Linear transformations and maps

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function/map/transformation. For $\vec{x} \in \mathbb{R}^n$, we call $T(\vec{x})$ the **image** of $\vec{x}$. The image of T is:

$$Im(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

Finally, the **kernel** of T is:

$$Ker(T) = \{\vec{x} \in \mathbb{R}^n : T(\vec{x}) = 0\}$$

For example, define $T : \mathbb{R}^n \to \mathbb{R}^m$ as $\vec{x} \mapsto \vec{A}\vec{x}$, where $\vec{A}$ is $m \times n$. This is a map. Therefore, $Im(T) = span\left\{columnsof\vec{A}\right\}$, and $Ker(T) = \left\{\vec{x} \in \mathbb{R}^n : \vec{A}\vec{x} = 0\right\}$; that is, $Ker(T)$ is the set of solutions to the homogenous system of equations!

Numerical example:

$$\vec{A} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$T : \mathbb{R}^2 \to \mathbb{R}^3$$

$$T : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \vec{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 6x_2 \end{bmatrix} \Rightarrow Ker(T) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \vec{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$Im(T) = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$$

The latter - the span of T - is the plane generated by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ from $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, i.e. the entirety of $\mathbb{R}^2$.

Linear maps

A map $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **linear** if it respects vector addition and scaling. That is:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \; \forall \; \vec{u}, \vec{v} \in \mathbb{R}^n$
2. $T(c\vec{u}) = cT(\vec{u}) \; \forall \; \vec{u} \in \mathbb{R}^n, c \in \mathbb{R}$

For example, assume, as before, that $T : \mathbb{R}^n \to \mathbb{R}^m$, $\vec{x} \mapsto \vec{A}\vec{x}$. T is linear, as $\vec{A}(\vec{x}+\vec{y}) = \vec{A}\vec{x}\vec{A}\vec{y}$, and $\vec{A}(c\vec{x}) = c(\vec{A}\vec{x})$, because matrix multiplication is distributive and associative.

1. $T(\vec{0}) = T(\vec{0}\,\vec{u}) = 0T(\vec{u}) = \vec{0}$
2. $T(\sum_{i=1}^{p} c_i \vec{u}_i) = \sum_{i=1}^{p} c_i T(\vec{u}_i)$ (result of the fact that this map is linear)

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then there exists a $m \times n$ matrix $\vec{A}$ such that $T(\vec{x}) = \vec{A}\vec{x}$

Proof. For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ we have $\vec{x} = \sum_{i=1}^{n} x_i \vec{e_i}$, where $\vec{e_i}$ is the i^{th} column of the identity matrix $\vec{I_n}$. Therefore:

$$T(\vec{x}) = T(\sum_{i=1}^{n} x_i \vec{e_i}) = \sum_{i=1}^{n} x_i T(\vec{e_i}) = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

which is in the form $\vec{A}\vec{x}$. $\square$

For example, for $T : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{x} \mapsto 3\vec{x}$, we can (1) show that it is linear and (2) find the matrix of T:

1. $3(\vec{x}+\vec{y}) = 3\vec{x} + 3\vec{y}$ and $3(c\vec{x}) = c(3\vec{x})$ (trivial)
2. Using the procedure outlined above, $\vec{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Surjective and injective maps

Given $T : \mathbb{R}^n \to \mathbb{R}^m$, we say that:

1. T is **surjective** or **onto** if $Im(T) = \mathbb{R}^m$.
2. T is **injective** or **one-to-one** if $T(\vec{x}) = T(\vec{y}) \implies \vec{x} = \vec{y}$.

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then T is injective iff $Ker(T) = \left\{\vec{0}\right\}$.

Proof. Assume T is injective. Then $\forall\ \vec{x} : T(\vec{x}) = \vec{0} \implies \vec{x} = \vec{0}$, since $\vec{0} = T(\vec{0})$. Therefore, $Ker(T) = \left\{\vec{0}\right\}$.

To prove the reverse, assume that $Ker(T) = \left\{\vec{0}\right\}$. Then, $T(\vec{x}) = T(\vec{y}) \implies \vec{x} = \vec{y}\ \forall\ \vec{x}, \vec{y}$, since $T(\vec{x}) = T(\vec{y}) \implies T(\vec{x}) - T(\vec{y}) = \vec{0} \implies T(\vec{x} - \vec{y}) = \vec{0} \implies \vec{x} - \vec{y} = \vec{0} \implies \vec{x} = \vec{y}$. Therefore, T is injective. $\square$

For example, consider $T(\vec{x}) = \vec{A}\vec{x}$. Given that

$$\vec{A} = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

1. Is T injective? No, since there is a free variable in $\vec{A}$, and so $Ker(T) \neq \vec{0}$.
2. Is T surjective? Yes, since there is a pivot element in every row, so $\vec{A}\vec{x} = \vec{b}$ has solutions regardless of the value of $\vec{b}$.

Theorem: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map with matrix $\vec{A}$, then, T is surjective iff the span of the columns of $\vec{A}$ is $\mathbb{R}^m$, and T is injective iff the columns of $\vec{A}$ are linearly independent.

Proof. To prove the first part, $Im(T) = \left\{\vec{A}\vec{x} : \vec{x} \in \mathbb{R}^n\right\} = span\left\{columnsof\vec{A}\right\}$. For the second part, if the columns of $\vec{A}$ are linearly independent then $\vec{A}\vec{x} = \vec{0}$ only has the trivial solution $\vec{x} = \vec{0}$, so T is injective. $\square$