MATH 3110 HOMEWORK #6

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Problem 7.2.5

Proof. Define the sequence $\{s_n\}$ as the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$: $s_n = \sum_{k=1}^{n} a_k$. Consider the partial sums of $\sum_{k=1}^{\infty} b_k$:

$$\sum_{k=1}^{n} b_k = \sum_{k=1}^{2n} a_k = s_{2n}$$

Therefore, the sequence of partial sums of $\sum_{k=1}^{\infty} b_k$ is just the subsequence of $\{s_n\}$ where $n = 2m$ for some $m \in \mathbb{N}$. By the subsequence theorem, since $\{s_n\}$ converges to S, all of its subsequences must also converge to S, and so $\sum_{k=1}^{\infty} b_k = S$. □

Problem 7.3.3

(a)

Proof. Define the subsequence $\{b_n\}$ as $b_n = |a_n|$. Since $\sum_{k=1}^{\infty} a_n$ is absolutely convergent, $\sum_{k=1}^{\infty} b_n$ is convergent. Define $s_i = \sum_{k=1}^{i} b_n$ as the partial sum of $\sum_{k=1}^{\infty} b_n$, and define $t_i = \sum_{k=1}^{n_i} b_{n_i}$ as the partial sum of $\sum_{k=1}^{\infty} b_{n_i}$. $t_i \leq s_{n_i}$, since all the terms of both sums are positive, and s_{n_i} includes every term found in t_i, and potentially more terms. But $\lim_{n\to\infty} s_n = L$, where L is finite, so by the limit location theorem, $\lim_{n\to\infty} t_n \leq L$. Since all the terms in t_i are positive, this also means that $\lim_{n\to\infty} t_n \geq 0$, and so t_n converges. Therefore, the partial sum of any subsequence of $\{b_n\}$ converges, and so the partial sum of any subsequence of $\{a_n\}$ is absolutely convergent. □

(b)

Proof. Define the sequence $\{a_n\}$ as $\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \cdots\}$. $\sum_{k=1}^{2m} a_n = \sum_{k=1}^{m} 0$, so $\sum_{k=1}^{\infty} a_n$ converges to 0. However, if we take the subsequence of $\{a_n\}$ where $n = 2m$ for some $m \in \mathbb{N}$, then the sum of this subsequence is the negative of the sum of the harmonic series, which does not converge. □

Problem 7.4.1

(d) Consider the ratio between two consecutive terms of the series:

$$\begin{aligned}\frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2} &= \left(\frac{(n+1)!}{n!}\right)^2 \cdot \frac{(2n)!}{(2n+2)!} \\ &= (n+1)^2 \cdot \frac{1}{(2n+1)(2n+2)} \\ &= \frac{(n+1)^2}{(2n+1)(2n+2)} \to \frac{1}{4}\end{aligned}$$

The series must therefore converge.

(e) Apply the root test:

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left(\frac{n+1}{2n+1}\right)^n} &= \lim_{n\to\infty}\frac{n+1}{2n+1}\\ &= \frac{1}{2}\end{aligned}$$

The series must therefore converge.

(f) The antiderivative of $\frac{1}{n\ln n}$ is $\ln(\ln n)$. Consider the value of $\int_2^\infty \frac{1}{n\ln n}$:

$$\begin{aligned}\int_2^\infty \frac{1}{n\ln n}\,\mathrm{d}n &= \lim_{k\to\infty}\Big|_2^k \ln(\ln n)\\ &= \lim_{k\to\infty}(\ln(\ln k)-\ln(\ln 2)) = \infty\end{aligned}$$

The integral diverges, and so the series must also diverge.

Problem 8.1.1

(b) As shown in (7.4.1)d, the limit of the ratio of two consecutive coefficients for this power series is $\frac{1}{4}$, so the limit of two consecutive terms is $\frac{x}{4}$. To converge, we must have $\left|\frac{x}{4}\right| < 1 \implies |x| < 4$.

(c) Consider the ratio between two consecutive terms of the series:

$$\begin{aligned}\lim_{n\to\infty}\frac{x^{n+1}}{\sqrt[n+1]{n+1}}\cdot\frac{\sqrt[n]{n}}{x^n} &= x\lim_{n\to\infty}\frac{\sqrt[n]{n}}{\sqrt[n+1]{n+1}}\\ &= x\end{aligned}$$

Therefore, the radius of convergence is simply $|x| < 1$.

(d)