Structural Analysis with Discrete Elastic Rods

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This document details how the discrete elastic rods model can be used to evaluate important structural aspects of a deployed gridshell such as the internal stresses in its beams and the forces acting on its bolts. We separately consider internal stresses due to stretching, bending, and twisting. The stretching and bending stresses can be naturally superimposed because they both take the form of a "z-stress" (i.e., tensile/compressive forces oriented along the tangential direction). We will see that twisting stresses are slightly different, and note that our analysis cannot capture the interplay of bending and twisting stresses with complete accuracy.

A note on units: we prefer N for force, mm for length, and MPa for stress. However no units are actually specified in the code or equations, so any compatible system of units can be chosen. For example, one could instead use N, m, and Pa.

1 Stretching Stress

The simplest energy term to analyze is stretching, as the strain and stress are just the familiar 1D quantities $\frac{\Delta l}{l}$ and $E\frac{\Delta l}{l}$, where l is the length of a small segment of rod and E is the Young's modulus. This quantity lives most naturally on the edges, but it could be averaged onto the vertices if desired.

2 Bending Stress

When a segment of rod is bent, it causes stretching in one "half" of the cross-section and compression in the other. We consider a cross-section Ω with a center of mass $\mathbf{c}$. We bend the rod segment so that its centerline centerline has the *curvature normal* vector $\boldsymbol{\kappa}$, arriving at the following configuration:

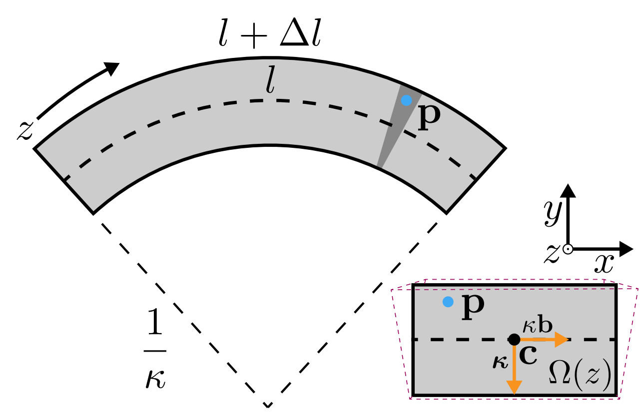

Figure 1 *Geometry of a bent rod.*

The dashed line parallel to the curvature binormal $\kappa\mathbf{b}$ and passing through the center of mass indicates the neutral surface, which experiences no strain under this bending deformation. This neutral surface divides Ω into two halves: the half pointed to by $\boldsymbol{\kappa}$ which experiences a compressive z strain, and the other half which experiences a tensile z strain.

Studying the deformation, we see that at a particular signed height h "above" (in the opposite direction of $\boldsymbol{\kappa}$) the neutral surface, the deformed length is $l + \Delta l = \frac{l}{1/\kappa}\left(\frac{1}{\kappa} + h\right) = l + h\kappa l$, indicating a z strain of $\frac{\Delta l}{l} = h\kappa$. The z strain at some point $\mathbf{p} \in \Omega(z)$ can be therefore written as $\varepsilon_{zz} = -\boldsymbol{\kappa} \cdot (\mathbf{p} - \mathbf{c}) = \boldsymbol{\kappa} \cdot (\mathbf{c} - \mathbf{p})$, and under our isotropic linear material model, the stress is $\sigma_{zz} = E\boldsymbol{\kappa} \cdot (\mathbf{c} - \mathbf{p})$.

For engineering applications, we want to know the maximum and minimum principal stresses occurring in each cross-section of the beam. The formula for σ_{zz} is linear in $\mathbf{p}$ and therefore must achieve its maximum and minimum on the boundary of $\Omega(z)$. However the particular boundary location of the maximum stress depends on the direction of $\boldsymbol{\kappa}$, and for general cross-section geometries we will need to check every boundary point of the mesh as the boundary may have concave regions.

In the discrete elastic rods model, we discretize the curvature quantities on the vertices and store the material frames/cross sections on the edges. So, to estimate the principal stresses, we will need to transfer quantities between vertices and edges. We compute two versions of the maximum/minimum principal stresses at a vertex, one using the material frame from the incoming edge and another using the frame from the outgoing edge. We then take a length-weighted average of these quantities to define the vertex stresses. The resulting vertex stress expressions are:

$$\sigma_{zz}^{\max} = \frac{1}{2\bar{l}_i}\left[\frac{\bar{l}^{i-1}}{\bar{l}_i}E^{i-1}\left(\max_{\mathbf{p}\in\Omega^{i-1}} \begin{bmatrix}(\kappa_1)_i^{i-1}\\(\kappa_2)_i^{i-1}\end{bmatrix} \cdot (\mathbf{c}-\mathbf{p})\right) + \frac{\bar{l}^{i}}{\bar{l}_i}E^{i}\left(\max_{\mathbf{p}\in\Omega^{i}} \begin{bmatrix}(\kappa_1)_i^{i}\\(\kappa_2)_i^{i}\end{bmatrix} \cdot (\mathbf{c}-\mathbf{p})\right)\right]$$
$$\sigma_{zz}^{\min} = \frac{1}{2\bar{l}_i}\left[\frac{\bar{l}^{i-1}}{\bar{l}_i}E^{i-1}\left(\min_{\mathbf{p}\in\Omega^{i-1}} \begin{bmatrix}(\kappa_1)_i^{i-1}\\(\kappa_2)_i^{i-1}\end{bmatrix} \cdot (\mathbf{c}-\mathbf{p})\right) + \frac{\bar{l}^{i}}{\bar{l}_i}E^{i}\left(\min_{\mathbf{p}\in\Omega^{i}} \begin{bmatrix}(\kappa_1)_i^{i}\\(\kappa_2)_i^{i}\end{bmatrix} \cdot (\mathbf{c}-\mathbf{p})\right)\right].$$

The multiplication by $\frac{\bar{l}^j}{\bar{l}_i}$ converts the integrated curvature normal (integrated over a vertex Voronoi region) into a pointwise value and then integrates it over half of the associated incident edge to perform the weighted average. Note that $(\kappa_a)_i^j$ is the curvature normal vector expressed in the material frame for edge j, so the diagram in Figure 1 applies (although $\boldsymbol{\kappa}$ can be rotated arbitrarily depending on the bending axis!). Also, in our implementation, the cross-section's center of mass is translated to the origin, so $\mathbf{c}$ vanishes from the formula.

3 Twisting Stress

Under an imposed rate of twist $\tau_i = \frac{m_i - \overline{m}_i}{\bar{l}_i}$, a cross-section will experience the stress distribution:

$$\sigma(x, y) = \mu\tau_i \begin{bmatrix} 0 & 0 & \frac{\partial\psi}{\partial x} - y \\ 0 & 0 & \frac{\partial\psi}{\partial y} + x \\ \frac{\partial\psi}{\partial x} - y & \frac{\partial\psi}{\partial y} + x & 0 \end{bmatrix},$$

where ψ is the out-of-plane warping function found as the solution to the Laplace equation $\triangle\psi = 0$ in Ω with Neumann boundary conditions $\mathbf{n} \cdot \nabla\psi = \mathbf{n} \cdot \begin{pmatrix} y \\ -x \end{pmatrix}$ on $\partial\Omega$ when computing the twisting stiffness. This stress tensor has two nonzero eigenvalues, $\sigma_{z*} := \pm \left\| \nabla\psi + \begin{bmatrix} -y \\ x \end{bmatrix} \right\|$, indicating the principal tensile and compressive stresses associated with the torsional shearing. We choose to report only the positive eigenvalue.

As with bending stresses, because the cross-sections live on the edges and the twists on the vertices, we must perform an averaging procedure. We compute a weighted average of the maximum shearing stress induced by a vertex's twist as measured in the incoming and outgoing edges:

$$\sigma_\tau = \tau_i \frac{S^{i-1}\bar{l}^{i-1} + S^i\bar{l}^i}{2\bar{l}_i}, \quad S^j := \mu^j \max_{\mathbf{p}\in\Omega^j} \left\| \nabla\psi + R^{90^\circ}(\mathbf{p} - \mathbf{c}) \right\|.$$

S^j is not load-dependent and can be pre-computed for each cross-section appearing in the design when computing the material parameters. Finally, we note that the "twisting" stress σ_τ is in the form of a shear traction acting on the cross-section.

4 Rivet Axial/Shearing Forces

We can compute the forces acting on the rivets/bolts binding the rods by summing the elastic forces over all the A rods while omitting the B rods. The components of this (partial) elastic energy gradient that correspond to the joints' degrees of freedom no longer vanish and inform us of the generalized forces applied by the A rods on the joint geometry. These will be balanced by equal and opposite generalized forces from the B rods. This force vector is computed by `RodLinkage::rivetForces`.

There are three types of joint variables leading to three types of generalized forces. The (negated) gradient with respect to the joint position tells us the net force acting on the joint's center of mass. The (negated) gradient with respect to the orientation variables ω tells us the net torques acting around the joint's frame vectors at its center of mass (in units N mm). Finally the gradient with respect to the two length variables relate to tensile forces acting on the rods, but these forces should be in balance (i.e., these components of the gradient should remain zero even when we drop the contributions from B rods). The elastic forces acting on a given variable are actually given by the

The net force acting on the joint can be decomposed into a component along the joint normal that applies a tensile/compressive load to the bolt and a component perpendicular to the normal that attempts to shear the bolt. The net torque acting on the joint's frame also introduces shearing forces on the joint.

5 Tutorial

A Jupyter notebook demonstrating these features is available in `python/examples/StructuralAnalysis.ipynb`.