MATH 3110 HOMEWORK #4

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Problem 5.1.4 Suppose $\frac{a_n}{b_n} \to L, b_n \neq 0 \;\forall\; n$ and $b_n \to 0$. Show that $a_n \to 0$.

Proof. $a_n = \frac{a_n}{b_n} \cdot b_n$. Therefore, $\lim a_n = \lim \left(\frac{a_n}{b_n} \cdot b_n \right) = \lim \frac{a_n}{b_n} \cdot \lim b_n = L \cdot 0 = 0$, by the limit product rule. □

Problem 5.3.4

(i)

Proof. Since b_n is convergent, it must also be bounded above. Since $a_n \leq b_n \;\forall\; n$, a_n must also be bounded above. By a theorem from earlier in the course, since a_n is also increasing, it must converge.

Since both a_n and b_n are convergent sequences, $a_n \leq b_n \;\forall\; n$ implies that $\lim a_n \leq \lim b_n$, by the limit location theorem (part *(15c)*). □

(ii) A simple counterexample is the pair of sequences $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$, for $n \geq 2$. Since $n \geq 2 \implies n^2 > n$, $a_n < b_n \;\forall\; n$. But, $\lim a_n = \lim b_n = 0$, so the statement $\lim a_n < \lim b_n$ is false.

Problem 5.4.1

(a) (b) Implied by part *(c)*

(c)

Proof. Suppose there are K colourings, and denote the different subsequences as a_k, and their terms as a_{k_m} for $0 < k \leq K, m \geq 1$. Suppose all of these subsequences converge to L, that is, $\forall\; k$, $\forall\; \epsilon > 0$ there exists M_k such that $|a_{k_m} - L| < \epsilon \;\forall\; m \geq M_k$.

Let $N = \max(\{M_i | 0 < i \leq K\})$. Then, one version of the above statement applies to all of the subsequences: $\forall\; k$, $\forall\; \epsilon > 0, |a_{k_m} - L| < \epsilon \;\forall\; m \geq N$.

Since every term of $\{a_n\}$ is coloured in some way, we know that for any a_n there exist k, m such that $a_n = a_{k_m}$. Therefore, for any $\epsilon > 0$, $|a_n - L| = |a_{k_m} - L| < \epsilon \;\forall\; m \geq N$, that is, $\{a_n\}$ converges to L. □

Problem 5.4.2

Suppose $s(n)$ is the sum of the prime factors of $n \in \mathbb{Z}^+$, and define the sequence $a_n = \frac{s(n)}{n}$. Show that $\lim_{n\to\infty} a_n$ does not exist.

Proof. There are infinitely many numbers that are greater than the sum of their prime factors. An example is $10 > 5 + 2$. To show that there are infinitely many, note that if $n > s(n), n > 2$ then $s(2n) = s(n) + 2 < n + 2 < 2n$. There are also infinitely many prime numbers (*Elements*, Euclid, c. 300BC), and if n is prime, then $n = s(n)$. Define two subsequences of $\{a_n\}$ as follows: $\{b_k\}$ as the members of $\{a_n\}$ where n was generated by the above rule, that is, $b_k = \frac{s(n)}{n}$ where $n = 10 \cdot 2^k$; and $\{c_p\}$ as $\frac{s(p)}{p}$ where p is prime. In the case of b_n, since $n > s(n)\ \forall\ n$, $\frac{s(n)}{n} < 1\ \forall\ n$. Consider the ratio between two consecutive terms:

$$\begin{aligned}
\frac{b_{k+1}}{b_k} &= \frac{s(10 \cdot 2^{k+1})}{10 \cdot 2^{k+1}} \cdot \frac{10 \cdot 2^k}{s(10 \cdot 2^k)} \\
&= \frac{20(k+1)}{10 \cdot 2^{k+1}} \cdot \frac{10 \cdot 2^k}{20k} \\
&= \frac{k+1}{2k}
\end{aligned}$$

Since for all $k \geq 2$, this ratio is less than 1, this means that for $k \geq 2$, $\{b_k\}$ is decreasing, and so its limit must be less than one. On the other hand, $p_n = \frac{s(p)}{p} = 1$, so $\lim_{n\to\infty} p_n = 1$. Since the two subsequences converge to different limits, a_n cannot have a limit. □

Problem 5-5

Suppose $a_n \to L$, and b_n lies between a_n and a_{n+1} for all n. Show that $b_n \to L$.

Proof. Define two subsequences of b_n: $\{b_\alpha\}$, consisting of terms of $\{b_n\}$ where $a_n \leq b_n \leq a_{n+1}$, and $\{b_\beta\}$, consisting of the terms of $\{b_n\}$ for which $a_{n+1} \leq b_n \leq a_n$. Since $a_n \to \infty$ and $a_{n+1} \to \infty$, by the Squeeze theorem, $\{b_\alpha\}$ must converge to L, and $\{b_\beta\}$ must also converge to L. Since all terms of $\{b_n\}$ are in exactly one of $\{b_\alpha\}$ or $\{b_\beta\}$, $\{b_n\}$ converges to L by the subsequence theorem proven in problem *(5.4.1)*. □