MATH 3110 HOMEWORK #11

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Problem 17.2.4

(a) By Rolle's theorem, there exists $c \in [a,b]$ such that $f'(c) = 0$. Then, again by Rolle's theorem, there exist $d \in [a,c]$ and $e \in [c,b]$ such that $f''(d) = f''(e) = 0$. Finally, by Rolle's theorem, there exists $n \in [d,e] \subset [a,b]$ such that $f'''(n) = 0$.

(b) Let $f(x) \equiv (x-a)^2(x-b)^2$; $f(a) = f(b) = 0$. $f'(x) = 2(x-a)^2(x-b)+2(x-a)(x-b)^2$, and so $f'(a) = f'(b) = 0$. Further derivatives are $f''(x) = 2(x-a)^2+4(x-a)(x-b)+4(x-a)(x-b)+2(x-b)^2$, and $f'''(x) = 4(x-a)+4(x-a)+4(x-b)+4(x-a)+4(x-b)+4(x-b) = 12(x-a)+12(x-b)$. Indeed, $f'''(x)\left(\frac{a+b}{2}\right)$.

Problem 17.3.4

If you take the Taylor series for $\cos x$ at 0, the error term is $R_n(x) = \frac{f^{n+1}(c)x^{n+1}}{(n+1)!}$ where $f(x) = \cos x$. Since any derivative of $\cos x$ is $\pm \sin x$ or $\pm \cos x$, $|f^{n+1}(x)| \leq 1$. Therefore, $|R_n(0.1)| \leq \frac{0.1^{n+1}}{(n+1)!}$. Taking the 5th power approximation, $|R_5(0.1)| \leq \frac{0.1^6}{(6)!} < 10^{-8}$; this approximation is therefore accurate to at least 7 decimal places.

$$\begin{aligned}\cos 0.1 &\approx 1 + 0 - \frac{0.01}{2} + 0 + \frac{0.0001}{24} + 0 \\ &\approx 0.9950042\end{aligned}$$

Problem 19.2.1

By definition:

$$\begin{aligned}\int_0^1 e^x \, \mathrm{d}x &= \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} e^{\frac{i}{n}} \frac{1}{n} \\ &= \lim_{n\longrightarrow\infty} \frac{1}{n} e^{\frac{1}{n}} \sum_{i=1}^{n} (e^{\frac{1}{n}})^{i-1}\end{aligned}$$

The sum is a geometric series, and therefore evaluates to $\frac{1-(e^{\frac{1}{n}})^n}{1-e^{\frac{1}{n}}} = \frac{1-e^{\frac{n}{n}}}{1-e^{\frac{1}{n}}} = \frac{1-e}{1-e^{\frac{1}{n}}}$. This gives

$$\lim_{n\longrightarrow\infty} \frac{e^{\frac{1}{n}}}{n} \frac{1-e}{1-e^{\frac{1}{n}}} = (1-e) \lim_{n\longrightarrow\infty} \frac{e^{\frac{1}{n}}}{n\left(1-e^{\frac{1}{n}}\right)}$$

Making the substitution $u = \frac{1}{n}$ and noting that $n \to \infty \implies u \to 0^+$, produces

$$
\begin{aligned}
(1-e)\lim_{u\longrightarrow 0}\frac{ue^u}{(1-e^u)} &= (1-e)\lim_{u\longrightarrow 0}\frac{u}{(e^{-u}-1)} \\
&= (1-e)\lim_{u\longrightarrow 0}\frac{1}{(-e^{-u})} = e-1
\end{aligned}
$$

Problem 19.3.2

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{k=0}^{2n}\frac{k}{n^2+k^2} &= \lim_{n\to\infty}\sum_{k=0}^{2n}\frac{\frac{k}{n^2}}{1+\left(\frac{k}{n}\right)^2} \\
&= \lim_{n\to\infty}\sum_{k=0}^{2n}\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\frac{1}{n}
\end{aligned}
$$

Let $f(x)=\frac{x}{1+x^2}$. Then:

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{k=0}^{2n}\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}\frac{1}{n} &= \lim_{n\to\infty}\sum_{k=0}^{2n} f\left(\frac{k}{n}\right)\frac{1}{n} \\
&= \int_0^2 \frac{x}{1+x^2}\,\mathrm{d}x \\
&= \int_1^5 \frac{1}{2u}\,\mathrm{d}u && (u=1+x^2;\mathrm{d}u=2x\,\mathrm{d}x) \\
&= \frac{\ln 5}{2}
\end{aligned}
$$

Problem 20.2.1

(a) By the fundamental theorem of calculus, $\int_0^{x^2} f(t)\,\mathrm{d}t = F(x^2)-F(0)$. Let $f(x)=\sqrt{1+x^2}$. Then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} f(t)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}(F(x^2)-F(0)) \\
&= 2xF'(x^2)-F'(0) \\
&= \frac{x}{1-sqrt(x^2)}-\frac{1}{2}
\end{aligned}
$$

(b) Let $f(x)=e^{-x^2}$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^3}^{1} f(t)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}(F(1)-F(x^3)) \\
&= F'(1)-3x^2F'(x^3) \\
&= -2e^{-1}+6x^5e^{-x^6}
\end{aligned}
$$

(c) Use the same f as before:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{x} f(t)\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}(F(x)-F(x^2))\\ &= F'(x) - 2xF'(x^2)\\ &= -2xe^{-x^2} + \cdot 4x^3e^{-x^4}\end{aligned}$$

Problem 20.3.1

(a)

$$\begin{aligned}I_k &= \int_0^{\frac{\pi}{2}} \sin^k\theta\,\mathrm{d}\theta\\ &= \int_0^{\frac{\pi}{2}} \sin^{k-1}\theta\sin\theta\,\mathrm{d}\theta\\ &= -\sin^{k-1}\theta\cos\theta\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}(k-1)\sin^{k-2}\theta\cos^2\theta\,\mathrm{d}\theta\\ &= 0 + \int_0^{\frac{\pi}{2}}(k-1)\sin^{k-2}\theta(1-\sin^2\theta)\,\mathrm{d}\theta\\ &= \int_0^{\frac{\pi}{2}}(k-1)\sin^{k-2}\theta\,\mathrm{d}\theta - \int_0^{\frac{\pi}{2}}(k-1)\sin^k\theta\,\mathrm{d}\theta\\ &= (k-1)\left(\int_0^{\frac{\pi}{2}}\sin^{k-2}\theta\,\mathrm{d}\theta - \int_0^{\frac{\pi}{2}}\sin^k\theta\,\mathrm{d}\theta\right)\\ &= (k-1)(I_{k-2}-I_k)\end{aligned}$$

(b)

Proof. Define $P(n)$ as the statement that $I_n = \frac{(n-1)!!}{n!!}c$ for constant c as described in the question. The inductive base cases are $P(0)$ and $P(1)$, which are trivial because $I_0 = \frac{\pi}{2}$ and $I_1 = 1$. There are two cases for the inductive step. Suppose $k = 2m+1$ for some m. Then, $I_{k+1} = I_{2m+2} = \frac{2m+1}{2m+2}I_{2m}$. Since $2m < 2m+1 = k$, by strong induction, $I_{2m} = \frac{(2m-1)!!}{2m!!}\frac{\pi}{2}$, and so $I_{2m+2} = \frac{2m+1}{2m+2}\frac{(2m-1)!!}{2m!!}\frac{\pi}{2} = \frac{(2m+1)!!}{2m+2!!}\frac{\pi}{2}$, implying $P(k+1)$. If instead $k = 2m$, the steps are similar, except the strong inductive step gives $c = 1$. □

Problem 20.6.4

Proof. Consider the limit $\lim_{x\to\infty}\frac{Lix}{\frac{x}{\ln x}}$. As $x\to\infty$, $Lix\to\infty$, since it is the integral of a function that is always positive, and $\frac{x}{\ln x}\to\infty$, (transforming to $\frac{x}{1}$ by L'Hopital's rule). Therefore, the limit is an indeterminate form, and L'Hopital's rule can be applied.

$\frac{\mathrm{d}}{\mathrm{d}x}Lix = \frac{\mathrm{d}}{\mathrm{d}x}\int_2^x\frac{\mathrm{d}t}{\ln t} = \frac{1}{\ln x}$, and $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{\ln x} = \frac{\ln x - 1}{(\ln x)^2}$. Therefore:

$$\begin{aligned}
\lim_{x\to\infty} \frac{Lix}{\frac{x}{\ln x}} &= \lim_{x\to\infty} \frac{1}{\ln x}\frac{(\ln x)^2}{\ln x - 1} \\
&= \lim_{x\to\infty} \frac{\ln x}{\ln x - 1} \\
&= \lim_{x\to\infty} \frac{\ln x + 1 - 1}{\ln x - 1} \\
&= \lim_{x\to\infty} \left(\frac{\ln x + 1}{\ln x - 1} - \frac{1}{\ln x - 1}\right) \\
&= 1
\end{aligned}$$

Therefore, $Lix \sim \frac{x}{\ln x}$. $\square$