

Analys i en variabel

SF1673 (HT25)

Contents

1. The Real Numbers

1.1. REALS

1.1.1. Prerequisites

1.1.2. Comparison

1.1.3. Bounds

1.2. CARDINALITY

1.3. TOPOLOGY

1.3.1. Points

1.3.2. Open and Closed Sets

1.3.3. Compactness

2. Limits

2.1. SEQUENCES

2.1.1. Bounded

2.1.2. Cauchy

2.2. SERIES

2.3. FUNCTIONS

2.4. CONTINUITY

2.4.1. Existence

2.4.2. Composition

2.4.3. Results

3. Calculus

3.1. THE DERIVATIVE

3.1.1. Differentiation

3.1.2. Function Character

3.1.3. The Mean Value Theorems

3.2. FUNCTION GRAPHS

3.2.1. Asymptotes

3.2.2. Convexity

3.2.3. Points

3.3. THE RIEMANN INTEGRAL

3.3.1. Definition

3.3.2. Integrability

3.3.3. Properties

3.3.4. Integration Techniques

3.4. TAYLOR'S THEOREM

3.4.1. Function Order

3.5. ORDINARY DIFFERENTIAL EQUATIONS

1. The Real Numbers

1.1. REALS

1.1.1. Prerequisites

Theorem 1.1.1 (Induction)

Let $S \subseteq \mathbb{N}$. If

- (i) $1 \in S$, and
- (ii) $n \in S \implies n + 1 \in S$ (inductive step),

then $S = \mathbb{N}$.

Definition 1.1.2 (Injective/Surjective/Bijective)

$f : X \rightarrow Y$ is *injective* (or one-to-one) if $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ or equivalently if $f(x_1) = f(x_2) \implies x_1 = x_2$.

f is *surjective* if $\forall y \exists x : f(x) = y$.

f is *bijective* if is both injective and surjective or equivalently if each y is mapped to exactly one x .

1.1.2. Comparison

Definition 1.1.3 (Equality)

$$a = b \iff (\forall \varepsilon > 0 \Rightarrow |a - b| < \varepsilon)$$

Theorem 1.1.4 (Triangle Inequalities)

- (i) $|a + b| \leq |a| + |b|$
- (ii) $|a - b| \leq |a - c| + |c - b|$
- (iii) $|a - b| \geq \big||a| - |b|\big|$

The reverse triangle inequality (iii) is seldom used.

1.1.3. Bounds

Axiom 1.1.5 (Supremum Property or Axiom of Completeness)

Every bounded, nonempty set of real numbers has a least upper bound.

Note

The same does not apply for the rationals.

Definition 1.1.6 (Least Upper Bound)

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then,

$$s = \sup A \iff \forall \varepsilon > 0 \exists a \in A : s - \varepsilon < a.$$

1.2. CARDINALITY

Definition 1.2.1 (Cardinality)

A has the same *cardinality* as B if there exists a bijective $f : A \rightarrow B$.

Definition 1.2.2 (Countable/Uncountable)

A is *countably infinite* if $\mathbb{N} \sim A$.

A is *countable* if it is finite or countably infinite.

Otherwise, A is *uncountable*.

Theorem 1.2.3 (Countability of \mathbb{Q} , \mathbb{R})

\mathbb{Q} is countable.

Proof. Let $A_1 = \{0\}$ and let

$$A_n = \{\pm p/q : p, q \in \mathbb{N}_+, \gcd(p, q) = 1, p + q = n\}$$

for all $n \geq 2$. Each A_n is finite and every rational numbers appears in exactly one set. □

\mathbb{R} is uncountable.

Proof. Cantor's diagonalization method. □

\mathbb{I} is uncountable.

Proof. $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ where \mathbb{Q} is countable. □

Theorem 1.2.4 (Density of \mathbb{Q} in \mathbb{R})

- (i) $\forall a < b \in \mathbb{R} \exists r \in \mathbb{Q} : a < r < b$
- (ii) $\forall y \in \mathbb{R} \exists (r_n) \in \mathbb{Q} : (r_n) \rightarrow y$

1.3. TOPOLOGY

1.3.1. Points

Definition 1.3.1 (Limit Point)

x is a *limit point* of A if every $V_\varepsilon(x)$ intersects A at some point other than x .

Theorem 1.3.2 (Sequential Limit Point)

x is a limit point of A if $x = \lim a_n$ for some $(a_n) \subseteq A : a_n \neq x \ \forall n \in \mathbb{N}$.

Theorem 1.3.3 (Nested Interval Property)

Let (I_n) be a nested sequence of nonempty closed and bounded intervals with

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

. Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

. In particular, there exists $a \in \bigcap_{n=1}^{\infty} I_n$.

1.3.2. Open and Closed Sets

Definition 1.3.4 (Open/Closed Set)

$A \subseteq \mathbb{R}$ is *open* if $\forall a \in A \exists V_\varepsilon(a) \subseteq A$ or equivalently if its complement is closed.

$A \subseteq \mathbb{R}$ is *closed* if it contains its limit points or equivalently if its complement is open.

Theorem 1.3.5 (Clopen Sets)

\mathbb{R} and \emptyset are *clopen* (both opened and closed).

Theorem 1.3.6 (Unions/Intersections)

- (i) Arbitrary unions of open sets are open; finite intersections of open sets are open.
- (ii) Arbitrary intersections of closed sets are closed; finite unions of closed sets are closed.

1.3.3. Compactness

Definition 1.3.7 (Compact)

A set K in a topological space is *compact* if every open cover has a finite subcover.

Theorem 1.3.8 (Heine–Borel)

A set $K \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Note

Compactness is like a generalization of closed intervals.

2. Limits

2.1. SEQUENCES

Definition 2.1.1 (Sequence)

A *sequence* is a function whose domain is \mathbb{N} .

Definition 2.1.2 (Convergence)

A sequence *converges* to a if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies |a_n - a| < \varepsilon$$

or equivalently if for any $V_\varepsilon(a)$ there exists a point in the sequence after which all terms are in $V_\varepsilon(a)$. In other words if every ε -neighborhood contains all but a finite number of the terms in (a_n) .

We write this $\lim_{n \rightarrow \infty} a_n = \lim a_n = a$ or $a_n \rightarrow a$.

Example. Template of a typical convergence proof:

- (i) Let $\varepsilon > 0$ be arbitrary.
- (ii) Propose an $N \in \mathbb{N}$ (found before writing the proof).
- (iii) Assume $n \geq N$.
- (iv) Show that $|a_n - a| < \varepsilon$.

Theorem 2.1.3 (Uniqueness of Limits)

The limit of a sequence, if it exists, is unique.

2.1.1. Bounded

Definition 2.1.4 (Bounded)

A sequence is *bounded* if $\exists M > 0 : |a_n| < M \ \forall n \in \mathbb{N}$.

Theorem 2.1.5 (Convergent)

Every convergent sequence is bounded.

If a sequence is monotone and bounded it converges.

Subsequences of a convergent sequence converge to the same limit.

Theorem 2.1.6 (Bolzano–Weierstrass)

In a compact set $K \subseteq \mathbb{R}$, every bounded sequence contains a convergent subsequence whose limit point is in K .

2.1.2. Cauchy

Definition 2.1.7 (Cauchy Sequence)

A sequence (a_n) is a *Cauchy sequence* if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : m, n \geq N \implies |a_n - a_m| < \varepsilon.$$

Theorem 2.1.8 (Cauchy Criterion)

A sequence converges if and only if it is a Cauchy sequence.

2.2. SERIES

Definition 2.2.1 (Infinite Series)

Let $(a_j)_{j=0}^\infty$ and let $(s_n)_{n=0}^\infty$. The sum of the infinite series is defined as

$$\sum_{j=0}^\infty a_j = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{j=0}^n a_j.$$

If $a_j \geq 0$ for every j we say that the series is *positive*.

⚠ Caution

Beware of treating infinite series like elementary algebra, e.g., by rearranging terms.

Theorem 2.2.2 (Cauchy Criterion for Series)

The series $\sum_{k=0}^\infty a_k$ converges if and only if

$$\forall \varepsilon > 0 \exists N : n > m > N \implies |a_m + a_{m+1} + \cdots + a_{n-1} + a_n| < \varepsilon.$$

Corollary 2.2.3 (Series Term Test)

If $\sum_{k=1}^\infty a_k$ converges, then $a_k \rightarrow 0$. However, the reverse implication is false.

Theorem 2.2.4

The series $\sum_{j=1}^\infty 1/j$ is divergent.

Theorem 2.2.5

The series $\sum_{j=1}^\infty 1/j^p$ converges if and only if $p > 1$.

Theorem 2.2.6 (Ratio Test)

Let (a_n) be a sequence of positive terms and define

$$L = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then:

- (i) If $L < 1$, the series $\sum_{n=1}^\infty a_n$ converges.
- (ii) If $L > 1$ (including $L = \infty$), the series diverges.
- (iii) If $L = 1$, the test is inconclusive.

Theorem 2.2.7 (Cauchy Condensation Test)

Let (a_n) be a decreasing sequence of nonnegative real numbers. Then $\sum_{n=1}^\infty a_n$ converges if and only if $\sum_{n=0}^\infty 2^n a_{2^n}$ converges.

Theorem 2.2.8

Let $\sum_{j=0}^\infty a_j$ and $\sum_{j=0}^\infty b_j$ be positive series with terms such that

$$\lim_{j \rightarrow \infty} \frac{a_j}{b_j} = K$$

for some $K \neq 0$. Then, $\sum_{j=0}^\infty a_j$ converges if and only if $\sum_{j=0}^\infty b_j$ converges.

Theorem 2.2.9 (Comparison Test)

Let (a_k) and (b_k) satisfy $0 \leq a_k \leq b_k$. Then,

- (i) $\sum_{k=1}^\infty (a_k)$ converges if $\sum_{k=1}^\infty (b_k)$ converges.
- (ii) $\sum_{k=1}^\infty (b_k)$ diverges if $\sum_{k=1}^\infty (a_k)$ diverges.

Theorem 2.2.10 (Alternating Series Test)

Let (a_n) satisfy

- (i) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and
- (ii) $(a_n) \rightarrow 0$.

Then, $\sum_{n=1}^\infty (-1)^{n+1} a_n$ converges.

Definition 2.2.11 (Absolutely Convergent)

A series $\sum_{j=0}^\infty a_j$ is *absolutely convergent* if $\sum_{j=0}^\infty |a_j|$ is convergent.

Theorem 2.2.12

If a series is absolutely convergent then it is convergent.

Theorem 2.2.13 (Geometric Series)

If $|x| < 1$, then

$$\sum_{j=0}^\infty x^j = \frac{1}{1-x}$$

since

$$s_n = \sum_{j=0}^n x^j = \frac{1-x^{n+1}}{1-x}.$$

2.3. FUNCTIONS

Theorem 2.3.1 (Function Limit)

Given $f : A \rightarrow \mathbb{R}$ with the limit point c ,

- (i) $\lim_{x \rightarrow c} f(x) = L$ is equivalent to
- (ii) if $\forall (x_n) \subseteq A : (x_n \neq c \text{ and } x_n \rightarrow c)$ it follows that $f(x_n) \rightarrow L$.

📌 Note

In the $\varepsilon\delta$ -definition of limits, the additional restriction that $0 < |x - a|$ is just a way to say $x \neq c$.

Definition 2.3.2 (Infinite Limit)

Given a limit point $c \in D_f$, we say that $\lim_{x \rightarrow c} f(x) = \infty$ if

$$\forall M \exists \delta > 0 : 0 < |x - c| < \delta \implies f(x) \geq M.$$

2.4. CONTINUITY

2.4.1. Existence

Theorem 2.4.1 (Continuity)

The following are equivalent:

- (i) $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is *continuous* at $c \in A$.
- (ii) $\forall \varepsilon > 0 \exists \delta > 0 : |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$, where $x \in A$.
- (iii) $\forall V_\varepsilon(f(c)) \exists V_\delta(c) : x \in V_\delta(c) \cap A \implies f(x) \in V_\varepsilon(f(c))$
- (iv) $x_n \rightarrow c$, where $(x_n) \subseteq A$, implies $f(x_n) \rightarrow f(c)$.

If c is a limit point of A :

- (v) $\lim_{x \rightarrow c} f(x) = f(c)$, also written $\lim_{h \rightarrow 0} f(c + h) - f(c) = 0$.

Note that (ii) defines (i). Mostly (v) is used in practice.

Corollary 2.4.2 (Isolated Continuity)

All functions are continuous at isolated points.

Theorem 2.4.3 (Dirichlet Discontinuous)

The Dirichlet function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \in \mathbb{I}$ is discontinuous everywhere.

Definition 2.4.4 (Uniform Continuity)

We say f is *uniformly continuous* on I if

$$\forall \varepsilon > 0 \exists \delta > 0 : x, y \in I, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

In particular, δ can be chosen independent of y .

Theorem 2.4.5

If a function is uniformly continuous, it is also continuous.

Theorem 2.4.6 (Heine–Cantor)

If f is continuous and defined on a compact set K , then it is also uniformly continuous on K .

Proof. Assume the opposite, that f is continuous but not uniformly. Since f is not uniformly continuous,

$$\exists \varepsilon_0 > 0 : \forall \delta > 0 \exists x, y \in K : |x - y| < \delta \text{ but } |f(x) - f(y)| \geq \varepsilon_0.$$

Now, choose (x_n) and (y_n) such that

$$|x_n - y_n| < \frac{1}{n} \text{ and } |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

Theorem 2.1.6 asserts that there exists some subsequence $x_{n_k} \rightarrow x_0$ for some $x_0 \in K$. From $|x_n - y_n| < \frac{1}{n}$ it follows that $y_{n_k} \rightarrow x_0$. Thus,

$$|x_{n_k} - y_{n_k}| \rightarrow 0,$$

and, because f is continuous with $f(x_{n_k}) \rightarrow x_0$ and $f(y_{n_k}) \rightarrow x_0$,

$$|f(x_{n_k}) - f(y_{n_k})| \rightarrow 0.$$

However, this contradicts our assumption that

$$|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0.$$

□

2.4.2. Composition

Theorem 2.4.7 (Composition)

Given $f : A \rightarrow B$ and $g : B \rightarrow \mathbb{R}$ with $f(A) \subseteq B$, if f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c .

Theorem 2.4.8 (Composition Limit)

If f is continuous at y and $\lim_{x \rightarrow c} g(x) = y$, then

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(y).$$

2.4.3. Results

Theorem 2.4.9 (Intermediate Value)

If f is continuous on $[a, b]$, then for any y between $f(a)$ and $f(b)$, there exists some $c \in (a, b)$ such that $f(c) = y$.

Theorem 2.4.10 (Weierstrass Extreme Value)

If f is continuous on the compact set K , then f attains a maximum and a minimum value on K .

Theorem 2.4.11 (Limit of Bounded Function)

If f is bounded then $\lim_{h \rightarrow 0} f(h)h = 0$.

3. Calculus

3.1. THE DERIVATIVE

3.1.1. Differentiation

Definition 3.1.1 (Derivative at a Point)
Let $f : A \rightarrow \mathbb{R}$ and c a limit point of A . If

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

exists (finite), we say f is *differentiable* at c .

Theorem 3.1.2 (Chain Rule)
Let $f : X \rightarrow Y$ and $g : Y \rightarrow \mathbb{R}$. If f is differentiable at $c \in X$ and g is differentiable at $f(c) \in Y$, then $g \circ f$ is differentiable at c with

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Theorem 3.1.3 (Basic Derivatives)

$$\begin{aligned} \frac{d}{dx}(\arcsin x) &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}(\sin x) &= \cos x \\ \frac{d}{dx}(\arccos x) &= -\frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}(\cos x) &= -\sin x \\ \frac{d}{dx}(\arctan x) &= \frac{1}{1+x^2} & \frac{d}{dx}(\tan x) &= \frac{1}{\cos^2 x} \\ \frac{d}{dx}(\operatorname{arccot} x) &= -\frac{1}{1+x^2} & \frac{d}{dx}(\ln|x|) &= \frac{1}{x} \\ \frac{d}{dx}(x^a) &= ax^{a-1} \quad (a \neq 0) & (f^{-1})'(y) &= \frac{1}{f'(x)} \quad (y = f(x), f'(x) \neq 0) \end{aligned}$$

Theorem 3.1.4 (L'Hôpital's Rule)

Let f and g be differentiable on an open interval containing c (except possibly at c), with $g'(x) \neq 0$ near c . Suppose
(i) $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$
(or both $\pm\infty$), and
(ii) $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L$ exists (or $\pm\infty$).

Then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = L.$$

Proof of the zero case. Assume the limits are zero.

Let the functions be differentiable on the open interval (c, x) . Then, rewriting and applying Theorem 3.1.1(i) gives

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x \rightarrow c} \frac{f'(p)}{g'(p)} = \lim_{p \rightarrow c} \frac{f'(p)}{g'(p)}$$

for some p between c and x . □

Proof of the infinity case. The proof is too complicated. □

Important

This is only an implication, not an equivalence, so there may exist some other solution if this method fails.

3.1.2. Function Character

Theorem 3.1.5 (Fermat's or Interior Extremum)

Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable at the local extremum $c \in (a, b)$. Then $f'(c) = 0$.

However, note that a zero-derivative point may also be a stationary point of inflection.

Theorem 3.1.6 (Darboux's)

If f is differentiable on $[a, b]$ and if y lies strictly between $f'(a)$ and $f'(b)$, then $\exists c \in (a, b) : f'(c) = y$.

In other words, if f is differentiable on an interval, then f' satisfies the Intermediate Value Property (IVP).

Proof. Assume that $f'(a) < y < f'(b)$.

Let $g(x) = f(x) - yx$ with $g'(x) = f'(x) - y$. Note that $f'(c) = y$ if $g'(c) = 0$ for some $c \in (a, b)$.

Theorem 2.4.1.0 states that g must have a minimum point $c \in [a, b]$. More precisely $c \in (a, b)$ since, from the assumption, $g'(a) < 0$ and $g'(b) > 0$. Furthermore, $g'(c) = 0$ according to Theorem 3.1.5. □

Theorem 3.1.7 (Newton's Method)

Find roots to a differentiable function $f(x)$.

Given x_n with the coordinates $(x_n, f(x_n))$, the tangent line is given by

$$T(x) = f'(x_n)(x - x_n) + f(x_n)$$

and intersects the x -axis at

$$T(x_{n+1}) = 0 \iff x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The method fails if it iterates endlessly or $f'(x_n) = 0$.

3.1.3. The Mean Value Theorems

Let f and g be continuous on $[a, b]$ and differentiable on (a, b) .

Theorem 3.1.8 (Rolle's)

$$f(a) = f(b) \implies \exists c \in (a, b) : f'(c) = 0$$

Proof. $f(x)$ is bounded and $f'(x) = 0$ at its interior extreme points. □

Theorem 3.1.9 (Mean Value)

$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Let the signed distance d between the function value f and the secant y through a and b be

$$d(x) = f(x) - y(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

and note that $d(a) = d(b) = 0$. Then apply Theorem 3.1.8. □

Theorem 3.1.10 (Generalized Mean Value)

$$\exists c \in (a, b) : [f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

If g' is never zero on (a, b) , then the above can be stated as

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Let $h = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$ and then apply Theorem 3.1.8. □

3.2. FUNCTION GRAPHS

Tip (Sketching Graphs)

Information

- (i) symmetries
- (ii) split into cases
- (iii) domain \rightarrow vertical asymptotes
- (iv) factorize \rightarrow oblique asymptotes & roots
- (v) first and second derivative and their roots
- (vi) sign tables
- (vii) calculate interesting points: intersection with y -axis, defined nondifferentiable points, local extremums, endpoints, inflection points

Sketching

- (i) axes
- (ii) symmetries
- (iii) asymptotes
- (iv) interesting points
- (v) curves

3.2.1. Asymptotes

Definition 3.2.1 (Asymptote)

The line $y = kx + m$ is an *oblique asymptote* of f if

$$\lim_{x \rightarrow \infty} (f(x) - (kx + m)) = 0.$$

The line $x = c$ is a *vertical asymptote* of f if

$$\lim_{x \rightarrow c^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow c^+} f(x) = \pm\infty.$$

The line $y = b$ is a *horizontal asymptote* of f if

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

Theorem 3.2.2 (Oblique Asymptote)

If $f(x)$ has an oblique asymptote $y = kx + m$, then

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$$

and

$$m = \lim_{x \rightarrow \infty} (f(x) - kx).$$

3.2.2. Convexity

Theorem 3.2.3 (Convexity)

Let f be twice differentiable on (a, b) . Then, $f''(x) \geq 0$ if and only if f is convex on (a, b) .

Definition 3.2.4 (Concave)

On $[a, b]$, a function $f : [a, b] \rightarrow \mathbb{R}$ is *concave* if $-f$ is convex.

3.2.3. Points

Definition 3.2.5 (Local Extremum)

A *local maximum* of $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a point c for which there exists an open neighborhood $N(c) \subseteq D$ such that

$$f(c) \geq f(x) \quad \forall x \in N(c).$$

Definition 3.2.6 (Stationary)

The point c is a *stationary point* of f if $f'(c) = 0$.

The *stationary order* is the smallest $n \geq 2$ such that

$$f'(c) = f''(c) = \dots = f^{(n-1)}(c) = 0 \quad \text{but} \quad f^{(n)}(c) \neq 0.$$

Definition 3.2.7 (Critical)

The point c is a *critical point* if $f(c)$ is stationary or undefined.

Definition 3.2.8 (Inflection)

A point c is an *inflection point* of f if f is continuous at c and if f is convex on one side of c and concave on the other side.

Theorem 3.2.9 (First Nonzero Derivative)

If f has stationary order n , then:

- If n is *even* $\rightarrow f$ has a local extremum at c .
Furthermore: $f^{(n)}(c) > 0 \rightarrow$ local minimum, $f^{(n)}(c) < 0 \rightarrow$ local maximum.
- If n is *odd* $\rightarrow c$ is a stationary inflection point.

Proof. The Taylor series with remainder simplifies to

$$f(c+h) = f(c) + \frac{f^{(n)}(c)}{n!} h^n + O(h^{n+1}).$$

Its change close to c is thus

$$f(c+h) - f(c) \approx \frac{f^{(n)}(c)}{n!} h^n,$$

which changes sign if and only if n is odd. Similarly,

$$f'(c+h) - f'(c) \approx \frac{f^{(n-1)}(c)}{(n-1)!} h^{n-1}$$

for the first derivative and

$$f''(c+h) - f''(c) \approx \frac{f^{(n-2)}(c)}{(n-2)!} h^{n-2}$$

for the second derivative. □

Corollary 3.2.10 (Second Derivative Test)

If f'' is continuous at c and $f'(c) = 0$, then:

- $f''(c) > 0 \rightarrow$ local minimum.
- $f''(c) < 0 \rightarrow$ local maximum.
- $f''(c) = 0$ and $f^{(3)}(c) \neq 0 \rightarrow$ stationary inflection point.

Note: $f''(c) = 0$ alone is insufficient for an inflection; the curvature must change sign.

Examples.

- $f(x) = x^3$: $f'(0) = f''(0) = 0$, $f^{(3)}(0) = 6 \neq 0$ (odd $n = 3$) \rightarrow stationary inflection at 0.
- $f(x) = x^4$: $f'(0) = f''(0) = f^{(3)}(0) = 0$, $f^{(4)}(0) = 24 > 0$ (even $n = 4$) \rightarrow local minimum at 0, no inflection.
- $f(x) = -x^4$: local maximum at 0, no inflection.

3.3. THE RIEMANN INTEGRAL

3.3.1. Definition

Definition 3.3.1 (Partition)

A *partition* of $[a, b]$ is a finite set

$$P = \{x_0, x_1, \dots, x_n\}$$

such that

$$a = x_0 < x_1 < \dots < x_n = b,$$

The partition P has *subintervals*

$$[x_{i-1}, x_i] \quad i = 1, 2, \dots, n$$

of which the length of the largest is its *mesh* or *norm*

$$\|P\| = \max_{1 \leq i \leq n} (x_i - x_{i-1}).$$

A smaller such is indicative of a finer partition.

Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. We now define its definite integral.

Definition 3.3.2 (Darboux Integral)

Define the *lower sum*

$$L(f, P) = \sum_{i=1}^n (\inf\{f(x) : x \in [x_{i-1}, x_i]\})(x_i - x_{i-1}).$$

and the *upper sum*

$$U(f, P) = \sum_{i=1}^n (\sup\{f(x) : x \in [x_{i-1}, x_i]\})(x_i - x_{i-1}).$$

The function f is *Darboux integrable* if $\sup_P L(f, P) = \inf_P U(f, P)$. The common value is denoted as the *definite integral* $\int_a^b f(x) dx$.

Definition 3.3.3 (Alternative Darboux Integral)

Let Φ and Ψ be the *lower* and *upper step functions* such that

$$\Phi(x) \leq f(x) \leq \Psi(x) \quad \forall x \in [a, b],$$

forming the *lower integral*

$$L(f) = \sup \left\{ \int_a^b \Phi(x) dx : \Phi \text{ is a lower step function to } f \right\}$$

and the *upper integral*

$$U(f) = \inf \left\{ \int_a^b \Psi(x) dx : \Psi \text{ is an upper step function to } f \right\}$$

which, if equal, give the definite integral.

Note that the integral of a step function is simply its signed area.

Definition 3.3.4 (Riemann Integral)

From a partition P of $[a, b]$ pick *sample points*

$$t_i \in [x_{i-1}, x_i], \quad i = 1, 2, \dots, n$$

and form the (tagged) *Riemann sum*

$$S(f, P, (t_i)) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}).$$

We say f is *Riemann integrable* if there exists $L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \exists \delta > 0 : \|P\| < \delta \implies |S(f, P, (t_i)) - L| < \varepsilon$$

for every choice of sample points (t_i) . In that case we write

$$L = \int_a^b f(x) dx.$$

Theorem 3.3.5

The Darboux and Riemann integrals are equivalent.

3.3.2. Integrability

Theorem 3.3.6 (Integrability)

Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded.

The function is integrable if and only if:

- (i) $\forall \varepsilon > 0 \exists P : U(f, P) - L(f, P) < \varepsilon$.
- (ii) $\forall (P_n) : U(P_n) \rightarrow 0 \implies U(f, P_n) - L(f, P_n) \rightarrow 0$.
- (iii) (Lebesgue Criterion for Riemann Integrability)
Its set of discontinuities has Lebesgue measure zero.
- (iv) $\forall \varepsilon > 0 \exists \Phi, \Psi : \int_a^b \Psi(x) dx - \int_a^b \Phi(x) dx < \varepsilon$,

where Φ and Ψ are lower and upper step functions.

The function is integrable if:

- (iii) f is *monotone* on $[a, b]$.
- (iv) f is continuous except at finitely many points, or at countably many points where it has only removable or jump discontinuities.

Theorem 3.3.7

Assume f is continuous on $[a, b]$. Let

$$M_i = \max_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \min_{x \in [x_{i-1}, x_i]} f(x).$$

Then,

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n M_i(x_i - x_{i-1}) = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n m_i(x_i - x_{i-1}) = \int_a^b f(x) dx.$$

Theorem 3.3.8 (Absolute Value / Triangle)

If f integrable, then $|f|$ integrable and

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Theorem 3.3.9 (Products and Composition)

If f, g integrable, then fg is integrable.

If f integrable and φ continuous on a set containing $f([a, b])$, then $\varphi \circ f$ is integrable.

Theorem 3.3.10 (Uniform Limit)

If (f_n) are integrable on $[a, b]$ and $f_n \rightarrow f$ uniformly, then f is integrable and

$$\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx.$$

3.3.3. Properties

Theorem 3.3.11 (Linearity)

If f, g are integrable and $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$$

Theorem 3.3.12 (Additivity of the Interval)

If $c \in (a, b)$ and f integrable on $[a, b]$, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

It follows that $\int_a^a f(x) dx = 0$ and $\int_b^a f(x) dx = -\int_a^b f(x) dx$.

Theorem 3.3.13 (Order / Comparison)

If f, g integrable and $f(x) \leq g(x)$ on $[a, b]$, then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Corollary 3.3.14 (Positivity)

If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x) dx \geq 0$. Moreover, if f is continuous and the integral is 0, then $f \equiv 0$.

Theorem 3.3.15 (Bounding by a Supremum)

If $|f(x)| \leq M$ on $[a, b]$, then

$$\left| \int_a^b f(x) dx \right| \leq M(b - a).$$

Theorem 3.3.16 (Mean Value for Integrals)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x) dx = f(\xi)(b - a).$$

for some $\xi \in [a, b]$ or, to be more strict if f is not constant, $\xi \in (a, b)$.