Homework 1 solutions

(1)

1. Let $Im(f)$ be the image of function f, and let $D(f)$ be its domain. Then, f_1 extends f_2 iff $Im(f_2) \subseteq Im(f_1)$ and, for any $i \in Im(f_2)$, $f_2(i) = f_1(i)$.
2. A **tentative Sudoku problem** is a function f_1 that extends the Sudoku board function such that there may or may not exist total function f_2 that extends f_1, follows the three properties of Sudoku solutions, and acts on the entire domain of f_1, such that, for any element i in $D(f_1)$, $f_2(i)$ is defined. A **solvable Sudoku problem** is a function f_1 that extends a tentative Sudoku problem, for which such a function f_2 *does* exist, but is not necessarily known. A **uniquely solvable Sudoku problem** is such a function f_1 that extends a solvable Sudoku problem, for which once again, such a function f_2 does exist. However, in this case, for any two total functions f_a, f_b that extend f_1, and any element $i \in D(f_1)$, $f_a(i) = f_b(i)$.

(2)

Proof. We know that, by definition, there are N distinct boxes, each numbered distinctly. Therefore, we know that for any $b, n \in N$, there exists a box with that number, that is, there exist $|N|$ $c \in C$ such that $box(c) = b$. We want to show that for one of these c, the equality $s(c) = n$ holds. Suppose that property $3'$ is false, that is, such a c does not necessarily exist. We know that the image of s is N, and the amount of elements in a box, that is, the amount of cs for which $box(x) = b$ holds, is $|N|$. However, if $3'$ is false then the image of s within this box is a subset of $|N|$ that is missing one element. By the pigeonhole principle, there must exist two distinct cs, c_1 and c_2, such that $box(c_1) = box(c_2)$ and $s(c_1) = s(c_2)$. This violates property 3, therefore $3'$ cannot be false if 3 is true. □

(3)

1. The domain of this function, $flip$, is the set F of partial functions that define a Sudoku board. The codomain of this function is the same set F. Let f_1, f_2 be Sudoku functions such that $flip(f_1) = f_2$. Then, for all $c \in C$, $f_1(c) = f_2(c')$. We define c' as the cell whose co-ordinates are the same as c but swapped; for instance, if the co-ordinates of c are $(3, 5)$ then the co-ordinates of c' are $(5, 3)$. This implies that $flip(flip(f_1)) = f_1$.

2. One can use the definition of the flip function to show that if all three properties of a solution hold for f_1, then they must hold for $flip(f_1)$. The first two properties are as follows:

   i. If two cells are in the same row, then they must contain different values. That is, for all j, if $i_1 \neq i_2$ then $s(i_1, j) \neq s(i_2, j)$.
   ii. If two cells are in the same column, then they must contain different values. That is, for all i, if $j_1 \neq j_2$ then $s(i, j_1) \neq s(i, j_2)$.

   From the definition of the function, it is trivial to see that if the first property holds in f_1, then the second property holds in $flip(f_1)$. This comes from the fact that the flipping function switches around the co-ordinates; therefore, if in f_1, $\forall\ i, j_1 \neq j_2 \implies s(i, j_1) \neq s(i, j_2)$ (the first property), then in $flip(f_1)$, $\forall\ i, j_1 \neq j_2 \implies s(j_1, i) \neq s(j_2, i)$ (the second property). Vice-versa, this shows that if the second property holds in f_1 then the first property must hold in $flip(f_1)$. Therefore, if f_1 satisfies both the properties then so does $flip(f_1)$.

   For the third property, we must first explicitly define the *box* function. If the boxes are designated co-ordinates (a, b) with a, b ranging from 1 through $k = \sqrt{|N|}$, then we can define \$box(i, j) $= (\lceil \frac{i}{k} \rceil, \lceil \frac{j}{k} \rceil)$. It is clear that applying the flipping function, i.e. swapping the co-ordinates of a cell, also swaps the co-ordinates of the cell's box: \$box(j, i) $= (\lceil \frac{j}{k} \rceil, \lceil \frac{i}{k} \rceil)$. Therefore, two cells will be in the same box in $flip(f_1)$ iff they are in the same box in f_1, since each cell's box simply has its co-ordinates swapped; the cells whose boxes had the same co-ordinates before will still have the same box co-ordinates after being flipped, and those with different box co-ordinates will still have different box co-ordinates. This is because flipping co-ordinates $(a, b) \mapsto (b, a)$ is a bijective function. Since cells will stay in the same boxes going from f_1 to $flip(f_1)$, this implies that property 3 will hold for $flip(f_1)$ iff it holds for f_1. Therefore, all three properties will hold for $flip(f_1)$ iff they hold for f_1; in other words, $flip(f_1)$ is a solution iff f_1 is a solution.