9. Let $f : [a, b] \to \mathbb{R}$ is a continuous function. If for each $x \in [a, b]$ there exists $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$, show that there exists a point $c \in [a, b]$ such that $f(c) = 0$.

(15 points) Suppose that $f : [0, 1] \to \mathbb{R}$ is a continuous function such that

$$\int_0^1 (3x + 1)^n f(x) \, \mathrm{d}x = 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Show that $f(x) = 0$ for all $x \in [0, 1]$.

(4pts) Let $R_n = \sum_{i=1}^{n} \frac{n}{4n^2+i^2}$, $M_n = \sum_{i=1}^{n} \frac{n}{4n^2+(i-\frac{1}{2})^2}$, and $T_n = \frac{1}{8n} + \sum_{i=1}^{n-1} \frac{n}{4n^2+i^2} + \frac{1}{10n}$. Recognize R_n, M_n, and T_n as approximations of a definite integral, I, and compute $\lim_{n\to\infty} R_n$, $\lim_{n\to\infty} M_n$, and $\lim_{n\to\infty} T_n$.

(6pts) List R_n, M_n, T_n, and I in increasing order for any $n \in \mathbf{N}$.

(6pts) Prove Taylor's inequality: If there are positive constants M and d such that $|f''(x)| \leq M$ for $|x-a| \leq d$, then $|f(x) - f(a) - f'(a)(x-a)| \leq \frac{M}{2}|x-a|^2$ for $|x-a| \leq d$.

(6pts) In the theory of special relativity an object moving with velocity $v(m/s)$ has kinetic energy $K = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$, where m_0 is the mass of the object when at rest and $c = 3 \times 10^8(m/s)$ is the speed of light. However, in the classical Newtonian physics the kinetic energy is $K = \dfrac{1}{2}m_0v^2$. Use Taylor's inequality to estimate the difference in these expressions for K when $|v| \lesssim 10^3(m/s)$.

2. (30 pts) Let $g(x,y)$ be a function satisfying $-1 < g(x,y) < 1$ and

$$\ln\left(\frac{1+g(x,y)}{1-g(x,y)}\right) + 2y\tan^{-1}(yg(x,y)) = 2(y^2+1)x$$

for $-\infty < x < \infty, y > 1$, where $\tan^{-1} = \arctan$ maps $(-\infty, \infty)$ to $(-\pi/2, \pi/2)$.

(a) Show that $g(x,y)$ is increasing in x.

(b) Find the limit function $\bar{g}(x) = \lim_{y\to\infty} g(x,y)$.

(c) Show that $g(x,y)$ is a differentiable function in (x,y).

(d) Find $\lim_{y\to\infty} \frac{\partial}{\partial x} g(x,y)$.

In another universe, the magnitude of the gravitational force $\mathbf{G}$ is inversely proportional to the *cube* of the distance from the origin. In other words,

$$\mathbf{G}(x,y,z) = \frac{K}{(x^2+y^2+z^2)^2}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$

where K is the gravitational constant. Let $R>0$ and S_R be the sphere $x^2+y^2+z^2=R^2$, oriented outward.

(a) (2%) Explain why the Divergence Theorem cannot be applied to compute the flux of $\mathbf{G}$ across the sphere S_R.

(b) (6%) Find the flux $\iint_{S_R} \mathbf{G}\cdot d\mathbf{S}$. Express your answer in terms of K and R.

(c) (3%) Suppose it is known that $\iint_{S_5} \mathbf{G}\cdot d\mathbf{S} = 8$. Find $\iint_{S_{10}} \mathbf{G}\cdot d\mathbf{S}$.

(d) (6%) Let $U = \{(x,y,z)\in\mathbb{R}^3 : 1\le x^2+y^2+z^2\le 4\}$. Compute directly $\iiint_U \operatorname{div}(\mathbf{G})\,\mathrm{d}V$ and explain how the value of this integral and your answer in (b) are consistent with the Divergence Theorem.

8. Let f_n, g_n, h_n be sequences of functions on $\mathbb{R}$ that satisfy $f_n \leq g_n \leq h_n$ on $\mathbb{R}$ for all $n \in \mathbb{N}$. If $\sum f_n$ and $\sum h_n$ converge, show that $\sum g_n$ converges.

Problem 6. *(20 points)* A student forgot the product rule by mistake, and thought that $(fg)' = f'g'$. However, in the problem the student was solving, this happened to give the right answer. The function f that the student used was $f(x) = e^{x^2}$, and the domain of the problem was the interval $(1/2, \infty)$. Moreover, of course g was not just the zero function – that would have been a silly problem. What is an example of a function g for which the student's calculation might have given the right answer?

Hints: write out a differential equation for g. Solve the differential equation. Partial fractions may help in the final steps.

Bessel functions obey the recurrence relation $J_{n+1}(x) = -J_n'(x) + \frac{n}{x} J_n(x)$. Show by mathematical induction（數學歸納法 " if correct for n, then also correct for $n+1$"）that $J_n(x) = (-1)^n x^n \left(\frac{1}{x} \frac{d}{dx} \right)^n J_0(x)$ for any integer n. (Hint: Calculate $J_n'(x)$, the derivative of $J_n(x)$.)

Vector **B** is formed by the product of two gradients: $\boldsymbol{B} = (\nabla u) \times (\nabla \mathrm{v})$, where u and v are scalar functions.

(a) Show that $\boldsymbol{B}$ is solenoidal. (10 points)

(b) Show that $\boldsymbol{A} = \frac{1}{2}(u \, \nabla \mathrm{v} - \mathrm{v} \, \nabla u)$ is a vector potential for **B** defined above. (15 points)

Use the generating function $e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$ to show that Bessel function $J_n(x)$ has odd or even parity according to whether *n* is odd or even, namely, $J_n(x) = (-1)^n J_n(-x)$ (10 points)

Let $x = r\cos\theta$, $y = r\sin\theta$. Let $f(x, y) = g(r, \theta)$ be a differentiable function. Prove

$$\frac{\partial f}{\partial x} = \cos\theta\frac{\partial g}{\partial r} - \frac{\sin\theta}{r}\frac{\partial g}{\partial \theta}.$$