Continuity

A function $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous at $x \in \mathbb{R}$ iff $\forall\ \epsilon > 0 \exists \delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

We can extend this definition to multivariate functions: a function $F : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ is continuous at $\vec{x} \in D$ iff $\forall\ \epsilon > 0 \exists \delta > 0$ such that whenever $\|\vec{x} - \vec{y}\| < \delta$, then $\|F(\vec{x}) - F(\vec{y})\| < \epsilon$.

Lemma

Linear functions are continuous.

Proof. Fix $\epsilon > 0$. Let L be a linear function, and $\vec{h} = \vec{x} - \vec{y}$; we want to show that $\exists \delta > 0$ such that if $\left\|\vec{h}\right\| < \delta$ then $\left\|L(\vec{x} + \vec{h}) - L(\vec{x})\right\| < \epsilon$.

Let $\vec{C}$ be the matrix representing L, that is, $L(\vec{y}) = \vec{C}\vec{y}\ \ \forall\ \vec{y}$. Then, $\left\|L(\vec{x} + \vec{h}) - L(\vec{x})\right\| = \left\|L(\vec{x}) + L(\vec{h}) - L(\vec{x})\right\| = \left\|L(\vec{h})\right\| = \left\|\vec{C}\vec{h}\right\| \leq \left\|\vec{C}\right\| \cdot \left\|\vec{h}\right\|$. We can then specify $\delta = \frac{\epsilon}{\|\vec{C}\|}$.

Suppose $\left\|\vec{h}\right\| < \frac{\epsilon}{\|\vec{C}\|}$. Then $\left\|L(\vec{x} + \vec{h}) - L(\vec{x})\right\| = \left\|L(-\vec{h})\right\| = \left\|-\vec{C}\vec{h}\right\| \leq \left\|\vec{C}\right\| \cdot \left\|-\vec{h}\right\| < \left\|\vec{C}\right\| \cdot \frac{\epsilon}{\|\vec{C}\|} = \epsilon$

Therefore L is continuous on $\vec{x}$. Since $\vec{x}$ was arbitrary, L is continuous. $\square$

Theorem

A function $F : \mathbb{R}^n \mapsto \mathbb{R}^m$ is written as $F((x_1, x_2 \cdots x_n)) = (f_1(x_1, x_2 \cdots x_n), f_2(x_1, x_2 \cdots x_n) \cdots f_m(x_1, x_2 \cdots x_n)$
F is continuous iff $f_1, f_2 \cdots f_m : \mathbb{R}^n \mapsto \mathbb{R}$ are continuous.

Proof. Let us first prove one direction of the theorem. Let us assume that F is continuous, and try to show that f_i is continuous $\forall\ i$. If we fix $\epsilon > 0$, $\vec{x}$, we know that $\exists \delta > 0$ such that if $\|\vec{x} - \vec{y}\| < \delta$ then $\|F(\vec{x}) - F(\vec{y})\| < \epsilon$.

We want to show that $|f_i(\vec{x}) - f_i(\vec{y})| < \epsilon$. We know that $\epsilon > \|F(\vec{x}) - F(\vec{y})\| = \sqrt{\sum_{j=1}^{m}(f_j(x_1, x_2 \cdots x_n) - f_j(y_1, y_2 \cdots y_n))^2} \geq \sqrt{(f_i(x_1, x_2 \cdots x_n) - f_i(y_1, y_2 \cdots y_n))^2} = |f(\vec{x} - f(\vec{y})|$. This proves the theorem in this direction.

To prove the other direction, we assume that $f_1, f_2 \cdots f_m$ are continuous, that is, $\forall\ \epsilon_i > 0 \exists \delta_i > 0$ such that if $\|\vec{x} - \vec{y}\| < \delta_i$ then $|f_i(\vec{x}) - f_i(\vec{y})| < \epsilon_i$.

Fix $\epsilon > 0$. Let $\epsilon_i = \frac{\epsilon}{\sqrt{m}}$ and use the above definition to define δ_i. Let $\delta = min(\delta_1, \delta_2 \cdots \delta_m)$. If $\|\vec{x} - \vec{y}\| < \delta$ then $\|F(\vec{x}) - F(\vec{y})\| = \sqrt{\sum_{i=1}^{m}(f_i(\vec{x}) - f_i(\vec{y}))^2} < \sqrt{\sum_{i=1}^{m}(\frac{\epsilon}{\sqrt{m}})^2} = \epsilon$. $\square$