Homework 3 solutions

(1)

(a)

This can be done by drawing the diagram with the line going from the x-axis to $(0, 1)$, touching the circle. There is a small right-angled triangle with catheti $1 - y$ and x, and a larger one with catheti 1 and t, the latter being the x-value of the point we want to find. Setting up the equation for similar triangles, we have $\frac{1}{1-y} = \frac{t}{x} \implies t = \frac{x}{1-y}$, which gives the point on the x-axis $\left(\frac{x}{1-y}, 0\right)$.

(b)

Once again set up the same diagram. We have the equation from before $t = \frac{x}{1-y} \implies x = t(1-y)$. This time we have two unknowns so we need two equations, the second one given by the circle: $x^2 + y^2 = 1 \implies x^2 = 1 - y^2 \implies t^2(1-y)^2 = 1 - y^2$, using the first equation. This equation will have two roots, but one is the root $y = 1$, which simply describes the topmost point of the circle and thus is useless to us. We can therefore divide through by $y - 1$, giving $t^2(1-y) = 1 + y \implies t + t^2 y = t^2 - 1 \implies y(t^2+1) = t^2 - 1 \implies y = \frac{t^2-1}{t^2+1}$. This is the value of y in the point (x, y). To find x, use $x^2 = 1 - y^2 \implies \frac{x^2}{1-y^2} = 1$. Using this and the fact that $t = \frac{x}{1-y}$, divide both sides by t: $\frac{1}{t} = \frac{x^2}{1-y^2} \cdot \frac{1-y}{x} = \frac{x}{1+y} \implies t = \frac{1+y}{x} \implies x = \frac{1+y}{t}$. Using the previous solution for $y = \frac{t^2-1}{t^2+1} \implies 1 + y = \frac{t^2+1+t^2-1}{t^2+1} = \frac{2t^2}{t^2+1}$, we have $x = \frac{1+y}{t} = \frac{2t}{t^2+1}$.

(c)

For every point on the unit circle about the origin other than $(0, 1)$, we have the function $(x, y) \mapsto \left(\frac{x}{1-y}, 0\right)$ that maps the point to a point on the x-axis (this function does not work for $(0, 1)$ since $y = 1 \implies 1 - y = 0$. Also, for every point $(t, 0)$ we have the function $t \mapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$ that maps the point to a point on the unit circle about the origin. Thus, excluding $(0, 1)$, we have a bijection between the x-axis and the points on the unit circle about the origin. The points on the x-axis inside the semicircle map to the southern hemisphere, since inside the semicircle we have $1 \geq t \geq -1 \implies t^2 \leq 1$ and therefore $y = \frac{t^2-1}{t^2+1} \leq 0$. All other points on the x-axis map to points in the northern hemisphere, since for them, $y > 0$.

(d)

We have the equation $x^2 + y^2 = 1$; using $x = \frac{2t}{t^2+1}$ and $y = \frac{t^2-1}{t^2+1}$ we have $\left(\frac{2t}{t^2+1}\right)^2 + \left(\frac{t^2-1}{t^2+1}\right) = 1 \implies (2t)^2 + (t^2-1)^2 = (t^2+1)^2$. This gives us a method of generating Pythagorean triples: sets of integers a, b and c such that $a^2 + b^2 = c^2$: for any $t \in \mathbb{Z}$, $a = 2t$, $b = t^2 - 1$ and $c = t^2 + 1$.

(2)

(a)

Proof. We are given that $g(\vec{x})$ is continuous on all values of $\vec{x}$, including $\vec{f}(\vec{x})$, that is, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\left\|\vec{f}(\vec{x}) - \vec{f}(\vec{y})\right\| < \delta \implies |g(\vec{f}(\vec{x})) - g(\vec{f}(\vec{y}))| < \epsilon$. We also know that $\vec{f}$ is continuous, that is, for all $\delta > 0$ there exists η such that $\|\vec{x} - \vec{y}\| < \eta \implies \left\|\vec{f}(\vec{x}) - \vec{f}(\vec{x})\right\| < \delta$. Therefore, for all $\epsilon > 0$ there exists $\eta > 0$ such that $\|\vec{x} - \vec{y}\| < \eta \implies \left\|\vec{f}(\vec{x}) - \vec{f}(\vec{x})\right\| < \delta \implies |g(\vec{f}(\vec{x})) - g(\vec{f}(\vec{y}))| < \epsilon$. This shows that $g \circ \vec{f}$ is continuous on all values. □

(b)

Proof. Let $f(\vec{x}) = x_1 + x_2$. Assume $\|\vec{y} - \vec{x}\| < \delta$. $\|\vec{y} - \vec{x}\| = \sqrt{(y_1 - x_1)^2 - (y_2 - x_2)^2} < y_1 - x_1, y_2 - x_2$. Therefore, $y_1 - x_1 < \delta, y_2 - x_2 < \delta$. $\|f(\vec{y}) - f(\vec{x})\| = |y_1 + y_2 - x_1 - x_2| \leq |y_1 - x_1| + |y_2 - x_2| < 2\delta$. Therefore, stting $\epsilon > 2\delta$, for example $\epsilon = 3\delta$ would satisfy the continuity property, therefore $f(\vec{x})$ is continuous. □

(c)

Proof. Let $f(\vec{x}) = x_1 \cdot x_2$. Fix $\epsilon > 0$. We need to find a δ such that $\|\vec{x} - \vec{y}\| < \delta$ implies that $\|f(\vec{x}) - f(\vec{y})\| > \epsilon$. $\|f(\vec{x}) - f(\vec{y})\| = |x_1x_2 - y_1y_2| = |(x_1x_2 - x_1y_2) + (x_1y_2 - y_1y_2)| \leq |x_1||x_2 - y_2| + |y_2||x_1 - y_1|$, using the triangle inequality. We also know that this is $\leq |x_1|\delta + |y_2|\delta$, since $\|\vec{x} - \vec{y}\| < \delta$, and $\|\vec{x} - \vec{y}\| \geq |x_1 - y_1|, |x_2 - y_2|$, using the definition of the norm. Applying another algebraic trick, $|y_2| = |x_2 + y_2 - x_2| \leq |x_2| + |y_2 - x_2| < |x_2| + \delta$. Apply the inequality for δ: $|x_1|\delta + |y_2|\delta < |x_1|\delta + |x_2|\delta + \delta^2$. If $\delta \leq 1$, $\delta^2 \leq \delta$. So $|x_1|\delta + |y_2|\delta < |x_1|\delta + |y_2|\delta + \delta^2 \leq \delta(|x_1| + |x_2| + 1) \leq \epsilon$. So $\delta = \min(\frac{\epsilon}{|x_1|+|x_2|+1}, 1)$. □

(d)

Proof. We know that if $\overrightarrow{f}$ and g are both continuous then so is $g \circ \overrightarrow{f}$. If $g(x,y) = x + y$(a continuous function) then $g \circ \overrightarrow{f} = f_1 + f_2$, which we know to be continuous. If $g(x,y) = xy$ (another continuous function), then, again, $g \circ \overrightarrow{f} = f_1 \cdot f_2$ is also continuous. $\square$

(e)

You would need to show that $g(x,y) = \frac{x}{y}$ is continuous on all points, which it is not, since it is undefined whenever $y = 0$. To show that $\frac{f_1}{f_2}$ is continuous in the same way as above, we would need to know that $f_2(x) > 0 \; \forall \; x \in \mathbb{R}$.

(3)

~~First, we must show that $sin(x)$ is continuous. For this, we can express $|sin(x) - sin(y)| = \left|2\cos\frac{x+y}{2}\sin\frac{x-y}{2}\right| \leq 2\left|\sin\frac{x-y}{2}\right|$, since $-1 \leq cos(x) \leq 1$. If we have $|x - y| < \delta$, it is clear that $\left|\frac{x-y}{2}\right| < \delta$. $2\left|\sin\frac{x-y}{2}\right| \leq 2\left|\frac{x-y}{2}\right| < 2\delta$, since $|sin(x)| \leq |x| \; \forall \; x \in \mathbb{R}$. Therefore, the continuity property is satisfied if, for example, $\delta = \frac{\epsilon}{2}$.~~

Proof. We know that $f(x) = sin(x)$, $f(x) = e^x$, $f(x) = x$ are continuous on all values of x, and $f(x) = \frac{1}{x}$ is continuous on all values of x other than $x = 0$. We also know, from question (2), that the composition of two continuous functions is also continuous. $f(x) = x^2$ is therefore continuous since it is the composition $g \circ \overrightarrow{h}$ where $g(x,y) = xy$ and $\overrightarrow{h}(x) = (x,x)$, both continuous functions ($\overrightarrow{h}$ is continuous since its component functions are continuous). By the same reasoning, $f(x,y,z) = x^2 + y^2 + z^2$ is continuous, since it is the composition $g(h(x), h(y), h(z))$ where $g(x,y,z) = x + y + z$, and $h(x) = x^2$, two continuous functions. The exact same reasoning gives us that $f(x,y,z) = e^{z-x+y}$ is continuous for all values. $f(x,y,z) = \frac{1}{e^{z-x+y}}$ is also continuous, since it is a composition of the former function and $g(x) = \frac{1}{x}$. $g(x)$ is not continuous on $x = 0$, but $e^x > 0 \; \forall \; x \in \mathbb{R}$, so we can still compose these two functions. $f(x,y,z) = sin(x^2 + y^2 + z^2)$ is continuous, as once again it is the composition of two continuous functions, $x \mapsto sin(x)$ and $(x,y,z) \mapsto x^2 + y^2 + z^2$. Finally, $f(x,y,z) = \frac{sin(x^2+y^2+z^2)}{e^{z-x+y}}$, since it is the product of the continuous functions $sin(x^2 + y^2 + z^2)$ and $\frac{1}{e^{z-x+y}}$; the product of two continuous functions is continuous, per (2), so $f(x,y,z)$ is continuous. $\square$

(4)

(a)

Any function $f(x) = \frac{k}{x}$ where k is a finite constant will satisfy this, since its values will become arbitrarily large as x tends to 0.

(b)

An example of this is $f(x) = tan^{-1}(x)$, since $\frac{-\pi}{2} < tan^{-1}(x) < \frac{\pi}{2}\ \forall\ x \in \mathbb{R}$.

(c)

An example of this is the sequence $f_n(x) = x^n$. For any finite n, f_n is a product of several functions $g(x) = x$, and therefore continuous. However, $h(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ \infty & x = 1 \end{cases}$. This function does not take any intermediate values between 0 and 1, and is therefore not continuous, since any continuous function that takes values a and b must also take all values between a and b.

(5)

For any point $\vec{a} = (a, b) \in S$, let $\epsilon < 1 - \sqrt{a^2 + b^2} < 0$. Then, the greatest norm of any point in $\mathrm{B}_\epsilon(\vec{x})$ is strictly less than $\|a\| + \epsilon = \sqrt{a^2 + b^2} + 1 - \sqrt{a^2 + b^2} = 1$, which means any point in $\mathrm{B}_\epsilon(\vec{x})$ is also in S, that is, S is open. Let C be the set of all points on the unit circle about the origin. The norm of any point in C is exactly 1; but the open ball around any point in C, with radius ϵ, will contain a point with norm $1 - \epsilon$, which is inside S, and a point with norm $1 + \epsilon$, which is outside S. Therefore, the points in C are the boundary points of S.

(6)

Such an example is the set $S = \left\{\frac{1}{n} | n \in \mathbb{Z}^+, n \ge 1\right\} \subseteq \mathbb{R}$. Any interval between two distinct rational numbers is of the form $\{n | a < n < b\}$, for some $a, b \in \mathbb{Q}$, which is clearly open. The union of infinitely many of these sets is also open, and is the complement to S, which is therefore closed. Any single point in $\mathbb{R}$, which is what S consists of, is also closed.