Linear Algebra

SF1672

Contents

1. Vector Spaces

1.1. Definitions

Definition 1.1.1 (Field)

A field is a set F with operations $+$ and $\cdot$ such that

(i) $0 + a = a + 0 = a$
(ii) $(a + b) + c = a + (b + c)$
(iii) $a + b = b + a$
(iv) there exists $(-a)$ with $a + (-a) = 0$
(v) $1 \cdot a = a \cdot 1$
(vi) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(vii) $a \cdot b = b \cdot a$
(viii) for all $a \neq 0$ there exists a^{-1} with $a \cdot a^{-1} = 1$

Definition 1.1.2 (Vector Space)

A vector space over a field F is a set V with two operations:

- Vector addition $+ : V \times V \to V$
- Scalar multiplication $\cdot : F \times V \to V$

These must satisfy

(i) $\mathbf{u} + \mathbf{v} \in V$
(ii) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
(iii) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
(iv) $a(b\mathbf{u}) = (ab)\mathbf{u}$
(v) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
(vi) $1\mathbf{u} = \mathbf{u}$
(vii) $\mathbf{u} + \mathbf{0} = \mathbf{u}$
(viii) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

1.2. Subspaces

Definition 1.2.1 (Subspace)

A subspace of a vector space V is a subset H that is

(i) nonempty (e.g. $\mathbf{0} \in H$),
(ii) closed under addition, and
(iii) closed under scalar multiplication.

Example. The set $\{\mathbf{0}\}$ (with $\mathbf{0} \in V$) is a subspace of every V.

⚠ **Warning**

$\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$. However, the set $\{(s, t, 0) : s, t \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Definition 1.2.2 (Span)

The span of a set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is the set of all linear combinations of the vectors in S. The span of a set of vectors is the smallest subspace that contains them.

Theorem 1.2.1

If $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$, then $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

1.3. Bases

Definition 1.3.1 (Basis)

A basis for a vector space V is a set $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of linearly independent vectors that span V.

Example. The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$, and the set $\{1, t, t^2, \ldots, t^n\}$ is the standard basis for P_n.

Theorem 1.3.1 (Basis Criterion)

Let V be an n-dimensional vector space with $n \geq 1$. Any set of n vectors in V is automatically a basis if

(i) it is linearly independent or, equivalently,
(ii) it spans V.

Definition 1.3.2 (Coordinates)

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V. Each $\mathbf{v} \in V$ can be expressed as

$$[\mathbf{v}]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_B,$$

the injective coordinate mapping $\mathbf{v} \to [\mathbf{v}]_B$,

$$P_B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n]$$

from B to the standard basis in V, with

$$\mathbf{v} = P_B[\mathbf{v}]_B.$$

Now let $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be another basis for V. Then there is a unique $n \times n$ matrix

$$P_{C \leftarrow B} = \left[[\mathbf{b}_1]_C \ \cdots \ [\mathbf{b}_n]_C\right]$$

$$[\mathbf{x}]_C = P_{C \leftarrow B}[\mathbf{x}]_B$$

and

$$(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C}.$$

Example. Let E be the standard basis and let $P_{E \leftarrow B}$ and $P_{E \leftarrow C}$ be given.

$$P_{C \leftarrow B} = P_{C \leftarrow E} P_{E \leftarrow B} = (P_{E \leftarrow C})^{-1} P_{E \leftarrow B}.$$

1.4. Dimension

Definition 1.4.1 (Dimension)

The dimension of a vector space is the number of vectors in every basis. A vector space is either finite-dimensional och infinite-dimensional.

Definition 1.4.2 (Rank and Nullity)

The rank of a linear transformation (or matrix) is the dimension of its image (column space) and is also given by the number of pivot columns, and the nullity is the dimension of its kernel.

Theorem 1.4.1 (Rank Theorem)

For an $m \times n$ matrix A it holds that

$$\operatorname{rank} A + \operatorname{nullity} A = n.$$

2. Matrices

2.1. Matrix Forms

Definition 2.1.1 (Matrix Forms)

- Row Echelon Form (REF):
  Pivots move to the right as you go down, with zeros below each pivot.
- Reduced Row Echelon Form (RREF):
  REF plus each pivot is 1 and is the only nonzero entry in its column. Canonical, i.e., unique.
- Upper/Lower Triangular Form:
  The diagonal entries of the triangular form are the eigenvalues of the original.

Definition 2.1.2 (Consistent)

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if it has at least one solution.

Theorem 2.1.1 (Echelon Test)

After reducing $[A \mid \mathbf{b}]$ to REF or RREF, a system is inconsistent if and only if you obtain a row of the form

$$[0 \ 0 \ \cdots \ 0 \mid c], \quad c \neq 0.$$

Definition 2.1.3 (Singular)

A matrix is singular if it is square and non-invertible.

Definition 2.1.4 (Elementary Matrix)

An elementary matrix is an $n \times n$ matrix obtained by applying one elementary row operation to the identity matrix I_n.

All elementary matrices are invertible.

Theorem 2.1.2 (Invertible)

Let A be an $n \times n$ matrix. Then it is equivalent that

(i) A is invertible
(ii) A^T is invertible
(iii) A has a left-hand inverse
(iv) A has a right-hand inverse

(v) A has n pivot positions
(vi) A is row equivalent to the $n \times n$ identity matrix
(vii) A can be expressed as a product of elementary matrices
(viii) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
(ix) $A\mathbf{x} = \mathbf{b}$ has at least (and thus exactly) one solution for each $\mathbf{b}$

(x) the columns of A are linearly independent, forming a basis for $\mathbb{R}^n$
(xi) $\operatorname{col} A = \mathbb{R}^n$
(xii) $\ker A = \{\mathbf{0}\}$
(xiii) $\operatorname{rank} A = n$
(xiv) $\operatorname{nullity} A = 0$
(xv) $(\operatorname{col} A)^\perp = \{\mathbf{0}\}$
(xvi) $(\ker A)^\perp = \mathbb{R}^n$
(xvii) $\operatorname{row} A = \mathbb{R}^n$

(xviii) $T(\mathbf{x}) = A\mathbf{x}$ is injective
(xix) $T(\mathbf{x}) = A\mathbf{x}$ is surjective
(xx) $T(\mathbf{x}) = A\mathbf{x}$ is bijective, having $\operatorname{im} T = \mathbb{R}^n$ and an inverse T^{-1}

(xxi) A has n nonzero singular values
(xxii) $\det A \neq 0$

2.2. Column and Row Spaces

Definition 2.2.1 (Column Space)

$$\operatorname{col}(A) = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \operatorname{im}(f_A),$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the columns of A.

Theorem 2.2.1 (Pivot Basis)

The pivot columns of a matrix form a basis for its column space.

2.3. Eigenvalues

Definition 2.3.1 (Eigenvalues and Eigenvectors)

An eigenvalue λ with eigenvector $\mathbf{x} \neq \mathbf{0}$ of a matrix A satisfies

$$A\mathbf{x} = \lambda\mathbf{x}.$$

The eigenvalues are found from the characteristic equation

$$\det(A - \lambda I) = 0.$$

3. Linear Maps

3.1. Definitions

Definition 3.1.1 (Linear Map)

A function $T : V \to W$ is linear if

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $a, b \in \mathbb{R}$. In other words, the function satisfies additivity and homogeneity.

Theorem 3.1.1

A linear map $T : V \to W$ is bijective if it is

(i) injective ($\ker T = \{\mathbf{0}\}$) and
(ii) surjective ($\operatorname{im} T = W$).

This is equivalent to T having a linear inverse $T^{-1} : W \to V$.

Definition 3.1.2 (Isomorphism)

A linear map $T : V \to W$ is an *isomorphism* if it is bijective.

If such a map exists, the spaces *isomorphic*, written $V \cong W$.

If $V = W$, then T is an *automorphism*.

Definition 3.1.3 (Affine)

A transformation f is affine if the function g defined by $g(x) = f(x) - f(0)$ is linear.

3.2. Properties

Theorem 3.2.1 (Left/Right Inverse)

If a linear transformation has a right inverse then it is surjective, and if it has a left inverse then it is injective.

4. Geometry

4.1. Inner Products

Inner product:

$$[1 \ 2 \ 3] \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = [1 + 2 + 6] = [9]$$

4.2. Intersections in $\mathbb{R}^3$

- line–plane

  Substitute the line's parametric form into the plane's general equation.
- line–line

  Solve the system formed by the two parametric equations.
- plane–plane

  Construct a vector between two arbitrary points (one on each plane) given in parametric form. Set its dot product with each plane's normal vector to zero. Solve the resulting system of equations.

4.3. Rotations in $\mathbb{R}^3$

Theorem 4.3.1

Rotation about the x-, y- and z-axes by θ (counterclockwise):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

⚠ **Warning**

Pay special attention to the placement of the minus sign for the y-axis rotation.