Function compositions

We have proven that a multivariate function is continuous iff all of its component functions are continuous. However, some cases can be more complicated. For example, consider $f(x, y) = sin(x + y)$. We know that $x + y$ and sin are both continuous functions.

Theorem

Define two continuous functions $F : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m, G : E \subseteq \mathbb{R}^m \mapsto \mathbb{R}^k, F \subseteq E$. Then, $G \circ F : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}^k$ is continuous.

Proof. Write $G(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}) \cdot g_k(\vec{x}))$. Is suffices to show that $g_i(F(\vec{x}))$ is continuous for all i.

Fix $\epsilon > 0, \vec{x} \in D$. Let $\vec{y} = F(\vec{x})$. We know that $\exists \delta > 0$ such that if $\left\|\vec{h}\right\| < \delta$ then $|g_i(\vec{y} + \vec{h}) - g_i(\vec{y})| < \epsilon$, since we know that g_i is continuous for all i. We also know that F is continuous, so $\exists \eta > 0$ such that if $\|h\| < \eta$ then $\left\|F(\vec{x} + \vec{h}) - F(\vec{x})\right\| < \delta$.

This η is what we want: we want to show that if $\|h\|' < \eta$ then $|g_i(F(\vec{x} + \vec{h}')) - g_i(F(\vec{x}))| < \epsilon$. Let $\vec{j} = F(\vec{x} + \vec{h}') - F(\vec{x})$. We know that $\left\|\vec{j}\right\| < \delta$.

$\left\|g_i(F(\vec{x} + \vec{h}')) - g_i(F(\vec{x}))\right\| = \left\|g_i(F(\vec{x}) + \vec{j}) - g_i(F(\vec{x}))\right\| < \epsilon$, since $\|J\| < \delta$. $\square$

Point-set topology

An *open ball* around $\vec{x} \in \mathbb{R}^n$ is defined as $B_\epsilon(\vec{x}) = \{\vec{y} \in \mathbb{R}^n | \, \|\vec{x} - \vec{y}\| < \epsilon\}$.

A subset $D \subseteq \mathbb{R}^n$ is *open* iff $\forall \, \vec{x} \in D \exists \epsilon > 0$ such that $B_\epsilon(\vec{x}) \leq D$.

For example: in $\mathbb{R}^1$, (a, b) is open.

Proof. If $x \in (a, b)$ then let $\epsilon = min(x - a, b - x)$. Then, $B_\epsilon(\vec{x}) = (x - \epsilon, x + \epsilon) \subseteq (a, b)$. $\square$

Another example: in $mathbbR^2$, $D_1 = \{(x, 0) | a < x < b\}$ is *not* open.

Proof. $\forall \, \epsilon > 0$, the point (x, ϵ) is not in D_1, so $B_\epsilon(\vec{x}) \nsubseteq D \, \forall \, \epsilon > 0$. $\square$

We also set the following definition: a point $\vec{x} \in \mathbb{R}^n$ is on the *boundary* of D iff $\forall\, \epsilon > 0$, $B_\epsilon(\vec{x}) \cap D \neq \emptyset$ *and* $B_\epsilon(\vec{x}) \cup D \neq \emptyset$.

For example, b is the boundary of (a, b): $B_\epsilon(\vec{x}) = (b - \epsilon, b + \epsilon)$.

Definition: a set is *closed* iff it contains all of its boundary points. For instance, we can take the *closure* of the ball around $\vec{x}$ $\bar{B_\epsilon}(\vec{x}) = \{\vec{y} \in \mathbb{R}^n |\, \|\vec{x} - \vec{y}\| \leq \epsilon\}$.