Analys i en variabel

SF1673 (HT25)

Theorem 1 (Induction)

If $s \in \mathbb{N}$ such that
(i) $1 \in S$ and
(ii) when $n \in S$ it follows that $n+1 \in S$

it follows that $S = \mathbb{N}$.

Definition 2 (Injective/Surjective/Bijective)

$f : X \to Y$ is *injective* (or one-to-one) if $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ or equivalently if $f(x_1) = f(x_2) \implies x_1 = x_2$.

f is *surjective* if $\forall y \; \exists x : f(x) = y$.

f is *bijective* if is both injective and surjective or equivalently if each y is mapped to exactly one x.

1. The Real Numbers

1.1. Reals

1.1.1. Comparison

Definition 1.1.1 (Equality)

$a = b \iff (\forall \varepsilon > 0 \Rightarrow |a - b| < \varepsilon)$

Theorem 1.1.2 (Triangle Inequalities)

(i) $|a+b| \leq |a| + |b|$
(ii) $|a-b| \leq |a-c| + |c-b|$
(iii) $|a-b| \geq \||a| - |b|\|$

The reverse triangle inequality (iii) is seldom used.

1.1.2. Bounds

Axiom 1.1.3 (Supremum Property or Axiom of Completeness)

Every bounded, non-empty set of real numbers has a least upper bound.

Note

The same does not apply for the rationals.

Definition 1.1.4 (Least Upper Bound)

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then,

$$s = \sup A \quad \iff \quad \forall \varepsilon > 0 \; \exists a \in A : s - \varepsilon < a.$$

1.2. Cardinality

Definition 1.2.1 (Cardinality)

A has the same *cardinality* as B if there exists a bijective $f : A \to B$.

Definition 1.2.2 (Countable/Uncountable)

A is *countable* if $\mathbb{N} \sim A$. Otherwise, A is *uncountable* if there are infinite elements or *finite* if there are finite elements.

Theorem 1.2.3 (Countability of $\mathbb{Q}$, $\mathbb{R}$)

$\mathbb{Q}$ is countable.

Proof. Let $A_1 = \{0\}$ and let

$$A_n = \{\pm p/q : p, q \in \mathbb{N}_+, \gcd(p,q) = 1, p + q = n\}$$

for all $n \geq 2$. Each A_n is finite and every rational numbers appears in exactly one set. □

$\mathbb{R}$ is uncountable.

Proof. Cantor's diagonalization method. □

$\mathbb{I}$ is uncountable.

Proof. $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ where $\mathbb{Q}$ is countable. □

Theorem 1.2.4 (Density of $\mathbb{Q}$ in $\mathbb{R}$)

(i) $\forall a < b \in \mathbb{R} \; \exists r \in \mathbb{Q} : a < r < b$
(ii) $\forall y \in \mathbb{R} \; \exists (r_n) \in \mathbb{Q} : (r_n) \to y$

1.3. Topology

1.3.1. Points

Definition 1.3.1 (Limit Point)

x is a *limit point* of A if every $V_\varepsilon(x)$ intersects A at some point other than x.

Theorem 1.3.2 (Sequential Limit Point)

x is a limit point of A if $x = \lim a_n$ for some $(a_n) \subseteq A : a_n \neq x \; \forall n \in \mathbb{N}$.

Theorem 1.3.3 (Nested Interval Property)

The intervals $\mathbb{R} \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ all contain a point $a = \bigcap_{n=1}^{\infty} I_n$.

1.3.2. Opened and Closed Sets

Definition 1.3.4 (Open/Closed Set)

$A \subseteq \mathbb{R}$ is *open* if $\forall a \in A \; \exists V_\varepsilon(a) \subseteq A$ or equivalently if its complement is closed.

$A \subseteq \mathbb{R}$ is *closed* if it contains its limit points or equivalently if its complement is open.

Theorem 1.3.5 (Clopen Sets)

$\mathbb{R}$ and $\emptyset$ are *clopen* (both opened and closed).

Theorem 1.3.6 (Unions/Intersections)

The union of open (closed) sets is open (closed).

The intersection of finitely many open (closed) sets is open (closed).

1.3.3. Compactness

Definition 1.3.7 (Compact)

A set K in a topological space is *compact* if every open cover has a finite subcover.

Theorem 1.3.8 (Heine–Borel)

A set $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Note

Compactness is like a generalization of closed intervals.

1.4. Sequences

Definition 1.4.1 (Sequence)

A *sequence* is a function whose domain is $\mathbb{N}$.

Definition 1.4.2 (Convergence)

A sequence *converges* to a if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} : n \geq N \implies |a_n - a| < \varepsilon$$

or equivalently if for any $V_\varepsilon(a)$ there exists a point in the sequence after which all terms are in $V_\varepsilon(a)$. In other words if every ε-neighborhood contains all but a finite number of the terms in (a_n).

We write this $\lim_{n\to\infty} a_n = \lim a_n = a$ or $a_n \to a$.

Example. Template of a typical convergence proof:
(i) Let $\varepsilon > 0$ be arbitrary.
(ii) Propose an $N \in \mathbb{N}$ (found before writing the proof).
(iii) Assume $n \geq N$.
(iv) Show that $|a_n - a| < \varepsilon$.

Theorem 1.4.3 (Uniqueness of Limits)

The limit of a sequence, if it exists, is unique.

1.4.1. Bounded

Definition 1.4.4 (Bounded)

A sequence is *bounded* if $\exists M > 0 : |a_n| < M \;\; \forall n \in \mathbb{N}$.

Theorem 1.4.5 (Convergent/Monotone)

Every convergent series is bounded.

If a sequence is monotone and bounded it converges.

Subsequences of a convergent series converge to the same limit.

Theorem 1.4.6 (Bolzano–Weierstrass)

Every bounded sequence contains a convergent subsequence.

1.4.2. Cauchy

Definition 1.4.7 (Cauchy Sequence)

A sequence (a_n) is a *Cauchy sequence* if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} : m, n \geq N \implies |a_n - a_m| < \varepsilon.$$

Theorem 1.4.8 (Cauchy Criterion)

A sequence converges if and only if it is a Cauchy sequence.

1.5. Series

Definition 1.5.1 (Infinite Series)

Let $(a_j)_{j=0}^{\infty}$ and let $(s_n)_{n=0}^{\infty}$. The sum of the infinite series is defined as

$$\sum_{j=0}^{\infty} a_j = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{j=0}^{n} a_j.$$

Caution

Beware of treating infinite series like elementary algebra, e.g., by rearranging terms.

Theorem 1.5.2 (Series Term Test)

If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$.

Theorem 1.5.3 (Cauchy Condensation Test)

Let (a_n) be a decreasing sequence of non-negative real numbers. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Theorem 1.5.4 (Comparison Test)

Let (a_k) and (b_k) satisfy $0 \leq a_k \leq b_k$. Then,
(i) $\sum_{k=1}^{\infty}(a_k)$ converges if $\sum_{k=1}^{\infty}(b_k)$ converges.
(ii) $\sum_{k=1}^{\infty}(b_k)$ diverges if $\sum_{k=1}^{\infty}(a_k)$ diverges.

Theorem 1.5.5 (Alternating Series Test)

Let (a_n) satisfy
(i) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and
(ii) $(a_n) \to 0$.

Then, $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

2. Real functions

2.1. Limits

Theorem 2.1.1 (Function Limit)

Given $f : A \to \mathbb{R}$ with the limit point c,
(i) $\lim_{x\to c} f(x) = L$ is equivalent to
(ii) if $\forall (x_n) \subseteq A : (x_n \neq c$ and $x_n \to c)$ it follows that $f(x_n) \to L$.

Note

In the $\varepsilon\delta$-definition of limits, the additional restriction that $0 < |x - a|$ is just a way to say $x \neq c$.

Definition 2.1.2 (Infinite Limit)

Given a limit point $c \in D_f$, we say that $\lim_{x\to c} f(x) = \infty$ if

$$\forall M \; \exists \delta > 0 : 0 < |x - c| < \delta \implies f(x) \geq M.$$

2.2. Continuity

Theorem 2.2.1 (Continuity)

The following are equivalent:
(i) $f : A \to \mathbb{R}$ is *continuous* at $c \in \mathbb{R}$.
(ii) $\forall \varepsilon > 0 \; \exists \delta > 0 : |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$, where $x \in A$.
(iii) $\forall V_\varepsilon(f(c)) \; \exists V_\delta(c) : x \in V_\delta \cap A \implies f(x) \in V_\varepsilon$
(iv) $x_n \to c$, where $(x_n) \subseteq A$, implies $f(x_n) \to f(c)$.

If c is a limit point of A:
(v) $\lim_{x\to c} f(x) = f(c)$, also written $\lim_{h\to 0} f(c+h) - f(c) = 0$.

Note that (ii) defines (i). Mostly (v) is used in practice.

Theorem 2.2.2 (Isolated Continuity)

All functions are continuous at isolated points.

Theorem 2.2.3 (Dirichlet Discontinuous)

The Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \in \mathbb{I}$ is discontinuous everywhere.

2.2.1. Composition

Theorem 2.2.4 (Composition)

Given $f : A \to B$ and $g : B \to \mathbb{R}$ with $f(A) \subseteq B$, if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Theorem 2.2.5 (Composition Limit)

If f is continuous at y and $\lim_{x\to c} g(x) = y$, then

$$\lim_{x\to c} f(g(x)) = f\Big(\lim_{x\to c} g(x)\Big) = f(y).$$

2.2.2. Results

Theorem 2.2.6 (Intermediate Value)

If f is continuous on $[a,b]$, then for any y between $f(a)$ and $f(b)$, there exists some $c \in (a,b)$ such that $f(c) = y$.

Theorem 2.2.7 (Weierstrass Extreme Value)

If f is continuous on the compact set K, then f attains a maximum and a minimum value on K.

2.3. Derivatives

2.3.1. Differentiation

Theorem 2.3.1 (Chain Rule)

Let $f : X \to Y$ and $g : Y \to \mathbb{R}$. If f is differentiable at $c \in X$ and g is differentiable at $f(c) \in Y$, then $g \circ f$ is differentable at c with

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Theorem 2.3.2 (Basic Derivatives)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \frac{1}{\cos^2 x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arccot} x) = -\frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\ln|x|) = \frac{1}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^a) = ax^{a-1} \quad (a \neq 0) \qquad (f^{-1})'(y) = -\frac{1}{f'(x)} \quad (f'(x) \neq 0)$$

Theorem 2.3.3 (L'Hôpital's Rule)

Let $f(x)$ and $g(x)$ be defined and, with the possible exception of at the limit point c, differentiable. If
(i) $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$ and
(ii) $g'(x) \neq 0$ for all $x \neq c$, then

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \implies \lim_{x\to c} \frac{f(x)}{g(x)} = L.$$

Proof of the zero case. Assume the limits are zero.

Let the functions be differentiable on the open interval (c, x). Then, rewriting and applying Theorem 2.3.9 gives

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x\to c} \frac{f'(p)}{g'(p)} = \lim_{p\to c} \frac{f'(p)}{g'(p)}$$

for some p between c and x. □

Proof of the infinity case. The proof is too complicated. □

Important

This is only an implication, not an equivalence, so there may exist some other solution if this method fails.

2.3.2. Function Character

Theorem 2.3.4 (Fermat's or Interior Extremum)

Let $f : (a,b) \to \mathbb{R}$ be differentiable at the local extremum $c \in (a,b)$. Then $f'(x) = 0$.

However, note that a zero-derivative point may also be a stationary point of inflection.

Theorem 2.3.5 (Darboux's)

If f is differentiable on $[a,b]$ and if y lies strictly between $f'(a)$ and $f'(b)$, then $\exists c \in (a,b) : f'(c) = y$.

In other words, if f is differentiable on an interval, then f' satisfies the Intermediate Value Property (IVP).

Proof. Assume that $f'(a) < y < f'(b)$.

Let $g(x) = f(x) - yx$ with $g'(x) = f'(x) - y$. Note that $f'(c) = y$ if $g'(c) = 0$ for some $c \in (a,b)$.

Theorem 2.2.7 states that g must have a minimum point $c \in [a,b]$. More precisely $c \in (a,b)$ since, from the assumption, $g'(a) < 0$ and $g'(b) > 0$. Furthermore, $g'(c) = 0$ according to Theorem 2.3.4. □

Theorem 2.3.6 (Newton's Method)

Find roots to a differentiable function $f(x)$.

Given x_n with the coordinates $(x_n, f(x_n))$, the tangent line is given by

$$T(x) = f'(x_n)(x - x_n) + f(x_n)$$

and intersects the x-axis at

$$T(x_{n+1}) = 0 \iff x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The method fails if it iterates endlessly or $f'(x_n) = 0$.

2.3.3. The Mean Value Theorems

Let f and g be continuous on $[a,b]$ and differentiable on (a,b).

Theorem 2.3.7 (Rolle's)

$$f(a) = f(b) \implies \exists c \in (a,b) : f'(c) = 0$$

Proof. $f(x)$ is bounded and $f'(x) = 0$ at its extreme points. □

Theorem 2.3.8 (Mean Value)

$$\exists c \in (a,b) : f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Let the signed distance d between the function value f and the secant y through a and b be

$$d(x) = f(x) - y(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

and note that $d(a) = d(b) = 0$. Then apply Theorem 2.3.7. □

Theorem 2.3.9 (Generalized Mean Value)

$$\exists c \in (a,b) : [f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If g' is never zero on (a,b), then the above can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Let $h = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$ and then apply Theorem 2.3.7. □

2.4. Function Graphs

Tip (Sketching Graphs)

— Information
(i) split into cases
(ii) symmetries
(iii) domain → vertical asymptotes
(iv) factorize → oblique asymptotes & roots
(v) first and second derivative and their roots
(vi) sign tables
(vii) calculate interesting points: intersection with y-axis, defined non-differentiable points, local extremums, endpoints, inflection points

— Sketching
(i) axes
(ii) symmetries
(iii) asymptotes
(iv) interesting points
(v) curves

2.4.1. Asymptotes

Definition 2.4.1 (Asymptote)

The line $y = kx + m$ is an *oblique* asymptote of f if

$$\lim_{x\to\infty} (f(x) - (kx + m)) = 0.$$

The line $x = c$ is a *vertical* asymptote of f if

$$\lim_{x\to c+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to c-} f(x) = \pm\infty.$$

The line $y = b$ is a *horizontal* asymptote of f if

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

Theorem 2.4.2 (Oblique Asymptote)

If $f(x)$ has an oblique asymptote $y = kx + m$, then

$$k = \lim_{x\to\infty} \frac{f(x)}{x}$$

and

$$m = \lim_{x\to\infty} (f(x) - kx).$$

2.4.2. Convexity

Theorem 2.4.3 (Convexity)

Let f be twice differentiable on (a,b). Then, $f''(x) \geq 0$ if and only if f is convex on (a,b).

Definition 2.4.4 (Concave)

On $[a,b]$, a function $f : [a,b] \to \mathbb{R}$ is *concave* if $-f$ is convex.

2.5. Taylor's Theorem

Theorem 2.5.1 (Taylor's)

Let $f(x)$ be $n+1$ times differentiable at a. Then,

$$f(x) = P_n(x) + R_n(x),$$

where the *Taylor polynomial* of degree n is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$$

and the *Lagrange remainder* of degree n is

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - a)^{n+1}$$

for some ξ between a and x.

Proof. Let $h = x - a$ be the deviation from the point. Then,

$$f(x) = f(a+h) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1} = p_n(h) + r_n(h),$$

where $p_n(h)$ and $r_n(h)$ are like $P_n(x)$ and $R_n(x)$.

We introduce a helper function

$$F_{n,h}(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(a + h - t)^k,$$

with $F_{n,h}(a) = p_{n(h)}$ and $F_{n,h}(a+h) = f(a+h)$, and the derivative

$$F'_{n,h}(\xi) = \frac{f^{(n+1)}(\xi)}{n!}(a + h - \xi)^n.$$

We introduce another helper function

$$g_{n,h}(t) = (a + h - t)^{n+1},$$

with $g_{n,h}(a) = h^{n+1}$ and $g_{n,h}(a+h) = 0$ as well as

$$g'_{n,h}(\xi) = -(n+1)(a + h - \xi)^n.$$

Theorem 2.3.9 states that

$$\frac{F_{n,h}(a+h) - F_{n,h}(a)}{g_{n,h}(a+h) - g_{n,h}(a)} = \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$$

for some ξ between a and $a+h$. Now we substitute the terms to get

$$\frac{f(a+h) - p_n(h)}{0 - h^{n+1}} = \frac{f^{(n+1)}(\xi)(a + h - \xi)^n/n!}{-(n+1)(a + h - \xi)^n}$$

which we simplify as

$$f(a+h) - p_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1},$$

finally yielding

$$f(a+h) = p_n(h) + r_n(h)$$

or, in the original notation,

$$f(x) = P_n(x) + R_n(x)$$

for some ξ between a and x. □