Determinants

A very coarse definition of the determinant of a matrix is a number that is calculated based on the matrix cells that contains information about it. Only square matrices have determinants. We will use the notation $det(\vec{A})$ or $\left|\vec{A}\right|$ for the determinant of $\vec{A}_{n\times n}$, and $\vec{A}_{ij}$ to denote the matrix $\vec{A}$ but with the row i and the column j removed. For example, given

$$\vec{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

then

$$\vec{A}_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

Definition

The deteminant for $n \times n$ matrices is defined recursively, as follows:

1. $det(a) = a$
2. $det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$
3. $det(\vec{A}) = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}det(\vec{A}_{ij})$, or $\sum_{j=1}^{n}(-1)^{i+j}a_{ij}det(\vec{A}_{ij})$

For example:

$$\vec{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

we can fix $j = 3$:

$$det(\vec{A}) = (-1)^{1+3}\times 3\times det(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix})+(-1)^{2+3}\times 6\times det(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix})+(-1)^{3+3}\times 9\times det(\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}) = 3(4\times 8-5\times 7)+6(1\times$$

This definition is not very good, since this calculation becomes much more difficult as the matrix dimension increases:

$$det(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}) = (-1)^{1+2}\times 5\times det(\begin{bmatrix} 2 & 3 & 4 \\ 10 & 11 & 12 \\ 14 & 15 & 16 \end{bmatrix})+(-1)^{2+2}\times 6\times det(\begin{bmatrix} 1 & 3 & 4 \\ 9 & 11 & 12 \\ 13 & 15 & 16 \end{bmatrix})$$

$$+(-1)^{3+2} \times 7 \times det(\begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}) + (-1)^{4+2} \times 8 \times det(\begin{bmatrix} 1 & 2 & 3 \\ 9 & 10 & 11 \\ 13 & 14 & 15 \end{bmatrix})$$

This is obviously very hard to compute, and this method becomes very impractical for larger matrices. There is a simplification of notation here: the (i, j)th **cofactor** of $\vec{A}$ is defined as $C_{ij} = (-1)^{i+j} det(\vec{A}_{ij})$, and so

$$det(\vec{A}) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Given a matrix with nothing below the main diagonal (an **upper triangular** matrix) or nothing above it (**lower triangular**), there is a faster way to calculate the determinant: $det(\vec{A}) = \prod_{i=0}^{n} a_{ii}$.

A better definition of the determinant

$det : \left\{ \vec{A}_{n \times n} \right\} \mapsto \mathbb{R}$ is the unique function satisfying the following properties:

1. $det(\vec{I}) = 1$
2. If a multiple of a row is added to another row, $det(\vec{A})$ will not change
3. Interchanging two rows will invert the sign of $det(\vec{A})$
4. If a row is multiplied by a scalar c, $det(\vec{A})$ will also increase by the same factor c

The existence and uniqueness of $det(\vec{A})$ follows from the existence and uniqueness of reduced echelon form. The formula from earlier satisfies all of these properties, so it can be shown to compute the same det function.

Theorem: $\vec{A}$ is invertible iff $det(\vec{A}) \neq 0$.

Proof. We know that $\vec{A}$ is invertible iff the reduced echelon form of $\vec{A}$ is $\vec{I}$. As $det(\vec{I}) \neq 0$, this is equivalent to saying that $\vec{A}$ is invertible iff $det(\vec{A}) \neq 0$. □

As well as the ones from before, the matrix determinant can be shown to follow the following properties:

1. $det(c\vec{A}) = c^n \times det(\vec{A})$
2. If two rows or two columns of $det(\vec{A})$ are equal or are multiples of each other, then $det(\vec{A}) = 0$
3. $det(\vec{A}) = det(\vec{A}^T)$

4. If $\vec{A}$ and $\vec{B}$ are both square, $det(\vec{A}\vec{B}) = det(\vec{A}) \times det(\vec{B}) \implies det(\vec{A}\vec{B}) = det(\vec{B}\vec{A})$

> Theorem (Cramer's rule): given an invertible matrix $\vec{A}_{n\times n}$, $\vec{b} \in \mathbb{R}^n$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that $\vec{A}\vec{x} = \vec{b}$, the unique solution $\vec{x} = \vec{A}^{-1}\vec{b}$ satisfies $x_i = \frac{det(\vec{A}_i(\vec{b}))}{det(\vec{A})}$, where $\vec{A}_i(\vec{b})$ is $\vec{A}$ with the ith column replaced with $\vec{b}$.

Proof. We claim that $\vec{A}\vec{I}_i(\vec{x}) = \vec{A}_i(\vec{b})$, which is trivial to prove. By the multiplicative property of the determinant, $det(\vec{A}) \times det(\vec{I}_i(\vec{x})) = det(\vec{A}_i(\vec{b}))$. We complete the proof by noticing that, clearly, $det(\vec{I}_i(\vec{x})) = x_i$. $\square$

As a numerical example, consider the following system:

$$\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$$

It is clear that $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. However, we can use Cramer's rule as an alternative method:

$$x = \frac{det(\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix})}{det(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix})} = \frac{-2}{-2} = 1$$

$$y = \frac{det(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix})}{det(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix})} = \frac{-2}{-2} = 1$$

Another method of inversion

The matrix determinant gives us another way to find the inverse of a matrix, using a matrix of its cofactors:

Theorem: Given a square matrix $\overrightarrow{A}_{n\times n}$, $\overrightarrow{A}^{-1} = \frac{1}{det(\overrightarrow{A})}\left[C_{ij}\right]^T = \frac{1}{det(\overrightarrow{A})}\left[C_{ij}\right]^T$

We can define the **adjugate** matrix of $\overrightarrow{A}$: $adj(\overrightarrow{A}) = \left[C_{ij}\right]^T$, so $\overrightarrow{A}^{-1} = \frac{1}{det(\overrightarrow{A})}adj(\overrightarrow{A})$.

Areas and volumes

\begin{framed} Theorem: If S is a parallelogram generated by the origin and the columns of $\overrightarrow{A}_{2\times 2}$, and V is the parallelepiped generated by the origin and the columns of $\overrightarrow{B}_{3\times 3}$, then the area of S is $|det(\overrightarrow{A})|$, and the volume of V is $|det(\overrightarrow{B})|$.