

Notes on Statistical Field Theory

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1 Thermodynamics & Statistical Mechanics

Closed system, Volume V , in thermodynamic equilibrium

Conserved quantities: energy E , particle no. N .

Comments: a) particle and anti-particle pairs can be created out of vacuum, only net particle no. or quantum numbers (e.g. net electric charge $N = N^+ - N^-$) are conserved.

b) different particle species, different conserved quantum numbers, N_1, N_2, \dots, N_M .

Thermodynamics

For closed system of volume V and M different conserved quantum numbers,

$$dE = T dS - p dV + \sum_i^M \mu_i dN_i. \quad (1)$$

Energy is an extensive quantity, it scales with volume V . If $V \rightarrow \alpha V$, then $E(\alpha S, \alpha V, \{\alpha N_i\}) = \alpha E(S, V, \{N_i\})$. Now let $\alpha = 1 + \epsilon$, where $|\epsilon| \ll 1$, we have

$$\begin{aligned} E(S, V, \{N_i\}) + \epsilon E(S, V, \{N_i\}) &= E(S + \epsilon S, V + \epsilon V, \{N_i + \epsilon N_i\}) \\ &= E(S, V, \{N_i\}) + \epsilon \left[\left. \frac{\partial E}{\partial S} \right|_{V, \{N_i\}} S + \left. \frac{\partial E}{\partial V} \right|_{S, \{N_i\}} V + \sum_i \left. \frac{\partial E}{\partial N_i} \right|_{S, V} N_i \right] \\ &= E(S, V, \{N_i\}) + \epsilon \left[TS - pV + \sum_i \mu_i N_i \right], \end{aligned} \quad (2)$$

where in the first line we just plug in $\alpha = 1 + \epsilon$, in the second line we used the Taylor expansion and only kept the first-order term, and in the third line we used the partial thermodynamic relations. Comparing LHS and RHS, we immediately get

$$E(S, V, \{N_i\}) = TS - pV + \sum_i \mu_i N_i. \quad (3)$$

This is the *fundamental relation of thermodynamics*.

By taking its total derivative we have

$$dE = T dS + S dT - p dV - V dp + \sum_i^M (\mu_i dN_i + N_i d\mu_i) \implies 0 = S dT - V dp + \sum_i^M N_i d\mu_i, \quad (4)$$

where we cancel terms due to the first law. This is the *Gibbs-Duham relation*.

Statistical Mechanics

micro-canonical partition function: $\Xi(E, V, \{N_i\})$ = numbers of all possible micro-states of the system.

relation with thermodynamics: $S(E, V, \{N_i\}) = \ln \Xi(E, V, \{N_i\})$

microscopic density of states is

$$\sigma(E, V, \{N_i\}) = \frac{\Xi(E, V, \{N_i\})}{dE} = \sum_{\mathcal{R}}' \delta(E - E_{\mathcal{R}}), \quad (5)$$

where $\frac{\Xi(E, V, \{N_i\})}{dE}$ denotes number of micro-states per energy interval, and \mathcal{R} denotes all micro-states, and prime for given V and $\{N_i\}$.

Calculation of equation of state $S(E, V, \{N_i\})$ is possible if $\Xi(E, V, \{N_i\})$ is known, but generally this is very hard to compute.

Canonical ensemble:
 Legendre transform of E with respect to S :

$$F = E - TS = E - \frac{\partial E}{\partial S} \quad (6)$$

Canonical partition function: Laplace transform of $\sigma(E, V, \{N_i\})$,

$$Z(T, V, \{N_i\}) = \int_0^\infty dE e^{-E/T} \sigma(E, V, \{N_i\}) = \int_0^\infty dE e^{-E/T} \sum_{\mathcal{R}}' \delta(E - E_{\mathcal{R}}) = \sum_{\mathcal{R}}' e^{-E_{\mathcal{R}}/T} \quad (7)$$

relation to thermodynamics: $F(T, V, \{N_i\}) = -T \ln Z(T, V, \{N_i\})$.

Grand canonical ensemble:
 Legendre transform of F with respect to N_i :

$$\Omega = F - \sum_i^M \mu_i N_i = E - TS - \sum_i^M \mu_i N_i = -pV \quad (8)$$

and

$$d\Omega = -S dT - p dV - \sum_i^M N_i d\mu_i \quad (9)$$

where

$$p = -\frac{\partial \Omega}{\partial V} = -\frac{\Omega}{V} \quad (10)$$

which implies $p = p(T, \{\mu_i\})$ does not depend on V .

Grand canonical partition function is the Laplace transform of the canonical partition function,

$$\begin{aligned} Z(T, V, \{\mu_i\}) &= \sum_{N_1} e^{\frac{\mu_1 N_1}{T}} \dots \sum_{N_M} e^{\frac{\mu_M N_M}{T}} Z(T, V, \{N_i\}) \\ &= \sum_{N_1} \dots \sum_{N_M} \sum_{\mathcal{R}}' e^{\sum_{i=1}^M \frac{\mu_i N_i}{T}} e^{-E_{\mathcal{R}}/T} \\ &= \sum_{\mathcal{R}} e^{-\frac{1}{T}(E_{\mathcal{R}} - \sum_{i=1}^M \mu_i N_i)}, \end{aligned} \quad (11)$$

where the unprimed sum denotes no restriction except fixing the volume.
 relation to thermodynamics:

$$\begin{aligned} \Omega(T, V, \{\mu_i\}) &= -T \ln Z(T, V, \{\mu_i\}) \\ p(T, \{\mu_i\}) &= \frac{T}{V} \ln Z(T, V, \{\mu_i\}). \end{aligned} \quad (12)$$

2 Quantum mechanics & Quantum Field Theory

completeness relation:

$$\int \prod_{\vec{x}} d\phi_0(\vec{x}) |\phi_0\rangle \langle \phi_0| = 1 \quad (13)$$

orthogonality:

$$\langle \phi_a | \phi_b \rangle = \prod_{\vec{x}} \delta(\phi_a(\vec{x}) - \phi_b(\vec{x})) \equiv \delta[\phi_a - \phi_b], \quad (14)$$

where the new delta notation is called the “functional δ -function”.

3 Fundamental Integral Representation of the Partition Function

3.1 The Partition Function as Fundamental Integral

3.2 Neutral Scalar Fields

3.3 Charged Scalar Fields: Bose-Einstein Condensation

3.4 Fermion Fields

3.5 Massive Vector Fields

3.6 The Electro-magnetic Fields

4 The Partition Function in Perturbation Theory

4.1 Expansion in Powers of the Interaction

4.2 ϕ^4 Theory

4.3 Yukawa Theory

4.4 QED

4.5 QCD

5 Resummation Techniques

5.1 IR Divergences & Breakdown of Naive Perturbation Series

5.2 1PI Effective Action

5.3 2PI Effective Action (C-J-T Formalism)

5.4 Restoration of Spontaneously ?? Symmetries

5.5 Symmetry Restoration in H&H-F Approximation

5.6 Fermions: the N-J-L Model

5.7 Superfluidity & Superconductivity

5.8 Gauge Theories: Hard Thermal Loop & Hard Dense Loop (Random Phase Approximation)

5.9 The Function Renormalization Group