Variation

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1 The principle of least action

Let the system occupy positions defined by two sets of values of the coordinates, q_1 and q_2, at the instants t_1 and t_2. The *action* of the system takes the form of integral of its *Lagrangian* as

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,\mathrm{d}t \tag{1}$$

The necessary condition for S to have an extremum is that the variation δS should be zero:

$$\begin{aligned} 0 = \delta S &= \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\,\mathrm{d}t \\ &= \delta \int_{t_1}^{t_2} (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q})\,\mathrm{d}t \end{aligned} \tag{2}$$

Since $\delta \dot{q} = \mathrm{d}\delta q / \mathrm{d}t$, we obtain, on integrating the second term by parts,

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} (\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}})\delta q\,\mathrm{d}t = 0 \tag{3}$$

For $t = t_1$ and $t = t_2$, the values of $q(t)$ should be fixed, i.e.

$$\delta q(t_1) = \delta q(t_2) = 0 \tag{4}$$

Thus the first term in (3) vanishes, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{5}$$

Considering the degrees of the system, we then obtain s equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, 2, ..., s). \tag{6}$$

These are called *Euler-Lagrange's equations* in mechanics.

2 Gauge invariance

Considering two functions $L'(q, \dot{q}, t)$ and $L(q, \dot{q}, t)$, differing by the total derivative with respect to time of some function $f(q, t)$ as

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{\mathrm{d}}{\mathrm{d}t} f(q, t) \tag{7}$$

The integrals are

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t)\,\mathrm{d}t = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,\mathrm{d}t + \int_{t_1}^{t_2} \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = S + f(q_2, t_2) - f(q_1, t_1)$$

Where $f(q_2, t_2)$ and $f(q_1, t_1)$ are fixed values, i.e. $\delta f(q_2, t_2) = \delta f(q_1, t_1) = 0$. This leads to

$$\delta S' = \delta S \tag{8}$$

So the conditions $\delta S' = 0$ and $\delta S = 0$ are equivalent.

3 Law of inertia

A frame of reference in which space is homogeneous and isotropic and time is homogeneous is called an *inertial frame*.

Considering a particle moving freely in such a frame. The homogeneity of space and time implies that the Lagrangian cannot contain explicitly either the radius vecter $\boldsymbol{r}$ or the time t, i.e. L must be a function of the velocity $\boldsymbol{v}$ only; Since space is isotropic, the Lagrangian must also be independent of the direction of $\boldsymbol{v}$, and is therefore a function only of its magnitude, i.e. of $\boldsymbol{v}^2 = v^2$:

$$L = L(v^2) \tag{9}$$

Bringing $\partial L/\partial \boldsymbol{r} = 0$ into Lagrange's equation (5), we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{v}}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) \tag{10}$$

whence $\partial L/\partial \boldsymbol{v}$ =constant. Since $\partial L/\partial \boldsymbol{v}$ is a function of the velocity only, it follows that

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = 0, \quad \text{or} \quad \boldsymbol{v} = \text{constant}. \tag{11}$$

Thus we conclude that, in an inertial frame, any free motion takes place with a velocity which is constant in both magnitude and direction. This is the *law of inertia*.