MATH 2220 SECTION 203
HW #8

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Problem 1 The area of a rectangle with sides x and y is $f(x,y) = xy$. Its perimeter is $g(x,y) = 2(x+y)$. To maximise f with the constraint $g(x,y) = p$ we need to solve $\nabla f(x,y) = \lambda \nabla g(x,y)$. $\nabla f(x,y) = (y,x)$, $\nabla g(x,y) = (2,2)$ so we have $(y,x) = \lambda(2,2) \implies x = y = 2\lambda$. Our constraint gives us $2(x+y) = 2(4\lambda) = 8\lambda = p$, so $x = y = \frac{p}{4}$, and the area is $f(\frac{p}{4}, \frac{p}{4}) = \left(\frac{p}{4}\right)^2$.

Problem 2 We want to find $\vec{x} \in \mathbb{R}^n$ such that the distance between $\vec{a}$ and $\vec{x}$ is as small as possible. This distance is $\|\vec{a} - \vec{x}\|$. Since the norm is always positive, we can treat this as minimising $f(\vec{x}) = \|\vec{a} - \vec{x}\|^2 = \sum_{i=1}^{n}(a_i - x_i)^2$, under the constraint $g(\vec{x}) = \vec{c} \cdot \vec{x} = 0$, reducing our problem to solving $\nabla f = \lambda \nabla g(\vec{x})$. $\nabla f(\vec{x}) = (-2(a_1 - x_1), -2(a_2 - x_2) \cdots - 2(a_n - x_n)) = -2((a_1 - x_1), (a_2 - x_2) \cdots (a_n - x_n)) = -2(\vec{a} - \vec{x})$, and $\nabla g = \vec{c}$, so we have $2(\vec{x} - \vec{a}) = \lambda \vec{c}$. We can use λ to encompass any multiplicative constants, so this is equivalent to $\vec{x} - \vec{a} = \lambda \vec{c} \implies \vec{x} = \lambda \vec{c} + \vec{a}$ for some other value of λ.

Using our constraint $\vec{c} \cdot \vec{x} = 0$, we get $\vec{c} \cdot (\lambda \vec{c} + \vec{a}) \implies \lambda \|\vec{c}\|^2 + \vec{c} \cdot \vec{a} = 0 \implies \lambda = -\frac{\vec{c} \cdot \vec{a}}{\|\vec{c}\|^2}$. Plugging this into our equation for $\vec{x}$ we get $\vec{x} = -\frac{\vec{c} \cdot \vec{a}}{\|\vec{c}\|^2}\vec{c} + \vec{a}$.

Problem 3

(a) The velocity is $\vec{v}(t) = \nabla \vec{x} = (-r\omega \sin(\omega t), r\omega \cos(\omega t))$. $\vec{x} \cdot \nabla \vec{x} = (r\cos(\omega t))(-r\omega \sin(\omega t)) + (r\sin(\omega t))(r\omega \cos(\omega t)) = r^2\omega(\sin(\omega t)\cos(\omega t) - \sin(\omega t)\cos(\omega t)) = 0$, so the velocity is orthogonal to the displacement. Because, the displacement is orthogonal to the circle, the velocity is tangent to the circle.

(b) The speed is $s = \|\vec{v}\| = \|(-r\omega \sin(\omega t), r\omega \cos(\omega t))\| = \sqrt{(-r\omega \sin(\omega t))^2 + (r\omega \cos(\omega t))^2} = \sqrt{r^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t))} = \sqrt{r^2\omega^2} = r\omega$.

(c) The acceleration is $\vec{a}(t) = \nabla \vec{v} = (-r\omega^2 \cos(\omega t), -r\omega^2 \sin(\omega t)) = -\omega^2(r\cos(\omega t), r\sin(\omega t)) = -\omega^2 \vec{x}(t)$. Since it is a negative multiple of the displacement, which is directed away from the origin, the acceleration must be directed towards the origin.

(d) $\|\vec{a}\| = \sqrt{(-r\omega^2 \cos(\omega t))^2 + (-r\omega^2 \sin(\omega t))^2} = \sqrt{r\omega^4(\cos^2(\omega t) + \sin^2(\omega t))} = \sqrt{r^2\omega^4} = r\omega^2$.

(e) First, calculate $(x^2+y^2)^{\frac{3}{2}} = (r^2(\cos^2(\omega t) + \sin^2(\omega t))^{\frac{3}{2}} = (r^2)^{\frac{3}{2}} = r^3$. Then, $x'' + \frac{x}{(x^2+y^2)^{\frac{3}{2}}} = -r\omega^2\cos(\omega t) + \frac{r\cos(\omega t)}{r^3}$. If $\omega^2 r^3 = 1 \implies r^3 = \frac{1}{\omega^2}$, this is equation becomes $-r\omega^2\cos(\omega t) + r\omega^2\cos(\omega t) = 0$. In exactly the same way but with sin instead of cos, we get $y'' + \frac{y}{(x^2+y^2)^{\frac{3}{2}}} = 0$.

(f) $x'y - xy' = (-r\omega \sin(\omega t))(r\sin(\omega t)) - (r\cos(\omega t))(r\omega\cos(\omega t)) = -r^2\omega(\sin^2(\omega t) + \cos^2(\omega t)) = -r^2\omega$.

Problem 4

(a) Using the definition $\vec{\omega} = (\vec{x}, \vec{y})$, we have $\vec{\omega}'' = (\vec{x}'', \vec{y}'') = \left(\frac{k(\vec{y}-\vec{x})}{m}, -\frac{k(\vec{y}-\vec{x})}{m}\right)$. Therefore, $m\vec{\omega}'' = (k(\vec{x}-\vec{y}), -k(\vec{x}-\vec{y}))$. $\nabla p(\vec{\omega}) = \frac{k}{2}\nabla \|\vec{x}-\vec{y}\|^2 = \frac{k}{2}\nabla((x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2) = \frac{k}{2}(2(x_1-y_1), 2(x_2-y_2), 2(x_3-y_3), -2(x_1-y_1), -2(x_2-y_2), -2(x_3-y_3)) = (k(\vec{x}-\vec{y}), -k(\vec{x}-\vec{y})) = -m\vec{\omega}''$.

(b) $\nabla(p(\vec{\omega}(t))) = \nabla p(\vec{\omega}(t)) \cdot \nabla\vec{\omega}(t) = -m\vec{\omega}'' \cdot \vec{\omega}'$, as per the equation derived above.

$\nabla \|\vec{\omega}'\|^2 = \nabla(x_1'^2 + x_2'^2 + x_3'^2 + y_1'^2 + y_2'^2 + y_3'^2) = (2x_1'x_1'', 2x_2'x_2'', 2x_3'x_3'', 2y_1'y_1'', 2y_2'y_2'', 2y_3'y_3'') = 2\vec{\omega}' \cdot \vec{\omega}''$

Therefore, $\nabla\left(\frac{m\|\vec{\omega}'\|^2}{2} + p(\vec{\omega}(t))\right) = m\vec{\omega}' \cdot \vec{\omega}'' - m\vec{\omega}'' \cdot \vec{\omega}' = 0$. This quantity therefore does not change with time.

Problem 5

(a) The linear approximation for $\vec{x}$ at t is $\vec{x}(t+h) \approx L_x(h) = \vec{x}(t) + \vec{x}'(t)h$, and that of $\vec{y}$ at t is $L_y(h) = \vec{y}(t) + \vec{y}'(t)h$. Therefore, $\vec{u}(t+h) \approx (\vec{x}(t) + \vec{x}'(t)h, \vec{y}(t) + \vec{y}'(t)h)$.

(b) The signed area of the triangle formed by two vectors (a, b) and (c, d), both originating at 0, is $\frac{bc-ad}{2}$, as this is the determinant of the matrix $\begin{bmatrix}\vec{u} & \vec{v}\end{bmatrix}$. In this case, this is equal to $\frac{x(y+y'h)-y(x+x'h)}{2} = \frac{h}{2}(xy' - yx')$.

(c) If A is the area, the change in area is $\Delta A = \frac{h}{2}(xy' - yx')$, and $\Delta t = h$. The rate of change of the area is $\frac{\mathrm{d}A}{\mathrm{d}t} = \lim_{h\to 0}\frac{\Delta A}{\Delta t} = \lim_{h\to 0}\frac{1}{2}(xy' - yx') = \frac{1}{2}(xy' - yx')$. If we parametrise like in (3) but for an ellipse instead of a circle, we get $\vec{x}(t) = (ra\cos(\omega t), rb\sin(\omega t))$. From this, if we push through the algebra as in (3c) we get $x'y - xy' = -r^2ab\omega \implies xy' - x'y = r^2ab\omega$, which is also constant given fixed a, b, r and ω. That means $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{r^2ab\omega}{2}$ is constant, which implies that planet that travels for time t will always sweep out the same area, regardless of where the planet is.

(d) The period is the time taken to sweep out the entire area: $T = \frac{A}{\frac{\mathrm{d}A}{\mathrm{d}t}} = \frac{2A}{r^2ab\omega}$.

Problem 6

(a) We know that $-p\sqrt{x^2+y^2} \geq 0$, since $p < -1$ and $\sqrt{n} \geq 0 \;\forall\, n \in \mathbb{R}$. Therefore, the equation only makes sense if the right hand side is also nonnegative: $q - y \geq 0 \implies q \geq y$. However, for motion to occur, we need this to be positive, so we have $y < q$.

(b) As per the equation derived in part (c), this function is the equation for an ellipse.

(c) Rearranging the equation, we get $p^2(x^2+y^2) = (q-y)^2 \implies p^2x^2 + (p^2-1)y^2 + 2qy = q^2 \implies p^2x^2 + (p^2-1)\left(y + \frac{q}{p^2-1}\right)^2 - \frac{q^2}{p^2-1} = q^2 \implies p^2x^2 + (p^2-1)\left(y + \frac{q}{p^2-1}\right)^2 = \frac{p^2q^2}{p^2-1}$. To make this into the traditional equation for an ellipse we divide both sides by the right-hand side, giving $x^2\frac{p^2-1}{q^2} + \left(y + \frac{q}{p^2-1}\right)^2\frac{(p^2-1)^2}{p^2q^2} = 1$. This gives the values of the two axes: $r_1 = \frac{q}{\sqrt{p^2-1}}$ and $r_2 = \frac{-pq}{p^2-1}$. The minus in the second equation comes from the fact that we want $r_2 \geq 0$, but $p < -1$, so we need to invert the value to get the unsigned magnitude.

(d) By Kepler's second law, we know that the time taken to "sweep" out the entire ellipse is T, the period of the orbit. The total area is $A = \pi r_1 r_2 = \pi\frac{q}{\sqrt{p^2-1}}\frac{-pq}{p^2-1} = \pi\frac{-pq^2}{(p^2-1)^{\frac{3}{2}}}$. We know that $\frac{\mathrm{d}A}{\mathrm{d}t}$ is constant and $T = A\frac{\mathrm{d}A}{\mathrm{d}t}$, so $T \propto A$, and therefore $T^2 \propto A^2$. $A^2 = \pi^2\frac{p^2q^4}{(p^2-1)^3}$