MATH 3110 HOMEWORK #2

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Problem 2.1.3 This can easily be disproven by means of a counterexample. Suppose $a_n = n$, and $b_n = n - 3$, both clearly increasing sequences. Then, $a_n b_n = n(n-4)$, that is, $a_1 b_1 = -3$, $a_2 b_2 = -4$ and $a_3 b_3 = -3$. Thus, $a_n b_n$ is neither increasing nor decreasing.

A better statement is that if $\{a_n\}$ and $\{b_n\}$ are increasing **and** $a_n, b_n > 0 \; \forall \; n$, then $\{a_n b_n\}$ is increasing.

Proof. Suppose $\{a_n\}$ and $\{b_n\}$ are increasing, that is, $a_{n+1} \geq a_n$ and $b_{n+1} \geq b_n$ for all n. Consider the ratio between two consecutive terms of $\{a_n b_n\}$:

$$\frac{a_{n+1} b_{n+1}}{a_n b_n} = \frac{a_{n+1}}{a_n} \cdot \frac{b_{n+1}}{b_n}$$

Since $\{a_n\}$ and $\{b_n\}$ are increasing with all positive terms, we know that $\frac{a_{n+1}}{a_n} \geq 0$ and $\frac{b_{n+1}}{b_n} \geq 0$. Therefore, $\frac{a_{n+1} b_{n+1}}{a_n b_n} \geq 0$. □

Problem 2.4.2 The contrapositive of this statement is as follows: if $|a_i| \leq 1 \; \forall \; i$, then

$$\left| \sum_{i=1}^{n} a_i \sin(ib) \right| \leq n$$

Proof. Assume that $|a_i| \leq 1 \; \forall \; i$. A fundamental fact is that $|\sin x| \leq 1 \; \forall \; x \in \mathbb{R}$. Therefore, $|a_i \sin(ib)| \leq 1 \; \forall \; i$.

By the triangle inequality, $|\sum_{i=1}^{n} a_i \sin(ib)| \leq \sum_{i=1}^{n} |a_i \sin(ib)|$. Since $|a_i \sin(ib)| \leq 1$, $\sum_{i=1}^{n} |a_i \sin(ib)| \leq \sum_{i=1}^{n} 1 = n$. □

Problem 2.6.1 To put this statement formally, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{a^{n+1}}{(n+1)!} > \frac{a^n}{n!}$.

Proof. Let N be the smallest integer such that $N \geq a$. Then, consider the ratio between two consecutive terms of the sequence:

$$\begin{aligned} \frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} &= \frac{a^{n+1}}{a^n} \cdot \frac{n!}{(n+1)!} \\ &= a \cdot \frac{1}{n+1} = \frac{a}{n+1} \end{aligned}$$

Since $n \geq N \geq a$, $n + 1 > a$, and since $a > 0$, $0 < \frac{a}{n+1} < 1$. Since the ratio between two consecutive terms is positive but less than 1, the sequence must be monotone strictly decreasing. □

Problem 2-1

(a)

Proof. Consider the difference between two consecutive terms:

$$
\begin{aligned}
b_{n+1} - b_n &= \frac{\sum_{i=1}^{n+1} a_i}{n+1} - \frac{\sum_{i=1}^{n} a_i}{n} \\
&= \frac{n\sum_{i=1}^{n+1} a_i - (n+1)\sum_{i=1}^{n} a_i}{n(n+1)} \\
&= \frac{na_{n+1} - \sum_{i=1}^{n} a_i}{n(n+1)} \\
&= \frac{\sum_{i=1}^{n} a_{n+1} - \sum_{i=1}^{n} a_i}{n(n+1)} = \frac{\sum_{i=1}^{n}(a_{n+1} - a_i)}{n(n+1)}
\end{aligned}
$$

Since it is given that $\{a_n\}$ is increasing, we know that $a_{n+1} \geq a_i \ \forall \ i \leq n$, and therefore, $a_{n+1} - a_i \geq 0 \ \forall \ i \leq n \implies \sum_{i=1}^{n}(a_{n+1} - a_i) \geq 0$. $n > 0$, so the denominator is also positive, which means the difference between two consecutive terms is positive, that is, $\{b_n\}$ is increasing. □

(b)

Proof. We are given that $\{a_n\}$ is bounded above, i.e. there exists $M \in \mathbb{R}$ such that $a_i \leq M \ \forall \ i$. Consider a term of the sequence $\{b_n\}$: $b_n = \frac{\sum_{i=1}^{n} a_i}{n}$. Given the former condition, we know that $\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} M = nM$. Thus, $b_n \leq \frac{nM}{n} = M \ \forall \ n$, i.e. $\{b_n\}$ is bounded above. □

Problem 2-2

Proof. Let L be the smallest upper bound for $\{a_n\}$. Then, $a_i \leq L \ \forall \ i$, and there is no such number M such that $M < L$ and $a_i \leq M \ \forall \ i$. Since $\{a_n\}$ is increasing, this means that after a certain point, a_i will get arbitrarily close to L (since every term is greater than or equal to the previous term); that is, there exists N such that if $n \geq N$ then for any $\epsilon > 0$, $L - a_n < \epsilon$. □