Homework 4 solutions

(1)

(a)

We are given that any two points in D can be joined by a continuous line. Let $g(x) = \vec{u} + x(\vec{v} - \vec{u})$. Then, $g(0) = \vec{u}$ and $g(1) = \vec{v}$. g is a composition of a sum, a subtraction and a product, which are all continuous, therefore g is continuous, and thus takes all values in between $\vec{u}$ and $\vec{v}$, which means it is a curve.

(b)

g is continuous and f is given as continuous, therefore $f \circ g$ is continuous.

(c)

$f \circ g$ is continuous, which means that for any y such that $f(g(0)) < y < f(g(1))$, there exists $n \in [0, 1]$ such that $f(g(n)) = y$. Let $\vec{x} = g(n)$. By definition, $\vec{x} \in D$, and we know that $f(\vec{x}) = f(g(n)) = y$.

(2)

(a)

Proof. On the interval $(-\infty, 0)$, the function always takes the value -1, and therefore is equal to the function $f(x) = -1$, which is constant and therefore continuous. Similarly, on the interval $(0, \infty)$ the function is equal to 1, and therefore is equal to $f(x) = 1$, so it is continuous. So, for $x \neq 0$ the function is continuous. At $x = 0$, however it is not. Take $0 < \epsilon < 1$. Then, for any $y \in \mathbb{R}$, $|\text{sgn}(x) - \text{sgn}(y)| = 1 > \epsilon$, regardless of the value of $|x - y|$. Therefore, the function is discontinuous at $x = 0$. $\square$

(b)

Proof. Any range (a, b) in $\mathbb{R}$ where $a \neq b$ has at least one rational number and at least one irrational number. For any $x \in \mathbb{R}$ there are two cases: either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$. Let $0 < \epsilon < 1$. In the former case, for any $\delta > 0$ we know that there is at least one irrational number y in $(x - \delta, x + \delta)$, and so $|D(x) - D(y)| = 1 > \epsilon$. In the latter case, for any $\delta > 0$ we know that there is at least one rational number y in $(x-\delta, x+\delta)$, and so once again $|D(x)-D(y)| = 1 > \epsilon$. Either way, $D(x)$ is discontinuous; therefore $D(x)$ is discontinuous for all $x \in \mathbb{R}$. $\square$

(c)

The function is discontinuous for $(x, y) \neq (0,0)$, and continuous everywhere else.

Proof. If $(x, y) \neq (0,0)$, then the function is a composition of functions that are continuous ($f(x) = \frac{1}{x}$ is discontinuous only on $x = 0$). Therefore, it is continuous for nonzero x and y. However, $\lim_{(x,y)\to(0,0)} f(x, y) = 0 \neq f(0,0)$, so the function is discontinuous on $(0,0)$. $\square$

(3)

If $f(x, y) = x^2 + 3y$, then we have $f_x(x, y) = 2x$, $f_y(x, y) = 3$. The linear approximation to f at $(2,4)$ is $L(x, y) = f(2,4)+f_x(2,4)(x-2)+f_y(2,4)(y-4) = 16+4(x-2)+3(y-4)$. $f(2.01, 4.03) \approx L(2.01, 4.03) = 16.13$.

(4)

$(a+u)^2(b+v)^3 = a^2b^3 + 3a^2b^2v + 3a^2bv^2 + a^2v^3 + 2ab^3u + 6ab^2uv + 6abuv^2 + 2auv^3 + b^3u^2 + 3b^2u^2v + 3bu^2v^2 + u^2v^3$. $u = x-a$ and $v = y-b$. The only term without u or v is a^2b^3, so $c_1 = a^2b^3$. The only terms with only one of v or u are $3a^2b^2v$ and $2ab^3u$, so $c_2 = 2ab^3$ and $c_3 = 3a^2b^2$.

(a)

$c_1 = f(a, b)$, since all other terms become 0.

(b)

$c_2 = f_x(a, b)$ and $c_3 = f_y(a, b)$. This is because after differentiating with respect to x, the only term what will not go to 0 as $(x-a)$ and $(y-b)$ go to 0 is c_2, since it will not be dependent on either x or y, and after differentiating with respect to y, the only term that will similarly not go to 0 is c_3.

(c)

$l(u, v) = c_2u + c_3v = \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v$. This is clearly bilinear in u and v.

(5)

(a)

Let $f(x, y) = e^x \cos y$. Then $\nabla f = \begin{bmatrix} f_x(x, y) & f_y(x, y) \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \end{bmatrix}$. $\nabla \cdot \nabla f = e^x \cos y + -e^x \cos y = 0$.

(b)

Let $f(x, y) = x^2 - y^2$. Then $\nabla f = \begin{bmatrix} f_x(x, y) & f_y(x, y) \end{bmatrix} = \begin{bmatrix} 2x & -2y \end{bmatrix}$. $\nabla \cdot \nabla f = 2 - 2 = 0$.

(c)

Let $f(x, y) = e^x$. $\nabla f = \begin{bmatrix} f_x(x, y) & f_y(x, y) \end{bmatrix} = \begin{bmatrix} e^x & 0 \end{bmatrix}$. $\nabla \cdot \nabla f = e^x - 0 = e^x \geq 0 \; \forall \; x \in \mathbb{R}$.

(6)

(a)

$f_x(x, y) = h(y) \frac{\mathrm{d}}{\mathrm{d}x} g(x)$ and $f_y(x, y) = g(x) \frac{\mathrm{d}}{\mathrm{d}y} h(y)$. $\frac{\partial}{\partial y} f_x(x, y) = \frac{\mathrm{d}}{\mathrm{d}y} h(y) \frac{\mathrm{d}}{\mathrm{d}x} g(x)$, and $\frac{\partial}{\partial x} f_y(x, y) = \frac{\mathrm{d}}{\mathrm{d}x} g(x) \frac{\mathrm{d}}{\mathrm{d}y} h(y)$.

(b)

The derivative of any term of the form $x_{mn} x^m y^n$ with respect to x and then y is $c_{mn} mn x^{m-1} y^{n-1}$, which is also the derivative with respect to y and then x. We also know that the derivative of $f(a+x, b+y)$ with respect to x and then y is the same as its derivative with respect to y and then x, from the previous question. $f(a, b) = f(a+x, b+y) - c_{10}x - c_{01}y - c_{20}x^2 - c_{11}xy - c_{02}y^2 - \cdots - c_{mn} mn x^m y^n$. All the terms on the right hand side satisfy the desired property, which means the entire right hand side satisfies the property since differentiation is linear. This means the left hand side, $f(a, b)$, also satisfies this property.

(c)

Let $(x, y) = (r \sin\theta, r \cos\theta)$. Then $f(x, y) = r^2 \cos\theta \sin\theta(\sin^2\theta - \cos^2\theta)$. Therefore, $\lim_{\|(x,y)\| \to 0} f(x, y) = \lim_{r \to 0} r^2 \cos\theta \sin\theta(\sin^2\theta - \cos^2\theta) = 0$. The direction/slope of the line we approach zero at does not matter, as the limit value is independent of the value of θ, and is always 0, which is equal to $f(0,0)$. This means f is continuous at the origin. On points other than the origin, f is continuous as it is the composition of functions that are all continuous on points that are not the origin.

$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(3x^2y-y^3)-2x^2y(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^4y+4x^2y^3-y^5}{x^4+2x^2y^2+y^4}$, and $\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(x^3-3xy^{)}-2xy^2(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^5-4x^3y^2-xy^4}{x^4+2x^2y^2+y^4}$. Both derivatives are 0 at $(0,0)$, by the original function definition. The limit as $(x,y) \to 0$ of both derivatives is also 0, so they are continuous at $(0,0)$. At other points they are continuous because they are rational functions, which are compositions of functions that are continuous at points that are not $(0,0)$ and therefore continuous on points other than the origin.

$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$, and $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$. They are undefined at $(0,0)$ and therefore not "equal" at that point.