Multiple Signal Classification (MUSIC) Algorithm for Angular Estimation

Multiple Signal Classification (MUSIC) is a classic algorithm for angular information estimation in wireless communications. In this article, I first model the far-field wireless channel in the view of angular-domain, and then describe the MUSIC algorithm for angular information extraction in the modeled angular-domain channel. The simulation codes will be attached for performance verification.

I. Far-field Channel Model

We consider a single-input-multiple-output (SIMO) system where a single-antenna transmitter sends pilot signal x to the receiver with N antennas. As shown in Fig. 1(a), each of the N antennas receives a copy of the signal x. When the distance between the transmitter and receiver is large enough, the signal x can be assumed to arrive at each receiver antenna in a parallel way, as illustrated in Fig. 1(b). Due to the separation d between two neighboring receiver antennas, length of the parallel path from the transmitter to each receiver antenna is different. As shown in Fig. 1(b), the distance d_n between transmitter and nth receiver antenna is given by

$$d_n = d_1 + (n-1)d\sin(\theta), \quad n = 1, 2, \ldots, N, \tag{1}$$

where d_1 is the distance from the transmitter to the 1st antenna of the receiver. On this basis, the received signal at the nth antenna is computed as[1]

$$y_n = a(d_n)e^{j2\pi f_c \frac{d_n}{c}}x = a(d_n)e^{j\frac{2\pi}{\lambda}f_c d_1}e^{j\frac{2\pi}{\lambda}(n-1)d\sin(\theta)}x \overset{(a)}{\approx} a(d_1)e^{j\frac{2\pi}{\lambda}f_c d_1}e^{j\frac{2\pi}{\lambda}(n-1)d\sin(\theta)}x, \tag{2}$$

where $a(d_n)$ is the large-scale attenuation (e.g., pathloss), $e^{j2\pi f_c \frac{d_n}{c}}$ denotes the phase change of the signal due to propagation time delay $\frac{d_n}{c}$, (a) is because of the fact that $a(d_1) \approx a(d_2) \approx \ldots a(d_N)$ for large-scale attenuation. Based on (2), the received signal at all the N antennas are given by

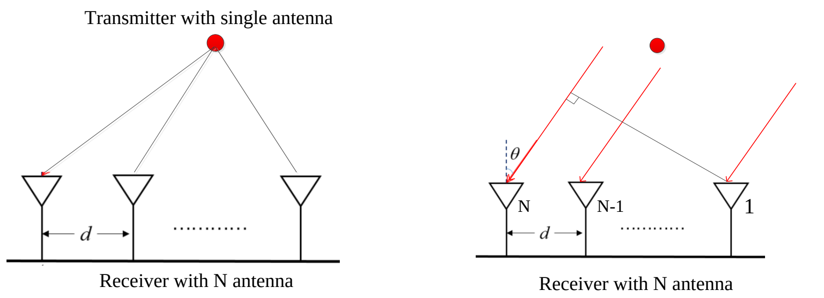
$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = a(d_1)e^{j\frac{2\pi}{\lambda}f_c d_1}x \underbrace{\begin{bmatrix} 1 \\ e^{j\frac{2\pi}{\lambda}d\sin(\theta)} \\ \vdots \\ e^{j\frac{2\pi}{\lambda}(N-1)d\sin(\theta)} \end{bmatrix}}_{\mathbf{e}(\theta)} = a(d_1)e^{j\frac{2\pi}{\lambda}f_c d_1}x\mathbf{e}(\theta) \tag{3}$$

Note that (3) represents the received signals along with a certain path with length d_1 and angle-of-arrival (AoA) θ. In practise, the signal from transmitter can arrive at the receiver along with many different paths under different length distance and AoA. An extension of the (3) to the multi-path scenario is given by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{e}(\theta_1) & \mathbf{e}(\theta_2) & \ldots \mathbf{e}(\theta_L) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_l \end{bmatrix}}_{\mathbf{x}} + \mathbf{n}, \tag{4}$$

where θ_l is the AoA of the lth path, and $x_l = a(d_{1,l})e^{j\frac{2\pi}{\lambda}f_c d_{1,l}}x$ is for the l the path, and $\mathbf{n} \in \mathcal{C}^{Nr\times 1}$ is the noise with norm distribution $\mathcal{CN}(\mathbf{0}, \mathbf{I})$.

[1] One can check my previous paper https://zhuanlan.zhihu.com/p/141475685 for the physical meaning in (2).



(a) True signal propagation model

(b) Far-field approximation, which holds when the distance between Tx and Rx is large enough.

Fig. 1: Far field channel model in free space with only line-of-site (LoS) path.

II. MUSIC Algorithm Theory

In this section, we theoretically describe the MUSIC algorithm. Note that the MUSIC can only work in the scenario with $N > L$, i.e., the number of receive antenna is larger than the number of paths in the channel $\mathbf{A}$ in (4).

According to (4), the received signal can be expressed as

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}. \tag{5}$$

Due to the randomness of noise $\mathbf{n}$, the $\mathbf{y}$ is a random vector with covariance matrix

$$\begin{aligned} \mathbf{R_y} = \mathbb{E}[\mathbf{y}\mathbf{y}^*] &= \mathbb{E}[(\mathbf{A}\mathbf{x}+\mathbf{n})(\mathbf{A}\mathbf{x}+\mathbf{n})^*] \\ &\overset{(b)}{=} \mathbf{A}\mathbb{E}[\mathbf{x}\mathbf{x}^*]\mathbf{A}^* + \mathbb{E}[\mathbf{n}\mathbf{n}^*] \\ &= \mathbf{A}\mathbf{R_x}\mathbf{A}^* + \mathbf{R_n}, \end{aligned} \tag{6}$$

where (b) is due to the fact that $\mathbb{E}[\mathbf{x}\mathbf{n}^*] = 0$. Since the $\mathbf{R_y} \in \mathcal{C}^{N\times N}$ in (6) is symmetric, we conduct the eigenvalue decomposition (EVD) for the $\mathbf{R_y}$ given by

$$\mathbf{R_y} = \begin{bmatrix} \mathbf{U_x} & \mathbf{U_n} \end{bmatrix} \begin{bmatrix} \Sigma_\mathbf{x} & \\ & \Sigma_\mathbf{n} \end{bmatrix} \begin{bmatrix} \mathbf{U_x^*} \\ \mathbf{U_n^*} \end{bmatrix} = \mathbf{U_x}\Sigma_\mathbf{x}\mathbf{U_x^*} + \mathbf{U_n}\Sigma_\mathbf{n}\mathbf{U_n^*}, \tag{7}$$

where $\mathbf{U_x} \in \mathcal{C}^{N\times L}$, $\mathbf{U_n} \in \mathcal{C}^{N\times(N-L)}$, $\Sigma_\mathbf{x} \in \mathcal{C}^{L\times L}$, and $\Sigma_\mathbf{n} \in \mathcal{C}^{(N-L)\times(N-L)}$. It is clear that $\mathbf{U_x}\Sigma_\mathbf{x}\mathbf{U_x^*}$ is the EVD of the $\mathbf{A}\mathbf{R_x}\mathbf{A}^*$ and $\mathbf{U_n}\Sigma_\mathbf{n}\mathbf{U_n^*}$ is the EVD of $\mathbf{R_n}$ in (6). Based on the properties of EVD, we have

$$\mathbf{A}\mathbf{R_x}\mathbf{A}^*\mathbf{U_n} = \mathbf{U_x}\Sigma_\mathbf{x}\mathbf{U_x^*}\mathbf{U_n} = \mathbf{0},$$

which yields[2]

$$\left(\mathbf{R_x}^{-1}(\mathbf{A}\mathbf{A}^*)^{-1}\mathbf{A}^*\right)\mathbf{A}\mathbf{R_x}\mathbf{A}^*\mathbf{U_n} = \mathbf{0} \Rightarrow \mathbf{A}^*\mathbf{U_n} = \mathbf{0}.$$

This implies

$$\mathbf{e}(\theta_l)\mathbf{U_n} = \mathbf{0} \tag{8}$$

because of the expression of $\mathbf{A}$ in (4). It means a path angle θ_l appears when

$$P(\theta) = \frac{1}{|\mathbf{e}(\theta_l)\mathbf{U_n}|^2} \to +\infty. \tag{9}$$

[2]Since different paths will independently affect the transmit signal x, the $\mathbf{R_x}$ is full rank and its inverse exists.

III. MUSIC ALGORITHM IMPLEMENTATION

In real system, one can sample the received signal $\mathbf{y}$ at different time $t \in \{1, 2, \ldots, T\}$ yielding $\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(T)$. Using these samples, the covariance matrix is computed as

$$\hat{\mathbf{R}}_{\mathbf{y}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}(t)\mathbf{y}(t)^{*}. \tag{10}$$

Taking the EVD for the $\hat{\mathbf{R}}_{\mathbf{y}}$ yields

$$\hat{\mathbf{R}}_{\mathbf{y}} = \hat{\mathbf{U}}\hat{\Sigma}\hat{\mathbf{U}}^{*}. \tag{11}$$

We sort the eigenvalues in $\hat{\Sigma}$ at a descending order and accordingly adjust the corresponding eigenvectors in $\hat{\mathbf{U}}$. In this way, we obtain the $\hat{\mathbf{U}}_{\mathbf{n}}$ as in (7) by taking the last $N - L$ columns of the adjusted $\hat{\mathbf{U}}$. Under different $\theta \in (0, \pi]$, we can obtain different value for the function $P(\theta)$ in (9). According to (9), the θ yielding a locally maximum $P(\theta)$ is a target AoA. Summaries of the MUSIC algorithm for AoA extraction is given in Algorithm 1.

Algorithm 1 MUSIC algorithm for AoA Extraction

Require: $\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(T)$, resolution for $\theta \in (0, \pi]$

1. Compute the covariance matrix $\mathbf{R}_{\mathbf{y}}$ in (10).
2. Find the noise space matrix $\hat{\mathbf{U}}_{\mathbf{n}}$.
3. Based on the resolution for $\theta \in (0, \pi]$, test each θ and find the ones yielding the locally maximum $P(\theta)$ in (9). These θs are the expected AoAs.
