

## MATH 3110 HOMEWORK #5

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**Problem 6.4.1** Prove that every convergent sequence is a Cauchy sequence.

*Proof.* Suppose  $\{a_n\}$  is convergent. Then, there exists  $L$  such that  $\lim_{n \rightarrow \infty} a_n = L$ , that is, for any  $\epsilon > 0$ , there exists  $N$  such that for all  $n \geq N$ ,  $|a_n - L| < \epsilon$ . Fix  $\epsilon$ , and pick any  $m, n \geq N$ . Then:

$$\begin{aligned} |a_n - a_m| &= |a_n - L - a_m + L| \\ &= |a_n - L - (a_m - L)| \leq |a_n - L| + |a_m - L| \leq 2\epsilon \end{aligned}$$

Therefore, by the  $K - \epsilon$  principle,  $\{a_n\}$  is a Cauchy sequence.  $\square$

**Problem 6.5.4** Let  $S$  and  $T$  be non-empty subsets of  $\mathbb{R}$ , such that for all  $s \in S, t \in T$ ,  $s \leq t$ . Prove that  $\sup S \leq \inf T$ .

*Proof.* Fix  $s \in S$ . Since for any  $t \in T$ ,  $s \leq t$ ,  $s$  is a lower bound for  $T$ . By definition,  $\inf T$  is the greatest lower bound of  $T$ , and so  $s \leq \inf T$ . Since we made no assumptions about  $s$ , it follows that for any  $s \in S$ ,  $s \leq \inf T$ , and therefore  $\inf T$  is an upper bound for  $S$ . Since  $\sup S$  is the least upper bound of  $S$ ,  $\sup S \leq \inf T$ .  $\square$

### Problem 6-1

(a) Show that  $\{x_n\}$  is a Cauchy sequence.

*Proof.* We are given that  $x_n = \frac{x_{n-1} + x_{n-2}}{2}$ . Consider the absolute difference between two consecutive terms:

$$\begin{aligned} |x_n - x_{n-1}| &= \left| \frac{x_{n-1} + x_{n-2}}{2} - x_{n-1} \right| \\ &= \left| \frac{x_{n-2} - x_{n-1}}{2} \right| \\ &= \frac{1}{2} |x_{n-2} - x_{n-1}| = \frac{1}{2} |x_{n-1} - x_{n-2}| \\ &= \frac{1}{2} \frac{1}{2} |x_{n-2} - x_{n-3}| = \cdots = \frac{1}{2^{n-1}} |x_1 - x_0| \end{aligned}$$

Now, consider the difference between two arbitrary terms,  $x_m$  and  $x_n$ , where  $m \geq n$ :

$$\begin{aligned}
|x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&= \frac{1}{2^{m-1}}|x_1 - x_0| + \frac{1}{2^{m-2}}|x_1 - x_0| + \cdots + \frac{1}{2^n}|x_1 - x_0| \\
&= |x_1 - x_0| \sum_{i=n}^{m-1} \frac{1}{2^i} \\
&= |x_1 - x_0| \frac{1}{2^n} \frac{1 - \frac{1}{2^{m-n}}}{1 - \frac{1}{2}} \\
&= |x_1 - x_0| \frac{1}{2^{n-1}} \left(1 - \frac{1}{2^{m-n}}\right) \\
&= |x_1 - x_0| \left(\frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}\right) \\
&< \frac{1}{2^{n-1}}|x_1 - x_0|
\end{aligned}$$

Fix  $\epsilon$ , and set  $N$  such that  $\epsilon > \frac{1}{2^{N-1}}$ , i.e.  $N > 1 - \log_2 \epsilon$ . Then, for all  $m \geq n \geq N$ ,  $|x_m - x_n| < \epsilon|x_1 - x_0|$ . Since  $|x_1 - x_0|$  is a constant, by the  $K - \epsilon$  principle,  $\{x_n\}$  is a Cauchy sequence.  $\square$

**(b)** The limit must exist, since  $\{x_n\}$  is a Cauchy sequence. As seen in part (a),  $x_n - x_{n-1} = -\frac{1}{2}|x_{n-1} - x_{n-2}| = \left(\frac{-1}{2}\right)^{n-1}(x_1 - x_0) = \left(\frac{-1}{2}\right)^{n-1}(b - a)$ . Now, consider a single term of the sequence:

$$\begin{aligned}
x_n &= \sum_{i=1}^n (x_i - x_{i-1}) + x_0 \\
&= \sum_{i=1}^n \left(\frac{-1}{2}\right)^{i-1} (b - a) + a \\
&= (b - a) \sum_{i=1}^n \left(\frac{-1}{2}\right)^{i-1} + a
\end{aligned}$$

Therefore, we can calculate the limit:

$$\begin{aligned}
\lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} \left[ (b - a) \sum_{i=1}^n \left(\frac{-1}{2}\right)^{i-1} + a \right] \\
&= \lim_{n \rightarrow \infty} \left[ (b - a) \sum_{i=1}^n \left(\frac{-1}{2}\right)^{i-1} \right] + \lim_{n \rightarrow \infty} [a] \\
&= \lim_{n \rightarrow \infty} [(b - a)] \cdot \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n \left(\frac{-1}{2}\right)^{i-1} \right] + \lim_{n \rightarrow \infty} [a] \\
&= (b - a) \left( \frac{1}{1 + \frac{1}{2}} \right) + a = \frac{2}{3}(b - a) + a
\end{aligned}$$

## Problem 6-2

(a) Let  $S \subseteq \mathbb{R}$  be a bounded non-empty set, and let  $\bar{m} = \sup S$ . Prove that there exists a sequence  $\{a_n\}$  such that for all  $n$ ,  $a_n \in S$ , and  $a_n \rightarrow \bar{m}$ .

*Proof.* Pick any sequence  $\{b_n\}$  such that  $b_n \rightarrow 0$ , and bounded such that  $\bar{m} - b_n \in S$  for all  $n$ . Since  $S$  is non-empty, this is always possible. Then, define  $\{a_n\}$  as  $a_n = \bar{m} - b_n$ . Then,  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (\bar{m} - b_n) = \lim_{n \rightarrow \infty} \bar{m} - \lim_{n \rightarrow \infty} b_n = \bar{m} - 0 = \bar{m}$ .  $\square$

(b) From exercise 6.5.3, we know that  $\sup(A + B) \leq \sup A + \sup B$ . Therefore, showing that  $\sup(A + B) \geq \sup A + \sup B$  will be sufficient to prove that  $\sup(A + B) = \sup A + \sup B$ .

*Proof.* By part (a), there exist sequences  $\{a_n\} \subseteq A$  and  $\{b_n\} \subseteq B$  such that  $a_n \rightarrow \sup A$ , and  $b_n \rightarrow \sup B$ . Since  $a_n \in A, b_n \in B \forall n$ ,  $a_n + b_n \in A + B$ , and therefore  $a_n + b_n \leq \sup(A + B)$ . By the limit location theorem,  $\lim a_n + b_n \leq \sup(A + B)$ . But  $\lim a_n + b_n = \lim a_n + \lim b_n = \sup A + \sup B$ , and therefore  $\sup A + \sup B = \sup(A + B)$ .  $\square$

**Problem 7.2.1** Evaluate  $\sum_{i=0}^{\infty} \frac{1}{(2n+1)^2}$ , given that  $\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ .

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{1}{n^2} &= \sum_{i=0}^{\infty} \frac{1}{(2n)^2} + \sum_{i=0}^{\infty} \frac{1}{(2n+1)^2} \\ &= \frac{1}{4} \sum_{i=0}^{\infty} \frac{1}{n^2} + \sum_{i=0}^{\infty} \frac{1}{(2n+1)^2} \end{aligned}$$

Therefore,  $\sum_{i=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4} \sum_{i=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$ . Line 1 above works because it is given that that  $\sum_{i=0}^{\infty} \frac{1}{n^2}$  and therefore  $\sum_{i=0}^{\infty} \frac{1}{(2n)^2}$  are both convergent, so  $\sum_{i=0}^{\infty} \frac{1}{(2n+1)^2}$  must also converge.