CONSTRAINED EXTREMA AND LAGRANGE MULTIPLIERS

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Let f and g be C^1 functions $E \subseteq \mathbb{R}^3 \to \mathbb{R}$. Let S be the level set $g(x, y, z) = c$. Let $\mathbf{p} \in \mathbf{S}$ be a vector such that $\nabla g(\mathbf{p}) \neq \mathbf{0}$ and $\mathbf{p}$ is a local extremum of f on S. Then, there exists a scalar λ such that $\nabla f(\mathbf{p}) = \lambda \nabla \mathbf{g}(\mathbf{p})$.

Proof. Assume that $\nabla g(\mathbf{p}) \neq \mathbf{0}$. This means there exists a co-ordinate in $\mathbf{p}$ that is nonzero. Without loss of generality we can assume that z is that co-ordinate. By the implicit function theorem, in an open set around $\mathbf{p}$ we can solve for z in terms of x and y. That is, there exists $\phi : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ such that $g(x, y, \phi(x, y)) = c$ and $\phi(p_1, p_2) = p_3$.

Define $h(x, y) = f(x, y, \phi(x, y))$. (p_1, p_2) is a local extremum of h, since $\mathbf{p}$ is an extremum of f. Therefore, $h_x(p_1, p_2) = 0$, which is equal to, using the chain rule, $\frac{\partial f}{\partial x}\mathbf{p} + \frac{\partial \mathbf{f}}{\partial \mathbf{z}} \cdot \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{p_1}, \mathbf{p_2})$. The same steps can be applied to h_y.

We know that $g(x, y, \phi(x, y)) = c$, so $g_x(x, y, \phi(x, y)) = \frac{\partial g}{\partial x}(\mathbf{p}) + \frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{p}) \cdot \frac{\partial \phi}{\partial \mathbf{x}}(\mathbf{p_1}, \mathbf{p_2}) = \mathbf{0}$, which in turn means that $\frac{\partial \phi}{\partial x}(p_1, p_2) = -\frac{\partial g}{\partial x}(\mathbf{p}) \cdot \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{p})\right)^{\mathbf{-1}}$.

Using the equation from before, we can plug in the value for $\frac{\partial \phi}{\partial x}(p_1, p_2)$, we have $\frac{\partial f}{\partial x}(\mathbf{p}) = \frac{\frac{\partial \mathbf{f}}{\partial \mathbf{z}}(\mathbf{p})}{\frac{\partial \mathbf{g}}{\partial \mathbf{z}}(\mathbf{p})} \cdot \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{p})$. This gives $\lambda = \frac{\frac{\partial f}{\partial z}(\mathbf{p})}{\frac{\partial g}{\partial z}(\mathbf{p})}$. $\square$

We can use this to find constrained extrema. For example, say we are given an ellipsoid in $\mathbb{R}^3$, and we want to find the box with the greatest volume that is inscribed in the ellipsoid. This is an example of constrained optimisation.

Let $3x^2 + 5y^2 + 7z^2 = 1$ be the level set that describes the ellipsoid. If one vertex of the box is (a, b, c) then all the other vertices will be of the form $(\pm a, \pm b, \pm c)$. Therefore, the volume of this box will be $(2a)(2b)(2c) = 8abc$. Therefore we have $f(x, y, z) = 8xyz$ as the function that describes the volume, and $g(x, y, z) = 3x^2 + 5y^2 + 7z^2 = 1$ as the level set constraint.

$\nabla f = (8yz, 8xz, 8xy)$ and $\nabla g = (6x, 10y, 14z)$. We need to find $\mathbf{p}$ such that $\nabla f(\mathbf{p}) = \lambda_{\mathbf{p}} \nabla \mathbf{f}(\mathbf{p})$:

$$\begin{aligned}
&3x^2 + 5y^2 + 7z^2 = 1 \\
&8yz = 6\lambda_p x \\
&8xz = 10\lambda_p y \\
&8xy = 14\lambda_p z
\end{aligned}$$

These four equations are in four variables, so they can be solved for x, y, z and λ_p (we can assume $x, y, z > 0$, and therefore $\lambda_p \neq 0$).

Dividing equation 2 by equation 3, we get $\frac{y}{x} = \frac{3x}{5y} \implies 5y^2 = 3x^2 \implies y = \sqrt{\frac{3x}{5}}$. Dividing equation 3 by equation 4, we get $\frac{z}{y} = \frac{5y}{7z} \implies 5y^2 = 7z^2 \implies z = \sqrt{\frac{5y}{7}} = \sqrt{\frac{3x}{7}}$. We can put this into the first equation to get $3x^2 + 3x^2 + 3x^2 = 1 \implies x = \frac{1}{3}, y = \frac{1}{\sqrt{15}}, z = \frac{1}{\sqrt{21}}$, and the volume is $8xyz = \frac{8}{9\sqrt{35}}$.