Matrix notation

Matrix notation can be used to summarise systems of equations such as this one:

$$\begin{cases} x - y + z = 1 \\ x + y + 2z = 5 \end{cases}$$

This system can be stored in a matrix containing its coefficients (1), or an augmented matrix (2) that includes the output values:

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{1}$$

$$M = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 5 \end{bmatrix} \tag{2}$$

Linear systems of equations like these can be "solved" efficiently using the following *elementary row operations*:

1. Replace a row with the sum of that row and a multiple of another row
2. Interchange (switch) two rows
3. Multiply a row by a nonzero constant

For example:

$$\begin{cases} x + y = 1 \\ 2x - y = 2 \end{cases} \implies \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \end{bmatrix}$$

The last step is replacing $\boldsymbol{R}_2$ with $\boldsymbol{R}_2 - 2 \cdot \boldsymbol{R}_1$. It is obvious from the last matrix that $y = 0, x = 1$. We want to use elementary row operations to find equivalent systems that are "simpler".

Definiton A: Row echelon form

A matrix is in row echelon form if:

1. All nonzero rows are above the zero rows.
2. The leading entry of a row is in a column to the right of the leading entry of rows above.
3. All entries *directly* below a leading entry are zero.

Definiton B: Reduced row echelon form

A matrix is in reduced row echelon form if the conditions of row echelon form are satisfied *and*:

1. All leading entries are 0.
2. Each leading entry is the only nonzero entry in its column.

Theorem:

Any matrix is row equivalent to infinite row echelon form matrices but only one reduced row echelon form matrix

Row reduction algorithm

1. Find the leftmost nonzero column (this is a pivot column)
2. Select a nonzero entry in the pivot column from (1) and make it into a pivot by interchanging rows to put it on top
3. Make everything in the column and below the pivot in (2) zero by replacement
4. Ignore the pivot row and the rows above and repeat (1), (2) and (3) on the remainder
5. For reduced row echelon form:
   - Starting with the rightmost pivot and moving left and up, create zeros above pivots and scale to make pivots = 1

Row reduction example

The task is to row-reduce the following matrix, into any row echelon form and then into its unique reduced row echelon form:

$$M = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \tag{3}$$

First, we need to find the leftmost nonzero column (step 1). This is the second column in the case of M. One nonzero entry in this column is item $M_{3,2} = 3$. Therefore, we need to interchange rows 1 and 3 to put this *pivot* entry on top (step 2):

$$M = \begin{bmatrix} *3* & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \tag{4}$$

For step 3, we need to make $M_{2,2}$ equal to 0 to satisfy condition **3** in the definition for row echelon form. We can do this by subtracting the first row from the second row:

$$M = \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \tag{5}$$

Now we ignore the first row, as it is the pivot row, and perform steps 1, 2 and 3 on the remainder of the matrix. The next pivot column is $\boldsymbol{C}_3$. As we are ignoring line 1, we make $M_{2,3} = 0$ the pivot element, as it is the topmost nonzero entry. To make $M_{3,3}$ equal to 0, subtract $\frac{3}{2} \cdot \boldsymbol{R}_2$ from $\boldsymbol{R}_3$:

$$M = \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \tag{6}$$

Now, all three conditions for row echelon form are satisfied. To make this into reduced row echelon form, continue by replacing $\boldsymbol{R}_2$ with $\boldsymbol{R}_2 - 2 \cdot \boldsymbol{R}_3$, and then replace $\boldsymbol{R}_1$ with $\boldsymbol{R}_1 - 6 \cdot \boldsymbol{R}_3$:

$$M = \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \tag{7}$$

Then, replace $\boldsymbol{R}_1$ with $\boldsymbol{R}_1 + \frac{9}{2} \cdot \boldsymbol{R}_2$:

$$M = \begin{bmatrix} *3* & 0 & -6 & 9 & 0 & -72 \\ 0 & *2* & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & *1* & 4 \end{bmatrix} \tag{8}$$

Now, all the pivot entries (highlighted with asterisks only have zeroes above and below them. Therefore, M is now almost in reduced row echelon form; all that remains is to scale the rows by multiplicative factors, to make the pivots equal to 1:

$$M = \begin{bmatrix} *1* & 0 & -2 & 3 & 0 & -24 \\ 0 & *1* & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & *1* & 4 \end{bmatrix} \tag{9}$$

The matrix is now in reduced row echelon form.

Application of row reduction example

The following system of linear equations can be summarised by M in its initial form:

$$\begin{cases} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

Therefore, by fully row reducing M, we know this is equivalent to:

$$\begin{cases} x_1 - 2x_3 + 3x_4 = -24 \\ x_2 - 2x_3 + 2x_4 = -7 \\ x_5 = 4 \end{cases}$$

As can be seen, $x_3 = t$ and $x_4 = s$ are *free*, as they are not pivot elements, and $x_5 = 4$. Thus, $x_1 = -24 + 2t - 3s$ and $x_2 = -7 + 2t - 2s$.

Consistency

Theorem: A linear system is consistent iff some echelon form of the augmented matrix has no rows of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & b \end{bmatrix}$$

where $b \neq 0$.

If it *is* consistent then either:

- there is at least one free variable and therefore there exist infinitely many solutions to the system, or
- there are no free variables and therefore there exists only one solution to the system