MATH 3110 HOMEWORK #5

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Problem 6.4.1 Prove that every convergent sequence is a Cauchy sequence.

Proof. Suppose $\{a_n\}$ is convergent. Then, there exists L such that $\lim_{n\to\infty} a_n = L$, that is, for any $\epsilon > 0$, there exists N such that for all $n \geq N$, $|a_n - L| < \epsilon$. Fix ϵ, and pick any $m, n \geq N$. Then:

$$\begin{aligned}|a_n - a_m| &= |a_n - L - a_m + L| \\ &= |a_n - L - (a_m - L)| \qquad \leq |a_n - L| + |a_m - L| \leq 2\epsilon\end{aligned}$$

Therefore, by the $K - \epsilon$ principle, $\{a_n\}$ is a Cauchy sequence. □

Problem 6.5.4 Let S and T be non-empty subsets of $\mathbb{R}$, such that for all $s \in S, t \in T$, $s \leq t$. Prove that $\sup S \leq \inf T$.

Proof. Fix $s \in S$. Since for any $t \in T$, $s \leq t$, s is a lower bound for T. By definition, $\inf T$ is the greatest lower bound of T, and so $s \leq \inf T$. Since we made no assumptions about s, it follows that for any $s \in S$, $s \leq \inf T$, and therefore $\inf T$ is an upper bound for S. Since $\sup S$ is the least upper bound of S, $\sup S \leq \inf T$. □

Problem 6-1

(a) Show that $\{x_n\}$ is a Cauchy sequence.

Proof. We are given that $x_n = \frac{x_{n-1}+x_{n-2}}{2}$. Consider the absolute difference between two consecutive terms:

$$\begin{aligned}|x_n - x_{n-1}| &= \left|\frac{x_{n-1}+x_{n-2}}{2} - x_{n-1}\right| \\ &= \left|\frac{x_{n-2}-x_{n-1}}{2}\right| \\ &= \frac{1}{2}|x_{n-2} - x_{n-1}| = \frac{1}{2}|x_{n-1} - x_{n-2}| \\ &= \frac{1}{2}\frac{1}{2}|x_{n-2} - x_{n-3}| = \cdots = \frac{1}{2^{n-1}}|x_1 - x_0|\end{aligned}$$

Now, consider the difference between two arbitrary terms, x_m and x_n, where $m \geq n$:

$$\begin{aligned}
|x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)| \\
&\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&= \frac{1}{2^{m-1}}|x_1 - x_0| + \frac{1}{2^{m-2}}|x_1 - x_0| + \cdots + \frac{1}{2^n}|x_1 - x_0| \\
&= |x_1 - x_0| \sum_{i=n}^{m-1} \frac{1}{2^i} \\
&= |x_1 - x_0| \frac{1}{2^n} \frac{1 - \frac{1}{2^{m-n}}}{1 - \frac{1}{2}} \\
&= |x_1 - x_0| \frac{1}{2^{n-1}} \left(1 - \frac{1}{2^{m-n}}\right) \\
&= |x_1 - x_0| \left(\frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}\right) \\
&< \frac{1}{2^{n-1}}|x_1 - x_0|
\end{aligned}$$

Fix ϵ, and set N such that $\epsilon > \frac{1}{2^{N-1}}$, i.e. $N > 1 - \log_2 \epsilon$. Then, for all $m \ge n \ge N$, $|x_m - x_n| < \epsilon|x_1 - x_0|$. Since $|x_1 - x_0|$ is a constant, by the $K - \epsilon$ principle, $\{x_n\}$ is a Cauchy sequence. □

(b) The limit must exist, since $\{x_n\}$ is a Cauchy sequence. As seen in part (a), $x_n - x_{n-1} = -\frac{1}{2}|x_{n-1} - x_{n-2}| = \left(\frac{-1}{2}\right)^{n-1}(x_1 - x_0) = \left(\frac{-1}{2}\right)^{n-1}(b - a)$. Now, consider a single term of the sequence:

$$\begin{aligned}
x_n &= \sum_{i=1}^{n}(x_i - x_{i-1}) + x_0 \\
&= \sum_{i=1}^{n}\left(\frac{-1}{2}\right)^{i-1}(b - a) + a \\
&= (b - a)\sum_{i=1}^{n}\left(\frac{-1}{2}\right)^{i-1} + a
\end{aligned}$$

Therefore, we can calculate the limit:

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left[(b - a)\sum_{i=1}^{n}\left(\frac{-1}{2}\right)^{i-1} + a\right] \\
&= \lim_{n\to\infty}\left[(b - a)\sum_{i=1}^{n}\left(\frac{-1}{2}\right)^{i-1}\right] + \lim_{n\to\infty}[a] \\
&= \lim_{n\to\infty}[(b - a)] \cdot \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{-1}{2}\right)^{i-1}\right] + \lim_{n\to\infty}[a] \\
&= (b - a)\left(\frac{1}{1 + \frac{1}{2}}\right) + a = \frac{2}{3}(b - a) + a
\end{aligned}$$

Problem 6-2

(a) Let $S \subseteq \mathbb{R}$ be a bounded non-empty set, and let $\overline{m} = \sup S$. Prove that there exists a sequence $\{a_n\}$ such that for all n, $a_n \in S$, and $a_n \to \overline{m}$.

Proof. Pick any sequence $\{b_n\}$ such that $b_n \to 0$, and bounded such that $\overline{m} - b_n \in S$ for all n. Since S is non-empty, this is always possible. Then, define $\{a_n\}$ as $a_n = \overline{m} - b_n$. Then, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(\overline{m} - b_n) = \lim_{n\to\infty} \overline{m} - \lim_{n\to\infty} b_n = \overline{m} - 0 = \overline{m}$. □

(b) From exercise 6.5.3, we know that $\sup(A + B) \leq \sup A + \sup B$. Therefore, showing that $\sup(A + B) \geq \sup A + \sup B$ will be sufficient to prove that $\sup(A + B) = \sup A + \sup B$.

Proof. By part (a), there exist sequences $\{a_n\} \subseteq A$ and $\{b_n\} \subseteq B$ such that $a_n \to \sup A$, and $b_n \to \sup B$. Since $a_n \in A, b_n \in B\ \forall\, n$, $a_n + b_n \in A + B$, and therefore $a_n + b_n \leq \sup(A+B)$. By the limit location theorem, $\lim a_n + b_n \leq \sup(A+B)$. But $\lim a_n + b_n = \lim a_n + \lim b_n = \sup A + \sup B$, and therefore $\sup A + \sup B = \sup(A + B)$. □

Problem 7.2.1 Evaluate $\sum_{i=0}^{\infty} \frac{1}{(2n+1)^2}$, given that $\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

$$\begin{aligned}
\sum_{i=0}^{\infty} \frac{1}{n^2} &= \sum_{i=0}^{\infty} \frac{1}{(2n)^2} + \sum_{i=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= \frac{1}{4} \sum_{i=0}^{\infty} \frac{1}{n^2} + \sum_{i=0}^{\infty} \frac{1}{(2n+1)^2}
\end{aligned}$$

Therefore, $\sum_{i=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4} \sum_{i=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$. Line 1 above works because it is given that that $\sum_{i=0}^{\infty} \frac{1}{n^2}$ and therefore $\sum_{i=0}^{\infty} \frac{1}{(2n)^2}$ are both convergent, so $\sum_{i=0}^{\infty} \frac{1}{(2n+1)^2}$ must also converge.