Continuity of functions

Definition of continuity

A function $f : D \subseteq \mathbb{R} \mapsto \mathbb{R}$ is continuous at $x \in X$ if $\forall\ \epsilon > 0 \exists \delta > 0$ such that for all $y : |x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. f is continuous on D if it is continuous on x for all $x \in D$.

For example, $f(x) = x$ is continuous.

Proof. Fix x, $\epsilon > 0$. Let $\delta = \epsilon$. If y is such that $|x-y| < \delta$, then $|f(x)-f(y)| = |x - y| < \delta = \epsilon$. Therefore, f is continuous. $\square$

Another example: $f(x) = 2x$ is continuous.

Proof. Fix x, $\epsilon > 0$. Assume $|x-y| < \delta$. $|f(x)-f(y)| = |2x-2y| = 2|x-y| < 2\delta$. Therefore, if we set $\delta = \frac{\epsilon}{2}$, then the continuity property is satisfied, therefore f is continuous. $\square$

A more complicated example is the continuity of $f(x) = x^2$.

Proof. Fix x, $\epsilon > 0$. Assume $|x - y| < \delta$. $|f(x) - f(y)| = |x^2 - y^2| = |(x - y)(x + y)| < \delta|x + y|$. We know that y is close to x, that is, $|x + y| \leq 2|x| + \delta$. Therefore, $|(x - y)(x + y)| < \delta|x + y| \leq \delta(2|x| + \delta)$. Now, we have two cases.

The first case is $x \neq 0$. Assume that $\delta < |x|$, since we are always allowed to make δ smaller. Then, our inequality becomes $\delta|x + y| \leq \delta(2|x| + \delta) \leq \delta(3|x|)$. Therefore setting $\delta(3|x|) = \epsilon$ we satisfy continuity by using $\delta = min(|x|, \frac{\epsilon}{3|x|}$ (we need the minimum since we can't assume that $|x| > \frac{\epsilon}{3|x|}$.

The second case is the single case $x = 0$. Then, $|f(x) - f(y)| = |y^2| < \epsilon$. If we set $\delta = \sqrt{\epsilon}$ then we have $|x - y| = |y| < \delta \implies |f(x) - f(y)| < \epsilon$. $\square$

To prove the continuity of a more complex function, such as $f(x) = x^5 - 15x^2 + x$, we can use the fact that the composition of two continuous functions is also continuous. For example, if we prove that $f(x, y) = x + y$ and $f(x, y) = xy$ are continuous, then we can prove that the polynomial above is continuous!

Let us prove that the product of two continuous functions is continuous.

Proof. Fix x, $\epsilon > 0$. We want to show that if $|x - y| < \delta$, then $|f(x)g(x) - f(y)g(y)| < \epsilon$. We know that $\exists \delta_1$ such that if $|x-y| < \delta_1$ then $|f(x)-f(y)| < \epsilon$, and $\exists \delta_2$ such that if $|x - y| < \delta_2$ then $|g(x) - g(y)| < \epsilon$. $|f(x) - f(y)||g(x)| = |f(x)g(x) - f(y)g(x)|$. Also, $f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(x)g(y)|$. Then, this is less than or equal to $|(f(x) - f(y))g(x)| + |f(y)(g(x) - g(y))| = |f(x) - f(y)||g(y)| + |f(x)||g(x) - g(y)| < \epsilon|g(y)| + \epsilon|f(x)$.

Since we can set the continuity inequality to be less than *anything*, not necessarily specifically ϵ, we can go back and set $|f(x) - f(y)| < \frac{\epsilon}{2|g(y)}$ and $|g(x) - g(y)| < \frac{\epsilon}{2|f(x)}$. This gets rid of the extra factors, and just leaves $|f(x) - f(y)||g(y)| + |f(x)||g(x) - g(y)| < \epsilon$. $\square$

Continuity in multivariate functions

The definition of continuity is similar for multivariate functions. A function $\vec{f} : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ is continuous on $\vec{x} \in D$ if $\forall\ \epsilon > 0 \exists \delta > 0$ such that for all $\vec{y}$ such that $\|\vec{x} - \vec{y}\| < \delta$, we have $\left\|\vec{f}(\vec{x}) - \vec{f}(\vec{y})\right\| < \epsilon$.

For example, let $f(x, y) = \frac{xy}{x^2+y^2}$ or 0 where $(x, y) = (0, 0)$. We know that $f(x, y) = x$, $f(x, y) = y$ and $f(x, y) = x^2 + y^2$ are all continuous, and the latter function is only 0 if $(x, y) = (0, 0)$. Therefore, we know that $f(x, y) = \frac{xy}{x^2+y^2}$ is continuous for all (x, y) other than the origin.

Proof. Let $(x, y) = (r \sin \theta, r \cos \theta)$. Then, $r = \sqrt{x^2 + y^2}$, $\cos \theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin \theta = \frac{y}{\sqrt{x^2+y^2}}$. $f(x, y) = \cos \theta \cdot \sin \theta = \frac{1}{2} \sin(2\theta)$. We see that, surprisingly, the output is independent of r. Also note that $\|(r \cos \theta, r \sin \theta\| = r$. Now, let $\epsilon = \frac{1}{4}$, and let $\vec{x} = (0, 0)$. For $\vec{y} = (\frac{\delta}{2} \cos \theta, \frac{\delta}{2} \sin \theta)$, $\|\vec{x} - \vec{y}\| = \frac{\delta}{2} < \delta$. However, $\|f\vec{x}) - f(\vec{y})\| = \left\|0 - \frac{1}{2} \sin(2\theta)\right\| = \frac{1}{2} > \epsilon$. Therefore, $f(x, y)$ is not continuous on $(0, 0)$. $\square$