Inner product spaces

Assuming V is a vector space over $\mathbb{R}$, we define an **inner product** on V as a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}, (\vec{u}, \vec{v}) \mapsto \langle \vec{u}, \vec{v} \rangle$ that satisfies the following:

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
3. $\langle \vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
4. $\langle \vec{u}, \vec{u} \rangle \geq 0$, equal iff $\vec{u} = \vec{0}$

We say that $(V, \langle \cdot, \cdot \rangle)$ is an **inner product space**. For example, $(\mathbb{R}^n, \cdot)$ is an inner product space, as the dot product is an example of an inner product that acts on $\mathbb{R}^n$. As another example, consider $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$ where $\langle \vec{u}, \vec{v} \rangle = 4u_1v_1 + 5u_2v_2 \; \forall \; \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. This function also satisfies the inner product properties, so the former space is also an inner product space. Note that we can express this inner product as a modification of the dot product: $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \vec{v}$.

We can use the notion of inned product spaces to generalise some definitions that we have discussed. Given an inner product space $(V, \langle \cdot, \cdot \rangle)$, we (re)define:

1. $||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}, ||\vec{v}|| = 0 \iff \vec{v} = 0$
2. A unit vector is a vector $\vec{u}$ such that $||\vec{u}|| = 1$
3. $dist(\vec{u}, \vec{w}) = ||\vec{u} - \vec{w}||$
4. $\vec{u} \perp \vec{w} \iff \langle \vec{u}, \vec{w} \rangle = 0$
5. Many formulae will carry over, such as the pythagorean theorem $||vva \pm \vec{b}||^2 = ||\vec{a}||^2 + ||\vec{b}||^2 \iff \vec{a} \perp \vec{b}$.

Using V from before, consider a finitely-dimensional subspace $W \subseteq V$. Given $\vec{b} \in V$, we define $\text{proj}_W \vec{b} = \hat{b}$ as satisfying $\vec{b} - \hat{b} \perp \vec{w} \; \forall \; \vec{w} \in W$. By the pythagorean theorem, $||\vec{b}||^2 = ||\hat{b}||^2 + ||\vec{b} - \hat{b}||^2$, and therefore $||\hat{b}|| \leq ||\vec{b}||$, with equality iff $\vec{b} \in W$.