Cochin University of Science And Technology
Department of Physics

PHY1604 - Mini Project

Chaos - Interpreting the Physical Pendulum and the Chaotic Ratchet.

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Abstract

In this study, we address the topic of Non-linearity and Chaos. We look at how nonlinear systems are interpreted and discuss methods of representing the system, such as Space-Time diagrams and Phase space Diagrams. We discuss numerically solving differential equations that would usually be impossible to solve using analytical methods. We then look up the route to chaos and understand how a system goes from order to disorder and maybe back to order again. We look upon bifurcation diagrams and discuss how poincarè sections of the Phase space Diagrams help us identify attractors and then understand the order in chaos. We then apply these ideas to understand a system of Chaotic Ratchets and see how the counterintuitive concept of negative currents work. All this study helps us set a base for further research into how entropy can be a quantitative measure of chaos and how we can use entropy to predict chaotic systems' behavior.

Acknowledgments

This project entitled **“Chaos - Interpreting the Physical Pendulum and the Chaotic Ratchet.”** is carried out from the Department of Physics, Cochin University of Science and Technology under the guidance of Dr. Ronald Benjamin.

I take this opportunity to thank all persons responsible for helping me complete this project well.

With great pleasure, I express my sincere gratitude to Dr. Ronald Benjamin, UGCFRP-Assistant professor, Department of Physics, Cochin University of Science and Technology, whose guidance paved the way for me to complete this project thoroughly.

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Balakrishna Prabhu B. N.

Declaration

I **Balakrishna Prabhu B. N.** hereby declare that the Project Report entitled **"Chaos - Interpreting the Physical Pendulum and the Chaotic Ratchet."** submitted towards the partial fulfillment of the requirement for the sixth semester course of Integrated M.Sc. in Science (Physics), at Cochin University of Science and Technology, is an original work written and composed by me under the supervision of Dr. Ronald Benjamin, UGCFRP Assistant Professor, Department of Physics, Cochin University of Science And Technology.

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CERTIFICATE

This is to certify that the Minor Project work entitled **"Chaos - Interpreting the Physical Pendulum and the Chaotic Ratchet."** is the bonafide work done by **Balakrishna Prabhu B. N. (Register No: 35218024)**, at Cochin University of Science And Technology, under the guidance and supervision of Dr. Ronald Benjamin, in partial fulfillment of the requirement for the sixth-semester course of Integrated M.Sc. in Sciences (Physics).

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Chapter 1

Introduction

In the 1960's Edward N. Lorenz, while working on a system of equations trying to describe the atmosphere, observed that a very slight change in the initial conditions vastly altered the system's trajectory. He simplified the system from 12 to 3 parameters and repeated his numerical analysis to study this further. He saw that the system had sensitive dependence on the initial conditions. He published his findings in the paper 'Deterministic non-periodic flow' [1].

In 1972 He presented a paper at the '139^{th} Meeting of the American Association For The Advancement Of Science', it was titled 'Predictability; Does the flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?'[2]. This was the birth of the butterfly effect and set off a chain reaction causing a surge in the study of non-linear systems which show chaotic behavior.

Edward Lorenz summarized this theory as Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

Chaos is a vital part of the non-linear system and a very significant branch in classical mechanics[3]. The theory[4] talks about the deterministic but unpredictable nature of non-linear systems. It aims to study the systems to make predictions for a collection of initial conditions by plotting the phase space diagram, identifying the attractors, and plotting graphs such as the Poincare diagram.

To study the evolution of Chaos, we can start from a simple, linear system such as a 'Simple Pendulum' and increase complexity, making it a non-linear, and then a physical pendulum with damping and driving forces. This allows us to observe how the different graphs work to interpret the system. We can then study complex systems, such as Chaotic Ratchets, to describe the motion of a particle in space, constrained by a potential function and driven by a driving force.

Once we have modeled a system, we can use numerical methods[5] to solve the differential equations and thus obtain the system's trajectory. Each method of solving a differential equation is used in particular situations. For example, for solving Hamiltonian systems, where energy is a constant, we cannot use methods such as Euler or Runge-Kutta. We have to opt for methods such as Euler-Cromer, or Velocity-Verlet in order to conserve the energy[6]. We can then use the attractors, Poincare maps, and Bifurcation diagrams to study the route to Chaos to help characterize the system.

To extend the study to further research, we can use the idea of microstates of a system to determine the entropy[7] and use it to predict the system possibly.

Chapter 2

Chaotic Systems and Interpretations

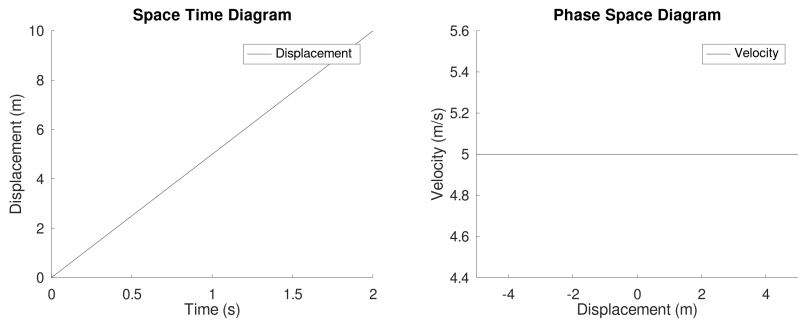
Every continuous system can be represented using a set of differential equations. Some of these differential equations can not be solved analytically due to the complexity in their nature. However, it can be solved numerically[1], and given an initial condition, the system will always follow a fixed trajectory.

2.1 Graphs representing a differential equation.

To understand the interpretation of the differential equation, let us consider a one-dimensional motion of a particle.

The equation $\frac{d^2x}{dt^2} = 0$ represents a particle moving with a constant velocity v in the x direction. The most common way of representing this is using a space-time graph of x vs t to show the motion of the particle as it varies with time(Fig : 2.1a).

[1]The time step for solving the equations numerically, unless specified is $0.01s$



(a) Space Time Graph. (b) Phase Space Diagram.

Figure 2.1: Graphs representing the system with v = 5 m/s

These position-time graphs do not adequately represent the system as a point on this graph cannot uniquely identify the system's state for more complex and nonlinear systems. Hence, we use the phase space diagram to describe the system uniquely, which is a graph between x and v. A point on the phase space diagram uniquely describes the state of the system. For our current equation, the phase space diagram is a straight line parallel and distance of v from the x-axis(Fig: 2.1b). This tells us that the system is in a never-ending trajectory that is not periodic.

2.2 Interpreting the Phase Space Diagram

Each point on the phase space diagram represents a unique state of the system completely. As the system is deterministic, the system's trajectory through a point will always be the same. i.e., the system's trajectory will evolve invariantly if we start from any point on the phase space diagram. This means that, for a given particle, no two lines would ever cross and travel in different trajectories over time. Therefore, we can characterize the possible categorize of diagrams to understand the system better.

If a system converges to a single point on the phase space diagram, we call it a fixed point attractor. This means that the system has found stability and that wherever we start the system from (any initial condition), the system will always end up in the same spot. For Example, a damped pendulum follows the trajectory of an inward spiral and always returns to rest($\theta = 0$, $\omega = 0$).

If the system forms a loop, then it means that the system is periodic. This would be the graph for a simple pendulum. The Graphs for a simple pendulum would show concentric circles/ellipses showing periodicity.

The third case would be that where the trajectory neither forms a loop nor converge to a point, this case shows an infinite trajectory. This means the system is infinite with no period. This could represent a system with linear, predictable motion or a system with nonlinear, chaotic motion.

To study the system further, we can take a section of the Phase Space diagram, usually by selecting points in period with the driving force, showing the Poincare map to determine the attractors, and collectively explain the possible future for a set of initial conditions.

Chapter 3

Physical Pendulum

3.1 Introduction

A pendulum is a weight suspended from a pivot such that it can swing freely. When displaced from its equilibrium position, it accelerates back to the equilibrium position under the restoring force of gravity. For a simple linear pendulum, there is no damping or driving force. Hence due to the conservation of energy, the system will keep oscillating back and forth.

To study chaotic behavior, we have to consider the case of a non-linear pendulum. We can do that by increasing the complexity of our system and then making fewer approximations.

We start with the case of a Linear Pendulum in an ideal scenario and then move on to more complex systems with friction, damping, and driving forces.

3.2 The Simple Pendulum(Linear)

The simple pendulum is an ideal case of a linear pendulum. It consists of a mass suspended from a frictionless pivot using a massless string.

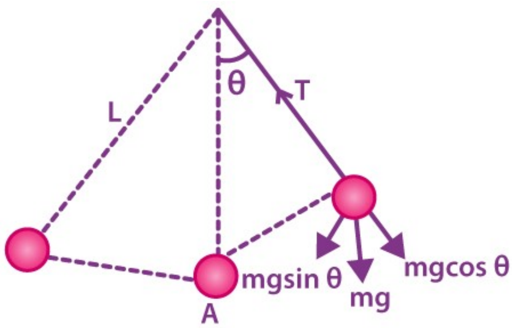


Figure 3.1: The Simple Pendulum.

To derive the equation, let us consider the pendulum with a bob of mass m, a string of length l, making an angle θ with the vertical. As the system is ideal, the only forces acting on the mass are gravity and the tension in the string. The forces on the pendulum's bob can be split into components perpendicular and parallel to the string. The parallel component of the force due to gravity balances the tension in the string. Thus, the resultant force acting on the bob at an angle θ is given by,

$$F_\theta = -mg\sin\theta$$

Here the $-ve$ sign signifies that the force is always opposite in direction to the displacement of the bob.

Using Newton's second law, we can write this in the form of a differential equation. We know, $F_\theta = m\dfrac{d^2s}{dt^2}$. The displacement of the bob is given by the arc of length, $s = l\theta$. For small oscillations, we can approximate $\sin\theta \approx \theta$.

Thus, the equation of motion becomes,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

This equation id linear and hence will not show chaotic behaviour. this is both deterministic and predictable. The solution to this system can be analytically determined as,

$$\theta = \theta_0 \sin(\Omega t + \Phi)$$

Where, $\Omega = \sqrt{\frac{g}{l}}$ is the frequency of oscillation, and θ_0 and Φ are initial conditions (initial displacement and velocity.) For a pendulum released from rest, $\Phi = 0$.

This equation can be solved numerically using algorithms such as Euler, Euler-Cromer, Runge-Kutta, Velocity-Verlet, etc[8].

To solve this second-order equation, it is easier to express this as a system of first order equations, such as,

$$\frac{d\omega}{dt} = -\frac{g}{l}\theta,$$

$$\frac{d\theta}{dt} = \omega,$$

Here, ω is the angular velocity of the pendulum.

Now that we have the differential equation we can rearrange the terms and use computational methods to calculate the values of the variables given initial conditions. The Algorithm to calculate the values in the next step can be written as ::

$$\omega_{i+1} = \omega_i - (g/l)\theta_i \Delta t,$$

$$\theta_{i+1} = \theta_i + \omega_i \Delta t$$

$$t_{i+1} = t_i + \Delta t$$

We calculate the values of θ and ω at any given t by repeating this step.

This algorithm is known as Euler's method, which does not conserve energy, and we see from the graph(Fig: 3.2) as an increase in amplitude with time.

To correct this, we use a method called Euler-Cromer, in which we use the $(i+1)^t h$ value of ω to calculate the new value of θ.

i.e.

$$\omega_{i+1} = \omega_i - (g/l)\theta_i \Delta t,$$

$$\theta_{i+1} = \theta_i + \omega_{i+1} \Delta t$$

$$t_{i+1} = t_i + \Delta t$$

The Euler-Cromer method produces a more accurate result while solving the equation. The value of Δt is significant while solving the equation. The smaller the value of Δt, the more accurate the result. In these simulations of the pendulum, we have used standard units for the parameters and variables. We have not converted them into their dimensionless form.

For simple differential equations, we can expend computational power to use these methods to solve the equation using a shorter time step (Δt). However, with the increase in complexity, this becomes difficult and time-consuming. To get accurate results, we will have to make the time step extremely small, which will result in a massive amount of computation. In such scenarios, we need algorithms like Runge-Kutta or Velocity-Verlet, where a comparatively larger time step would still provide accurate results. In our current case, we limit our study to the Euler-Cromer Algorithm.

3.2.1 Space Time diagram

One of the most intuitive and easy ways to visualize the pendulum's motion is to plot the space-time graph in which we plot the object's position as it evolves with time.

The space-time graph shows the trajectory of the objects. In this case, we plot the angle with the normal, θ as a function of time. To compare the accuracy of the Euler and Euler-Cromer Algorithms, we plot them both on the same graph[1] [2] [3]

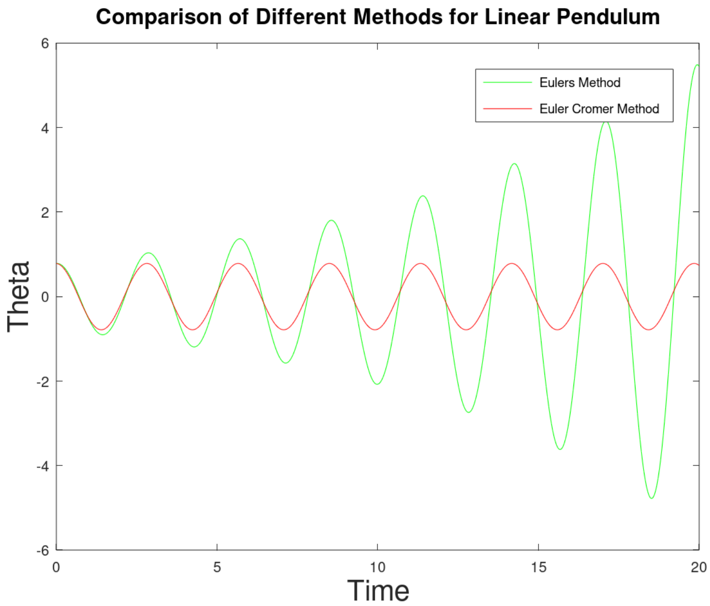


Figure 3.2: Comparison of Euler and Euler-Cromer Methods of Integration.

In the graph(Fig: 3.2), the green plot shows the solution using Euler Method, and the red plot shows the solution using the Euler-Cromer Method. It is visible that the amplitude of oscillation is increasing in the Euler Method.

[1]We use GNU Octave/MATLAB to solve and plot the graphs.

[2]Initial conditions for the simulation :: $\theta_0 = \frac{\pi}{4}$, $\omega_0 = 0$, $l = 1$ $t_0 = 0$, all in SI Units.

[3]The time step used for this simulation is 0.04s

Hence, we will be proceeding with the Euler-Cromer Method to solve the upcoming equations.

Comparison of different initial conditions.

As the system is linear, the variations in initial conditions[4] [5]will not affect the periodicity of the system, this can be seen from the graph(Fig: 3.3)

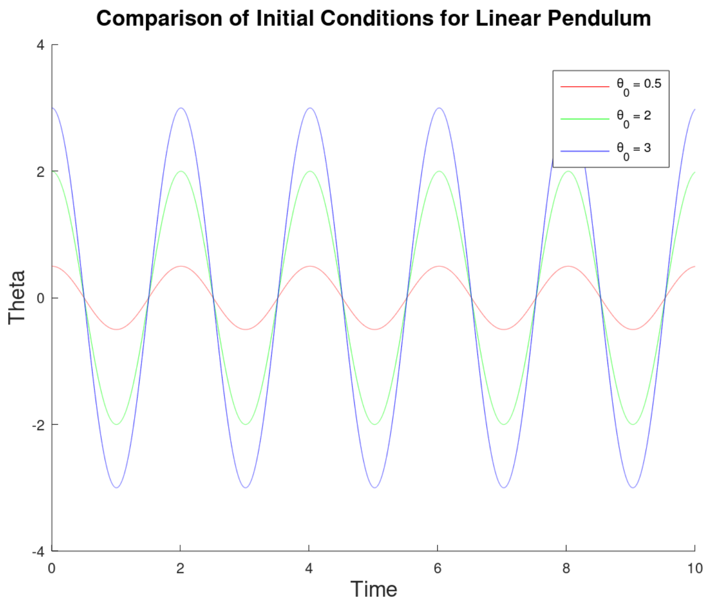


Figure 3.3: Comparison of Different Initial conditions for a linear pendulum.

From the analytical solution to the equation, we know that the period and frequency of oscillation is independent of the initial conditions, as $\Omega = \sqrt{\frac{g}{l}}$. If we vary the initial angular velocity(ω_0) of the system, the graphs would appear shifted, but the frequency of oscillations would remain the same.

[4]θ_0 is varied and $\omega_0 = 0$ is constant

[5]For comparison of initial conditions, a finer time step of 0.001s is used.

3.2.2 Phase Space Diagram

Another way of representing the system is to plot the values of ω as a function of θ. In this graph, each point represents a unique state of the system. Starting from any point would evolve the system forward deterministically irrespective of how the system reached that point.

In this case, we start from the initial conditions[6] [7], and observe that the phase-space diagram(Fig: 3.4) for the simple linear pendulum is an ellipse. This means that the system is trapped in a loop and it is periodic.

We can also see that as the Euler Method fails to conserve energy, the graph spirals outward, indicating increase in total energy.

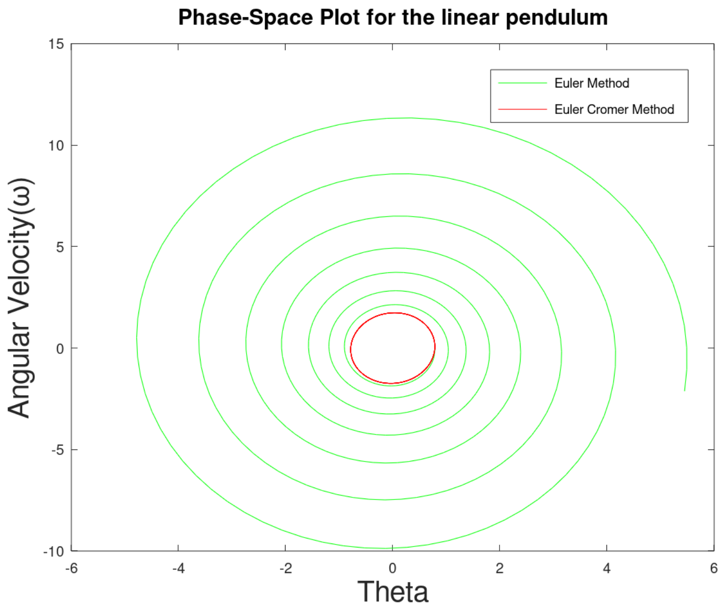


Figure 3.4: Phase-Space Diagram for the Simple Pendulum.

[6]Initial conditions for the simulation :: $\theta_0 = \frac{\pi}{4}$, $\omega_0 = 0$, $t_0 = 0$

[7]The time step used for this simulation is 0.04s

Comparison of different initial conditions.

As the system is linear, the variations in initial conditions[8] [9]will not affect overall symmetry of the phase-space plot, only the size of the loop changes, this can be seen from the graph(Fig: 3.5)

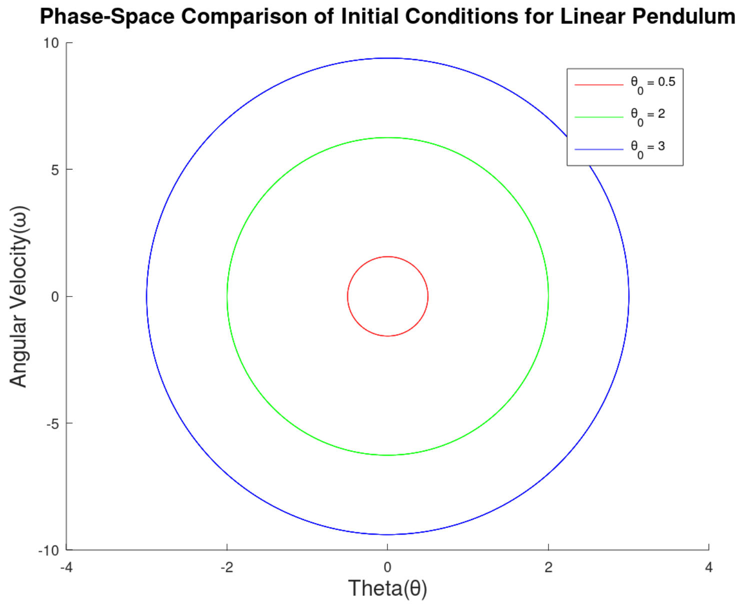


Figure 3.5: Phase-Space Comparison of Different Initial conditions for a linear pendulum.

3.3 The Non-Linear Simple Pendulum

If we remove our approximation $\sin\theta \approx \theta$, then the system becomes non-linear. This nonlinearity causes the system to be heavily dependent on initial conditions. We see this from the comparison graphs(Fig: 3.6 and Fig: 3.7) of the Non-Linear Pendulum[10], where we vary the initial conditions of the system.

[8] θ_0 is varied and $\omega_0 = 0$ is constant

[9] For comparison of initial conditions, a finer time step of 0.001s is used.

[10] For comparison of initial conditions, a finer time step of 0.001s is used.

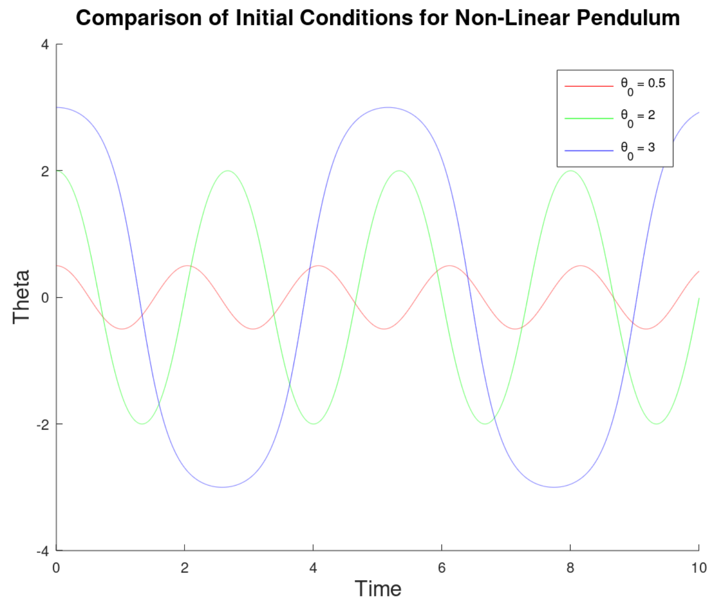


Figure 3.6: Comparison of Different Initial conditions for a Non-linear pendulum.

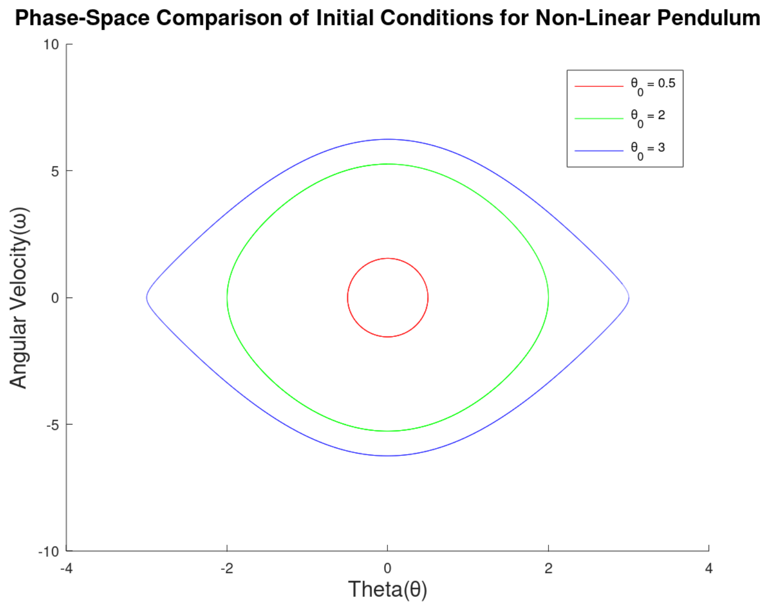


Figure 3.7: Phase-Space Comparison of Different Initial conditions for a non-linear pendulum.

In this, we see that even slight variations in the initial conditions can cause considerable deviations in the system's trajectory. It causes shifts in the frequency of oscillation as well as distorts the symmetry of the phase-space diagram. However, as the system is is still under no external influence, the system does not go into chaos. In order to study chaos, we introduce factors such as damping and driving forces into the non-linear system. The simple non-linear pendulum can be considered a subclass of the physical pendulum where the driving and damping forces are 0.

3.4 The Physical Pendulum

In order to make a more realistic version of the pendulum, we consider a pendulum that is driven by an external driving force, and undergoes dissipation(damping). The equation of motion of such a pendulum can be written as,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} - q\frac{d\theta}{dt} + F_D \sin(\Omega_D t)$$

Where, $q\frac{d\theta}{dt}$ is the damping force which is proportional to velocity(like friction) with a damping constant q. $F_D \sin(\Omega_D t)$ is the driving force with F_D being the amplitude of the force and Ω_D the driving frequency.

For the ease of plotting and to study the evolution of the system in a convenient manner, we can constrain the value of θ between $-\pi$ and π. Thus when the case of the pendulum swinging around the pivot arises, we consider it as a reset rather than as a continuous motion. Including the constrain to θ the algorithm using the Euler-Cromer method to solve this equation numerically can be written as,

$$\omega_{i+1} = \omega_i + [-(g/l)\sin\theta_i - q\omega_i + F_D \sin(\Omega_D t)]\Delta t,$$

$$\theta_{i+1} = \theta_i + \omega_{i+1}\Delta t$$

$$\theta_{i+1} \pm 2\pi$$

$$t_{i+1} = t_i + \Delta t$$

We use this algorithm to calculate the values of θ and ω with progress in time[11] [12].

We can study how the system behaves as we change the driving force by observing their space-time and phase-space diagrams.

In order to study the variations in the system we fix all the quantities except for the value of the driving force F_D. i.e. $q = 0.5$, $l = g = 9.8$, $\Omega_D = 2/3, dt = 0.04$ and initial conditions as $\theta_0 = 0.2$ and $\omega_0 = 0$

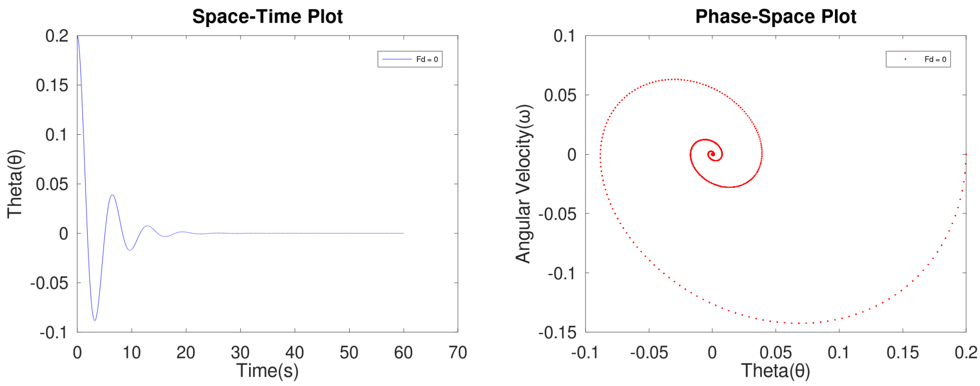
3.4.1 Damped Oscillations, $F_D = 0$

We begin our analysis with the case where, $F_D = 0$. This means that the only forces acting on the pendulum is gravity and the damping force. Hence the system continuously loses energy and eventually comes to rest.

[*] $\theta_{i+1} \pm 2\pi$ such that :: $-\pi \leq \theta i + 1 \leq \pi$

[11] For solving the equations faster, we use FORTRAN.

[12] We use Gnuplot and GNU Octave/MATLAB to plot the graphs



(a) Space Time Graph.

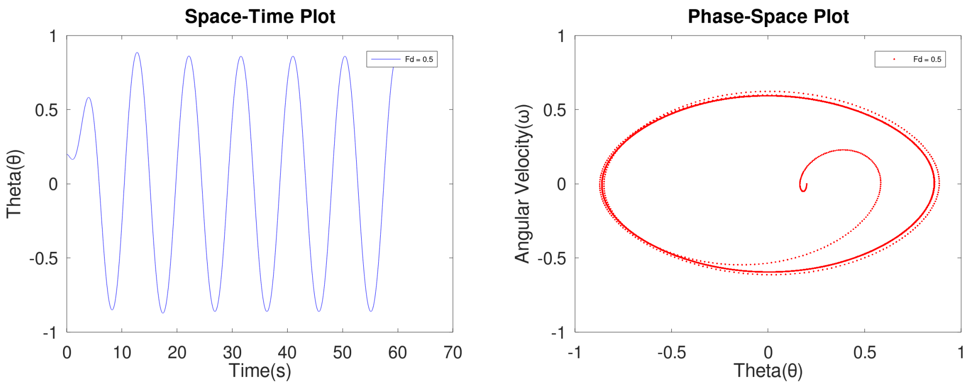
(b) Phase Space Diagram.

Figure 3.8: Graphs representing the Damped Oscillating System, $F_D = 0$

Looking at the Space-Time(Fig: 3.8a) and Phase-Space diagrams(Fig: 3.8b), we can observe the motion of the pendulum. The Space-Time graph shows the decrease in amplitude of the pendulum, which eventually comes to 0. The Phase-Space graph shows an inward spiral which also comes to rest at 0.

3.4.2 Stable, Driven Oscillations, $F_D = 0.5$

The next case is that of a Driven Oscillator, The system initially takes some time to overcome the transient state that depends on the initial condition. this is soon settles into a regular orbit. This final orbit is independent of the initial conditions.



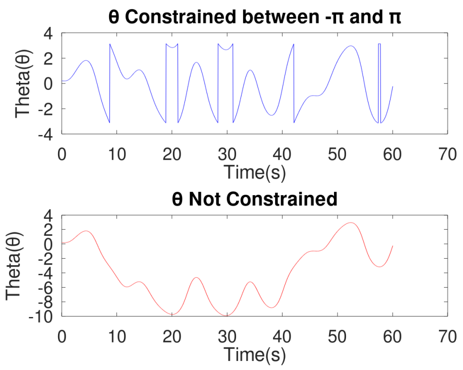
(a) Space Time Graph. (b) Phase Space Diagram.

Figure 3.9: Graphs representing the Stable Driven Oscillating System, $F_D = 0.5$

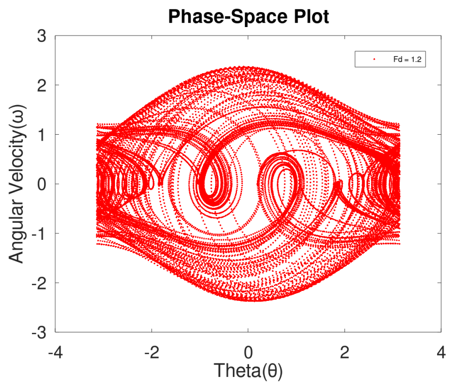
The Space-Time graph(Fig: 3.9a) shows clearly how the amplitude initially varies during the transient stage and then eventually settles into a regular oscillation with a fixed amplitude. The Phase-Space graph(Fig: 3.9b) is an outward spiral initially which settles into a fixed loop.

3.4.3 Chaotic Oscillations, $F_D = 1.2$

The most interesting case is that of the chaotic system In which we cannot determine the trend in which the pendulum oscillates. The trajectory is perfectly deterministic but unpredictable. Given information about any state, it is possible to determine the exact path that the system would follow. However, it is impossible to predict the future without calculating all the steps.



(a) Space Time Graph.



(b) Phase Space Diagram.

Figure 3.10: Graphs representing the Chaotic Oscillating System, $F_D = 1.2$

From the Space-Time graph(Fig: 3.10a), it is pretty evident that we cannot specify a pattern that the system follows. It might seem that it is random, but looking at the Phase-Space diagram(Fig: 3.10b), we can see that there, in fact, is a pattern followed by the system. Every Initial condition will eventually flow towards the surface that we can see in the graph.

3.4.4 Poincarè Map

The Phase-Space Gives us a magnificent structure and a pattern where the possible trajectories of the system lie. However, if we can reduce the number of points we draw on the Phase-Space diagram and plot only those points that might be more significant, we can see a much better pattern. To do this, we plot all points of the Phase-Space diagram that are in phase with the driving force. i.e we plot all points where, $\Omega_D t = 2n\pi$, where n is an integer. This special plot is called the Poincarè Section of the Phase-Space diagram, also called the Poincarè Map of the system.

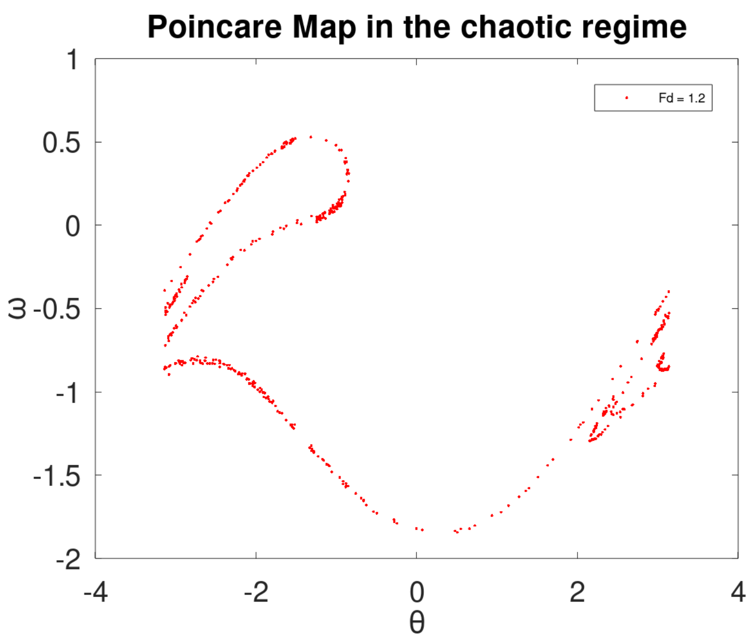


Figure 3.11: Poincarè Section of the Phase-Space Diagram for $F_D = 1.2$.

This means that if we are looking at a periodic system, then the Poincarè map would be a point. If the system is oscillating with a period twice to that of the driving force, then we would see 2 points, and if the system is chaotic, we would see the points being attracted to a specific surface of the Phase-Space. Hence The Poincarè map for the chaotic system is called its attractor.

3.5 Period Doubling - The Route To Chaos

We know that the system is periodic when the driving force is low and is chaotic at some higher value of the driving force. This means that there is some transition that is taking place from Periodic to Chaotic motion.

We saw that from $F_D = 0$ to $F_D = 1.2$ The system undergoes one of these transitions. But to better understand this process, we can look at the case from $F_D = 1.35$ to $F_D = 1.485$. We can start by observing the Space-Time Graphs at different values of F_D[13] [14].

3.5.1 Space Time Diagrams

We Start by observing the Space-Time diagram at $F_D = 1.35$, where The system is periodic. The period of the driving force is 3π as $\Omega = 2/3$. We can see that the pendulum oscillates with the same frequency as that of the driving force. We can call this Period-1(Fig: 3.12).

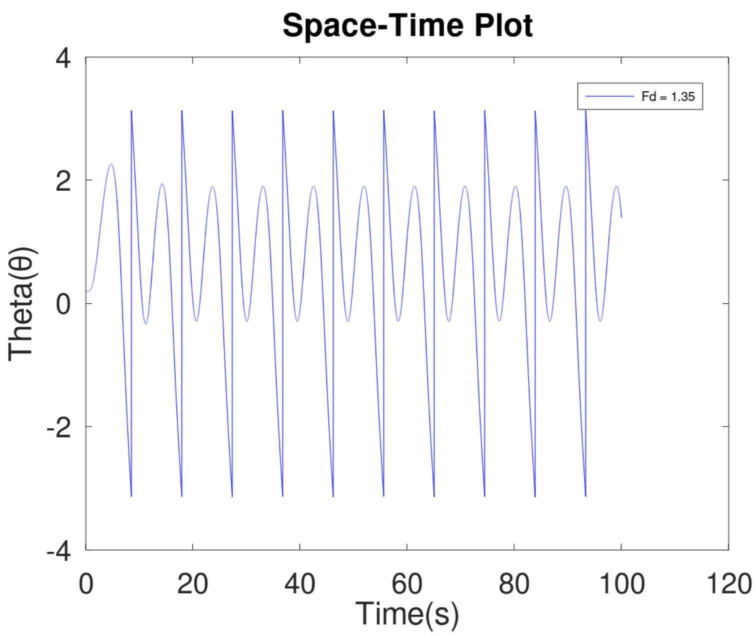


Figure 3.12: Period-1

Next, we can look at a higher value of the driving force, $F_D = 1.44$. Here, we can see that the time period of the pendulum is twice that of the driving force. We call this Period-2(Fig: 3.13)

At an even higher value of $F_D = 1.465$, we can see that the period of

[13]The time step used for simulating the plots is 0.01s
[14]Other parameters used to calculate these plots are as same as before.

oscillation of the pendulum is four times that of the driving force. We call this Period-4(Fig: 3.14).

From These Figures(Fig: 3.12, 3.13, 3.14), it is clear as we increase the value of the driving force, the period is doubling at every tum and then it turns into chaos. If we observe closely, the driving forces at Period-2 and Period-4 are closer than Period-1 and Period-2. That is the distance between the doubling decreases.

The decrease in distance between doublings and the increase in the number of periods together cause what we call chaos, where we can no longer find order within the system.

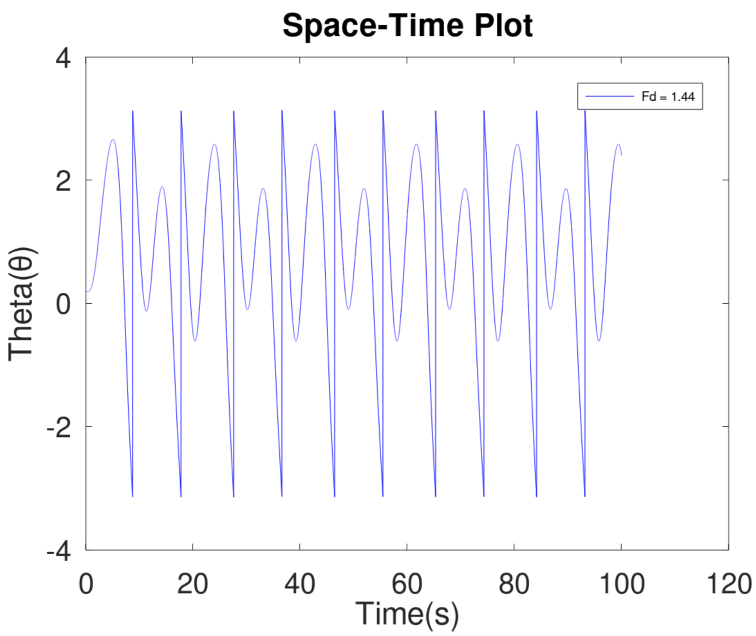


Figure 3.13: Period-2

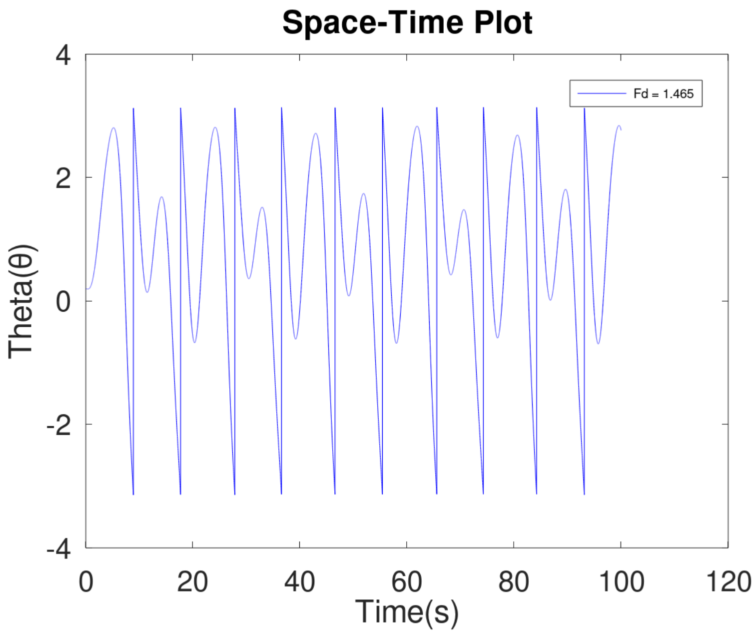


Figure 3.14: Period-3

3.5.2 The Bifurcation Diagram

Plotting a bifurcation diagram is a clear way to show this period-doubling and transition from order to chaos.

The Bifurcation diagram is a plot of θ as a function of F_D. The values of θ that are in period with the driving force are plotted corresponding to the driving force.

As discussed in the case of a periodic Poincarè map, we will get the points corresponding to the values of theta in the way that it oscillates. i.e., for a period-2 system, we will get two values of θ, and for a period-4 system, we will get four values of θ. thus we can show how the doubling/branching of the system occurs.

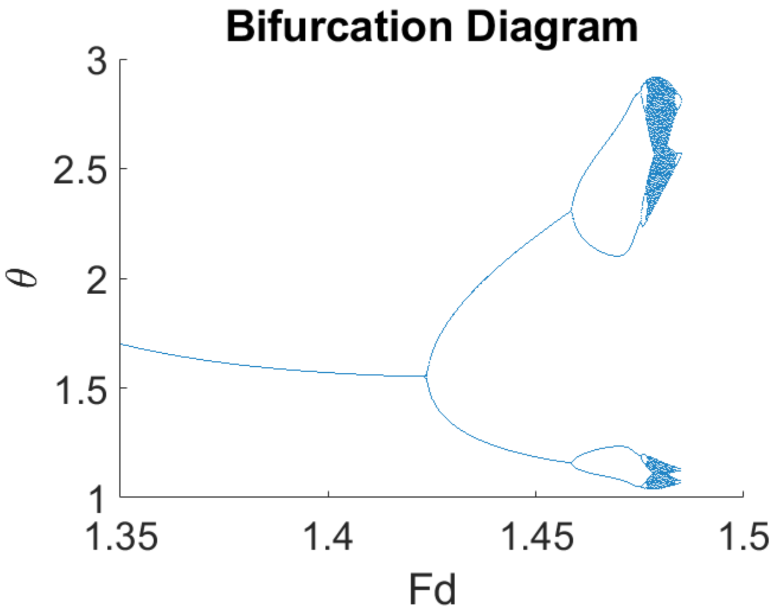


Figure 3.15: Bifurcation Diagram for the Physical Pendulum.

From the bifurcation diagram(Fig: 3.15)[15 16], we can see how the diagram forks into two, and then into four and so on, eventually, the spacing between 2 forking F_D decreases beyond what is resolvable and the number of periods increases exponentially and overlap with each other. This is where chaos originates in the system.

3.5.3 The Feigenbaum(δ)

The bifurcation diagram gives us a qualitative understanding of the transition to chaos. We had discussed that the spacing between period-doubling transitions becomes smaller as we approach chaos.

We define F_n as the driving force where the transition to period-2^n takes place.

[15]In order to have a clear bifurcation diagram, the time step(dt) was selected such that, every period of the driving force is 1200 time steps. i.e., $dt = \dfrac{2\pi}{1200 \times \Omega_D} = 0.00785$

[16]An F_d step of 0.00005 is used for the x-step.

Now we define a parameter δ_n as the measure of shrinkage of the size of the periodic window, i.e.

$$\delta_n = \frac{F_n - F_{n-1}}{F_{n+1} - F_n}$$

It has been found that as $n \to \infty$, δ converges to a value $\delta \approx 4.669$.[8] this δ is called the Feigenbaum. This is a universal parameter that is linked with the transition to chaos.

Chapter 4

Chaotic Ratchet

4.1 Introduction

A Ratchet is a device that allows continuous motion in only one direction. A ratchet is mechanically build using gears(Fig: 4.1) which are not cut symmetrically, and this allows a spring-loaded finger called a pawl to fall onto the gear, as the tooth of the gear is not symmetric, it allows effortless motion in one direction and massive resistance in the other direction. In the figure: 4.1, the gear can rotate counter-clockwise, but not clockwise due to the ratchet mechanism.

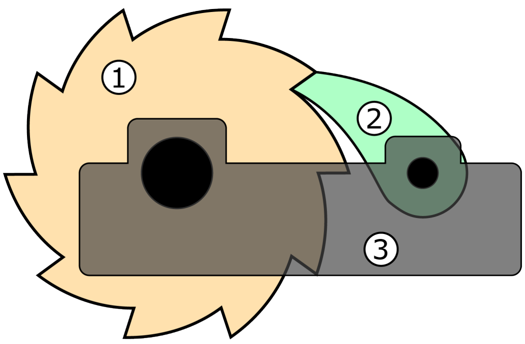


Figure 4.1: A Mechanical Ratchet[9], (1:Gear 2:Pawl 3:Mounting Base).

In our case, instead of using mechanical gears and pawls, we will be considering a one-dimensional potential field. A particle in this potential field will tend to move towards minimizing the energy stored by it. We can apply a driving force to push and pull the particle such that the force, along with force due to the potential, will cause the particle to undergo a motion. As the ratchet potential will not be symmetric, the system should only move in one direction. However, it is seen that the particle also moves in the reverse direction. i.e., with a reverse current.

In order to study this Chaotic Ratchet system, we start by defining the ratchet potential[10]

4.2 The Ratchet Potential

The Asymmetric Periodic Ratchet Potential $V(x)$ is defined as,

$$V(x) = V_1 - V_0 \sin \frac{2\pi(x - x_0)}{L} - \frac{V_0}{4} \sin \frac{4\pi(x - x_0)}{L}$$

Where, L is the periodicity of the potential, V_0 is the amplitude, and V_1 is an arbitrary constant. x_0 is the shift applied to the potential such that the minimum potential is located at the origin.

4.3 Particle in the Ratchet Potential

Let us now consider a particle which is driven by a periodic, external force whose time average is zero. Under the influence of the asymmetric ratchet potential we defined the equation of motion of the particle can be derived as,

$$m\ddot{x} + \gamma\dot{x} + \frac{dV(x)}{dx} = F_0 \cos(\omega_D t)$$

Where, m is the mass of the particle, γ is the friction coefficient, $V(x)$ is the ratchet potential, F_0 is the amplitude of the external driving force, and ω is the frequency of the external driving force.[10]

4.4 Dimensionless equation

In order to efficiently solve the equation, it is preferable to convert the equation into a dimensionless form and simplify the equation[10].

We define the dimensionless units, $x' = x/L$, $x'_o = x_0/L$, $t' = \omega_0 t$, $w = \omega_D/\omega_0$, $b = \gamma/m\omega_0$, $a - F_0/mL\omega_0^2$, where $\omega_0^2 = 4\pi^2 V_0\delta/mL^2$, and δ is defined as $\delta = \sin(2\pi|x'_0|) + \sin(4\pi|x'_0|)$

Now that we have our dimensionless variables, we replace them with the equation of motion and remove the primes for simplicity.

Then, the equation of motion becomes,

$$\ddot{x} + b\dot{x} + \frac{dV(x)}{dx} = a\cos(\omega t)$$

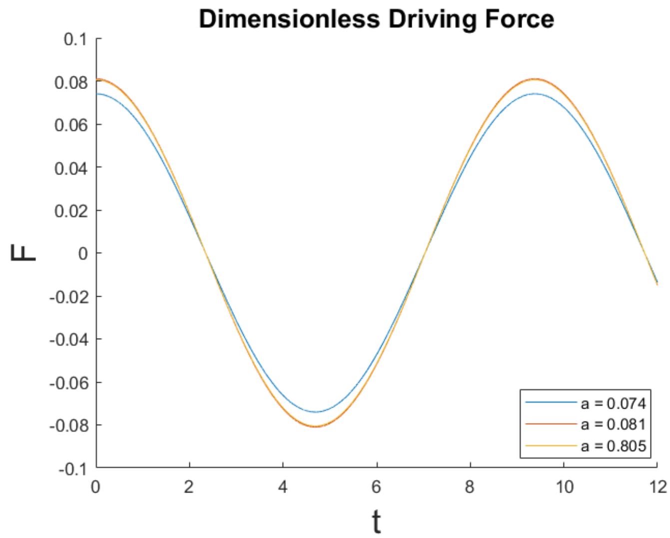


Figure 4.2: Time Varying Driving Force.

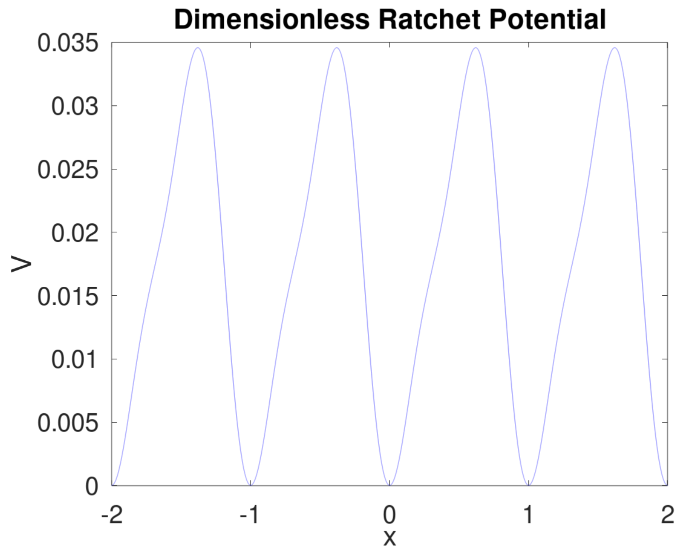


Figure 4.3: Ratchet Potential.

where the dimensionless potential(Fig: 4.3) is given by[1],

$$V(x) = C - [\sin 2\pi(x - x_0) + 0.25 \sin 4\pi(x - x_0)]/4\pi^2\delta$$

Where, C is a constant such that $V(0) = 0$.

$\therefore C = -(\sin 2\pi x_0 + 0.25 \sin 4\pi x_0)/4\pi^2\delta$.

$x_0 \simeq -0.19$, $\delta \simeq 1.6$, $C \simeq 0.0173$

4.5 Solving the Equation

Now that we have all the necessary data to solve the equation of motion, We can use Runge-Kutta or Velocity-Verlet and solve the equations.

The dimensionless equation of motion of the system has 3 parameters, a, b, and w. for our study, we vary the parameter a and fix $b = 0.1$ and $w = 0.67$.

The equation can be converted into a system of equations and then solved using the RK-4 Method. We also use the derivative of the potential function within the equation to not increase complications. the system of equations can be written as,

$$\frac{dx}{dt} = v \qquad \therefore dx = vdt$$

$$\frac{dv}{dt} = a\cos(\omega t) - bv - \frac{dV(x)}{dx} \qquad \therefore dv = \dot{v}dt$$

Where,

$\dot{v} = a\cos(\omega t) - bv - \dfrac{dV(x)}{dx} = F(t, x, v)$

[1] For simulating the graphs, a dimensionless distance step of 0.0001 was used for the Ratchet Potential, and a time step of 0.001s was used for the driving force. All other parameters are exactly as specified in the report.

$$\frac{dV(x)}{dx} = \frac{2\cos(2\pi(x - x0) + \cos(4\pi(x - x0)))}{4\pi\delta}$$

Now, we can use the RK-4 algorithm to solve this system using a step size of h as follows.

4.5.1 The Runge-Kutta(4^{th} order) Algorithm

$$
\begin{aligned}
dx_1 &= hv & dv_1 &= hF(t, x, v) \\
dx_2 &= h(v + \frac{dv_1}{2}) & dv_2 &= hF(t + \frac{h}{2}, x + \frac{dx_1}{2}, v + \frac{dv_1}{2}) \\
dx_3 &= h(v + \frac{dv_2}{2}) & dv_3 &= hF(t + \frac{h}{2}, x + \frac{dx_2}{2}, v + \frac{dv_2}{2}) \\
dx_4 &= h(v + dv_3) & dv_4 &= hF(t + h, x + dx_3, v + dv_3) \\
dx &= \frac{dx_1 + 2dx_2 + 2dx_3 + dx_4}{6} & dv &= \frac{dv_1 + 2dv_2 + 2dv_3 + dv_4}{6} \\
x(t + h) &= x(t) + dx & v(t + h) &= v(t) + dv
\end{aligned}
$$

This can now be repeated to calculate the values of x and v at any t.

4.6 The Bifurcation Diagram

As in the case of the physical pendulum, we can draw v, as a function of a, to show how the period doubles and view the bifurcation diagram. This bifurcation diagram helps us to select the points for further observation by providing information about the periodicity of the particle in the ratchet.

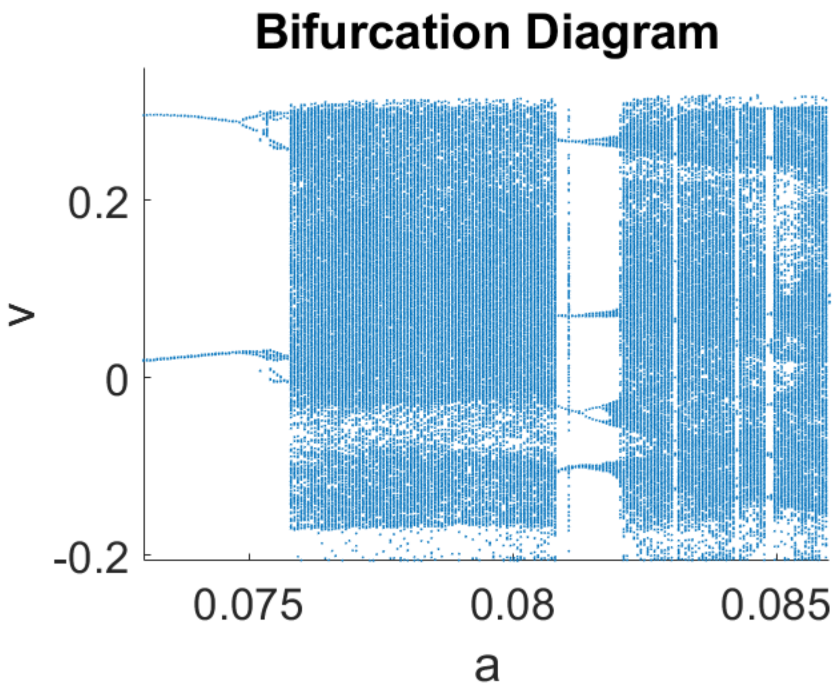


Figure 4.4: Bifurcation Diagram of the particle in the chaotic ratchet.

from the bifurcation diagram(Fig: 3.15)[2] [3] it is clear that the system is in its period-2 form at 0.074, and in its period-4 form at 0.081.

4.7 Space Time Diagrams

In order to study specific states of the system, we can plot the Space-Time graph[4] of the particle in the ratchet at different driving forces.

[2]The time step used for the simulation of the bifurcation diagram is 0.01s

[3]The a step used for the bifurcation diagram is 0.000065 in order to obtain 200 values of a between 0.073 and 0.086

[4]The time step used for the simulation of the space-time plots is 0.001s.

4.7.1 Positive Current. $a = 0.074$

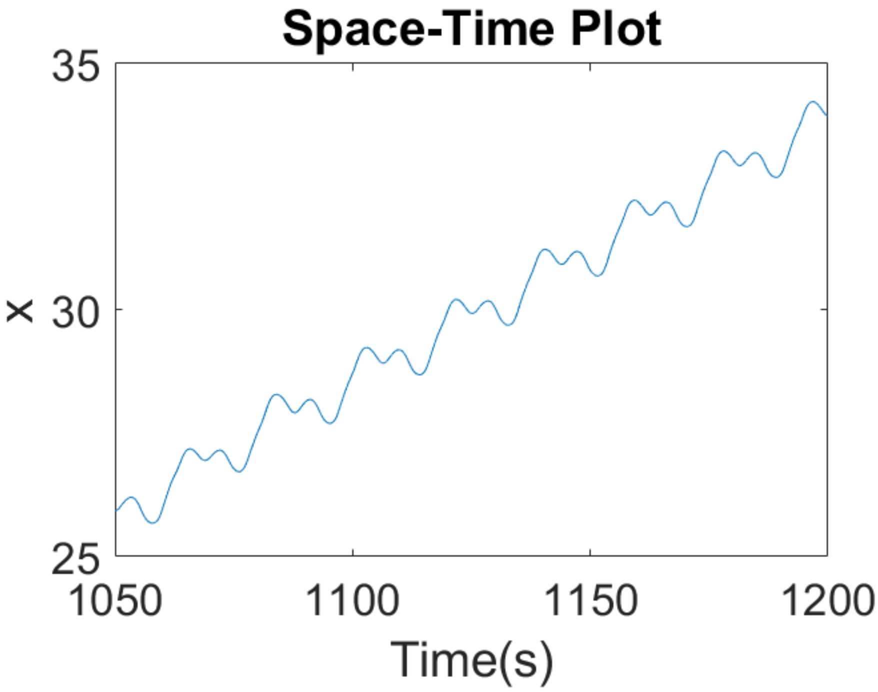


Figure 4.5: Space-Time diagram of the system showing positive current.

We can see that when the system is in the period-2 region, the particle has a positive current. This is something that we can intuitively understand. The oscillation helps the particle move forward in the ratchet.

4.7.2 Negative Current. $a = 0.081$

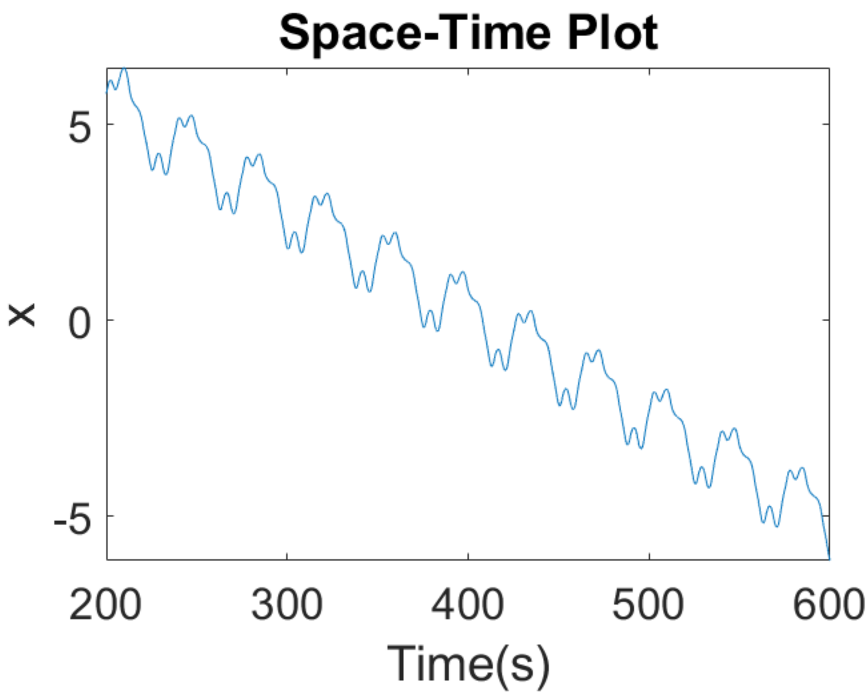


Figure 4.6: Space-Time diagram of the system showing negative current.

It is the Negative current that is counterintuitive. However, upon finer inspection, we can see that it is the 4-period that helps the particle climb in the negative direction. The particle requires twice the time to cross a potential well. In order to advance one step towards the left, the particle moves one step to the right and then two steps to the left. Thus, the net current is negative.

4.7.3 Chaotic Regime. $a = 0.0805$

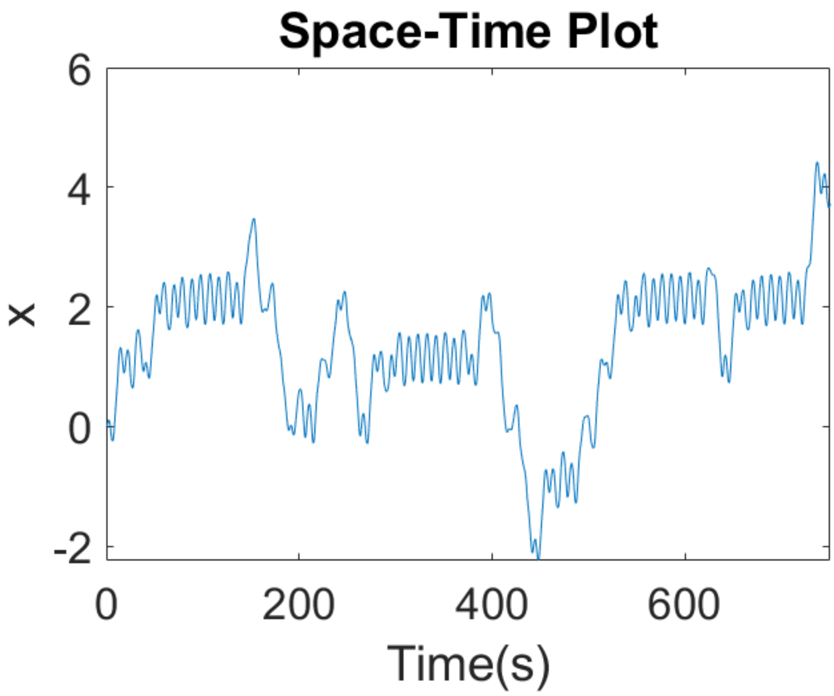


Figure 4.7: Space-Time diagram of the system showing Chaos.

As expected, the chaotic regime does not show any periodicity. Although at times it may appear that the system has settled into an orbit, it breaks out of it and continues its motion. We can understand better about the chaotic regime by plotting the poincarè map.

4.8 Poincarè Map

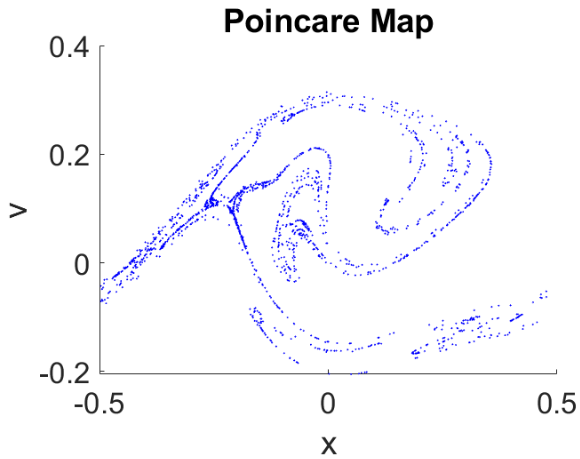


Figure 4.8: Poincarè Map showing the Chaotic Attractor at $a = 0.0805$.

The Poincarè Map[5] gives us a plot of the chaotic attractor(Fig: 4.8) for the system. This attractor, along with the periodic attractors(Fig: 4.9) at the Period-4 and Period-2 regimes, helps us understand how the chaotic ratchet system works.

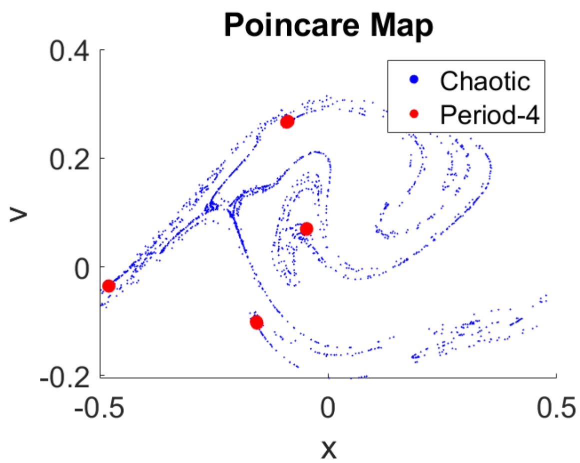


Figure 4.9: Poincarè Map showing the Chaotic($a = 0.0805$) and the Period-4 Attractor($a = 0.081$).

[5]The simulation of the poincarè map was done with a time step of 0.01s

We can see that The periodic attractor lies on top of the chaotic attractor at points where it forms closed loops. It is also observable that the period-4 attractor is in the vicinity of the chaotic regime such that any disturbance would send it into a chaotic trajectory.

Chapter 5

Results and discussions

During this study, we were able to understand different representations of systems and their interpretations. The simulation and interpretation of the Physical pendulum help us develop the basic intuition needed to interpret more complex systems. We explored different methods of numerically solving differential equations and understood the necessity of each method according to the circumstances.

We appreciate the concept of non-linearity, which allows the system to take on chaotic trajectories. We were also able to see that there is order in chaos. Moreover, even though we cannot predict what trajectory the system would be taking, we can still predict the sense of its behavior.

The next stage of analysis is to identify patterns in the data distribution within a bifurcation diagram and determine why some values of the phase of space occur more often than others.

The study of Non-linearity and chaos and the ability to create accurate predictions are necessary as the world we live in is non-linear. Understanding such systems would help us efficiently solve weather predictions, encryption, understanding ECGs, etc., efficiently.

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