Homework 1

(1)

Proof. Refer to the four vectors as $\vec{v}_i$, with $1 \leq i \leq 4$, respectively. $\vec{v}_1$ is the only vector whose 1st term is 1; the rest of the vectors have a 0 first term. There is no linear combination of $\vec{v}_2, \vec{v}_3$ and $\vec{v}_4$ that can make a vector with a nonzero first term; thus there is no linear combination of these three vectors that can make $\vec{v}_1$. □

(2)

Proof. The two linear properties are linearity in addition and multiplication; that is, for any function $f(x)$ and numbers a, b, the properties $f(a+b) = f(a) + f(b)$ and $f(a \cdot b) = a \cdot f(b)$ hold iff the function is linear.

In the case of l, the input is the vector $\vec{u} = \begin{bmatrix} u_1, u_2 \cdots u_n \end{bmatrix}$ and the function can be described simply as the dot product $\vec{u} \cdot \vec{c}$, where $\vec{c} = \begin{bmatrix} c_1, c_2 \cdots c_n \end{bmatrix}$, i.e. a vector of the coefficients in the function. We know that the dot product of two vectors is linear in both addition and multiplication, and therefore is linear. Thus, the function l is also linear. □

(3)

We can think of the function d taking $P(x)$ to $P'(x)$ (differentiation) as a map with the n-dimensional vector $\vec{u} = \begin{bmatrix} u_0, u_1 \cdots u_n \end{bmatrix}$ as the input. The output of this function is the n-dimensional vector $\begin{bmatrix} u_1, 2u_2, 3u_3 \cdots nu_n, 0 \end{bmatrix}$. The multiplicative property is easy to prove: a vector $c\vec{u}$ would be mapped to $\begin{bmatrix} cu_1, 2cu_2, 3cu_3 \cdots ncu_n, 0 \end{bmatrix} = c \begin{bmatrix} u_1, 2u_2, 3u_3 \cdots nu_n, 0 \end{bmatrix} = cd(\vec{u})$. To show the additive property, consider the vectors $\vec{u} = \begin{bmatrix} u_0, u_1 \cdots u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_0, v_1 \cdots v_n \end{bmatrix}$. $d(\vec{u}) + d(\vec{v}) = \begin{bmatrix} u_1, 2u_2 \cdots nu_n, 0 \end{bmatrix} + \begin{bmatrix} v_1, 2v_2 \cdots nv_n, 0 \end{bmatrix} = \begin{bmatrix} u_1 + v_1, 2u_2 + 2v_2 \cdots nu_n + nv_n, 0 \end{bmatrix} = d(\vec{u} + \vec{v})$.

To calculate the matrix we can simply apply the transformation d to each of the identity vectors. We end up with:

$$\vec{A}_{(n+1)\times(n+1)} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

We know that the determinant of this matrix is 0, since differentiation is not reversible.

(4)

1. The output of the function is independent of its inputs, i.e. it is constant. Therefore, the function is linear in both arguments.
2. We know that the dot product is bilinear. In this case, $b(\vec{u}, \vec{v}) = \vec{u} \cdot \vec{w}$, where $\vec{w} = \begin{bmatrix} v_2, v_3 \cdots v_n, v_1 \end{bmatrix}$. Thus it is trivial to see that the function is bilinear.
3. The function is obviously linear in the second argument, since it has no effect on the output. However, if you multiply the first vector by a number c, the output is multiplied by c^n, where n is the dimension of the input vectors. Therefore, the function is not linear in the first argument, and thus not bilinear.
4. It is clear that the multiplicative property holds for both vectors: $b(c\vec{u}, d\vec{v}) = cu_1 \cdot dv_n - cu_n \cdot dv_1 = cd(u_1v_n - u_nv_1) = cd \cdot b(\vec{u}, \vec{v})$. As for the additive property, consider the vector $\vec{w}$. $b(\vec{u} + \vec{w}, \vec{v}) = (u_1+w_1)v_n-(u_n+w_n)v_1 = u_1v_n-u_nv_1+w_1v_n-w_nv_1 = b(\vec{u}, \vec{v})+b(\vec{w}, \vec{b})$. The same steps can be followed for the second argument, $\vec{v}$. Therefore, this function is bilinear.
5. Without loss of generality, $b(\vec{u} + \vec{w}, \vec{v}) = (u_1 + w_1)v_1 + 2(u_2 + w_2)v_2 + \cdots + n(u_n + w_n)v_n = u_1v_1 + 2u_2v_2 + \cdots + nu_nv_n + w_1v_1 + 2w_2v_2 + \cdots + nw_nv_n = b(\vec{u}, \vec{v}) + b(\vec{w}, \vec{v})$. Again without loss of generality, $b(c\vec{u}, \vec{v}) = cu_1v_2 + 2cu_2v_2 + \cdots + ncu_nv_n = c(u_1v_1 + 2u_2v_2 + \cdots + nu_nv_n) = c \cdot b(\vec{u}, \vec{v})$.

(5)

$$det(\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}) = det(\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}) = 24$$

$$det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) = -2$$

$$det(\begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}) = det(\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}) = -6$$

$$det(\begin{bmatrix} n & 1 & 1 & \cdots & 1 \\ 0 & n-1 & 1 & \cdots & 1 \\ 0 & 0 & n-2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}) = n \cdot det(\begin{bmatrix} n-1 & 1 & \cdots & 1 \\ 0 & n-2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}) = n(n-1) \cdot det(\begin{bmatrix} n-2 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}) \cdots$$

$$= \frac{n!}{2} \cdot det(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}) = n!$$

(6)

Proof.

$$\|\vec{u} + \vec{v}\|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} + \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{x}$$

Since the dot product of two orthogonal vectors is 0, this therefore shows that $(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} \implies \|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$. □

(7)

1. The point with the largest norm must be $(c, c \cdots c)$, whose norm is $\sqrt{nc^2}$.
2. All points on the unit sphere have unit norm. Therefore, $\sqrt{nc^2} = 1 \implies nc^2 = 1 \implies c = \frac{1}{\sqrt{n}}$.
3. As n goes to infinity, $\sqrt{n}$ goes to ∞, so this value of c goes to 0.

(8)

1. The co-ordinates of C and D are, similarly, $(1-h, 1, 2)$ and $(1+h, 1, 2)$, respectively.
2. $\|A - B\| = \|(2-2, 1-h-(1+h), 1-1)\| = \|(0, 2h, 0)\| = 2h$.
3. $\|A - D\| = \|(2-(1+h), 1-h-1, 1-2)\| = \|(1-h, -h, -1)\| = \sqrt{(1-h)^2 + h^2 + 1^2} = \sqrt{2h^2 - 2h + 2}$.
4. $2h = \sqrt{2h^2 - 2h + 2} \implies h^2 + h - 1 = 0$. By the quadratic formula, we have $h = \frac{\sqrt{5}}{2} - \frac{1}{2}, h = -\frac{\sqrt{5}}{2} - \frac{1}{2}$. Since we know h must be positive, $h = \frac{\sqrt{5}}{2} - \frac{1}{2}$.