Green's function in Quantum Scattering Problem

1 Generating Green's function

Schrodinger's equation:

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi \tag{1}$$

ding tai:

$$E\psi = \hat{H}\psi = (\frac{p^2}{2m} + V)\psi = (-\frac{\hbar^2}{2m}\nabla^2 + V)\psi \tag{2}$$

In 3-dimensional coordinate biao xiang

$$(\nabla^2 + k^2)\psi(\vec{r}) = \frac{2\mu}{\hbar^2}V(r)\psi(\vec{r}) \tag{3}$$

The right-hand side:

$$\frac{2\mu}{\hbar^2}V(r)\psi(\vec{r}) = \frac{2\mu}{\hbar^2}\int \delta(\vec{r} - \vec{r}')V(r')\psi(\vec{r}')d\vec{r}' \tag{4}$$

Here we assume

$$\psi(\vec{r}) = \frac{2\mu}{\hbar^2}\int G(\vec{r}, \vec{r}')V(r')\psi(\vec{r}')d\vec{r}' \tag{5}$$

Thus we bring (5) and (4) into (3), we get

$$\boxed{\boldsymbol{(\nabla^2 + k^2)G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}')}} \tag{6}$$

The general form of $\psi(\vec{r})$ can be written as

$$\psi(\vec{r}) = \psi_i(\vec{r}) + \frac{2\mu}{\hbar^2}\int G(\vec{r}, \vec{r}')V(r')\psi(\vec{r}')d\vec{r}' \tag{7}$$

where $\psi_i(\vec{r})$ satisfies

$$(\nabla^2 + k^2)\psi_i(\vec{r}) = 0 \tag{8}$$

In scattering problem we choose $\psi_i(\vec{r}) = e^{ikz}$, so

$$\psi(\vec{r}) = e^{ikz} + \frac{2\mu}{\hbar^2}\int G(\vec{r}, \vec{r}')V(r')\psi(\vec{r}')d\vec{r}' \tag{9}$$

2 Solving Green's function

2.1 Fourier's transformation

In order to solve equation(6), we execute Fourier's transformation first,

$$G(\vec{r}-\vec{r}') = \int \tilde{G}(\vec{q}) e^{i\vec{q}(\vec{r}-\vec{r}')} d^3q \tag{10}$$

then we bring it into (6), we get

$$\int (\nabla^2 + k^2) e^{i\vec{q}(\vec{r}-\vec{r}')} \tilde{G}(\vec{q}) d^3q = \delta(\vec{r}-\vec{r}') \tag{11}$$

i.e.

$$\int (-q^2 + k^2) e^{i\vec{q}(\vec{r}-\vec{r}')} \tilde{G}(\vec{q}) d^3q = \frac{1}{(2\pi)^3} \int e^{i\vec{q}(\vec{r}-\vec{r}')} d^3q \tag{12}$$

So we have

$$\boxed{\tilde{G}(\vec{q}) = \frac{1}{(2\pi)^3} \frac{1}{(-q^2 + k^2)}} \tag{13}$$

Bringing this into (10), we have

$$G(\vec{r}-\vec{r}') = -\frac{1}{(2\pi)^3} \int e^{i\vec{q}(\vec{r}-\vec{r}')} \frac{1}{(q^2 - k^2)} d^3q \tag{14}$$

2.2 Complex integral