1　变量定义

1.1　Voigt 表记法

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \xRightarrow{\text{Voigt Notation}} \{\sigma\} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \underbrace{\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}}_{\text{Engineering Notation}}$$

Figure 1-1 应力张量的 Voigt 表记

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \xRightarrow{\text{Voigt Notation}} \{\varepsilon\} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \underbrace{\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}}_{\text{Engineering Notation}}$$

Figure 1-2 应变张量的 Voigt 表记

$$\left[\begin{array}{ccc:ccc:ccc}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1113} & C_{1121} & C_{1132} & C_{1131} \\
C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2213} & C_{2221} & C_{2232} & C_{2231} \\
C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3313} & C_{3321} & C_{3332} & C_{3331} \\ \hdashline
C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1213} & C_{1221} & C_{1232} & C_{1231} \\
C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2313} & C_{2321} & C_{2332} & C_{2331} \\
C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1323} & C_{1313} & C_{1321} & C_{1332} & C_{1331} \\ \hdashline
C_{2111} & C_{2122} & C_{2133} & C_{2112} & C_{2123} & C_{2113} & C_{2121} & C_{2132} & C_{2131} \\
C_{3211} & C_{3222} & C_{3233} & C_{3212} & C_{3223} & C_{3213} & C_{3221} & C_{3232} & C_{3231} \\
C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3113} & C_{3121} & C_{3132} & C_{3131}
\end{array}\right]$$

取出红色方框框选部分，并且将
前两个指标和后两个指标按下表替换

11	22	33	12	23	13
1	2	3	4	5	6

$$[\mathbf{C}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}$$

Figure 1-3 弹性张量的 Voigt 表记

总结：不仅应力应变张量以及弹性张量可以用 Voigt 表记形式，**任何具有相同对称性的二阶张量和四阶张量都可以采用 Voigt 表记**。用到的对称性有二阶张量的对称性 $\sigma_{ij}=\sigma_{ji},\varepsilon_{ij}=\varepsilon_{ji}$ 以及四阶张量的副对称（minor symtric）关系

1.2　四阶单位张量

3 种基本四阶单位张量：

一般四阶单位张量（fourth order unit tensor）：　$\mathbb{I}_{ijkl}=\delta_{ik}\delta_{jl}$　　$\mathbb{I}:\mathrm{A}=\mathrm{A}$

转置四阶单位张量（transpositional fourth order unit tensor）：$(\mathbb{I}_{\mathrm{T}})_{ijkl}=\delta_{il}\delta_{jk}$　　$\mathbb{I}_{\mathrm{T}}:\mathrm{A}=\mathrm{A}^{T}$

球形四阶单位张量（spherical fourth order unit tensor ）：　$(\mathrm{I}\otimes\mathrm{I})_{ijkl}=\delta_{ij}\delta_{kl}$　　$(\mathrm{I}\otimes\mathrm{I}):\mathrm{A}=\mathrm{Tr}(\mathrm{A})\mathrm{I}$

与弹性力学相关的四阶单位张量：

对称四阶单位张量（ symmetric fourth order unit tensor）：　$\mathbb{I}^{\mathrm{sym}}=\frac{1}{2}(\mathbb{I}+\mathbb{I}_{\mathrm{T}})$　　$\mathbb{I}^{\mathrm{sym}}:\mathrm{A}=\mathrm{A}^{\mathrm{sym}}$

反对称四阶单位张量（ skew-symmetric fourth order unit tensor）：$\mathbb{I}^{\mathrm{skew}}=\frac{1}{2}(\mathbb{I}-\mathbb{I}_{\mathrm{T}})$　　$\mathbb{I}^{\mathrm{skew}}:\mathrm{A}=\mathrm{A}^{\mathrm{skew}}$

体积四阶单位张量（ volumetric fourth order unit tensor）：　$\mathbb{I}^{\mathrm{vol}}=\frac{1}{3}\mathrm{I}\otimes\mathrm{I}$　　$\mathbb{I}^{\mathrm{vol}}:\mathrm{A}=\mathrm{A}^{\mathrm{sph}}=\frac{1}{3}\mathrm{Tr}(\mathrm{A})\mathrm{I}$

偏差四阶单位张量（deviatoric fourth order unit tensor）：　$\mathbb{I}^{\mathrm{dev}}=\mathbb{I}^{\mathrm{sym}}-\mathbb{I}^{\mathrm{vol}}$　　$\mathbb{I}^{\mathrm{dev}}:\mathrm{A}=\mathrm{A}^{\mathrm{dev}}$

1.3　四阶单位张量的 Voigt 表记

一般四阶单位张量 $\mathbb{I}_{ijkl}=\delta_{ik}\delta_{jl}$　　转置四阶单位张量 $(\mathbb{I}_{\mathrm{T}})_{ijkl}=\delta_{il}\delta_{jk}$　　球形四阶单位张量 $(\mathrm{I}\otimes\mathrm{I})_{ijkl}=\delta_{ij}\delta_{kl}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}$$

7

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

对称四阶单位张量 $\mathbb{I}^{\mathrm{sym}}=\frac{1}{2}(\mathbb{I}+\mathbb{I}_{\mathrm{T}})$　　反对称四阶单位张量 $\mathbb{I}^{\mathrm{skew}}=\frac{1}{2}(\mathbb{I}-\mathbb{I}_{\mathrm{T}})$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1}{2} & 0 & 0 \\ & & & & \frac{1}{2} & 0 \\ & & & & & \frac{1}{2} \end{bmatrix}$$

无

体积四阶单位张量 $\mathbb{I}^{\mathrm{vol}} = \frac{1}{3}\mathrm{I} \otimes \mathrm{I}$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ & & \frac{1}{3} & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

偏差四阶单位张量 $\mathbb{I}^{\mathrm{dev}} = \mathbb{I}^{\mathrm{sym}} - \mathbb{I}^{\mathrm{vol}}$

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ & & \frac{2}{3} & 0 & 0 & 0 \\ & & & \frac{1}{2} & 0 & 0 \\ & & & & \frac{1}{2} & 0 \\ & & & & & \frac{1}{2} \end{bmatrix}$$

1.4 运动学变量

$$E = \frac{1}{2}(C - I) \quad \text{Green-Lagrange strain tensor}$$
$$e = \frac{1}{2}(I - F^{-T} \cdot F^{-1}) \quad \text{Euler-Almansi Strain Tensor (refer to zhihu)} \tag{1.1}$$
$$\varepsilon = \frac{1}{2}(B - I) \quad \text{Eulerian strain tensors (refer to CMFP)}$$

$$\begin{gathered} F = F^{iso} \cdot F^{v} \\ F^{v} = J^{\frac{1}{3}} I \qquad F^{iso} = J^{-\frac{1}{3}} F \\ B^{iso} = F^{iso} \cdot (F^{iso})^{T} = J^{-\frac{2}{3}} F F^{T} = J^{-\frac{2}{3}} B \\ \overline{I}_1 = \mathrm{tr} B^{iso} \qquad \overline{I}_2 = \frac{1}{2}\{(\overline{I}_1)^2 - \mathrm{tr}[(\mathrm{B}^{iso})^2]\} \end{gathered} \tag{1.2}$$

2　张量求导

2.1　对称张量的标量各向同性函数

在三维空间中，一个对称张量的标量函数是各向同性的，当且仅当它满足表示定理（(2.1)(a)(b)任意满足一个另一个自动满足）：

$$\begin{aligned} &\phi(X)=\bar{\phi}(I_1(X),I_2(X),I_3(X)) \text{ principal invarients representation} &&\text{(a)}\\ &\phi(X)=\hat{\phi}(x_1,x_2,x_3) \text{ eigenvalues representation} &&\text{(b)} \end{aligned} \tag{2.1}$$

对于上述各向同性标量函数，具有以下性质：

$$\hat{\phi}(x_1,x_2,x_3)=\hat{\phi}(x_2,x_1,x_3)=\hat{\phi}(x_1,x_3,x_2) \tag{2.2}$$

对于对称张量 X 各向同性标量函数 $\phi(X)$，$\phi(X),X$ 共轴，从而他们可交换，即：

$$\frac{\partial\phi}{\partial X}\cdot X=X\cdot\frac{\partial\phi}{\partial X} \tag{2.3}$$

且其导数有如下表达式（由 Ogden(1984)证明）：

$$\frac{\partial\phi}{\partial X}=\sum_i\frac{\partial\hat{\phi}}{\partial x_i}e_i\otimes e_i \qquad (e_i\otimes e_i \text{ is eigenprojection corresponding to } x_i) \tag{2.4}$$

也就是说，$\dfrac{\partial\phi}{\partial X}$ 与 X 共主轴，共谱阵，且特征值 $y_i=\partial\hat{\phi}/\partial x_i$

!值得注意的是 $\dfrac{\partial\phi}{\partial X},\dfrac{\partial\phi}{\partial X}\cdot X$ 仍然是关于对称张量 X 的二阶对称各向同性函数

2.2　对称张量的各向同性对称张量函数

在三维空间中，一个对称张量的对称张量函数是各向同性的，当且仅当它满足表示定理：

$$Y(X)=\sum_{i=1}^{p}y_iE_i \qquad y_i \text{ is eigenvalues of } Y, E_i \text{ is eigenprojection of } (X \text{ and } Y) \tag{2.5}$$

其中：

$$\begin{cases} y_1=y(x_1,x_2,x_3)\\ y_2=y(x_1,x_2,x_3)\\ y_3=y(x_1,x_2,x_3) \end{cases} \qquad y \text{ satisfy } y(a,b,c)=y(a,c,b) \tag{2.6}$$

那么这样的对称张量的各向同性对称张量函数的求导有以下表达式：

Box A.3. Computation of the derivative of a general isotropic tensor function in two dimensions.

HYPLAS procedure:	DGISO2

(i) Given $\boldsymbol{X}$, compute its eigenvalues, x_α, and eigenprojections, $\boldsymbol{E}_\alpha$ $(\alpha = 1, 2)$ – GOTO Box A.2

(ii) Compute the eigenvalues y_α of $\boldsymbol{Y}$ and their derivatives $\partial y_\alpha / \partial x_\beta$ for $\alpha = 1, 2$ and $\beta = 1, 2$

(iii) Assemble the derivative

$$
\mathsf{D}(\boldsymbol{X}) := \begin{cases} \dfrac{y_1 - y_2}{x_1 - x_2} \left[\mathsf{I}_S - \boldsymbol{E}_1 \otimes \boldsymbol{E}_1 - \boldsymbol{E}_2 \otimes \boldsymbol{E}_2\right] \\ \quad + \displaystyle\sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \frac{\partial y_\alpha}{\partial x_\beta} \boldsymbol{E}_\alpha \otimes \boldsymbol{E}_\beta & \text{if } x_1 \neq x_2 \\ \left(\dfrac{\partial y_1}{\partial x_1} - \dfrac{\partial y_1}{\partial x_2} \right) \mathsf{I}_S + \dfrac{\partial y_1}{\partial x_2} \boldsymbol{I} \otimes \boldsymbol{I} & \text{if } x_1 = x_2 \end{cases}
$$

Box A.6. Computation of the derivative of a general isotropic tensor function in three dimensions.

(i) Given $\boldsymbol{X}$, compute its eigenvalues, x_i, and eigenprojections, $\boldsymbol{E}_i$ $(i = 1, 2, 3)$ – GOTO Box A.5

(ii) Compute the eigenvalues y_i of $\boldsymbol{Y}$ and their derivatives $\partial y_i / \partial x_j$ for $i, j = 1, 2, 3$

(iii) Assemble the derivative

$$
\mathsf{D}(\boldsymbol{X}) = \begin{cases} \displaystyle\sum_{a=1}^{3} \frac{y_a}{(x_a - x_b)(x_a - x_c)} \Big\{ \frac{\mathrm{d}\boldsymbol{X}^2}{\mathrm{d}\boldsymbol{X}} - (x_b + x_c)\, \mathsf{I}_S \\ \qquad - [(x_a - x_b) + (x_a - x_c)] \boldsymbol{E}_a \otimes \boldsymbol{E}_a \\ \qquad - (x_b - x_c)(\boldsymbol{E}_b \otimes \boldsymbol{E}_b - \boldsymbol{E}_c \otimes \boldsymbol{E}_c) \Big\} \\ \qquad + \displaystyle\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial y_i}{\partial x_j} \boldsymbol{E}_i \otimes \boldsymbol{E}_j \\ \qquad\qquad \text{if } x_1 \neq x_2 \neq x_3 \\ s_1 \dfrac{\mathrm{d}\boldsymbol{X}^2}{\mathrm{d}\boldsymbol{X}} - s_2\, \mathsf{I}_S - s_3\, \boldsymbol{X} \otimes \boldsymbol{X} + s_4\, \boldsymbol{X} \otimes \boldsymbol{I} \\ \qquad + s_5\, \boldsymbol{I} \otimes \boldsymbol{X} - s_6\, \boldsymbol{I} \otimes \boldsymbol{I} \\ \qquad\qquad \text{if } x_a \neq x_b = x_c \\ \left(\dfrac{\partial y_1}{\partial x_1} - \dfrac{\partial y_1}{\partial x_2} \right) \mathsf{I}_S + \dfrac{\partial y_1}{\partial x_2} \boldsymbol{I} \otimes \boldsymbol{I} \qquad \text{if } x_1 = x_2 = x_3 \end{cases}
$$

$$
\begin{aligned}
s_1 &= \frac{y(x_a) - y(x_c)}{(x_a - x_c)^2} - \frac{y'(x_c)}{x_a - x_c} \\
s_2 &= 2x_c \frac{y(x_a) - y(x_c)}{(x_a - x_c)^2} - \frac{x_a + x_c}{x_a - x_c} y'(x_c) \\
s_3 &= 2 \frac{y(x_a) - y(x_c)}{(x_a - x_c)^3} - \frac{y'(x_a) + y'(x_c)}{(x_a - x_c)^2} \\
s_4 &= s_5 = x_c\, s_3 \\
s_6 &= x_c^2\, s_3 .
\end{aligned} \tag{A.53}
$$

2.3 各向同性主值表示张量函数

2.3.1 谱分解定理

当可对角化二阶张量 $X=\sum_i x_i E_i$ 具有互相相异的特征值时，普阵 E_i 有如下表达式：

$$E_i=\prod_{j\neq i}\frac{X-x_j I}{x_i-x_j} \tag{2.7}$$

当 X 的特征值不同时有：

$$\frac{dx_i}{dX}=E_i \tag{2.8}$$

2.3.2 基本定义

所谓的主值表示函数是指：

$$Y(X)=\sum_{i=1}^{p} y_i E_i \qquad y_i=y(x_i,x_j,x_k) \tag{2.9}$$

其中 X, Y 都是对称二阶张量，$y_i=y(x_i,x_j,x_k)$ 表示 Y 的特征值是 X 的特征值的标量函数，i,j,k 满足指标循环轮换，且 $y(x_i,x_j,x_k)=y(x_i,x_k,x_j)$，如此 $Y(X)$ 自然也是一个各向同性函数。

2.3.3 二维主值表示函数求导

$$\frac{dY}{dX}=\sum_i\{E_i\frac{dy_i}{dX}+y_i\frac{dE_i}{dX}\}=\sum_i\{y_i\frac{dE_i}{dX}+\frac{dy_i}{dx_j}E_i\frac{dx_j}{dX}\} \tag{2.10}$$

在二维的情况下有：

$$\begin{cases} I_1=\mathrm{tr}[X]=x_1+x_2 \\ I_2=\det[X]=x_1x_2 \end{cases} \Rightarrow \begin{cases} x_1=\dfrac{I_1+\sqrt{I_1^2-4I_2}}{2} \\ x_2=\dfrac{I_1-\sqrt{I_1^2-4I_2}}{2} \end{cases} \tag{2.11}$$

当特征值 $x_1\neq x_2$ 时，根据(2.7)有：

$$E_\alpha=\frac{1}{2x_\alpha-I_1}[X+(x_\alpha-I_1)I] \tag{2.12}$$

$$\frac{dY}{dX}=\sum_\alpha\{y_\alpha\frac{dE_\alpha}{dX}+\frac{dy_\alpha}{dx_\beta}E_\alpha\frac{dx_\beta}{dX}\} \tag{2.13}$$

将(2.11)(2.12)(2.8)带入(2.13)有：

$$\begin{cases} \dfrac{dY}{dX}=\dfrac{y_1-y_2}{x_1-x_2}[\mathbb{I}^{sym}-E_iE_i]+\dfrac{\partial y_\alpha}{\partial x_\beta}E_\alpha E_\beta & \text{if } x_1\neq x_2 \\ \dfrac{dY}{dX}=\lim\limits_{x_1\to x_2}(\dfrac{dY}{dX}(x_1\neq x_2))=(\dfrac{\partial y_1}{\partial x_1}-\dfrac{\partial y_1}{\partial x_2})\mathbb{I}^{sym}+\dfrac{\partial y_1}{\partial x_2}I\otimes I & \text{if } x_1=x_2 \end{cases} \tag{2.14}$$

3 虚功原理的线化

3.1 小变形虚功原理线化

虚功原理表达式：

$$G(u,\eta)=\int_{\Omega}(\sigma:\nabla^s\eta-b\cdot\eta)dv-\int_{\partial\Omega_t}t\cdot\eta da=0$$
$$(u \text{ is a kinematically admissible fild}) \tag{3.1}$$

线化虚功原理表达式：

$$L(\delta u,\eta)=G(u^*,\eta)+DG(u^*,\eta)[\delta u]=0 \qquad DG(u^*,\eta)[\delta u]=\frac{d}{dc}\Big|_{c=0}DG(u^*+c\delta u,\eta) \tag{3.2}$$

(3.3)的公式推导见(5.1)~(5.4)：

$$DG(u^*,\eta)[\delta u]=\frac{d}{dc}\Big|_{c=0}\int_{\Omega}\sigma(u^*+\delta u):\nabla^s\eta dv=\int_{\Omega}D:\nabla^s(\delta u):\nabla^s\eta dv \tag{3.3}$$

将(3.3)带入(3.2)得小变形线化虚功表达式：

$$\int_{\Omega}D:\nabla^s(\delta u):\nabla^s\eta dv=-\int_{\Omega}(\sigma:\nabla^s\eta-b\cdot\eta)dv+\int_{\partial\Omega_t}t\cdot\eta da \tag{3.4}$$

3.2 有限应变虚功原理线化

3.2.1 Material description

$$P=P(F) \tag{3.5}$$

虚功原理表达式（证明过程见(5.9)~(5.12)）：

$$G(u,\eta)=\int_{\Omega}(P:\nabla_p\eta-\overline{b}\cdot\eta)dv_0-\int_{\partial\Omega_t}\overline{t}\cdot\eta dA=0$$
$$(u \text{ is a kinematically admissible fild}) \tag{3.6}$$

线化虚功原理表达式：

$$L(\delta u,\eta)=G(u^*,\eta)+DG(u^*,\eta)[\delta u]=0 \tag{3.7}$$

(3.8)推导见(5.5)~(5.8)

$$DG(u^*,\eta)[\delta u]=\int_{\Omega}A:\nabla_p\delta u:\nabla_p\eta dv \qquad A=\frac{\partial P}{\partial F}\Big|_{u^*} \qquad \text{material tangent modulus} \tag{3.8}$$

带入(3.7)得：

$$\int_{\Omega}\mathrm{A}:\nabla_p\delta u):\nabla_p\eta dv=-\int_{\Omega}(P:\nabla_p\eta-b\cdot\eta)dv+\int_{\partial\Omega_t}t\cdot\eta da \tag{3.9}$$

3.2.2 Spatial description

$$\sigma=\sigma(F) \tag{3.10}$$

虚功原理表达式：

$$G(u,\eta)=\int_{\varphi(\Omega)}(\sigma:\nabla_x\eta-b\cdot\eta)dv-\int_{\varphi(\partial\Omega_t)}t\cdot\eta da=0$$
$$(u \text{ is a kinematically admissible fild}) \tag{3.11}$$

线化虚功原理表达式：

$$L(\delta u,\eta)=G(u^*,\eta)+DG(u^*,\eta)[\delta u]=0 \tag{3.12}$$

由(3.11)进行积分坐标变换可得数值相等得虚功得表达式（证明过程见(5.13)(5.14)）：

$$
\begin{aligned}
&DG(u^*,\eta)[\delta u]=\int_{\varphi(\Omega)}\mathrm{a}:\nabla_x\delta u:\nabla_x\eta dv \\
&\mathrm{a}_{ijkl}=J^{-1}A_{imkn}F_{ln}F_{jm} \qquad A=\frac{\partial P}{\partial F} \qquad \text{a is spatial tangent modulus}
\end{aligned}
\tag{3.13}
$$

将(3.13)带入(3.12)得到(3.14)空间描述虚功原理的线化方程：

$$
\int_{\varphi(\Omega)}\mathrm{a}:\nabla_x\delta u:\nabla_x\eta dv=-\int_{\varphi(\Omega)}(\sigma:\nabla_x\eta-b\cdot\eta)dv+\int_{\varphi(\partial\Omega_t)}t\cdot\eta da \tag{3.14}
$$

a 的另一个表达形式如下(推导过程见(5.15)(5.16))：

$$
\mathrm{a}_{ijkl}=J^{-1}\frac{\partial\tau_{ij}}{\partial F_{km}}F_{lm}-\sigma_{il}\delta_{jk} \tag{3.15}
$$

4　超弹性本构

4.1　通用公式

4.1.1　应力计算公式

应力计算通用公式：

$$
\begin{aligned}
P&=\frac{\partial\psi}{\partial F}=2F\cdot\frac{\partial\psi}{\partial C} \\
\tau&=\frac{\partial\psi}{\partial F}\cdot F^T=2F\cdot\frac{\partial\psi}{\partial C}\cdot F^T \\
\sigma&=J^{-1}\frac{\partial\psi}{\partial F}\cdot F^T=2J^{-1}F\cdot\frac{\partial\psi}{\partial C}\cdot F^T
\end{aligned}
\tag{4.1}
$$

各向同性假设前提下的公式：

$$
\begin{aligned}
P&=2\frac{\partial\psi}{\partial B}\cdot F \\
\tau&=2\frac{\partial\psi}{\partial B}\cdot B=2B\cdot\frac{\partial\psi}{\partial B} \\
\sigma&=2J^{-1}\frac{\partial\psi}{\partial B}\cdot B=2J^{-1}B\cdot\frac{\partial\psi}{\partial B}
\end{aligned}
\tag{4.2}
$$

各向同性假设前提下的主量表达形式：

$$
\begin{aligned}
&\tau=\sum_i\frac{\partial\psi}{\partial\lambda_i}\lambda_i\hat{e}_i\hat{e}_i \\
&\{\hat{e}_i\}\ \text{is an orthonormal basis of eigenvector of } \mathbf{V}(\text{or } \mathbf{B})
\end{aligned}
\tag{4.3}
$$

各向同性假设前提下的主伸长量表达形式：

$$
\psi(V)=\hat{\psi}(\lambda_1,\lambda_2,\lambda_3)=\tilde{\psi}(b_1,b_2,b_3) \tag{4.4}
$$

$$
\tau=\sum_i\frac{\partial\hat{\psi}}{\partial\lambda_i}\lambda_i e_i\otimes e_i=\sum_i 2\frac{\partial\tilde{\psi}}{\partial b_i}b_i e_i\otimes e_i \tag{4.5}
$$

4.1.2　Tangent modulus 推导

由(3.15)的 spatial tangent modulus 的表达式：

$$a_{ijkl} = J^{-1}\frac{\partial \tau_{ij}}{\partial F_{km}}F_{lm} - \sigma_{il}\delta_{jk} = 2\frac{\partial \tau_{ij}}{\partial B_{km}}B_{ml} - \sigma_{il}\delta_{jk} \tag{4.6}$$

$$\begin{aligned}
&\frac{\partial \tau_{ij}}{\partial F_{km}} = \frac{\partial \tau_{ij}}{\partial B_{rs}}\frac{\partial F_{rq}F_{sq}}{\partial F_{km}} = \frac{\partial \tau_{ij}}{\partial B_{rs}}(\delta_{rk}F_{sm} + F_{rm}\delta_{sk}) \\
&\frac{\partial \tau_{ij}}{\partial F_{km}}F_{lm} = \frac{\partial \tau_{ij}}{\partial B_{rs}}(\delta_{rk}F_{sm} + F_{rm}\delta_{sk})F_{lm} \\
&= \frac{\partial \tau_{ij}}{\partial B_{rs}}(\delta_{rk}B_{sl} + B_{rl}\delta_{sk}) = \frac{\partial \tau_{ij}}{\partial B_{rs}}\delta_{rk}B_{sl} + \frac{\partial \tau_{ij}}{\partial B_{sr}}B_{sl}\delta_{rk} \\
&= 2\frac{\partial \tau_{ij}}{\partial B_{rm}}B_{ml}\delta_{rk} = 2\frac{\partial \tau_{ij}}{\partial B_{km}}B_{ml}
\end{aligned} \tag{4.7}$$

该四阶矩阵（spatial tangent modulus）具有对称性，即：$\mathbf{a}_{ijkl} = \mathbf{a}_{klij}$

4.2 The general elastic predictor/return-mapping algorithm

4.2.1 有限应变弹塑性初始值问题

运动学变量定义：

$$F^e = F\cdot(F^p)^{-1} \qquad V^e = [F^e\cdot F^{eT}]^{1/2} \qquad \varepsilon^e = \ln V^e \qquad R^e = (V^e)^{-1}\cdot F^e \tag{4.8}$$

$$F_\Delta = F_{n+1}\cdot F_n^{-1} = I + \nabla_{x_n}u_{n+1} = I + \nabla_{x_n}[\Delta u]$$

$$\text{incremental deformation gradient: maps the configuration of time } t_{\text{n}} \text{ onto the configuration of } t_{\text{n+1}} \tag{4.9}$$

在各向同性本构的基础上有：

$$\tau = 2\frac{\partial \psi}{\partial B}\cdot B = 2B\cdot\frac{\partial \psi}{\partial B} \tag{4.10}$$

$$\frac{\partial \psi}{\partial B^e} = \frac{\partial \psi}{\partial(\ln V^e)} : \frac{\partial(\frac{1}{2}\ln B^e)}{\partial B^e} = \frac{1}{2}\frac{\partial \psi}{\partial \varepsilon^e} : \frac{\partial(\ln B^e)}{\partial B^e} \tag{4.11}$$

将(4.11)带入(4.10)得：

$$\tau = \frac{\partial \psi}{\partial \varepsilon^e} : \frac{\partial(\ln B^e)}{\partial B^e}\cdot B = \frac{\partial \psi}{\partial \varepsilon^e} \tag{4.12}$$

问题描述：给定初始值 $F^p(t_0),\alpha(t_0)$ 和一段时间得变形历史 $F(t), t\in[t_0,T]$，找出内变量函数 $F^p(t),\alpha(t),\dot{\gamma}(t)$ 以满足(4.13)

$$\begin{cases}
\dot{F}^p(t)[F^p(t)]^{-1} = \dot{\gamma}(t)R^e(t)^T\frac{\partial \Phi}{\partial \tau}\big|_t R^e(t) & \text{(a)} \\
\dot{\alpha}(t) = \dot{\gamma}(t)H(\tau(t),A(t)) & \text{(b)} \\
\dot{\gamma}(t)\ge 0, \qquad \Phi(\tau(t),A(t))\le 0, \qquad \dot{\gamma}(t)\Phi(\tau(t),A(t)) = 0 & \text{(c)}
\end{cases} \tag{4.13}$$

其中 $H = -\dfrac{\partial \Phi}{\partial A}$

4.2.2　有限应变弹塑性初始值问题指数映射的积分

采用后向欧拉法对(4.13)(b~c)进行离散得到:

$$\begin{cases}\alpha_{n+1}=\alpha_n+\Delta\gamma H_{n+1}\\ \Delta\gamma\ge 0,\qquad \Phi(\tau_{n+1},A_{n+1})\le 0,\qquad \Delta\gamma\Phi(\tau_{n+1},A_{n+1})=0\end{cases}\tag{4.14}$$

利用后向指数积分法对(4.13)(a)进行离散得到：

$$\begin{aligned}F^p_{n+1}&=\exp[\Delta\gamma\frac{\partial\Phi}{\partial\tau}|_{n+1}\,R^e_{n+1}]F^p_n\\&=R^{eT}_{n+1}\exp[\Delta\gamma\frac{\partial\Phi}{\partial\tau}|_{n+1}]R^e_{n+1}F^p_n\end{aligned}\tag{4.15}$$

Remark:采用一般的后向欧拉积分不能满足塑性不可压本构的保体积特征，因此采用后向指数积分法。

$$\begin{aligned}F^e_{n+1}&=F_{n+1}\cdot(F^p_{n+1})^{-1}=F_{n+1}\cdot(F^p_n)^{-1}\cdot R^{eT}_{n+1}\exp[-\Delta\gamma\frac{\partial\Phi}{\partial\tau}|_{n+1}]R^e_{n+1}\\&=F_{n+1}\cdot(F^p_n)^{-1}\cdot R^{eT}_{n+1}\exp[-\Delta\gamma\frac{\partial\Phi}{\partial\tau}|_{n+1}]R^e_{n+1}\\&=F_{n+1}\cdot(F_n)^{-1}\cdot F^e_n\cdot R^{eT}_{n+1}\exp[-\Delta\gamma\frac{\partial\Phi}{\partial\tau}|_{n+1}]R^e_{n+1}\\&=F_\Delta\cdot F^e_n\cdot R^{eT}_{n+1}\exp[-\Delta\gamma\frac{\partial\Phi}{\partial\tau}|_{n+1}]R^e_{n+1}\end{aligned}\tag{4.16}$$

4.2.3　有限应变弹塑性初始值问题的增量形式

与有限应变弹塑性初始值问题不同的是通过采用(4.16)得到缩减形式的方程（F^e而不是F^p作为基本变量）

$$\begin{cases}F^e_{n+1}=F_\Delta\cdot F^e_n\cdot R^{eT}_{n+1}\exp[-\Delta\gamma\frac{\partial\Phi}{\partial\tau}|_{n+1}]R^e_{n+1} & \text{(a)}\\ \alpha_{n+1}=\alpha_n+\Delta\gamma H_{n+1} & \text{(b)}\\ \Delta\gamma\ge 0,\qquad \Phi(\tau_{n+1},A_{n+1})\le 0,\qquad \Delta\gamma\Phi(\tau_{n+1},A_{n+1})=0 & \text{(c)}\end{cases}\tag{4.17}$$

其中$\tau_{n+1}=\frac{\partial\psi}{\partial\varepsilon^e}|_{n+1}\qquad A_{n+1}=\frac{\partial\psi}{\partial\alpha}|_{n+1}\qquad H=-\frac{\partial\Phi}{\partial A}$

4.2.4　The elastic predictor/return-mapping scheme

首先令(4.17)中$\Delta\gamma=0$得到elastic trial state：

$$F^{e\ \text{trial}}_{n+1}=F_\Delta\cdot F^e_n\qquad \alpha^{\text{trial}}_{n+1}=\alpha_n\tag{4.18}$$

$$\begin{cases}F^e_{n+1}=F^{e\ \text{trial}}_n\cdot R^{eT}_{n+1}\exp[-\Delta\gamma\frac{\partial\Phi}{\partial\tau}|_{n+1}]R^e_{n+1} & \text{(a)}\\ \alpha_{n+1}=\alpha^{\text{trial}}_n+\Delta\gamma H_{n+1} & \text{(b)}\quad\Rightarrow F^e_{n+1},\alpha_{n+1},\Delta\gamma\\ \Phi(\tau_{n+1},A_{n+1})=0 & \text{(c)}\end{cases}\tag{4.19}$$

通过用$\varepsilon^e=\ln V$代替F^e作为基本变量，(4.19)还可以进行进一步的简化：

首先由(4.19)(a)可以得到（再各向同性弹塑性的基础假设下，推导过程用到了$V^e,\partial\Phi/\partial\tau$共轴可交换）：

$$\varepsilon_{n+1}^{e}=\varepsilon_{n+1}^{e\ \mathrm{trial}}-\Delta\gamma\frac{\partial\Phi}{\partial\tau}\Big|_{n+1} \tag{4.20}$$

the return-mapping equation system of the finite strain incremental problem is reduced to：

$$\begin{cases}\varepsilon_{n+1}^{e}=\varepsilon_{n+1}^{e\ \mathrm{trial}}-\Delta\gamma\dfrac{\partial\Phi}{\partial\tau}\Big|_{n+1} & \text{(a)}\\ \alpha_{n+1}=\alpha_{n}^{\mathrm{trial}}+\Delta\gamma H_{n+1} & \text{(b)}\quad\Rightarrow\varepsilon_{n+1}^{e},\alpha_{n+1},\Delta\gamma\\ \Phi(\tau_{n+1},A_{n+1})=0 & \text{(c)}\end{cases} \tag{4.21}$$

确定了ε_{n+1}^{e}相当于确定了V_{n+1}^{e}，但是$F_{n+1}^{e}=V_{n+1}^{e}R_{n+1}^{e}$，因此还需要确定$R_{n+1}^{e}$，实际上有（推导见 CMFP）：

$$R_{n+1}^{e}=R_{n+1}^{e\ \mathrm{trail}} \tag{4.22}$$

Box 14.3. General integration algorithm for isotropic multiplicative finite strain elastoplasticity.

HYPLAS procedure:	MATISU

(i) Given incremental displacement $\Delta\boldsymbol{u}$, update the deformation gradient

$$\boldsymbol{F}_{\Delta}:=\boldsymbol{I}+\nabla_n[\Delta\boldsymbol{u}],\quad \boldsymbol{F}_{n+1}:=\boldsymbol{F}_{\Delta}\,\boldsymbol{F}_n$$

(ii) Compute elastic trial state

$$\begin{aligned}\boldsymbol{B}_n^e\ &:=\exp[2\,\boldsymbol{\varepsilon}_n^e]\\ \boldsymbol{B}_{n+1}^{e\ \mathrm{trial}}&:=\boldsymbol{F}_{\Delta}\,\boldsymbol{B}_n^e\,(\boldsymbol{F}_{\Delta})^T\\ \boldsymbol{\varepsilon}_{n+1}^{e\ \mathrm{trial}}&:=\ln[\boldsymbol{V}_{n+1}^{e\ \mathrm{trial}}]=\tfrac{1}{2}\ln[\boldsymbol{B}_{n+1}^{e\ \mathrm{trial}}]\\ \boldsymbol{\alpha}_{n+1}^{\mathrm{trial}}&:=\boldsymbol{\alpha}_n\end{aligned}$$

(iii) GOTO BOX 14.4 – small-strain algorithm (update $\boldsymbol{\tau}$, $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\alpha}$)

(iv) Update the Cauchy stress

$$\boldsymbol{\sigma}_{n+1}:=\det[\boldsymbol{F}_{n+1}]^{-1}\,\boldsymbol{\tau}_{n+1}$$

Due to the use of the logarithmic elastic strain measure in conjunction with the backward exponential approximation(4.16) tothe plastic flow rule, the essential material-related stress-updating procedure, shown in Box 14.4, preserves the format of the general elastic predictor/return-mapping algorithm for infinitesimal plasticity described in Chapter 7 and summarised in Box 7.1.

It is important to emphasise that the simplicity of the integration algorithm of Boxes 14.3 and 14.4 came as a result of the assumptions of elastoplastic isotropy!!!1

Box 14.4. General integration procedure – small strains.

(i) Given $\varepsilon_{n+1}^{e\ \text{trial}}$ and $\boldsymbol{\alpha}_{n+1}^{\text{trial}}$, compute

$$\boldsymbol{\tau}_{n+1}^{\text{trial}} = \bar{\rho}\left.\frac{\partial \psi}{\partial \varepsilon^e}\right|_{n+1}^{\text{trial}}, \qquad \boldsymbol{A}_{n+1}^{\text{trial}} = \bar{\rho}\left.\frac{\partial \psi}{\partial \boldsymbol{\alpha}}\right|_{n+1}^{\text{trial}}$$

(ii) Check plastic admissibility

$$\text{IF } \Phi(\boldsymbol{\tau}_{n+1}^{\text{trial}}, \boldsymbol{A}_{n+1}^{\text{trial}}) \le 0$$

$$\text{THEN set} \quad (\cdot)_{n+1} := (\cdot)_{n+1}^{\text{trial}} \quad \text{and EXIT}$$

(iii) Return mapping. Solve the algebraic system

$$\begin{Bmatrix} \varepsilon_{n+1}^e - \varepsilon_{n+1}^{e\ \text{trial}} + \Delta\gamma \left.\dfrac{\partial \Psi}{\partial \boldsymbol{\tau}}\right|_{n+1} \\ \boldsymbol{\alpha}_{n+1} - \boldsymbol{\alpha}_n - \Delta\gamma\, \boldsymbol{H}(\boldsymbol{\tau}_{n+1}, \boldsymbol{A}_{n+1}) \\ \Phi(\boldsymbol{\tau}_{n+1}, \boldsymbol{A}_{n+1}) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

for ε_{n+1}^e, $\boldsymbol{\alpha}_{n+1}$ and $\Delta\gamma$, with

$$\boldsymbol{\tau}_{n+1} = \bar{\rho}\left.\frac{\partial \psi}{\partial \varepsilon^e}\right|_{n+1}, \qquad \boldsymbol{A}_{n+1} = \bar{\rho}\left.\frac{\partial \psi}{\partial \boldsymbol{\alpha}}\right|_{n+1}$$

(iv) EXIT

4.3 Hencky material

The Hency model is the finite logarithmic strain-based extension of the standard linear elastic material.

能量方程：

$$\varepsilon = \ln V = \frac{1}{2}\ln B$$

$$\psi = \frac{1}{2}\varepsilon : D : \varepsilon \qquad\qquad D = 2G\mathbb{I}^{sys} + (K - \frac{2}{3}G)I \otimes I \tag{4.23}$$

$$\frac{1}{2}\varepsilon : (2G\mathbb{I}^{sym}) : \varepsilon = G\varepsilon_{ij}\varepsilon_{ij} = G\sum_i (\ln\lambda_i)^2$$

$$\frac{1}{2}\varepsilon : (K - \frac{2}{3}G)I \otimes I : \varepsilon = \frac{1}{2}(K - \frac{2}{3}G)\varepsilon_{ii}\varepsilon_{jj} = \frac{1}{2}(K - \frac{2}{3}G)(\ln\lambda_1\lambda_2\lambda_3)^2 \tag{4.24}$$

$$\psi = \frac{1}{2}\varepsilon : D : \varepsilon = G\sum_i (\ln\lambda_i)^2 + \frac{1}{2}(K - \frac{2}{3}G)(\ln\lambda_1\lambda_2\lambda_3)^2$$

$$\tau = \frac{\partial \psi}{\partial \varepsilon} = \frac{1}{2}\frac{\partial \varepsilon_{rs} D_{rsmn}\varepsilon_{mn}}{\partial \varepsilon_{ij}} = D_{ijmn}\varepsilon_{mn} = D : \varepsilon \tag{4.25}$$

4.4 Neo-Hookean

4.4.1 Abaqus version with N=1

The user can request that Abaqus calculate the C_{10} and D_1 values from measurements of nominal stress and strain in simple experiments. The basis of this calculation is described in Fitting of hyperelastic and hyperfoam constants.

$$\begin{aligned}\psi &= C_{10}(\bar{I}_1-3)+\frac{1}{D_1}(J-1)^2 \\ &= \frac{G}{2}(\bar{I}_1-3)+\frac{K}{2}(J-1)^2\end{aligned} \tag{4.26}$$

$$\begin{aligned}\tau &= 2C_{10}\mathrm{dev}(B^{iso})+\frac{2}{D_1}J(J-1)I \\ &= G\mathrm{dev}(B^{iso})+KJ(J-1)I\end{aligned} \tag{4.27}$$

Spatial tangent modulus 推导见

$$\frac{\partial\tau}{\partial B}=G\mathbb{I}^{\mathrm{dev}}:\frac{\partial B^{iso}}{\partial B}+K(2J-1)I\frac{\partial J}{\partial B} \tag{4.28}$$

$$\frac{\partial B_{iso}}{\partial B}=-\frac{2}{3}J^{-\frac{5}{3}}B\frac{1}{2}JB^{-1}+J^{-\frac{2}{3}}\mathbb{I}^{sym}=-\frac{1}{3}J^{-\frac{2}{3}}BB^{-1}+J^{-\frac{2}{3}}\mathbb{I}^{sym} \qquad \frac{\partial J}{\partial B}=\frac{1}{2}JB^{-1} \tag{4.29}$$

将(4.29)带入(4.28)得到应力应变导数表达式：

$$\begin{aligned}\frac{\partial\tau}{\partial B} &= G\mathbb{I}^{\mathrm{dev}}:(-\frac{1}{3}J^{-\frac{2}{3}}BB^{-1}+J^{-\frac{2}{3}}\mathbb{I}^{sym})+K(2J-1)I\frac{1}{2}JB^{-1} \\ &= GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}}-\frac{1}{3}Gdev[B^{iso}]B^{-1}+\frac{KJ(2J-1)}{2}IB^{-1} \\ &= GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}}-\frac{1}{3}\tau^{dev}B^{-1}+\frac{KJ(2J-1)}{2}IB^{-1}\end{aligned} \tag{4.30}$$

将(4.30)带入(4.6)得到 spatial tangent modulus（推导见(5.23)~(5.26)）：

$$\begin{aligned}\mathrm{a}_{ijkl} &= 2\frac{\partial\tau}{\partial B}\cdot B-\sigma_{il}\delta_{jk} \\ &= (2GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}}-\frac{2}{3}\tau^{dev}B^{-1}+KJ(2J-1)IB^{-1})\cdot B-\sigma_{il}\delta_{jk}\end{aligned} \tag{4.31}$$

$$\begin{aligned}&\mathbf{a}_{ijkl}=C_{ijkl}++\delta_{ik}\sigma_{jl} \\ &C=\frac{2G}{3J}\mathrm{tr}[B^{iso}]\mathbb{I}^{dev}-2p\mathbb{I}^{sym}-\frac{2}{3}(I\otimes s+s\otimes I)+K(2J-1)I\otimes I \qquad p=\mathrm{tr}(\sigma) \qquad \mathrm{s}{=}\sigma^{dev}\end{aligned} \tag{4.32}$$

4.4.2 CMFP version

应力表达式：

$$\begin{aligned}&\psi=\frac{G}{2}(\bar{I}_1-3)+\frac{K}{2}(\ln J)^2 \\ &\tau=G\mathrm{dev}(B^{iso})+K(\ln J)I=G\mathbb{I}^{\mathrm{dev}}:B^{iso}+K(\ln J)I\end{aligned} \tag{4.33}$$

Spatial tangent modulus 推导((4.35)推导过程见(5.17)~(5.19))：

$$\frac{\partial\tau}{\partial B}=G\mathbb{I}^{\mathrm{dev}}:\frac{\partial B^{iso}}{\partial B}+\frac{K}{J}I\frac{\partial J}{\partial B} \tag{4.34}$$

$$\frac{\partial B_{iso}}{\partial B}=-\frac{2}{3}J^{-\frac{5}{3}}B\frac{\partial J}{\partial B}+J^{-\frac{2}{3}}\mathbb{I}^{sym} \qquad \frac{\partial J}{\partial B}=\frac{1}{2}JB^{-1} \tag{4.35}$$

将(4.35)带入(4.34)得到应力应变导数：

$$
\begin{aligned}
\frac{\partial \tau}{\partial B} &= G\mathbb{I}^{\mathrm{dev}} : (-\frac{1}{3}J^{-\frac{2}{3}}BB^{-1} + J^{-\frac{2}{3}}\mathbb{I}^{sym}) + \frac{1}{2}KIB^{-1} \\
&= GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} - \frac{1}{3}Gdev[B^{iso}]B^{-1} + \frac{1}{2}KIB^{-1} \\
&= GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} - \frac{1}{3}\tau^{dev}B^{-1} + \frac{1}{2}KIB^{-1}
\end{aligned}
\tag{4.36}
$$

将(4.36)带入(4.6)得到 spatial tangent modulus（推导见(5.20)~(5.22)）：

$$
\begin{aligned}
&\mathbf{a}_{ijkl} = C_{ijkl} + +\delta_{ik}\sigma_{jl} \\
&C = \frac{2G}{3J}\mathrm{tr}[B^{iso}]\mathbb{I}^{dev} - 2p\mathbb{I}^{sym} - \frac{2}{3}(I\otimes s + s\otimes I) + \frac{K}{J}I\otimes I \qquad p = \mathrm{tr}(\sigma) \qquad \mathrm{s}=\sigma^{dev}
\end{aligned}
\tag{4.37}
$$

4.5 Regularised ogden matrial (refer to CMFP)

At very large strains, it is a well-known fact that the neo-Hookean and Mooney–Rivlin models fail to represent the behaviour of rubbery materials

首先，令 $\{\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3\}$ 是*principal isochoric stretche*，即 $V^{iso} = \sqrt{B^{iso}}$ 的特征值。

$$
\bar{\lambda}_i = J^{-\frac{1}{3}}\lambda_i = \lambda_i^{2/3}(\lambda_j\lambda_k)^{-1/3} \tag{4.38}
$$

自由能表达式为：

$$
\psi = \sum_{p=1}^{N}\frac{\mu_p}{\alpha_p}[\bar{\lambda}_1^{\alpha_p} + \bar{\lambda}_2^{\alpha_p} + \bar{\lambda}_3^{\alpha_p} - 3] + \frac{1}{2}K(\ln J)^2 \tag{4.39}
$$

应力推导由公式(4.3)可得应力主量表达式：

应力 τ 与 B 与 V 具有相同的单位正交特征向量

$$
\begin{aligned}
\tau_i &= \frac{\partial \psi}{\partial \lambda_i}\lambda_i = \lambda_i(\frac{\partial \psi}{\partial \bar{\lambda}_i}\frac{\partial \bar{\lambda}_i}{\partial \lambda_i} + \frac{\partial \psi}{\partial \bar{\lambda}_j}\frac{\partial \bar{\lambda}_j}{\partial \lambda_{ii}} + \frac{\partial \psi}{\partial \bar{\lambda}_k}\frac{\partial \bar{\lambda}_k}{\partial \lambda_{ii}} + \frac{\partial \psi}{\partial J}\frac{\partial J}{\partial \lambda_i}) \\
&= \lambda_i\{\sum_{p=1}^{N}\mu_p[\bar{\lambda}_i^{\alpha_p-1}\frac{\partial \bar{\lambda}_i}{\partial \lambda_i} + \bar{\lambda}_j^{\alpha_p-1}\frac{\partial \bar{\lambda}_j}{\partial \lambda_i} + \bar{\lambda}_k^{\alpha_p-1}\frac{\partial \bar{\lambda}_k}{\partial \lambda_i}] + K\frac{\ln J}{J}\frac{\partial J}{\partial \lambda_i}\} \\
&= \lambda_i\{\sum_{p=1}^{N}\mu_p[\bar{\lambda}_i^{\alpha_p-1}\frac{\partial \bar{\lambda}_i}{\partial \lambda_i} + \bar{\lambda}_j^{\alpha_p-1}\frac{\partial \bar{\lambda}_j}{\partial \lambda_i} + \bar{\lambda}_k^{\alpha_p-1}\frac{\partial \bar{\lambda}_k}{\partial \lambda_i}] + K\frac{\ln J}{\lambda_i}\} \\
&= \lambda_i\{\sum_{p=1}^{N}\mu_p[\bar{\lambda}_i^{\alpha_p-1}\frac{2}{3}(\lambda_i\lambda_j\lambda_k)^{-1/3} + \bar{\lambda}_j^{\alpha_p-1}(-\frac{1}{3}\lambda_i^{-4/3}\lambda_j^{2/3}\lambda_k^{-1/3}) + \bar{\lambda}_k^{\alpha_p-1}(-\frac{1}{3}\lambda_i^{-4/3}\lambda_k^{2/3}\lambda_j^{-1/3})] + K\frac{\ln J}{\lambda_i}\} \\
&= \lambda_i\{\sum_{p=1}^{N}\mu_p[\bar{\lambda}_i^{\alpha_p}\bar{\lambda}_i^{-1}\frac{2}{3}(\lambda_i\lambda_j\lambda_k)^{-1/3} + \bar{\lambda}_j^{\alpha_p}\bar{\lambda}_i^{-1}(-\frac{1}{3}\lambda_i^{-1/3}\lambda_j^{-1/3}\lambda_k^{-1/3}) + \bar{\lambda}_k^{\alpha_p}\bar{\lambda}_i^{-1}(-\frac{1}{3}\lambda_i^{-1/3}\lambda_k^{-1/3}\lambda_j^{-1/3})] + K\frac{\ln J}{\lambda_i}\} \\
&= \lambda_i\{\sum_{p=1}^{N}\mu_p[J^{-\alpha_p/3}\lambda_i^{\alpha_p}\lambda_i^{-1} + \bar{\lambda}_i^{\alpha_p}(-\frac{1}{3}\lambda_i^{-1}) + \bar{\lambda}_j^{\alpha_p}(-\frac{1}{3}\lambda_i^{-1}) + \bar{\lambda}_k^{\alpha_p}(-\frac{1}{3}\lambda_i^{-1})] + K\frac{\ln J}{\lambda_i}\} \\
&= \sum_{p=1}^{N}\mu_p[J^{-\alpha_p/3}\lambda_i^{\alpha_p} + J^{-\alpha_p/3}\lambda_i^{\alpha_p}(-\frac{1}{3}) + J^{-\alpha_p/3}\lambda_j^{\alpha_p}(-\frac{1}{3}) + J^{-\alpha_p/3}\lambda_k^{\alpha_p}(-\frac{1}{3})] + K\ln J \\
&= \sum_{p=1}^{N}\mu_p J^{-\alpha_p/3}[\lambda_i^{\alpha_p} - \frac{1}{3}(\lambda_i^{\alpha_p} + \lambda_j^{\alpha_p} + \lambda_k^{\alpha_p})] + K\ln J
\end{aligned}
$$

(4.40)

spatial tangent modulus 推导：

$$\tau_i = \hat{\tau}(\lambda_i, \lambda_j, \lambda_k) = \tilde{\tau}(b_i, b_j, b_k) \qquad b_i = \lambda_i^2 \tag{4.41}$$

$$\begin{aligned}
\tau_i &= \frac{\partial \psi}{\partial \lambda_i}\lambda_i = \sum_{p=1}^{N} \mu_p J^{-\alpha_p/3}[\lambda_i^{\alpha_p} - \frac{1}{3}(\lambda_i^{\alpha_p} + \lambda_j^{\alpha_p} + \lambda_k^{\alpha_p})] + K \ln J \\
\frac{\partial \tau_i}{\partial b_j} &= \frac{\partial \tau_i}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial b_j} = \sum_{p=1}^{N} \frac{\mu_p \alpha_p J^{-\alpha_p/3}}{6\lambda_j^2}[\frac{1}{3}(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p}) - \lambda_i^{\alpha_p} - \lambda_j^{\alpha_p} + 3\lambda_i^{\alpha_p}\delta_{ij}] + \frac{K}{2\lambda_j^2}
\end{aligned} \tag{4.42}$$

有了(4.42)，通过章节 2.2 的 Box A.3，Box A.6 就可求得$\partial \tau / \partial B$

5 公式推导

5.1 虚功原理的线化

小应变虚功原理线化：

$$DG(u^*, \eta)[\delta u] = \frac{d}{dc}\Big|_{c=0} \int_\Omega \sigma(u^* + c\delta u) : \nabla^s \eta dv \tag{5.1}$$

$$\frac{d}{dc}[\sigma(u^* + c\delta u) : \nabla^s \eta] = \frac{d\sigma(u^* + c\delta u)}{dc} : \nabla^s \eta \tag{5.2}$$

$$\begin{aligned}
&\frac{\partial \varepsilon}{\partial u}\cdot \delta u = \frac{\partial}{\partial c}(\frac{\partial(u_i + c\delta u_i)}{\partial x_j} + \frac{\partial(u_j + c\delta u_j)}{\partial x_i}) = \frac{\partial(\delta u_i)}{\partial x_j} + \frac{\partial(\delta u_j)}{\partial x_i} = \nabla^s(\delta u) \\
&\frac{d\sigma(u^* + c\delta u)}{dc} = \frac{\partial \sigma}{\partial \varepsilon} : \frac{\partial \varepsilon}{\partial u}\cdot \delta u = D : \nabla^s(\delta u)
\end{aligned} \tag{5.3}$$

(5.3)带入(5.2)(5.1)得：

$$DG(u^*, \eta)[\delta u] = \int_\Omega D : \nabla^s(\delta u) : \nabla^s \eta dv \tag{5.4}$$

有限应变材料描述虚功原理线化：

$$DG(u^*, \eta)[\delta u] = \frac{d}{dc}\Big|_{c=0} DG(u^* + c\delta u, \eta) = \int_\Omega \frac{d}{dc}\Big|_{c=0} P(u^* + c\delta u) : \nabla_p \eta dv \tag{5.5}$$

$$\frac{d}{dc}\Big|_{c=0} P(u^* + c\delta u) = \frac{\partial P}{\partial F} : \frac{\partial F}{\partial u}\cdot \delta u \tag{5.6}$$

$$\begin{aligned}
&\frac{\partial F}{\partial u}\cdot \delta u = \frac{\partial}{\partial c}(I_{ij} + \frac{\partial(u_i + c\delta u_i)}{\partial p_j}) = \frac{\partial(\delta u_i)}{\partial p_j} = \nabla_p \delta u \\
&\frac{dP(u^* + c\delta u)}{dc} = \frac{\partial P}{\partial F} : \frac{\partial F}{\partial u}\cdot \delta u = A : \nabla_p \delta u
\end{aligned} \tag{5.7}$$

带入(5.5)得到：

$$DG(u^*, \eta)[\delta u] = \int_\Omega A : \nabla_p \delta u : \nabla_p \eta dv \tag{5.8}$$

空间描述和材料描述虚功原理等价证明：

$$\begin{aligned}
G(u, \eta) &= \int_{\varphi(\Omega)} (\sigma : \nabla_x \eta - b\cdot \eta) dv - \int_{\varphi(\partial \Omega_t)} t\cdot \eta da \\
&= \int_\Omega (\sigma : \nabla_x \eta J - b\cdot \eta J) dv_0 - \int_{\varphi(\partial \Omega_t)} t\cdot \eta da
\end{aligned} \tag{5.9}$$

$$
\begin{aligned}
\sigma : \nabla_x \eta J &= J^{-1}(P \cdot F^T) : (\nabla_p \eta \cdot F^{-1}) J \\
&= P_{ik} F_{jk} \frac{\partial \eta_i}{\partial p_m} (F^{-1})_{mj} = P_{ik} \frac{\partial \eta_i}{\partial p_k} = P : \nabla_p \eta
\end{aligned} \tag{5.10}
$$

$$
b \cdot \eta J = \overline{b} \cdot \eta \qquad (b dV = \overline{b} dV_0 \rightarrow \overline{b} = Jb)
$$

$$
\begin{aligned}
& da = \sqrt{\mathbf{da} \cdot \mathbf{da}} = \sqrt{(JF^{-T} \cdot \mathbf{dA}) \cdot (JF^{-T} \cdot \mathbf{dA})} = J \| F^{-T} \cdot \hat{N} \| dA \qquad (da = \| \mathbf{da} \|,\ dA = \| \mathbf{dA} \|) \\
& t da = \overline{t} dA \\
& \hat{n} = \frac{\mathbf{da}}{da} = \frac{JF^{-T} \cdot \mathbf{dA}}{J \| F^{-T} \cdot \hat{N} \| dA} \qquad \hat{N} = \frac{\mathbf{dA}}{dA} \\
& \overline{t} dA = t da \\
& \overline{t} = tJ \| F^{-T} \cdot \hat{N} \| = J \| F^{-T} \cdot \hat{N} \| \sigma \cdot \hat{n} \\
& \quad = J \| F^{-T} \frac{\mathbf{dA}}{dA} \| \sigma \cdot \frac{F^{-T} \cdot \mathbf{dA}}{\| F^{-T} \cdot \frac{\mathbf{dA}}{dA} \| dA} = P \cdot \frac{\mathbf{dA}}{dA} \\
& t = \sigma \cdot \hat{n}, \qquad \overline{t} = P \cdot \hat{N}
\end{aligned} \tag{5.11}
$$

将(5.10)(5.11)带入(5.9)得到：

$$
G(u,\eta) = \int_\Omega (P : \nabla_p \eta - \overline{b} \cdot \eta) dv_0 - \int_{\partial \Omega_t} \overline{t} \cdot \eta dA \tag{5.12}
$$

有限应变空间描述虚功原理线化：

由空间描述和材料描述虚功原理等价可得：

$$
\begin{aligned}
& DG(u^*,\eta)[\delta u] = \int_\Omega A : \nabla_p \delta u : \nabla_p \eta dv \\
& = \int_{\varphi(\Omega)} J^{-1} A : \nabla_p \delta u : \nabla_p \eta dv = \int_{\varphi(\Omega)} J^{-1} A : (\nabla_x \delta u \cdot F) : (\nabla_x \eta \cdot F) dv \\
& = \int_{\varphi(\Omega)} J^{-1} A_{ijrt} (\nabla_x \delta u)_{rs} F_{st} (\nabla_x \eta)_{ik} F_{kj} dv \\
& = \int_{\varphi(\Omega)} J^{-1} A_{imkn} F_{ln} F_{jm} (\nabla_x \delta u)_{kl} (\nabla_x \eta)_{ij} dv
\end{aligned} \tag{5.13}
$$

$$
\mathrm{a}_{ijkl} = J^{-1} A_{imkn} F_{ln} F_{jm} \tag{5.14}
$$

a 的等效形式推导：

$$
\begin{aligned}
& \frac{d(F \cdot F^{-1})}{dF} = 0 \\
& \frac{dF_{in}}{dF_{kl}} (F^{-1})_{nj} + F_{in} \frac{d(F^{-1})_{nj}}{dF_{kl}} = 0 \\
& \delta_{ik} (F^{-1})_{lj} + F_{in} \frac{d(F^{-1})_{nj}}{dF_{kl}} = 0 \\
& [\delta_{ik} (F^{-1})_{lj} + F_{in} \frac{d(F^{-1})_{nj}}{dF_{kl}}] (F^{-1})_{mi} = 0 \\
& (F^{-1})_{mk} (F^{-1})_{lj} + \frac{d(F^{-1})_{mj}}{dF_{kl}} = 0 \qquad \rightarrow \qquad \frac{d(F^{-1})_{ij}}{dF_{kl}} = -(F^{-1})_{ik} (F^{-1})_{lj}
\end{aligned} \tag{5.15}
$$

$$P = J\sigma \cdot F^{-T} = \tau \cdot F^{-T}$$

$$A_{imkn} = \frac{\partial P_{im}}{\partial F_{kn}} = \frac{\partial \tau_{ip}(F^{-1})_{mp}}{\partial F_{kn}} = \frac{\partial \tau_{ip}}{\partial F_{kn}}(F^{-1})_{mp} + \tau_{ip}\frac{\partial (F^{-1})_{mp}}{\partial F_{kn}}$$

$$= \frac{\partial \tau_{ip}}{\partial F_{kn}}(F^{-1})_{mp} - \tau_{ip}(F^{-1})_{mk}(F^{-1})_{np} \tag{5.16}$$

$$\mathrm{a}_{ijkl} = J^{-1}A_{imkn}F_{ln}F_{jm} = J^{-1}[\frac{\partial \tau_{ip}}{\partial F_{kn}}(F^{-1})_{mp}F_{ln}F_{jm} - \tau_{ip}(F^{-1})_{mk}(F^{-1})_{np}F_{ln}F_{jm}]$$

$$= 2J^{-1}[\frac{\partial \tau_{ij}}{\partial F_{kn}}F_{ln} - \tau_{il}\delta_{jk}] = 2J^{-1}\frac{\partial \tau_{ij}}{\partial F_{km}}F_{lm} - \sigma_{il}\delta_{jk}$$

5.2 超弹性本构

$\dfrac{\partial B^{iso}}{\partial B}$ 推导：

$$\frac{\partial B^{iso}}{\partial B} = -\frac{2}{3}J^{-\frac{5}{3}}B\frac{\partial J}{\partial B} + J^{-\frac{2}{3}}\mathbb{I}^{sym} \qquad \frac{\partial J}{\partial B} = \frac{1}{2}JB^{-1} \tag{5.17}$$

$$\frac{\partial B^{iso}}{\partial B} = \frac{\partial J^{-\frac{2}{3}}B}{\partial B} = \frac{\partial J^{-\frac{2}{3}}}{\partial B}B + J^{-\frac{2}{3}}\frac{\partial B}{\partial B} = \frac{\partial J^{-\frac{2}{3}}}{\partial J^2}\frac{\partial J^2}{B}B + J^{-\frac{2}{3}}\frac{\partial B}{\partial B} = -\frac{1}{3}(J^2)^{-\frac{4}{3}}J^2B^{-1}B + J^{-\frac{2}{3}}\mathbb{I}^{sym}$$
$$= -\frac{1}{3}(J^2)^{-\frac{4}{3}}J^2B^{-1} + J^{-\frac{2}{3}}\mathbb{I}^{sym} = -\frac{1}{3}J^{-\frac{2}{3}}B^{-1}B + J^{-\frac{2}{3}}\mathbb{I}^{sym} \tag{5.18}$$

$$\begin{aligned}&\frac{\partial B_{iso}}{\partial B} = -\frac{2}{3}J^{-\frac{5}{3}}B\frac{\partial J}{\partial B} + J^{-\frac{2}{3}}\mathbb{I}^{sym} \qquad \frac{\partial J}{\partial B} = \frac{1}{2}JB^{-1}\\ &\frac{\partial B_{iso}}{\partial B} = -\frac{2}{3}J^{-\frac{5}{3}}B\frac{1}{2}JB^{-1} + J^{-\frac{2}{3}}\mathbb{I}^{sym} = -\frac{1}{3}J^{-\frac{2}{3}}BB^{-1} + J^{-\frac{2}{3}}\mathbb{I}^{sym}\end{aligned} \tag{5.19}$$

Neo-Hookean CMFP version spatial modulus:

$$\mathrm{a}_{ijkl} = \frac{2}{J}\frac{\partial \tau_{ij}}{\partial B_{km}}B_{ml} - \sigma_{il}\delta_{jk}$$

$$\frac{\partial \tau_{ij}}{\partial B_{km}}B_{ml} = \frac{\partial \tau}{\partial B}\cdot B = (GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} - \frac{1}{3}\tau^{dev}B^{-1} + \frac{1}{2}KIB^{-1})\cdot B$$

$$\frac{2}{J}\frac{\partial \tau_{ij}}{\partial B_{km}}B_{ml} = \frac{2}{J}GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}}\cdot B - \frac{2G}{3J}\mathrm{dev}(B^{iso})I + \frac{K}{J}I\otimes I \tag{5.20}$$

$$GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}}\cdot B = [\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}]GB^{iso}_{lm} = \frac{1}{2}(\delta_{ik}GB^{iso}_{jm} + GB^{iso}_{im}\delta_{jk}) - \frac{1}{3}\delta_{ij}GB^{iso}_{km}$$

$$B^{iso} = \mathrm{dev}(B^{iso}) + \frac{1}{3}tr[B^{iso}]I$$

$$GB^{iso} = \tau^{dev} + \frac{1}{3}G\mathrm{tr}[B^{iso}]I$$

$$\frac{2}{J}GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} \cdot B = (\delta_{ik}(\sigma^{dev}_{jm} + \frac{G}{3J}\mathrm{tr}[B^{iso}]\delta_{jm}) + (\sigma^{dev}_{im} + \frac{G}{3J}\mathrm{tr}[B^{iso}]\delta_{im})\delta_{jk}) - \frac{2}{3}\delta_{ij}(\sigma^{dev}_{km} + \frac{G}{3J}\mathrm{tr}[B^{iso}]\delta_{km})$$

$$= \delta_{ik}\sigma^{dev}_{jl} + \sigma^{dev}_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\sigma^{dev}_{kl} + \frac{G}{3J}\mathrm{tr}[B^{iso}](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})$$

$$\sigma_{ij} = \sigma^{dev}_{ij} + p\delta_{ij} \qquad p = \frac{1}{3}\mathrm{tr}(\sigma)$$

$$-\sigma_{il}\delta_{jk} = -(\sigma^{dev}_{il} + p\delta_{il})\delta_{jk}$$

$$\frac{2}{J}GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} \cdot B - \sigma_{il}\delta_{jk} = \delta_{ik}\sigma^{dev}_{jl} - \frac{2}{3}\delta_{ij}\sigma^{dev}_{kl} + \frac{G}{3J}\mathrm{tr}[B^{iso}](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}) - p\delta_{il}\delta_{jk}$$

$$= \delta_{ik}\sigma_{jl} - p\delta_{ik}\delta_{jl} - \frac{2}{3}\delta_{ij}\sigma^{dev}_{kl} + \frac{G}{3J}\mathrm{tr}[B^{iso}](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}) - p\delta_{il}\delta_{jk}$$

$$= \delta_{ik}\sigma_{jl} - \frac{2}{3}\delta_{ij}\sigma^{dev}_{kl} + \frac{G}{3J}\mathrm{tr}[B^{iso}](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}) - p(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$$

$$= \delta_{ik}\sigma_{jl} - \frac{2}{3}\delta_{ij}\sigma^{dev}_{kl} + \frac{2G}{3J}\mathrm{tr}[B^{iso}]\mathbb{I}^{dev}_{ijkl} - 2p\mathbb{I}^{sym}_{ijkl}$$

(5.21)

$$\begin{aligned}
\mathrm{a}_{ijkl} &= \frac{2}{J}\frac{\partial \tau_{ij}}{\partial B_{km}}B_{ml} - \sigma_{il}\delta_{jk} = \frac{2}{J}GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} \cdot B - \sigma_{il}\delta_{jk} - \frac{2G}{3J}\mathrm{dev}(B^{iso})I + \frac{K}{J}I \otimes I \\
&= \delta_{ik}\sigma_{jl} - \frac{2}{3}\delta_{ij}\sigma^{dev}_{kl} + \frac{2G}{3J}\mathrm{tr}[B^{iso}]\mathbb{I}^{dev}_{ijkl} - 2p\mathbb{I}^{sym}_{ijkl} - \frac{2}{3}\sigma^{dev}_{ij}\delta_{kl} + \frac{K}{J}I \otimes I \\
&= \frac{2G}{3J}\mathrm{tr}[B^{iso}]\mathbb{I}^{dev} - 2p\mathbb{I}^{sym} - \frac{2}{3}\delta_{ij}\sigma^{dev}_{kl} - \frac{2}{3}\sigma^{dev}_{ij}\sigma_{kl} + \frac{K}{J}I \otimes I + \delta_{ik}\sigma_{jl} \\
&= \frac{2G}{3J}\mathrm{tr}[B^{iso}]\mathbb{I}^{dev} - 2p\mathbb{I}^{sym} - \frac{2}{3}(I \otimes s + s \otimes I) + \frac{K}{J}I \otimes I + \delta_{ik}\sigma_{jl}
\end{aligned} \tag{5.22}$$

Neo-Hookean Abaqus version spatial tangent modulus a 推导：

$$\begin{aligned}
\mathrm{a}_{ijkl} &= 2\frac{\partial \tau}{\partial B} \cdot B - \sigma_{il}\delta_{jk} \\
&= (2GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} - \frac{2}{3}\tau^{dev}B^{-1} + KJ(2J-1)IB^{-1}) \cdot B - \sigma_{il}\delta_{jk}
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
&\mathrm{a}_{ijkl} = \frac{2}{J}\frac{\partial \tau_{ij}}{\partial B_{km}}B_{ml} - \sigma_{il}\delta_{jk} \\
&2\frac{\partial \tau_{ij}}{\partial B_{km}}B_{ml} = \frac{\partial \tau}{\partial B} \cdot B = (2GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} - \frac{2}{3}\tau^{dev}B^{-1} + KJ(2J-1)IB^{-1}) \cdot B \\
&\frac{2}{J}\frac{\partial \tau_{ij}}{\partial B_{km}}B_{ml} = \frac{2}{J}GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} \cdot B - \frac{2G}{3J}\mathrm{dev}(B^{iso})I + K(2J-1)I \otimes I \\
&GJ^{-\frac{2}{3}}\mathbb{I}^{\mathrm{dev}} \cdot B = [\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}]GB^{iso}_{lm} = \frac{1}{2}(\delta_{ik}GB^{iso}_{jm} + GB^{iso}_{im}\delta_{jk}) - \frac{1}{3}\delta_{ij}GB^{iso}_{km}
\end{aligned} \tag{5.24}$$

$$\frac{2}{J}GJ^{-\frac{2}{3}}\mathbb{I}^{\text{dev}}\cdot B=(\delta_{ik}(\sigma_{jm}^{dev}+\frac{G}{3J}\text{tr}[B^{iso}]\delta_{jm})+(\sigma_{im}^{dev}+\frac{G}{3J}\text{tr}[B^{iso}]\delta_{im})\delta_{jk})-\frac{2}{3}\delta_{ij}(\sigma_{km}^{dev}+\frac{G}{3J}\text{tr}[B^{iso}]\delta_{km})$$

$$=\delta_{ik}\sigma_{jl}^{dev}+\sigma_{il}^{dev}\delta_{jk}-\frac{2}{3}\delta_{ij}\sigma_{kl}^{dev}+\frac{G}{3J}\text{tr}[B^{iso}](\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl})$$

$$\sigma_{ij}=\sigma_{ij}^{dev}+p\delta_{ij}\qquad p=\frac{1}{3}\text{tr}(\sigma)$$

$$-\sigma_{il}\delta_{jk}=-(\sigma_{il}^{dev}+p\delta_{il})\delta_{jk}$$

$$\frac{2}{J}GJ^{-\frac{2}{3}}\mathbb{I}^{\text{dev}}\cdot B-\sigma_{il}\delta_{jk}=\delta_{ik}\sigma_{jl}^{dev}-\frac{2}{3}\delta_{ij}\sigma_{kl}^{dev}+\frac{G}{3J}\text{tr}[B^{iso}](\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl})-p\delta_{il}\delta_{jk}$$

$$=\delta_{ik}\sigma_{jl}-p\delta_{ik}\delta_{jl}-\frac{2}{3}\delta_{ij}\sigma_{kl}^{dev}+\frac{G}{3J}\text{tr}[B^{iso}](\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl})-p\delta_{il}\delta_{jk}$$

$$=\delta_{ik}\sigma_{jl}-\frac{2}{3}\delta_{ij}\sigma_{kl}^{dev}+\frac{G}{3J}\text{tr}[B^{iso}](\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl})-p(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl})$$

$$=\delta_{ik}\sigma_{jl}-\frac{2}{3}\delta_{ij}\sigma_{kl}^{dev}+\frac{2G}{3J}\text{tr}[B^{iso}]\mathbb{I}_{ijkl}^{dev}-2p\mathbb{I}_{ijkl}^{sym}$$

(5.25)

$$\begin{aligned}\mathrm{a}_{ijkl}&=\frac{2}{J}\frac{\partial\tau_{ij}}{\partial B_{km}}B_{ml}-\sigma_{il}\delta_{jk}=\frac{2}{J}GJ^{-\frac{2}{3}}\mathbb{I}^{\text{dev}}\cdot B-\sigma_{il}\delta_{jk}-\frac{2G}{3J}\text{dev}(B^{iso})I+K(2J-1)I\otimes I\\&=\delta_{ik}\sigma_{jl}-\frac{2}{3}\delta_{ij}\sigma_{kl}^{dev}+\frac{2G}{3J}\text{tr}[B^{iso}]\mathbb{I}_{ijkl}^{dev}-2p\mathbb{I}_{ijkl}^{sym}-\frac{2}{3}\sigma_{ij}^{dev}\delta_{kl}+K(2J-1)I\otimes I\\&=\frac{2G}{3J}\text{tr}[B^{iso}]\mathbb{I}^{dev}-2p\mathbb{I}^{sym}-\frac{2}{3}\delta_{ij}\sigma_{kl}^{dev}-\frac{2}{3}\sigma_{ij}^{dev}\sigma_{kl}+K(2J-1)I\otimes I+\delta_{ik}\sigma_{jl}\\&=\frac{2G}{3J}\text{tr}[B^{iso}]\mathbb{I}^{dev}-2p\mathbb{I}^{sym}-\frac{2}{3}(I\otimes s+s\otimes I)+K(2J-1)I\otimes I+\delta_{ik}\sigma_{jl}\end{aligned}\tag{5.26}$$

6 编程对应公式

6.1 Hyplas

6.1.1 Ifstd2

计算单元内力 ELOAD：

$$\{\mathrm{f}^{\text{int e}}\}=\sum_{i=1}^{\text{nqp}}\det(\frac{dx}{dp})w_i\{\mathbf{B}^T\}\{\sigma\}\tag{5.27}$$

6.1.2 Stdstd2

计算刚度矩阵 ESTIF

$$\{\mathrm{K}^e\}=\sum_{i=1}^{\text{nqp}}\det(\frac{dx}{dp})w_i\{\mathrm{G}^T\}\{\mathrm{a}\}\{\mathrm{G}\}$$

a is Consistent spatial tangent modulus (5.28)

Getgmx cal G

6.1.3 Shpfun

计算 SHAPE（$N_i(r)$：单元每个结点的形函数在每个积分点的值，所有单元都相同）

计算 DERIV（$\partial N_i(r)/\partial r_j$：单元每个结点的形函数对自然坐标的导数在每个积分点的值，所有单元都相同）

6.1.4 Jacob2

计算 XJACM：积分点处当前坐标对自然坐标的导数

$$x^d = N_i \hat{x}_i^d \qquad d-\text{dim id} \qquad i-\text{node id in a element}$$

$$\frac{\partial x^d}{\partial p_j} = N_{i,j}\hat{x}_i^d\,(\text{XJACM}) \qquad j-\text{dim id to deriv} \tag{6.1}$$

计算 DATJAC=det(XJACM)

计算 XJACI=inverse(XJACM)

计算 CARTD：单元每个结点的形函数对当前坐标的导数在每个积分点的值

$$\frac{\partial N_i}{\partial x_j} = \frac{\partial N_i}{\partial p_k}\frac{\partial p_k}{\partial x_j} \qquad i-\text{node id in a element} \qquad j-\text{id of coord dof} \tag{6.2}$$

6.1.5 Getbmx

Computes the discrete symmetric gradient (in current configuration) operator for 2-D elements.

计算 BMATX：

$$\mathbf{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & & \frac{\partial N_2}{\partial x_1} & & \cdots \\ & \frac{\partial N_1}{\partial x_2} & & \frac{\partial N_2}{\partial x_2} & \cdots \\ \frac{\partial N_1}{\partial x_2} & \frac{\partial N_1}{\partial x_1} & \frac{\partial N_2}{\partial x_2} & \frac{\partial N_2}{\partial x_1} & \cdots \end{bmatrix} \qquad \mathbf{B}\cdot\hat{\mathbf{u}}^e = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \end{bmatrix} \tag{6.3}$$

6.1.6 Getgmx

Computes the discrete (full) gradient (in current configuration) operator for 2-D elements

计算 GMATX：

$$\mathbf{G} = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & & \frac{\partial N_2}{\partial x_1} & & \cdots \\ & \frac{\partial N_1}{\partial x_1} & & \frac{\partial N_2}{\partial x_1} & \cdots \\ \frac{\partial N_1}{\partial x_2} & & \frac{\partial N_2}{\partial x_2} & & \cdots \\ & \frac{\partial N_1}{\partial x_2} & & \frac{\partial N_2}{\partial x_2} & \end{bmatrix} \qquad \mathrm{G}\cdot\hat{\mathbf{u}}^e = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{bmatrix} \tag{6.4}$$

6.1.7 defgra

Deformation gradient for 2D isoparametric finite element.

计算 FINCIN (inverse of incremental deformation gradient tensor)

$$(F_\Delta)^{-1} = \frac{\partial x_i}{\partial x_j} = \frac{\partial x_i - \Delta u_i}{\partial x_j} = \delta_{ij} - \frac{\partial \Delta u_i}{\partial x_j} \qquad (F_\Delta)^{-1}_{33} = \begin{cases} 0 & \text{plane stress} \\ 1 & \text{plane strain} \end{cases} \tag{6.5}$$

计算 FINV (inverse of total deformation gradient tensor)

$$F^{-1}=\frac{\partial X_i}{\partial x_j}=\frac{\partial x_i-u_i}{\partial x_j}=\delta_{ij}-\frac{\partial u_i}{\partial x_j} \qquad (F)^{-1}_{33}=\begin{cases}0 & \text{plane stress}\\ 1 & \text{plane strain}\end{cases} \tag{6.6}$$

6.1.8 Invf2

Inverts the deformation gradient for 2-D problems

计算 FINCR=inverse(FINCIN) F_Δ : incremental deformation gradient tensor $(F_\Delta)_{33}=\begin{cases}0 & \text{plane stress}\\ 1 & \text{plane strain}\end{cases}$

6.1.9 Listra

Computes the incremental infinitesimal strain components in 2-D

$$\begin{aligned}&\{\Delta\varepsilon\}_i=\{B\}_{ik}\{\Delta\hat{u}\}_k^e\\ &\Delta\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial\Delta u_i}{\partial x_j}+\frac{\partial\Delta u_j}{\partial x_i}\right)\end{aligned} \tag{6.7}$$

6.1.10 Logstr

Computes the incremental Logarithmic strain

首先谱分解得到 $B^{e\ \mathrm{trial}}$ 的特征值 x_α 和谱阵 E_α （具体过程参照(6.10)~(6.13)）

$$\begin{aligned}&\varepsilon^{e\ \mathrm{trial}}=\frac{1}{2}\ln B^{e\ \mathrm{trial}}=\frac{1}{2}\ln(x_i)E_i\ (\text{as for }\{\mathrm{B}^e\}_{1、2、3})\\ &\text{remark:}\{\varepsilon^{e\ \mathrm{trial}}\}_4=\frac{1}{2}\ln(\{B^{e\ \mathrm{trial}}\}_4) \qquad \text{so that }\ \varepsilon_{33}^{e\ \mathrm{trial}}=\frac{1}{2}\ln B_{33}^{e\ \mathrm{trial}}\end{aligned} \tag{6.8}$$

6.1.11 Instd2

Call SHPFUN, JACOB2, GETBMX, GETGMX, DEFGRA, INVF2, LISTRA

DETFIN is the determinate of F^{-1}

DATF is the determinate of F

6.1.12 Matisu

Material interface for state update routine calls

小应变下计算 STRAT(elastic trial strain)的方法:

$$\text{if small strain: }\{\varepsilon_i^{e\ \mathrm{trail}}\}=\{\varepsilon_i^e\}+\{\Delta\varepsilon_i\} \tag{6.9}$$

大应变下计算 STRAT(elastic trial strain)的方法:

首先对上次收敛的 logarithmic elastic strain tensor ε^e 进行谱分解：

$$\text{EIGX(1 or 2)} \quad x_{1,2}=\frac{I_1\pm\sqrt{I_1-4I_2}}{2} \qquad (I_1=\mathrm{tr}[\varepsilon^e],\quad I_2=\det[\varepsilon^e]) \tag{6.10}$$

如果计算出的特征值满足$|x_1-x_2|/\max(|x_1|,|x_2|)<1\mathrm{E}\text{-}5$ 则认为 2 个特征值相等：

$$\begin{aligned}&\text{EIGPGJ(1 or 2)} \qquad E_\alpha=\hat{e}_\alpha\hat{e}_a\\ &\hat{e}_\alpha\ \text{ is eigen vector obtained by jacob iteration}\end{aligned} \tag{6.11}$$

如果计算出的特征值满足$|x_1 - x_2| / \max(|x_1|, |x_2|) \geq$ 1E-5 则认为 2 个特征值不相等：

$$\text{EIGPGJ(1 or 2)} \qquad E_\alpha = \frac{1}{2x_\alpha - I_1}[\varepsilon^e + (x_\alpha - I_1)I] \tag{6.12}$$

接下来计算 Elastic left cauchy-green tensor B^e 的特征值：

$$y_{1\text{ or }2} = \exp(2x_{1\text{ or }2}) \tag{6.13}$$

接下来计算得到$\{B^e\}$

$$\begin{aligned} &B^e = \exp(2\varepsilon^e) = \exp(2x_i)E_i \quad (\text{as for } \{\mathrm{B}^e\}_{1、2、3}) \\ &\text{remark:} \{\mathrm{B}^e\}_4 = \exp(2\{\varepsilon^e\}_4) \end{aligned} \tag{6.14}$$

接下来计算$\{B^{e\text{ trial}}\}$

$$\begin{aligned} &\{B^{e\text{ trial}}\} = F_\Delta B^e (F_\Delta)^T \quad (\text{as for } \{\mathrm{B}^{e\text{ trial}}\}_{1、2、3}) \\ &\{B^{e\text{ trial}}\}_4 = \begin{cases} \{\mathrm{B}^e\}_4 & \text{as for plane strain} \\ \{B^e\}_4 F_{\Delta 33}{}^2 & \text{as for axisymmetric analysis} \end{cases} \end{aligned} \tag{6.15}$$

接下来根据不同的本构计算出当前高斯点处的$\{\sigma\}$

6.2 asfem

6.2.1 Shapefun::calc

计算积分点处的形函数数值和自然坐标导数：

$$N_i(r) \qquad dN_i(r)/dr \tag{6.16}$$

计算积分点坐标（参考构型/当前构型）对自然坐标导数：

$$\begin{aligned} &\frac{dx_i}{dr_j} = \frac{dN_k}{dr_j}\hat{x}_{ki} \quad i, j\text{-dim type id} \qquad k\text{-node id in element} \\ &\qquad x \text{ could be } x \text{ or } \mathrm{X}\text{, depending on param t\_nodes} \end{aligned} \tag{6.17}$$

计算相对自然坐标的雅可比矩阵的行列式

$$JTJ = \frac{dx_k}{dr_i}\frac{dx_k}{dr_j} \qquad \text{jacdet} = \sqrt{\det(JTJ)} \tag{6.18}$$

$$\begin{aligned} &JTJ = \frac{dx_k}{dr_i}\frac{dx_k}{dr_j} \\ &\because JTJ_{ij}^{-1} = \frac{dr_i}{dx_k}\frac{dr_j}{dx_k} \qquad \therefore \frac{dr_i}{dx_j} = JTJ^{-1} \cdot J^T = \frac{dr_i}{dx_n}\frac{dr_k}{dx_n}\frac{dx_j}{dr_k} \\ &J = \frac{dN_i}{dx_j} = \frac{dN_i}{dr_k}\frac{dr_k}{dx_j} \end{aligned} \tag{6.19}$$