STSCI 3080 HW #6

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Problem 5.9.6 We are given $p_3 = 0.38, p_4 = 0.32$. Therefore, $\mathrm{E}(X_3) = 15 \cdot 0.38 = 5.7, \mathrm{E}(X_4) = 15 \cdot 0.32 = 4.8, \mathrm{Var}(X_3) = 15 \cdot 0.38(1-0.38) = 3.534, \mathrm{Var}(X_4) = 15 \cdot 0.32(1-0.32) = 3.264$. By definition we know that $\mathrm{E}(X_3 - X_4) = \mathrm{E}(X_3) - \mathrm{E}(X_4) = 0.9$, and $\mathrm{Var}(X_3 - X_4) = \mathrm{Var}(X_3) + \mathrm{Var}(X_4) = 6.798$.

Problem 5.10.2 The random variable $(B|a)$, where $a = 80$ is a score on test A is normally distributed. Its parameters are

$$\mu = \mu_B + \rho\sigma_B \left(\frac{a - \mu_A}{\sigma_A}\right) = 90 + 0.8 \cdot 16 \left(\frac{80 - 85}{10}\right) = 83.6$$
$$\sigma^2 = (1 - \rho^2)\sigma_B^2 = (1 - 0.8^2)16^2 = 92.16$$

The result $(B|80) = 90$ is 6.4 away from the mean; given $\sigma = \sqrt{92.16} = 9.6$, this is $\frac{2}{3}$ standard deviations. Therefore, $\mathbb{P}(B|80 > 90) = 1 - \Phi(0.6667) = 1 - 0.7474 = 0.2525$.

Problem 5.10.5 The prediction with the smallest mean standard error is $\mathrm{E}(A|100) = \mu_A + \rho \cdot \sigma_A \left(\frac{100-\mu_B}{\sigma_B}\right) = 85 + 0.8 \cdot 10 \left(\frac{100-90}{16}\right) = 90$. The mean standard error is $\mathrm{Var}(A|100) = (1 - \rho^2)\sigma_A^2 = (1 - 0.8^2)10^2 = 36$.

Problem 6.2.6 We know that $\mathrm{E}(\bar{X}) = \mu$ and $\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$ where n is the number of observations. Using Chebyshev's inequality, $\mathbb{P}(6 \le \bar{X} \le 7) = \mathbb{P}(|\bar{X} - 6.5| \le 0.5) = 1 - \mathbb{P}(|\bar{X} - 6.5| \ge 0.5) \ge 1 - \frac{\mathrm{Var}\bar{X}}{0.5} = 1 - \frac{8}{n} = 0.8$; therefore, the smallest n that satisfies this is $n = 40$.

Problem 6.3.1 Let X be the length of rope produced in one minute, in feet, and let T be the total length of rope produced in 60 minutes. Then, $T \sim N(60 \cdot 4, 60 \cdot \frac{5^2}{12^2}) = N(240, 10.42)$, and $\mathbb{P}(T \ge 250) = 1 - \Phi\left(\frac{250-240}{\sqrt{10.42}}\right) = 1 - \Phi(3.098) = 1 - 0.9990 = 0.001$.

Problem 6.3.5 Define X as being 1 if an item is defective and 0 if it is not defective, with $\mu = 0.1, \sigma^2 = 0.1 \cdot 0.9 = 0.09$. Then, $\bar{X}$ is the proportion of defective items out of a total sample of n items. It follows from the central limit theorem that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) = N(0.1, \frac{0.09}{n})$. We need to find the smallest n such that $\mathbb{P}(\bar{X} < 0.13) \ge 0.99$.

$$\mathbb{P}(\bar{X} < 0.13) = \Phi\left(\frac{0.13 - 0.1}{\frac{0.09}{n}}\right) = \Phi\left(\frac{n}{3}\right) \ge 0.99$$
$$\frac{n}{3} \ge \Phi^{-1}(0.99) = 2.3264$$
$$n \ge 6.9792$$

Therefore, the smallest number of samples that satisfies this property is $n = 7$.

Problem 6.4.2 We are given that $X \sim B(15, 0.3)$. Suppose $Y \sim N(4.5, 3.15)$; applying a continuity correction, $\mathbb{P}(X = 4) = \mathbb{P}(3.5 < Y < 4.5) = \mathbb{P}(Y < 4.5) - \mathbb{P}(Y < 3.5) = 0.5 - (1 - \Phi\left(\frac{4.5-3.5}{\sqrt{3.15}}\right)) = 0.5 - (1 - \Phi(0.5634)) = 0.5 - (1 - 0.7134) = 0.2134$.

Problem 8.3.6

(a) The M.L.E. of the standard deviation of the normal distribution has the chi-squared distribution with n degrees of freedom, where n is the sample size. Since

$$\mathbb{P}\left(\frac{\sigma^2}{2} \leq \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2 \leq 2\sigma^2\right) = \mathbb{P}\left(\frac{n}{2} \leq \sum_{i=1}^{n}\frac{(X_i - \mu)^2}{\sigma^2} \leq 2n\right)$$

use $n = 16$ and the chi-squared distribution with 16 degrees of freedom. $\mathbb{P}(8 \leq \chi^2_{16} \leq 32) = \mathbb{P}(\chi^2_{16} \leq 32) - \mathbb{P}(\chi^2_{16} \leq 8) = 0.990 - 0.0511 = 0.9389$.

(b) The value of $\sum_{i=1}^{n}\frac{(X_i - \bar{X}_n)^2}{\sigma^2}$ has the chi-squared distribution with $n-1$ degrees of freedom. Since

$$\mathbb{P}\left(\frac{\sigma^2}{2} \leq \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \leq 2\sigma^2\right) = \mathbb{P}\left(\frac{n}{2} \leq \sum_{i=1}^{n}\frac{(X_i - \bar{X}_n)^2}{\sigma^2} \leq 2n\right)$$

use $n = 16$ and the chi-squared distribution with 15 degrees of freedom. $\mathbb{P}(8 \leq \chi^2_{15} \leq 32) = \mathbb{P}(\chi^2_{15} \leq 32) - \mathbb{P}(\chi^2_{15} \leq 8) = 0.994 - 0.0762 = 0.9178$.