Intermediate function theorem

The intermediate function theorem is as follows:

Theorem

Suppose a function $f : D \subseteq \mathbb{R}^{n+k} \to \mathbb{R}^k$ is C^1. We denote a point in $\mathbb{R}^{n+k}$ as $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^k$. If for $(\mathbf{a}, \mathbf{b}) \in D$, the matrix $\left[\frac{\partial f_i}{\partial y_j}\right]$ is invertible and $f(\mathbf{a}, \mathbf{b}) = 0$, then there exists an open set $E \subseteq \mathbb{R}^n$, $\mathbf{a} \in E$, and function $g : E \to \mathbb{R}^k$ that is C^1 such that $g(\mathbf{a}) = \mathbf{b}$ and $f(\mathbf{x}, g(\mathbf{x})) = 0$ for $\mathbf{x} \in E$.

Proof. Let $F : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$, $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, f(\mathbf{x}, \mathbf{y})$. We want to use the inverse function theorem. For that we need $\mathbf{D}F$ to be invertible.

$$\mathbf{D}F = \begin{bmatrix} & \mathbf{I}_n & & & & 0 \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} & \frac{\partial f_k}{\partial y_1} & \cdots & \frac{\partial f_k}{\partial y_n} \end{bmatrix}$$

This can be row reduced to

$$\begin{bmatrix} & \mathbf{I}_n & & & & 0 \\ 0 & \cdots & 0 & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \frac{\partial f_k}{\partial y_1} & \cdots & \frac{\partial f_k}{\partial y_n} \end{bmatrix}$$

Therefore, $\mathbf{D}F$ is invertible if and only if $\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_k}{\partial y_1} & \cdots & \frac{\partial f_k}{\partial y_k} \end{bmatrix}$ is invertible.

By the inverse function theofem, if $F(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{0})$ there exists an open set $E' \subseteq \mathbb{R}^{n+k}$ and a C^1 function $G : E' \to \mathbb{R}^{n+k}$ which is a local inverse to F. $G(\mathbf{a}, \mathbf{0}) = (\mathbf{a}, fb)$. Let $E = E' \cup \mathbb{R}^n \times \{\mathbf{0}\}$, an open subset of $\mathbb{R}^n$. $\mathbf{a} \in E$ because $(\mathbf{a}, \mathbf{0}) \in E'$. Define $g(\mathbf{x}) = \operatorname{proj} \mathbf{y} G(\mathbf{x}, \mathbf{0})$. g is C^1, since G and proj are C^1.

$f(\mathbf{x}, g(\mathbf{x})) = f(\mathbf{x}, \operatorname{proj} \mathbf{y} G(\mathbf{x}, \mathbf{0}))$. By definition, $G(\mathbf{x}, \mathbf{0}) = (\mathbf{x}, \mathbf{y})$ such that $F(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{0})$. Therefore $f(\mathbf{x}, \operatorname{proj} \mathbf{y} G(\mathbf{x}, \mathbf{0})) = f(\mathbf{x}, \mathbf{y})$ with $\mathbf{y}$ satisfying this property. Therefore, $f(\mathbf{x}, \operatorname{proj} \mathbf{y} G(\mathbf{x}, \mathbf{0})) = f(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. $\square$