Homework 2 solutions

2.1.16

There are two possible series of events leading up to the shopper purchasing B on his second purchase: he purchases A and then B, or he purchases B and then B. The probability of the former is equal to the probability of him choosing A and then switching: $\frac{1}{4}\frac{2}{3} = \frac{1}{6}$. The probability of the latter is equal to the probability of him choosing B and then not switching: $\frac{3}{4}\frac{1}{3} = \frac{1}{4}$. Thus the total probability of him choosing B is $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$.

2.2.5

The probability of the ship being under the control of the original system is that of it not failing, i.e. $1 - 0.001 = 0.999$. The probability of it being under the control of the second system is the probability of the first system failing and the second system not failing. Since the two systems are independent, this is a product: $0.001(1 - 0.001) = 0.000999$.

2.2.6

My assumption, since it isn't 100% clear, is that exactly one ticket per lottery will win the first prize. Let A be the event of winning the first prize in the first lottery, and B be the event of winning the first prize in the second lottery. Since A and B are independent, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{100}{10000} + \frac{100}{5000} - \frac{100}{10000}\frac{100}{5000} = \frac{149}{5000} = 0.0298$.

2.3.2

Let the event that the item is defective be D, and let the event that the item was produced by machine i be M_i. We are given that $\mathbb{P}(M_1) = 0.2$, $\mathbb{P}(M_2) = 0.3$ and $\mathbb{P}(M_3) = 0.5$, and that $\mathbb{P}(D|M_1) = 0.01$, $\mathbb{P}(D|M_2) = 0.02$ and $\mathbb{P}(D|M_3) = 0.03$. By the law of total probability, we have $\mathbb{P}(D) = \mathbb{P}(D|M_1) \cdot \mathbb{P}(M_1) + \mathbb{P}(D|M_2) \cdot \mathbb{P}(M_2) + \mathbb{P}(D|M_3) \cdot \mathbb{P}(M_3) = 0.023$, since $M_1 \cup M_2 \cup M_3 = 1$ and the three events are mutually exclusive. By Bayes' theorem, $\mathbb{P}(M_i|D) = \frac{\mathbb{P}(D|M_i) \cdot \mathbb{P}(M_i)}{\mathbb{P}(D)}$. We thus have $\mathbb{P}(M_1|D) = \frac{0.01 \cdot 0.2}{0.023} = 0.0870$, $\mathbb{P}(M_2|D) = \frac{0.02 \cdot 0.3}{0.023} = 0.261$ and $\mathbb{P}(M_3|D) = \frac{0.03 \cdot 0.5}{0.023} = 0.652$. The only machine that satisfies the desired property $\mathbb{P}(M_i) < \mathbb{P}(M_i|D)$ is M_3.

2.3.4

Let the event of reacting positively to the test be T, and the event of having this kind of cancer be C. We are given the following values: $\mathbb{P}(C) = 0.00001$, $\mathbb{P}(T|C) = 0.95$, $\mathbb{P}(T^{\complement}|C) = 0.05$, $\mathbb{P}(T|C^{\complement}) = 0.05$ and $\mathbb{P}(T^{\complement}|C^{\complement}) = 0.95$. By Bayes' theorem, $\mathbb{P}(C|T) = \frac{\mathbb{P}(T|C)\cdot\mathbb{P}(C)}{\mathbb{P}(T)}$. By the law of total probability, $\mathbb{P}(T) = \mathbb{P}(T|C)\cdot\mathbb{P}(C) + \mathbb{P}(T|C^{\complement})\cdot P(C^{\complement}) = 0.95\cdot 0.00001 + 0.05\cdot 0.99999 = 0.050009$. Therefore, $\mathbb{P}(C|T) = \frac{\mathbb{P}(T|C)\cdot\mathbb{P}(C)}{\mathbb{P}(T)} = \frac{0.95\cdot 0.00001}{0.050009} = 0.000190$.

2.3.6

(a)

Given that the machine is adjusted properly(event P), the probability of four items being of high quality and one being of medium quality (event A) is $\mathbb{P}(A|P) = C_1^5 0.5^4\cdot 0.5 = 0.15625$. Given that the machine is not adjusted properly, this probability is $\mathbb{P}(A|P^{\complement}) = C_1^5 0.25^4\cdot 0.75 = 0.014648$. We are also given that $\mathbb{P}(P) = 0.9$. By Bayes' theorem, $\mathbb{P}(P|A) = \frac{\mathbb{P}(A|P)\cdot\mathbb{P}(P)}{\mathbb{P}(A)}$. By the law of total probability, $\mathbb{P}(A) = mathbbP(A|P)\cdot\mathbb{P}(P) + mathbbP(A|P^{\complement})\cdot\mathbb{P}(P^{\complement}) = 0.15625\cdot 0.9 + 0.014648\cdot 0.1 = 0.14209$. Therefore, $\mathbb{P}(P|A) = \frac{0.15625\cdot 0.9}{0.14209} = 0.9897$.

(b)

This question is the same except the values of $\mathbb{P}(A|P)$ and $\mathbb{P}(A|P^{\complement})$ are different: $\mathbb{P}(A|P) = C_2^6 0.5^4\cdot 0.5^2 = 0.23438$, and $\mathbb{P}(A|P^{\complement}) = C_2^6 0.25^4\cdot 0.75^2 = 0.32959$. Therefore $\mathbb{P}(A) = 0.23438\cdot 0.9 + 0.32959\cdot 0.1 = 0.24390$, and $\mathbb{P}(P|A) = \frac{0.23438\cdot 0.9}{0.24390} = 0.86487$.

2.3.14

Let A be the event that 8 out of 11 programs compiled. $\mathbb{P}(A|B) = C_3^{11} 0.8^8\cdot 0.2^3 = 0.22146$, and $\mathbb{P}(A|B^{\complement}) = 0.023357$. By the law of total probability, $\mathbb{P}(A) = \mathbb{P}(A|B)\cdot\mathbb{P}(B) + \mathbb{P}(A|B^{\complement})\cdot\mathbb{P}(B^{\complement}) = 0.22146\cdot 0.4 + 0.023357\cdot 0.6 = 0.10260$. Finally, by Bayes' theorem, $\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\cdot\mathbb{P}(B)}{\mathbb{P}(A)} = \frac{0.22146\cdot 0.4}{0.10260} = 0.8634$.

3.1.6

$\mathbb{P}(X < 6) = \mathbb{P}(X = 0, 1, 2, 3, 4, 5) = \sum_{i=0}^{5}\mathbb{P}(X = i) = 0.5^{15}(C_1^{15} + C_2^{15} + C_3^{15} + C_4^{15} + C_5^{15}) = 4943\cdot 0.5^{15} = 0.1508$.

3.1.10

(a)

By one of the axioms of probability, we know that $\mathbb{P}(X = 0, 1, 2, 3, 4, 5, 6, 7) = \sum_{i=0}^{7} \mathbb{P}(X = i) = 1$. Expressing the function as $\mathbb{P}(X = x) = c(x+1)(8-x)$, we can solve for c:

$$\sum_{i=0}^{7} c(i+1)(8-i) = c\sum_{i=0}^{7}(i+1)(8-i) = 1$$

$$c\sum_{i=0}^{7}(i+1)(8-i) = c(1 \cdot 8 + 2 \cdot 7 + \cdots + 8 \cdot 1) = 120c$$

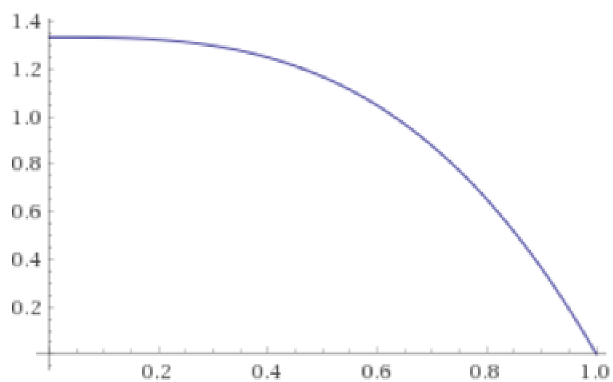
$$c = \frac{1}{120}$$

Therefore, $\mathbb{P}(X = x) = \frac{(x+1)(8-x)}{120}$.

(b)

$\mathbb{P}(X \geq 5) = \mathbb{P}(X = 5, 6, 7) = \sum_{i=5}^{7} \mathbb{P}(X = i) = \sum_{i=5}^{7} \frac{(i+1)(8-i)}{120} = \frac{1}{3}$.

3.2.2



(a)

$\mathbb{P}\left(X < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x = \frac{-4}{3}\left(\frac{x^4}{4} - x\right)\Big|_0^{\frac{1}{2}} = 0.645833.$

(b)

$\mathbb{P}\left(\frac{1}{4} < X < \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} f(x)\,\mathrm{d}x = \frac{-4}{3}\left(\frac{x^4}{4} - x\right)\Big|_{\frac{1}{4}}^{\frac{3}{4}} = 0.5625.$

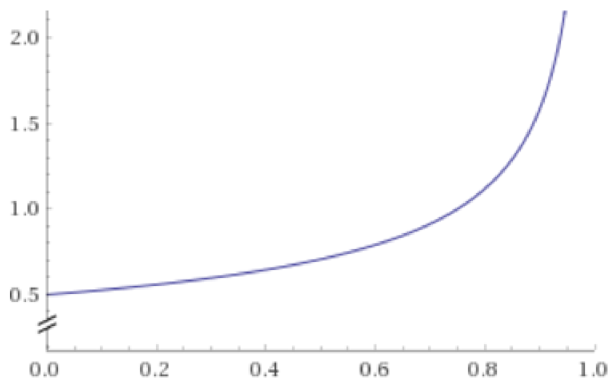
(c)

$\mathbb{P}\left(X > \frac{1}{3}\right) = \int_{\frac{1}{3}}^{1} f(x)\,\mathrm{d}x = \frac{-4}{3}\left(\frac{x^4}{4} - x\right)\Big|_{\frac{1}{3}}^{1} = 0.5597.$

3.2.10

(a)

By one of the axioms of probability, we know that $\int_0^1 f(x)\,\mathrm{d}x = 1$. $\int_0^1 \frac{c}{(1-x)^{\frac{1}{2}}}\,\mathrm{d}x = c\int_0^1 \frac{1}{(1-x)^{\frac{1}{2}}}\,\mathrm{d}x = 2c$. Therefore $c = \frac{1}{2}$.

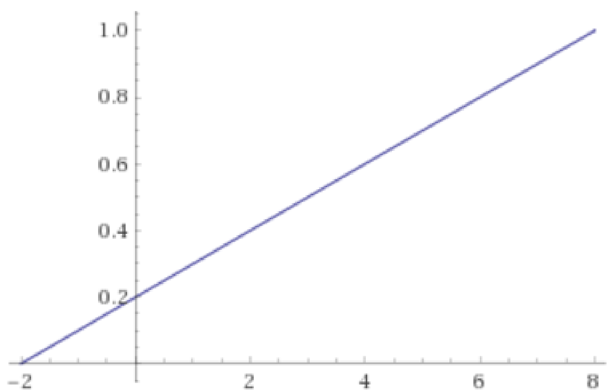


(b)

$\mathbb{P}\left(X \leq \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{1}{2(1-x)^{\frac{1}{2}}}\,\mathrm{d}x = 0.2929.$

3.3.7

The probability density function of X is $f(x) = \begin{cases} \frac{1}{10} & -2 < x < 8 \\ 0 & \text{otherwise} \end{cases}$. $\int_{-2}^{n} \frac{1}{10}\, \mathrm{d}x = \frac{n+2}{10}$. Therefore, the cumulative distribution function of X is $F(x) = \begin{cases} fracx + 210 & -2 \leq x \leq 8 \\ 1 & x > 8 \\ 0 & x < -2 \end{cases}$.



3.3.12

The quantile function is defined as $F^{-1}(p)$. Since $F(x) = \begin{cases} e^{(}x-3) & x \leq 3 \\ 1 & x > 3 \end{cases}$, we have $F^{-1}(p) = \begin{cases} ln(p) + 3 & 0 < p \leq 1 \\ 1 & p > 1 \end{cases}$.

3.3.14

By manual calculation:

1. The largest value x such that $\mathbb{P}(X \leq x) < 0.25$ is $x = 0$ ($\mathbb{P}(X \leq 0) = 0.1074, \mathbb{P}(X \leq 1) = 0.3758$). This is the first quartile of X.
2. The largest value x such that $\mathbb{P}(X \leq x) < 0.5$ is $x = 1$ ($\mathbb{P}(X \leq 1) = 0.3758, \mathbb{P}(X \leq 2) = 0.6778$). This is the median of X.
3. The largest value x such that $\mathbb{P}(X \leq x) < 0.75$ is $x = 2$ ($\mathbb{P}(X \leq 2 = 0.6778, \mathbb{P}(X \leq 3) = 0.8791$). This is the third quartile of X.