Let $f(x,y,z)$ be a scalar field and $\mathbf{G}(x,y,z)$ be a vector field, both smooth (that is, partial derivatives exist in any order). Let D be a solid region in $\mathbb{R}^3$ with boundary surface ∂D oriented outward.

(a) (4%) Prove that $\operatorname{div}(f\mathbf{G}) = \nabla f \cdot \mathbf{G} + f \operatorname{div}(\mathbf{G})$.

(b) (2%) Prove that $\iiint_D \nabla f \cdot \mathbf{G}\, \mathrm{d}V = \iint_{\partial D} f\mathbf{G} \cdot \mathrm{d}\mathbf{S} - \iiint_D f \operatorname{div}(\mathbf{G})\, \mathrm{d}V$.

(c) (5%) Let $f(x,y,z) = 4 - x^2 - y^2 - z^2$ and $\mathbf{G}(x,y,z) = \sin(y+1)\,\mathbf{i} + e^{x+1}\,\mathbf{j} + z^3\,\mathbf{k}$. Use (b) to evaluate

$$\iiint_D \nabla f \cdot \mathbf{G}\, \mathrm{d}V \text{ where } D = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 4\}.$$

[8%] Suppose $y > 0$. Show that $\int_0^\infty e^{-xy} \sin x \, dx = \frac{1}{1+y^2}$. (Hint: use the integration by parts.)

[7%] Use (i) and $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ to find the function $F(y) = \int_0^\infty e^{-xy} \frac{\sin x}{x} dx$.

(20%) For a second continuously differentiable function, $f(x,y,z)$, defined on the space $\mathbb{R}^3$, its Laplacian is defined to be

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,.$$

Derive the formula of Δf in the spherical coordinate system.

(10%) Let D be the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the xy-plane. Find

$$\int\int_S F \cdot \vec{n}\, dS$$

where S is the boundary of D with the unit normal vector $\vec{n}$ directed outward from D and $F(x,y,z) = (x^2 + \sin z)i + (xy + \cos z)j + e^y k$.

Consider the 'pringle' surface S (see **Figure 1**) which is part of the graph $z = xy$ inside the cylinder $x^2 + y^2 = 2$. Let C be the boundary of S, counterclockwisely oriented when viewed from above.

(a) (4%) Set up, but do not evaluate, a definite integral that computes the line integral $\int_C z^2\, \mathrm{d}s$.

(b) (6%) Find the surface area of S.

(c) (7%) Use Stokes' Theorem to evaluate

$$\oint_C (x^3 + y)\, \mathrm{d}x - 2xy\, \mathrm{d}y + (1 + \sin(z^{10}))\, \mathrm{d}z.$$

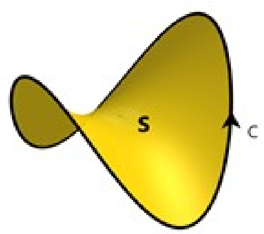

Figure 1. Pringle surface

9.(10 pts) let B be the region in the first quadrant of $\mathbb{R}^2$ bounded by the curve $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$. Find the value of the double integral

$$\iint_B (x^2 + y^2) dxdy.$$

设函数$f(x)$具有 2 阶导数，且$f'(0) = f'(1), |f''(x)| \le 1$，证明:

(1) 当$x \in (0,1)$时，$|f(x) - f(0)(1-x) - f(1)x| \le \frac{x(1-x)}{2}$

(2) $\left| \int_0^1 f(x)dx - \frac{f(0)+f(1)}{2} \right| \le \frac{1}{12}$

Given the iterated triple integral

$$\int_0^1 \int_0^1 \int_0^1 f(x, y, z)\, dx\, dy\, dz,$$

please rewrite the integral as an equivalent iterated integral in cylindrical coordinates (*6 points*) and spherical coordinates (*16 points*), respectively.

Let f be continuous ob $\mathbb{R}^3$. Rewrite the following integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x, y, z)\, dy\, dx\, dz$$

as an equivalent iterated integral in the five other orders. (*10 points*)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } x = y = 0. \end{cases}$$

(a) (12 points) Show that all second order partial derivatives of f exist everywhere.

(b) (3 points) Is it true that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$? Justify your answer.

Evaluate

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-x^2}}{1 + x^2 + 2xy + y^2} dxdy.$$