Homework 2 solutions

(1)

$3x - 2y = z \implies 3x - 2y - z = 0 = l(x, y, z)$. All level sets of this function are a flat plane in $\mathbb{R}^3$.

(2)

$\vec{f}(x) = (sin(x), cos(x))$, $\vec{g}(x) = (cos(x), sin(x))$, $\vec{h}(x) = (-sin(x), cos(x))$

(3)

1. For $c = 0$, the level set is a pair of intersecting lines, one covering the x-axis and one covering the y-axis. For all other values, the level set is a hyperbola, with values being in the top right and bottom left quadrants for $c = 1, 2$ and in the top left and bottom right quadrants for $c = -1$.
2. If $c = 0$, then the equation reduces to $x^2 + y^2 = 1$, therefore the level set is a unit circle about the origin. If $c = \frac{1}{2}$ the equation becomes $x^2 + y^2 = \frac{3}{4}$, so the level set is a circle with radius $\frac{\sqrt{3}}{2}$ about the origin. If $c = 1$ then the equation reduces to $x^2 + y^2 = 0$, so the level set is just the origin.
3. It is clear that this function never evaluates to 0 for any finite values of x and y, therefore the level set at $c = 0$ does not exist. If $c = 1$, the equation reduces to $x^2 + y^2 = 1$ which means the level set is a unit circle about the origin, and if $c = 2$ the equation becomes $x^2 + y^2 = \frac{1}{2}$, so similarly this level set is a circle with radius $\frac{1}{4}$ about the origin.
4. This is a linear combination of the elements of $\vec{x}$, so any level set will be a plane.

(4)

1. Theorem 2.2 states that $\left\|\vec{C}\vec{x}\right\| \leq \left\|\vec{C}\right\| \|\vec{x}\|$. The norm of a matrix is the square root of the sum of the norms of its rows or columns. Since $\vec{C}$ is an orthogonal matrix, all of its columns have norm 1. Therefore the norm of $\vec{C}$ is $\sqrt{n}$, and so $\left\|\vec{C}\vec{x}\right\| \leq \sqrt{n}\,\|\vec{x}\|$.

2. For any matrix $\vec{C}$, $\vec{C}\vec{x} = \Sigma_{i=1}^{n} x_i \vec{c}_i$, where x_i is the ith element of $\vec{x}$ and $\vec{c}_i$ is the ith column of $\vec{C}$. Therefore, using the fact that the columns of $\vec{C}$ are orthonormal, we have $\left\|\vec{C}\vec{x}\right\|^2 = \|\Sigma_{i=1}^{n} x_i \vec{c}_i\|^2 = \Sigma_{i=1}^{n} \|x_i \vec{c}_i\| = \Sigma_{i=1}^{n} x_i^2 = \|\vec{x}\|^2$. Since norms are always ≥ 0, we have $\left\|\vec{C}\vec{x}\right\| = \|\vec{x}\|$.

(7)

1. True: one of the values must clearly be smaller than all the others, since the function is not constant (known since two different values of the function have been provided).
2. True
3. False: the minimum value does not necessarily have to be -2.
4. True. By the definition of continuity, we know that $\forall\ \epsilon \exists \delta$ such that $\|\vec{x} - \vec{y}\| < \delta \implies \|f(\vec{x}) - f(\vec{y})\| < \epsilon$. If we set $\vec{y} = (\frac{-1}{2}, 0, 0)$ and $\epsilon = 0.02$, we can write $\vec{x} = (x, y, z)$, $\|\vec{x} - \vec{y}\| = |(x + \frac{1}{2})^2 + y^2 + z^2| = (x + \frac{1}{2})^2 + y^2 + z^2$ and $\|f(\vec{x}) - f(\vec{y})\| = \|f(\vec{x}) + 2\|$. We know that there exists a δ such that if it is greater than $(x + \frac{1}{2})^2 + y^2 + z^2$, that is, if $(x + \frac{1}{2})^2 + y^2 + z^2$ is sufficiently small, then the difference between -2 and $f(\vec{x})$ is less than 0.02.

(8)

1. This set is clearly bounded. Set $M > 5$, then, by the function definition, we have the fact that the norms of all points in the set are $\leq \sqrt{25} = 5$.
2. This set is the set of all points inside the intersection of $y = x^2 - 2$ and $y = \frac{x^2}{2} + 3$, which is where the values of $x^2 - 2$ are smaller than the values of $\frac{x^2}{2} + 3$, as required by the condition. Since the area of this intersection is finite, there must exist a subset of points in the set whose norms are finite and greater than the norms of all other points. Since the greatest norm is finite, there exists a finite $M > 0$ such that it is greater than the norm of any point in the set. Thus, the set is bounded.
3. The greatest value of y is clearly $\frac{1}{x}$, regardless of the value of x. That means that the square of the norm of any point (x, y) in the set is $(x, y) \cdot (x, y) = x^2 + y^2 = x^2 + \frac{1}{x^2} = \frac{x^4+1}{x^2}$. The degree of the polynomial on top of the fraction is greater than the degree of the polynomial on the bottom, which means this term tends to infinity as x tends to infinity, and it tends to 0 as x tends to 0. Therefore, the largest value of this term is at the largest value of x, i.e. $x = 2$, where this term evaluates to $\frac{17}{4}$. Since this is

the largest value of the square of the largest norm, the largest norm must be finite, and therefore the set is bounded.

(9)

1. $\left\{\vec{x} \in \mathbb{R}^2 | 5 < \|\vec{x}\| \leq 6\right\}$.
2. An open set: $\left\{\vec{x} \in \mathbb{R}^2 | 5 < \|\vec{x}\| < 6\right\}$. A closed set: $\left\{\vec{x} \in \mathbb{R}^2 | 5 \leq \|\vec{x}\| \leq 6\right\}$.
3. Any open 2-dimensional disk (not sphere) in $\mathbb{R}^3$ is not open, since every point in it is not an interior point. This is because points inside the disk are not interior points if the disk is considered as part of $\mathbb{R}^3$, since only a ball of radius 0 around them would not include any points from $\mathbb{R}^3$ that are not also inside the disk. Therefore those points are not interior points and this set is not open. However, it is open when intersected with the x-axis and considered part of $\mathbb{R}^1$, since their intersection is an open segment in $\mathbb{R}^1$.
4. As before, any open 2-dimensional disk is open in $\mathbb{R}^2$, but it is not open in $\mathbb{R}^3$, since the points inside the disk are no longer interior points.
5. Take all possible closed sections in $mathbbR$, of the form $\{x | a \leq x \leq b, a \neq b\}$ for any arbitrary constants a and b. While individually, these sets are closed, their union is $\mathbb{R}$, which is not closed.