Matrix operations

Definitions

- A **diagonal entry** is a matrix element a_{ij} where $i = j$.
- A **diagonal matrix** is a matrix where $ineqj \implies a_{ij} = 0$.
- The **identity matrix** I_n is a diagonal matrix where all the nonzero elements are 1.

Addition and scalar multiplication

Given that $\vec{A}$ and $\vec{B}$ are both $m \times n$ matrices, then:

1. $\vec{A} = \vec{B}$ iff $a_{ij} = b_{ij} \; \forall \; i, j$.
2. $\vec{A} + \vec{B} = [a_{ij} + b_{ij}]$.
3. Given $c \in \mathbb{R}$, $c\vec{A} = [ca_{ij}]$.
4. Matrix addition is associative and commutative.
5. Any matrix plus its own negative equals $\vec{0}$.
6. $r(\vec{A} + \vec{B}) = r\vec{A} + r\vec{B}$
7. $(r + s)\vec{A} = r\vec{A} + s\vec{A}$
8. $r(s\vec{A}) = (rs)\vec{A}$
9. $1\vec{A} = \vec{A}$

Matrix multiplication

Given two matrices $\vec{A}_{p\times m}$ and $\vec{B}_{m\times n}$, we can create two linear maps:

1. $T : \mathbb{R}^n \to \mathbb{R}^m, \vec{x} \mapsto \vec{B}\vec{x}$
2. $S : \mathbb{R}^m \to \mathbb{R}^p, \vec{x} \mapsto \vec{A}\vec{x}$

Let

$$\vec{B} = \begin{bmatrix} \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \end{bmatrix} \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then $\vec{B}\vec{x} = \sum_{i=1}^n x_i\vec{b_i}$ and $\vec{A}(\vec{B}\vec{x}) = \vec{A}(\sum_{i=1}^n x_i\vec{b_i}) = \sum_{i=1}^n x_i(\vec{A}\vec{b_i})$, which can be expressed in matrix form:

$$\sum_{i=1}^{n} x_i(\vec{A}\vec{b_i}) = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{A}\vec{b_1} & \vec{A}\vec{b_2} & \cdots & \vec{A}\vec{b_n} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We can define this as the product $\vec{A}\vec{B}$; if $\vec{A} = (a_{ij})_{p\times m}, \vec{B} = (b_{ij})_{m\times n}$ then $\vec{C} = \vec{A}\vec{B} = (c_{ij})_{p\times n}$ where $c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$.

Rules of matrix multiplication

1. $\vec{A}(\vec{B}\vec{C}) = (\vec{A}\vec{B})\vec{C}$
2. $\vec{A}(\vec{B}+\vec{C}) = \vec{A}\vec{B} + \vec{A}\vec{C}$
3. $(\vec{B}+\vec{C})\vec{A} = \vec{B}\vec{A} + \vec{C}\vec{A}$
4. $r(\vec{A}\vec{B}) = (r\vec{A})\vec{B} = \vec{A}(r\vec{B}), r \in \mathbb{R}$
5. $\vec{I}_m\vec{A}_{m\times n} = \vec{A}_{m\times n}\vec{I}_n = \vec{A}$

Proof. To prove the first property, use

$$\vec{C} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{c_1} & \vec{c_2} & \cdots & \vec{c_p} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and so

$$\vec{B}\vec{C} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{B}\vec{c_1} & \vec{B}\vec{c_2} & \cdots & \vec{B}\vec{c_n} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\vec{A}(\vec{B}\vec{C}) = \vec{A}\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{B}\vec{c_1} & \vec{B}\vec{c_2} & \cdots & \vec{B}\vec{c_n} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{A}(\vec{B}\vec{c_1}) & \vec{A}(\vec{B}\vec{c_2}) & \cdots & \vec{A}(\vec{B}\vec{c_n}) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ (\vec{A}\vec{B})\vec{c_1} & (\vec{A}\vec{B})\vec{c_2} & \cdots & (\vec{A}\vec{B})\vec{c_n} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = (\vec{A}\vec{B})\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{c_1} & \vec{c_2} & \cdots & \vec{c_p} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = (\vec{A}\vec{B})\vec{C}$$

The final four properties come from the fact that matrix multiplication is a linear combination of products; recall that $c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$. □

However, matrix multiplication does not always behave like that of the real numbers. For example, $ab = ac \implies b = c \ \forall \ a, b, c \in \mathbb{R}, a \neq 0$, but the same is **not** true for $\vec{A}\vec{B} = \vec{A}\vec{C}$. Another case of this is that $\forall \ a, b \in \mathbb{R}, ab = 0 \implies a = 0 \ or \ b = 0$, but $\vec{A}\vec{B} = \vec{0} \not\Longrightarrow \vec{A} = \vec{0} \ or \ \vec{B} = \vec{0}$. Also, matrix multiplication is **not commutative**; in fact, $\vec{A}\vec{B}$ and $\vec{B}\vec{A}$ can only both exist iff $\vec{A}$ and $\vec{B}$ are both square matrices, that is, they both have the same amount of rows as columns. Even if both products do exist, in the vast majority of cases $\vec{A}\vec{B} \neq \vec{B}\vec{A}$. For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$$

but

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$$

Given a square matrix $\vec{A}_{n \times n}$, we can also define exponentiation as $\vec{A}^0 = \vec{I}_n, \vec{A}^k = \prod_{i=1}^{k} \vec{A} \ \forall \ k > 0$, e.g. $\vec{A}^2 = \vec{A}\vec{A}$. Finally, the **transpose** of a matrix $\vec{A} = (a_{ij})_{m \times n}$ is $\vec{A}^T = (b_{ij})_{n \times m} \ni b_{ij} = a_{ji}$. The transpose of a matrix can be used to referse the direction of a mapping between two vector spaces: if $T : \mathbb{R}^n \to \mathbb{R}^m, \vec{x} \mapsto \vec{A}\vec{x}$ then we can say that $S : \mathbb{R}^m \to \mathbb{R}^n, \vec{x} \mapsto \vec{A}^T\vec{x}$.