THE EXPECTATION OF A RANDOM VARIABLE

1. The expectation function

The expectation function E gives the mean value of a random variable. It follows the following rules:

- $\mathrm{E}(aX+b) = a\mathrm{E}(X)+b$
- $\mathrm{E}(X+Y) = \mathrm{E}(X)+\mathrm{E}(Y)$
- $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$ iff X and Y are independent

2. The expectation of a discrete random variable

Suppose we have a variable X such that $X : S \to D$ for some sample space S. As a result of the application of the formula for the expectation, we have

$$\mathrm{E}(X) = \sum_{D} x f(x)$$

2.0.1. *Example.* Suppose we define X such that $\mathbb{P}(X=x) = \frac{1}{x(x+1)}$ for $x \in \mathbb{Z}^+$. This is a valid probability distribution, since $\sum_{i=1}^{\infty} \mathbb{P}(X=i) = 1$. Using the formula above, we have

$$\mathrm{E}(X) = \sum_{i=1}^{\infty} \frac{i}{i(i+1)} = \sum_{i=1}^{\infty} \frac{1}{i+1} = \infty$$

2.0.2. *Example.* To calculate the expected value of a binomially distributed variable, represent it as follows. Suppose there are n Bernoulli distributed variables X_i such that $\mathbb{P}(X=1)=p$ and $\mathbb{P}(X=0)=1-p$. Then, $X = \sum_{i=0}^{n} X_i$ is binomially distributed with $X \sim B(n,p)$, since it is the sum of successes from n trials with probability of success p. Using the rules listed above, we have

$$\mathrm{E}(X) = \mathrm{E}\left(\sum_{i=0}^{n} X_i\right) = \sum_{i=0}^{n} \mathrm{E}(X_i) = np$$

3. The expectation of a continuous random variable

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

if at least one of $\int_{-\infty}^{0} x f(x)\,\mathrm{d}x$ and $\int_{0}^{\infty} x f(x)\,\mathrm{d}x$ is finite.

3.0.1. *Example.* Suppose we have a device that will fail after a certain period of time, and that the time until the device fails is modelled by $X \sim f(x)$ where

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

In this case, $\mathrm{E}(X) = \int_0^1 (x \cdot 2x)\,\mathrm{d}x = 2\int_0^1 x^2\,\mathrm{d}x = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$.

3.0.2. *Example.* An example of a pathological case is the Cauchy distribution: a special kind of continuous distribution whose p.d.f. is defined as $f(x) = \frac{1}{\pi(1+x^2)}$. Suppose we have $X \sim f(x)$. We know that

$$\int_{-\infty}^{0} \frac{x}{\pi(1+x^2)} = -\infty$$
$$\int_{0}^{\infty} \frac{x}{\pi(1+x^2)} = \infty$$

Therefore, $\mathrm{E}(X) = -\infty + \infty$, which is undefined, so this distribution *does not have a mean*!.

4. The expectation of a function of random variables

4.1. **Theorem.** Let X be a random variable, and let $Y = r(X)$ for $r : A \subseteq \mathbb{R} \to B \subseteq \mathbb{R}$. We have

$$\mathrm{E}(Y) = \sum_{B} r(x)f(x)$$

if X is discrete, and

$$\mathrm{E}(Y) = \int_{-\infty}^{\infty} r(x)f(x)\,\mathrm{d}x$$

if X is continuous. These results follow directly from the *Law of the Unconscious Statistician.*

4.1.1. *Example.* Let X represent the rate of failure of a certain machine per year, and let $Y = \frac{1}{X}$ be the time taken to fail. Suppose $X \sim f(x)$ where

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

In order to find the expected amount of time until the first failure, we calculate

$$\mathrm{E}(Y) = \int_{0}^{1} \frac{1}{x} \cdot 3x^2\,\mathrm{d}x = 3\int_{0}^{1} x\,\mathrm{d}x = 3\frac{x^2}{2}\bigg|_{0}^{1} = \frac{3}{2}$$

4.2. **Theorem.** Let $(X_1, X_2) \sim f(x_1, x_2)$, and let $Y = r(X_1, X_2)$ for $r : A \subseteq \mathbb{R}^2 \to B \subseteq \mathbb{R}^2$. Then, for continuous X

$$\mathrm{E}(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(x_1, x_2)f(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

4.2.1. *Example.* Let $(X, Y) \sim f(x, y)$ where, for some $S \subseteq \mathbb{R}^2$

$$f(x, y) = \begin{cases} 1 & (x, y) \in S \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = X^2+Y^2$, i.e. the squared distance of (X, Y) from the origin. Suppose $S = \{(x, y) | 0 < x < 1, 0 < y < 1\}$. To calculate the mean squared distance of a point within S from the origin, we use

$$\mathrm{E}(Z) = \int_0^1 \int_0^1 (x^2 + y^2) f(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 \int_0^1 (x^2 + y^2) \, \mathrm{d}x \, \mathrm{d}y = \frac{2}{3}$$