MATH 2220 HW #13

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Problem 1

(a) We need to find $F(x, y, z)$ that satisfies the following equalities:

$$\frac{\partial F}{\partial x} = \frac{-2x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$
$$\frac{\partial F}{\partial y} = \frac{-2y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$
$$\frac{\partial F}{\partial z} = \frac{-2z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Integrating with the substitution $u = x^2 + y^2 + z^2$, we get

$$\begin{aligned} F(x, y, z) &= \frac{2}{\sqrt{x^2 + y^2 + z^2}} + g(y, z) \\ &= \frac{2}{\sqrt{x^2 + y^2 + z^2}} + h(x, z) \\ &= \frac{2}{\sqrt{x^2 + y^2 + z^2}} + i(x, y) \end{aligned}$$

for some functions g, h, i. Canceling terms gives $g(y, z) = h(x, z) = i(x, y)$; the only way this equality can hold is if all the functions are constant. The simplest case is $g(y, z) = h(x, z) = i(x, y) = 0$, giving $F(x, y, z) = \frac{2}{\sqrt{x^2+y^2+z^2}}$.

(b) This time, the equalities for F are:

$$\frac{\partial F}{\partial x} = \frac{3(x + 1)}{((x + 1)^2 + (y - 5)^2 + z^2)^{\frac{3}{2}}} + x^3$$
$$\frac{\partial F}{\partial y} = \frac{3(y - 5)}{((x + 1)^2 + (y - 5)^2 + z^2)^{\frac{3}{2}}} - y^6$$
$$\frac{\partial F}{\partial z} = \frac{3z}{((x + 1)^2 + (y - 5)^2 + z^2)^{\frac{3}{2}}} + z + y$$

To integrate the first fraction, we first substitute $u = x + 1, \mathrm{d}u = \mathrm{d}x$ and then $s = u^2 + (y - 5)^2 + z^2, \mathrm{d}s = 2u\,\mathrm{d}u$:

$$\begin{aligned}\int \frac{3(x+1)}{((x+1)^2+(y-5)^2+z^2)^{\frac{3}{2}}}\,\mathrm{d}x &= \int \frac{3u}{(u^2+(y-5)^2+z^2)^{\frac{3}{2}}}\,\mathrm{d}u \\ &= \int \frac{3}{2}\frac{1}{s^{\frac{3}{2}}}\,\mathrm{d}s \\ &= -\frac{3}{\sqrt{s}}+C \\ &= -\frac{3}{\sqrt{(x+1)^2+(y-5)^2+z^2}}+C\end{aligned}$$

Similarly for the second and third, we get

$$\begin{aligned}\int \frac{3(y-5)}{((x+1)^2+(y-5)^2+z^2)^{\frac{3}{2}}}\,\mathrm{d}y &= -\frac{3}{\sqrt{(x+1)^2+(y-5)^2+z^2}}+C \\ \int \frac{3z}{((x+1)^2+(y-5)^2+z^2)^{\frac{3}{2}}}\,\mathrm{d}z &= -\frac{3}{\sqrt{(x+1)^2+(y-5)^2+z^2}}+C\end{aligned}$$

Note that these are equal. Therefore, we have:

$$\begin{aligned}F(x,y,z) &= -\frac{3}{\sqrt{(x+1)^2+(y-5)^2+z^2}}+\frac{x^4}{4}+g(y,z) \\ &= -\frac{3}{\sqrt{(x+1)^2+(y-5)^2+z^2}}-\frac{y^7}{7}+h(x,z) \\ &= -\frac{3}{\sqrt{(x+1)^2+(y-5)^2+z^2}}+\frac{z^2}{2}+yz+i(x,y)\end{aligned}$$

Canceling terms gives $\frac{x^4}{4}+g(y,z)=-\frac{y^7}{7}+h(x,z)=\frac{z^2}{2}+yz+i(x,y)$. There is no solution to this because I am a massive idiot and should have checked that the curl was 0 first. The curl is not 0, so this vector field is not conservative and has no potential function.

Problem 2

(a) The parametrisation here is $\mathbf{r}(t)=(\cos t,\sin t,t)$, so $\mathbf{r}'(t)=(-\sin t,\cos t,1)$. Suppose D is the region of $\mathbf{r}$ from $(1,0,0)$ to $(1,0,2\pi)$ (i.e. $0\le t\le 2\pi$). To rewrite $\mathbf{F}$ as a function of t, note that $x=\cos t$, $y=\sin t$ and $z=t$, so $\mathbf{F}(x,y,z)=(a\sin t+b\cos t\sin^2 t, 2\cos^2 t\sin t, \cos^2 t-t^2)$. Then

$$\begin{aligned}\int_D \mathbf{F}\cdot\mathbf{t}\,\mathrm{d}\sigma &= \int_0^{2\pi}(a\sin t+b\cos t\sin^2 t, 2\cos^2 t\sin t,\cos^2 t-t^2)\cdot\mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi}(-a\sin^2 t-b\cos t\sin^3 t+2\cos^3 t\sin t+\cos^2 t-t^2)\,\mathrm{d}t\end{aligned}$$

Using a variety of u-substitutions, we arrive at

$$\int_0^{2\pi}(-a\sin^2 t-b\cos t\sin^3 t+2\cos^3 t\sin t+\cos^2 t-t^2)\,\mathrm{d}t = \frac{-6at+3a\sin(2t)-2b\sin^4 t-4t^3+6t+3\sin(2t)-6}{12}$$

(b) The curl is equal to

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

For this $\mathbf{F}$, the derivatives are:

$$\begin{aligned}
\frac{\partial F_x}{\partial y} &= 2bxy \\
\frac{\partial F_x}{\partial z} &= a \cos z \\
\frac{\partial F_y}{\partial x} &= 4xy \\
\frac{\partial F_y}{\partial z} &= 0 \\
\frac{\partial F_z}{\partial x} &= \cos z \\
\frac{\partial F_z}{\partial y} &= 0
\end{aligned}$$

so

$$\begin{aligned}
\nabla \times \mathbf{F} &= (0 - 0, a \cos z - \cos z, 4xy - 2bxy) \\
&= (0, \cos z(a-1), 2xy(2-b))
\end{aligned}$$

For $\mathbf{F}$ to be conservative, i.e. for its curl to be 0, we need $a - 1 = 0 \implies a = 1$ and $2 - b = 0 \implies b = 2$.

(c) Suppose $\mathbf{F} = \nabla G$. Then

$$\begin{aligned}
\frac{\partial G}{\partial x} &= \sin z + 2xy^2 \\
\frac{\partial G}{\partial y} &= 2x^2 y \\
\frac{\partial G}{\partial z} &= x \cos z - z^2
\end{aligned}$$

Therefore

$$\begin{aligned}
G(x, y, z) &= x \sin z + x^2 y^2 + h(y, z) \\
&= x^2 y^2 + i(x, z) \\
&= x \sin z - \frac{z^3}{3} + j(x, y)
\end{aligned}$$

One solution for this is $G(x, y, z) = x \sin z + x^2 y^2 - \frac{z^3}{3} = \cos t \sin t + \cos^2 t \sin^2 t - \frac{t^3}{3}$. By the Fundamental Theorem of Calculus, the integral from before is equal to $G(2\pi) - G(0) = -\frac{8\pi^3}{3}$), which is exactly the same as the value calculated above when $a = 1$.

Problem 3

(a) By the Fundamental Theorem of Calculus, $\int_{\partial D} f \,\mathrm{d}y = \int_D f_x \,\mathrm{d}x\,\mathrm{d}y$, and $\int_{\partial D} f \,\mathrm{d}x = -\int_D f_y \,\mathrm{d}x\,\mathrm{d}y$. Therefore

$$\int_{\partial D} x \,\mathrm{d}y = \int_D 1 \,\mathrm{d}x\,\mathrm{d}y$$
$$\int_{\partial D} y \,\mathrm{d}x = -\int_D 1 \,\mathrm{d}x\,\mathrm{d}y$$

Also, $\int_D 1 \,\mathrm{d}x\,\mathrm{d}y$ is simply the area of region D, by definition.

(b) If $\mathbf{F}(x, y) = (x, 0)$, then $\nabla \cdot \mathbf{F}(x, y) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}0 = 1$.

(c) The area of a polygon mapped by set D is by definition $\int_D 1 \,\mathrm{d}x\,\mathrm{d}y$. Suppose $\mathbf{F}(x, y) = (x, 0)$. Then, by the divergence theorem, $\int_D 1 \,\mathrm{d}x\,\mathrm{d}y = \int_D \nabla \cdot \mathbf{F} \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial D} \mathbf{F} \cdot \hat{\mathbf{n}} \,\mathrm{d}\sigma$. In this case, ∂D is the piecewise curve enclosing the polygon. Here, we can parametrise one "face" C of the polygon (going from (a, b) to (c, d)) as $\mathbf{r}(t) = (a + t(c - a), b + t(d - b)), 0 \le t \le 1$. $\mathbf{r}'(t) = (c - a, d - b)$, and so the normal vector is **hatn** $= (d - b, a - c)$. Therefore

$$\begin{aligned}
\int_C \mathbf{F} \cdot \hat{\mathbf{n}} \,\mathrm{d}\sigma &= \int_0^1 (a + t(c - a), 0) \cdot (d - b, a - c) \,\mathrm{d}t \\
&= \int_0^1 (a + t(c - a))(d - b) \,\mathrm{d}t \\
&= \int_0^1 (a(d - b) + t(c - a)(d - b)) \,\mathrm{d}t \\
&= a(d - b) + \frac{(c - a)(d - b)}{2}
\end{aligned}$$

Therefore, the total area is the sum of these terms for all pairs of adjacent vertices (a, b) and (c, d).

Problem 4

The parametrisation of a circle with radius r is $\mathbf{s}(t) = r(\cos t, \sin t), \mathbf{s}'(t) = r(-\sin t, \cos t)$. On this circle, $\mathbf{F}(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = (-\sin t, \cos t)$. Therefore

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{t} \,\mathrm{d}\sigma &= \int_0^{2\pi} (-\sin t, \cos t) \cdot r(-\sin t, \cos t) \,\mathrm{d}t \\
&= \int_0^{2\pi} r(\sin^2 t + \cos^2 t) \,\mathrm{d}t = \int_0^{2\pi} r \,\mathrm{d}t = 2\pi r
\end{aligned}$$

Problem 5

Assume that X is indeed a closed curve. Then, we can apply the divergence theorem, like so:

$$\begin{aligned}\iint_C \nabla\cdot\mathbf{F}\,\mathrm{d}A &= \int_X \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\sigma\\ &= \int_X \mathbf{F}(X)\cdot(y',-x')\,\mathrm{d}t\\ &= \int_X (x',y')\cdot(y',-x')\,\mathrm{d}t\\ &= \int_X (x'y'-y'x')\,\mathrm{d}t = 0\end{aligned}$$

where C is the region bounded by X. If $\nabla\cdot\mathbf{F}>0$ everywhere on C, then $\iint_C \nabla\cdot\mathbf{F}\,\mathrm{d}A > 0$, and so X cannot possibly be a closed (periodic) curve.

Problem 6

(a) As in (4), parametrise the circle C as $\mathbf{r}(t)=4(\cos t,\sin t), \mathbf{r}'(t)=4(-\sin t,\cos t)$, and convert $\mathbf{F}(x,y)=(3x+4y,2x-3y)=(3\cos t+4\sin t,2\cos t-3\sin t)$:

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{t}\,\mathrm{d}\sigma &= \int_0^{2\pi}(3\cos t+4\sin t,2\cos t-3\sin t)\cdot 4(-\sin t,\cos t)\,\mathrm{d}t\\ &= \int_0^{2\pi}(-12\sin t\cos t-16\sin^2 t+8\cos^2 t-12\sin t\cos t)\,\mathrm{d}t\\ &= \int_0^{2\pi}(8\cos^2 t-16\sin^2 t-24\sin t\cos t)\,\mathrm{d}t\\ &= (6(\sin(2t)+\cos(2t))-4t)\Big|_0^{2\pi}\\ &= (6-8\pi)-(6-0) = -8\pi\end{aligned}$$

(b) Suppose D is the region enclosed by C. The area of D is $\iint_D 1\,\mathrm{d}A$. Given F such that $\nabla\times\mathbf{F}=1$, this is equal to $\iint_D \nabla\times\mathbf{F}\,\mathrm{d}A = \int_C \mathbf{F}\cdot\mathbf{t}\,\mathrm{d}\sigma$. For example, $\mathbf{F}(x,y)=(x,0)$ satisfies this property. On C, $F(x,y)=(2\sin t,0)$, therefore

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{t}\,\mathrm{d}\sigma &= \int_0^{2\pi}(2\sin t,0)\cdot(2\sin t,\sin t)\,\mathrm{d}t\\ &= \int_0^{2\pi} 4\sin^2 t\,\mathrm{d}t = 4\pi\end{aligned}$$

(c) By Green's Theorem, $\int \partial D\mathbf{F}\cdot\mathbf{t}\,\mathrm{d}\sigma = \iint_D \nabla\times\mathbf{F}\,\mathrm{d}A$. Suppose D is the unit disk and $\mathbf{F}(x,y)=(-y^3+\log(2+\sin x),x^3+\arctan y)$; then

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{t}\,\mathrm{d}\sigma &= \iint_D \nabla\times\mathbf{F}\,\mathrm{d}A\\ &= \iint_D (3x^2-(-3y^2))\,\mathrm{d}A = 3\iint_D (x^2+y^2)\,\mathrm{d}A\end{aligned}$$

Transforming to polar co-ordinates, we get $x=r\cos\theta$, $y=r\sin\theta$ and $D: 0\le r\le 1, 0\le\theta\le 2\pi$.

$$\begin{aligned} 3\iint_D (x^2+y^2)\,\mathrm{d}A &= 3\int_0^{2\pi}\int_0^1 r^3\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 3\int_0^{2\pi}\frac{1}{4}\,\mathrm{d}\theta = \frac{3\pi}{2} \end{aligned}$$

Problem 7

(a)

$$\begin{aligned} \int_D \nabla\cdot\mathbf{F}\,\mathrm{d}A &= \int_D 0\,\mathrm{d}A = 0 \\ &= \int_{\partial D}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}\sigma \\ &= \int_{\partial D}(1,0,0)\cdot(n_1,n_2,n_3)\,\mathrm{d}\sigma \\ &= \int_{\partial D} n_1\,\mathrm{d}\sigma \end{aligned}$$

(b) Set $\mathbf{G} = (0,1,0), \mathbf{H} = (0,0,1)$. As above, $\int_D \nabla\cdot\mathbf{G}\,\mathrm{d}A = \int_D \nabla\cdot\mathbf{H}\,\mathrm{d}A = 0$. However, the dot product is n_2 for $\mathbf{G}$ and n_3 for $\mathbf{H}$. Therefore, $\int_{\partial D} n_1\,\mathrm{d}\sigma = \int_{\partial D} n_2\,\mathrm{d}\sigma = \int_{\partial D} n_3\,\mathrm{d}\sigma = 0$, and so $\int_{\partial D}\hat{\mathbf{n}}\,\mathrm{d}\sigma = 0$.

(c) Suppose $C \subseteq \partial D$ is some face of D; then, its area is $\int_C 1\,\mathrm{d}s$. The sum of the areas of all the faces is just $\int_{\partial D} 1\,\mathrm{d}\sigma$. Therefore

$$\begin{aligned} \sum \text{Area(face)}\hat{\mathbf{n}}\,\mathrm{d}\sigma &= \int_{\partial D} 1\hat{\mathbf{n}}\,\mathrm{d}\sigma \\ &= 0 \end{aligned}$$