9. Let $f:[a,b]\to\mathbb{R}$ is a continuous function. If for each $x\in[a,b]$ there exists $y\in[a,b]$ such that $|f(y)|\leq\frac{1}{2}|f(x)|$, show that there exists a point $c\in[a,b]$ such that f(c)=0.

(15 points) Suppose that $f:[0,1]\to\mathbb{R}$ is a continuous function such that

$$\int_0^1 (3x+1)^n f(x) dx = 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Show that f(x) = 0 for all $x \in [0, 1]$.

(4pts) Let $R_n = \sum_{i=1}^n \frac{n}{4n^2+i^2}$, $M_n = \sum_{i=1}^n \frac{n}{4n^2+(i-\frac{1}{2})^2}$, and $T_n = \frac{1}{8n} + \sum_{i=1}^{n-1} \frac{n}{4n^2+i^2} + \frac{1}{10n}$. Recognize R_n , M_n , and T_n as approximations of a definite integral, I, and compute $\lim_{n\to\infty} R_n$, $\lim_{n\to\infty} M_n$, and $\lim_{n\to\infty} T_n$.

(6pts) List R_n , M_n , T_n , and I in increasing order for any $n \in \mathbb{N}$.

(6pts) Prove Taylor's inequality: If there are positive constants M and d such that $|f''(x)| \le M$ for $|x-a| \le d$, then $|f(x)-f(a)-f'(a)(x-a)| \le \frac{M}{2}|x-a|^2$ for $|x-a| \le d$.

(6pts) In the theory of special relativity an object moving with velocity v(m/s) has kinetic energy $K = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2$, where m_0 is the mass of the object when at rest and $c = 3 \times 10^8 (m/s)$ is the speed of light. However, in the classical Newtonian physics the kinetic energy is $K = \frac{1}{2}m_0v^2$. Use Taylor's inequality to estimate the difference in these expressions for K when $|v| \leq 10^3 (m/s)$.

2. (30 pts) Let g(x,y) be a function satisfying -1 < g(x,y) < 1 and

$$\ln(\frac{1+g(x,y)}{1-g(x,y)}) + 2y \tan^{-1}(yg(x,y)) = 2(y^2+1)x$$

for $-\infty < x < \infty, y > 1$, where $\tan^{-1} = \arctan \max (-\infty, \infty)$ to $(-\pi/2, \pi/2)$.

(a) Show that g(x, y) is increasing in x.

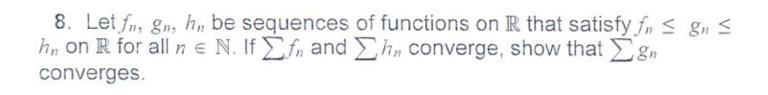
(b) Find the limit function ḡ(x) = lim_{y→∞} g(x, y).
(c) Show that g(x, y) is a differentiable function in (x, y).
(d) Find lim_{y→∞} ∂/∂x g(x, y).

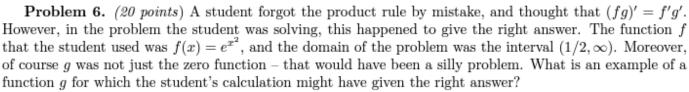
In another universe, the magnitude of the gravitational force G is inversely proportional to the *cube* of the distance from the origin. In other words,

$$G(x, y, z) = \frac{K}{(x^2 + y^2 + z^2)^2} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

where K is the gravitational constant. Let R > 0 and S_R be the sphere $x^2 + y^2 + z^2 = R^2$, oriented outward.

- (a) (2%) Explain why the Divergence Theorem cannot be applied to compute the flux of G across the sphere S_R .
- (b) (6%) Find the flux $\iint_{S_R} \mathbf{G} \cdot d\mathbf{S}$. Express your answer in terms of K and R.
- (c) (3%) Suppose it is known that $\iint_{S_5} \mathbf{G} \cdot d\mathbf{S} = 8$. Find $\iint_{S_{10}} \mathbf{G} \cdot d\mathbf{S}$.
- (d) (6%) Let $U = \{(x, y, z) \in \mathbb{R}^3 : 1 \le x^2 + y^2 + z^2 \le 4\}$. Compute directly $\iiint_U \operatorname{div}(\mathbf{G}) \, dV$ and explain how the value of this integral and your answer in (b) are consistent with the Divergence Theorem.





Hints: write out a differential equation for g. Solve the differential equation. Partial fractions may help in the final steps.

Bessel functions obey the recurrence relation $J_{n+1}(x) = -J'_n(x) + \frac{n}{x}J_n(x)$. Show by mathematical induction (數學歸納法 "if correct for n, then also correct for n+1") that $J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$ for any integer n. (Hint: Calculate $J'_n(x)$, the derivative of $J_n(x)$.)

Vector B is formed by the product of two gradients: $B = (\nabla u) \times (\nabla v)$, where u and v are scalar functions.

- (a) Show that B is solenoidal. (10 points)
- (b) Show that $A = \frac{1}{2}(u \nabla v v \nabla u)$ is a vector potential for B defined above. (15 points)



Let $x=r\cos\theta$, $\,y=r\sin\theta.$ Let $f(x,y)=g(r,\theta)$ be a differentiable function. Prove

$$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial g}{\partial r} - \frac{\sin\theta}{r} \frac{\partial g}{\partial \theta}.$$