Differential Equations: Condensed Notes

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May 2025

1 Introduction

Definition 1.1 (Differential Equations)

1. An Ordinary Differential Equation is an equation of the form:

$$\frac{dx}{dt} = f(t, x) \text{ with } f : I \times \Omega^d \to \Omega^d,\ t \in I,\ x \in \Omega^d$$

2. An ODE is called Autonomous if the function f does not depend on t.

3. We can write any n_{th} order ODE:

$$x^{(n)} = f(t, x, \dot{x}, ..., x^{(n-1)})$$

as a system first order ODEs, with appropriate substitutions. (So we only have to worry about ODEs in the form $\dot{x} = f(t, x)$)

Theorem 1.2 (Solutions to IVPs)

A solution to an Initial Value Problem $\dot{x} = f(t, x),\ x(t_0) = x_0$ is a differentiable function $\lambda(t) : I \to \mathbb{R}^d$ which satisfies the solution identity:

$$\frac{d\lambda}{dt} = f(t, \lambda(t)) \quad \forall t \in I \text{ and } \lambda(t_0) = x_0$$

I is the time interval which $\lambda(t)$ is defined on with $t_0 \in I$. By the fundamental theorem of calculus we have:

$$\int_{t_0}^{t} \frac{d\lambda}{ds} ds = \int_{t_0}^{t} f(s, \lambda(s))\, ds \implies \lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))\, ds$$

2 Existence and Uniqueness

2.1 The Picard-Lindeloff Theorem

Definition 2.1 (Lipschitz Continuity)

A function f is called Lipschitz Continuous if $\exists K > 0$ st:

$$\|f(x) - f(y)\| \leq K\|x - y\| \;\; \forall x, y \in D$$

Theorem 2.2 (Banach's Fixed Point Theorem)

A function f is called a contraction if $\exists K \in (0, 1)$ st:

$$\|f(x) - f(y)\| \leq K\|x - y\| \;\; \forall x, y \in D$$

Definition 2.3 (Picard's Iterates)

To find a solution λ to the solution identity:

$$\lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))\, ds$$

we see if the sequence of functions:

$$\lambda_0(t) = x_0, \;\; \lambda_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\, ds$$

Converges uniformly to some $\lambda_\infty(t)$, then $\lambda_\infty(t)$ satisfies the solution identity:

$$\lambda_\infty(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_\infty(s))\, ds$$

and hence, it is a unique solution to the IVP.

Theorem 2.4 (Picard-Lindeloff Theorem)

1. If $f(t, x)$ is globally Lipschitz Continuous ie: $\exists K > 0$ st

   $$\|f(t, x) - f(t, y)\| \leq K\|x - y\| \;\; \forall (t, x), (t, y) \in D$$

   Then every IVP $x(t_0) = x_0$ has a unique solution on the interval:

   $$I = [t_0 - \frac{1}{2K}, t_0 + \frac{1}{2K}]$$

   We can actually extend each of these unique solutions to a solution on the whole time interval by applying this very theorem to the IVP $x(t_0 + \frac{1}{2K}) = \lambda(t_0 + \frac{1}{2K})$.

2. If $f(t,x)$ is only locally Lipschitz Continuous ie: $\exists U \subset D$ around (t_0, x_0) and $K > 0$ st

$$\|f(t,x) - f(t,y)\| \leq K\|x - y\| \ \ \forall (t,x),(t,y) \in U$$

   Then we have a solution $\lambda(t) : I \to \mathbb{R}^d$ which depending on the initial condition. ie. we cant find a solution to $\dot{x} = \frac{1}{x}$ local to $x_0 = 0$

3. If $f(t,x)$ is continuously differentiable, then it is locally Lipschitz Continuous somewhere. The derivative is continuous so we can just take any interval where it is bounded.

2.2 Applications

Theorem 2.5 (Constant Solutions)

$$\dot{x} = f(x) \text{ has a constant solution } \lambda(t) = a \iff f(a) = 0$$

Theorem 2.6 (Solutions do different IVPs cannot touch)

Consider a differential equation $\dot{x} = f(t,x)$ with two solutions to different initial value problems $\lambda_1(t)$ and $\lambda_2(t)$ then:

$$\text{either } \lambda_1(t) = \lambda_2(t) \ \forall t \in I \text{ or } \lambda_1(t) \neq \lambda_2(t) \ \forall t \in I$$

ie. we're either talking about the same solution or the two solutions don't touch.

Cor 2.7 (Bounding a solution between constant solutions)

Consider the IVP:

$$\dot{x} = (x^2 - 1)g(x), \ \ x(0) = 0$$

Looking at the Differential Equation alone, we can see two constant solutions: $x = 1$ and $x = -1$. Now, any maximal solution to the IVP (passes through (0,0)) can't attain the values 1 or -1 otherwise it would be 'absorbed into' the constant solutions.

Theorem 2.8 (Concatenating Solutions)

- Let $\dot{x} = f(x)$ reach an equilibrium at x_0. ie. $f(x_0) = 0$ st $f(x)$ is not Lipschitz continuous at x_0.
- Then, if there are many solutions to the IVP $f(x_0) = 0$, then we can construct infinitely many solutions via concatenation

Consider: $\dot{x} = \sqrt{|x|}$ with $x(t_0) = 0$

- We have a trivial solution $\lambda(t) = 0$ and a general solution attained via seperation of variables: $\lambda(t) = \frac{1}{4}(t - t_0)^2$ for $t_0 > 0$

- we can concatenate the two solutions like so:

$$\lambda(t) = \begin{cases} 0 & t < b \\ \frac{1}{4}(t-b)^2 & t \geq b \end{cases}$$

- the two parts match in value and derivative at $t = b$

Definition 2.9 (Maximal Existence Interval)

Consider an IVP $\dot{x} = f(t, x)$ with $x(t_0) = x_0$. We define:

$$I^+ = \sup\{t \geq t_0 : \text{ a solution exists on } [t_0, t]\}$$

$$I^- = \inf\{t \leq t_0 : \text{ a solution exists on } [t, t_0]\}$$

We define the Maximal Existence Interval:

$$I_{max}(t_0, x_0) = (I^-, I^+)$$

Furthermore, any solution $\lambda_{max}(t, t_0, x_0)$ on $I_{max}(t_0, x_0)$ is called a maximal solution to the IVP (t_0, x_0).

Theorem 2.10 (Extending to a Maximal Solution)

If $I^+ < \infty$ then we have two possible behaviours as $t \to \infty$

1. $\lambda(t)$ approaches the boundary of D (sometimes denoted as ∂D). ie. if t were to increase beyond I^+, the solution would go out of bounds.

2. $\lambda(t) \to \infty$ as $t \to I^+$ ie. the solution has a vertical asymptote at I^+

The same applies to I^- of course.

3 General Solutions

Definition 3.1 (General Solutions)

Typically, we define a general solution as a function of t with constants t_0 and x_0. Here, we define our general solution:

$$\lambda(t; t_0, x_0) = \lambda_{max}(t, t_0, x_0)$$

Note the following 'co-cycle property':

$$\lambda(t; s, \lambda(s, t_0, x_0)) = \lambda(t; t_0, x_0)$$

Definition 3.2 (Flows)

For an autonomous ODE, we can use the co-cycle property to classify each IVP solution by x(0):

$$\lambda(t; 0, \lambda(0; t_0, x_0)) = \lambda(t; t_0, x_0)$$

This allows us to clean up the notation. The flow of a differential equation φ is defined by:

$$\varphi(t, x) = \lambda(t; 0, x)$$

A more intuitive syntax would be $\varphi_x(t)$ which is the solution that 'starts at x' as a function of t

Theorem 3.3 (Flows are Group Actions on D)

The flow $\varphi : \mathbb{R} \times X \to X$ for $X \subset \mathbb{R}^d$ is, in fact, a group action on the Group $(\mathbb{R}, +)$ (the time parameter):

1. $\varphi(0, x) = x$ (Identity)
2. $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$ (Compatibility)

3.1 Orbits

Definition 3.4 (Orbits)

An orbit $O(x)$ is the locus/set of points attained by a flow starting at (or even containing) x:

$$O(x) = \{\varphi(t, x) \in D : t \in I_{max}\}$$

Note the following properties:

1. Different IVP orbits do not touch:

$$O(x) = O(y) \iff y = \varphi(t, x) \text{ for some } t \in I_{max}$$

2. Orbits are only defined for Autonomous differential equations. Otherwise $O(x)$ wouldn't be well defined.

3. Positive and Negative Orbits:

$$O^+ = \{\varphi(t,x) \in D : t \in I_{max} \cap \mathbb{R}_{\geq 0}\}$$

$$O^- = \{\varphi(t,x) \in D : t \in I_{max} \cap \mathbb{R}_{\leq 0}\}$$

Definition 3.5 (homoclinic and heteroclinic orbits)

- An orbit $O(x)$ is called ***homoclinic*** if:

$$\exists x^* \text{ st } \lim_{t\to\infty} \varphi(t,x) = \lim_{t\to-\infty} \varphi(t,x) = x^*$$

  ie. a particle starts and ends at the same equilibrium point. (This does not include the case where $O(x) = \{x^*\}$).

- An orbit $O(x)$ is called ***heteroclinic*** if:

$$\exists x_1^* \neq x_2^* \text{ st } \lim_{t\to\infty} \varphi(t,x) = x_2^* \text{ and } \lim_{t\to-\infty} \varphi(t,x) = x_1^*$$

  ie. a particle starts at an equilibrium and ends at a different equilibrium.

Definition 3.6 (Invariant Sets)

A set $M \subset D$ is called invariant if:

$$x \in M \implies O(x) \subset M$$

ie. M is closed under $\varphi(t,.)$ for arbitrary t

- We say that M is positively invariant if:

$$x \in M \implies O^+(x) \subset M$$

  'a particle in M stays in M'

- similarly, M is negatively invariant if:

$$x \in M \implies O^-(x) \subset M$$

  'a particle in M has always been in M'

4 Linear Systems

4.1 Matrix Norms

Definition 4.1 (The Operator Norm of A Matrix)

Let $A \in \mathbb{R}^{n\times m}$:

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^n,\ \|x\| = 1\}$$

We define the operator norm of a matrix as how far the linear transformation can scale a unit vector

Theorem 4.2 (Properties of matrix Norms)

1. $\|Ax\| \leq \|A\|\|x\|$
2. $\|AB\| \leq \|A\|\|B\|$

Theorem 4.3 (Linear Systems have unique IVP solutions)

- Let $f(x) = Ax$, then $Df(x) = A$ for all $x \in \mathbb{R}$
- It follows that $Df(x)$ is bounded by $\|A\|$ and hence, f globally Lipschitz Continuous.

4.2 The matrix Exponential

Definition 4.4 (The Matrix Exponential)

Recall the Maclaurin series for e^{tx}:

$$e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$$

Hence, we can raise e to the power of a matrix by defining e^{tA} as the matrix polynomial:

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

Theorem 4.5 (Properties of e^{tA})

1. If $A \sim B$, then $e^{tA} \sim e^{tB}$. In fact, $A = P^{-1}BP \implies e^{tA} = P^{-1}e^{tB}P$.
2. $e^{t(A_1 \oplus A_2 \oplus \ldots \oplus A_n)} = e^{tA_1} \oplus e^{tA_2} \oplus \ldots \oplus e^{tA_n}$
3. $e^{t(-A)} = e^{-tA} = (e^{tA})^{-1}$ (Matrix Exponential is always invertable)
4. If $AB = BA$, then $e^{t(A+B)} = e^{tA}e^{tB}$

Theorem 4.6 (e^{tA} as a solution)

- Let $\dot{x} = Ax$ with $x(0) = x_0$. (It follows that Ax is Lipschitz continuous and hence there is a unique solution)
- Then consider the Picard Iterates for $\dot{x} = Ax$ with $\lambda_0 = x_0$:
$$\lambda_{n+1} = x_0 + \int_0^t A\lambda_n(s)\, ds$$
- This produces the power series:
$$\lambda_n = x_0 + tAx_0 + \frac{t^2}{2!}A^2x_0 + ... + \frac{t^n}{n!}A^n x_0$$
$$\implies \lim_{n\to\infty} \lambda_n(t) = e^{tA}x_0$$
- Hence, we've found a solution local to $t = 0$. It follows that this solution exists for all $t \in \mathbb{R}$
$$\varphi(t, x) = e^{tA}x \text{ is the flow generated by } \dot{x} = Ax$$

Theorem 4.7 (Similar matrices represent the same linear system)

- Let $A \sim B$, then $\exists$ invertible P st $A = P^{-1}BP$. Through a substitution $x = Px$:
$$\dot{x} = Bx \implies \dot{Py} = BPy \implies \dot{y} = P^{-1}BPy \implies \dot{y} = Ay$$
Similar matrices represent the same linear transformation under different bases. Thus, a particle traveling through the phase plane of $\dot{x} = Ax$ will have the same trajectory as $\dot{x} = Bx$.
- Hence, we can characterize every linear system $\dot{x} = Ax$ by A's Jordan Normal Form. This will also give us clear information about the phase portrait of the system.

Theorem 4.8 (Flow of a linear system through a change of basis)

- Let $x(t) = \varphi(t, x_0) = e^{tA}x_0$, and note that $J = PAP^{-1}$ and $e^{tJ} = Pe^{tA}P^{-1}$
- Then consider the substitution $\mathbf{x}(t) = P^{-1}\mathbf{y}(t)$:
$$P^{-1}y(t) = e^{tA}P^{-1}y_0 \implies y(t) = Pe^{tA}P^{-1}y_0 \implies \varphi(t, y_0) = y(t) = e^{tJ}y_0$$
Where vector y is the same as x but with respect to the Jordan Basis.
- A special case of this when A is diagonalisable with distinct eigenvalues λ_i and eigenvectors v_i:
$x(t) = \varphi(t, x_0) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + ... + c_ie^{\lambda_i t}$ where $x_0 = c_1v_1 + c_2v_2 + ... + c_iv_i$
We rewrite a point on the phase plane in terms of the eigenvector basis and we can see the flow component for each of these eigen vectors.

4.3 Jordan Normal Forms

Theorem 4.9 (Matrix Exponential of a Jordan Block)

- Let $J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & ... & ... & 0 \\ 0 & \lambda & 1 & 0 & ... & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & ... & ... & ... & ... & 0 \end{pmatrix} = \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

- Then, $e^{tJ_n(\lambda)} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & ... & ... & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & ... & \frac{t^{n-2}}{(n-2)!} \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & \frac{t^2}{2!} \\ \vdots & & & & & t \\ 0 & ... & ... & ... & ... & 1 \end{pmatrix}$

Each row r is the first $n-(r-1)$ terms of the maclaurin expansion of e^t.

Theorem 4.10 (Real form of a complex Diagonal)

- let $A = \begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix}$, be a transformation with respect to the basis of eigen vectors: $\{u+iv, u-iv\}$

- then $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ represents the same transformation with respect to $\{u, v\}$

- Thus, $\begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix} \sim \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

Proposition 4.11 (Attaining oscillators from complex eigenvalues)

Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, then:

$$e^{tA} = e^{ta} \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$$

Theorem 4.12 (The matrix exponential of the Real form of a complex J-block)

- Let $J_n(\lambda)$ be an $n \times n$ Jordan block for a complex eigenvalue $\lambda = a + ib$
- then:

$$e^{tJ_n(\lambda)} = e^{\lambda t} \begin{pmatrix} 1 & G(t) & \frac{G(t)^2}{2!} & \dots & \dots & \frac{G(t)^{n-1}}{(n-1)!} \\ 0 & 1 & t & \frac{G(t)^2}{2!} & \dots & \frac{G(t)^{n-2}}{(n-2)!} \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & & \frac{G(t)^2}{2!} \\ \vdots & & & & & G(t) \\ 0 & \dots & \dots & \dots & \dots & 1 \end{pmatrix}$$

where $G(t) = \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$

4.4 Exponential Growth

Definition 4.13 (Rate of Exponential Growth)

For a 1-dimensional function $\mu(t) = e^{at}$, we might want to evaluate the rate of exponential growth as $t \to \infty$:

$$\lim_{t\to\infty} \frac{\ln(\mu(t))}{t} = \frac{t\ln(e^a)}{t} = a$$

Now, this is still true if we have a function that is not purely exponential $\mu(t) = t^n e^{at}$:

$$\lim_{t\to\infty} \frac{\ln(\mu(t))}{t} = \frac{n\ln(t) + t\ln(e^a)}{t} = a$$

Definition 4.14 (Lyapunov Exponents)

- Let $\lambda(t) = e^{tA}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ be a solution to an IVP where $(x_0, y_0) \neq (0, 0)$
- then, the Lyapunov Exponent:

$$\sigma_{Lyap}(\lambda) = \lim_{t\to\infty} \frac{\ln||\lambda(t)||}{t}$$

This will always equal one of the real parts of the eigenvalues of A.

Definition 4.15 (Simple and Semi-Simple Eigenvalues)

- An eigenvalue λ is Simple if $a(\lambda) = 1$, ie. λ corresponds to a single eigenvector
- An eigenvalue λ is Semi-Simple if $a(\lambda) = g(\lambda)$ ie. each J-block for λ has dimension 1 (no trailing 1s above λ).

Theorem 4.16 (EigenSpace Decomposition)

- For a Matrix $A_{d\times d}$ we can write $\mathbb{R}^d$ as the direct sum of A's Eigen spaces:

$$\mathbb{R}^d = E_1 \oplus E_2 \oplus ... \oplus E_q$$

  Where each E_i is invariant under flow:

$$x \in E_i \implies \varphi(t,x) = e^{\lambda_i t}(1 + t + t^2 + ...) \in E_i$$

  Note: This polynomial in t will have degree equal to the $\dim(E_i)+1$. This arises in the case that λ_i is semi-simple but has algebraic multiplicity ?1

- Hence, if $x \in E_i$, $\sigma_{Lyap} = Re(\lambda_i)$. If a particle is on the eigen-vector line, it's only factor of exponential growth is the eigenvalue λ_i

Definition 4.17 (Exponential Estimate)

Choose $\gamma > \max\{Re\lambda : \lambda \text{ is an eigenvalue for } A\}$

- Then $\exists K > 0$ st

$$||e^{tA}|| \leq Ke^{\gamma t} \ \ \forall t \geq 0$$

- If each eigenvalue λ which shares a maximum real part is semi-simple. ie:

$$\forall\lambda \text{ st } Re(\lambda) = \max\{Re\lambda : \lambda \text{ is an eigenvalue for } A\}, \ \lambda \text{ is semi-simple.}$$

  Then, we can let $\gamma = \max\{Re\lambda : \lambda \text{ is an eigenvalue for } A\}$

This means that we can always bound our matrix exponential by an exponential function taking the largest Lyapunov Constant.

5 Non-Linear Systems

5.1 Equilibria and Stability

Definition 5.1 (Equilibrium Point)

We say that the ODE $\dot{x} = f(x)$ has an equilibrium at x^* if $f(x^*) = 0$. ie. no movement at $x = x^*$. We now want to characterise the different types of equilibira:

Definition 5.2 (Stability) *Also called Lyapunov Stability.*

An equilibrium x^* is stable if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ st } x \in B_\delta(x^*) \implies \varphi(t, x) \in B_\epsilon(x^*) \;\; \forall t \geq 0$$

If a particle gets within δ of the equilibrium point x^*, then it stays in the ϵ neighborhood of x^* forever.

Definition 5.3 (Attractive)

x^* is called attractive if:

$$\exists \delta > 0 \text{ st } x \in B_\delta(x^*) \implies \lim_{t \to \infty} \varphi(t, x) = x^*$$

If a particle gets within δ of the equilibrium point x^* (passes the event horizon), then it gets sucked into x^*.

Definition 5.4 (Repulsive)

The opposite of attractive; x^* reppels particles instead of attracting them:

$$\exists \delta > 0 \text{ st } x \in B_\delta(x^*) \implies \lim_{t \to -\infty} \varphi(t, x) = x^*$$

Definition 5.5 (Asymptotically Stable)

x^* is called Asymptotically Stable if it is both stable and attractive

- A linear system $\dot{x} = Ax$ is Asymptotically Stable $\iff$ all the eigenvalues of A all have real part ≤ 0. And the ones $= 0$ are semi-simple.

Definition 5.6 (Exponential Stability)

An equilibrium x^* is called exponentially stable if: $\exists \delta > 0, K \geq 1, \gamma < 0$ st $x \in B_\delta(x^*) \implies$

$$|\varphi(t, x) - x^*| \leq K e^{\gamma t} |x - x^*| \;\; \forall t \geq 0$$

- This means that a particle 'accelerates' towards the equilibrium point once it's passed the event horizon.

- Exponential Stability $\implies$ Asymptotic Stability. The converse is not true.
- A linear system $\dot{x} = Ax$ with $x^* = 0$ has Exponential Stability $\iff$ all the eigenvalues of A have negative real parts $\iff$ x^* is Asymptotically stable.

5.2 Linearised Stability

Definition 5.7 (Hyperbolicity)

A matrix A is called hyperbolic if each of its eigenvalues have a non-zero real part. For $\dot{x} = f(x)$, we say that an equilibrium x^* is hyperbolic if the Jabcobian $Df(x^*)$ is hyperbolic.

Theorem 5.8 (Linearised Stability)

- Let f be continuously differentiable in the ODE $\dot{x} = f(x)$.
- If x^* is a hyperbolic equilibrium, with $Re(\lambda_i) < 0 \;\; \forall$ eigenvalues λ_i of $Df(x^*)$ then, the equilibrium x^* is Exponentially Stable

5.3 Stable Sets and Invariance

Definition 5.9 (Stable and Unstable Sets)

For an equilibrium x^*:

- The stable set is defined as:

$$W^s(x^*) = \{x \in D : \lim_{t \to \infty} \varphi(t, x) = x^*\}$$

- The unstable set is defined as:

$$W^u(x^*) = \{x \in D : \lim_{t \to -\infty} \varphi(t, x) = x^*\}$$

- Note: If x^* is attractive, then W^s denotes the domain of attraction which is an open set in D.

Theorem 5.10 (Linear System Decomposition)

Let $\dot{x} = Ax$ where A is hyperbolic. We have:

$$\mathbb{R}^d = W^s(0) \oplus W^s(0)$$

This means that each $x \in \mathbb{R}^d$ is either being repelled by or attracted to 0. Note: this is not true if A is not hyperbolic, then we introduce the case where we have 0 Lyapunov exponents and hence circular orbits.

Theorem 5.11 (Asymptotic Stability of Linear Systems)

1. The only point that can be attractive in a linear system is $x^* = 0$
2. $x^* = 0$ is attractive
$\iff$ the real parts of each eigenvalue are all negative
$\iff$ $x^* = 0$ is exponentially stable
3. x^* is attractive $\implies$ x^* is globally attractive:

$$\forall y \in D, \exists \gamma > 0 \text{ st } \gamma y \in B_\delta(x)$$

$$\text{hence: } \lim_{t\to\infty} e^{tA} y = \frac{1}{\gamma} \lim_{t\to\infty} e^{tA} \gamma y = 0$$

5.4 Limit Points and Invariance

Definition 5.12 (α and ω sets)

let $x \in \mathbb{R}^d$, we define:

- $\omega(x) = \{x_\omega : \exists \text{ sequence } t_n \text{ st } \lim_{t_n\to\infty} \varphi(t_n, x) = x_\omega\}$
ie. The set of xs that occur infinitely many times in the flow defined by x (as time goes forwards)
- $\alpha(x) = \{x_\alpha : \exists \text{ sequence } t_n \text{ st } \lim_{t_n\to-\infty} \varphi(t_n, x) = x_\alpha\}$
ie. The set of xs that occur infinitely many times in the flow defined by x (as time goes backwards)
- Alternatively: $\omega(x) = \bigcap_{t\geq 0} \overline{O^+(\varphi(t,x))}$ and $\alpha(x) = \bigcap_{t\leq 0} \overline{O^-(\varphi(t,x))}$
(Note: we need to take the intersection of the closures to catch the values x_w which are not attained in finite time)

Theorem 5.13 (Invariance of α and ω)

1. $\alpha(x)$ and $\omega(x)$ are invariant.
2. If $O^+(x)$ is bounded and $\overline{O^+(X)} \subset D$, then $\omega(x)$ is non-empty and compact.
3. Analogously, ff $O^-(x)$ is bounded and $\overline{O^-(X)} \subset D$, then $\alpha(x)$ is non-empty and compact.

5.5 Lyapunov Functions

Definition 5.14 (Lyapunov Function)

Lyapunov functions are used to describe the energy of a particle local to an equilibrium. A Lyapunov function $V(x)$ is continuously differentiable and has the following properties:

- $V(x^*) = 0$ (A particle has zero energy at the equilibrium point)
- $V(x) > 0 \ \ \forall x \in D$ st $x \neq x^*$
- $\dot{V}(x) \leq 0 \ \ \forall x \in D$ (see below)

If such a function exists we can use it to verify stability of an equilibrium. Note: To find a Lyapunov function, we typically try an ansatz of the form: $V(x) = \alpha x^2 + \beta y^2$ such that $\dot{V}(x) \leq 0$.

Definition 5.15 (Orbital Derivative)

We define a shorthand for the dot product between the gradient of V with f

$$\dot{V}(x) = \nabla V(x)^T f(x) = \sum \frac{\partial V}{\partial x_i} f_i(x)$$

Note that $\dot{V}(x)$ describes the derivative of V along solutions $\mu(t)$ to $\dot{x} = f(x)$:

$$\frac{dV}{dt}(\mu(t)) = V'(\mu(t))\dot{\mu}(t) = \dot{V}(\mu(t)) \quad \text{(by the chain rule)}$$

Theorem 5.16 (Lyapunov Stability Theorem)

If there exists a Lyapunov function local to an equilibrium x^*, then x^* is Lyapunov Stable.

Theorem 5.17 (Lyapunov Asymptotic Stability Theorem)

If a Lyapunov Function exists and:

- $\dot{V}(x^*) = 0$
- $\dot{V}(x) < 0 \ \ \forall x \in D$ st $x \neq x^*$

Then x^* is asymptotically stable.

Definition 5.18 (Sublevels of a Lyapunov Function)

- Let $S_c = \{x : V(x) \leq c\}$
- then S_c is positively invariant.

Once a particle drops below the energy level c, it cannot 'climb back up.'

- Furthermore, if S_c is compact, then $S_c \subset W^s(x^*)$ (The domain of attraction for x^*)

Theorem 5.19 (LaSalle's Theorem)

$$w(x) \subset \{x : \dot{V}(x) = 0\} \;\; \forall x \in D$$

This means that any periodic orbit in the system $p \subset \omega(x) \subset \{x : \dot{V}(x) = 0\}$. Particles that end up in a periodic orbit have constant energy as $t \to \infty$.

5.6 Poncare-Bendixon Theorem

When considering a two-dimensional Non-Linear Autonomous ODE. We can expect relatively predictable behaviour. We can usually classify the different types of ω sets that occur.

Theorem 5.20 (Poncare-Bendixon Theorem)

Let $\dot{x} = f(x)$, $f : D \to \mathbb{R}^d$ with open D

- If $O^+(x) \subset \mathcal{K} \subset D$ for compact $\mathcal{K}$ and $O^+(x)$ has finitely many equilibria:
- Then $\omega(x)$ is either:
    1. a singleton equilibria $\{x^*\}$
    2. a periodic orbit
    3. a set of equilibria and homo/hetero-clinic orbits.

Hence, if $\omega(x)$ contains no equilibria, then it must be a periodic orbit.