# BIVARIATE RANDOM VARIABLES

## 1. Discrete joint distributions of bivariate random vectors

$(X, Y)$ has a joint distribution if:

- The set of possible values $(x, y)$ of $(X, Y)$ is countable
- The joint probability function of $(X, Y)$, denoted by $f(x, y)$, is given by $f(x, y) = \mathbb{P}(X = x, Y = y)$.

Example: let $X$ be the number of cars in a household ($X \in \{1, 2, 3\}$) and let $Y$ be the number of TVs in a household ($Y \in \{1, 2, 3, 4\}$). Then, the set of outcomes of $(X, Y)$ is the cartesian product of the outcome sets of $X$ and $Y$. The distribution of $(X, Y)$ can be described in a *contingency table*: a table of all the possible combinations of the values of $X$ and $Y$ and the probability of each of those outcomes.

For any such discrete joint distribution, we have the following properties, much like for univariate distributions:

- $f(a, b) = 0$ iff $(a, b)$ is not a possible value of $(X, Y)$
- $\sum f(x, y) = 1$
- $\mathbb{P}[(X, Y) \in C] = \sum_{(x,y)\in C} f(x, y)$

## 2. Continuous joint distributions

$(X, Y)$ has a continuous joint distribution iff there exists a function $f : \mathbb{R}^2 \to \mathbb{R}_+$ such that for any $C \subseteq \mathbb{R}$ we can calculate $\mathbb{P}[(X, Y) \in C)] = \int\int_C f(x, y)\,\mathrm{d}x\,\mathrm{d}y$. Such a distribution must satisfy $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = 1$, much like a univariate continuous distribution.

### 2.1. Independence and conditional probabilities.

If $(X, Y)$ has a continuous joint distribution, then $X$ and $Y$ are independent iff $f(x, y) = f_X(x)f_Y(y)$, where $f_X(x)$ and $f_Y(y)$ are the *marginal* p.d.f.s of $X$ and $Y$ respectively. In general, if $f(x, y)$ can be expressed as the product $h_1(x)h_2(y)$, for any functions $h_1$ and $h_2$, then $X$ and $Y$ are independent.

Using the definition for conditional probabilities, we can also work out $\mathbb{P}(a < X < b|Y = y)$ and $\mathbb{P}(a < Y < b|X = x)$. This is because $f(x, y) \equiv f(x|y)f_Y(y) = f(y|x)f_X(x)$. We define $\mathbb{P}(a < X < b|Y = y) = \int_a^b f(x|y)\,\mathrm{d}x$.

## 3. Transformation of a continuous random variable

For some random variable $X$ with p.d.f. $f(x)$, define $Y = r(X)$. We can calculate the p.d.f. and c.d.f. of $Y$ as follows. By definition, the c.d.f. of $Y$ $G(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(r(X) \leq y) = \int_{\{x:r(x)\leq y\}} f(x)\,\mathrm{d}x$. Then, the p.d.f. of $Y$ is $g(y) = \frac{\mathrm{d}}{\mathrm{d}y}G(y)$.

3.0.1. *Example.* Suppose $X \sim U[-1, 1]$, and let $Y = X^2$. To find the p.d.f. of $Y$, we must first find its c.d.f.. The range of $Y$ is that of $X^2$, i.e. $0 \leq Y \leq 1$. Then, $G(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,\mathrm{d}x = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\,\mathrm{d}x = \sqrt{y}$. Finally, the p.d.f. of $Y$ is $\frac{\mathrm{d}}{\mathrm{d}y}\sqrt{y} = \frac{1}{2\sqrt{y}}$, $0 < y < 1$.

### 3.1. Deriving the p.d.f. of $Y = r(X)$ where $k$ is injective and differentiable.

This is a special case of the above. In this case, pushing through the algebra we have $g(y) = f(s(y)) \cdot |s'(y)|$, where $s(y) = k^{-1}(y)$.

3.1.1. *Example.* Suppose $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$, and let $Y = -2X + 3$. Then, the p.d.f. of $Y$ is $g(y) = \frac{1}{2} f\left(\frac{3-y}{2}\right) = \begin{cases} \frac{1}{2} e^{-\frac{3-y}{2}} & y < 3 \\ 0 & y \geq 3 \end{cases}$.

3.2. **Transformation of a bivariate vector.** Suppose $(X_1, X_2) \sim f(x_1, x_2)$, $r : \mathbb{R}^2 \to \mathbb{R}$ and $Y = r(X_1, X_2)$. Then, the c.d.f. of $Y$ is still denoted by $G(y) = \mathbb{P}(Y \leq y) = \int\int_{\{(x_1,x_2)\in\mathbb{R}^2 | r(x_1,x_2) \leq y\}} f(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2$.

In particular, if $Y = a_1 X_1 + a_2 X_2 + b$, then it can be shown that $g(y) = \frac{1}{|a_1|} \int_{-\infty}^{\infty} f\left(\frac{y-b-a_2 x_2}{a_1}, x_2\right) \mathrm{d}x_2$.

## 4. The distribution of the maximum of $n$ independent variables

Suppose $X_1, X_2 \cdots X_n \sim f(x)$, and let $Y_n = \max(X_1, X_2 \cdots X_n)$.