MATH 2220 HW #11

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Problem 1

If two integals are calculated over sets C_1 and C_2 respectively, such that their intersection is either the empty set or a subset of their boundary points, then their sum is the integral over $C_1 \cup C_2$. In this case, $C_1 \cup C_2 = C$, and all points on a line are boundary points.

Problem 2

(a) Let $r(t) = (R\cos t, R\sin t)$. Then, $r'(t) = (-R\sin t, R\cos t)$. By definition:

$$\begin{aligned}
\int_C \vec{f} \cdot \vec{t}\,\mathrm{d}S &= \int_0^{2\pi} \vec{f}(r(t)) \cdot r'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \vec{f}((R\cos t, R\sin t)) \cdot (-R\sin t, R\cos t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \left(\frac{-\sin t}{R}, \frac{\cos t}{R}\right) \cdot (-R\sin t, R\cos t)\,\mathrm{d}t \\
&= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,\mathrm{d}t = \int_0^{2\pi} \mathrm{d}t = 2\pi
\end{aligned}$$

(b) ~~$\vec{f}$ voted for Obama~~ C starts and ends at the same point, since $r(0) = r(2\pi) = (R, 0)$. In a conservative vector field, any line integral over a curve that loops back to the same point would be 0, as a line integral in such a field is path-independent. However, the integral over C is $2\pi \neq 0$. Therefore, $\vec{f}$ is not conservative.

Problem 3

(a) Since the set C_1 is symmetric about both x and y, $\int_{C_1} x^2\,\mathrm{d}S = \int_{C_1} y^2\,\mathrm{d}S$. Therefore, $\int_C (x^2+y^2)\,\mathrm{d}S = \int_{C_1} x^2\,\mathrm{d}S + \int_{C_1} y^2\,\mathrm{d}S = 2\int_{C_1} x^2\,\mathrm{d}S$.

(b) For the same reason, this works for C_2.

(c) On the unit circle, every point has the property that $x^2 + y^2 = 1$. Therefore, $\int_{C_1} (x^2 + y^2)^{10}\,\mathrm{d}S = \int_C \mathrm{d}S = \text{length}(C_1)$.

(d) Once again, $\text{length}(C_2) = \int_{C_2} \mathrm{d}S$. On the unit square, the closest points to the origin are the points of intersection with the x- and y-axes, where $x = \pm 1, y = 0$ or $x = 0, y = \pm 1$. Their squared norm is their distance to the origin squared, i.e. 1. This means that $(x^2+y^2)^{10} \geq x^2+y^2 \geq 1\ \forall\, x, y \in C_2$, so $\int_{C_2} (x^2+y^2)^{10}\,\mathrm{d}S > \int_{C_2} \mathrm{d}S$.

Problem 4 The function that parametrises the set of points C where $x^2 + y^2 = r^2$ is $s(t) = (r\cos t, r\sin t)$, for $0 \leq r \leq 2\pi$. and $s'(t) = (-r\sin t, r\cos t)$. Therefore:

$$\begin{aligned}
\int_C (-3y, 2x) \cdot \vec{t}\, \mathrm{d}S &= \int_0^{2\pi} (-3r\sin t, 2r\cos t) \cdot (-r\sin t, r\cos t)\, \mathrm{d}t \\
&= \int_0^{2\pi} (3r^2\sin^2 t + 2r^2\cos^2 t)\, \mathrm{d}t = \int_0^{2\pi} (r^2\sin^2 t + 2r^2(\sin^2 t + \cos^2 t)\, \mathrm{d}t = \int_0^{2\pi} (r^2\sin^2 t + 2r^2\, \mathrm{d} \\
&= r^2 \int_0^{2\pi} (\sin^2 t + 2)\, \mathrm{d}t = r^2 \left(\frac{5t}{2} - \frac{\sin(2t)}{4} \right) \Big|_0^{2\pi} = 5\pi r^2
\end{aligned}$$

Since the second vector field is the first vector field multiplied by -5, its rotation is $\int_C -5(-3y, 2x)\cdot \vec{t}\, \mathrm{d}S = -5\int_C (-3y, 2x) \cdot \vec{t}\, \mathrm{d}S = -25\pi r^2$.

Problem 5

Given this information, we have the following equalities:

$$\begin{aligned}
\frac{\partial}{\partial x} g &= a \\
\frac{\partial}{\partial y} g &= b \\
\frac{\partial}{\partial z} g &= c
\end{aligned}$$

Consider for example a and b. Differentiating one with respect to y and one with respect to x, we have $\frac{\partial}{\partial y} a = \frac{\partial}{\partial y}\frac{\partial}{\partial x} g$ and $\frac{\partial}{\partial x} b = \frac{\partial}{\partial x}\frac{\partial}{\partial y} g$. Since the order of differentiation does not matter, we have $\frac{\partial}{\partial y} a = \frac{\partial}{\partial x} b$. Similarly, we can derive the fact that $\frac{\partial}{\partial z} b = \frac{\partial}{\partial y} c$ and $\frac{\partial}{\partial z} a = \frac{\partial}{\partial x} c$.

Problem 6

(a) By definition, using the chain rule and then the fundamental theorem of calculus:

$$\begin{aligned}
\int_C \nabla g \cdot \vec{T}\, \mathrm{d}S &= \int_0^1 \nabla g(\vec{\phi}(t)) \cdot \vec{\phi}'(t)\, \mathrm{d}t \\
&= \int_0^1 \nabla (g \circ \vec{\phi})\, \mathrm{d}t = (g \circ \vec{\phi})(1) - (g \circ \vec{\phi})(0)
\end{aligned}$$

(b) Suppose $C = \bigcup_{i=0}^k C_i$ for some k, where the endpoints of the curves $C_1, C_2 \cdots C_k$ are $t = a, t = b$, $t = b, t = c$ et cetera. Then

$$\begin{aligned}
\int_C \nabla g \cdot \vec{T}\, \mathrm{d}S &= \sum_{i=0}^k \int_{C_i} \nabla g \cdot \vec{T}\, \mathrm{d}S \\
&= (g \circ \vec{\phi})(1) - (g \circ \vec{\phi})(a) + (g \circ \vec{\phi})(a) - (g \circ \vec{\phi})(b) + \cdots + (g \circ \vec{\phi})(n) - (g \circ \vec{\phi})(0) \\
&= (g \circ \vec{\phi})(1) - (g \circ \vec{\phi})(0)
\end{aligned}$$

(c) Gravitational force $\vec{F}$ is defined in terms of gravitational potential $\vec{U}$ as $\vec{F} = -\nabla\vec{U}$. Therefore, gravity is a conservative vector field. This means that the total work done walking in a path that starts and ends at the same point, i.e. the line integral along the path, is 0, assuming no friction etc. This is why it's impossible to go uphill both ways, as that would imply that there is a positive amount of work being done.

(d) what even is this question