MATH 3110 HOMEWORK #9

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Problem 14.1.3 I will consider the right- and left-hand derivatives of f.

$$\begin{aligned}
f'(0^+) &= \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{-y\to 0^+} \frac{f(-y)-f(0)}{-y} && (y=-x) \\
&= \lim_{y\to 0^-} \frac{f(-y)-f(0)}{-y} && (-y\to 0^+ \iff y\to 0^-) \\
&= -\lim_{y\to 0^-} \frac{f(y)-f(0)}{y} && (f \text{ is even}) \\
&= -f'(0^-)
\end{aligned}$$

But f is differentiable, so $f'(0^+) = f'(0^-)$. Therefore, $f'(0) = f'(0^+) = f'(0^-) = 0$.

Problem 14.1.4 (b)

We have that $|f(x)| \le x^2$ for $x \approx 0$; that is, there exists $d > 0$ such that for all $|x| < d$, $|f(x)| \le x^2$. By definition, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x}$; I will show that this limit is 0.

Proof. Pick $\epsilon > 0$, let $\delta = \min(d, \epsilon)$, and suppose $|x| < \delta$. Then:

$$\begin{aligned}
\left|\frac{f(x)-f(0)}{x}\right| &\le \frac{|f(x)|+|f(0)|}{x} \\
&\le \frac{x^2}{|x|} && (\text{since } |x| < \delta \le d) \\
&= \frac{|x^2|}{|x|} && (\text{since } x^2 \ge 0\ \forall\, x\in\mathbb{R}) \\
&= \left|\frac{x^2}{x}\right| \\
&= |x| < \delta \le \epsilon
\end{aligned}$$

□

Problem 14.2.4

(a)

Proof.

$$\begin{aligned}
f'(-a) &= \lim_{x\to -a} \frac{f(x)-f(-a)}{x+a} \\
&= \lim_{x\to -a} \frac{f(x)-f(a)}{x+a} && (f \text{ is even}) \\
&= \lim_{y\to a} \frac{f(y)-f(a)}{-y+a} && (y=-x, f \text{ is even}) \\
&= -f'(a)
\end{aligned}$$

□

(b)

Proof.

$$\begin{aligned}
f'(-a) &= \lim_{x\to -a} \frac{f(x)-f(-a)}{x+a} \\
&= \lim_{x\to -a} \frac{f(x)+f(a)}{x+a} && (f \text{ is odd}) \\
&= \lim_{y\to a} \frac{-f(y)+f(a)}{-y+a} && (y=-x, f \text{ is odd}) \\
&= \lim_{y\to a} \frac{f(y)-f(a)}{y-a} \\
&= f'(a)
\end{aligned}$$

□

Problem 14-1 (a)

$$\begin{aligned}
\lim_{\Delta x\to 0} F(\Delta x) &= \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a-\Delta x)}{2\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)+f(a)-f(a-\Delta x)}{2\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{2\Delta x} + \lim_{\Delta x\to 0} \frac{f(a)-f(a-\Delta x)}{2\Delta x} \\
&= \frac{f'(a)}{2} + \lim_{\Delta x\to 0} \frac{f(a)-f(a-\Delta x)}{2\Delta x} \\
&= \frac{f'(a)}{2} + \lim_{-h\to 0} \frac{f(a)-f(a+h)}{-2h} && (h=-\Delta x) \\
&= \frac{f'(a)}{2} + \lim_{-h\to 0} \frac{f(a+h)-f(a)}{2h} \\
&= \frac{f'(a)}{2} + \lim_{h\to 0} \frac{f(a+h)-f(a)}{2h} && (-h\to 0 \iff h\to 0) \\
&= f'(a)
\end{aligned}$$

(b) Yes, this limit can exist. An example is $f(x)=|x|$. f is left- and right-differentiable at $a=0$, but it is not differentiable there. However, at $a=0$, $\frac{f(a+\Delta x)-f(a-\Delta x)}{2\Delta x} = \frac{|\Delta x|-|-\Delta x|}{2\Delta x} = 0$, so the limit exists and is equal to 0, by the limit location theorem.

(c) I will consider separately the right- and left-hand limits (skipping some steps as they are very similar to those in part (a)):

$$\begin{aligned}
\lim_{\Delta x\to 0^+} F(\Delta x) &= \lim_{\Delta x\to 0^+} \frac{f(a+\Delta x)-f(a)}{2\Delta x} + \lim_{\Delta x\to 0+} \frac{f(a)-f(a-\Delta x)}{2\Delta x} \\
&= \frac{f'(a^+)}{2} + \lim_{\Delta x\to 0^+} \frac{f(a)-f(a-\Delta x)}{2\Delta x} \\
&= \frac{f'(a^+)}{2} + \lim_{-h\to 0^+} \frac{f(a)-f(a+h)}{-2h} && (h=-\Delta x) \\
&= \frac{f'(a^+)}{2} + \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{2h} && (-h\to 0^+ \iff h\to 0^-) \\
&= \frac{f'(a^+)}{2} + \frac{f'(a^-)}{2}
\end{aligned}$$

The steps for $\lim_{\Delta x\to 0^-} F(\Delta x)$ are exactly the same as above, except 0^+ becomes 0^- and 0^- becomes 0^+:

$$\begin{aligned}
\lim_{\Delta x\to 0^-} F(\Delta x) &= \lim_{\Delta x\to 0^-} \frac{f(a+\Delta x)-f(a)}{2\Delta x} + \lim_{\Delta x\to 0+} \frac{f(a)-f(a-\Delta x)}{2\Delta x} \\
&= \frac{f'(a^-)}{2} + \lim_{\Delta x\to 0^-} \frac{f(a)-f(a-\Delta x)}{2\Delta x} \\
&= \frac{f'(a^-)}{2} + \lim_{-h\to 0^-} \frac{f(a)-f(a+h)}{-2h} && (h=-\Delta x) \\
&= \frac{f'(a^-)}{2} + \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{2h} && (-h\to 0^- \iff h\to 0^+) \\
&= \frac{f'(a^-)}{2} + \frac{f'(a^+)}{2}
\end{aligned}$$

Since both the right- and left-hand limits are equal, $\lim_{\Delta x\to 0} F(\Delta x) = \frac{f'(a^+)}{2} + \frac{f'(a^-)}{2}$.

Problem 14-3

Proof. First, note that $f(a+0) = f(a) + f(0)$, and so $f(0) = f(a) - f(a+0) = f(a) - f(a) = 0$. Then, using the definition of the derivative, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$. Now we can calculate $f'(x)$ in terms of this:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x)+f(h)+2xh-f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(h)+2xh}{h} \\
&= \lim_{h\to 0} \frac{f(h)}{h} + \lim_{h\to 0} \frac{2xh}{h} \\
&= f'(0) + 2x
\end{aligned}$$

□

Functions that have this property are of the form $x^2 + kx$ for some $k \in \mathbb{R}$. Two examples are x^2 and $x^2 + 5x$.