Spark Spread Option Pricing

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Outline

Introduction to Spark Spread Options

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$$C_T = \max(F(T, T+\tau) - h \cdot G(T, T+\tau) - K, 0)$$

where:

- T: Option's maturity.
- $F(T, T+\tau)$: τ-Forward price of electricity observed at T.
- $G(T, T+\tau)$: τ-Forward price of gas observed at T.
- h: The heat rate, representing the efficiency of converting gas into electricity.
- K: The strike

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  where:
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  - $G(T, T+\tau)$: τ-Forward price of gas observed at T.
  - h: The heat rate, representing the efficiency of converting gas into electricity.
  - K: The strike
- Under the risk-neutral probability measure:

$$\boxed{C_0 = e^{-rT}\mathbb{E}^Q\left[(F(T) - h \cdot G(T) - K)^+\right]}$$

Model Assumptions

We assume that forward prices follow a driftless log-normal dynamic under the risk-neutral measure Q. We simplify the notation by setting:

- $F(t) = F(t, \tilde{T})$
- $G(t) = G(t, \tilde{T})$

Dynamics of the Underlyings

$$dF(t) = \sigma_F \cdot F(t)\, dW_F(t)$$
$$dG(t) = \sigma_G \cdot G(t)\, dW_G(t)$$

with

- $d\langle W_F, W_G \rangle_t = \rho\, dt$
- $\rho \in [-1, 1]$
- $\sigma_F, \sigma_G \in \mathbb{R}^+$

Special Case: Zero Strike Price

When $K = 0$, the option becomes an exchange option. The payoff simplifies to:

$$C_T = \max(F(T) - h \cdot G(T), 0)$$

r is supposed to be constant.

Margrabe's Formula for Exchange Options

$$\boxed{C_0 = e^{-rT} \left(F(0) \cdot N(d) - hG(0) \cdot N(d - \sigma\sqrt{T}) \right)}$$

with:

$$d = \frac{\ln\left(\frac{F(0)}{G(0)}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$\sigma = \sqrt{\sigma_F^2 + \sigma_G^2 - 2\rho\sigma_F\sigma_G}$$

$N(.)$: Cumulative standard normal density function

Special Case: Zero Strike Price

Proof:

$$C_0 = e^{-rT}\mathbb{E}^Q\left[\max(F(T) - h \cdot G(T), 0)\right]$$

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- Apply a change of measure:

$$\frac{dQ'}{dQ} = \exp\left(-\frac{1}{2}\sigma_G^2 t + \sigma_G W_G(t)\right)$$

This gives:

$$C_0 = e^{-rT}G(0)\mathbb{E}^{Q'}\left[\max\left(\frac{F(T)}{G(T)} - h, 0\right)\right]$$

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- Using Ito's Lemma for $X(t) := \frac{F(T)}{G(T)}$:

$$dX(t) = X(t)\sqrt{\sigma_F^2 + \sigma_G^2 - 2\rho\sigma_F\sigma_G}\, dW = X(t)\sigma \cdot dW$$

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$$dX(t) = X(t)\sqrt{\sigma_F^2 + \sigma_G^2 - 2\rho\sigma_F\sigma_G}\, dW = X(t)\sigma \cdot dW$$

- Finally:

$$C_0 = e^{-rT} G(0)\mathbb{E}^{Q'}\left[\max(X(T) - h, 0)\right]$$

Non-Zero Strike Price: Kirk's Approximation

Ewan Kirk (1995) extended the exchange option model for $K \neq 0$:

- Define the adjusted gas price as $G'(t) := hG(t) + K$.
- Approximate $G'(t)$ as a lognormal process.

Non-Zero Strike Price: Kirk's Approximation

Kirk's Approximation

$$\boxed{C_0 = e^{-rT}\left[F(0)N(\tilde{d}) - (hG(0)+K)N(\tilde{d} - \tilde{\sigma}\sqrt{T})\right]}$$

where:

$$\tilde{d} = \frac{\ln\left(\frac{F(0)}{hG(0)+K}\right) + \frac{1}{2}\tilde{\sigma}^2 T}{\tilde{\sigma}\sqrt{T}}$$

$$\tilde{\sigma} = \sqrt{\sigma_F^2 + \tilde{\sigma}_G^2 - 2\sigma_F\tilde{\sigma}_G\rho}$$

$$\tilde{\sigma}_G = \frac{hG(0)}{hG(0)+K}\sigma_G$$

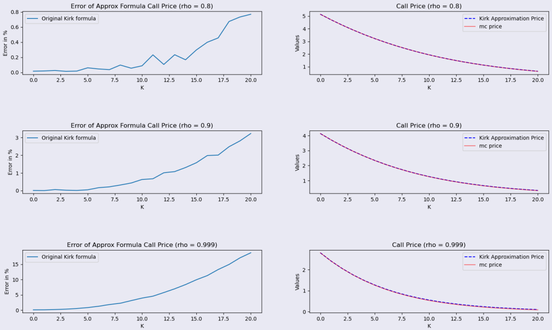
Non-Zero Strike Price: Kirk's Approximation

Numerical Results

2D Plots: Kirk Approximation Relative Errors vs. MC Simulation

Kirk's approximation becomes less accurate when the strike price K is large and the correlation ρ is close to 1.

- $F(0) = G(0) = 100$
- $h = 1$
- $\sigma_F = 0.3$
- $\sigma_G = 0.2$
- $T = 0.5$



Non-Zero Strike Price: Kirk's Approximation

Numerical Results

3D Plots: Kirk Approximation Relative Errors vs. MC Simulation

As shown once again, relative errors increase with correlation ρ and strike price K.

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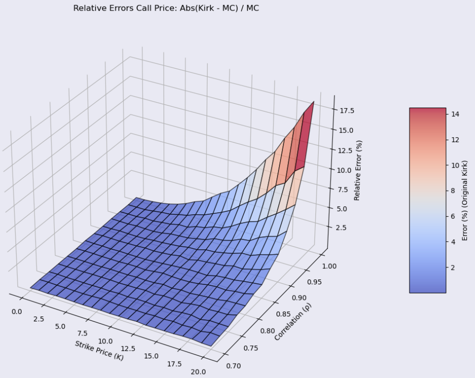

Modified Kirk Approximation

Alòs and León (2015)

In 2015, Elisa Alòs and Jorge Alberto León introduced a modification to Kirk's approximation to improve accuracy, particularly when K is large or ρ is close to 1.

Modified Kirk Approximation

$$\boxed{C_0 = e^{-rT}\left[F(0)\, N(\tilde{d}_I) - (h\, G(0) + K)\, N(\tilde{d}_I - I\sqrt{T})\right]}$$

where

$$\tilde{d}_I := \frac{\ln\left(\frac{F(0)}{h\, G(0)+K}\right) + \frac{1}{2} I^2 T}{I\sqrt{T}}$$

$$I := \tilde{\sigma} + \frac{1}{2} \cdot (\tilde{\sigma}_G - \rho\sigma_F)^2 \cdot \frac{1}{\tilde{\sigma}^3} \cdot \sigma_G^2 \cdot \frac{hG(0)K}{(hG(0)+K)^2} \cdot ln\left(\frac{F(0)}{hG((0)+K}\right)$$

and $\tilde{\sigma}$ is the effective volatility:

$$\tilde{\sigma} = \sqrt{\sigma_F^2 + \tilde{\sigma}_G^2 - 2\sigma_F\tilde{\sigma}_G\rho}$$

K	ρ	Kirk	Mod Kirk	Monte-Carlo (95% CI)	Error (%)	Mod Err (%)
5	0.7	3.9906	3.9905	3.9935 (3.9900, 3.9971)	0.0741	0.0753
	0.8	3.2254	3.2249	3.2241 (3.2211, 3.2272)	0.0396	0.0258
	0.9	2.3412	2.3392	2.3393 (2.3369, 2.3417)	0.0807	0.0055
	0.999	1.2735	1.2642	1.2629 (1.2614, 1.2645)	0.8324	0.0959
10	0.7	2.5702	2.5699	2.5689 (2.5659, 2.5718)	0.0505	0.0389
	0.8	1.9403	1.9385	1.9380 (1.9356, 1.9404)	0.1203	0.0281
	0.9	1.2618	1.2555	1.2554 (1.2536, 1.2572)	0.5140	0.0095
	0.999	0.5560	0.5360	0.5358 (0.5348, 0.5369)	3.7635	0.0353
20	0.7	0.9785	0.9771	0.9790 (0.9772, 0.9808)	0.0534	0.1948
	0.8	0.6401	0.6354	0.6349 (0.6335, 0.6363)	0.8228	0.0844
	0.9	0.3358	0.3251	0.3252 (0.3243, 0.3262)	3.2405	0.0464
	0.999	0.1052	0.0882	0.0884 (0.0880, 0.0889)	18.9102	0.2287

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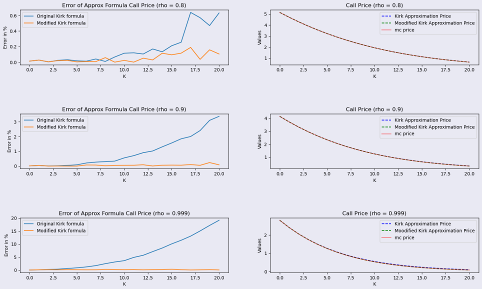
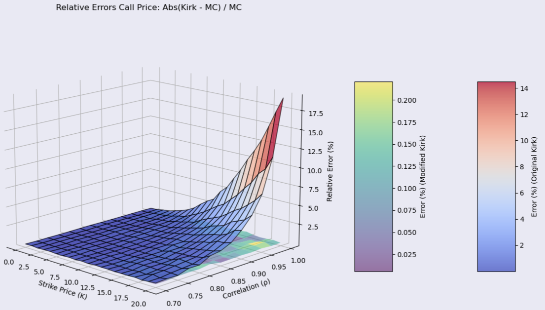
2D plots: Modified Kirk Approximation relative errors / MC

As the correlation increases, Kirk's Approximation becomes less accurate, while the Modified Kirk's Approximation remains reliable, demonstrating the effectiveness of the new method



3D plots: Modified Kirk Approximation relative errors / MC

the Modified Kirk's Approximation method greatly reduces the errors in percentage terms of the Monte-Carlo simulation



Delta Hedging for Spark Spread Options

For a spark spread option, the delta can be defined with respect to both the forward price of electricity F (T) and the forward price of gas G(T), as the option's payoff depends on the difference between these two.

$$\Delta_F = \frac{\partial C_0}{\partial F(0)} \qquad \Delta_G = \frac{\partial C_0}{\partial G(0)}.$$

Hedging involves dynamically adjusting positions in electricity and gas forwards

Kirk's Approximation for Deltas:

- Δ_F:

$$\boxed{\Delta_F = e^{-rT} \cdot N(\tilde{d})}$$

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- Δ_G:

$$\boxed{\Delta_G = e^{-rT} \left[-hN(\tilde{d} - \tilde{\sigma}\sqrt{T}) + (hG(0) + K) \cdot n(\tilde{d} - \tilde{\sigma}\sqrt{T}) \cdot \sqrt{T}\xi_G \right]}$$

with :

$$\xi_G := \frac{\partial \tilde{\sigma}}{\partial G(0)} = \frac{\sigma_G \cdot K}{\tilde{\sigma}} \cdot \frac{\tilde{\sigma}_G - \rho\sigma_F}{(hG(0) + K)^2}$$

Modified Kirk's Approximation for Deltas:

- Δ_F:

$$\boxed{\Delta_F = e^{-rT}\left[\cdot N(\tilde{d}) + F(0)\sqrt{T}\frac{\partial I}{\partial F(0)}\right]}$$

with :

$$\frac{\partial I}{\partial F(0)} = \frac{1}{2}\cdot(\tilde{\sigma}_G - \rho\sigma_F)^2\cdot\frac{1}{\tilde{\sigma}^3}\cdot\sigma_G^2\cdot\frac{hG(0)K}{(hG(0)+K)^2}\cdot\frac{1}{F(0)}$$

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Modified Kirk's Approximation for Deltas:

- Δ_G:

$$\boxed{\Delta_G = e^{-rT}\left(-N(\tilde{d}_l - I\sqrt{T}) + (hG(0)+K)\cdot N(\tilde{d}_l - I\sqrt{T})\cdot\sqrt{T}\frac{\partial I}{\partial G(0)}\right)}$$

with :

$$I := \tilde{\sigma} + \frac{1}{2}\cdot(\tilde{\sigma}_G - \rho\sigma_F)^2\cdot\frac{1}{\tilde{\sigma}^3}\cdot\sigma_G^2\cdot\frac{hG(0)K}{(hG(0)+K)^2}\cdot ln\left(\frac{F(0)}{hG((0)+K}\right)$$

Modified Kirk's Approximation for Deltas:

- Δ_G:

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Numerical Results

Δ_F

K	ρ	Kirk	Mod Kirk	Monte-Carlo (95% CI)	Error (%)	Mod Err (%)
5	0.7	0.3993	0.3993	0.3991 (0.3988, 0.3993)	0.0712	0.0733
	0.8	0.3769	0.3770	0.3766 (0.3764, 0.3769)	0.0821	0.1024
	0.9	0.3423	0.3427	0.3425 (0.3422, 0.3427)	0.0337	0.0595
	0.999	0.2755	0.2766	0.2760 (0.2758, 0.2762)	0.1918	0.2091
10	0.7	0.2892	0.2892	0.2892 (0.2889, 0.2894)	0.0015	0.0076
	0.8	0.2567	0.2568	0.2565 (0.2563, 0.2567)	0.0714	0.1038
	0.9	0.2101	0.2102	0.2100 (0.2098, 0.2102)	0.0315	0.1010
	0.999	0.1356	0.1348	0.1345 (0.1343, 0.1347)	0.7877	0.2268
20	0.7	0.1338	0.1338	0.1335 (0.1333, 0.1337)	0.2428	0.2153
	0.8	0.1032	0.1030	0.1029 (0.1027, 0.1030)	0.2933	0.0793
	0.9	0.0677	0.0668	0.0666 (0.0665, 0.0668)	1.5577	0.2292
	0.999	0.0297	0.0269	0.0268 (0.0268, 0.0269)	10.5586	0.2897

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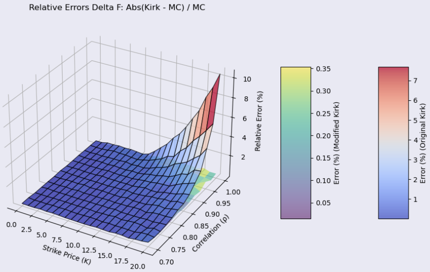
Numerical Results

Δ_F

Δ_F 3D plots: Modified Kirk Approximation relative errors / MC

Once again, the Modified Kirk's Approximation method greatly reduces the errors in percentage terms of the Monte-Carlo simulation



Numerical Results

Δ_G

K	ρ	Kirk	Mod Kirk	Monte-Carlo (95% CI)	Error (%)	Mod Err (%
5	0.7	-0.3425	-0.3426	-0.3426 (-0.3428, -0.3424)	0.0089	0.0067
	0.8	-0.3289	-0.3290	-0.3288 (-0.3290, -0.3286)	0.0284	0.0504
	0.9	-0.3050	-0.3053	-0.3051 (-0.3054, -0.3049)	0.0599	0.0398
	0.999	-0.2523	-0.2534	-0.2534 (-0.2536, -0.2532)	0.4212	0.0011
10	0.7	-0.2400	-0.2400	-0.2399 (-0.2401, -0.2397)	0.0380	0.0441
	0.8	-0.2169	-0.2169	-0.2170 (-0.2172, -0.2168)	0.0760	0.0422
	0.9	-0.1814	-0.1815	-0.1815 (-0.1816, -0.1813)	0.0437	0.0308
	0.999	-0.1207	-0.1200	-0.1198 (-0.1199, -0.1196)	0.8106	0.2692
20	0.7	-0.1042	-0.1042	-0.1041 (-0.1043, -0.1040)	0.0490	0.0139
	0.8	-0.0820	-0.0818	-0.0818 (-0.0819, -0.0817)	0.2409	0.0083
	0.9	-0.0552	-0.0544	-0.0545 (-0.0546, -0.0544)	1.3001	0.0611
	0.999	-0.0252	-0.0228	-0.0227 (-0.0228, -0.0226)	10.8119	0.5431

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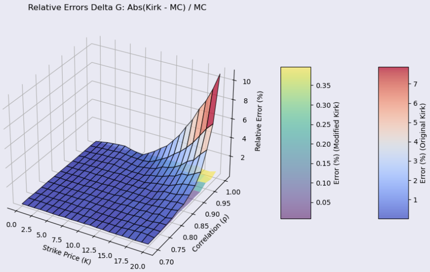

Delta Hedging in Energy Markets

- **Goal:** Maintain a delta-neutral position to minimize risk from price fluctuations.
- **Process:**
  - If electricity price rises: Sell electricity forwards
  - If gas price rises: Buy gas forwards
- **Key points**:
  - Continuously adjust positions in electricity and gas forwards
  - Aim to create a locally risk-free portfolio
  - Balance option value changes with underlying asset positions

This strategy helps protect against small price movements in electricity and gas throughout the option's lifetime.

Limitations of Delta Hedging in Energy Markets

Market Liquidity

- Low liquidity in electricity markets due to non-storability
- Volatile prices and illiquid conditions during peak demand
- Difficulty in executing trades, especially for large positions

Transaction Costs

- Frequent portfolio adjustments incur costs

Price Jumps and Spikes

- Sudden price movements, especially in electricity
- Continuous price movement assumption often invalid

Correlation Breakdown

- Unstable correlation between electricity and gas prices
- Reduced hedging effectiveness during market stress

Model Risk

- Reliance on simplified pricing models
- Potential for suboptimal or risky hedging if models are inaccurate

These challenges can significantly impact the effectiveness and practicality of delta hedging strategies in energy markets, particularly for spark spread options. Market participants must carefully consider these factors when implementing hedging strategies in these complex and volatile markets.

Conclusion

- Kirk's approximation is a valuable method for pricing spark spread options but has limitations when K is large or ρ is near 1.
- The modified Kirk Approximation enhances the accuracy in these challenging scenarios.
- Further research could focus on extending these models for other complex energy derivatives.