MATH 3110 - SEQUENCES

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1. Sequences

1.1. **Definition.** A **sequence** is an ordered collection of rational or real numbers $\{a_n\}_{n=1}^{\infty}$. Essentially, it is a map $a : \mathbb{N} \to \mathbb{Q}$ or $\mathbb{R}$, that is, $n \mapsto a_n$.

1.1.1. *Example.* The set of numbers $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \cdots$, i.e. $a_n = \frac{1}{n^2}$ is a sequence.

1.1.2. *Example.* Another example of a sequence is $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3} \cdots$, where $b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$.

Any sequence that can be expressed in the form

$$b_n = \sum_{k=1}^{n} s_k$$

is called a **series**.

1.2. **Increasing sequences.** A sequence $\{a_n\}$ is **increasing** iff $a_{n+1} \geq a_n \ \forall\ n$. Note that by this definition, $a_n = 1$ is an increasing sequence, since the inequality is not strict. Therefore, we can also define a sequence as **strictly increasing** iff $a_{n+1} > a_n \ \forall\ n$. An example of a strictly increasing sequence is the sequence $b_n = \sum_{k=1}^{n} \frac{1}{k}$ from above, since $b_{n+1} - b_n = \frac{1}{n+1} > 0$. We can also define **decreasing** and **strictly decreasing** sequences in a similar way. A sequence that is either increasing **or** decreasing is called a **monotonic** sequence.

1.3. **Bounded sequences.** A sequence $\{a_n\}$ is **bounded above** iff there exists $M \in \mathbb{R}$ such that $a_n \leq M \ \forall\ n$. An example of a bounded sequence is $a_n = \frac{1}{n^2}$ from earlier, since $a_n \leq 1 \ \forall\ n$. Note that M does not necessarily have to be the *best* upper bound, but simply *an* upper bound; for example, setting $M = 100$ would also be sufficient to show that this sequence is bounded above.

Similarly, a sequence $\{a_n\}$ is **bounded below** iff there exists $P \in \mathbb{R}$ such that $a_n \geq P \ \forall\ n$. Finally, a sequence is **bounded above and below** iff it is bounded above and bounded below.

1.3.1. *Example.* Once again examine the sequence $b_n = \sum_{k=1}^{n} \frac{1}{k}$. This sequence is bounded below, but not bounded above

Proof. Trivially, $b_1 \geq 1$. We know that the sequence is increasing, that is, $b_{n+1} > b_n \ \forall\ n$. Therefore, $b_n > b_1 \geq 1$, so the sequence is bounded below with $P = 1$.

Proving that it is not bounded above is harder. Take any $M \in \mathbb{R}$. It can be seen by analysing the series that for $k > 2M$, $b_{2^k} > 1 + \frac{k}{2} > 1 + \frac{2M}{2} = 1 + M > M$. Therefore, the sequence is not bounded above by any M. □

2. Convergence (an imprecise definition)

We say a sequence $\{a_n\}$ **converges to** L iff for all integers $k \geq 0$, there exists an integer N, such that for all $n \geq N$, a_n and L are equal up to k decimal places.

2.0.1. *Example.* Consider the sequence $a_1 = 1, a_2 = 1.1, a_3 = 1.11 \cdots$. This sequence is bounded by $L = 1.\overline{1} = \frac{10}{9}$.

Proof. $\square$