Chaos - Interpreting the Physical Pendulum and the Chaotic Ratchet

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Linear Dynamics

- In Classical mechanics, Newton's equations can be used to represent the systems[1].
- For linear systems, it is possible to analytically solve the differential equations and obtain the required information about the system.
- In the 17^{th} century when newton had discovered his laws of motion, it was used along with his theory of gravity to explain Kepler's laws of the planetary motion[2].
- Newton was able to successfully solve the two-body problem, but his tools fell short in regards to the three-body problem.

Non-Linear Dynamics

- Systems where the output is not directly proportional to the input are called Non-linear systems.
- The non-linearity in the system makes it difficult for us to solve the differential equation of the complex system analytically.
- For the three-body problem, Poincarè suggested an alternative approach to interpreting the system.
- Instead of the quantitative solution to the system, he suggested looking at the overall picture and discussed the stability of the systems[2].
- He developed geometrical tools which help us understand such characteristics of the system.

Introduction to chaos

- With the advancement in technology and development of high-speed computers in the 1950's[2], scientists were able to solve the equations numerically and with great precision.
- In the 1960's Edward N. Lorenz, while working on a system of equations describing the atmosphere, observed that a very slight change in initial conditions altered the system's trajectory vastly.
- He used a simplified system with 3 parameters and observed a similar result.
- He saw that the system had very sensitive dependence on initial conditions and published his findings in the paper 'Deterministic non-periodic flow'[3].

The Butterfly Effect

- In 1972, Lorenz presented a paper at the '139th Meeting of the American Association or The Advancement Of Science' titled 'Predictability; Does the flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?' [4].
- The title of this paper inspired many to call this sensitive dependence on initial conditions the 'Butterfly Effect.'
- Coincidentally, the Phase-space diagram of Lorenz's paper from 1963 for the simplified 3 parameter system, in the three dimensions resembles a butterfly[3].

The Butterfly Effect

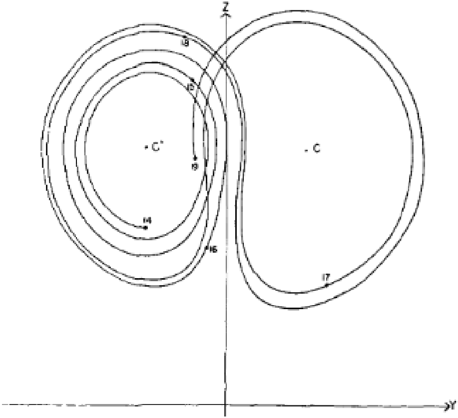

Figure 1: Phase-Space diagram, using 2 of the 3 parameters from Lorenz's simulation.[3]

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Introduction

- Every continuous system can be represented using a set of differential equations.
- These differential equations represent the relations between different variables of the system.
- We can study the relationships between these variables more efficiently by using graphs.
- different types of graphs can give us different ideas about the characteristics of the system.

Space-Time Diagram

- The most common and intuitive graph representing a system.
- Describes how the system evolves with time.

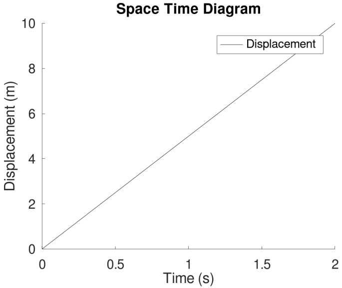

Figure 2: Shows the motion of a particle which is traveling along the x axis with a uniform velocity of 5 m/s

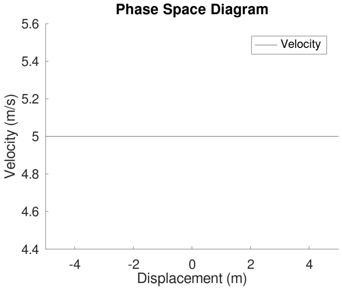

Figure 3: Shows the phase-space diagram of the same particle. The straight line represents a never-ending trajectory, which is non periodic.

Phase-Space Diagram

- The phase-space diagram explores the relation between the variables that perfectly define the particle.
- Each point in the phase-space uniquely describes a state of the system.
- The trajectory for a given initial condition would be unique, and hence lines in the phase-space diagram never intersect.
- The phase-space diagram can have different possible characteristics, which can be broadly categorized as follows.

Interpreting the phase-space diagram

- The system converges to a single point.
  - This is called the fixed point attractor.
  - This represents a stable system which has attained equilibrium.
  - Example :: A damped oscillator.
- The system forms a loop
  - This represents a stable system which is periodic.
  - Example :: A simple pendulum.
- The system neither converges to a point nor forms a loop
  - This represents a non periodic system.
  - This system can be predictable like in the case of the particle with a constant velocity or be unpredictable and chaotic.

The Poincarè Map

- A Poincarè map is the intersection of the phase-space diagram of a system with a lower-dimensional subspace called the Poincarè section.
- In usual cases, we use the driving force or a periodic function that is related to the system to generate the Poincarè map.
- In our case, to generate a Poincarè map, we plot the phase-space diagram by selecting the points that are in phase with the driving force.

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The Simple Pendulum

- The simple pendulum is an ideal case of a linear pendulum which consists of a mass suspended from a frictionless pivot.
- The only forces acting on the bob of the pendulum are gravity and tension in the string.
- The net force acting on the bob at an angle θ can be given as $F_\theta = -mg\sin\theta$ where the negative sign signifies the the force is always opposite in direction to the displacement of the bob.

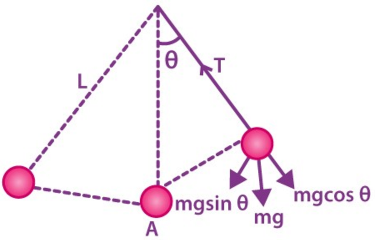

Figure 4: A Simple Pendulum.

Modeling the system

- The equation of motion of the system is represented as $\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$, with the approximation of $\sin\theta \approx \theta$
- This can be simplified into a system of first order equations and solved using numerical methods to calculate values of the variables as follows,
- $\frac{d\omega}{dt} = -\frac{g}{l}\theta$ and $\frac{d\theta}{dt} = \omega$,

- Euler Method
  - $\omega_{i+1} = \omega_i - (g/l)\theta_i\Delta t$,
  - $\theta_{i+1} = \theta_i + \omega_i\Delta t$
  - $t_{i+1} = t_i + \Delta t$
- Euler-Cromer Method
  - $\omega_{i+1} = \omega_i - (g/l)\theta_i\Delta t$,
  - $\theta_{i+1} = \theta_i + \omega_{i+1}\Delta t$
  - $t_{i+1} = t_i + \Delta t$

Method Comparison

- The Euler Method of integration does not conserve energy[5] [6].
- As seen in the graph the amplitude of Oscillation keeps increasing.
- Euler-Cromer Method conserves energy and hence is better suited to solve System with constant energy.

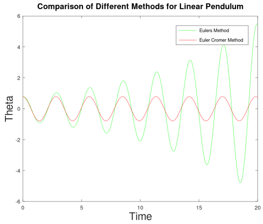

Figure 5: Comparison of Euler and Euler-Cromer Methods of Integration.

Method Comparison

- The same can be seen from the phase-space diagram of the pendulum.
- The Euler method takes the system into an outward spiral which is an unstable system.
- The Euler-Cromer method keeps the system in a closed loop which is a conserved energy state.

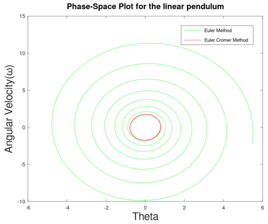

Figure 6: Comparison of Euler and Euler-Cromer Methods of Integration.

Non-Linear Simple Pendulum

- If we remove our approximation $\sin\theta \approx \theta$, the system becomes non-linear.
- This non-linearity causes the system to be heavily dependent on initial conditions.
- We can use the Euler-Cromer method to Solve the system for different initial conditions and then study its variations.
- We can also compare the system with that of the linear pendulum to see how non-linearity affects the system.

Comparison Using the space-time diagram

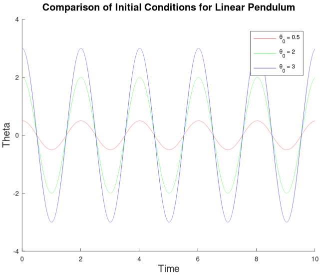

Figure 7: Phase-Space Comparison of Different

Initial conditions for a linear pendulum.

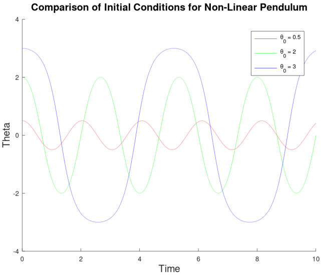

Figure 8: Phase-Space Comparison of Different

Initial conditions for a non-linear pendulum.

Comparison Using the phase-space diagram

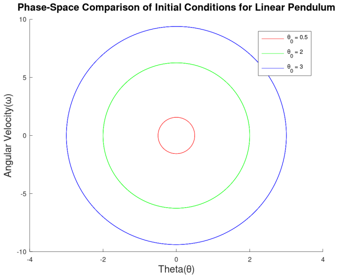

Figure 9: Phase-Space Comparison of Different Initial conditions for a linear pendulum.

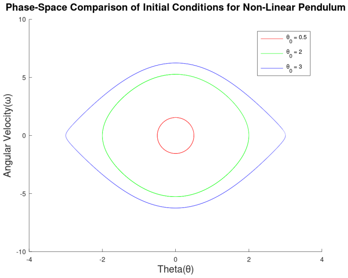

Figure 10: Phase-Space Comparison of Different Initial conditions for a non-linear pendulum.

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The Physical Pendulum

- We can make the system less ideal and more realistic by introducing factors such as damping, friction and driving forces.
- The system of the physical pendulum can be represented by the differential equation : $\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{l} - q\dfrac{d\theta}{dt} + F_D \sin(\Omega_D t)$
- Where, $q\dfrac{d\theta}{dt}$ is the damping force which is proportional to velocity(like friction) with a damping constant q. $F_D \sin(\Omega_D t)$ is the driving force with F_D being the amplitude of the force and Ω_D the driving frequency.
- In order to view the graphs more efficiently we can also constrain the value of θ between $-\pi$ and π.

Numerical Solution

- The differential equation can be solved using the Euler-Cromer method using the following algorithm.
  - $\omega_{i+1} = \omega_i + [-(g/l)\sin\theta_i - q\omega_i + F_D \sin(\Omega_D t)]\Delta t,$
  - $\theta_{i+1} = \theta_i + \omega_{i+1}\Delta t$
  - $\theta_{i+1} \pm 2\pi$
  - $t_{i+1} = t_i + \Delta t$
- This can be repeated to calculate the values of the variable and hence the trajectory of the system.
- In order to study the variations in the system we fix all the quantities except for the value of the driving force F_D. i.e. $q = 0.5$, $l = g = 9.8$, $\Omega_D = 2/3, dt = 0.04$ and initial conditions as $\theta_0 = 0.2$ and $\omega_0 = 0$

Damped Oscillations, $F_D = 0$

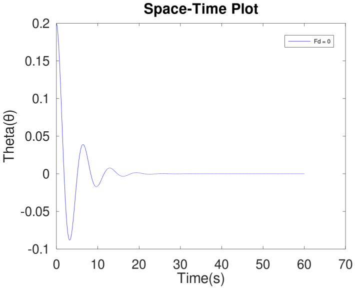

Figure 11: Space-Time Diagram for a damped oscillator.

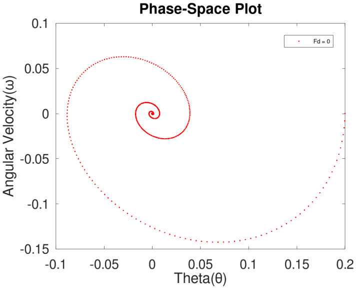

Figure 12: Phase-Space Diagram for a Damped Oscillator.

Stable, Driven Oscillations, $F_D = 0.5$

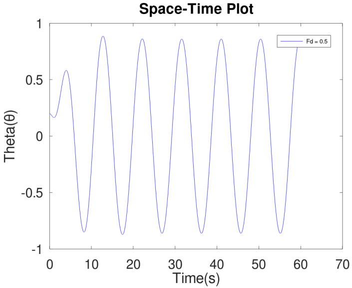

Figure 13: Space-Time Diagram for a Stable, Driven Oscillator.

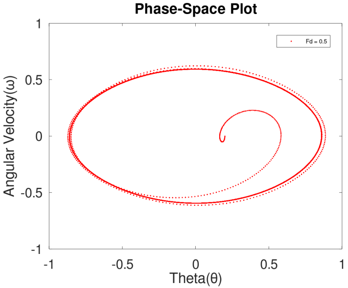

Figure 14: Phase-Space Diagram for a Stable, Driven Oscillator.

Chaotic Oscillations, $F_D = 1.2$

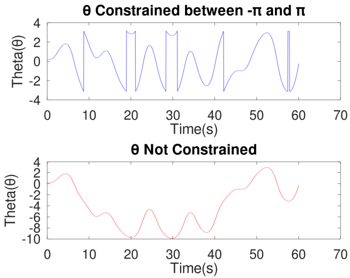

Figure 15: Space-Time Diagram for a Chaotic Oscillator.

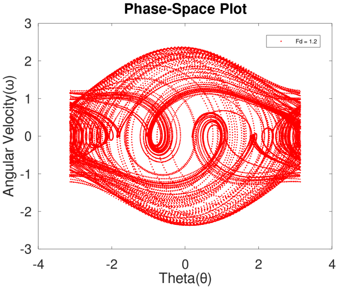

Figure 16: Phase-Space Diagram for a Chaotic Oscillator.

Poincarè Map

- From the Space-Time graph it is pretty evident that we cannot predict the trajectory that the system follows.
- It might seem that it is random, but looking at the Phase-Space diagram, a pattern is evident.
- Every Initial condition will eventually flow towards the surface that we can see in the graph.
- We plot all points in the Phase-Space diagram that are in phase with the driving force. (i.e points where, $\Omega_D t = 2n\pi$, where n is an integer.)
- This is called the Poincare Map of the system.

Poincarè Map

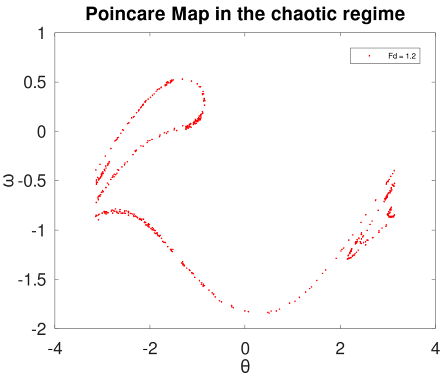

Figure 17: Poincarè Map of the Physical Pendulum at the chaotic regime at $F_D = 1.2$

Route to Chaos

- The system is periodic when the driving force is low and is chaotic at some higher value of the driving force
- We can look at the case from $F_D = 1.35$ to $F_D = 1.485$, to identify the transition from order to chaos.
- We then plot the bifurcation diagram to represent the number of periods of the system to qualitatively explain how the transition to chaos occurs.

Period-1

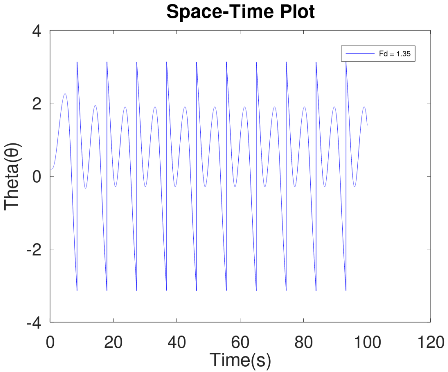

Figure 18: Space-Time graph at $F_D = 1.35$ showing a Period-1 Oscillation, where the Time period of oscillation of the pendulum is equal to the period of the driving force.

Period-2

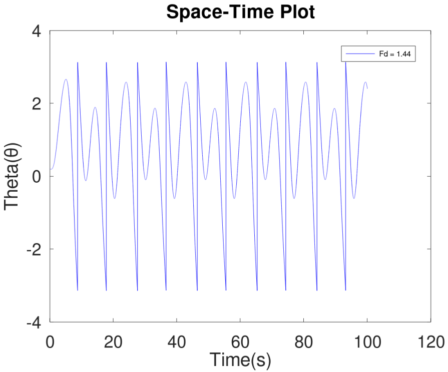

Figure 19: Space-Time graph at $F_D = 1.44$ showing a Period-2 Oscillation, where the Time period of oscillation of the pendulum is twice that of the driving force.

Period-4

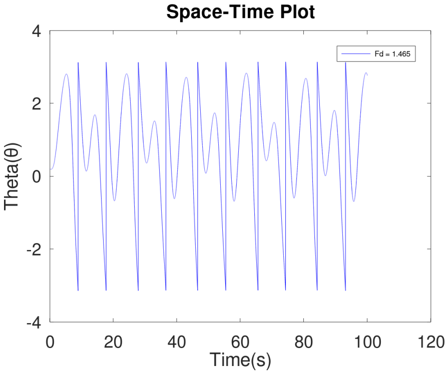

Figure 20: Space-Time graph at $F_D = 1.465$ showing a Period-2 Oscillation, where the Time period of oscillation of the pendulum is four times that of the driving force.

Bifurcation Diagram

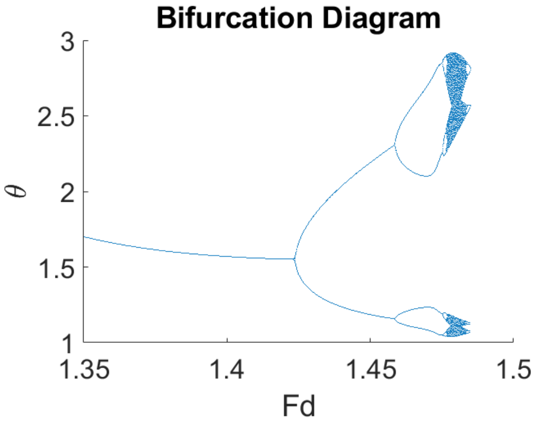

Figure 21: Bifurcation diagram showing the Route to chaos.

The Feigenbaum

- The bifurcation diagram gives us a qualitative understanding of the transition to chaos.
- The spacing between period-doubling transitions becomes smaller as we approach chaos.
- We define F_n as the driving force where the transition to period-2^n takes place.
- Now we define $\delta_n = \dfrac{F_n - F_{n-1}}{F_{n+1} - F_n}$
- As $n \to \infty$, δ converges to a value $\delta \approx 4.669$.[7]
- This δ is called the Feigenbaum.

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Introduction

- A Ratchet is a device that allows continuous motion in only one direction.

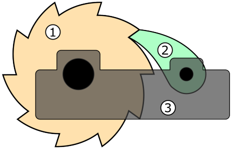

Figure 22: A Mechanical Ratchet[8], (1:Gear 2:Pawl 3:Mounting Base).

- In our case, instead of using mechanical gears and pawls, we will be considering a particle in a one-dimensional potential field, driven by a periodic driving force[9].

The Chaotic Ratchet System

- The dimensionless equation of motion of the particle in the ratchet is given as, $\ddot{x} + b\dot{x} + \dfrac{dV(x)}{dx} = a\cos(\omega t)$
- Where the Dimensionless potential is given as

$$V(x) = C - [\sin 2\pi(x - x_0) + 0.25 \sin 4\pi(x - x_0)]/4\pi^2\delta$$

- δ is defined as $\delta = \sin(2\pi|x_0'|) + \sin(4\pi|x_0'|)$
- The parameters for the simulation are ::
  - $x_0 \simeq -0.19,\ \delta \simeq 1.6,\ C \simeq 0.0173$
  - We vary the parameter a and fix $b = 0.1$ and $w = 0.67$.

Numerical Solution

- The dimensionless equation is written as,

$$\frac{dx}{dt} = v \qquad \therefore dx = v dt$$
$$\frac{dv}{dt} = a\cos(\omega t) - bv - \frac{dV(x)}{dx} \qquad \therefore dv = \dot{v} dt$$

- Where,
  - $\dot{v} = a\cos(\omega t) - bv - \frac{dV(x)}{dx} = F(t, x, v)$
  - $\frac{dV(x)}{dx} = \frac{2\cos(2\pi(x - x0) + \cos(4\pi(x - x0)))}{4\pi\delta}$

Runge-Kutta(4^{th} order) Algorithm

- We use the RK4 Algorithm to solve the system of equations.

$$
\begin{aligned}
dx_1 &= hv & dv_1 &= hF(t, x, v) \\
dx_2 &= h\left(v + \frac{dv_1}{2}\right) & dv_2 &= hF\left(t + \frac{h}{2}, x + \frac{dx_1}{2}, v + \frac{dv_1}{2}\right) \\
dx_3 &= h\left(v + \frac{dv_2}{2}\right) & dv_3 &= hF\left(t + \frac{h}{2}, x + \frac{dx_2}{2}, v + \frac{dv_2}{2}\right) \\
dx_4 &= h(v + dv_3) & dv_4 &= hF(t + h, x + dx_3, v + dv_3) \\
dx &= \frac{dx_1 + 2dx_2 + 2dx_3 + dx_4}{6} & dv &= \frac{dv_1 + 2dv_2 + 2dv_3 + dv_4}{6} \\
x(t + h) &= x(t) + dx & v(t + h) &= v(t) + dv
\end{aligned}
$$

Bifurcation Diagram

- We draw v, as a function of a, to show how the period doubles and view the bifurcation diagram.

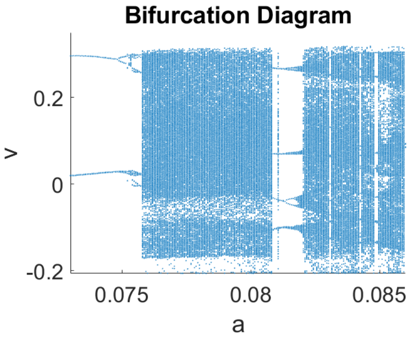

Figure 23: Bifurcation Diagram of the particle in the chaotic ratchet.

Space-Time Diagrams

- This bifurcation diagram helps us to select the points for further observation by providing information about the periodicity of the particle in the ratchet.
- In order to study specific states of the system, we can plot the Space-Time graph of the particle in the ratchet at different driving forces.

Positive Current, $a = 0.074$

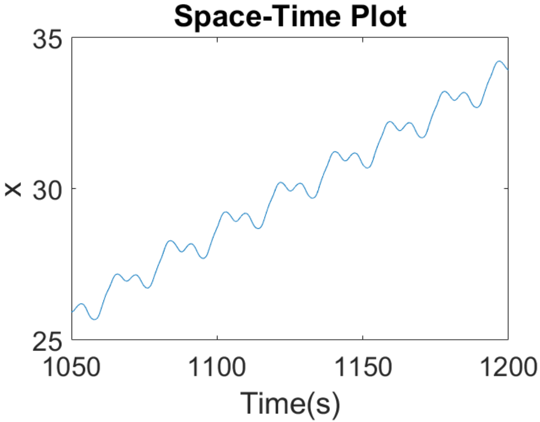

Figure 24: Space-Time diagram of the system showing positive current.

Negative Current, $a = 0.081$

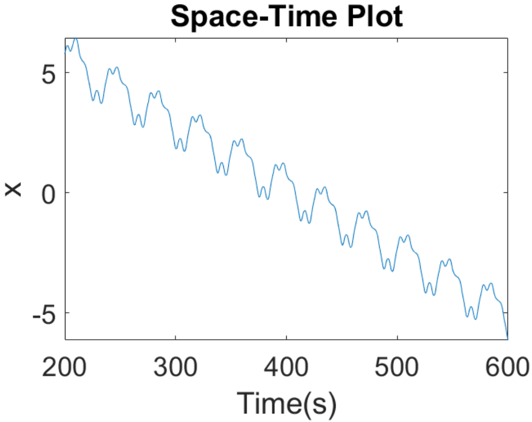

Figure 25: Space-Time diagram of the system showing negative current.

Chaotic Regime, $a = 0.0805$

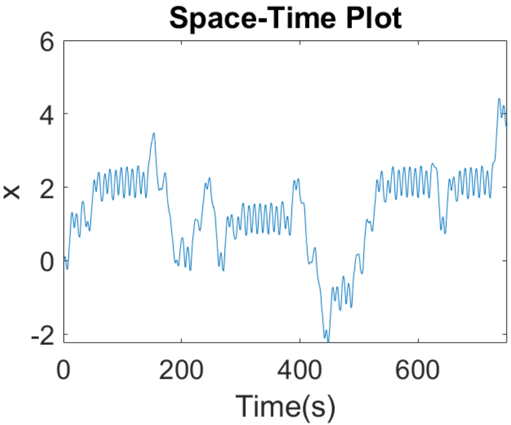

Figure 26: Space-Time diagram of the system showing Chaos.

Poincarè Map, $a = 0.0805$

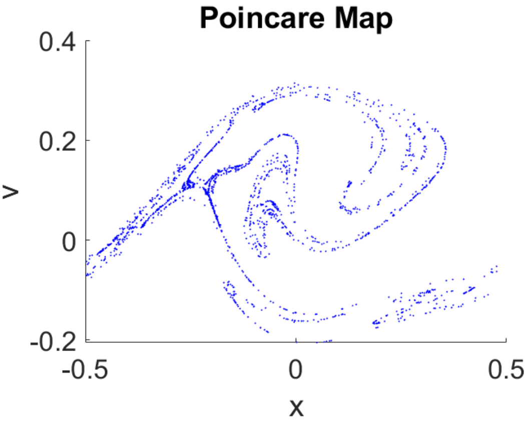

Figure 27: Poincarè Map showing the Chaotic Attractor at $a = 0.0805$.

Poincarè Map

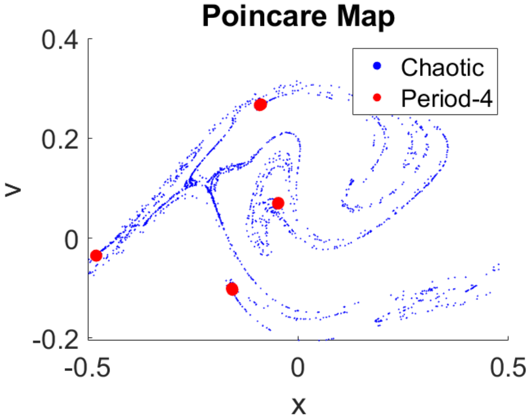

Figure 28: Poincarè Map showing the Chaotic($a = 0.0805$) and the Period-4 Attractor($a = 0.081$).

Observations

- We can see that it is the 4-period that helps the particle climb in the negative direction.
- In order to advance one step towards the left, the particle moves one step to the right and then two steps to the left. Thus, the net current is negative.
- Although counter-intuitive, we see that the transport of particles through a ratchet in the negative direction is possible due to the close vicinity of the Period-4 system to that of the chaotic system.
- We also can see that The periodic attractor lies on top of the chaotic attractor at points where it forms closed loops.

Contents

Conclusions

- The study of Non-linearity and chaos and the ability to create accurate predictions are necessary as the world we live in is non-linear.
- Counter-intuitive phenomena such as negative current in the chaotic ratchets occur due to the non-linearity in the system.
- Understanding such systems would help us efficiently solve weather predictions, encryption, understanding ECGs, etc., efficiently.

Future Plans

- The next stage of analysis is to identify patterns in the data distribution within a bifurcation diagram.
- It can be seen that some values occur more often than others.
- We also plan to study the bifurcation diagram quantitatively and apply the statistical concept of microstates and entropy to understand the diagram and the system better[10].

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