Multiplication

Assume that $\overrightarrow{A}$ is a $m \times n$ matrix, and $\vec{x}$ is an n-vector (i.e. a $n \times 1$ matrix). We can define a law of multiplication for them as follows:

$$\overrightarrow{A} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & \ddots & & \\ \vdots & & & \\ u_{m1} & & & \end{bmatrix} \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \overrightarrow{A}\vec{x} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & \ddots & & \\ \vdots & & & \\ u_{m1} & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i \overrightarrow{A}_i \in \mathbb{R}^m$$

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For example:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

The following system of equations can be denoted as a product of two matrices since:

$$\begin{cases} 2x + y + z = 1 \\ 3x + y + 5x = 2 \end{cases} \implies x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \implies \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Which is in the form $\overrightarrow{A}\vec{x} = \vec{b}$. The product $\overrightarrow{A}\vec{x}$ is only *possible* if $\vec{x}$ is a n-vector and $\vec{b}$ is a m-vector, where the dimensions of $\overrightarrow{A}$ are $m \times n$. That is:

$$\overrightarrow{A}\vec{x} = \vec{b} \iff \sum_{i=1}^{n} x_i \overrightarrow{A}_i = \vec{b}$$

If valid, $\vec{A}\vec{x} = \vec{b}$ has solutions regardless of the value of $\vec{b}$, iff any $\vec{b}$ is a linear combination of columns of $\vec{A}$, iff the span of the columns of $\vec{A}$ is $\mathbb{R}^m$, and iff every row of $\vec{A}$ has a pivot position.

Another example, skipping the intermediate step:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{bmatrix}$$

Is there such a matrix $\vec{I}$ such that $\vec{I}\vec{x} = \vec{x} \; \forall \; \vec{x} \in \mathbb{R}^n$? Yes there is! It is called the identity matrix, and consists of a main diagonal of 1s, with 0s everywhere else:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Properties

1. $\vec{A}(\vec{u} + \vec{v}) = \vec{A}\vec{u} + \vec{A}\vec{v}$
2. $\vec{A}(c\vec{u}) = c(\vec{A}\vec{u})$

Linear systems

If a linear system $\vec{A}\vec{x} = \vec{b}$ has $\vec{b} = \vec{0}$, it is called **homogenous**. In this case,

$$x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is always a solution. Therefore, it is called the *trivial* solution, and normally ignored: the question normally is whether $\vec{A}\vec{x} = 0$ has *non-trivial* solutions. To do this, one needs to solve the augmented matrix

$$M = \begin{bmatrix} \vec{A} & \vec{0} \end{bmatrix}$$

For example, to find the *non-trivial* solutions to

$$\begin{cases} x + y + z = 0 \\ x - y + z = 0 \end{cases}$$

we would need to row reduce the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} y = 0 \\ z = t \\ x + y + z = 0 \Rightarrow x = -t \end{cases}$$

Therefore, the solution set can be described as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \Longrightarrow t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : t \in \mathbb{R} = span \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

However, not all equation systems are homogenous, that is, they are in the form $\vec{A}\vec{x} = \vec{b}$ where $\vec{b} \neq \vec{0}$. For example:

$$\begin{cases} x + y + z = 3 \\ x - y + z = 1 \end{cases}$$

If we assume we know a particular solution $\vec{x}_p$, then the *general* solution of the system of equations will be $\vec{x}_p + \vec{x}_h$, where $\vec{x}_h$ is the solution to $\vec{A}\vec{x} = \vec{0}$. In the case of the system above, we know that the solution to the homogenous version is $t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Given that $\vec{x}_p = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a particular solution, we know that all solutions will be of the form $\vec{x}_g = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, t \in \mathbb{R}$. But, equally, since $\vec{x}_p = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is also a particular solution, the general solution can also be expressed as $\vec{x}_g = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, t \in \mathbb{R}$.