MATH 3110 - LIMITS

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1. Limits

1.1. **Definition.** We define an infinite limit of a sequence as follows:
$\lim_{n\to\infty} a_n = L$ means that $\forall\, \epsilon > 0$, $\exists N \in \mathbb{R}$ such that $\forall\, n \geq N$, $|a_n - L| < \epsilon$.

1.2. **Uniqueness of limit.** Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$. Then, $L = M$.

Proof. Suppose that instead, $L \neq M$. Let $\epsilon' = \frac{|L-M|}{2} > 0$. $\lim_{n\to\infty} a_n = L$ means that $|a_n - L| < \epsilon'$ for all $n \geq N_1$. Similarly, $\lim_{n\to\infty} a_n = M$ means that $|a_n - M| < \epsilon'$ for all $n \geq N_2$. Combining these inequalities, we can say that $-\epsilon' < a_n - L < \epsilon'$ **and** $-\epsilon' < a_n - M < \epsilon'$ for all $n \geq \max(N_1, N_2)$. Subtracting, we get the largest and smallest values of the difference between the two limits: $-2\epsilon' < M - L < 2\epsilon'$, that is $|M - L| < 2\epsilon'$.

But, we defined ϵ' as $\frac{|M-L|}{2}$. Substituting, we get the inequality $|M - L| < |M - L|$, which obviously cannot hold. Therefore, the initial premise that $\frac{|L-M|}{2} > 0$, that is, $L \neq M$, cannot be true. $\square$