CS 2800 HW #6

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Problem 1

(a) Let $D = \{0, 1 \cdots b-1\}$. Define n as follows: $n(\epsilon) \equiv 0$, $n(ax) \equiv b \cdot n(a) + x$ where $a \in D^*$ and $x \in D$.

Proof. Let $P(k)$ be the statement that $n(k) = (k)_b$. We have the definition of n above, and the definition $(d_j \cdots d_1 d_0)_b = \sum_i d_i b^i$.

Proving the base case, $P(\epsilon)$, is trivial: $n(\epsilon) \equiv 0$, and $(\epsilon)_b \equiv 0$ by our definitions.

The inductive step is as follows: assume $P(a)$ holds for all $a \in D^*$, and show that as a result $P(ax)$ holds for all $x \in D$. Suppose $a = d_j \cdots d_1 d_0$; then, by $P(a)$ and the definition of n:

$$\begin{aligned} n(ax) \equiv b \cdot n(a) + x &= b \sum_i d_i b^i + x \\ &= \sum_i d_i b^{i+1} + x \\ &= \sum_i d_i b^{i+1} + x b^0 \end{aligned}$$

Let $j = i + 1$ $d_i = g_j$ and $x = g_0$. Then, $\sum_i d_i b^{i+1} + x b^0 = \sum_j g_j b^j$, which is the definition of $(ax)_b$ (since $ax = g_j \cdots g_1 g_0$). $\square$

(b)

Let $P(k)$, where k is a string of digits, be the statement that, if $n = (k)_b$, then k is the unique representation of n in base b. The base case is $P(d)$, where d is a single digit; this is true, since $n = (d)_b \equiv db$, so if $(d_1)_b = (d_2)_b$ then $d_1 b = d_2 b \implies d_1 = d_2$ as long as d_1 and d_2 are nonzero (which we are allowed to assume since we are given that there are no leading zeros), i.e. the representation of n is unique in this case.

The inductive step: assume $P(k)$ holds for some string of digits k, and show that $P(kd)$ holds, where d is a single digit. In this case, suppose l is the length of (amount of digits in) k; then $n = (kd)_b \equiv d + \sum_{i=1}^{l} k_i b^i = d + b \sum_{i=1}^{l} k_i b^{i-1} = d + b \sum_{i=0}^{l-1} k_{i-1} b^i = d + b(k)_b$. By $P(k)$, we know that $(k)_b$ is unique, and by Euclidean division, we know that d is unique, as $n = d + b(k)_b$, i.e. d is the remainder when dividing n by b. Therefore, all the digits in kd must take on a certain value and cannot be anything else, i.e. the base b representation of n is unique.

Problem 2

Suppose $\mathbb{Z}_m$ is the set of all states. Suppose there are m groups of states: $\mathbb{Z}_m = \{[0], [1] \cdots [m-1]\}$, with the equivalence relation of numbers $\mod m$, that is, there are m equivalence classes for states, and $[ma + b] = [b]$, i.e. it wraps back around. Define δ as follows: $\delta([a], 0) = [2a]$, and $\delta([a], 1) = [2a + 1]$, and let $[0]$ be the starting and accepting states, with all other states being rejecting.

Proof. We want to prove that the language of M is strings whose value interpreted as binary is [0]. Essentially, we want to show that $\hat{\delta}([0], x) = [n(x)] \; \forall \; x \in \Sigma^*$. Let $P(k)$ be the statement that the former holds for some $k \in Sigma^*$.

The base case is trivial: $P(\epsilon) = [0]$, and $n(\epsilon) = 0$.

Now, suppose we have $P(k)$; we wish to show $P(ka)$ for some $a \in Sigma$ also holds. By $P(k)$, $\hat{\delta}([0], k) = [n(k)]$, so $\hat{\delta}([0], ka) \equiv \delta(\hat{\delta}([0], k), a) = \delta([n(k)], a)$. From our definition of δ above, this is equal to $[2n(k) + a]$, since a is either 1 or 0. But, by the definition of n, if $k \in Sigma^*, a \in \Sigma$ then $n(ka) = 2n(k) + a$. Therefore, $P(ka)$ holds given $P(k)$ and this is the inductive step.

This language fits the specification. This is because if $k \in \mathbb{Z}$ (integers not states) is divisible by m then by this definition, $\hat{\delta}([0], k) = [n(k)] = [0]$, so k is accepted, as it should be. Conversely, if k is not divisible by m, then $\hat{\delta}([0], k) = [n(k)] \neq [0]$, so k is rejected, as it should be. □

Problem 3

Proof. Let $P(n)$ be the statement that $\exists a, b \in \mathbb{Z}$ such that $n = 4a + 5b$. The base cases are $P(12)$, $P(13)$, $P(14)$ and $P(15)$; we know they are true since $12 = 3 \cdot 4$, $13 = 2 \cdot 4 + 1 \cdot 5$, $14 = 1 \cdot 4 + 2 \cdot 5$ and $15 = 3 \cdot 5$. The inductive step is as follows: for any $n > 15$, we know that $P(n-4)$ holds, since we proved the base cases. Therefore, we know that $n - 4 = 4a + 5b$ for $a, b \in \mathbb{Z}$. Then, $n = 4a + 5b + 4 = 4(a + 1) + 5b$. Let $c = a + 1$; since $a \in \mathbb{Z}$, $c \in \mathbb{Z}$, and we have $n = 4c + 5b$, i.e. $P(n)$ holds. Therefore, $P(n)$ holds for all $n \geq 12$, that is, any postage of 12 cents or more can be formed using just 4- and 5-cent stamps. □

Problem 4

Encode the lights like so: red is 0, orange is 1, yellow is 2, green is 3 and blue is 4. To cycle a light, we add 1 to its value, but we are only interested in the answer mod 5 since there are 5 lights and lights wrap back around from blue to red, and we want this value to be 4, for blue. After button each button n is pressed b_n times, the colours of the lights are as follows:

Light	Colour
1	$0 + b_1 + b_2 + b_4 + b_5 \equiv 4 \mod 5$
2	$1 + b_1 + b_2 \equiv 4 \mod 5$
3	$2 + b_3 + b_4 + b_5 \equiv 4 \mod 5$
4	$3 + b_1 + b_3 + b_4 \equiv 4 \mod 5$
5	$4 + b_2 + b_4 + 2b_5 \equiv 4 \mod 5$

If we push through the algebra, we can solve for $b_3 \equiv 1$, and simplify the above equations down to the following: $b_1 + b_4 \equiv 0$, $b_4 + b_5 \equiv 1$ and $b_2 + b_5 \equiv 4$. These equations are consistent with infinite solutions; that is, any combination satisfying them will work; for instance, $b_1 \equiv 2$, $b_4 \equiv 3$, $b_5 \equiv 3$ and $b_2 \equiv 1$ would satisfy the table above. Therefore, all the lights will turn blue when, for example, you press button 1 twice, button 2 once, button 3 once, button 5 thrice and button 5 thrice.