Analys i en variabel

SF1673 (HT25)

1. The real numbers

Theorem 1.1 (Triangle Inequalities)

(i) $|a+b| \leq |a| + |b|$
(ii) $|a-b| \leq |a-c| + |c-b|$
(iii) $|a-b| \geq \||a| - |b|\|$

Definition 1.2

$a = b \iff (\forall \varepsilon > 0 \Rightarrow |a-b| < \varepsilon)$

Theorem 1.3 (Induction)

If $s \in \mathbb{N}$ such that

(i) $1 \in S$ and
(ii) when $n \in S$ it follows that $n+1 \in S$

it follows that $S = \mathbb{N}$.

Theorem 1.4 (Nested Interval Property)

The intervals $\mathbb{R} \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ all contain a point $a = \bigcap_{n=1}^{\infty} I_n$.

Axiom 1.5 (Supremum Property or Axiom of Completeness)

Every bounded, non-empty set of real numbers has a least upper bound.

Note

The same does not apply for the rationals.

Definition 1.6 (Least Upper Bound)

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then,

$$s = \sup A \iff \forall \varepsilon > 0 \ \exists a \in A : s - \varepsilon < a.$$

Theorem 1.7 (Density of $\mathbb{Q}$ in $\mathbb{R}$)

(i) $\forall a < b \in \mathbb{R} \ \exists r \in \mathbb{Q} : a < r < b$
(ii) $\forall y \in \mathbb{R} \ \exists (r_n) \in \mathbb{Q} : (r_n) \to y$

Definition 1.8

$f : X \to Y$ is injective (or one-to-one) if $x_1 \neq x_2 \implies y_1 \neq y_2$ or equivalently if $y_1 = y_2 \implies x_1 = x_2$.

f is surjective if $\forall y \ \exists x : x = y$.

f is bijective if is both injective and surjective or equivalently if each y is mapped to exactly one x.

1.1. Cardinality

Definition 1.1.1 (Cardinality)

A has the same cardinality as B if there exists a bijective $f : A \to B$.

Definition 1.1.2

A is countable if $\mathbb{N} \sim A$. Otherwise, A is uncountable if there are infinite elements or finite if there are finite elements.

Theorem 1.1.3

$\mathbb{Q}$ is countable.

Proof. Let $A_1 = \{0\}$ and let

$$A_n = \{\pm p/q : p, q \in \mathbb{N}_+, \gcd(p,q) = 1, p + q = n\}$$

for all $n \geq 2$. Each A_n is finite and every rational numbers appears in exactly one set. □

$\mathbb{I}$ is uncountable.

Proof. $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ where $\mathbb{Q}$ is countable. □

$\mathbb{R}$ is uncountable.

Proof. Cantor's diagonalization method. □

1.2. Sequences and Series

Theorem 1.2.1

Let $(a_j)_{j=0}^{\infty}$ and let $(s_n)_{n=0}^{\infty}$. The sum of the infinite series is defined as

$$\sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{j=0}^{n} a_j.$$

Caution

Beware of treating infinite series like elementary algebra, e.g., by rearranging terms.

1.2.1. Convergence

Definition 1.2.2

A sequence is a function whose domain is $\mathbb{N}$.

Definition 1.2.3 (Convergence)

A sequence converges to a if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : n \geq \implies |a_n - a| < \varepsilon$$

or equivalently if for any $V_\varepsilon(a)$ there exists a point in the sequence after which all terms are in $V_\varepsilon(a)$. In other words if every ε-neighborhood contains all but a finite number of the terms in (a_n).

We write this $\lim_{n\to\infty} a_n = a$ or $\lim a_n = a$ or $a_n \to a$.

Example. Template of a typical convergence proof:

(i) Let $\varepsilon > 0$ be arbitrary.
(ii) Propose an $N \in \mathbb{N}$ (found before writing the proof).
(iii) Assume $n \geq N$.
(iv) Show that $|a_n - a| < \varepsilon$.

Theorem 1.2.4

The limit of a sequence, if it exists, is unique.

Definition 1.2.5

A sequence is bounded if $\exists M > 0 : |a_n| < M \ \forall n \in \mathbb{N}$.

Theorem 1.2.6

Every convergent series is bounded.

If a sequence is monotone and bounded it converges.

Subsequences of a convergent series converge to the same limit.

Theorem 1.2.7 (Bolzano–Weierstrass)

Every bounded sequence contains a convergent subsequence.

Theorem 1.2.8

If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$.

Theorem 1.2.9 (Cauchy Condensation Test)

Let (a_n) be a decreasing sequence of non-negative real numbers. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Definition 1.2.10 (Cauchy Sequence)

A sequence (a_n) is a Cauchy sequence if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : m, n \geq N \implies |a_n - a_m| < \varepsilon.$$

Theorem 1.2.11 (Cauchy Criterion)

A sequence converges if and only if it is a Cauchy sequence.

Theorem 1.2.12

Let (a_k) and (b_k) satisfy $0 \leq a_k \leq b_k$. Then,

(i) $\sum_{k=1}^{\infty} (a_k)$ converges if $\sum_{k=1}^{\infty} (b_k)$ converges.
(ii) $\sum_{k=1}^{\infty} (b_k)$ diverges if $\sum_{k=1}^{\infty} (a_k)$ diverges.

Theorem 1.2.13 (Alternating Series Test)

Let (a_n) satisfy

(i) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and
(ii) $(a_n) \to 0$.

Then, $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

1.3. Topology

Definition 1.3.1 (Limit Point)

x is a limit point of A if every $V_\varepsilon(x)$ intersects A at some point other than x.

Theorem 1.3.2

x is a limit point of A if $x = \lim a_n$ for some $(a_n) \subseteq A : a_n \neq x \; \forall n \in \mathbb{N}$.

Definition 1.3.3

$A \subseteq \mathbb{R}$ is open if $\forall a \in A \; \exists V_\varepsilon(a) \subseteq A$ or equivalently if its complement is closed.

$A \subseteq \mathbb{R}$ is closed if it contains its limit points or equivalently if its complement is open.

Theorem 1.3.4

$\mathbb{R}$ and $\emptyset$ are clopen (both opened and closed).

Theorem 1.3.5

The union of open (closed) sets is open (closed).

The intersection of finitely many open (closed) sets is open (closed).

Definition 1.3.6

A set K in a topological space is compact if every open cover has a finite subcover.

Theorem 1.3.7 (Heine–Borel)

A set $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Note

Compactness is like a generalization of closed intervals.

2. Real functions

2.1. Limits

Theorem 2.1.1

Given $f : A \to \mathbb{R}$ with the limit point c,
(i) $\lim_{x\to c} f(x) = L$ is equivalent to
(ii) if $\forall (x_n) \subseteq A : (x_n \neq c \text{ and } x_n \to c)$ it follows that $f(x_n) \to L$.

Note

In the $\varepsilon\delta$-definition of limits, the additional restriction that $0 < |x - a|$ is just a way to say $x \neq c$.

2.2. Continuity

Definition 2.2.1 (Continuity)

A function $f : A \to R$ is continuous at $c \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : |x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon,$$

where $x \in A$.

Theorem 2.2.2

The following are equivalent:

(i) f is continuous (see Definition 2.2.1).
(ii) $\forall V_\varepsilon(f(c)) \; \exists V_\delta(c) : x \in V_\delta \cap A \Longrightarrow f(x) \in V_\varepsilon$.
(iii) $x_n \to c$, where $(x_n) \subseteq A$, implies $f(x_n) \to f(c)$.

If c is a limit point of A:
(iv) $\lim_{x\to c} f(x) = f(c)$.

Note

f is continuous if $f(x + h) - f(x) \to 0$ as $h \to 0$.

Theorem 2.2.3

All functions are continuous at isolated points.

Theorem 2.2.4

The Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \in \mathbb{I}$ is discontinuous everywhere.

Theorem 2.2.5 (Composition of Continuous Functions)

Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $f(A) \subseteq B$, if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Theorem 2.2.6

If f is continuous at y and $\lim_{x\to c} g(x) = y$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(y).$$

Theorem 2.2.7 (Darboux's)

If f is differentiable on $[a, b]$ and if α lies between $f'(a)$ and $f'(b)$, then $\exists c \in (a, b) : f'(c) = \alpha$.

2.3. The Mean Value Theorems

Let f and g be continuous on $[a, b]$ and differentiable on (a, b).

Theorem 2.3.1 (Rolle's)

$$f(a) = f(b) \Longrightarrow \exists c \in (a, b) : f'(c) = 0$$

Proof. $f(x)$ is bounded and $f'(x) = 0$ at its extreme points. □

Theorem 2.3.2 (Mean Value)

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Let the signed distance d between the function value f and the secant y through a and b be

$$d(x) = f(x) - y(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

and note that $d(a) = d(b) = 0$. Then apply Theorem 2.3.1. □

Theorem 2.3.3 (Generalized Mean Value)

$$\exists c \in (a, b) : [f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If g' is never zero on (a, b), then the above can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Let $h = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$ and then apply Theorem 2.3.1. □

2.3.1. L'Hôpital's Rule

Definition 2.3.4

Given a limit point $c \in D_f$, we say that $\lim_{x\to c} f(x) = \infty$ if

$$\forall M \; \exists \delta > 0 : 0 < |x - c| < \delta \Longrightarrow f(x) \geq M$$

Theorem 2.3.5

Let $f(x)$ and $g(x)$ be defined and, with the possible exception of at the limit point c, differentiable. If
(i) $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$ and
(ii) $g'(x) \neq 0$ for all $x \neq c$, then

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \Longrightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = L.$$

Proof of the zero case. First, assume the limits are zero.

Let the functions be differentiable on the open interval (c, x). Then, rewriting and applying Theorem 2.3.3 gives

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x\to c} \frac{f'(p)}{g'(p)} = \lim_{p\to c} \frac{f'(p)}{g'(p)}$$

for some p between c and x. □

Proof of the infinity case. Second, assume the limits are infinite. We will only prove the right-hand limit.

Let $c < a < b$. The Theorem 2.3.3 states that there exists a $p \in (a, b)$ such that

$$f'(p)[g(b) - g(a)] = g'(p)[f(b) - f(a)].$$

Solving for $f(a)$, we get

$$f(a) = f(b) + \frac{f'(p)(g(a) - g(b))}{g'(p)}.$$

We divide by $g(a)$ and get

$$\frac{f(a)}{g(a)} = \frac{f'(p)}{g'(p)} + \frac{1}{g(a)}\left(f(b) - g(b)\frac{f'(p)}{g'(p)}\right)$$

which we rewrite as

$$\frac{f(a)}{g(a)} - L = \frac{f'(p)}{g'(p)} - L + \frac{1}{g(a)}\left(f(b) - g(b)\frac{f'(p)}{g'(p)}\right)$$

□

Important

This is only an implication, not an equivalence, so there may exist some other solution if this method fails.