MATH 3110 HOMEWORK #8

KYRYLO CHERNYSHOV

Problem 1

Proof. Construct two sequences: $\{a_n\} : a_n = \frac{\pi}{2} + 2n\pi$, and $\{b_n\} : b_n = 2n\pi$. Clearly, $a_n \to \infty$ and $b_n \to \infty$. Suppose there exists $L \in \mathbb{R}$ such that $\lim_{x\to\infty} \sin x = L$. Then, by the limit form of sequential continuity, $\lim_{n\to\infty} \sin a_n = \lim_{n\to\infty} \sin b_n = L$. But $\sin a_n = 1, \sin b_n = -1 \ \forall\ n \implies 1 = -1$. Therefore, by contradiction, $\lim_{x\to\infty} \sin x$ cannot exist. □

Problem 2

Proof. Suppose that there exists a continuous f such that $f(x) \in \mathbb{Q} \ \forall\ x$. Pick $a, b \in \mathbb{R}, f(a) \neq f(b)$; then we have $f(a), f(b) \in \mathbb{Q}$. By a fundamental theorem, there exists $x : f(a) < x < f(b)$ such that $x \notin \mathbb{Q}$. But by the intermediate value theorem, there must exist $c : a < c < b$ such that $f(c) = x$. Therefore, the condition $f(a) \neq f(b)$ cannot hold, i.e. the value of $f(x)$ is constant for all x. □

Problem 3

Proof. Let $f(x) = y^2 \cos x - e^x$ for some $y \geq 1$. $f(0) = y^2 - 1 \geq 0$, and $f\left(\frac{\pi}{2}\right) = -e^{\frac{\pi}{2}} < 0$. Since f changes sign on $\left[0, \frac{\pi}{2}\right]$, by Bolzano's theorem, it must have a zero in $\left[0, \frac{\pi}{2}\right]$. This zero cannot be $\frac{\pi}{2}$, since as above, $f\left(\frac{\pi}{2}\right) < 0$, so the zero must be in $\left[0, \frac{\pi}{2}\right)$. □

Problem 4

$k(-4) = -19$, $k(-2) = 13$, $k(0) = -3$, $k(2) = -19$ and $k(4) = 13$. There are, then, at least three intervals on which k changes sign: $[-4, -2]$, $[-2, 0]$ and $[2, 4]$. k is continuous, since it is a linear combination of continuous functions. Therefore, by Bolzano's theorem, it has roots in all three of these intervals. Since $k(-2) \neq 0$, -2 is not a root, so the roots in the intervals $[-4, -2]$ and $[-2, 0]$ must be distinct (they cannot both be -2), and thus $k(x) = 0$ has at least three roots. But since $k(x) = 0$ is cubic, it can have at most three roots; therefore, $k(x) = 0$ has exactly 3 roots.

Problem 5

(a) E

(b) C

(c) $[A, C]$ and $[E, I]$, because the function is continuous on both these intervals.

(d) Same as part (c), and $[A, F]$, since on $[A, F]$ $p(x)$ takes all values between $p(A)$ and $p(F)$.