$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2n\sqrt{1-(\frac{i}{2n})^2}} = \underline{\quad (3) \quad}.$$

Let $n \geq 2$ be an integer.

(i) (5 %) Show that the formula

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx.$$

(ii) (6 %) Find $\int_0^{\pi/2} \sin^6 x dx$ and $\int_0^{\pi/2} \sin^7 x dx$.

Problem 3. a) Show that $\mathbf{F} = (3x^2 - 6y^2)\hat{\mathbf{i}} + (-12xy + 4y)\hat{\mathbf{j}}$ is conservative.

b) Find a potential function for $\mathbf{F}$.

c) Let C be the curve $x = 1 + y^3(1-y)^3$, $0 \leq y \leq 1$. Calculate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

A vector field $\mathbf{V}$ on $\mathbb{R}^3$ is called *conservative* if $\mathbf{V} = \nabla f$ for some differentiable function f.

(a) (5 points) Prove that if a smooth vector field $\mathbf{V} = (V_1, V_2, V_3)$ (i.e. V_i's are smooth) is conservative on some domain $U \subset \mathbb{R}^3$, then

$$curl\ \mathbf{V} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) = (0, 0, 0)$$

on U.

2. 设 $P=P(x,y,z)$， $Q=Q(x,y,z)$ 均为连续函数，$\sum$ 为曲面 $z=\sqrt{1-x^2-y^2}\ (x\le 0, y\ge 0)$的上侧，则$\iint\limits_{\Sigma} Pdydz+Qdzdx=(\quad)$

A. $\iint\limits_{\Sigma}\left(\frac{x}{z}P+\frac{y}{z}Q\right)dxdy$　　B. $\iint\limits_{\Sigma}\left(-\frac{x}{z}P+\frac{y}{z}Q\right)dxdy$

C. $\iint\limits_{\Sigma}\left(\frac{x}{z}P-\frac{y}{z}Q\right)dxdy$　　D. $\iint\limits_{\Sigma}\left(-\frac{x}{z}P-\frac{y}{z}Q\right)dxdy$

Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^4y^2}{x^4+y^6} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

(a) (4%) Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$. Is $f(x,y)$ continuous at $(0,0)$?

(b) (4%) Let $\mathbf{u} = \langle a,b\rangle$ be a unit vector (i.e. $a^2+b^2=1$). Use the definition of directional derivative to find $D_{\mathbf{u}}f(0,0)$.

(c) (2%) Write down the linearization $L(x,y)$ of $f(x,y)$ at $(0,0)$.

(d) (4%) Determine whether $f(x,y)$ is differentiable at $(0,0)$ by considering the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}}.$$

(15 points) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a map so that

$$\|\varphi(\mathbf{x}) - \varphi(\mathbf{y})\| < c\|\mathbf{x} - \mathbf{y}\|$$

for some $c \in (0, 1)$. Prove that there exists a unique point $\mathbf{x}_0 \in \mathbb{R}^n$ so that

$$\varphi(\mathbf{x}_0) = \mathbf{x}_0.$$

導出 Leibniz 公式 $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{(2n+1)} + \cdots$

Let $I_n = \int_0^{\infty} \frac{x^{2n-1}}{(x^2+1)^{n+3}} dx$, for $n \geq 1$.

(a) (10%) Prove that $I_{n+1} = \frac{n}{n+3} I_n$, for $n \geq 1$.

(b) (5%) Evaluate $\int_0^{\infty} \frac{x^5}{(x^2+1)^6} dx$.

Let $f(x)$ be a continuous function on the interval $[0,1]$ and $f(0) = f(1)$. Show that, for any integer $n \geq 2$, there exists some $a \in [0, 1 - \frac{1}{n}]$ such that $f(a) = f(a + \frac{1}{n})$. (12pts)

(12%) Evaluate the integral $\iint\limits_R (x+y)^2\, dA$ where R is the region bounded by the ellipse $x^2 + xy + y^2 = 1$.

Suppose that $f(u)$ is continuous and $f(u) > 0$ for all u. Let $g(x) = \int_0^x (t \int_{t^2}^1 f(u)\,du)\,dt$.

Then $g(x)$ obtains local maximum at $x =$ ___(3)___. $g(1) = \int_0^1 h(u)f(u)\,du$, where

$h(u) =$ ___(4)___.

Let $f(x) = \dfrac{a\sin^2 x}{x^3+3x+4} + \dfrac{b\cos^2 x}{x^3+x-2} + \dfrac{ab\sin x\cos x}{x^3+2x^2-x-2}$, $a > 0$, $b > 0$. Use the Intermediate Value Theorem to show that $f(x) = 0$ has at least one root in $(-1, 1)$.(*8 points*)

Let $f : (0, \infty) \to \mathbb{R}$ be differentiable. If $f(x) \to 5$ and $f'(x) \to \lambda$ as $x \to \infty$, prove that $\lambda = 0$.