Homework 3 solutions

(1)

(a)

Let C be the event that the two coins produce different results, i.e. one is heads and the other is tails. Knowing how one coin landed doesn't tell you anything about how the other coin will land, and therefore tells you nothing about whether it will land with a different result.

Proof. $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2}$. After one coin is tossed, the other coin will land heads or tails with a $50:50$ probability; exactly one of these will make the two coins land differently, so $\mathbb{P}(C|A) = \mathbb{P}(C|B) = \frac{1}{2}$. $\mathbb{P}(A \cap C) = \mathbb{P}(C|A) \cdot \mathbb{P}(A) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = \mathbb{P}(A) \cdot \mathbb{P}(C)$, and similarly $\mathbb{P}(B \cap C) = \frac{1}{4} = \mathbb{P}(B) \cdot \mathbb{P}(C)$. However, $\mathbb{P}(A \cap B \cap C) = 0$, since if the two coins both land on heads then they cannot be different, whereas $\mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C) = \frac{1}{8}$. Therefore, A, B and C are pairwise but not mutually independent. $\square$

(b)

Proof. $\mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C) = \mathbb{P}(A \cup B \cup C) = \mathbb{P}((A \cup B) \cup C) = \mathbb{P}(A \cup B|C) \cdot \mathbb{P}(C)$. Assume that C is some event with nonzero probability. Then, divide by $\mathbb{P}(C)$: $\mathbb{P}(A) \cdot \mathbb{P}(B) = \mathbb{P}(A \cup B|C)$. Assume A, B and C are mutually independent; then we have $\mathbb{P}(A \cup B) = \mathbb{P}(A) \cdot \mathbb{P}(B) = \mathbb{P}(A \cup B|C)$. The equality $\mathbb{P}(A \cup B) = \mathbb{P}(A \cup B|C)$ holds iff $(A \cup B)$ and C are independent, which we cannot assume. Therefore, $\mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C) = \mathbb{P}(A \cup B \cup C)$ by itself does not imply pairwise independence. $\square$

(2)

(a)

This statement is true.

Proof. $PMF_{X,Y}(x, y) = \mathbb{P}(X = x \cap Y = y)$, $PMF_{X+Y}(x + y) = \mathbb{P}(X = x \cup Y = y)$, by definition. We know that for two sets A, B, $A \subseteq A \cup B$, and $A \cap B \subseteq A$. Therefore, $\mathbb{P}(A) \leq \mathbb{P}(A \cup B)$, since $\mathbb{P}(A) = \sum_{a \in A} \mathbb{P}(a)$ and $\mathbb{P}(A \cup B) = \sum_{a \in A \cup B} \mathbb{P}(a)$: we know that $A \subseteq A \cup B$ and all probabilities are positive, which means $\sum_{a \in A \cup B} \mathbb{P}(a) \geq \sum_{a \in A} \mathbb{P}(a) \implies \mathbb{P}(A) \leq \mathbb{P}(A \cup B)$. Similarly, $\mathbb{P}(A) \geq \mathbb{P}(A \cap B)$. Therefore, since $A \cap B \subseteq A \cup B$, we have $\mathbb{P}(A \cap B) \leq \mathbb{P}(A \cup B)$. $PMF_{X+Y}(x + y) - PMF_{X,Y}(x, y) = \mathbb{P}(X = x \cup Y = y) - \mathbb{P}(X =$

$x \cap Y = y) \geq 0$, since $\mathbb{P}(X = x \cup Y = y) \geq \mathbb{P}(X = x \cap Y = y)$. Therefore, $\mathbb{P}(X = x \cup Y = y) \geq \mathbb{P}(X = x \cap Y = y)$. $\square$

(b)

This statement is false.

Proof. I will disprove this statement by counterexample. Toss two coins, and let X be the value of the first coin toss and Y be the value of the second coin toss. $PMF_{X,Y}(\text{heads}, \text{heads}) = \mathbb{P}(X = \text{heads} \cap Y = \text{heads}) = \mathbb{P}(X = \text{heads}) \cdot \mathbb{P}(Y = \text{heads})$, since the two coin tosses are independent. Therefore, $PMF_{X,Y}(\text{heads}, \text{heads}) = \frac{1}{4}$. However, $PMF_{X+Y}(\text{heads} + \text{heads}) = \frac{3}{4}$, since there are 4 possible events (HH, HT, TH, TT) and only one of them (TT) does not involve one of the coins landing heads-up. $\frac{3}{4} > \frac{1}{4}$, which runs contrary to the original statement. $\square$

(3)

(a)

This statement is true.

Proof. Pick any $x \in A$. Then, $P(x)$ holds. $\square$

(b)

This statement is false.

Proof. The statement implies that there exist two sets $B \subseteq A$, $C \subseteq A$ such that $B \cup C = \varnothing$, $B \cap C = A$, $B \neq \varnothing$, and such that $P(x)$ holds for all $x \in B$ but does not hold for all $x \in C$. If $C \neq \varnothing$, then there exists at least one x such that $x \in C$ and therefore $x \in A$ *and* $P(x)$ is false. $\square$

A restriction that would make this statement hold is $|A| = 1$.

Proof. The statement implies that there exists one or more $x \in A$ such that $P(x)$ holds. Since A only has one member, $P(x)$ must hold for that member, and therefore holds for all $x \in A$. $\square$

(c)

This statement is false.

Proof. "$P(x)$ does not imply $Q(x)$" means that the value (true or false) of $Q(x)$ is "independent" of $P(x)$: both can be true, both can be false or one can be true and one can be false. According to the statement, this is true only for some $x \in A$, but for some $x \in A$, $P(x)$ does imply $Q(x)$. For any $x \in A$, there are two possibilities. Either for this x $P(x)$ implies $Q(x)$, or it does not. In both cases, the fact that $Q(x)$ holds says *nothing* about $P(x)$. Therefore, for any x, whether $Q(x)$ holds or not, $P(x)$ can still be false, since if $Q(x)$ holds, $P(x)$ can either be true or false, and if $Q(x)$ does *not* hold, we know that $P(x)$ *cannot* hold. Thus, regardless of whether there exists $x \in A$ such that $Q(x)$ holds, there is a possibility of $P(x)$ being false for all $x \in A$. $\square$

(4)

(a)

The second step uses the fact that $\alpha = \beta \implies \alpha + \gamma = \beta + \gamma \; \forall \; \alpha, \beta, \gamma \in \mathbb{R}$. The third step uses the same fact, noting that γ does not have to be positive. The fourth step uses the fact that multiplication is commutative over addition in $\mathbb{R}$, that is, $\gamma \cdot \alpha + \gamma \cdot \beta = \gamma \cdot (\alpha + \beta) \; \forall \; \alpha, \beta, \gamma \in \mathbb{R}$. The last step uses the fact that $\alpha = \beta \iff \alpha \cdot \gamma = \beta \cdot \gamma \; \forall \; \alpha, \beta, \gamma \in \mathbb{R}, \gamma \neq 0$.

(b)

The last step is invalid, since $a = b \implies a^2 = ab \implies a^2 - ab = 0$, and therefore the mathematical property that the last step uses does not apply.

(c)

Let P be the set of all primes: $P = \{p_1, p_2 \cdots p_n\}$. We want to show that the cardinality of P is infinite, that is, there are infinitely many primes.

Proof. This can be proven by contradiction. Suppose that P is finite, that is, there exists $n \in \mathbb{R}$ such that $|P| \leq n$. Let $N = \prod_{i=1}^{n} p_i + 1$. N is clearly not in P, since it is larger than any $p \in P$, therefore it is a composite number, which means it is divisible by at least one prime number: $p_i | N$ for some $0 < i \leq n$. However, it is clear that, for all $0 < i \leq n$, $p_i \nmid N$, since N is a number multiplied by p_i plus 1, which means that $N \equiv 1 \pmod{p_i}$. Thus, we have that there exists $p \in P$ such that $p|N$, but also that, for all $p \in N$, $p \nmid N$. This is a contradiction, and therefore our initial assumption (P is finite) is false. $\square$