MATH 2220 HW #10

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Problem 1

(a) If we transform to polar co-ordinates, $x^2+y^2=r$, so we have $a \le r \le b$. The limits for θ are $0 \le \theta \le 2\pi$, since we are integrating across the entire annulus. Therefore, using the substitution $u = -r^2$:

$$\begin{aligned}
\int_D e^{-x^2-y^2}\,\mathrm{d}A &= \int_0^{2\pi}\int_a^b re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_a^b\int_0^{2\pi} re^{-r^2}\,\mathrm{d}\theta\,\mathrm{d}r = 2\pi\int_a^b re^{-r^2}\,\mathrm{d}r \\
&= 2\pi\int_{-a^2}^{-b^2} -\frac{e^u}{2}\,\mathrm{d}u = \pi\int_{-b^2}^{-a^2} e^u\,\mathrm{d}u \\
&= \pi(e^{-a^2} - e^{-b^2})
\end{aligned}$$

(b) This is equal to the integral from above, with $a = 0, b = \infty$:

$$\begin{aligned}
\int_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A &= \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \lim_{b\to\infty}\pi(e^{-0^2} - e^{-b^2}) = \pi
\end{aligned}$$

(c) By the theorem proved in problem 3 from homework 9, we know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)^2$. Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A = \pi$, it follows that $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.

Problem 2

The limits of this integral are $0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x^2$. To solve this problem we must rearrange these inequalities, so that the limits for z and then y are absolute.

We have $0 \le x \le 1, 0 \le z \le 1-x^2$, which is equivalent to $0 \le z \le 1, 0 \le x \le \sqrt{1-z}$. Therefore, the integral becomes

$$\int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$

Similarly, we have $0 \le x \le 1, 0 \le y \le 1-x$, which is equivalent to $0 \le y \le 1, 0 \le x \le 1-y$, so the integal becomes

$$\int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

Problem 3

(a) $\int_D e^{-x}\,\mathrm{d}A = \int_0^\infty \int_0^1 e^{-x}\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty e^{-x}\,\mathrm{d}x = 1.$

(b) $\int_0^\infty e^{-xy}\,\mathrm{d}x = -e^{-xy}\Big|_{x=0}^{x=\infty} = \frac{e^0 - e^{-\infty}}{y} = \frac{1}{y}.$

(c) $\int_D e^{-xy}\,\mathrm{d}A = \int_0^1 \int_0^\infty e^{-xy}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \frac{1}{y}\,\mathrm{d}y$. $\frac{1}{y}$ is not integrable over any domain that icludes $y = 0$, so e^{-xy} is not integrable over D.

Problem 4
A circle with radius R is defined by the curve $x^2 + y^2 = R^2$. If we transform to polar co-ordinates using $u(x, y) = (r\cos\theta, r\sin\theta)$, we get $r^2 = R^2$, or $r = R$ (there is no need for a minus sign, as r cannot be negative), for $0 \le \theta \le 2\pi$. So, the area of the circle is the integral of r (the determinant of $\vec{D}u$) over all points inside the circle ($C : x^2 + y^2 \le R$):

$$\begin{aligned}\int_C \mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^R r\,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi} \frac{R^2}{2}\,\mathrm{d}\theta \\ &= 2\pi\frac{R^2}{2} = \pi R^2\end{aligned}$$

Similarly, a sphere with radius R is defined by $x^2+y^2+z^2 = R^2$. If we transform to spherical co-ordinates using $v(x, y, z) = (\rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)$, we have $x^2+y^2+z^2 = \rho^2\sin^2\theta\cos^2\phi + \rho^2\sin^2\theta\sin^2\phi + \rho^2\cos^2\theta = \rho^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta) = \rho^2(\sin^2\theta[\cos^2\phi + \sin^2\phi] + \cos^2\theta) = \rho^2(\sin^2\theta + \cos^2\theta) = \rho^2$, and therefore $\rho = R$ (no minus sign for the same reason). The limits for a complete sphere are, therefore, $0 \le \rho \le R, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$. The Jacobian matrix for v is

$$\vec{D}v = \begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \rho\cos\theta\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\theta & -\rho\sin\theta & 0 \end{bmatrix}$$

Therefore, $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \det\vec{D}v\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi$. To find the volume of the sphere, we need to integrate on the set $S : x^2 + y^2 + z^2$. Using the transformation rules derived above, we have

$$\begin{aligned}\int_S \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^\pi \int_0^{2\pi} \int_0^R \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \int_0^\pi \int_0^{2\pi} \frac{R^3}{3}\sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \int_0^\pi \frac{2\pi R^3}{3}\sin\phi\,\mathrm{d}\phi \\ &= \frac{4\pi R^3}{3}\end{aligned}$$

Problem 5
The limits given by the definition of W are $0 \le z \le 25 - x^2 - y^2, -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}, -2 \le x \le 2$. Therefore:

$$
\begin{aligned}
\int_W (x^2+y^2+2z)\,\mathrm{d}V &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{25-x^2-y^2} (x^2+y^2+2z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (zx^2+zy^2+z^2)\Big|_{z=0}^{z=25-x^2-y^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^2(25-x^2-y^2)+y^2(25-x^2-y^2)+(25-x^2-y^2)^2\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 25(25-x^2-y^2)\,\mathrm{d}y\,\mathrm{d}x = 25\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(25-x^2-y^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= 25\int_{-2}^{2}(25y-yx^2-\frac{y^3}{3})\Big|_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\,\mathrm{d}x = -25\int_{-2}^{2}\frac{2}{3}\sqrt{4-x^2}(2x^2-71)\,\mathrm{d}x
\end{aligned}
$$

This somehow evaluates to 2300π.

Problem 6

Mass can be evaluated as the integral of density with respect to volume. In this case, the cube has side length 2 and density $\rho(x,y,z) = x^2+y^2$. Assuming the cube is centered at the origin, the limits of integration are $-1 \le x,y,z \le 1$. Therefore, the total mass is

$$
\begin{aligned}
\int_C \rho(x,y,z)\,\mathrm{d}V &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^{1}\int_{-1}^{1} 2(x^2+y^2)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^{1} 4x^2+\frac{4}{3}\,\mathrm{d}x \\
&= \frac{16}{3}
\end{aligned}
$$

If the density is instead $\rho(x,y,z) = x^2+y^2+z^2$, then the total mass is

$$
\begin{aligned}
\int_C \rho(x,y,z)\,\mathrm{d}V &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2+z^2)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^{1}\int_{-1}^{1}\left(2(x^2+y^2)+\frac{2}{3}\right)\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-1}^{1}\left(4x^2+\frac{8}{3}\right)\mathrm{d}x \\
&= \frac{4}{3}+\frac{8}{3}+\frac{4}{3}+\frac{8}{3} = 8
\end{aligned}
$$

Problem 7 To find the volume of a cylinder with radius r and height h, we can compute the following integral:

$$
\int_0^h\int_{-r}^{r}\int_{-\sqrt{r-x^2}}^{\sqrt{r-x^2}}\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z
$$

However, in this case we need to find the volume of a part of the cylinder. In this case, there is a variable lower bound on the value of x, which varies as a linear function of z. When $z = h$, $r \geq x \geq r$, and when $z = 0$, $r \geq x \geq -r$. Therefore, $r \geq x \geq r(\frac{2z}{h} - 1)$, and we need to find the following:

$$\int_0^h \int_{r(\frac{2z}{h}-1)}^r \int_{-\sqrt{r-x^2}}^{\sqrt{r-x^2}} \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z$$

Since this function is linear, and since, as mentioned, at $z = h$, $r \geq x \geq r \Rightarrow x = r$, and at $z = 0$, $r \geq x \geq -r$, this integral is equal to half of the integral above, i.e. half of the volume of the cylinder with radius r and height h. Its value is therefore $\frac{\pi r^2 h}{2}$.

Problem 8

(a) The unit disk is defined as the set $C : \{(x, y) | x^2 + y^2 \leq 1\}$. The distance from the centre of a point (x, y) is $\sqrt{x^2 + y^2}$; transforming to polar co-ordinates, we use $u(x, y) = (r \cos \theta, r \sin \theta)$, $\det \vec{D}u = r$, so:

$$\begin{aligned}\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2 + y^2} \, \mathrm{d}y \, \mathrm{d}x &= \int_0^{2\pi} \int_0^1 r^2 \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{1}{3} \, \mathrm{d}\theta = \frac{2\pi}{3}\end{aligned}$$

To get the mean value we need to divide by the area, which is π, so we have $\frac{2}{3}$.

(b) Similarly, the unit sphere is defined as the set $S : \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$. The distance from the centre of (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$. To transform to spherical co-ordinates, use $v(x, y, z) = (\rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi, \rho \cos \theta)$, $\det \vec{D}v = \rho^2 \sin \phi$; the distance to the centre becomes $\sqrt{x^2 + y^2 + z^2} = \rho$. Therefore, the integral becomes:

$$\begin{aligned}&\int_0^\pi \int_0^{2\pi} \int_{0^1} \rho^3 \sin \phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \int_0^\pi \int_0^{2\pi} \frac{\sin \phi}{4} \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \int_0^\pi \frac{\pi \sin \phi}{2} \, \mathrm{d}\phi = \pi\end{aligned}$$

The volume of the unit sphere is $\frac{4\pi}{3}$, so the average distance is $\frac{3\pi}{4\pi} = \frac{3}{4}$.