Let d_n be the determinant of the n-by-n matrix

$$\begin{bmatrix} a+b & ab & 0 & \cdots & 0 \\ 1 & a+b & ab & & \\ & 1 & a+b & ab & \\ & & \ddots & \ddots & ab \\ & & & 1 & a+b \end{bmatrix}, \text{ where } a \neq b.$$

(a) Find d_1, d_2 and a recurrence relation of d_n. (10%)

(b) Use the diagonalization of a 2-by-2 matrix to find a general expression of d_n. (10%)

(14%) Lct V be a complex vector space of finite dimension with a fixed positive definite hcrmitian inner product $<,>$. Lct $f : V \rightarrow V$ bc a lincar transformation ovcr $\mathbb{C}$ such that $< f(v), v >= 0$. Show that f is a zcro map.

(20%) Let V denote a vector space over the complex number field $\mathbb{C}$ endowed with an inner product $\langle\, ,\, \rangle$. Recall that the norm of a vector $v \in V$ is defined as

$$||v|| = \sqrt{\langle v, v \rangle}.$$

Let W denote a finite-dimensional subspace of V. Prove that there exists a unique linear operator $P : V \to W$ such that

$$||v - P(v)|| \le ||v - w|| \qquad \text{for all } v \in V \text{ and } w \in W.$$

Let the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -9 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

(a) (15 %) Find a Jordan canonical form J of A and an invertible matrix P such that $P^{-1}AP = J$.

(b) (10 %) Consider $\mathcal{A} = \{A^n \mid n = 0, 1, 2, \ldots\}$ and let Span($\mathcal{A}$) be the subspace of $M_{n \times n}(\mathbb{R})$ spanned by $\mathcal{A}$ over the field of real numbers $\mathbb{R}$. Determine the dimension of Span($\mathcal{A}$).

$\mathcal{S}_n$ is the space of $n \times n$ square matrices.

[20%] $A \in \mathcal{S}_n(\mathbb{C})$. Over $\mathbb{C}$, show the following two statements are equivalent.

a. The characteristic polynomial of A is equal to minimal polynomial of A.

b. For any $X \in \mathcal{S}_n(\mathbb{C})$ satisfies $XA = AX$, X is a polynomial of A.

Let A be a complex square matrix with

nullity $(A - 2I) = 3$,
nullity $(A - 2I)^2 = 5$,
nullity $(A - 2I)^3 = 7$,
nullity $(A - 2I)^k = 8$ for $k \geqslant 4$,
nullity $(A + 3iI) = 3$, and
nullity $(A + 3iI)^k = 5$ for $k \geqslant 2$.

(a) Find the algebraic and geometric multiplicities for each eigenvalue of A. (10%)

(b) Find the Jordan canonical form of A. (5%)

(c) Find the determinant and trace of A. (5%)