Vectors

A matrix consisting of m rows and n columns looks like this:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A vector is a one-dimensional matrix, that looks like this:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

In this example, the vector is $m \times 1$, i.e. m items (1 column means it is a vector and not a matrix). Also, for the purpose of this course we assume that $a_i \in \mathbb{R} \; \forall \; i \in \mathbb{R}$. $\mathbb{R}^m$ represents the collection of all $m \times 1$ real vectors.

Rules

We say that

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

iff $a_i = b_i \; \forall \; 1 \leq i \leq m$. Addition and multiplication of vectors works like this:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{bmatrix} \tag{1}$$

$$c \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_m \end{bmatrix} \tag{2}$$

For example:

$$2\begin{bmatrix}1\\1\end{bmatrix} - 3\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$$

Properties

$\forall\ \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^m$ *and* $c, d \in \mathbb{R}$:

1. $\vec{u} + \vec{v} \in \mathbb{R}^m$
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
3. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
4. $c\vec{u} \in \mathbb{R}$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1\vec{u} = \vec{u}$

Assume $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_p} \in \mathbb{R}^m$ and $c_1, c_2, \cdots, c_p \in \mathbb{R}$. Then

$$s = \sum_{i=1}^{p} c_i\vec{v_i} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_p\vec{v_p}$$

is called a linear combination of $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_p}$ with weights $c_1, c_2, \cdots, c_p$.

For example, is $\vec{b} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$ a linear combination of $u = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $v = \begin{bmatrix}1\\0\\-1\end{bmatrix}$?

Answer: does there exist $x, y \in R$ such that $x\vec{u} + y\vec{v} = b$? That is:

$$x\begin{bmatrix}1\\-1\\0\end{bmatrix} + y\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix} \implies \begin{bmatrix}x+y\\-x\\-y\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix} \implies \begin{cases}x + y = 0\\-x = 1\\-y = -1\end{cases}$$

This system is clearly consistent, so $\vec{b}$ *is* a linear combination of $\vec{u}$ and $\vec{v}$.

Fact: $\vec{b} \in R^m$ is a linear combination of $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_p} \in \mathbb{R}^m$ if

$$\begin{bmatrix}\vec{v_1} & \vec{v_2} & \cdots & \vec{v_p} & \vec{b}\end{bmatrix}$$

has a solution (by row reduction).