Notes on The Theoretical Minimum
— Classical Mechanics

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0.1 The Euler-Lagrangian equation

The Euler-Lagrangian equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \tag{1}$$

and the canonical momentum is defined as

$$p_i := \frac{\partial L}{\partial \dot{q}_i}. \tag{2}$$

0.2 The Noether Theorom

If the Lagragian L remains invariant under the infinitesimal change of coordinates,

$$q_i \to q_i' = q_i + \epsilon f_i(q) \quad \text{or} \quad \delta q_i = \epsilon f_i(q), \tag{3}$$

we have

$$\begin{aligned} 0 = \delta L(q_i, \dot{q}_i) &= \sum_i \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right] \\ &= \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta q_i\right)\right] \\ &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right] \\ &= \epsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_i p_i f_i(q)\right] \\ &\equiv \epsilon\frac{\mathrm{d}Q}{\mathrm{d}t}, \end{aligned} \tag{4}$$

where in the second row we used the Euler-Lagrangian equation, and thus the quantity $Q \equiv \sum_i p_i f_i(q)$ is conserved. This is the Noether's Theorem, and we restate the mathematical form as follows:

If the Lagrangian is invariant ($\delta L = 0$) under the transformations $\delta q_i = \epsilon f_i(q)$, then the charge $Q = \sum_i p_i f_i(q)$ is a conserved quantity.

0.3 The Hamiltonian mechanics

The time derivative of the most general Lagrangian $L(q_i, \dot{q}_i, t)$is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}L(q, \dot{q}, t) &= \sum_i \left[\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right] + \frac{\partial L}{\partial t} \\ &= \sum_i \left[\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i\right] + \frac{\partial L}{\partial t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i p_i\dot{q}_i + \frac{\partial L}{\partial t}, \end{aligned} \tag{5}$$

where in the second line we used the Leibniz rule for derivatives, and in the third line we used the Euler-Lagrange equation, and in the fourth line the definition of the canonical momentum. Thus we have

$$-\frac{\partial L}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i p_i\dot{q}_i - L\right) \equiv \frac{\mathrm{d}H}{\mathrm{d}t}, \tag{6}$$

which says that if the Lagrangian does not explicitly contains time, the new quantity H, called the Hamiltonian, is conserved.

We rewrite the definition of the Hamiltonian as

$$H := \sum_i p_i \dot{q}_i - L = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L. \tag{7}$$

The change in the Hamiltonian is

$$\begin{aligned} \delta H &= \sum_i \left(\delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial L}{\partial q_i} \delta q_i - \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) \\ &= \sum_i \left(\dot{q}_i \delta p_i - \frac{\partial L}{\partial q_i} \delta q_i \right) \\ &= \sum_i \left(\dot{q}_i \delta p_i - \dot{p}_i \delta q_i \right), \end{aligned} \tag{8}$$

where the second and the fourth term cancel due to the definition of the canonical momentum, and the third row is due to the Euler-Lagrangian equation.
Generally, the change in the Hamiltonian, treated as a function of multiple variables, is given by

$$\delta H(p_i, q_i) = \sum_i \left(\frac{\partial H}{\partial p_i} \delta p_i + \frac{\partial H}{\partial q_i} \delta q_i \right), \tag{9}$$

by matching the terms in the above two equations, we arrive at

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}, \end{cases} \tag{10}$$

this is the Hamilton equaitons.

0.4 The Poisson Brackets

The time derivative of a quantity $F(q_i, p_i)$ is given by

$$\begin{aligned} \dot{F}(q_i, p_i) &= \sum_i \left(\frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i \right) \\ &= \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\ &\equiv \{F, H\}, \end{aligned} \tag{11}$$

where in the second row we used the Hamiltonian equations, and the notation in the third row is called the Poisson bracket.

The amazing thing about the above equation is that it summarizes so much. The time derivative of anything is given by the Poisson bracket of that thing with the Hamiltonian.
Firstly it even contains Hamilton's equations themselves,

$$\begin{cases} \dot{q}_i = \{q_i, H\} \\ \dot{p}_i = \{p_i, H\}, \end{cases} \tag{12}$$

Secondly it contains the energy conservation law, only by letting F to be H,

$$\dot{H} = \{H, H\} = 0. \tag{13}$$

Now let's abstract a set of rules that enable one to manipulate PB(Poisson Brackets) without all the effort of explicitly calculating them.
The first property is antisymmetry,

$$\{A, C\} = -\{C, A\}. \tag{14}$$

Second is linearity,

$$\{aA + bB, C\} = a\{A, C\} + b\{B, C\}. \tag{15}$$

Next is a little similar to Leibniz's rule,

$$\{AB, C\} = A\{B, C\} + \{A, C\}B. \tag{16}$$

Finally some specific PBs,

$$\begin{aligned} \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij}. \end{aligned} \tag{17}$$

Then it's easy by mathematical induction to derive an equation,

$$\{q^n, p\} = nq^{n-1} = \frac{\mathrm{d}(q^n)}{\mathrm{d}q}, \tag{18}$$

so that any polynomial P(q) of q will have the following result,

$$\{P(q), p\} = \frac{\mathrm{d}P(q)}{\mathrm{d}q}, \tag{19}$$

and due to any smooth function can be arbitrarily approximated by a polynomial, the above applies to any function $F(q)$ of q. In fact, it even goes further, for any function of q and p, we have

$$\{F(q, p), p_i\} = \frac{\mathrm{d}F(q, p)}{\mathrm{d}q_i}, \tag{20}$$

which is easy to prove by the definition of PB. Thus we have discovered a new fact: *Taking the PB of any function with p_i has the effect of differentiating the function with respect to q_i.*
The same goes with PB with q, except for a minus sign,

$$\{F(q, p), q_i\} = -\frac{\mathrm{d}F(q, p)}{\mathrm{d}p_i}. \tag{21}$$