

Analys i en variabel

SF1673 (HT25)

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1. The Real Numbers

1.1. REALS

1.1.1. Prerequisites

Theorem 1.1.1 (Induction)

Let $S \subseteq \mathbb{N}$. If

- (i) $1 \in S$, and
- (ii) $n \in S \implies n + 1 \in S$ (inductive step),

then $S = \mathbb{N}$.

Definition 1.1.2 (Injective/Surjective/Bijective)

$f : X \rightarrow Y$ is *injective* (or one-to-one) if $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ or equivalently if $f(x_1) = f(x_2) \implies x_1 = x_2$.

f is *surjective* if $\forall y \exists x : f(x) = y$.

f is *bijective* if is both injective and surjective or equivalently if each y is mapped to exactly one x .

1.1.2. Comparison

Definition 1.1.3 (Equality)

$$a = b \iff (\forall \varepsilon > 0 \Rightarrow |a - b| < \varepsilon)$$

Theorem 1.1.4 (Triangle Inequalities)

- (i) $|a + b| \leq |a| + |b|$
- (ii) $|a - b| \leq |a - c| + |c - b|$
- (iii) $|a - b| \geq \big||a| - |b|\big|$

The reverse triangle inequality (iii) is seldom used.

1.1.3. Bounds

Axiom 1.1.5 (Supremum Property or Axiom of Completeness)

Every bounded, nonempty set of real numbers has a least upper bound.

Note

The same does not apply for the rationals.

Definition 1.1.6 (Least Upper Bound)

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then,

$$s = \sup A \iff \forall \varepsilon > 0 \exists a \in A : s - \varepsilon < a.$$

1.2. CARDINALITY

Definition 1.2.1 (Cardinality)

A has the same *cardinality* as B if there exists a bijective $f : A \rightarrow B$.

Definition 1.2.2 (Countable/Uncountable)

A is *countably infinite* if $\mathbb{N} \sim A$.

A is *countable* if it is finite or countably infinite.

Otherwise, A is *uncountable*.

Theorem 1.2.3 (Countability of \mathbb{Q} , \mathbb{R})

\mathbb{Q} is countable.

Proof. Let $A_1 = \{0\}$ and let

$$A_n = \{\pm p/q : p, q \in \mathbb{N}_+, \gcd(p, q) = 1, p + q = n\}$$

for all $n \geq 2$. Each A_n is finite and every rational numbers appears in exactly one set. □

\mathbb{R} is uncountable.

Proof. Cantor's diagonalization method. □

\mathbb{I} is uncountable.

Proof. $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ where \mathbb{Q} is countable. □

Theorem 1.2.4 (Density of \mathbb{Q} in \mathbb{R})

- (i) $\forall a < b \in \mathbb{R} \exists r \in \mathbb{Q} : a < r < b$
- (ii) $\forall y \in \mathbb{R} \exists (r_n) \in \mathbb{Q} : (r_n) \rightarrow y$

1.3. TOPOLOGY

1.3.1. Points

Definition 1.3.1 (Limit Point)

x is a *limit point* of A if every $V_\varepsilon(x)$ intersects A at some point other than x .

Theorem 1.3.2 (Sequential Limit Point)

x is a limit point of A if $x = \lim a_n$ for some $(a_n) \subseteq A : a_n \neq x \forall n \in \mathbb{N}$.

Theorem 1.3.3 (Nested Interval Property)

Let (I_n) be a nested sequence of nonempty closed and bounded intervals with

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots.$$

Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

In particular, there exists $a \in \bigcap_{n=1}^{\infty} I_n$.

1.3.2. Open and Closed Sets

Definition 1.3.4 (Open/Closed Set)

$A \subseteq \mathbb{R}$ is *open* if $\forall a \in A \exists V_\varepsilon(a) \subseteq A$ or equivalently if its complement is closed.

$A \subseteq \mathbb{R}$ is *closed* if it contains its limit points or equivalently if its complement is open.

Theorem 1.3.5 (Clopen Sets)

\mathbb{R} and \emptyset are *clopen* (both opened and closed).

Theorem 1.3.6 (Unions/Intersections)

- (i) Arbitrary unions of open sets are open; finite intersections of open sets are open.
- (i) Arbitrary intersections of closed sets are closed; finite unions of closed sets are closed.

1.3.3. Compactness

Definition 1.3.7 (Compact)

A set K in a topological space is *compact* if every open cover has a finite subcover.

Theorem 1.3.8 (Heine–Borel)

A set $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Theorem 1.3.9

\mathbb{R} is not compact.

\emptyset is compact.

Note

Compactness is like a generalization of closed intervals.

2. The Basics of Limits

2.1. SEQUENCES

Definition 2.1.1 (Sequence)

A *sequence* is a function whose domain is \mathbb{N} .

Definition 2.1.2 (Convergence)

A sequence *converges* to a if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies |a_n - a| < \varepsilon$$

or equivalently if for any $V_\varepsilon(a)$ there exists a point in the sequence after which all terms are in $V_\varepsilon(a)$. In other words, if every ε -neighborhood of some point contains all but a finite number of the terms in (a_n) .

We write this $\lim_{n \rightarrow \infty} a_n = \lim a_n = a$ or $a_n \rightarrow a$.

Example. Template of a typical convergence proof:

- (i) Let $\varepsilon > 0$ be arbitrary.
- (ii) Propose an $N \in \mathbb{N}$ (found before writing the proof).
- (iii) Assume $n \geq N$.
- (iv) Show that $|a_n - a| < \varepsilon$.

Theorem 2.1.3 (Uniqueness of Limits)

The limit of a sequence, if it exists, is unique.

2.1.1. Bounded

Definition 2.1.4 (Bounded)

A sequence is *bounded* if $\exists M > 0 : |a_n| < M \ \forall n \in \mathbb{N}$.

Theorem 2.1.5 (Convergent)

Every convergent sequence is bounded.

If a sequence is monotone and bounded it converges.

Subsequences of a convergent sequence converge to the same limit.

Theorem 2.1.6 (Bolzano–Weierstrass)

In a compact set $K \subset \mathbb{R}$, every bounded sequence contains a convergent subsequence whose limit point is in K .

2.1.2. Cauchy

Definition 2.1.7 (Cauchy Sequence)

A sequence (a_n) is a *Cauchy sequence* if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : m, n \geq N \implies |a_n - a_m| < \varepsilon.$$

Theorem 2.1.8 (Cauchy Criterion)

A sequence converges if and only if it is a Cauchy sequence.

2.2. FUNCTIONS

Theorem 2.2.1 (Function Limit)

Given $f : A \rightarrow \mathbb{R}$ with the limit point c ,

- (i) $\lim_{x \rightarrow c} f(x) = L$ is equivalent to
- (ii) if $\forall (x_n) \subseteq A : (x_n \neq c \text{ and } x_n \rightarrow c)$ it follows

that $f(x_n) \rightarrow L$.

Note

In the $\varepsilon\delta$ -definition of limits, the additional restriction that $0 < |x - a|$ is just a way to say $x \neq c$.

Definition 2.2.2 (Infinite Limit)

Given a limit point $c \in D_f$, we say that $\lim_{x \rightarrow c} f(x) = \infty$ if

$$\forall M \exists \delta > 0 : 0 < |x - c| < \delta \implies f(x) \geq M.$$

2.3. CONTINUITY

2.3.1. Existence

Theorem 2.3.1 (Continuity)

The following are equivalent:

- (i) $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is *continuous* at $c \in A$.
- (ii) $\forall \varepsilon > 0 \exists \delta > 0 : |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$, where $x \in A$.
- (iii) $\forall V_\varepsilon(f(c)) \exists V_\delta(c) : x \in V_\delta(c) \cap A \implies f(x) \in V_\varepsilon(f(c))$
- (iv) $x_n \rightarrow c$, where $(x_n) \subseteq A$, implies $f(x_n) \rightarrow f(c)$.

If c is a limit point of A :

- (v) $\lim_{x \rightarrow c} f(x) = f(c)$, also written $\lim_{h \rightarrow 0} f(c + h) - f(c) = 0$.

Note that (ii) defines (i). Mostly (v) is used in practice.

Corollary 2.3.2 (Isolated Continuity)

All functions are continuous at isolated points.

Theorem 2.3.3 (Dirichlet Discontinuous)

The Dirichlet function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \in \mathbb{I}$ is discontinuous everywhere.

Definition 2.3.4 (Uniform Continuity)

We say f is *uniformly continuous* on I if

$$\forall \varepsilon > 0 \exists \delta > 0 : x, y \in I, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

In particular, δ can be chosen independent of y .

Theorem 2.3.5

If a function is uniformly continuous, it is also continuous.

Theorem 2.3.6 (Heine–Cantor)

If f is continuous and defined on a compact set K , then it is also uniformly continuous on K .

Proof. Assume the opposite, that f is continuous but not uniformly. Since f is not uniformly continuous,

$$\exists \varepsilon_0 > 0 : \forall \delta > 0 \exists x, y \in K : |x - y| < \delta \text{ but } |f(x) - f(y)| \geq \varepsilon_0.$$

Now, choose (x_n) and (y_n) such that

$$|x_n - y_n| < \frac{1}{n} \text{ and } |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

Theorem 2.1.6 asserts that there exists some subsequence $x_{n_k} \rightarrow x_0$ for some $x_0 \in K$. From $|x_n - y_n| < \frac{1}{n}$ it follows that $y_{n_k} \rightarrow x_0$. Thus,

$$|x_{n_k} - y_{n_k}| \rightarrow 0,$$

and, because f is continuous with $f(x_{n_k}) \rightarrow x_0$ and $f(y_{n_k}) \rightarrow x_0$,

$$|f(x_{n_k}) - f(y_{n_k})| \rightarrow 0.$$

However, this contradicts our assumption that

$$|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0.$$

□

2.3.2. Composition

Theorem 2.3.7 (Composition)

Given $f : A \rightarrow B$ and $g : B \rightarrow \mathbb{R}$ with $f(A) \subseteq B$, if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c .

Theorem 2.3.8 (Composition Limit)

If f is continuous at y and $\lim_{x \rightarrow c} g(x) = y$, then

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(y).$$

2.3.3. Results

Theorem 2.3.9 (Intermediate Value)

If f is continuous on $[a, b]$, then for any y between $f(a)$ and $f(b)$, there exists some $c \in (a, b)$ such that $f(c) = y$.

Theorem 2.3.10 (Weierstrass Extreme Value)

If f is continuous on the compact set K , then f attains a maximum and a minimum value on K .

Theorem 2.3.11 (Limit of Bounded Function)

If f is bounded then $\lim_{h \rightarrow 0} f(h)h = 0$.

3. Calculus

3.1. THE DERIVATIVE

3.1.1. Differentiation

Definition 3.1.1 (Derivative at a Point)
Let $f : A \rightarrow \mathbb{R}$ and c a limit point of A . If

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

exists (finite), we say f is *differentiable* at c .

Theorem 3.1.2 (Chain Rule)

Let $f : X \rightarrow Y$ and $g : Y \rightarrow \mathbb{R}$. If f is differentiable at $c \in X$ and g is differentiable at $f(c) \in Y$, then $g \circ f$ is differentiable at c with

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Theorem 3.1.3 (Basic Derivatives)

$$\begin{aligned} \frac{d}{dx}(\arcsin x) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}(\sin x) &= \cos x \\ \frac{d}{dx}(\arccos x) &= -\frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}(\cos x) &= -\sin x \\ \frac{d}{dx}(\arctan x) &= \frac{1}{1+x^2} & \frac{d}{dx}(\tan x) &= \frac{1}{\cos^2 x} \\ \frac{d}{dx}(\operatorname{arccot} x) &= -\frac{1}{1+x^2} & \frac{d}{dx}(\ln|x|) &= \frac{1}{x} \\ \frac{d}{dx}(x^a) &= ax^{a-1} \quad (a \neq 0) & (f^{-1})'(y) &= \frac{1}{f'(x)} \quad (y = f(x), f'(x) \neq 0) \end{aligned}$$

Theorem 3.1.4 (L'Hôpital's Rule)

Let f and g be differentiable on an open interval containing c (except possibly at c), with $g'(c) \neq 0$ near c . Suppose

- (i) $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} g(x) = 0$ (or both $\pm\infty$), and
- (ii) $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L$ exists (or $\pm\infty$).

Then,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = L.$$

Proof of the zero case. Assume the limits are zero.

Let the functions be differentiable on the open interval (c, x) . Then, rewriting and applying Theorem 3.1.10 gives

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x \rightarrow c} \frac{f'(p)}{g'(p)} = \lim_{p \rightarrow c} \frac{f'(p)}{g'(p)}$$

for some p between c and x . □

Proof of the infinity case. The proof is too complicated. □

Important

This is only an implication, not an equivalence, so there may exist some other solution if this method fails.

3.1.2. Function Character

Theorem 3.1.5 (Fermat's or Interior Extremum)

Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable at the local extremum $c \in (a, b)$. Then $f'(c) = 0$.

However, note that a zero-derivative point may also be a stationary point of inflection.

Theorem 3.1.6 (Darboux's)

If f is differentiable on $[a, b]$ and if y lies strictly between $f'(a)$ and $f'(b)$, then $\exists c \in (a, b) : f'(c) = y$. Let $g(x) = f(x) - yx$ with

$g'(x) = f'(x) - y$. Note that $f'(c) = y$ if $g'(c) = 0$ for some $c \in (a, b)$. In other words, if f is differentiable on an interval, then f' satisfies the Intermediate Value Property (IVP).

Proof. Assume that $f'(a) < y < f'(b)$.

Let $g(x) = f(x) - yx$ with $g'(x) = f'(x) - y$. Note that $f'(c) = y$ if $g'(c) = 0$ for some $c \in (a, b)$.

Theorem 2.3.10 states that g must have a minimum point $c \in [a, b]$. More precisely $c \in (a, b)$ since, from the assumption, $g'(a) < 0$ and $g'(b) > 0$. Furthermore, $g'(c) = 0$ according to Theorem 3.1.5. More precisely $c \in (a, b)$ since, per assumption, $g'(a) < 0$ and $g'(b) > 0$. □

Theorem 3.1.7 (Newton's Method)

Find roots to a differentiable function $f(x)$.

Given x_n with the coordinates $(x_n, f(x_n))$, the tangent line is given by

$$T(x) = f'(x_n)(x - x_n) + f(x_n)$$

and intersects the x -axis at

$$T(x_{n+1}) = 0 \iff x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The method fails if it iterates endlessly or $f'(x_n) = 0$.

3.1.3. The Mean Value Theorems

Let f and g be continuous on $[a, b]$ and differentiable on (a, b) .

Theorem 3.1.8 (Rolle's)

$$f(a) = f(b) \implies \exists c \in (a, b) : f'(c) = 0$$

Proof. $f(x)$ is bounded and $f'(x) = 0$ at its interior extreme points. □

Theorem 3.1.9 (Mean Value)

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Let the signed distance d between the function value f and the secant y through a and b be

$$d(x) = f(x) - y(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

and note that $d(a) = d(b) = 0$. Then apply Theorem 3.1.8. □

Theorem 3.1.10 (Generalized Mean Value)

$$\exists c \in (a, b) : [f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If g' is never zero on (a, b) , then the above can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Let $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$ and then apply Theorem 3.1.8. □

3.2. FUNCTION GRAPHS

Tip (Sketching Graphs)

— **Information**

- (i) symmetries
 - (ii) split into cases
 - (iii) domain \rightarrow vertical asymptotes
 - (iv) factorize \rightarrow oblique asymptotes & roots
 - (v) first and second derivative and their roots
 - (vi) sign tables
 - (vii) calculate interesting points: intersection with y -axis, defined nondifferentiable points, local extremums, endpoints, inflection
- **Sketching**
- (i) axes
 - (ii) symmetries
 - (iii) asymptotes
 - (iv) interesting points
 - (v) curves

3.2.1. Asymptotes

Definition 3.2.1 (Asymptote)

The line $y = kx + m$ is an *oblique* asymptote of f if

$$\lim_{x \rightarrow \infty} (f(x) - (kx + m)) = 0.$$

The line $x = c$ is a *vertical* asymptote of f if

$$\lim_{x \rightarrow c^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow c^-} f(x) = \pm\infty.$$

The line $y = b$ is a *horizontal* asymptote of f if

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

Theorem 3.2.2 (Oblique Asymptote)

If $f(x)$ has an oblique asymptote $y = kx + m$, then

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$$

and

$$m = \lim_{x \rightarrow \infty} (f(x) - kx).$$

3.2.2. Convexity

Theorem 3.2.3 (Convexity)

Let f be twice differentiable on (a, b) . Then, $f''(x) \geq 0$ if and only if f is convex on (a, b) .

Definition 3.2.4 (Concave)

On $[a, b]$, a function $f : [a, b] \rightarrow \mathbb{R}$ is *concave* if $-f$ is convex.

3.2.3. Points

Definition 3.2.5 (Local Extremum)

A *local maximum* of $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a point c for which there exists an open neighborhood $N(c) \subseteq D$ such that

$$f(c) \geq f(x) \quad \forall x \in N(c).$$

Definition 3.2.6 (Stationary)

The point c is a *stationary point* of f if $f'(c) = 0$.

The *stationary order* is the smallest $n \geq 2$ such that

$$f'(c) = f''(c) = \dots = f^{(n-1)}(c) = 0 \quad \text{but} \quad f^{(n)}(c) \neq 0.$$

Definition 3.2.7 (Critical)

The point c is a *critical point* if $f'(c)$ is stationary or undefined.

Definition 3.2.8 (Inflection)

A point c is an *inflection point* of f if f is continuous at c and if f is convex on one side of c and concave on the other side.

Theorem 3.2.9 (First Nonzero Derivative)

If f has stationary order n , then:

- If n is *even* $\implies f$ has a local extremum at c .
Furthermore: $f^{(n)}(c) > 0 \implies$ local minimum, $f^{(n)}(c) < 0 \implies$ local maximum.
 - If n is *odd* $\implies c$ is a stationary inflection point.
- Proof.* The Taylor series with remainder simplifies to

$$f(c+h) = f(c) + \frac{f^{(n)}(c)}{n!}h^n + O(h^{n+1}).$$

Its change close to c is thus

$$f(c+h) - f(c) \approx \frac{f^{(n)}(c)}{n!}h^n,$$

which changes sign if and only if n is odd. Similarly,

$$f'(c+h) - f'(c) \approx \frac{f^{(n-1)}(c)}{(n-1)!}h^{n-1}$$

for the first derivative and

$$f''(c+h) - f''(c) \approx \frac{f^{(n-2)}(c)}{(n-2)!}h^{n-2}$$

for the second derivative. □

Corollary 3.2.10 (Second Derivative Test)

If f'' is continuous at c and $f'(c) = 0$, then:

- $f''(c) > 0 \implies$ local minimum.
- $f''(c) < 0 \implies$ local maximum.
- $f''(c) = 0$ and $f^{(3)}(c) \neq 0 \implies$ stationary inflection point.

Note: $f''(c) = 0$ alone is insufficient for an inflection; the curvature must change sign.

Examples.

- $f(x) = x^3$: $f'(0) = f''(0) = 0$, $f^{(3)}(0) = 6 \neq 0$ (odd $n = 3$) \implies stationary inflection at 0.
- $f(x) = x^4$: $f'(0) = f''(0) = f^{(3)}(0) = 0$, $f^{(4)}(0) = 24 > 0$ (even $n = 4$) \implies local minimum at 0, no inflection.
- $f(x) = -x^4$: local maximum at 0, no inflection.

3.3. ORDINARY DIFFERENTIAL EQUATIONS

3.4. THE RIEMANN INTEGRAL

3.4.1. Definition

Definition 3.4.1 (Partition)

A partition of $[a, b]$ is a finite set

$$P = \{x_0, x_1, \dots, x_n\}$$

such that

$$a = x_0 < x_1 < \dots < x_n = b,$$

The partition P has *subintervals*

$$[x_{i-1}, x_i] \quad i = 1, 2, \dots, n$$

of which the length of the largest is its *mesh* or *norm*

$$\|P\| = \max_{1 \leq i \leq n} (x_i - x_{i-1}).$$

A smaller such is indicative of a finer partition.

Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. We now define its definite integral.

Definition 3.4.2 (Darboux Integral)

Define the *lower sum*

$$L(f, P) = \sum_{i=1}^n (\inf\{f(x) : x \in [x_{i-1}, x_i]\})(x_i - x_{i-1}).$$

and the *upper sum*

$$U(f, P) = \sum_{i=1}^n (\sup\{f(x) : x \in [x_{i-1}, x_i]\})(x_i - x_{i-1})$$

The function f is *Darboux integrable* if $\sup_P L(f, P) = \inf_P U(f, P)$. The common value is denoted as the *definite integral* $\int_a^b f(x) dx$.

Definition 3.4.3 (Alternative Darboux Integral)

Let Φ and Ψ be the *lower* and *upper step functions* such that

$$\Phi(x) \leq f(x) \leq \Psi(x) \quad \forall x \in [a, b],$$

forming the *lower integral*

$$L(f) = \sup \left\{ \int_a^b \Phi(x) dx : \Phi \text{ is a lower step function to } f \right\}$$

and the *upper integral*

$$U(f) = \inf \left\{ \int_a^b \Psi(x) dx : \Psi \text{ is an upper step function to } f \right\}$$

which, if equal, give the definite integral.

Note that the integral of a step function is simply its signed area.

Definition 3.4.4 (Riemann Integral)

From a partition P of $[a, b]$ pick *sample points*

$$t_i \in [x_{i-1}, x_i], \quad i = 1, 2, \dots, n$$

and form the (tagged) *Riemann sum*

$$S(f, P, (t_i)) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}).$$

We say f is *Riemann integrable* if there exists $L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \exists \delta > 0 : \|P\| < \delta \implies |S(f, P, (t_i)) - L| < \varepsilon$$

for every choice of sample points (t_i) . In that case we write

$$L = \int_a^b f(x) dx.$$

Theorem 3.4.5

The Darboux and Riemann integrals are equivalent.

3.4.2. Integrability

Theorem 3.4.6 (Integrability)

Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded.

The function is integrable if and only if:

- (i) $\forall \varepsilon > 0 \exists P : U(f, P) - L(f, P) < \varepsilon$.
- (ii) $\forall (P_n) : \|P_n\| \rightarrow 0 \implies U(f, P_n) - L(f, P_n) \rightarrow 0$.
- (iii) (Lebesgue Criterion for Riemann Integrability)
Its set of discontinuities has Lebesgue measure zero.
- (iv) $\forall \varepsilon > 0 \exists \Phi, \Psi : \int_a^b \Psi(x) dx - \int_a^b \Phi(x) dx < \varepsilon$,

where Φ and Ψ are lower and upper step functions.

- The function is integrable if:
 - (i) f is *monotone* on $[a, b]$
 - (ii) f is continuous except at finitely many points, or at countably many points where it has only removable or jump discontinuities.

Theorem 3.4.7

Assume f is continuous on $[a, b]$. Let

$$M_i = \max_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \min_{x \in [x_{i-1}, x_i]} f(x).$$

Then,

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n M_i(x_i - x_{i-1}) = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n m_i(x_i - x_{i-1}) = \int_a^b f(x) dx.$$

Theorem 3.4.8 (Absolute Value / Triangle)

If f integrable, then $|f|$ integrable and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Theorem 3.4.9 (Products and Composition)

If f, g integrable, then fg is integrable.

If f integrable and φ continuous on a set containing $f([a, b])$, then $\varphi \circ f$ is integrable.

Theorem 3.4.10 (Uniform Limit)

If (f_n) are integrable on $[a, b]$ and $f_n \rightarrow f$ uniformly, then f is integrable and

$$\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx.$$

3.4.3. Properties

Theorem 3.4.11 (Linearity)

If f, g are integrable and $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$$

Theorem 3.4.12 (Additivity of the Interval)

If $c \in (a, b)$ and f integrable on $[a, b]$, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

It follows that $\int_a^a f(x) dx = 0$ and $\int_a^b f(x) dx = -\int_b^a f(x) dx$.

Theorem 3.4.13 (Order / Comparison)

If f, g integrable and $f(x) \leq g(x)$ on $[a, b]$, then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Corollary 3.4.14 (Positivity)

If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x) dx \geq 0$. Moreover, if f is continuous and the integral is 0, then $f \equiv 0$.

Theorem 3.4.15 (Bounding by a Supremum)

If $|f(x)| \leq M$ on $[a, b]$, then

$$\left| \int_a^b f(x) dx \right| \leq M(b - a).$$

Theorem 3.4.16 (Mean Value for Integrals)

If f is continuous on $[a, b]$, then</

4. Infinite Series

4.1. SERIES

Definition 4.1.1 (Infinite Series)

Let $(a_j)_{j=0}^\infty$ and let $(s_n)_{n=0}^\infty$. The sum of the infinite series is defined as

$$\sum_{j=0}^\infty a_j = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{j=0}^n a_j.$$

If $a_j \geq 0$ for every j we say that the series is *positive*.

⚠ Warning

Beware of treating infinite series like elementary algebra, e.g., by rearranging terms.

Theorem 4.1.2 (Geometric Series)

If $|x| < 1$, then

$$\sum_{j=0}^\infty x^j = \frac{1}{1-x}$$

since

$$s_n = \sum_{j=0}^n x^j = \frac{1-x^{n+1}}{1-x}.$$

4.1.1. Convergence

Theorem 4.1.3 (Cauchy Criterion for Series)

The series $\sum_{k=0}^\infty a_k$ converges if and only if

$$\forall \varepsilon > 0 \ \exists N : n > m > N \implies |a_m + a_{m+1} + \cdots + a_{n-1} + a_n| < \varepsilon.$$

Corollary 4.1.4 (Series Term Test)

If $\sum_{k=1}^\infty a_k$ converges, then $a_k \rightarrow 0$. However, the reverse is not implied.

Lemma 4.1.5

The series $\sum_{j=1}^\infty 1/j$ is divergent.

Theorem 4.1.6 (Inverse Power Series)

The series $\sum_{j=1}^\infty 1/j^p$ converges if and only if $p > 1$.

Theorem 4.1.7 (Ratio Test)

Let (a_n) be a sequence of positive terms and define

$$L = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then:

- (i) If $L < 1$, the series $\sum_{n=1}^\infty a_n$ converges.
- (ii) If $L > 1$ (including $L = \infty$), the series diverges.
- (iii) If $L = 1$, the test is inconclusive.

Theorem 4.1.8 (Direct Comparison Test)

Let (a_k) and (b_k) satisfy $0 \leq a_k \leq b_k$. Then,

- (i) $\sum_{k=1}^\infty (a_k)$ converges if $\sum_{k=1}^\infty (b_k)$ converges.
- (ii) $\sum_{k=1}^\infty (b_k)$ diverges if $\sum_{k=1}^\infty (a_k)$ diverges.

Theorem 4.1.9 (Limit Comparison Test)

Let $\sum_{j=0}^\infty a_j$ and $\sum_{j=0}^\infty b_j$ be positive series with terms such that

$$\lim_{j \rightarrow \infty} \frac{a_j}{b_j} = K$$

for some finite $K \neq 0$. Then, $\sum_{j=0}^\infty a_j$ converges if and only if $\sum_{j=0}^\infty b_j$ converges.

Theorem 4.1.10 (Alternating Series Test)

Let (a_n) satisfy

- (i) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and
- (ii) $(a_n) \rightarrow 0$.

Then, $\sum_{n=1}^\infty (-1)^{n+1} a_n$ converges.

Definition 4.1.11 (Absolutely Convergent)

A series $\sum_{j=0}^\infty a_j$ is *absolutely convergent* if $\sum_{j=0}^\infty |a_j|$ is convergent.

Theorem 4.1.12

If a series is absolutely convergent then it is convergent.

Theorem 4.1.13 (Cauchy Condensation Test)

Let (a_n) be a decreasing sequence of nonnegative real numbers. Then $\sum_{n=1}^\infty a_n$ converges if and only if $\sum_{n=0}^\infty 2^n a_{2^n}$ converges.

Corollary 4.2.5 (Integral Test)

Let f be continuous, positive, and decreasing on $[m, \infty)$, where $m \in \mathbb{N}$. Then,

$$\sum_{n=m}^\infty f(n)$$

converges if and only if

$$\int_m^\infty f(x) \, dx$$

converges.

4.2. INDEFINITE INTEGRALS

4.2.1. Unlimited Intervals

Definition 4.2.1

Let f be integrable on $[a, R]$ for all $R > a$. Then the integral is defined

$$\int_a^\infty f(x) \, dx = \lim_{R \rightarrow \infty} \int_a^R f(x) \, dx.$$

If this limit exists, then the integral is said to be convergent.

Definition 4.2.2

Let f be integrable on every closed and bounded interval. If *both*

$$\int_{-\infty}^a f(x) \, dx \quad \text{and} \quad \int_{-\infty}^a f(x) \, dx$$

are convergent, then for any real a we define the convergent integral

$$\int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^a f(x) \, dx + \int_{-\infty}^a f(x) \, dx.$$

Theorem 4.2.3 (Properties)

Let f and g be integrable on $[a, R]$. The following applies.

Theorem 4.1.6 (Inverse Power Series)

The series $\sum_{j=1}^\infty 1/j^p$ converges if and only if $p > 1$.

Theorem 4.1.8 (Direct Comparison Test)

Let (a_k) and (b_k) satisfy $0 \leq a_k \leq b_k$. Then,

- (i) $\sum_{k=1}^\infty (a_k)$ converges if $\sum_{k=1}^\infty (b_k)$ converges.
- (ii) $\sum_{k=1}^\infty (b_k)$ diverges if $\sum_{k=1}^\infty (a_k)$ diverges.

Theorem 4.1.9 (Limit Comparison Test)

Let $\sum_{j=0}^\infty a_j$ and $\sum_{j=0}^\infty b_j$ be positive series with terms such that

$$\lim_{j \rightarrow \infty} \frac{a_j}{b_j} = K$$

for some finite $K \neq 0$. Then, $\sum_{j=0}^\infty a_j$ converges if and only if $\sum_{j=0}^\infty b_j$ converges.

Theorem 4.1.12

If a series is absolutely convergent then it is convergent.

⚠ Warning

The following does not apply.

Corollary 4.1.4 (Series Term Test)

If $\sum_{k=1}^\infty a_k$ converges, then $a_k \rightarrow 0$. However, the reverse is not implied.

Theorem 4.1.7 (Ratio Test)

Let (a_n) be a sequence of positive terms and define

$$L = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then:

- (i) If $L < 1$, the series $\sum_{n=1}^\infty a_n$ converges.
- (ii) If $L > 1$ (including $L = \infty$), the series diverges.
- (iii) If $L = 1$, the test is inconclusive.

Theorem 4.1.10 (Alternating Series Test)

Let (a_n) satisfy

- (i) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and
- (ii) $(a_n) \rightarrow 0$.

Then, $\sum_{n=1}^\infty (-1)^{n+1} a_n$ converges.

Theorem 4.2.4

Let f be decreasing on $[m, n]$, where $m < n$ are integers. Then,

$$\sum_{j=m+1}^n f(j) \leq \int_m^n f(x) \, dx \leq \sum_{j=m}^{n-1} f(j)$$

and

$$f(n) + \int_m^n f(x) \, dx \leq \sum_{j=m}^n f(j) \leq f(m) + \int_m^n f(x) \, dx.$$

Let f instead be increasing. Then,

$$\sum_{j=m}^{n-1} f(j) \leq \int_m^n f(x) \, dx \leq \sum_{j=m+1}^n f(j)$$

and

$$f(m) + \int_m^n f(x) \, dx \leq \sum_{j=m}^n f(j) \leq f(n) + \int_m^n f(x) \, dx.$$

Corollary 4.2.5 (Integral Test)

Let f be continuous, positive, and decreasing on $[m, \infty)$, where $m \in \mathbb{N}$. Then,

$$\sum_{n=m}^\infty f(n)$$

converges if and only if

$$\int_m^\infty f(x) \, dx$$

converges.

4.2.2. Open Intervals

4.3. TAYLOR'S THEOREM

4.3.1. Statement

Theorem 4.3.1 (Taylor's)

Suppose f is continuously differentiable n times on $[a, b]$ and $n + 1$ times on (a, b) . Fix $c \in [a, b]$. Then,

$$f(x) = P_n(x) + R_n(x),$$

where the *Taylor polynomial* of degree n around c is

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x-c)^k$$

and the *Lagrange remainder* of degree n around c is

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-c)^{n+1}$$

for some ξ strictly between c and x .

Note that other remainder forms exist.

Proof. Let $h = x - c$ be the distance from the point. Then,

$$f(x) = f(c+h) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1} = p_n(h) + r_n(h),$$

where $p_n(h)$ and $r_n(h)$ correspond to $P_n(x)$ and $R_n(x)$.

Define

$$F_{n,h}(t) = \sum_{k=0}^n \frac{f^{(k)}(t)}{k!} (c+h-t)^k,$$

with $F_{n,h}(c) = p_n(h)$ and $F_{n,h}(c+h) = f(c+h)$, and derivative

$$F'_{n,h}(\xi) = \frac{f^{(n+1)}(\xi)}{n!} (c+h-\xi)^n.$$

Also let

$$g_{n,h}(t) = (c+h-t)^{n+1},$$

with $g_{n,h}(c) = h^{n+1}$ and $g_{n,h}(c+h) = 0$ and

$$g'_{n,h}(\xi) = -(n+1)(c+h-\xi)^n.$$

Theorem 3.1.10 gives

$$\frac{F_{n,h}(c+h) - F_{n,h}(c)}{g_{n,h}(c+h) - g_{n,h}(c)} = \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$$

for some ξ between c and $c+h$. Substituting,

$$\frac{f(c+h) - p_n(h)}{0 - h^{n+1}} = \frac{f^{(n+1)}(\xi)(c+h-\xi)^n/n!}{-(n+1)(c+h-\xi)^n}$$

so

$$f(c+h) - p_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}.$$

Hence

$$f(c+h) = p_n(h) + r_n(h)$$

or in x -notation

$$f(x) = P_n(x) + R_n(x)$$

with ξ strictly between c and x . □

Proof using integrals. From Theorem 3.4.18 (ii) we have

$$\int_c^x f'(t) \, dt = f(t) - f(c)$$

which we expand using Theorem 3.4.20 as

$$\begin{aligned} f(x) &= f(c) + \int_c^x 1 \cdot f'(t) \, dt \\ &= f(c) + [(t-x)f'(t)]_c^x - \int_c^x (t-x)f''(x) \, dt \\ &= f(c) + f'(c)(x-c) - \left(\left[\frac{(t-x)^2}{2} f''(t) \right]_c^x - \int_c^x \frac{(t-x)^2}{2} f^{(3)}(t) \, dt \right) \\ &= f(c) + f'(c)(x-c) + \frac{f''(t)}{2} (x-c)^2 + \int_c^x \frac{(t-x)^2}{2} f^{(3)}(t) \, dt \\ &= \dots \\ &= P_n(x) + (-1)^n \int_c^x \frac{(t-x)^n}{n!} f^{(n+1)}(t) \, dt \end{aligned}$$

□

Definition 4.3.2 (Radius of Convergence)

Let $R_n(x)$ be the remainder to the Taylor polynomial around a point c . The *radius of convergence* R is the supremum of $r \geq 0$ such that

$$\forall x : |x - c| < r \implies \lim_{n \rightarrow \infty} R_n(x) = 0,$$

which implies that the Taylor series converges to $f(x)$ for all such x (so $f(x) = P_\infty(x)$).

Theorem 4.3.3 (Common Maclaurin Series)

The following functions have a Maclaurin series with radius of convergence $r = \infty$:

$$\begin{aligned} e^x &= \sum_{k=0}^\infty \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ \sin x &= \sum_{k=0}^\infty (-1)^k \frac{x^{\{2k+1\}}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ \cos x &= \sum_{k=0}^\infty (-1)^k \frac{x^{\{2k\}}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \arctan x &= \sum_{k=0}^\infty (-1)^k \frac{x^{\{2k+1\}}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (|x| \leq 1) \\ \ln(1+x) &= \sum_{k=1}^\infty (-1)^{\{k+1\}} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (|x| < 1) \\ (1+x)^a &= \sum_{k=0}^\infty \binom{a}{k} x^k \quad (|x| < 1) \end{aligned}$$

4.3.2. Function Order

Definition 4.3.4 (Big O at Infinity)

Let f and g be defined on (c, ∞) . We say that f belongs to the set O of g as $x \rightarrow \infty$, writing $O(g(x))$, if there exists M and x_0 such that

$$|f(x)| \leq M|g(x)|,$$

for every $x > x_0$.

Definition 4.3.5 (Big O at a Point)

Let f and g be defined on a neighborhood of c . We say that f belongs to the set O of g around c , writing $O(g(x))$, if there exists M and $\delta > 0$ such that

$$|f(x)| \leq M|g(x)|$$

for every $x \in (c - \delta, c + \delta)$.

Theorem 4.3.6 (Big O Behavior)

If $h(x) = O(f(x))$ and $k(x) = O(g(x))$ (same limiting regime), then $h(x)k(x) = O(f(x)g(x))$.

If $m \leq n$ then as $x \rightarrow 0$, $x^n = O(x^m)$ so $O(x^m) + O(x^n) = O(x^m)$. As $x \rightarrow \infty$, $x^m = O(x^n)$ so $O(x^m) + O(x^n) = O(x^n)$.

Theorem 4.3.7

Let $f(x) : [a, b] \rightarrow \mathbb{R}$ and fix $c \in [a, b]$. Suppose f is continuously differentiable n times on $[a, b]$ and $n + 1$ times on (a, b) . Then,

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x-c)^k + O(|x-c|^{n+1}) \text{ as } x \rightarrow c.$$

Furthermore, the coefficients $f^{(k)}(c)/k!$ are unique to each $(x-c)^k$.