Linear Algebra

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Notation **Vectors** v, w **Bases** $v_1, \ldots, v_n$ **Linear map** T **Matrix of T** A **Eigenvalues** $\lambda_1, \ldots, \lambda_n$ **Eigenvectors** $v_1, \ldots, v_n$ **Minimal polynomial** q **Characteristic polynomial** p **Inner product** $\langle v, w \rangle$ **Norm** $\|v\|$ **Adjoint** T^* **Dimension** $\dim(V)$ **Null space** $\operatorname{null}(T)$ **Range** $\operatorname{range}(T)$ **Trace** $\operatorname{tr}(T)$ **Determinant** $\det(T)$ **Complex numbers** $\mathbb{C}$ **Real numbers** $\mathbb{R}$ **Positive integers** $\mathbb{Z}^+$ **Complement** $U^\perp$ **Orthogonal projection onto U** P_U **Tensor product** $v \otimes w$ **Exterior power** $v_1 \wedge \cdots \wedge v_k$ **Symmetric power** $v_1 \cdot \cdots \cdot v_k$

1 Vector Spaces

1.1 Basics

Definition of vector space A set V with addition and scalar multiplication satisfying:

- Commutativity: $u + v = v + u$, $ab = ba$
- Associativity: $(u + v) + w = u + (v + w)$, $(ab)c = a(bc)$
- Identity: $v + 0 = v$, $1v = v$
- Distributivity: $a(u + v) = au + av$
- Additive inverse: $v + (-v) = 0$
- Multiplicative inverse: $av = 1$ for $a \neq 0$

Definition of field $\mathbb{R}$ or $\mathbb{C}$

Definition of $\mathbb{C}$ $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

Definition of $\mathbb{R}^n$ $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$

Definition of $\mathbb{C}^n$ $\mathbb{C}^n = \{(z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C}\}$

Definition of $\mathbb{F}^n$ $\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{F}\}$

Definition of $\mathbb{F}^\infty$ $\mathbb{F}^\infty = \{(x_1, x_2, \ldots) : x_1, x_2, \ldots \in \mathbb{F}\}$

Definition of $\mathbb{F}^S$ $\mathbb{F}^S = \{f : f \text{ is a function from } S \text{ to } \mathbb{F}\}$

Definition of $\mathbb{F}^n$ as $\mathbb{F}^S$ $\mathbb{F}^n = \mathbb{F}^{\{1, \ldots, n\}}$

Definition of $\mathbb{F}^\infty$ as $\mathbb{F}^S$ $\mathbb{F}^\infty = \mathbb{F}^{\mathbb{Z}^+}$

Definition of subspace $U \subseteq V$ is a subspace if U is closed under addition and scalar multiplication.

Subspace test $U \subseteq V$ is a subspace if:

- $0 \in U$
- $u, v \in U \Rightarrow u + v \in U$
- $a \in \mathbb{F}, v \in U \Rightarrow av \in U$

Intersection of subspaces is a subspace If $U_1, \ldots, U_n$ are subspaces of V, then $U_1 \cap \cdots \cap U_n$ is a subspace of V.

Union of subspaces is not a subspace in general $U_1 \cup U_2$ is not a subspace of V in general.

Sum of subspaces $U_1 + \cdots + U_n = \{u_1 + \cdots + u_n : u_1, \ldots, u_n \in U_1, \ldots, U_n\}$

Sum of subspaces is smallest containing subspace $U_1 + \cdots + U_n$ is the smallest subspace containing $U_1, \ldots, U_n$.

Direct sum $U_1 + \cdots + U_n$ is a direct sum if $u_1 + \cdots + u_n = 0$ implies $u_1 = \cdots = u_n = 0$.

Direct sum of two subspaces $U + W$ is a direct sum if $U \cap W = \{0\}$.

Direct sum of two subspaces is unique $U \oplus W$ is the unique decomposition $v = u + w$ for $v \in U + W$.

Direct sum of two subspaces is commutative $U \oplus W = W \oplus U$.

Direct sum of two subspaces is associative $(U \oplus W) \oplus X = U \oplus (W \oplus X)$.

Direct sum of two subspaces is closed under addition and scalar multiplication $U \oplus W$ is closed under addition and scalar multiplication.

Direct sum of two subspaces is a subspace $U \oplus W$ is a subspace of V.

Direct sum of two subspaces is the smallest containing subspace $U \oplus W$ is the smallest subspace containing U and $W