Homework 2 solutions

(1)

1. The first axiom is that $\mathbb{P}(E) \geq 0 \;\forall\; \mathbb{P} \subseteq S$. This is satisfied by our function, since it is defined as $\mathbb{P}(E) = \Sigma_{s\in E}P(s)$; we are given that $P(s) \geq 0 \;\forall\; s \in S$, therefore $\Sigma_{s\in E}P(s) \geq 0 \;\forall\; E \subseteq S$. The second axiom states that $\mathbb{P}(S) = 1$. Using our function definition we can evaluate this as $\mathbb{P}(S) = \Sigma_{s\in S}P(s)$, which is given to us as being equal to 1. The third axiom states that $\mathbb{P}(E_1 \cup E_2) = P(E_1) + P(E_2)$, if $E_1 \cap E_2 = \varnothing$. We can show this by again applying the function definition: $\mathbb{P}(E_1 \cup E_2) = \Sigma_{s\in(E_1\cup E_2)}P(s) = \Sigma_{s\in E_1}P(s) + \Sigma_{s\in E_2}P(s) - \Sigma_{s\in(E_1\cap E_2)}P(s) = \Sigma_{s\in E_1}P(s) + \Sigma_{s\in E_2}P(s) - \Sigma_{s\in\varnothing}P(s) = \Sigma_{s\in E_1}P(s) + \Sigma_{s\in E_2}P(s) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$.
2. Suppose there exists a function $\mathbb{P}'$ such that $\mathbb{P}'(\{s\} = P(s)$, that satisfies the three axioms of a probability measure. Then, for any set $E = \{s_1, s_2 \cdots s_n\}$, $\mathbb{P}'(E) = \mathbb{P}'(\{s_1, s_2 \cdots s_n\}) = \Sigma_{i=0}^{n}\mathbb{P}'(\{s_i\})$, by the third axiom. This, using our function definition, is equal to $\Sigma_{i=0}^{n}P(s_i) = \Sigma_{s\in E}P(s) = \mathbb{P}(E)$.
3. The cardinality of a set $\{s\}$ is 1, which means $\mathbb{P}(s) = \frac{1}{|S|}$. By the function definition, given a set $E = \{s_1, s_2 \cdots s_n\}$, we have $\mathbb{P}(E) = \mathbb{P}(\{s_1, s_2 \cdots s_n\}) = \frac{|E|}{|S|} = \frac{n}{|S|} = \Sigma_{i=0}^{n}\frac{1}{|S|} = \Sigma_{i=0}^{n}P(s_i) = \Sigma_{s\in E}P(s)$. Per part 1, this satisfies the three probability axioms, and is therefore a probability measure.

(2)

1. Given that the contestant opens door 1 every time, there are four possible outcomes: $\{(1,1,2),(1,1,3),(1,2,3),(1,3,2)\}$, with the notation (door chosen by contestant, door hiding the car, door the host opens). The first argument is always 1. If the second argument is also 1 (the door the contestant chose is correct), the host chooses with equal probability either of the remaining doors; in the remaining cases the host cannot choose. The probability of the car being behind any door is $\frac{1}{3}$. Therefore, the probability for the first two cases is $\frac{1}{3}\frac{1}{2} = \frac{1}{6}$ and the probability of the last two cases is $\frac{1}{3}$: there is no need to multiply by $\frac{1}{2}$ since the host no longer has the $50:50$ choice. In the first two cases, the contestant wins by sticking, and in the second two cases the contestant wins by switching. Therefore, $P(\text{win by sticking}) = P((1,1,2)\cup(1,1,3)) = P((1,1,2)) + P((1,1,3)) = \frac{1}{3}$ (we can add the probabilities since the four events are mutually exclusive), and $P(\text{win by switching}) = P((1,2,3)\cup(1,3,2)) = P((1,2,3)) + P((1,3,2)) = \frac{2}{3}$.

2. The incorrect assumption is contained in this phrase: "since the car was equally likely to be behind doors 1 and 3 to begin with, it must be equally likely after Monty opens a door". The sentence is claiming that $P(\text{car behind door 1}) = P(\text{car behind door 1}|\text{Monty opens door 2}) = P(\text{car behind door 1}|\text{Monty opens door 3})$. In reality this is wrong, since the law of total probability states that $P(\text{car behind door 1}) = P(\text{car behind door 1}|\text{Monty opens door 2}) + P(\text{car behind door 1}|\text{Monty opens door 3})$.

(3)

$x \in A \cap (B \cup C) \iff x \in A \cap B \parallel x \in A \cap C \iff x \in (A \cap B) \cup (A \cap C) \iff A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C), (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \iff A \cap (B \cup C) = (A \cap C), (A \cap B)$.

(4)

1. f is not $O(g)$ if for any c and $n_0 \in \mathbb{N}$, there exists an $n > n_0$ such that $f(n) > cg(n)$.

2. Fix n_0. For any value of $n > n_0$, let $k = \frac{f(n)}{g(n)}$. If $f(n) \leq 0$, we have $f(n) \leq cg(n)$ for $c = 1$, since we are given that $g(n) > 0 \, \forall \, n$. If not, then it follows that $f(n) = kg(n)$; let $c = k + 1$ and we have $f(n) \leq cg(n)$. Since we assumed nothing about n_0 or n, this shows that given any n_0, for any $n > n_0$ there exists c such that $f(n) \leq cg(n)$.

3. By definition, we have the following: $\exists c_g, n_g \in \mathbb{N}$ such that $\forall \ n > n_g, f(n) \leq c_g g(n)$ and $\exists c_h, n_h \in \mathbb{N}$ such that $\forall \ n > n_h, f(n) \leq c_h h(n)$. Let $2c_0 = max(c_g, c_h)$ and $n_0 = max(n_g, n_h)$. Then, for all $n > n_0$, we have $f(n) \leq 2c_0 g(n)$ and $f(n) \leq 2c_0 h(n)$. Therefore, $2f(n) \leq 2c_0 g(n) + 2c_0 h(n) = 2c_0(g(n) + h(n)) \implies f(n) \leq c_0(g(n) + h(n))$. By definition, this shows that f is $O(g + h)$.