(2) (10%) Let $S = \{(1,1,0),(1,0,1),(0,1,1)\}$ be a subset of the vector space $\mathbb{R}^3$. Prove that $S$ is linearly independent.

(7) (4%) Prove or disprove that the product of two matrices always has rank equal to the lesser of the ranks of the two matrices.

(25%) Let $T$ be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_1, x_1)$.

(a) What is $[T]_\alpha$? Her $\alpha$ is the standard ordered basis for $\mathbb{R}^2$.

(b) What is $[T]_\beta$? Here $\beta = \{(1,2),(1,-1)\}$.

Let $I$ be the identity transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, i.e., $I(x_1, x_2) = (x_1, x_2)$.

(c) What is $[I]_\alpha^\beta$?

(d) Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if there exists an $n \times n$ matrix $B$ such that $AB = BA = I$. Prove that if $A$ is invertible, then $B$, described as above, is unique. (Such $B$ is to be denoted by $A^{-1}$).

(e) Find a $2 \times 2$ matrix $Q$ such that $Q^{-1}[T]_\alpha Q = [T]_\beta$. (Hint: $T = I(TI)$.)

Let $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by $T(a + bx + cx^2) = (a + b, c, a - b)$. Let $\beta$ and $\gamma$ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively. Compute $[U]_\beta^\gamma$.

(15%) Assume matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and $|A| = -5$. Find

(1) $|3A|$ (3 分)； (2) $|A^{-1}|$ (3 分); (3) $|2A^{-1}|$ (3 分);

(4) $|(2A)^{-1}|$ (3 分); (5) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$ (3 分)

(5%) Find the projection matrix $P$ onto the column space of $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}$.

(12%) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(5) Assume that $A \in \mathbb{R}^{m \times n}$

a. Let $x \in \mathbb{R}^n$, show that if $(A^T A)x = 0$, then $Ax = 0$. (10 points)

b. Show that Rank $(A^T A)$ =Rank $A$. (10 points)

Which of the following statements are true?

(A) $tr(AB) = tr(A)tr(B)$

(B) $rank(A) = rank(A'A)$

(C) $det(cA) = c\,det(A)$

(D) If $A^2 = A$, then the eigenvalues of $A$ have to be 1, where $A$ is an $n \times n$ matrix over real numbers.

(E) If the columns of $A$ are linearly independent, then the rows of $A$ are also linearly independent, where $A$ is an $n \times n$ matrix.

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $A$ be its standard matrix, that is, $T\vec{x} = A\vec{x}$ for each $\vec{x} \in \mathbb{R}^n$. Please show that the range of $T$ is just the column space of $A$. (10%)

Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

(a) (5%) Find the rank of $A$.

(b) (5%) Find the nullspace of $A$.

(10 pts) Let $\mathbf{u}$ and $\mathbf{v}$ be two orthogonal unit vectors.

(a) (5 pts) Compute the inner product of $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

(b) (5 pts) Find the length of $\mathbf{u} - \mathbf{v}$.

Let A,B,C are $n x n$ matrices and BA=CA. Show that if $\det(A) \neq 0$, then B=C. (7%)

If a linear transformation from a vector space V to another vector space W is one-to-one, which of the following statements is/are true?

(A) It is onto;
(B) $\dim(V) = \dim(W)$;
(C) The null space of this transformation contains only the zero vector;
(D) It is invertible;
(E) All of the above.

Determine whether the two matrices below are similar to each other. Give your reasons.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(20%) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$. Find (1) the characteristic polynomial (5 分);

(2) eignevalues and associated eignevectors (10 分);

(3) Find the transformation matrix P and diagonal matrix D such that $P^{-1}AP = D$ (5 分).