(7) (10%) Prove or disprove that if $\{A_1, A_2, A_3\}$ is a linearly independent subset of $M_{2\times 2}(\mathbb{R})$, then $\{A_1^2 . A_2^2, A_3^2\}$ is also linear independent.

(15%) Describe explicitly a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ which has its range the subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$. Give a basis for the null space for such linear transformation.

(3) Let $A \in M_{m\times n}(F)$ and $B \in M_{n\times n}(F)$. Prove that

(a) (5%) $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

(b) (5%) $\operatorname{rank}(AB) = \operatorname{rank}(B)$ provided that B is invertible.

Prove that if $x_1, x_2, \ldots, x_n$ are mutually orthogonal nonzero vectors in an inner-product space, then they are linearly independent.

(12 points) An inconsistent system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ is given where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. Use the method of least squares to obtain the vector $\boldsymbol{z}$ for which $\|A\boldsymbol{z} - \boldsymbol{b}\|$ is a minimum.

(7%) Consider an $m \times m$ matrix P, an $n \times n$ matrix Q, an $m \times n$ matrix A, and an $n \times p$ matrix B. Which of the following statements are true?

(A) Null A is a subspace of Null PA.
(B) Null PA is a subspace of Null A.
(C) Null AQ may not be a subspace of Null A and Null A may not be a subspace of Null AQ.
(D) Col A is contained in Col AB.
(E) The nullity of B is no greater than the nullity of AB.

(10%) Find points on the surface $x^2 + 6x + y^2 + z^2 - 2z = 15$ at which the tangent plane is horizontal.

(10%) Assume that A, B, and C are $n \times n$ matrices.

(A) trace((A+B)C) = trace(AC) + trace(BC)

(B) trace(AB) = trace(A)trace(B)

(C) trace(AB)=trace(BA)

(D) Given a 2×2 matrix, $\begin{bmatrix} a & b+c \\ b-c & -a \end{bmatrix}$ with trace zero has real eigenvalues if $a^2 + b^2 \geq c^2$.

(E) Suppose a 2×2 matrix other than the identity matrix satisfies $A^3 = I$. Then we have trace(A) = 1.

Determine whether

$$\lim_{m \to \infty} A^m$$

exists for each of the following matrices A, and compute the limit if it exists.

(6%) (a)

$$A = \begin{pmatrix} -2 & -1 \\ 4 & 3 \end{pmatrix}$$

(6%) (b)

$$A = \begin{pmatrix} 0.1 & 0.7 \\ 0.7 & 0.1 \end{pmatrix}$$

(12%) Let W_1 and W_2 be subspaces of a vector space. Then V is called the direct sum of W_1 and W_2 provided that

$$V = W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1 \text{ and } w_2 \in W_2\}$$

and

$$W_1 \cap W_2 = \{0\}.$$

Prove that V is the direct sum of W_1 and W_2 if and only if each vector $v \in V$ can be uniquely written as $v = w_1 + w_2$, where $w_i \in W_i$, $i = 1$ and 2.

(15%) Let

$$A = \begin{pmatrix} -3 & -2 & 0 & 0 \\ -2 & 1 & 8 & 0 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

denote a 4×4 matrix over $\mathbb{Q}$. Determine if A is diagonalizable and give a proof for your answer.

Let

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(a) (6%) Find the eigen values of A.

(b) (6%) Find the eigen vectors of A.

(c) (5%) Find an orthogonal matrix B that diagonalizes A.

(d) (4%) Find the inverse matrix of B obtained in (c).

(e) (4%) Find A^5.