Analys i en variabel

SF1673 (HT25)

Contents

1. The Real Numbers

1.1. Reals

1.1.1. Prerequisites

Theorem 1.1.1 (Induction)

Let $S \subseteq \mathbb{N}$. If

(i) $1 \in S$, and

(ii) $n \in S \Longrightarrow n+1 \in S$ (inductive step),

then $S = \mathbb{N}$.

Definition 1.1.2 (Injective/Surjective/Bijective)

$f : X \to Y$ is *injective* (or one-to-one) if $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$ or equivalently if $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$.

f is *surjective* if $\forall y\ \exists x : f(x) = y$.

f is *bijective* if is both injective and surjective or equivalently if each y is mapped to exactly one x.

1.1.2. Comparison

Definition 1.1.3 (Equality)

$a = b \Longleftrightarrow (\forall \varepsilon > 0 \Rightarrow |a - b| < \varepsilon)$

Theorem 1.1.4 (Triangle Inequalities)

(i) $|a + b| \leq |a| + |b|$

(ii) $|a - b| \leq |a - c| + |c - b|$

(iii) $|a - b| \geq \| a| - |b\|$

The reverse triangle inequality (iii) is seldom used.

1.1.3. Bounds

Axiom 1.1.5 (Supremum Property or Axiom of Completeness)

Every bounded, nonempty set of real numbers has a least upper bound.

Note

The same does not apply for the rationals.

Definition 1.1.6 (Least Upper Bound)

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then,

$$s = \sup A \quad \Longleftrightarrow \quad \forall \varepsilon > 0\ \exists a \in A : s - \varepsilon < a.$$

1.2. Cardinality

Definition 1.2.1 (Cardinality)

A has the same *cardinality* as B if there exists a bijective $f : A \to B$.

Definition 1.2.2 (Countable/Uncountable)

A is *countably infinite* if $\mathbb{N} \sim A$.

A is *countable* if it is finite or countably infinite.

Otherwise, A is *uncountable*.

Theorem 1.2.3 (Countability of $\mathbb{Q}$, $\mathbb{R}$)

$\mathbb{Q}$ is countable.

Proof. Let $A_1 = \{0\}$ and let

$$A_n = \{\pm p/q : p, q \in \mathbb{N}_+, \gcd(p, q) = 1, p + q = n\}$$

for all $n \geq 2$. Each A_n is finite and every rational numbers appears in exactly one set. □

$\mathbb{R}$ is uncountable.

Proof. Cantor's diagonalization method. □

$\mathbb{I}$ is uncountable.

Proof. $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ where $\mathbb{Q}$ is countable. □

Theorem 1.2.4 (Density of $\mathbb{Q}$ in $\mathbb{R}$)

(i) $\forall a < b \in \mathbb{R}\ \exists r \in \mathbb{Q} : a < r < b$

(ii) $\forall y \in \mathbb{R}\ \exists (r_n) \in \mathbb{Q} : (r_n) \to y$

1.3. Topology

1.3.1. Points

Definition 1.3.1 (Limit Point)

x is a *limit point* of A if every $V_\varepsilon(x)$ intersects A at some point other than x.

Theorem 1.3.2 (Sequential Limit Point)

x is a limit point of A if $x = \lim a_n$ for some $(a_n) \subseteq A : a_n \neq x\ \forall n \in \mathbb{N}$.

Theorem 1.3.3 (Nested Interval Property)

Let (I_n) be a nested sequence of nonempty closed and bounded intervals with

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots.$$

Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

In particular, there exists $a \in \bigcap_{n=1}^{\infty} I_n$.

1.3.2. Open and Closed Sets

Definition 1.3.4 (Open/Closed Set)

$A \subseteq \mathbb{R}$ is *open* if $\forall a \in A\ \exists V_\varepsilon(a) \subseteq A$ or equivalently if its complement is closed.

$A \subseteq \mathbb{R}$ is *closed* if it contains its limit points or equivalently if its complement is open.

Theorem 1.3.5 (Clopen Sets)

$\mathbb{R}$ and $\emptyset$ are *clopen* (both opened and closed).

Theorem 1.3.6 (Unions/Intersections)

(i) Arbitrary unions of open sets are open; finite intersections of open sets are open.

(i) Arbitrary intersections of closed sets are closed; finite unions of closed sets are closed.

1.3.3. Compactness

Definition 1.3.7 (Compact)

A set K in a topological space is *compact* if every open cover has a finite subcover.

Theorem 1.3.8 (Heine–Borel)

A set $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Theorem 1.3.9

$\mathbb{R}$ is not compact.

$\emptyset$ is compact.

Note

Compactness is like a generalization of closed intervals.

2. The Basics of Limits

2.1. Sequences

Definition 2.1.1 (Sequence)

A *sequence* is a function whose domain is $\mathbb{N}$.

Definition 2.1.2 (Convergence)

A sequence *converges* to a if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} : n \geq N \Longrightarrow |a_n - a| < \varepsilon$$

or equivalently if for any $V_\varepsilon(a)$ there exists a point in the sequence after which all terms are in $V_\varepsilon(a)$. In other words, if every ε-neighborhood of some point contains all but a finite number of the terms in (a_n).

We write this $\lim_{n\to\infty} a_n = \lim a_n = a$ or $a_n \to a$.

Example. Template of a typical convergence proof:

(i) Let $\varepsilon > 0$ be arbitrary.
(ii) Propose an $N \in \mathbb{N}$ (found before writing the proof).
(iii) Assume $n \geq N$.
(iv) Show that $|a_n - a| < \varepsilon$.

Theorem 2.1.3 (Uniqueness of Limits)

The limit of a sequence, if it exists, is unique.

2.1.1. Bounded

Definition 2.1.4 (Bounded)

A sequence is *bounded* if $\exists M > 0 : |a_n| < M \; \forall n \in \mathbb{N}$.

Theorem 2.1.5 (Convergent)

Every convergent sequence is bounded.

If a sequence is monotone and bounded it converges.

Subsequences of a convergent sequence converge to the same limit.

Theorem 2.1.6 (Bolzano–Weierstrass)

In a compact set $K \subset \mathbb{R}$, every bounded sequence contains a convergent subsequence whose limit point is in K.

2.1.2. Cauchy

Definition 2.1.7 (Cauchy Sequence)

A sequence (a_n) is a *Cauchy sequence* if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} : m, n \geq N \Longrightarrow |a_n - a_m| < \varepsilon.$$

Theorem 2.1.8 (Cauchy Criterion)

A sequence converges if and only if it is a Cauchy sequence.

2.2. Functions

Theorem 2.2.1 (Function Limit)

Given $f : A \to \mathbb{R}$ with the limit point c,

(i) $\lim_{x\to c} f(x) = L$ is equivalent to
(ii) if $\forall (x_n) \subseteq A : (x_n \neq c$ and $x_n \to c)$ it follows that $f(x_n) \to L$.

Note

In the $\varepsilon\delta$-definition of limits, the additional restriction that $0 < |x - a|$ is just a way to say $x \neq c$.

Definition 2.2.2 (Infinite Limit)

Given a limit point $c \in D_f$, we say that $\lim_{x\to c} f(x) = \infty$ if

$$\forall M \; \exists \delta > 0 : 0 < |x - c| < \delta \Longrightarrow f(x) \geq M.$$

2.3. Continuity

2.3.1. Existence

Theorem 2.3.1 (Continuity)

The following are equivalent:

(i) $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous* at $c \in A$.
(ii) $\forall \varepsilon > 0 \; \exists \delta > 0 : |x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon$, where $x \in A$.
(iii) $\forall V_\varepsilon(f(c)) \; \exists V_\delta(c) : x \in V_\delta(c) \cap A \Longrightarrow f(x) \in V_\varepsilon(f(c))$
(iv) $x_n \to c$, where $(x_n) \subseteq A$, implies $f(x_n) \to f(c)$.

If c is a limit point of A:

(v) $\lim_{x\to c} f(x) = f(c)$, also written $\lim_{h\to 0} f(c+h) - f(c) = 0$.

Note that (ii) defines (i). Mostly (v) is used in practice.

Corollary 2.3.2 (Isolated Continuity)

All functions are continuous at isolated points.

Theorem 2.3.3 (Dirichlet Discontinuous)

The Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \in \mathbb{I}$ is discontinuous everywhere.

Definition 2.3.4 (Uniform Continuity)

We say f is *uniformly continuous* on I if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : x, y \in I, |x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon.$$

In particular, δ can be chosen independent of y.

Theorem 2.3.5

If a function is uniformly continuous, it is also continuous.

Theorem 2.3.6 (Heine–Cantor)

If f is continuous and defined on a compact set K, then it is also uniformly continuous on K.

Proof. Assume the opposite, that f is continuous but not uniformly. Since f is not uniformly continuous,

$$\exists \varepsilon_0 > 0 : \forall \delta > 0 \; \exists x, y \in K : \; |x - y| < \delta \text{ but } |f(x) - f(y)| \geq \varepsilon_0.$$

Now, choose (x_n) and (y_n) such that

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

Theorem 2.1.6 asserts that there exists some subsequence $x_{n_k} \to x_0$ for some $x_0 \in K$. From $|x_n - y_n| < \frac{1}{n}$ it follows that $y_{n_k} \to x_0$. Thus,

$$\left|x_{n_k} - y_{n_k}\right| \to 0,$$

and, because f is continuous with $f\left(x_{n_k}\right) \to x_0$ and $f\left(y_{n_k}\right) \to x_0$,

$$\left|f\left(x_{n_k}\right) - f\left(x_{n_k}\right)\right| \to 0.$$

However, this contradicts our assumption that

$$\left|f\left(x_{n_k}\right) \to f\left(y_{n_k}\right)\right| \geq \varepsilon_0.$$

□

2.3.2. Composition

Theorem 2.3.7 (Composition)

Given $f : A \to B$ and $g : B \to \mathbb{R}$ with $f(A) \subseteq B$, if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Theorem 2.3.8 (Composition Limit)

If f is continuous at y and $\lim_{x\to c} g(x) = y$, then

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(y).$$

2.3.3. Results

Theorem 2.3.9 (Intermediate Value)

If f is continuous on $[a, b]$, then for any y between $f(a)$ and $f(b)$, there exists some $c \in (a, b)$ such that $f(c) = y$.

Theorem 2.3.10 (Weierstrass Extreme Value)

If f is continuous on the compact set K, then f attains a maximum and a minimum value on K.

Theorem 2.3.11 (Limit of Bounded Function)

If f is bounded then $\lim_{h\to 0} f(h)h = 0$.

3. Calculus

3.1. The Derivative

3.1.1. Differentiation

Definition 3.1.1 (Derivative at a Point)

Let $f : A \to \mathbb{R}$ and c a limit point of A. If

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists (finite), we say f is *differentiable* at c.

Theorem 3.1.2 (Chain Rule)

Let $f : X \to Y$ and $g : Y \to \mathbb{R}$. If f is differentiable at $c \in X$ and g is differentiable at $f(c) \in Y$, then $g \circ f$ is differentiable at c with

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Theorem 3.1.3 (Basic Derivatives)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \frac{1}{\cos^2 x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arccot} x) = -\frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\ln|x|) = \frac{1}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^a) = ax^{a-1} \quad (a \neq 0) \qquad (f^{-1})'(y) = \frac{1}{f'(x)} \quad (y = f(x), f'(x) \neq 0)$$

Theorem 3.1.4 (L'Hôpital's Rule)

Let f and g be differentiable on an open interval containing c (except possibly at c), with $g'(x) \neq 0$ near c. Suppose

(i) $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ (or both $\pm\infty$), and
(ii) $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ exists (or $\pm\infty$).

Then,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L.$$

Proof of the zero case. Assume the limits are zero.

Let the functions be differentiable on the open interval (c, x). Then, rewriting and applying Theorem 3.1.10 gives

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x\to c} \frac{f'(p)}{g'(p)} = \lim_{p\to c} \frac{f'(p)}{g'(p)}$$

for some p between c and x. □

Proof of the infinity case. The proof is too complicated. □

Important

This is only an implication, not an equivalence, so there may exist some other solution if this method fails.

3.1.2. Function Character

Theorem 3.1.5 (Fermat's or Interior Extremum)

Let $f : (a, b) \to \mathbb{R}$ be differentiable at the local extremum $c \in (a, b)$. Then $f'(x) = 0$.

However, note that a zero-derivative point may also be a stationary point of inflection.

Theorem 3.1.6 (Darboux's)

If f is differentiable on $[a, b]$ and if y lies strictly between $f'(a)$ and $f'(b)$, then $\exists c \in (a, b) : f'(c) = y$. Let $g(x) = f(x) - yx$ with $g'(x) = f'(x) - y$. Note that $f'(c) = y$ if $g'(c) = 0$ for some $c \in (a, b)$.

In other words, if f is differentiable on an interval, then f' satisfies the Intermediate Value Property (IVP).

Proof. Assume that $f'(a) < y < f'(b)$.

Let $g(x) = f(x) - yx$ with $g'(x) = f'(x) - y$. Note that $f'(c) = y$ if $g'(c) = 0$ for some $c \in (a, b)$.

Theorem 2.3.10 states that g must have a minimum point $c \in [a, b]$. More precisely $c \in (a, b)$ since, from the assumption, $g'(a) < 0$ and $g'(b) > 0$. Furthermore, $g'(c) = 0$ according to Theorem 3.1.5. More precisely $c \in (a, b)$ since, per assumption, $g'(a) < 0$ and $g'(b) > 0$. □

Theorem 3.1.7 (Newton's Method)

Find roots to a differentiable function $f(x)$.

Given x_n with the coordinates $(x_n, f(x_n))$, the tangent line is given by

$$T(x) = f'(x_n)(x - x_n) + f(x_n)$$

and intersects the x-axis at

$$T(x_{n+1}) = 0 \iff x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The method fails if it iterates endlessly or $f'(x_n) = 0$.

3.1.3. The Mean Value Theorems

Let f and g be continuous on $[a, b]$ and differentiable on (a, b).

Theorem 3.1.8 (Rolle's)

$$f(a) = f(b) \implies \exists c \in (a, b) : f'(c) = 0$$

Proof. $f(x)$ is bounded and $f'(x) = 0$ at its interior extreme points. □

Theorem 3.1.9 (Mean Value)

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Let the signed distance d between the function value f and the secant y through a and b be

$$d(x) = f(x) - y(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

and note that $d(a) = d(b) = 0$. Then apply Theorem 3.1.8. □

Theorem 3.1.10 (Generalized Mean Value)

$$\exists c \in (a, b) : [f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If g' is never zero on (a, b), then the above can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Let $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$ and then apply Theorem 3.1.8. □

3.2. Function Graphs

Tip (Sketching Graphs)

— Information

(i) symmetries
(ii) split into cases
(iii) domain → vertical asymptotes
(iv) factorize → oblique asymptotes & roots
(v) first and second derivative and their roots
(vi) sign tables
(vii) calculate interesting points: intersection with y-axis, defined nondifferentiable points, local extremums, endpoints, inflection

— Sketching

(i) axes
(ii) symmetries
(iii) asymptotes
(iv) interesting points
(v) curves

3.2.1. Asymptotes

Definition 3.2.1 (Asymptote)

The line $y = kx + m$ is an *oblique* asymptote of f if

$$\lim_{x\to\infty} (f(x) - (kx + m)) = 0.$$

The line $x = c$ is a *vertical* asymptote of f if

$$\lim_{x\to c+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to c-} f(x) = \pm\infty.$$

The line $y = b$ is a *horizontal* asymptote of f if

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

Theorem 3.2.2 (Oblique Asymptote)

If $f(x)$ has an oblique asymptote $y = kx + m$, then

$$k = \lim_{x\to\infty} \frac{f(x)}{x}$$

and

$$m = \lim_{x\to\infty} (f(x) - kx).$$

3.2.2. Convexity

Theorem 3.2.3 (Convexity)

Let f be twice differentiable on (a, b). Then, $f''(x) \geq 0$ if and only if f is convex on (a, b).

Definition 3.2.4 (Concave)

On $[a, b]$, a function $f : [a, b] \to \mathbb{R}$ is *concave* if $-f$ is convex.

3.2.3. Points

Definition 3.2.5 (Local Extremum)

A *local maximum* of $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a point c for which there exists an open neighborhood $N(c) \subseteq D$ such that

$$f(c) \geq f(x) \quad \forall x \in N(c).$$

Definition 3.2.6 (Stationary)

The point c is a *stationary point* of f if $f'(c) = 0$.

The *stationary order* is the smallest $n \geq 2$ such that

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0 \ \text{ but } \ f^{(n)}(c) \neq 0.$$

Definition 3.2.7 (Critical)

The point c is a *critical point* if $f(c)$ is stationary or undefined.

Definition 3.2.8 (Inflection)

A point c is an *inflection point* of f if f is continuous at c and if f is convex on one side of c and concave on the other side.

Theorem 3.2.9 (First Nonzero Derivative)

If f has stationary order n, then:

- If n is *even* $\implies f$ has a local extremum at c.

  Furthermore: $f^{(n)}(c) > 0 \implies$ local minimum, $f^{(n)}(c) < 0 \implies$ local maximum.
- If n is *odd* $\implies c$ is a stationary inflection point.

Proof. The Taylor series with remainder simplifies to

$$f(c+h) = f(c) + \frac{f^{(n)}(c)}{n!}h^n + O(h^{n+1}).$$

Its change close to c is thus

$$f(c+h) - f(c) \approx \frac{f^{(n)}(c)}{n!}h^n,$$

which changes sign if and only if n is odd. Similarly,

$$f'(c+h) - f'(c) \approx \frac{f^{(n-1)}(c)}{(n-1)!}h^{n-1}$$

for the first derivative and

$$f''(c+h) - f''(c) \approx \frac{f^{(n-2)}(c)}{(n-2)!}h^{n-2}$$

for the second derivative. □

Corollary 3.2.10 (Second Derivative Test)

If f'' is continuous at c and $f'(c) = 0$, then:

- $f''(c) > 0 \implies$ local minimum.
- $f''(c) < 0 \implies$ local maximum.
- $f''(c) = 0$ and $f^{(3)}(c) \neq 0 \implies$ stationary inflection point.

Note: $f''(c) = 0$ alone is insufficient for an inflection; the curvature must change sign.

Examples.

- $f(x) = x^3$: $f'(0) = f''(0) = 0$, $f^{(3)}(0) = 6 \neq 0$ (odd $n = 3$) $\implies$ stationary inflection at 0.
- $f(x) = x^4$: $f'(0) = f''(0) = f^{(3)}(0) = 0$, $f^{(4)}(0) = 24 > 0$ (even $n = 4$) $\implies$ local minimum at 0, no inflection.
- $f(x) = -x^4$: local maximum at 0, no inflection.

3.3. Ordinary Differential Equations

3.4. The Riemann Integral

3.4.1. Definition

Definition 3.4.1 (Partition)

A *partition* of $[a, b]$ is a finite set

$$P = \{x_0, x_1, \cdots, x_n\}$$

such that

$$a = x_0 < x_1 < \cdots < x_n = b,$$

The partition P has *subintervals*

$$[x_{i-1}, x_i] \quad i = 1, 2, \ldots, n$$

of which the length of the largest is its *mesh* or *norm*

$$\|P\| = \max_{1\leq i\leq n} (x_i - x_{i-1}).$$

A smaller such is indicative of a finer partition.

Let $f : [a, b] \to \mathbb{R}$ be bounded. We now define its definite integral.

Definition 3.4.2 (Darboux Integral)

Define the *lower sum*

$$L(f, P) = \sum_{i=1}^{n} (\inf\{f(x) : x \in [x_{i-1}, x_i]\})(x_i - x_{i-1}).$$

and the *upper sum*

$$U(f, P) = \sum_{i=1}^{n} (\sup\{f(x) : x \in [x_{i-1}, x_i]\})(x_i - x_{i-1})$$

The function f is *Darboux integrable* if $\sup_P L(f, P) = \inf_P U(f, P)$. The common value is denoted as the *definite integral* $\int_a^b f(x)\,\mathrm{d}x$.

Definition 3.4.3 (Alternative Darboux Integral)

Let Φ and Ψ be the *lower and upper step functions* such that

$$\Phi(x) \leq f(x) \leq \Psi(x) \quad \forall x \in [a, b],$$

forming the *lower integral*

$$L(f) = \sup\left\{\int_a^b \Phi(x)\,\mathrm{d}x : \Phi \text{ is a lower step function to } f\right\}$$

and the *upper integral*

$$U(f) = \inf\left\{\int_a^b \Psi(x)\,\mathrm{d}x : \Psi \text{ is an upper step function to } f\right\}$$

which, if equal, give the definite integral.

Note that the integral of a step function is simply its signed area.

Definition 3.4.4 (Riemann Integral)

From a partition P of $[a, b]$ pick *sample points*

$$t_i \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n$$

and form the (tagged) *Riemann sum*

$$S(f, P, (t_i)) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

We say f is *Riemann integrable* if there exists $L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0\ \exists \delta > 0 : \|P\| < \delta \implies |S(f, P, (t_i)) - L| < \varepsilon$$

for every choice of sample points (t_i). In that case we write

$$L = \int_a^b f(x)\,\mathrm{d}x.$$

Theorem 3.4.5

The Darboux and Riemann integrals are equivalent.

3.4.2. Integrability

Theorem 3.4.6 (Integrability)

Let $f : [a, b] \to \mathbb{R}$ be bounded.

The function is integrable if and only if:

(i) $\forall \varepsilon > 0\ \exists P : U(f, P) - L(f, P) < \varepsilon$.
(ii) $\forall (P_n) : \|P_n\| \to 0 \implies U(f, P_n) - L(f, P_n) \to 0$.
(iii) (Lebesgue Criterion for Riemann Integrability) Its set of discontinuities has Lebesgue measure zero.
(iv) $$\forall \varepsilon > 0\ \exists \Phi, \Psi : \int_a^b \Psi(x)\,\mathrm{d}x - \int_a^b \Phi(x)\,\mathrm{d}x < \varepsilon,$$ where Φ and Ψ are lower and upper step functions.

The function is integrable if:

(iii) f is *monotone* on $[a, b]$
(iv) f is continuous except at finitely many points, or at countably many points where it has only removable or jump discontinuities.

Theorem 3.4.7

Assume f is continuous on $[a, b]$. Let

$$M_i = \max_{x\in[x_{i-1}, x_i]} f(x) \ \text{ and } \ m_i = \min_{x\in[x_{i-1}, x_i]} f(x).$$

Then,

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \int_a^b f(x)\,\mathrm{d}x.$$

Theorem 3.4.8 (Absolute Value / Triangle)

If f integrable, then $|f|$ integrable and

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Theorem 3.4.9 (Products and Composition)

If f, g integrable, then fg is integrable.

If f integrable and φ continuous on a set containing $f([a, b])$, then $\varphi \circ f$ is integrable.

Theorem 3.4.10 (Uniform Limit)

If (f_n) are integrable on $[a, b]$ and $f_n \to f$ uniformly, then f is integrable and

$$\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x.$$

3.4.3. Properties

Theorem 3.4.11 (Linearity)

If f, g are integrable and $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b (\alpha f(x) + \beta g(x))\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x.$$

Theorem 3.4.12 (Additivity of the Interval)

If $c \in (a, b)$ and f integrable on $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

It follows that $\int_a^a f(x)\,\mathrm{d}x = 0$ and $\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$.

Theorem 3.4.13 (Order / Comparison)

If f, g integrable and $f(x) \leq g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

Corollary 3.4.14 (Positivity)

If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,\mathrm{d}x \geq 0$. Moreover, if f is continuous and the integral is 0, then $f \equiv 0$.

Theorem 3.4.15 (Bounding by a Supremum)

If $|f(x)| \leq M$ on $[a, b]$, then

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq M(b - a).$$

Theorem 3.4.16 (Mean Value for Integrals)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,\mathrm{d}x = f(\xi)(b - a).$$

for some $\xi \in [a, b]$ or, to be more strict if f is not constant, $\xi \in (a, b)$.

Theorem 3.4.17 (Generalized Mean Value for Integrals)

If f is continuous and g is integrable and does not change sign on $[a, b]$,

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(\xi) \int_a^b g(x)\,\mathrm{d}x$$

for some $\xi \in [a, b]$ or, to be more strict if f is not constant, $\xi \in (a, b)$.

Proof. Let $m = \min f(x)$ and $M = \max f(x)$ for $x \in [a, b]$. Then,

$$m \int_a^b g(x) \leq \int_a^b f(x)g(x) \leq M \int_a^b g(x)$$

by Theorem 3.4.13, or rewritten,

$$m \leq \frac{1}{\int_a^b g(x)} \int_a^b f(x)g(x) \leq M.$$

Since $m \leq f(x) \leq M$, Theorem 2.3.9 gives that

$$f(\xi) = \frac{1}{\int_a^b g(x)} \int_a^b f(x)g(x)$$

for some $\xi \in [a, b]$. Rewritten, this is the theorem. □

Theorem 3.4.18 (Fundamental Theorems of Calculus)

If f is continuous on $[a, b]$, then the two theorems follow:

(i) Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$ for $x \in [a, b]$. Then, F is continuous on $[a, b]$, differentiable on (a, b), and $F'(x) = f(x)$.

Proof. We want to show that $F'(x) = f(x)$.

Applying the definition of derivatives,

$$F'(x) = \lim_{h\to 0} \frac{1}{h}(F(x+h) - F(x)) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(x)\,\mathrm{d}x,$$

where x and $x + h$ are in (a, b). By Theorem 3.4.16,

$$\int_x^{x+h} f(t)\,\mathrm{d}t = f(\xi)h$$

for some ξ between x and $x + h$, which in our previous result gives

$$F'(x) = \lim_{h\to 0} f(\xi) = f(x)$$

since f is continuous. □

(ii) If $F'(x) = f(x)$ for $x \in (a, b)$, then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Proof. Let $G(x)$ have $G'(x) = f(x) = F'(x)$ for all $x \in (a, b)$. Then, $G'(x) - F'(x) = 0$ gives that $G(x) - F(x) = C$ for some constant. We have $G(a) - F(a) = C$, but

$$G(a) = \int_a^a f(t)\,\mathrm{d}t = 0,$$

so $C = -F(a)$ and hence $G(b) = F(b) - F(a)$, but by definition

$$G(b) = \int_a^b f(t)\,\mathrm{d}t,$$

so the statement holds. □

3.4.4. Integration Techniques

Theorem 3.4.19 (Integration by Substitution)

Also known as *change of variables* or *u-substitution*.

Let g be injective and continuously differentiable on $[a, b]$ and let f be continuous on $g([a, b])$. Then, with $u = g(x)$ and $\mathrm{d}u = g'(x)\,\mathrm{d}x$,

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u.$$

Equivalently, if g is strictly monotonic and thus invertible as $x = g^{-1}(u)$,

$$\int_a^b f(x)\,\mathrm{d}x = \int_{g^{-1}(a)}^{g^{-1}(b)} f'(g(u))g'(u)\,\mathrm{d}u.$$

Proof. We prove the first formulation of the theorem. We have,

$$\int_a^b f(g(x))g'(x) = [f(g(x))]_a^b = [f(u)]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

according to Theorem 3.4.18 (ii) and Theorem 3.1.2. □

Theorem 3.4.20 (Integration by Parts)

If f, g are continuously differentiable on $[a, b]$, then

$$\int_a^b f(x)g(x)\,\mathrm{d}x = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)\,\mathrm{d}x.$$

LIATE

The LIATE rule helps choose $f(x)$ and $g(x)$ for integration by parts:

- Logarithmic: $\ln(x)$, $\log_{a(x)}$
- Inverse trigonometric: $\arctan(x)$, $\arcsin(x)$, $\arccos(x)$
- Algebraic: x, x^2, x^3, etc.
- Trigonometric: $\sin(x)$, $\cos(x)$, $\tan(x)$, etc.
- Exponential: e^x, a^x

Choose $g(x)$ as the function that appears first in this list.

Arctangent Rules

(i) Addition: ($ab < 1$, otherwise add or subtract $\pi/2$)

$$\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$$

(ii) Subtraction: ($ab > 1$, otherwise add or subtract $\pi/2$)

$$\arctan(a) - \arctan(b) = \arctan\left(\frac{a-b}{1+ab}\right)$$

(iii) Inverse:

$$\arctan(x) = -\arctan(-x)$$

(iv) Integration:

$$\begin{aligned}
\int \frac{a}{b^2 + c^2x^2}\,\mathrm{d}x &= \frac{a}{b^2} \int \frac{1}{1 + c^2x^2/b^2} \\
&= \left\{\begin{matrix} u = cx/b \\ \mathrm{d}u = c/b \end{matrix}\right\} \\
&= \frac{a}{bc} \int \frac{1}{1+u^2}\,\mathrm{d}u \\
&= \frac{a}{bc} \arctan\left(\frac{cx}{b}\right)
\end{aligned}$$

4. Infinite Series

4.1. Series

Definition 4.1.1 (Infinite Series)

Let $(a_j)_{j=0}^{\infty}$ and let $(s_n)_{n=0}^{\infty}$. The sum of the infinite series is defined as

$$\sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{j=0}^{n} a_j.$$

If $a_j \geq 0$ for every j we say that the series is *positive*.

⚠ Warning

Beware of treating infinite series like elementary algebra, e.g., by rearranging terms.

Theorem 4.1.2 (Geometric Series)

If $|x| < 1$, then

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

since

$$s_n = \sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x}.$$

4.1.1. Convergence

Theorem 4.1.3 (Cauchy Criterion for Series)

The series $\sum_{k=0}^{\infty} a_k$ converges if and only if

$$\forall \varepsilon > 0 \ \exists N : n > m > N \implies \left|a_m + a_{m+1} + \cdots + a_{n-1} + a_n\right| < \varepsilon.$$

Corollary 4.1.4 (Series Term Test)

If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$. However, the reverse is not implied.

Lemma 4.1.5

The series $\sum_{j=1}^{\infty} 1/j$ is divergent.

Theorem 4.1.6 (Inverse Power Series)

The series $\sum_{j=1}^{\infty} 1/j^p$ converges if and only if $p > 1$.

Theorem 4.1.7 (Ratio Test)

Let (a_n) be a sequence of positive terms and define

$$L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then:
(i) If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.
(ii) If $L > 1$ (including $L = \infty$), the series diverges.
(iii) If $L = 1$, the test is inconclusive.

Theorem 4.1.8 (Cauchy Condensation Test)

Let (a_n) be a decreasing sequence of nonnegative real numbers. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Theorem 4.1.9 (Constant Ratio Test)

Let $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ be positive series with terms such that

$$\lim_{j\to\infty} \frac{a_j}{b_j} = K$$

for some $K \neq 0$. Then, $\sum_{j=0}^{\infty} a_j$ converges if and only if $\sum_{j=0}^{\infty} b_j$ converges.

Theorem 4.1.10 (Comparison Test)

Let (a_k) and (b_k) satisfy $0 \leq a_k \leq b_k$. Then,
(i) $\sum_{k=1}^{\infty}(a_k)$ converges if $\sum_{k=1}^{\infty}(b_k)$ converges.
(ii) $\sum_{k=1}^{\infty}(b_k)$ diverges if $\sum_{k=1}^{\infty}(a_k)$ diverges.

Theorem 4.1.11 (Alternating Series Test)

Let (a_n) satisfy
(i) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and
(ii) $(a_n) \to 0$.

Then, $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Definition 4.1.12 (Absolutely Convergent)

A series $\sum_{j=0}^{\infty} a_j$ is *absolutely convergent* if $\sum_{j=0}^{\infty} |a_j|$ is convergent.

Theorem 4.1.13

If a series is absolutely convergent then it is convergent.

4.2. Indefinite Integrals

4.2.1. Unlimited Intervals

Definition 4.2.1

Let f be integrable on $[a, R]$ for all $R > a$. Then the integral is defined

$$\int_a^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_a^R f(x)\,\mathrm{d}x.$$

If this limit exists, then the integral is said to be convergent.

Definition 4.2.2

Let f be integrable on every closed and bounded interval. If *both*

$$\int_{-\infty}^{a} f(x)\,\mathrm{d}x \quad\text{and}\quad \int_{-\infty}^{a} f(x)\,\mathrm{d}x$$

are convergent, then for any real a we define the convergent integral

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_{-\infty}^{a} f(x).$$

Theorem 4.2.3 (Properties)

Let f and g be integrable on $[a, R]$. The following applies.

Theorem 4.1.6 (Inverse Power Series)
The series $\sum_{j=1}^{\infty} 1/j^p$ converges if and only if $p > 1$.

Theorem 4.1.9 (Constant Ratio Test)
Let $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ be positive series with terms such that

$$\lim_{j\to\infty} \frac{a_j}{b_j} = K$$

for some $K \neq 0$. Then, $\sum_{j=0}^{\infty} a_j$ converges if and only if $\sum_{j=0}^{\infty} b_j$ converges.

Theorem 4.1.10 (Comparison Test)
Let (a_k) and (b_k) satisfy $0 \leq a_k \leq b_k$. Then,
(i) $\sum_{k=1}^{\infty}(a_k)$ converges if $\sum_{k=1}^{\infty}(b_k)$ converges.
(ii) $\sum_{k=1}^{\infty}(b_k)$ diverges if $\sum_{k=1}^{\infty}(a_k)$ diverges.

Theorem 4.1.13
If a series is absolutely convergent then it is convergent.

⚠ Warning

The following does not apply.

Corollary 4.1.4 (Series Term Test)
If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$. However, the reverse is not implied.

Theorem 4.1.7 (Ratio Test)
Let (a_n) be a sequence of positive terms and define

$$L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Then:
(i) If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges.
(ii) If $L > 1$ (including $L = \infty$), the series diverges.
(iii) If $L = 1$, the test is inconclusive.

Theorem 4.1.11 (Alternating Series Test)
Let (a_n) satisfy
(i) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and
(ii) $(a_n) \to 0$.

Then, $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Theorem 4.2.4

Let f be decreasing on $[m, n]$, where $m < n$ are integers. Then,

$$\sum_{j=m+1}^{n} f(j) \leq \int_m^n f(x)\,\mathrm{d}x \leq \sum_{j=m}^{n-1} f(j)$$

and

$$f(n) + \int_m^n f(x)\,\mathrm{d}x \leq \sum_{j=m}^{n} f(j) \leq f(m) + \int_m^n f(x)\,\mathrm{d}x.$$

Let f instead be increasing. Then,

$$\sum_{j=m}^{n-1} f(j) \leq \int_m^n f(x)\,\mathrm{d}x \leq \sum_{j=m+1}^{n} f(j)$$

and

$$f(m) + \int_m^n f(x)\,\mathrm{d}x \leq \sum_{j=m}^{n} f(j) \leq f(n) + \int_m^n f(x)\,\mathrm{d}x.$$

4.2.2. Open Intervals

4.3. Taylor's Theorem

4.3.1. Statement

Theorem 4.3.1 (Taylor's)

Suppose f is continuously differentiable n times on $[a, b]$ and $n + 1$ times on (a, b). Fix $c \in [a, b]$. Then,

$$f(x) = P_n(x) + R_n(x),$$

where the *Taylor polynomial* of degree n around c is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

and the *Lagrange remainder* of degree n around c is

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

for some ξ strictly between c and x.

Note that other remainder forms exist.

Proof. Let $h = x - c$ be the deviation from the point. Then,

$$f(x) = f(c+h) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1} = p_n(h) + r_n(h),$$

where $p_n(h)$ and $r_n(h)$ correspond to $P_n(x)$ and $R_n(x)$.

Define

$$F_{n,h}(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(c+h-t)^k,$$

with $F_{n,h}(c) = p_{n(h)}$ and $F_{n,h}(c+h) = f(c+h)$, and derivative

$$F'_{n,h}(\xi) = \frac{f^{(n+1)}(\xi)}{n!}(c+h-\xi)^n.$$

Also let

$$g_{n,h}(t) = (c+h-t)^{n+1},$$

with $g_{n,h}(c) = h^{n+1}$ and $g_{n,h}(c+h) = 0$ and

$$g'_{n,h}(\xi) = -(n+1)(c+h-\xi)^n.$$

Theorem 3.1.10 gives

$$\frac{F_{n,h}(c+h) - F_{n,h}(c)}{g_{n,h}(c+h) - g_{n,h}(c)} = \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$$

for some ξ between c and $c + h$. Substituting,

$$\frac{f(c+h) - p_n(h)}{0 - h^{n+1}} = \frac{f^{(n+1)}(\xi)(c+h-\xi)^n/n!}{-(n+1)(c+h-\xi)^n}$$

so

$$f(c+h) - p_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}.$$

Hence

$$f(c+h) = p_n(h) + r_n(h)$$

or in x-notation

$$f(x) = P_n(x) + R_n(x)$$

with ξ strictly between c and x. □

Proof using integrals. From Theorem 3.4.18 (ii) we have

$$\int_c^x f'(t)\,\mathrm{d}t = f(t) - f(c)$$

which we expand using Theorem 3.4.20 as

$$\begin{aligned}
f(x) &= f(c) + \int_c^x 1 \cdot f'(t)\,\mathrm{d}t \\
&= f(c) + [(t-x)f'(t)]_c^x - \int_c^x (t-x)f''(x)\,\mathrm{d}t \\
&= f(c) + f'(c)(x-c) - \left(\left[\frac{(t-x)^2}{2}f''(t)\right]_c^x - \int_c^x \frac{(t-x)^2}{2}f^{(3)}(t)\,\mathrm{d}t\right) \\
&= f(c) + f'(c)(x-c) + \frac{f''(t)}{2}(x-c)^2 + \int_c^x \frac{(t-x)^2}{2}f^{(3)}(t)\,\mathrm{d}t \\
&= \cdots \\
&= P_n(x) + (-1)^n \int_c^x \frac{(t-x)^n}{n!}f^{(n+1)}(t)\,\mathrm{d}t
\end{aligned}$$

□

Definition 4.3.2 (Radius of Convergence)

Let $R_n(x)$ be the remainder to the Taylor polynomial around a point c. The *radius of convergence* R is the supremum of $r \geq 0$ such that

$$\forall x : |x-c| < r \implies \lim_{n\to\infty} R_n(x) = 0,$$

which implies that the Taylor series converges to $f(x)$ for all such x (so $f(x) = P_\infty(x)$).

Theorem 4.3.3 (Common Maclaurin Series)

The following functions have a Maclaurin series with radius of convergence $r = \infty$:

$$\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
\sin x &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{\{2k+1\}}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos x &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{\{2k\}}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
\arctan x &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{\{2k+1\}}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (|x| \leq 1) \\
\ln(1+x) &= \sum_{k=1}^{\infty} (-1)^{\{k+1\}} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (|x| < 1) \\
(1+x)^a &= \sum_{k=0}^{\infty} \binom{a}{k} x^k \quad (|x| < 1)
\end{aligned}$$

4.3.2. Function Order

Definition 4.3.4 (Big O at Infinity)

Let f and g be defined on (c, ∞). We say that f belongs to the set O of g as $x \to \infty$, writing $O(g(x))$, if there exists M and x_0 such that

$$|f(x)| \leq M|g(x)|,$$

for every $x > x_0$.

Definition 4.3.5 (Big O at a Point)

Let f and g be defined on a neighborhood of c. We say that f belongs to the set O of g around c, writing $O(g(x))$, if there exists M and $\delta > 0$ such that

$$|f(x)| \leq M|g(x)|$$

for every $x \in (c - \delta, c + \delta)$.

Theorem 4.3.6 (Big O Behavior)

If $h(x) = O(f(x))$ and $k(x) = O(g(x))$ (same limiting regime), then $h(x)k(x) = O(f(x)g(x))$.

If $m \leq n$ then as $x \to 0$, $x^n = O(x^m)$ so $O(x^m) + O(x^n) = O(x^m)$. As $x \to \infty$, $x^m = O(x^n)$ so $O(x^m) + O(x^n) = O(x^n)$.

Theorem 4.3.7

Let $f(x) : [a, b] \to \mathbb{R}$ and fix $c \in [a, b]$. Suppose f is continuously differentiable n times on $[a, b]$ and $n + 1$ times on (a, b). Then,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + O(|x-c|^{n+1}) \text{ as } x \to c.$$

Furthermore, the coefficients $f^{(k)}(c)/k!$ are unique to each $(x-c)^k$.