Homework 6 solutions

1

(a)

$x^2 + y^2 = 1 \implies f(x, y) = e^{-1}, g(x, y) = e^{-1}$.

(b)

$\nabla f = \begin{bmatrix} -2xe^{-x^2-y^2} & -2ye^{-x^2-y^2} \end{bmatrix}$, $\nabla g = \begin{bmatrix} \frac{-2e^{-1}x}{(x^2+y^2)^2} & \frac{-2e^{-1}y}{(x^2+y^2)^2} \end{bmatrix}$.

(c)

If $a^2+b^2 = 1$, then $\nabla f(a, b) = \begin{bmatrix} -2ae^{-1} & -2be^{-1} \end{bmatrix}$, and $\nabla g = \begin{bmatrix} -2ae^{-1} & -2be^{-1} \end{bmatrix}$. Since $\nabla f(a, b) = \nabla g(a, b)$ and $f(a, b) = g(a, b)$, the graphs are tangent.

2

(a)

$\frac{\partial}{\partial x}(y + x^2)^3 = 6x(y + x^2)^2$.

(b)

$\frac{\partial}{\partial x} g(x + x^2) = 2x \cdot g_x(y + x^2)$.

3

(a)

$f_1 = r^2 \cos 2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2(\frac{x^2}{r^2} - \frac{y^2}{r^2}) = x^2 - y^2$.

$f_2 = r^{-2} \sin 2\theta = r^{-2}(2 \sin\theta \cos\theta) = 2r^{-2} \frac{y}{r} \frac{x}{r} = \frac{2xy}{(x^2+y^2)^2}$

$f_3 = r^3 \sin 3\theta = r^3(3 \sin\theta \cos^2\theta - \sin^3\theta) = r^3(3 \frac{y}{r} \frac{x^2}{r^2} - \frac{y^3}{r^3}) = 3x^2y - y^3$

(b)

$f_{1xx} = \frac{\partial}{\partial x} 2x = 2$, $f_{1yy} = \frac{\partial}{\partial y} - 2y = -2$, so $f_{1xx} + f_{1yy} = 0$.

$f_{2xx} = \frac{\partial}{\partial x} \frac{y^3 - 3x^2 y}{(x^2+y^2)^3} = \frac{12xy(x^2-y^2)}{(x^2+y^2)^4}$. $f_{2yy} = \frac{12xy(y^2-x^2)}{(x^2+y^2)^4}$ since $f(x,y) = f(y,x)$. Therefore $f_{2xx} = -f_{2yy}$.

$f_{3xx} = \frac{\partial}{\partial x} 6xy = 6y$. $f_{3yy} = \frac{\partial}{\partial y}(3x^2 - 3y^2) = -6y = -f_{3xx}$.

4

(a)

For $f(x,y) = x^4 + y^4$, $\Delta f = f_{xx} + f_{yy} = 12(x^2+y^2)$. For $f(x,y) = (x^2+y^2)^p$, $f_{1xx} = \frac{\partial}{\partial x} 2px(x^2+y^2)^{p-1} = 2p(x^2+y^2)^{p-2}(x(p-1) + (x^2+y^2))$ $f_{yy} = 2p(x^2+y^2)^{p-2}(y(p-1) + (x^2+y^2))$ since $f(x,y) = f(y,x)$ $\Delta f = f_{xx} + f_{yy} = 2p(x^2+y^2)^{p-2}(x(p-1) + (x^2+y^2) + y(p-1) + (x^2+y^2)) = 2p(x^2+y^2)^{p-2}((x+y)(p-1) + 2x^2 + 2y^2)$.

(b)

$\Delta f = 2a + 2b$.

(c)

$\frac{\partial^2}{\partial x^2} ln(x^2+y^2) = \frac{\partial}{\partial x} \frac{2x}{x^2+y^2} = \frac{2(y^2-x^2)}{x^2+y^2}$. Since this function is symmetric, $\frac{\partial^2}{\partial y^2} ln(x^2+y^2) = \frac{2(x^2-y^2)}{x^2+y^2} = -\frac{\partial^2}{\partial x^2} ln(x^2+y^2)$. Therefore $\Delta ln(x^2+y^2) = 0$.

The domain of $ln(x)$ is $x > 0$, so the domain of $ln(x^2+y^2)$ is $x^2+y^2 > 0$, that is, either $x \neq 0$ or $y \neq 0$.

5

(a)

$t_r = \frac{\partial t}{\partial r} = \frac{\partial u}{\partial x} \frac{\mathrm{d}}{\mathrm{d}r} r\cos\theta + \frac{\partial u}{\partial y} \frac{\mathrm{d}}{\mathrm{d}r} r\sin\theta = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$. Similarly, $t_\theta = \frac{\partial t}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\mathrm{d}}{\mathrm{d}\theta} r\cos\theta + \frac{\partial u}{\partial y} \frac{\mathrm{d}}{\mathrm{d}\theta} r\sin\theta = -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\sin\theta$.

(b)

From (a) we know that $t_r = u_x \cos\theta + u_y \sin\theta$. By the product rule, we have $t_{rr} = u_{xr} \cos\theta + u_{yr} \sin\theta$, since $\frac{\partial}{\partial r}\cos\theta = \frac{\partial}{\partial r}\sin\theta = 0$. We can work out u_{xr} and u_{yr} by differentiating u_x and u_y with respect to r, as in (a): $u_{xr} = u_{xx}\cos\theta + u_{xy}\sin\theta$, and $u_{yr} = u_{yx}\cos\theta + u_{yy}\sin\theta$. Thus, we have $t_{rr} = (u_{xx}\cos\theta + u_{xy}\sin\theta)\cos\theta + (u_{yx}\cos\theta + u_{yy}\sin\theta)\sin\theta$. Similarly, we can apply the above steps to get $t_{\theta\theta} = \frac{\partial}{\partial\theta}t_\theta = \frac{\partial}{\partial\theta}(u_x(-r\sin\theta) + u_y(r\cos\theta))$. This comes out to $u_{x\theta}(-r\sin\theta) + u_x(-r\cos\theta) + u_{y\theta}(r\cos\theta) + u_y(-r\sin\theta)$, using the product rule. In a similar way we work out $u_{x\theta}$ and $u_{y\theta}$ by differentiating: $u_{x\theta} = u_{xx}(-r\sin\theta) + u_{yx}(r\cos\theta)$ and $u_{y\theta} = u_{xy}(-r\sin\theta) + u_{yy}(r\cos\theta)$. We can simplify by applying the change of variables specified in the question: $x = r\cos\theta$ and $y = r\sin\theta$, giving $t_{\theta\theta} = u_{x\theta}(-y) + u_x(-x) + u_{y\theta}(x) + u_y(-y) = -y(-yu_{xx} + xu_{yx}) + x(-yu_{yx} + xu_{yy}) + -xu_x - yu_y$. Note that $rt_r = u_x r\cos\theta + u_y r\sin\theta = xu_x + yu_y$ and $x^2 + y^2 = r^2$, so $t_{\theta\theta} = -rt_r + r^2(u_{xx} + u_{yy})$. Therefore, $\frac{t_{\theta\theta}}{r^2} + t_{rr} = -\frac{u_r}{r} + u_{xx} + u_{yy}$, i.e. $u_{xx} + u_{yy} = t_{rr} + \frac{t_r}{r} + \frac{t_{\theta\theta}}{r^2}$.

6

(a)

$$\overrightarrow{Df} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

(b)

$\det\overrightarrow{Df} = e^x\cos^2 y + e^x \sin^2 y = e^x$. $e^x > 0 \;\forall\, x \in \mathbb{R}$, so $\det\overrightarrow{Df}$ is nonzero for all (x, y), that is, $\overrightarrow{Df}^{-1}$ exists everywhere.

(c)

f contains y only as $\sin y$ and $\cos y$ and is therefore periodic in y. Therefore, there are infinite examples of such points, such as $f(1,0) = f(1,\pi) = \begin{bmatrix} e, 0 \end{bmatrix}$.