MATH 2220 HW #12

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Problem 1

First we need to parametrise each of the planes in the form $(x, y, z) = \mathbf{r}(s, t)$, and then, to find the area, evaluate the integral $\iint_S \mathrm{d}\sigma = \iint_D \|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s\,\mathrm{d}t$.

For S_1, we have the boundaries $y, z \in [0, 1]$. We can set $y = s, z = t$, keeping the same boundaries, and then solve for $x = \frac{7}{2}\left(10 - \frac{3y}{4} - \frac{6z}{7}\right) = \frac{7}{2}\left(10 - \frac{3s}{4} - \frac{6t}{7}\right)$. Then, $\mathbf{r}_s = (x_s, y_s, z_s) = (-\frac{21}{8}, 1, 0)$ and $\mathbf{r}_t = (x_t, y_t, z_t) = (-3, 0, 1)$. Then

$$\begin{aligned} A(S_1) = \iint_{S_1} \mathrm{d}\sigma &= \iint_D \|\mathbf{r}_s \times \mathbf{r}_t\| \,\mathrm{d}s\,\mathrm{d}t \\ &= \iint_D \left\|(-\frac{21}{8}, 1, 0) \times (-3, 0, 1)\right\| \mathrm{d}s\,\mathrm{d}t \\ &= \iint_D \left\|(1, \frac{21}{8}, 3)\right\| \mathrm{d}s\,\mathrm{d}t \\ &= \iint_D \frac{\sqrt{1081}}{8} \,\mathrm{d}s\,\mathrm{d}t = \frac{\sqrt{1081}}{8} \int_0^1 \int_0^1 \mathrm{d}s\,\mathrm{d}t = \frac{\sqrt{1081}}{8} \end{aligned}$$

The process is similar for S_2 and S_3. S_2 has the bounds $x, z \in [0, 1]$, so to parametrise it we set $x = s, z = t$, then solve for $y = \frac{4}{3}\left(10 - \frac{2x}{7} - \frac{6z}{7}\right) = \frac{4}{3}\left(10 - \frac{2s}{7} - \frac{6t}{7}\right)$. $\mathbf{r}_s = (1, -\frac{8}{21}, 0)$ and $\mathbf{r}_t = (0, -\frac{8}{7}, 1)$. $\|\mathbf{r}_s \times \mathbf{r}_t\| = \left\|(-\frac{8}{21}, -1, -\frac{8}{7})\right\| = \frac{\sqrt{1081}}{21}$. Since there is nothing else in the integrand, the surface area is simply $\frac{\sqrt{1081}}{21}$.

For S_3, we have $x, y \in [0, 1]$, so set $x = s, y = t$ and solve for $z = \frac{7}{6}\left(10 - \frac{2s}{7} - \frac{3t}{4}\right)$. $\mathbf{r}_s = (1, 0, -\frac{1}{3})$ and $\mathbf{r}_t = (0, 1, -\frac{7}{8})$, and $\|\mathbf{r}_s \times \mathbf{r}_t\| = \left\|(\frac{1}{3}, \frac{7}{8}, 1)\right\| = \frac{\sqrt{1081}}{24}$, which is also the surface area.

Actually, I am so dumb. The two derivative vectors completely form the parallelogram, so the area is simply the norm of their cross-product, no need to integrate in the first place.

Problem 2

(a) The surface of a sphere with radius R is the set of points (x, y, z) such that $x^2 + y^2 + z^2 = R$. The cap of this sphere with height h is, as defined, the subset of this set subject to the additional constraint $R - h \leq z \leq R$.

To parametrise this former set, we can use spherical co-ordinates, with $\rho = R$. Set $\mathbf{r}(\theta, \phi) = (R \sin\phi \cos\theta, R \sin\phi \sin\theta, R \cos\phi)$, for $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$. However, we need to incorporate the additional constraint, that is, $R - h \leq R\cos\phi \leq R$. The upper bound is already satisfied as the maximum value of $\cos\phi$ is 1. Let $n = \arccos\left(\frac{R-h}{R}\right)$. For the lower bound, we have $\frac{R-h}{R} \leq \cos\phi \implies \phi \leq \arccos\left(\frac{R-h}{R}\right)$. So, our bounds for ϕ are $0 \leq \phi \leq n$.

Now we just integrate using spherical co-ordinates:

$$\begin{aligned}A = \iint_D R^2 \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\theta &= R^2 \int_0^{2\pi}\int_0^n \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= R^2\int_0^{2\pi}(1-\cos n)\,\mathrm{d}\theta \\ &= -2\pi R^2(\cos n - 1) = -2\pi R^2\left(\frac{R-h}{R}-1\right) = 2\pi R^2\frac{h}{R} = 2\pi Rh\end{aligned}$$

(b) The surface area of a cylinder with radius R and height h (not counting the top and bottom) is simply the area of the rectangle around it, with sides equal to the height (h) and the circumference of the base circle ($2\pi R$); evaluating to $2\pi Rh$.

(c) Suppose the slice starts at $z = z_0$. Then, its endpoints in the z-axis are z_0 and $z_0 + h$. This gives us the range of ϕ: $\arccos\frac{z_0+h}{R} \le \phi \le \arccos\frac{z_0}{R}$. Denote these bounds as a and b, respectively. This time, we compute the same integral but with these bounds:

$$\begin{aligned}A = \iint_D R^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta &= R^2\int_0^{2\pi}\int_a^b \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \int_0^{2\pi}(\cos a - \cos b)\,\mathrm{d}\theta = \int_0^{2\pi}\left(\frac{z_0+h}{R}-\frac{z_0}{R}\right)\mathrm{d}\theta \\ &= \int_0^{2\pi}\frac{h}{R}\,\mathrm{d}\theta\end{aligned}$$

This last integral is independent of z_0, and so it does not matter where the slice is: as long as its height is h, its surface area will be constant.

Problem 3

(a) This is just the surface area of the unit sphere, 4π.

(b) Let U be the upper hemisphere of S (where $z \ge 0$) and L be the lower hemisphere (where $z \le 0$). Then, $\int_S z\,\mathrm{d}\sigma = \int_{U\cup L} z\,\mathrm{d}\sigma = \int_U z\,\mathrm{d}\sigma + \int_L z\,\mathrm{d}\sigma = \int_U z\,\mathrm{d}\sigma - \int_L -z\,\mathrm{d}\sigma$. Because the values of z are opposite in the two hemispheres, $\int_L -z\,\mathrm{d}\sigma = \int_U z\,\mathrm{d}\sigma$, so $\int_U z\,\mathrm{d}\sigma - \int_L -z\,\mathrm{d}\sigma = 0$.

(c) Any vector $\mathbf{x}$ on the unit sphere by definition satisfies the property $\|\mathbf{x}\| = 1$. Therefore, $\int_S \|\mathbf{x}\|^2\,\mathrm{d}\sigma = \int_S \mathrm{d}\sigma = 4\pi$.

(d) The unit sphere is symmetric along every axis going through the origin, which includes the x-, y- and z-axis. This means that x, y and z can be used interchangeably, that is, $\int_S x^2\,\mathrm{d}\sigma = \int_S y^2\,\mathrm{d}\sigma = \int_S z^2\,\mathrm{d}\sigma$. Moreover, $\int_S \|\mathbf{x}\|^2\,\mathrm{d}\sigma = \int_S (x^2+y^2+z^2)\,\mathrm{d}\sigma = \int_S x^2\,\mathrm{d}\sigma + \int_S y^2\,\mathrm{d}\sigma + \int_S z^2\,\mathrm{d}\sigma = 4\pi$. Since the three are equal, we have $\int_S x^2\,\mathrm{d}\sigma = \int_S y^2\,\mathrm{d}\sigma = \int_S z^2\,\mathrm{d}\sigma = \frac{4\pi}{3}$.

Problem 4

S can be thought of as the union of the following: A, a plane along the x- and z-axes, with $0 \le x \le 2, 0 \le z \le 2$, B, a plane along the x- and y-axes, with $0 \le x \le 2, 0 \le y \le 2$ and C, a plane along $y = 2$, with $0 \le x \le 2, 0 \le z \le 1$.

(a) $(0, 1, 0)$ is orthogonal to A and C and lies in B. Therefore, its flux over B is 0. A is oriented opposite to B, so the vector's flux over them will have opposing signs; however, A has a larger area, so A's flux outweighs that of C, and the total flux is negative.

(b) The flux across A is 0, since on any point in A the vector $(0, 3y, 0) = (0, 0, 0)$. The vector lies in B so once again the flux with B is 0, and on C it is equal to $(0, 6, 0)$ and it is orthogonal, in the direction of C, so the flux with C, and the total flux, is positive.

(c) The flux across A is 0 since the vector $(1, 0, 0)$ on A will lie in A. The vector has zero z-component, so it will also lie in B. On C the vector is $(1, 6, 0)$, and does not lie within C, so the flux is positive (since it points in the direction of C). The total flux is positive.

(d) This vector has zero y-component, so its flux on both A and C is zero. On B, it faces "upwards", since $x^2 > 0 \; \forall \, x \in \mathbb{R}$ and $5 > 0$. B faces "downwards", so its flux on B, and therefore the total flux, is negative.

Problem 5

A unit cylinder with arbitrary height lying along the x-axis is defined as the set of points (x, y, z) such that $y^2 + z^2 \leq 1$, and a unit cylinder lying along the y-axis is the set of points where $x^2 + z^2 \leq 1$. Their intersection is where both of these constraints are met.

We are concerned with the boundary of their intersection. Solving for y and z in terms of x we get $D : y = \pm x, z = \pm\sqrt{1 - x^2}$. The surface area is, therefore

$$\int_D y \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2} \, \mathrm{d}z$$

$$= \int_D x \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x$$

$$= \int_D \frac{x}{\sqrt{1 - x^2}} \, \mathrm{d}z$$

for one quarter of one of the four faces. We can integrate this from 0 to 1 to get the surface area of a quarter of one of the faces, and then multiply by 16 to get the total surface area:

$$\frac{A}{16} = \int_0^1 \frac{x}{\sqrt{1 - x^2}} \, \mathrm{d}x = -\sqrt{1 - x^2}\Big|_0^1 = 1$$

So the total surface area is 16.

For the volume, we can take a different approach, using Calc II. If we take a slice through the solid along the z-axis, we get a square, with side length $2\sqrt{1 - x^2}$. Its volume is, therefore the integral of the area of this square over the full range of x, which is $-1 \leq x \leq 1$:

$$V = \int_{-1}^1 (2\sqrt{1 - x^2})^2 \, \mathrm{d}x = 4 \int_{-1}^1 (1 - x^2) \, \mathrm{d}x$$

$$= 4(x - \frac{x^3}{3})\Big|_{-1}^1 = \frac{16}{3}$$

Problem 6

(a) A smooth surface is a surface whose defining functions have infinitely many derivatives. We are given that the surface parametrised by $X(u, v)$ is smooth; that is, if we denote $X(u, v) = (f_1(u, v), f_2(u, v), f_3(u, v))$, then each f_i is C^∞. T is defined as being parametrised by $kX(u, v) = (kf_1(u, v), kf_2(u, v), kf_3(u, v))$. If f_i is C^∞ then kf_i where $k \in \mathbb{R}$ is also C^∞: all of its derivatives are the corresponding derivatives of f_i, multiplied by k. Thus, T parametrised by $Y(u, v) = kX(u, v)$ is smooth.

(b) The area of S can be calculated by surface integration:

$$A_S = \iint_S \mathbf{n}\,\mathrm{d}\sigma = \iint_D \left\| \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \right\| \mathrm{d}u\,\mathrm{d}v$$

Similarly, we can calculate the area of T, using the properties of the cross product:

$$\begin{aligned} A_T = \iint_T \mathbf{n}\,\mathrm{d}\sigma &= \iint_D \left\| \frac{\partial Y}{\partial u} \times \frac{\partial Y}{\partial v} \right\| \mathrm{d}u\,\mathrm{d}v \\ &= \iint_D \left\| \frac{\partial kX}{\partial u} \times \frac{\partial kX}{\partial v} \right\| \mathrm{d}u\,\mathrm{d}v \\ &= \iint_D \left\| k\frac{\partial X}{\partial u} \times k\frac{\partial X}{\partial v} \right\| \mathrm{d}u\,\mathrm{d}v \\ &= \iint_D k^2 \left\| \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \right\| \mathrm{d}u\,\mathrm{d}v \\ &= k^2 \iint_D \left\| \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \right\| \mathrm{d}u\,\mathrm{d}v = k^2 A_S \end{aligned}$$

(c) By definition, the flux of $\mathbf{F}$ through S is $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\sigma$. We have $\frac{\partial \mathbf{X}}{\partial u} = (1,0,0)$ and $\frac{\partial \mathbf{X}}{\partial v} = (0,1,0)$, and $\frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} = (0,0,1)$. Thus:

$$\begin{aligned} \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\sigma &= \iint_D \mathbf{F}(\mathbf{X}(u,v))\cdot(0,0,1)\,\mathrm{d}u\,\mathrm{d}v \\ &= \iint_D (u,v,1)\cdot(0,0,1)\,\mathrm{d}u\,\mathrm{d}v = \iint_D 1\,\mathrm{d}u\,\mathrm{d}v \\ &= \pi \end{aligned}$$

since $\iint_D 1\,\mathrm{d}u\,\mathrm{d}v$ is simply the area of the unit disk. If we repeat the same calculation for T, the only difference is that the vector product of the partial derivatives is now $(0,0,k^2)$, and so the integral at the end is instead $\iint_D k^2\,\mathrm{d}u\,\mathrm{d}v = k^2 \iint_D \mathrm{d}u\,\mathrm{d}v = k^2\pi$.

(d)

Problem 7

(a) We know that the total flux through the entire cube is 0, since the vector field is conservative and the cube is symmetric. Suppose the cube is aligned with the x-, y- and z-axes, in the positive octant (?) with one of its vertices at the origin. Then, we can form normal vectors to each of its faces, all of which have ±1 as one of the components, and 0 as the other two. Therefore, the flux through each face will plus/minus be one of the components of $\mathbf{c} = (c_1, c_2, c_3)$, integrated between two of $x, y, z \in [0,1]$, which will simply multiply it by 1. Therefore, the fluxes are $\pm c_1$, $\pm c_2$ and $\pm c_3$.

(b) This time, we are integrating the dot product of the normal vector and (c_1+x, c_2+y, c_3+z). For example, in the case of the face for which the normal vector is $(1,0,0)$:

$$\begin{aligned}\int_S (c_1 + x, c_2 + y, c_3 + z) \cdot \hat{\mathbf{n}} \, \mathrm{d}A &= \int_0^1 \int_0^1 (c_1 + x) \, \mathrm{d}y \, \mathrm{d}z \\ &= c_1 + x\end{aligned}$$

The others are similar, with the variables of integration changing depending on the face. For example, for the vector $(1, 0, 0)$, since the x-component is 1 and the others are 0, we know that this plane lies along the y- and z-axes, so we integrate $\mathrm{d}y \, \mathrm{d}z$, and ditto for the other 5 faces. For all 6 faces, we have fluxes $\pm(c_1 + x)$, $\pm(c_2 + y)$ and $\pm(c_3 + z)$.

(c) Again, I will illustrate using an example with $\hat{\mathbf{n}} = (1, 0, 0)$:

$$\begin{aligned}\int_S (c_1 y, c_2 z, c_3 x) \cdot \hat{\mathbf{n}} \, \mathrm{d}A &= \int_0^1 \int_0^1 c_1 y \, \mathrm{d}y \, \mathrm{d}z \\ &= \frac{c_1}{2}\end{aligned}$$

Similarly, we have fluxes $\pm\frac{c_i}{2}$ for $i = 1, 2, 3$.

Problem 8

(a) The boundary is a rectangle (a square) and so it can be thought of as having four distinct components, for each of its four sides. The components are $x = 0, y \in [0, 1]$, $x = 1, y \in [0, 1]$, $y = 0, x \in [0, 1]$ and $y = 1, x \in [0, 1]$. Each component has a constant normal vector, oriented outwards: $(-1, 0)$, $(1, 0)$, $(0, -1)$ and $(0, 1)$, respectively. Therefore, we can split the integral up into these four distinct parts:

$$\begin{aligned}\int_{\partial D} \mathbf{f} \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= \int_0^1 f(0, y) \cdot (-1, 0) \, \mathrm{d}y + \int_0^1 f(1, y) \cdot (1, 0) \, \mathrm{d}y \\ &\quad + \int_0^1 f(x, 0) \cdot (0, -1) \, \mathrm{d}x + \int_0^1 f(x, 1) \cdot (0, 1) \, \mathrm{d}x \\ &= \int_0^1 -f_1(0, y) \, \mathrm{d}y + \int_0^1 f_1(1, y) \, \mathrm{d}y + \int_0^1 -f_2(x, 0) \, \mathrm{d}x + \int_0^1 f_2(x, 1) \, \mathrm{d}x \\ &= \int_0^1 (f_1(1, y) - f_1(0, y)) \, \mathrm{d}y + \int_0^1 (f_2(x, 1) - f_2(x, 0)) \, \mathrm{d}x\end{aligned}$$

(b) From the fundamental theorem of calculus, we know that $\int_a^b f_x(x) \, \mathrm{d}x = f(b) - f(a)$, where $f_x = \frac{\mathrm{d}}{\mathrm{d}x} f$. Therefore, $f_1(1, y) - f_1(0, y) = \int_0^1 \frac{\partial}{\partial x} f_1(x, y) \, \mathrm{d}x$, and similarly $f_2(x, 1) - f_2(x, 0) = \int_0^1 \frac{\partial}{\partial y} f_2(x, y) \, \mathrm{d}y$. Therefore

$$\begin{aligned}\int_{\partial D} \mathbf{f} \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= \int_0^1 \int_0^1 \frac{\partial}{\partial x} f_1(x, y) \, \mathrm{d}x \, \mathrm{d}y + \int_0^1 \int_0^1 \frac{\partial}{\partial y} f_2(x, y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_D \frac{\partial}{\partial x} f_1(x, y) \, \mathrm{d}A + \int_D \frac{\partial}{\partial y} f_2(x, y) \, \mathrm{d}A \\ &= \int_D \left(\frac{\partial}{\partial x} f_1(x, y) + \frac{\partial}{\partial y} f_2(x, y) \right) \mathrm{d}A\end{aligned}$$