Notes on The Theoretical Minimum — Classical Mechanics

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The Euler-Lagrangian equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}. \tag{1}$$

If the Lagragian L remains invariant under the infinitesimal change of coordinates,

$$q_i \to q_i' = q_i + \epsilon f_i(q) \quad \text{or} \quad \delta q_i = \epsilon f_i(q), \tag{2}$$

we have

$$\begin{aligned} 0 = \delta L(q_i, \dot{q}_i) &= \sum_i \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right] \\ &= \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta q_i\right)\right] \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right] \\ &= \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i p_i f_i(q)\right] \\ &\equiv \epsilon\frac{\mathrm{d}Q}{\mathrm{d}t}, \end{aligned} \tag{3}$$

where in the second row we used the Euler-Lagrangian equation, and thus the quantity $Q \equiv \sum_i p_i f_i(q)$ is conserved. This is the Noether's Theorem, and we restate the mathematical form as follows:

If the Lagrangian is invariant ($\delta L = 0$) under the transformations $\delta q_i = \epsilon f_i(q)$, then the charge $Q = \sum_i p_i f_i(q)$ is a conserved quantity.

The Hamiltonian is defined as

$$H := \sum_i p_i \dot{q}_i - L = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L, \tag{4}$$

and the Hamiltonian equations are given by

$$\begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \\ \dot{q}_i = \dfrac{\partial H}{\partial p_i}. \end{cases} \tag{5}$$

The time derivative of a quantity $F(q_i, p_i)$