Schwarzschild tunneling with HJ equation in PG-like coordinates

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Abstract

Firstly I derive HJ equation from Lagrangian, then put it in the PG-like coordinates to calculate the emission rate of Hawking radiation. We will find some convenience of using this coordinate system during the process.

1 From Lagrangian to HJ equation

The Lagrangian of a free particle in a curved spacetime is given as

$$L = \frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \ . \tag{1}$$

It's easy to compute the covariant four-momentum

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = m g_{\mu\nu} \dot{x}^\nu, \tag{2}$$

So we can get the Hamiltonian* of the particle

$$H = P_\mu \dot{x}^\mu - L = \frac{m}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2m} g^{\mu\nu} P_\mu P_\nu \ . \tag{3}$$

The classical HJ equation can be generalized into the curved spacetime as

$$\frac{\partial S}{\partial \tau} + H(\dot{x}^\mu, \frac{\partial S}{\partial x^\mu}) = 0 \tag{4}$$

where τ denotes the proper time of the freely moving particle,and $\frac{\partial S}{\partial x^\mu}$ is a simple substitution of P_μ.
Thus,

$$\frac{\partial S}{\partial \tau} + \frac{1}{2m} g^{\mu\nu} (\frac{\partial S}{\partial x^\mu})(\frac{\partial S}{\partial x^\nu}) = 0 \tag{5}$$

*We should reexpress the velocity in terms of the momentum as $\dot{x}^\mu = \frac{1}{m} g^{\mu\nu} P_\nu$.

Note that due to the normalization condition, the Hamilton $H = -m/2$ is a conservative quantity itself, the equation above can be separated into

$$\begin{cases} \frac{1}{2m} g^{\mu\nu} (\frac{\partial S}{\partial x^\mu})(\frac{\partial S}{\partial x^\nu}) + \frac{m}{2} = 0 \\ \frac{\partial S}{\partial \tau} = \frac{m}{2} \end{cases} \tag{6}$$

Generally, the proper time τ of a particle does not apparently show up in a coordinate system, so we neglect the second separated equation; moreover, in tunneling process of Hawking radiation, we care only about the imaginary part of the action S, while the second equation just contributes a real factor.
The first equation can be rearranged like

$$g^{\mu\nu} (\frac{\partial S}{\partial x^\mu})(\frac{\partial S}{\partial x^\nu}) + m^2 = 0 \tag{7}$$

or a simpler form

$$\boxed{\boldsymbol{\partial^\mu S \ \partial_\mu S + m^2 = 0}} \tag{8}$$

This is the standard Hamilton-Jaccobi equation of a free particle with mass m.

2 Schwarzschild tunneling analysis in PG-like coordinates

The line element of a Schwarzschild metric $g_{\mu\nu}$ is expressed as

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{9}$$

where $f = 1 - 2M/r$, and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$.
When transformed into the PG-like coordinate family, it becomes

$$\mathrm{d}s^2 = -f\mathrm{d}T^2 + 2\sqrt{1-pf}\mathrm{d}T\mathrm{d}r + p\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{10}$$

or

$$\mathrm{d}s^2 = -\frac{1}{p}\mathrm{d}T^2 + p(\mathrm{d}r^2 + \frac{1}{p}\sqrt{1-pf})^2 + r^2\mathrm{d}\Omega^2, \tag{11}$$

the transformation relationship is

$$\mathrm{d}T = \mathrm{d}t + \frac{\sqrt{1-pf}}{f}\mathrm{d}r \tag{12}$$

where the single parameter p symbolizing the coordinate choosing, is defined as $p = 1/\tilde{E}^2$, with $\tilde{E}$ representing energy per unit mass of the particle.†
We denote the new form of the metric as $g'_{\mu\nu}$, and get its covariant matrix form,

$$(g'_{\mu\nu}) = \begin{bmatrix} -f & \sqrt{1-pf} & 0 & 0 \\ \sqrt{1-pf} & p & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \tag{13}$$

†More about this coordinate system family, see Ref[?] in detail.

and contravariant form,

$$(g'^{\mu\nu}) = \begin{bmatrix} -p & \sqrt{1-pf} & 0 & 0 \\ \sqrt{1-pf} & f & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix} \tag{14}$$

Taking the contravariant components into the equation (**??**), and considering the case of radial motions, we get the HJ equation in PG-like coordinate family[‡],

$$-p(\frac{\partial S}{\partial T})^2 + 2\sqrt{1-pf}(\frac{\partial S}{\partial T})(\frac{\partial S}{\partial r}) + f(\frac{\partial S}{\partial r})^2 = -m^2. \tag{15}$$

It seems hard to separate the variants, but we will see one of them is a conservative quantity,

$$\begin{aligned} \frac{\partial S}{\partial T} = P_T = \frac{\partial L}{\partial \dot{T}} &= mg'_{TT}\dot{T} + mg'_{Tr}\dot{r} = m(-f\dot{T} + \sqrt{1-pf}\ \dot{r}) \\ &\overset{①}{=} -mf\ \dot{t} \overset{②}{=} -m\tilde{E} \overset{③}{=} -\omega, \end{aligned} \tag{16}$$

the equal sign ① is the result of substituting transformation relationship (**??**) into the equation, the equal sign ② is the expression of $\tilde{E}$ in the original coordinate system,[§] and ω represents the total energy of the particle.
So, equation (**??**) becomes

$$-p\ \omega^2 - 2\omega\sqrt{1-pf}(\frac{\partial S}{\partial r}) + f(\frac{\partial S}{\partial r})^2 = -m^2 \tag{17}$$

Note that the definition of the parameter $p = 1/\tilde{E}^2$ and the equal sign ③ together give us a new relationship as $p\ \omega^2 = m^2$, which makes the equation above even simpler,

$$(\frac{\partial S}{\partial r})(f\frac{\partial S}{\partial r} - 2\omega\sqrt{1-pf}) = 0. \tag{18}$$

Neglecting the trivial solution, we get the partial differentiation of S with respect to r,

$$\partial_r S \equiv \frac{\partial S}{\partial r} = 2\ \omega\frac{\sqrt{1-pf}}{f}. \tag{19}$$

And the imaginary part of the action across the event horizon can be directly computed,[¶]

$$ImS = -i\int_{r_{in}}^{r_{out}} 2\ \omega\frac{\sqrt{1-pf}}{f}\ \mathrm{d}r = 4\pi M \tag{20}$$

[‡] For radial motions, we neglect angular terms for the reason that

$$\begin{cases} \frac{\partial S}{\partial \theta} = P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mg'_{\theta\theta}\dot{\theta} = 0 \\ \frac{\partial S}{\partial \phi} = P_\phi = \frac{\partial L}{\partial \dot{\phi}} = mg'_{\phi\phi}\dot{\phi} = 0. \end{cases}$$

[§] In the original coordinate system, $\tilde{E} = g_{\mu\nu}(\frac{\mathrm{d}x^\mu}{\mathrm{d}t})(\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}) = g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\tau} = f\ \dot{t}$

[¶] This can be achieved by calculating the residue, $Res(\frac{\sqrt{1-pf}}{f})|_{r=2M} = 2\pi M$

where $r_{in} = 2M - \varepsilon$ and $r_{out} = 2M + \varepsilon$, with ε an infinitesimal.
Considering the self-gravitation effect, the result above ought to make such replacements:

$$\begin{cases} \omega \to \int_0^{\omega} \mathrm{d}\omega' \\ M \to M - \omega' \end{cases} \tag{21}$$

Thus, modification should be made as,

$$\begin{aligned} ImS &= 4\pi \int_0^{\omega} (M - \omega')\mathrm{d}\omega' \\ &= 4\pi\omega(M - \frac{\omega}{2}) \end{aligned} \tag{22}$$

According to WKB approximation, the tunneling rate should be like

$$\Gamma \propto e^{-2ImS} \tag{23}$$

So finally we have our result,

$$\Gamma \propto e^{-8\pi\omega(M - \frac{\omega}{2})} \tag{24}$$

which well accorded with previous works, being consistent with the underlying unitary theory.

References

[1] Regular coordinate systems for Schwarzschild and other spherical spacetimes, Karl Martel and Eric Poisson