Linear dependence and independence

A collection of vectors $\{\vec{u_1}, \vec{u_2}, \cdots \vec{u_p}\} \subseteq \mathbb{R}^n$ is called **linearly independent** if $\sum_{i=1}^{p} \vec{x}_i \vec{u_i} = 0$ only has the trivial solution $\vec{x}_1 = \vec{x}_2 = \cdots = \vec{x}_p = 0$. Conversely, if $\sum_{i=1}^{p} \vec{x}_i \vec{u_i} = 0$ has non-trivial solutions, then the collection of vectors is called **linearly dependent**. For example, the following collection of vectors is linearly dependent:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$$

since solving

$$x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + z \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} z = t \\ y = -t \\ x = 2t \end{cases}$$

as z is a free variable. Therefore, the solution is

$$2t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As another example, $\{\vec{0}\}$ is linearly dependent, since $c\vec{0} = \vec{0}\ \forall\ c \neq 0$. However, $\{\vec{v}\} \neq \{\vec{0}\}$ is linearly *in*dependent. This is because from $c\vec{v} = \vec{0}$, we get $c\vec{v}_i = 0\ \forall\ i$ **but** $\vec{v}_i \neq 0\ \forall\ i$, which is clearly impossible unless $c = 0$, i.e. the trivial solution is the only solution. Finally, $\{\vec{0}, \vec{u_1}, \vec{u_2}, \cdots \vec{u_p}\}$ is linearly dependent, since for any $c \neq 0$, $c\vec{0} + \sum_{i=1}^{p} 0\vec{u_i} = \vec{0}$, and so we have infinite non-trivial linear combinations giving $\vec{0}$.

> Theorem 1: $S = \{\vec{u_1}, \vec{u_2}, \cdots \vec{u_p}\} \subseteq \mathbb{R}^n$ is linearly dependent iff at least one $\vec{u_i}$ is a linear combination of other vectors in the collection.

Proof. Assume that $\sum_{i=1}^{p} c_i \overrightarrow{u_i} = 0$, where $c_k \neq 0$. Rearranging, we get $-c_k \overrightarrow{u_k} = \sum_{i=0}^{k-1} c_i \overrightarrow{u_i} + \sum_{i=k+1}^{p} c_i \overrightarrow{u_i}$, and so $\overrightarrow{u_k} = \sum_{i=0}^{k-1} \frac{-c_i}{c_k} \overrightarrow{u_i} + \sum_{i=k+1}^{p} \frac{-c_i}{c_k} \overrightarrow{u_i}$. Thus, it is clear that if any $\overrightarrow{u_k}$ is a linear combination of the other vectors, then the system must have non-trivial solutions. $\square$

$\boxed{\text{Theorem 2: If } p > n \text{ then } \{\overrightarrow{u_1}, \overrightarrow{u_2}, \cdots \overrightarrow{u_p}\} \subseteq \mathbb{R}^n \text{ is linearly dependent.}}$

Proof. This is analogous to solving $\vec{A}\vec{x} = \vec{0}$, where $\vec{A}$ has more columns than rows. Therefore, the row-reduced augmented matrix cannot have pivot elements in all columns, as each row can only have one pivot element, and so there must be at least one free variable. Thus, there must exist at least one set of non-trivial solutions, and so the set is linearly dependent. $\square$