# STSCI 3080 HW #4

KIRILL CHERNYSHOV

**Problem 3.8.4** The c.d.f. of $X$ is $F(x) = \int_0^x f(t)\,\mathrm{d}t = \int_0^x \frac{t}{2}\,\mathrm{d}t = \frac{t^2}{4}\Big|_0^x = \frac{x^2}{2}$, for $0 \le x \le 2$. By definition, $F(x) = \mathbb{P}(X \le x)$. Similarly, the c.d.f. of $Y$ is $G(y) = \mathbb{P}(Y \le y) = \mathbb{P}(4 - X^3 \le y) = \mathbb{P}(4 - y \le X^3) = \mathbb{P}(\sqrt[3]{4-y} \le X) = 1 - \mathbb{P}(X \le \sqrt[3]{4-y}) = 1 - F(\sqrt[3]{4-y}) = 1 - \frac{(4-y)^{\frac{2}{3}}}{2}$, for $-4 \le y \le 0$. The p.d.f. of $Y$ is, therefore,

$$\begin{aligned} g(y) &= \frac{\mathrm{d}}{\mathrm{d}y} G(y) \\ &= \frac{\mathrm{d}}{\mathrm{d}y}\left(1 - \frac{(4-y)^{\frac{2}{3}}}{2}\right) \\ &= \frac{1}{3\sqrt[3]{4-y}} \end{aligned}$$

**Problem 3.8.6** We have $Y = r(X)$ where $r(x) = 3x + 2$. Let $s(y) = r^{-1}(y) = \frac{y-2}{3}$. Then, the p.d.f. of $y$ is

$$\begin{aligned} g(y) &= f(s(y)) \cdot \frac{\mathrm{d}s}{\mathrm{d}y} \\ &= \left(\frac{y-2}{3}\right)^2 \frac{1}{2}\frac{1}{3} \\ &= \frac{(y-2)^2}{54} \end{aligned}$$

**Problem 3.9.8** For each $X_i$, its c.d.f. is $F(x) = \mathbb{P}(X \le x) = x$, as it is a uniform distribution on the interval $[0,1]$. By definition, the c.d.f. of $Y$ is $G(y) = \mathbb{P}(Y \le y) = \mathbb{P}(X_1 \le y, X_2 \le y \cdots X_n \le y) = \prod_{i=1}^n \mathbb{P}(X_i \le y) = \prod_{i=1}^n F(y) = F(y)^n = y^n$. $\mathbb{P}(Y_n \ge 0.99) = 1 - \mathbb{P}(Y_n \le 0.99) = 1 - G(0.99) = 1 - 0.99^n$. We need to find the smallest $n$ such that $\mathbb{P}(Y_n \ge 0.99) \ge 0.95$, so:

$$\begin{aligned} \mathbb{P}(Y_n \ge 0.99) &\ge 0.95 \\ 1 - 0.99^n &\ge 0.95 \\ 0.99^n &\le 0.05 \\ n &\ge \log_{0.99} 0.05 \\ &\ge 298.073 \end{aligned}$$

So the smallest integral value is $n = 299$.

**Problem 4.1.9** Let $X$ be the distance from a certain end of the stick to the point where it is broken; $X$ is therefore uniformly distributed on $[0,1]$. Define $Y$ as the length of the longer piece of the stick: $Y = \max(X, 1-X)$. Since $Y = X$ if $X \ge 1 - X$ and $Y = 1 - x$ if $X < 1 - X$, this means $Y$ takes on uniformly distributed values in the interval $[0.5, 1]$, and so its expected value is $\frac{0.5+1}{2} = 0.75$.

**Problem 4.2.3** Because they are uniformly distributed on $[0, 1]$, $\mathrm{E}(X_1) = \mathrm{E}(X_2) = \mathrm{E}(X_3) = 0.5$. Moreover, $\mathrm{E}(X_1^2) = \mathrm{E}(X_2^2) = \mathrm{E}(X_3^2) = \int_0^1 x^2\,\mathrm{d}x = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$. Therefore:

$$\begin{aligned}
\mathrm{E}[(X_1 - 2X_2 + X_3)^2] &= \mathrm{E}[X_1^2 - 2X_1X_2 + X_1X_3 - 2X_2X_1 + 4X_2^2 - 2X_2X_3 + X_3X_1 - 2X_2X_3 + X_3^2] \\
&= \mathrm{E}[X_1^2 + 4X_2^2 + X_3^2 - 4X_1X_2 - 4X_2X_3 + 2X_3X_1] \\
&= \mathrm{E}(X_1^2) + 4\mathrm{E}(X_2^2) + \mathrm{E}(X_3^2) - 4\mathrm{E}(X_1X_2) - 4\mathrm{E}(X_2X_3) + 2\mathrm{E}(X_3X_1) \\
&= \mathrm{E}(X_1^2) + 4\mathrm{E}(X_2^2) + \mathrm{E}(X_3^2) - 4\mathrm{E}(X_1)\mathrm{E}(X_2) - 4\mathrm{E}(X_2)\mathrm{E}(X_3) + 2\mathrm{E}(X_3)\mathrm{E}(X_1) \\
&= \frac{1}{3} + 4\frac{1}{3} + \frac{1}{3} - 4\frac{1}{2}\frac{1}{2} - 4\frac{1}{2}\frac{1}{2} + 2\frac{1}{2}\frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

**Problem 4.3.3** Given the p.d.f. $f(x) = \frac{1}{b-a}$, the variance can be calculated as

$$\begin{aligned}
\mathrm{Var}(X) &= \int_a^b \frac{x^2}{b-a}\,\mathrm{d}x - \left(\int_a^b \frac{x}{b-a}\,\mathrm{d}x\right)^2 \\
&= \frac{x^3}{3(b-a)}\Big|_a^b - \left(\frac{x^2}{2(b-a)}\Big|_a^b\right)^2 \\
&= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b^2 - a^2}{2(b-a)}\right)^2 \\
&= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

**Problem 4.3.7** $\mathrm{Var}(X - Y) = \mathrm{Var}(X) + (-1)^2\mathrm{Var}(Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 6$. $\mathrm{Var}(2X3Y + 1) = 2^2\mathrm{Var}(X) + (-3)^2\mathrm{Var}(Y) = 4\mathrm{Var}(X) + 9\mathrm{Var}(Y) = 12 + 27 = 39$.

**Problem 4.5.6** By Theorem 4.5.2 in the book, the value of $d$ that minimises $\mathrm{E}[(X - d)^2]$ is the mean of $X$, that is, $\int_0^1 2x^2\,\mathrm{d}x = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$. By Theorem 4.5.3 in the book, the value of $d$ that minimises $\mathrm{E}[|X - d|]$ is the median of $X$, that is, the value $d$ such that $\mathbb{P}(X \le d) = F(d) = \frac{1}{2}$. $\mathbb{P}(X \le d) = \int_0^d 2x\,\mathrm{d}x = x^2\Big|_0^d = d^2$. The equality is, therefore, $d^2 = \frac{1}{2}$, so $d = \frac{\sqrt{2}}{2}$.