is trivial to show that for any 2×2 matrix $\vec{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\forall\, a, b, c, d \ni ad \neq bc$, $\vec{M}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. The number $ad - bc$ is called the **determinant** of $\vec{M}$, written as $det(\vec{M})$ or $|\vec{M}|$.

> Theorem: Given $\vec{A}_{n\times n}$, if $\vec{A}^{-1}$ exists then the system $\vec{A}\vec{x} = \vec{b}$ always has a unique solution.

Proof. $\vec{A}\vec{x} = \vec{b} \implies \vec{A}^{-1}\vec{A}\vec{x} = \vec{A}^{-1}\vec{b} \implies \vec{I}\vec{x} = \vec{A}^{-1}\vec{b} \implies \vec{x} = \vec{A}^{-1}\vec{b}$ $\square$

Properties of the inverse

1. If $\vec{A}$ is invertible, then $\vec{A}^{-1}$ is also invertible. In fact, $(\vec{A}^{-1})^{-1} = \vec{A}$.
2. If $\vec{A}_{n\times n}$, $\vec{B}_{n\times n}$ are invertible, then $\vec{A}\vec{B}$ is also invertible. In fact, $(\vec{A}\vec{B})^{-1} = \vec{B}^{-1}\vec{A}^{-1}$.
3. If $\vec{A}$ is invertible, then $\vec{A}^T$ is also invertible: $(\vec{A}^T)^{-1} = (\vec{A}^{-1})^T$.

Proof. For the first property, if $\vec{A}$ is invertible, $\vec{A}\vec{A}^{-1} = \vec{A}^{-1}\vec{A} = \vec{I}$, so $\vec{A}^{-1}$ is invertible and $\vec{A}$ is its inverse. Next, $(\vec{A}\vec{B})(\vec{B}^{-1}\vec{A}^{-1}) = \vec{A}(\vec{B}\vec{B}^{-1})\vec{A}^{-1} = \vec{A}\vec{I}\vec{A}^{-1} = \vec{A}\vec{A}^{-1} = \vec{I}$. By a similar process, $(\vec{B}^{-1}\vec{A}^{-1})(\vec{A}\vec{B}) = \vec{I}$. Finally, to prove the third property [put this in] $\square$

If $\vec{A}_{n\times n}$ is invertible, then:

1. $\vec{A}\vec{x} = \vec{b}$ has unique solutions for all $\vec{b}$.

2. $\overrightarrow{A}$ has pivots in all rows.
3. The reduced row echelon form of $\overrightarrow{A}$ is $\overrightarrow{I}_n$.

Elementary row operations

In fact, elementary row operations can be modelled by matrix multiplication. This can be illustrated by using

$$\overrightarrow{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$$

$$\overrightarrow{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overrightarrow{E}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given $\overrightarrow{A}_{3\times n}$, the product $\overrightarrow{E}_1\overrightarrow{A}$ is equivalent to replacing R_3 with $R_3 + kR_1$. $\overrightarrow{E}_2\overrightarrow{A}$ multiplies R_2 by a factor k, and finally $\overrightarrow{E}_3\overrightarrow{A}$ interchanges R_2 and R_1. We know that elementary row operations are reversible, therefore $\overrightarrow{E}_1$, $\overrightarrow{E}_2$ and $\overrightarrow{E}_3$ are all invertible. Now assume that there is some sequence of elementary row operations $\left\{\overrightarrow{E}_1, \overrightarrow{E}_2, \overrightarrow{E}_3, \cdots, \overrightarrow{E}_n\right\}$ that reduces $\overrightarrow{A}$ to $\overrightarrow{I}$, the identity matrix. This can be expressed as:

$$\left(\prod_{i=0}^{n} \overrightarrow{E}_i\right)\overrightarrow{A} = \overrightarrow{I}$$

but we know that, by definition, $\overrightarrow{A}^{-1}\overrightarrow{A} = \overrightarrow{I}$. Therefore, we get the amazing inference that $\prod_{i=0}^{n} \overrightarrow{E}_i = \overrightarrow{A}^{-1}$! Also, wknow that, by the definition of the identity matrix $\overrightarrow{I}$, that $(\prod_{i=0}^{n} \overrightarrow{E}_i)\overrightarrow{I} = \prod_{i=0}^{n} \overrightarrow{E}_i = \overrightarrow{A}^{-1}$. Therefore, if we find out the set of row operations needed to reduce $\overrightarrow{A}$ to $\overrightarrow{I}$, and apply them to $\overrightarrow{I}$, the result will be $\overrightarrow{A}^{-1}$! Applying this fact, we get a straightforward way of finding out, for any matrix $\overrightarrow{A}$, whether it is invertible, and if it is, finding $\overrightarrow{A}^{-1}$: construct the augmented matrix $\begin{bmatrix}\overrightarrow{A} & \overrightarrow{I}\end{bmatrix}$, and row reduce the left half that is $\overrightarrow{A}$. If the end result is $\overrightarrow{I}$, then the right half of the matrix will have transformed into $\overrightarrow{A}^{-1}$; if the result is different then $\overrightarrow{A}$ is not invertible.

To see this from another point of view, assume that for a matrix $\overrightarrow{A}_{n\times n}$ there exists $\overrightarrow{A}^{-1}$, where

$$\vec{A}^{-1} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

therefore, $\vec{A}\vec{A}^{-1} = \vec{I}$ can be written as

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{A}\vec{x}_1 & \vec{A}\vec{x}_2 & \cdots & \vec{A}\vec{x}_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

So, to find $\vec{A}^{-1}$ we need to solve the linear systems $\vec{A}\vec{x}_i = \vec{e}_i \; \forall \; 0 < i \leq n$. This requires creating a set of augmented matrices $\left\{ \begin{bmatrix} \vec{A} & \vec{e}_1 \end{bmatrix}, \begin{bmatrix} \vec{A} & \vec{e}_2 \end{bmatrix}, \cdots, \begin{bmatrix} \vec{A} & \vec{e}_n \end{bmatrix} \right\}$. This can be solved in one computation, simply by row reducing the following matrix:

$$\begin{bmatrix} \vec{A} & \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix} = \begin{bmatrix} \vec{A} & \vec{I} \end{bmatrix}$$

Is a matrix invertible?

A square matrix $\vec{A}_{n\times n}$ can be invertible (non-singular) or non-invertible (singular). For any $\vec{A}_{n\times n}$, **all** of the following statements are equivalent, that is, if one is true then all of them are true:

1. $\vec{A}$ is invertible (non-singular).
2. $\vec{A}$ is row equivalent to $\vec{I}$.
3. $\vec{A}$ has n pivot positions.
4. $\vec{A}\vec{x} = \vec{0}$ has only the trivial solution.
5. The columns of $\vec{A}$ are linearly independent.
6. $\vec{x} \mapsto \vec{A}\vec{x}$ is injective.
7. $\vec{A}\vec{x} = \vec{b}$ has solutions $\forall \; \vec{b} \in \mathbb{R}^n$.
8. $\vec{x} \mapsto \vec{A}\vec{x}$ is surjective.
9. The span of the columns of $\vec{A}$ is $\mathbb{R}^n$.
10. $\exists \vec{C}_{n\times n} \ni \vec{C}\vec{A} = \vec{I}$.
11. $\exists \vec{D}_{n\times n} \ni \vec{A}\vec{D} = \vec{I}$.
12. $\vec{A}^T$ is invertible.

The concept of matrix inverses can be interpreted in the context of linear maps. Given a linear map $T : \mathbb{R}^m \to \mathbb{R}^n, \vec{x} \mapsto \vec{A}\vec{x}$, iff there exists a map $T : \mathbb{R}^n \to \mathbb{R}^m \ni S \circ T = \vec{x} \; \forall \; \vec{x} \in \mathbb{R}^n, T \circ S = \vec{x} \; \forall \; \vec{x} \in \mathbb{R}^m$, then $\vec{A}$ is invertible. This is easy to show, as the inverse map to $T : \vec{x} \mapsto \vec{A}\vec{x}$ is $S : \vec{x} \mapsto \vec{A}^{-1}\vec{x}$, which exists iff $\vec{A}$ is invertible.