Suppose that A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ where $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$ and eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n$. Let $\boldsymbol{w}$ be a vector in R^n and $\boldsymbol{w}^T \boldsymbol{u}_1 \neq 0$ and $\boldsymbol{v}_0 = a_1 \boldsymbol{u}_1 + a_2 \boldsymbol{u}_2 + \cdots + a_n \boldsymbol{u}_n$, where $a_1 \neq 0$. Let $\beta_i = \dfrac{\boldsymbol{w}^T \boldsymbol{v}_{i+1}}{\boldsymbol{w}^T \boldsymbol{v}_i}$ $i = 0, 1, \cdots$, where $\boldsymbol{v}_k = A \boldsymbol{v}_{k-1}$ $k = 1, 2, \cdots$. Find (a) $\lim\limits_{k \to \infty} \beta_k$ and (b) $\lim\limits_{k \to \infty} \dfrac{\beta_{k+1} - \lambda}{\beta_k - \lambda_1}$ when $a_2 \neq 0$ and $|\lambda_2| > |\lambda_3|$. (10 分)

[15%] Let A and B be two $n \times n$ matrices. Then $p(t) = \det(A + tB)$ is a polynomial in t. Suppose B is invertible. Show that $p(t)$ is a polynomial of degree n.

2. Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

(a) Compute the LU factorization of $\mathbf{A}$. (10%)

(b) Explain why $\mathbf{A}$ must be positive definite. (10%)

(12%) Let $f : V \to W$ and $g : W \to Z$ be linear transformations on finite-dimensional real vector spaces V, W and Z. Show that:

(a) $\text{rank}(g \circ f) \leq \text{rank}(g)$.

(b) $\text{rank}(g \circ f) \leq \text{rank}(f)$.

2. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$, (本題共 25 分)

(1) find the eigenvalues and eigenvectors of the matrix A, (5 分)

(2) find P and P^{-1}, then diagonalize the matrix A by $D = P^{-1}AP$, (5 分)

(3) $\left[A^3 - 6A^2 + 11A - 7I\right]^3 = ?$ (5 分)

(4) If $X(t) = \Omega(t)\,C$ is the general solution of the system $X' = AX$, find the fundamental matrix $\Omega(t) = ?$ (10 分)

Let W be the subspace of R^3 defined by $W = \{(x_1\ x_2\ x_3) \mid x_1 + x_2 - 3x_3 = 0\}$. Then, a basis for W is $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, where $\boldsymbol{u}_1 = (1\ -1\ 0)^T$ and $\boldsymbol{u}_2 = (3\ 0\ 1)^T$. Let $\boldsymbol{v} = (1\ -2\ -4)^T$. Find $\boldsymbol{w}^*$ in W such that $(\boldsymbol{v} - \boldsymbol{w}^*)^T \boldsymbol{w} = 0$ for all $\boldsymbol{w}$ in W. (10 分)

(20 points)
A $n \times n$ matrix P is idempotent and symmetric. Prove that
(a) (5 points) $I_n - P$ is also an idempotent matrix, where I_n is the $n \times n$ identity matrix.
(b) (5 points) P is a positive semi-definite matrix.
(c) (5 points) If $\text{rank}[P] = r \leq n$, then it has r eigenvalues equal to unity and $n-r$ eigenvalues equal to zero.
(d) (5 points) $\text{tr}[P] = \text{rank}[P]$.

(15%) Let V be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y \rangle z$ for all $x \in V$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.