MATH 2220 SECTION 203
HW #9

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Problem 1

(a) We can calculate this integral by applying the domain boundaries for D: $1 \leq x \leq 2$ and $0 \leq y \leq \log x$:

$$\begin{aligned}\int_D \mathrm{d}A &= \int_0^1 \int_0^{\log x} \mathrm{d}y\,\mathrm{d}x \\ &= \int_0^1 \log x\,\mathrm{d}x \\ &= (x\log x - x)\Big|_0^1 \\ &= (\log 1 - 1) - (0 - 0) \\ &= -1\end{aligned}$$

This represents the volume of the region of points (x, y) where $1 \leq x \leq 2$ and $0 \leq y \leq \log x$.

(b) Similarly:

$$\begin{aligned}\int_D x\,\mathrm{d}A &= \int_0^3 \int_{-1}^1 x\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^3 2x\,\mathrm{d}x \\ &= x^2\Big|_0^3 \\ &= 9\end{aligned}$$

This represents the volume of a triangular prism, with length 2, and with its cross-section being the isosceles right-angled triangle with catheti of length 3, i.e. half of the cuboid with sides 2, 3 and 3.

(c) Here, D is the unit disk in $\mathbb{R}^2$, that is, $D = \{(x, y) | x^2 + y^2 \leq 1\}$. We can express the domain boundaries as $0 \leq y^2 \leq 1 - x^2 \implies -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$, and $0 \leq x^2 \leq 1 \implies -1 \leq x \leq 1$; this is because the square of a number can never be less than 0. Therefore:

$$\int_D \sqrt{1 - x^2 - y^2}\,\mathrm{d}A = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\,\mathrm{d}y\,\mathrm{d}x$$

To compute the first integral we can apply the substitution $y = \sqrt{1 - x^2}\sin\theta$. Then $\mathrm{d}y = \sqrt{1 - x^2}\cos\theta\,\mathrm{d}\theta$, and $\sqrt{1 - x^2 - y^2} = \sqrt{1 - x^2 - (1 - x^2)\sin^2\theta} = \sqrt{(1 - x^2)(1 - \sin^2\theta)} = \sqrt{(1 - x^2)(\cos^2\theta)} =$

$\cos\theta\sqrt{(1-x^2)}$. Adjusting the bounds, we have $y = \sqrt{1-x^2} = \sqrt{1-x^2}\sin\theta \implies \sin\theta = 1 \implies \theta = \frac{\pi}{2}$; similarly, $y = -\sqrt{1-x^2} \implies \theta = -\frac{\pi}{2}$. So:

$$\begin{aligned}
\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}y &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\sqrt{(1-x^2)}\sqrt{1-x^2}\,\mathrm{d}\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta(1-x^2)\,\mathrm{d}\theta \\
&= (1-x^2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(2\theta)+1}{2}\,\mathrm{d}\theta \\
&= (1-x^2)\left(\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{2}(1-x^2)
\end{aligned}$$

Substituting that back in, we get

$$\begin{aligned}
\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}y\,\mathrm{d}x &= \frac{\pi}{2}\int_{-1}^{1}(1-x^2)\,\mathrm{d}x \\
&= \frac{\pi}{2}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} \\
&= \frac{\pi}{2}\frac{4}{3} \\
&= \frac{2\pi}{3}
\end{aligned}$$

This represents the volume of one unit hemisphere.

(d) Once again we need to express our constraints. We are given that $z \geq 0 \implies z^2 \geq 0$, and $x^2+y^2+z^2 \leq 1$. We have $0 \leq z \leq 1$. Then, fixing z, we get $y^2 \leq 1-z^2$ and fixing y, $x^2 \leq 1-y^2-z^2$. Then, using the result from (c):

$$\begin{aligned}
\int_H 2\,\mathrm{d}V &= \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} 4\sqrt{1-y^2-z^2}\,\mathrm{d}y\,\mathrm{d}z \\
&= 4\frac{2\pi}{3}\frac{1}{2} = \frac{4\pi}{3}
\end{aligned}$$

The last step comes from the fact that $\int_0^1\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \sqrt{1-y^2-z^2}\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{2}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}y\,\mathrm{d}x$. This is equal to twice the volume of the unit sphere.

Problem 2 Using the linearity of the integral, we know this is equal to $\int_D y^5\,\mathrm{d}A - \int_D xy\,\mathrm{d}A + \int_D 3\,\mathrm{d}A$. Since $f(y) = y^5$ is an odd function, we know that $\int_{-a}^{a} y^5\,\mathrm{d}y = 0$. Therefore

$$\begin{aligned}\int_D y^5 \,\mathrm{d}A &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^5 \,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-1}^{1} 0 \,\mathrm{d}x \\ &= 0\end{aligned}$$

The same argument can be made for $\int_D -xy \,\mathrm{d}A$. this is because, once again, $f(x) = x$ is an odd function, and therefore $-\int_{-1}^{1} y \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x \,\mathrm{d}x\,\mathrm{d}y = -\int_{-1}^{1} 0y \,\mathrm{d}y = 0$.

Finally, since D is the unit disk, $\int_D \mathrm{d}A$ is the volume of the unit cylinder $(h = r = 1)$, π, and $\int_D 3 \,\mathrm{d}A = 3\int_D \mathrm{d}A = 3\pi$. Therefore, $\int_D (y^5 - xy + 3)\,\mathrm{d}A = 3\pi$.

Problem 3 By definition, we can calculate the integrals as follows:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{\Delta x} f(x_i)\Delta x$$

$$\int_a^b g(y)\,\mathrm{d}y = \lim_{\Delta y \to 0} \sum_{\Delta y} g(y_j)\Delta y$$

Therefore, multiplying the integrals, we get, using the properties of limits and linear sums:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x \cdot \int_a^b g(y)\,\mathrm{d}y &= \left(\lim_{\Delta x \to 0} \sum_{\Delta x} f(x_i)\Delta x\right)\left(\lim_{\Delta y \to 0} \sum_{\Delta y} g(y_j)\Delta y\right) \\ &= \lim_{(\Delta x, \Delta y)\to(0,0)} \left(\sum_{\Delta x} f(x_i)\Delta x\right)\left(\sum_{\Delta y} g(y_j)\Delta y\right) \\ &= \lim_{(\Delta x, \Delta y)\to(0,0)} \sum_{\Delta x}\sum_{\Delta y} f(x_i)g(y_j)\Delta x\Delta y \\ &= \int_a^b \int_c^d f(x)g(y)\,\mathrm{d}A\end{aligned}$$

This is because, by definition, $\mathrm{d}A$ is the infinitesimal change in the area of the "square", that is, $\lim_{(\Delta x,\Delta y)\to(0,0)} \Delta x\Delta y$.

Problem 4

(a) The constraint for x is given as $0 \le x \le 1$. Since $y \in D \iff x^2 \le y \le x^4$, this gives the constraint for y. This means that

$$\int_D f(x,y)\,\mathrm{d}A = \int_0^1 \int_{x^2}^{x^4} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

To write this integral in another way, we need to make it so that the constraint for y is absolute, and not relative to x:

$$\int_D f(x,y)\,\mathrm{d}A = \int_0^1 \int_{sqrty}^{\sqrt[4]{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

(b) An example is any the region between the graphs $y = x^2$ and $y = x^4$, but for all x. In this case, the integral would have to be split into several regions, if integrating first with respect to x and then with respect to y. This is because in the region $0 \leq x \leq 1$, we are guaranteed that $x^2 \geq x^4$, so that can be expressed as a single condition. However, if we look at the entire range of the two functions, then in areas where $|x| > 1$ we have $x^4 > x^2$. Therefore, the integral would have to be done separately over several areas to accomodate this.

Problem 5 Since the bounding functions are lines, the bounded region is a triangle, with vertices $(1,1)$, $(2,2)$ and $(3,1)$. We can split this into two disjoint regions: the set $A \subseteq D : 1 \leq x \leq 2$ and the set $B \subseteq D : 2 < x \leq 3$. A is the space between $y = 1$ and $x = y$, while B is the space between $x = 4 - y$. This is because in A, $x \geq 4 - x$, and in B, $x < 4 - x$, and the upper bound is the smaller of the two functions. So:

$$
\begin{aligned}
\int_D e^{x+y}\,\mathrm{d}x\,\mathrm{d}y &= \int_A e^{x+y}\,\mathrm{d}x\,\mathrm{d}y + \int_B e^{x+y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_1^2 \int_1^x e^{x+y}\,\mathrm{d}y\,\mathrm{d}x + \int_2^3 \int_{4-x}^1 e^{x+y}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_1^2 e^{x+y}\Big|_{y=1}^{y=x}\,\mathrm{d}x + \int_2^3 e^{x+y}\Big|_{y=4-x}^{y=1}\,\mathrm{d}x \\
&= \int_1^2 (e^{2x} - e^{x+1})\,\mathrm{d}x + \int_2^3 (e^{x+1} - e^4)\,\mathrm{d}x \\
&= (2e^{2x} - e^{x+1})\Big|_1^2 + (e^{x+1} - xe^4)\Big|_2^3 \\
&= (2e^4 - e^3) - (2e^2 - e^2) + (e^4 - 3e^4) - (e^3 - 2e^4) \\
&= -e^2 - 2e^3 + 2e^4
\end{aligned}
$$

Problem 6

(a) Since $f(x,y) = e^{-xy}$, we have

$$
\begin{aligned}
\frac{\partial}{\partial y} f(x,y) &= -xe^{-xy} \\
\frac{\partial^2}{\partial y^2} f(x,y) &= x^2 e^{-xy} \\
\frac{\partial^3}{\partial y^3} f(x,y) &= -x^3 e^{-xy} \\
&\cdots \\
\frac{\partial^{n-1}}{\partial y^{n-1}} f(x,y) &= (-1)^{n-1} x^{n-1} e^{-xy}
\end{aligned}
$$

(b) This is a simple single-variable integral:

$$\int_0^\infty f(x,y)\,\mathrm{d}x = \int_0^\infty e^{-xy}\,\mathrm{d}x$$
$$= -\frac{e^{-xy}}{y}\Big|_0^\infty$$
$$= 0 - (-\frac{1}{y})$$
$$= \frac{1}{y}$$

(c) Taking the result from part (a) and integrating, we get:

$$(-1)^{n-1}\int_0^\infty x^{n-1}e^{-xy}\,\mathrm{d}x = \int_0^\infty \frac{\partial^{n-1}}{\partial y^{n-1}} e^{-xy}\,\mathrm{d}x \qquad = \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}\int_0^\infty e^{-xy}\,\mathrm{d}x$$
$$= \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}\frac{1}{y}$$
$$= (-1)^{n-1}\frac{(n-1)!}{y^n}$$

Therefore, $(-1)^{n-1}\int_0^\infty x^{n-1}e^{-xy}\,\mathrm{d}x = (-1)^{n-1}\frac{(n-1)!}{y^n} \implies \int_0^\infty x^{n-1}e^{-xy}\,\mathrm{d}x = \frac{(n-1)!}{y^n}$. Setting $y = 1$ we get

$$\int_0^\infty x^{n-1}e^{-x}\,\mathrm{d}x = (n-1)!$$

Problem 7

(a) As in (1c), we can use $f(x,y) = \sqrt{1-x^2-y^2}$, and the integral will be equal to the volume of one hemisphere. Since the volume of a sphere is twice that of a hemisphere with the same radius, if $f(x,y) = 2\sqrt{1-x^2-y^2}$ then $\int_D f(x,y)\,\mathrm{d}A$ will be the volume of the unit sphere.

(b) Transforming to polar co-ordinates, $g(x,y) = 2x\sqrt{1-x^2}$. Then, $\int_0^1\int_0^{2\pi} g(x,y)\,\mathrm{d}y\,\mathrm{d}x = \frac{4\pi}{3}$.

(c) If $u(r,\theta) = (r\cos\theta, r\sin\theta)$ then

$$\vec{D}u = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Therefore, $\det \vec{D}u = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$.

(d) Using our previous definitions for f and g, we have

$$f(r\cos\theta, r\sin\theta) = 2\sqrt{1 - r^2\cos^2\theta - r^2\sin^2\theta}$$
$$= 2\sqrt{1-(r^2\cos^2\theta + r^2\sin^2\theta)}$$
$$= 2\sqrt{1-(r^2[\cos^2\theta+\sin^2\theta])} = 2\sqrt{1-r^2}$$
$$g(r,\theta) = 2r\sqrt{1-r^2}$$

The ratio between them is r, which is also the determinant of the derivative matrix of the transformation to polar co-ordinates.

Problem 8 There are two possible ways to decompose this function into partial fractions:

$$\begin{aligned}\frac{x-y}{(x+y)^3} &= \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \\ &= \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\end{aligned}$$

To calculate $\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}y$ I will use the first one, and to calculate $\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}x$ I will use the second one, for reasons that will become obvious. We have:

$$\begin{aligned}\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}y &= \int_0^1 \left(\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right)\mathrm{d}y \\ = \left(-\frac{2x}{2(x+y)^2} + \frac{1}{x+y}\right)\Big|_{y=0}^{y=1} &= \left(\frac{1}{x+y} - \frac{x}{(x+y)^2}\right)\Big|_{y=0}^{y=1} \\ &= \left(\frac{1}{x+1} - \frac{x}{(x+1)^2}\right) - \left(\frac{1}{x} - \frac{x}{(x)^2}\right) \\ &= \frac{1}{x+1} - \frac{x}{(x+1)^2} - \frac{1}{x} + \frac{1}{x} \\ &= \frac{1}{x+1} - \frac{x}{(x+1)^2} \\ &= \frac{1}{(x+1)^2}\end{aligned}$$

Similarly:

$$\begin{aligned}\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}x &= \int_0^1 \left(\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\right)\mathrm{d}x \\ &= \left(\frac{y}{(x+y)^2} - \frac{1}{x+y}\right)\Big|_{x=0}^{x=1} \\ &= \left(\frac{y}{(y+1)^2} - \frac{1}{y+1}\right) - \left(\frac{y}{y^2} - \frac{1}{y}\right) \\ &= \frac{y}{(y+1)^2} - \frac{1}{y+1} \\ &= -\frac{1}{(y+1)^2}\end{aligned}$$

Therefore, we have $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \frac{1}{(x+1)^2}\,\mathrm{d}x = -\frac{1}{x+1}\Big|_0^1 = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$. Similarly, $\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}x\,\mathrm{d}y = -\int_0^1 \frac{1}{(y+1)^2}\,\mathrm{d}y = \frac{1}{y+1}\Big|_0^1 = -\frac{1}{2}$. This is interesting, since the two are not equal, but are negatives of each other.