Notes on Manifolds in Physics

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These notes are my way of learning Riemannian Geometry and better understanding Lagrangian Mechanics and General Relativity. As a physicist one usually learns all of this in a rather practical way without understanding the basic mathematical concepts. For example a physicist usually does not learn that the Lagrangian lives on the tangent bundle, because one implicitly always identifies some spaces (here the space and the tangent space), which is possible on a flat manifold. Another example: To understand that for vector fields on a manifold there generally does not exist a global basis one has to understand what a module is. I start from the lectures about General Relativity. This somehow turned into my fancy lecture notes. The plan is to go on and write more stuff after I have finished the lectures. I started this in Frankfurt, but then moved to Alabama, since I am not sure what affiliation to put here I just put all of them, probably nobody will care anyways.

Contents

1 Manifolds

1.1 Topology

This chapter is basically taken from (Schuller, 2015) with our remarks to it.

We start with a set M which is supposed to be the space where physics happens. The weakest structure one needs in order to talk about *continuity* (of curves or fields) is called a topology.

***Definition* 1.1** (Power set $\mathcal{P}$)
The set of all subsets of M.

***Definition* 1.2** (Topology)
A Topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$ satisfying:

1. $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$,
2. $U \in \mathcal{O}, \quad V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$
3. $U_\alpha \in \mathcal{O}, \quad \alpha \in A \Rightarrow \left(\bigcup_{\alpha \in A} U_\alpha \right) \in \mathcal{O}$

Every set has the *chaotic topology*

$$\mathcal{O}_{\text{chaotic}} := \{\emptyset, M\} \,, \tag{1}$$

and the *discrete topology*

$$\mathcal{O}_{\text{discrete}} := \mathcal{P}(M)\,, \tag{2}$$

which are both useless. The special case $M = \mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ has a standard topology for which we need the definition of a soft ball.

***Definition* 1.3** (Soft Ball in $\mathbb{R}^d$)

$$B_r(p) := \left\{ (q_1, \cdots, q_d) \middle| \sum_{i=1}^{d} (p_i - p_i)^2 < r^2 \right\} \,, \tag{3}$$

with $r \in \mathbb{R}^+$, $p \in \mathbb{R}^d$. Note: This does not need a norm or vector space structure on $\mathbb{R}^d$.

***Definition* 1.4** ($\mathcal{O}_{\text{standard}}$ on $\mathbb{R}^d$)

$$U \in \mathcal{O}_{\text{standard}} :\Leftrightarrow \forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U \tag{4}$$

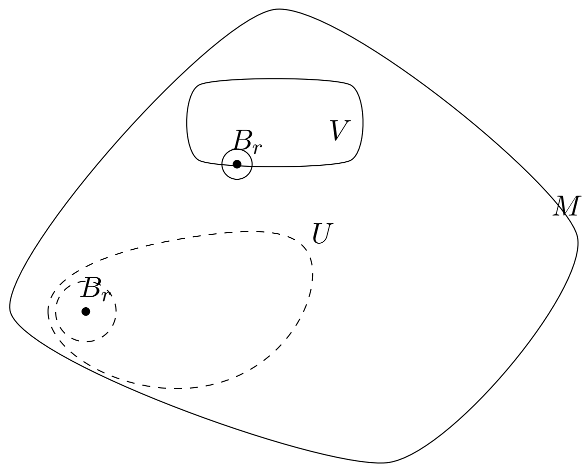


Figure 1: *The set U is in the standard topology, V not.*

Some terminology: Let M be a set with a topology $\mathcal{O} =$: set of open sets. We call $(M, \mathcal{O}$ a *topological space* and:

- $U \in \mathcal{O} \Leftrightarrow$: call $U \subseteq M$ an *open set*
- $M \backslash A \in \mathcal{O} \Leftrightarrow$: call $A \subseteq M$ a *closed set*

Note: The empty set is open and closed. If a set is open we cannot directly follow that it is not closed or vise versa. For $M = \{1, 2\}$ and $\mathcal{O}_M = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ the set $\{2\}$ is open and closed.

1.2 Continuous Maps

A map

$$f : M \to N \,, \tag{5}$$

takes every point from the domain M (a set) to the target N (a set). If at least one point $p \in N$ is not reached, the map is not *surjective*. If at least one point is hit twice, the map is not *injective*. A map that is injective and surjective is called *surjective*.

***Definition* 1.5** (Preimage)

$$\begin{aligned} f : M \to N \supseteq V \\ \text{preim}_f(V) := \{m \in M \mid f(m) \in V\} \end{aligned} \tag{6}$$

***Definition* 1.6** (Continuity)
$(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ topological spaces. Then a map $f : M \to N$ is called *continuous with respect to $\mathcal{O}_M$ and $\mathcal{O}_N$* if

$$\boxed{\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M} \,. \tag{7}$$

"A map is open iff the preimages of all open sets are open sets."

Note: If a map is not surjective there are sets with preimage $\emptyset$, thus we need to have $\emptyset$ in $\mathcal{O}$, otherwise only surjective maps could be continuous.

Note: The inverse of a continuous function does not need to be continuous.

***Definition* 1.7** (Composition of maps)
For f and g

$$f : M \to N, \quad g : N \to P \,,$$

we define the *composition* as

$$\begin{aligned} g \circ f : M \to P \\ m \mapsto (g \circ f)(m) := g(f(m)) \end{aligned} \tag{8}$$

***Theorem* 1.8** (Composition of continuous maps)
For f, g continuous also $g \circ f$ is continuous (if space match, *i.e.* $g \circ f$ is defined).

***Definition* 1.9** (Subset topology, Inherited topology)
A set M with topology $\mathcal{O}_M$. Given any subset $S \subseteq M$ we can construct the inherited topology $\mathcal{O}\big|_S \subseteq \mathcal{P}(S)$

$$\mathcal{O}\big|_S := \{U \cap S \mid U \in \mathcal{O}M\} \,. \tag{9}$$

Note: For $S \subseteq M$, if f is continuous then $f\big|_S$ is also continuous if $\mathcal{O}\big|_S$ is chosen. This is for example important if you are on a trajectory γ through $\mathbb{R}^n$ and measure the temperature $T\big|_\gamma$.

***Definition* 1.10** (Topological manifold)
A topological space $(\mathcal{M}, \mathcal{O})$ is called a *d-dimensional topological manifold* if

$$\forall p \in \mathcal{M} : \exists U \in \mathcal{O},\ p \in U : \exists x : U \to x(U) \subseteq \mathbb{R}^d \,, \tag{10}$$

with the following properties (wrt. $\mathcal{O}_{\text{std}}$ on $\mathbb{R}^d$):

1. x intervitble: $x^{-1} : x(U) \to U$,
2. x continuous,
3. x^{-1} continuous.

"Invertible, in both directions continuous map to $\mathbb{R}^n$."

Note: Thus in the above definition $x(U)$ is also open (from the definition of continuity).

Terminology:

- (U, x) is a *chart* of $\mathcal{M}, \mathcal{O}$,
- $\mathcal{A} = \{(U_{(\alpha)}, x_{(\alpha)} | \alpha \in A\}$ is an *atlas* of $(\mathcal{M}, \mathcal{O})$ if $\bigcup_{\alpha \in A} U_{(\alpha)}$ covers the whole manifold $\mathcal{M}$,
- $x : U \to x(U) \subseteq \mathbb{R}^d$ is a *chart map* $x(p) = (x^1(p), \ldots, x^d(p))$, where the *component maps* $x^i : U \to \mathbb{R}$ are called *coordinate maps*,
- $p \in U$, then $x^1(p)$ is the first coordinate of the point p wrt. the chosen chart (U, x).

Note: The choice of the chart (choice of coordinates) has nothing to do with the physics. Physics is chart independent. $\mathcal{M}$ is "the real world".

1.3 Chart Transition Maps

Given (U, x) and (V, y) charts, on $U \cup V$ one can transition from one chart to the other by (see figure 2)

$$y \circ x^{-1} : \mathbb{R}^d \supseteq x(U \cap V) \to y(U \cap V) \subseteq \mathbb{R}^d \,, \tag{11}$$

which is called the *chart transition map*.

Note: As a physicist one talks about a "change in coordinates".

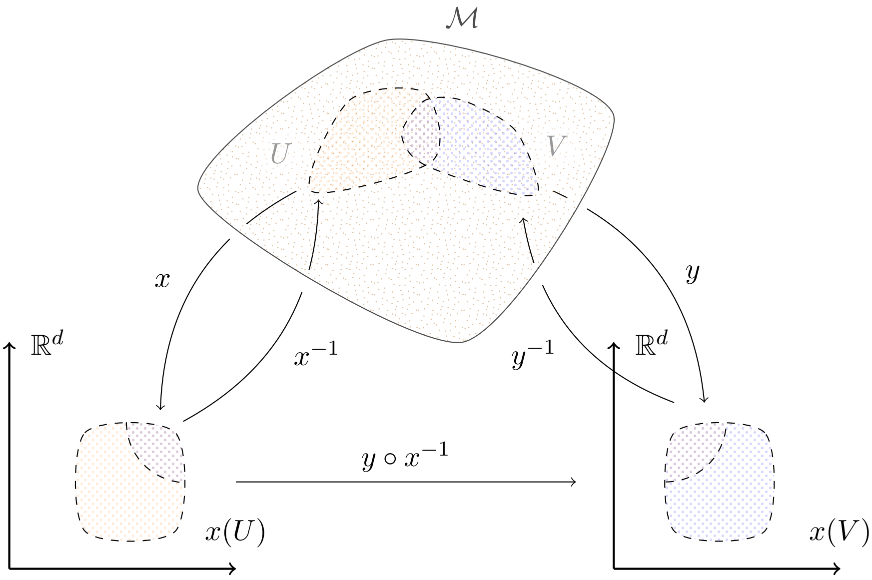


Figure 2: *Visualization of chart transition maps. "How to glue together the charts of an atlas." Plot modified from (QuantumMechanic, n.d.)*

1.4 Manifold Philosophy

The idea is to define properties of some object in the real world $\mathcal{M}$ by at a chart-representative of it. For example the continuity of a curve $\gamma : [0,1] \to \mathcal{M}$ can be judged by looking at $x \circ \gamma : [0,1] \to \mathbb{R}^d$, because x is invertible and in both directions continuous and the composition of two continuous maps is also continuous.

Note: One needs to make sure that the property of the object on $\mathcal{M}$ does not depend on the map x or y. For continuity this is the case, since $y \circ \gamma = (y \circ x^{-1}) \circ x \circ \gamma$ and the chart transition map $y \circ x^{-1}$ is also continuous.

Other properties like "differentiability" are not even defined on $\mathcal{M}$ a priori, so one can only talk about the chart representative. Here the definition that γ is differentiable iff $x \circ \gamma : [0,1] \to \mathbb{R}^d$ is differentiable has the problem that x and y only need to be continuous and so the chart transition map $y \circ x^{-1}$ does not need to be differentiable unless one restricts oneself to only differentiable charts.

2 Vector Spaces

2.1 Vectors and Linear Maps

In order to understand the tangent space we need to understand vector spaces.

***Definition* 2.1** (Vector space $(V, +, \cdot)$)
A vector space $(V, +, \cdot)$ is a set V with

- an "addition" $+ : V \times V\ toV$,
- an "S-multiplication" $\cdot : \mathbb{R} \times V\ toV$

and the properties CANI ADDU:
$\forall v, w, u \in V, \lambda, \mu \in \mathbb{R}$

C$^+$: $v + w = w + v\,,$
A$^+$: $(u + v) + w = u + (v + w)\,,$
N$^+$: $\exists 0 \in V : \forall v \in V : v + 0 = v\,,$
I$^+$: $\forall v \in V : \exists (-v) \in V : v + (-v) = 0\,,$
A: $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v\,,$
D: $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v\,,$
D: $\lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w)\,,$
U: $1 \cdot v = v\,.$

An element of a vector space is called a *vector*.

Note: The addition $+$ in definition 2.1 sometimes is between vectors and sometimes between scalars. It is important to know the difference.

***Definition* 2.2** (Linear maps)
(Structure respecting maps between vector spaces)
$(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ vector spaces. A map

$$\phi : V \to W \tag{12}$$

is called *linear* if $\forall v, \tilde{v} \in V, \forall \lambda \in \mathbb{R}$

1. $\phi(v +_V + \tilde{v}) = \phi(v) +_W \phi(\tilde{v})$
2. $\phi(\lambda \cdot_V v) = \lambda \cdot_W \phi(v)$

We write:

$$\phi : V \to W \text{ linear} \Leftrightarrow: \phi : V \xrightarrow{\sim} W \,. \tag{13}$$

***Theorem* 2.3** (Transitivity of linearity of maps)
V, W, U vector spaces, $\psi : V \xrightarrow{\sim} W$, $\phi : W \xrightarrow{\sim} U$ then $\phi \circ \psi$ is also linear: $\phi \circ \psi : V \xrightarrow{\sim} U$.

***Definition* 2.4** (Homomorphisms $\mathrm{Hom}(V, W)$)

$$\mathrm{Hom}(V, W) := \left\{ \phi : V \xrightarrow{\sim} W \right\} \,. \tag{14}$$

Note: $\mathrm{Hom}(V, W)$ can be made into a vector space by defining an addition and a multiplication

- $(\phi + \psi)(v) := \phi(v) + \psi(v)\,,$
- $(\lambda\psi)(v) := \lambda(\psi(v))\,.$

***Definition* 2.5** (Dual vector space $V^\star$)

$$V^\star := \left\{ \phi : V \xrightarrow{\sim} \mathbb{R} \right\} = \mathrm{Hom}(V, \mathbb{R}) \,. \tag{15}$$

The vector space $(V^\star, +, \cdot)$ is the *dual vector space* to V. $\phi \in V^\star$ is informally called a *covector*.

***Definition* 2.6** ((r, s) - Tensors)
$(V, +, \cdot)$ vector space, $r, s \in \mathbb{N}_0$. An (r, s)-tensor T over V is a multi-linar map

$$T : \overbrace{V^\star \times \cdots \times V^\star}^{r} \times \overbrace{V \times \cdots \times V}^{s} \xrightarrow{\sim} \mathbb{R} \tag{16}$$

***Theorem* 2.7** (Covector is (0,1)-tensor)

$$\phi \in V^\star \Leftrightarrow \phi : V \xrightarrow{\sim} \mathbb{R} \Leftrightarrow \phi \, (0,1) \text{ tensor} \,. \tag{17}$$

***Theorem* 2.8** (Vector is (1,0)-tensor)
If $\dim V < \infty$

$$v \in V = (V^\star)^\star \Leftrightarrow v : V^\star \xrightarrow{\sim} \mathbb{R} \Leftrightarrow v \text{ is } (1,0)\text{−tensor} \,. \tag{18}$$

2.2 Bases

***Definition* 2.9** (Hamel-basis)
$(V, +, \cdot))$ vector space. A subset $B \subset V$ is called a Hamel-basis if

$$\forall v \in V \; \exists! \text{ finite } F = \{f_1, \ldots, f_n\} \subset B : \exists! \underbrace{v^1, \cdots, v^n}_{\in \mathbb{R}}, \tag{19}$$

such that

$$v = v^1 f_1 + \cdots + v^n f_n \,. \tag{20}$$

(and all f_i linearly independent).

***Definition* 2.10** (Dimension of a vector space)
If a basis B with $d < \infty$ many elements, then we call $d =: \dim V$.

If we have chosen a basis $\{e_1, \ldots, e_n\}$ of $(V, +, \cdot)$ then $(v^1, \ldots, v^n)$ are called the *components of* V w.r.t. the chosen basis if

$$v = v^1 e_1 + \cdots v^n e_n \,. \tag{21}$$

***Definition* 2.11** (Dual basis)
Choose basis $\{e_1, \cdots, e_n\}$ for V. The basis $\{\epsilon^1, \cdots, \epsilon^n\}$ for $V^\star$ can be chosen that

$$\epsilon^a(e_b) = \delta^a_b \quad \forall a, b = 1, \ldots, n \,. \tag{22}$$

$\{\epsilon^1, \ldots, \epsilon^n\}$ is then called *the dual basis* of the dual space.

2.3 Components of a tensor

T an (r, s)-tensor. Then the real numbers

$$T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} = T\left(\epsilon^{i_1}, \ldots, \epsilon^{i_r}, e_{j_1}, \ldots e_{j_s}\right) \tag{23}$$

are the components of T with respect to the chosen basis. From the components and the basis one can reconstruct the entire tensor: Example for (1, 1)-tensor:

$$T(\varphi, v) = T^i{}_j \, \varphi_i v^j \tag{24}$$

where φ_i are the components of $\varphi \in V^\star$ and v^j the components of $v \in V$ with respect to the chosen basis. In equation (24) the *Einstein summation convention* is used, *i.e.* an index that appears up and down in an expression is summed over.

Note: The Einstein summation convention is only useful because we are working with *linear maps*, otherwise the expression

$$\varphi\left(\sum_i v^i e_i\right) = \sum_i \varphi(v^i e_i) \,, \tag{25}$$

would not hold and with the summation index we would not know where the sum sign goes.

3 Differentiable Manifolds

So far we only had topological manifolds. We also want to be able to talk about the velocity of curves. The problem is that the notion of a topological manifold is not enough to define differentiability of curves. In this section we will find out what additional structure we need to be able to talk about the differentiability of

- curves: $\mathbb{R} \to \mathcal{M}$
- functions: $\mathcal{M} \to \mathbb{R}$
- maps: $\mathcal{M} \to \mathcal{N}$

Strategy: Choose a chart (U, x) and consider portion of the curve in the domain of the chart: $\gamma : \mathbb{R} \to U$ (see figure 3). Since $x \circ \gamma : \mathbb{R} \to \mathbb{R}^d$ we can try to "lift" the notion of differentiability of a curve on $\mathbb{R}^d$ to that of a curve on $\mathcal{M}$. The problem is to make this independent of the chart.

$$\begin{tikzcd}
& y(U \cup V) \subseteq \mathbb{R}^d & \\
\gamma : \mathbb{R} \arrow[ur, "y \circ \gamma"] \arrow[r, "\gamma"] \arrow[dr, "x \circ \gamma"'] & U \cup V \neq \emptyset \arrow[u, "y"'] \arrow[d, "x"] & \\
& x(U \cup V) \subseteq \mathbb{R}^d \arrow[u, bend right=60, "y \circ x^{-1}"'] &
\end{tikzcd}$$

Figure 3: *Curve γ in chart.*

$$y \circ \gamma = \underbrace{(y \circ x^{-1})}_{\mathbb{R}^d \to \mathbb{R}^d} \circ \underbrace{(x \circ \gamma)}_{\substack{\mathbb{R} \to \mathbb{R}^d \\ \text{differentiable}}} , \tag{26}$$

but we only know that the *chart transition map* $y \circ x^{-1}$ is continuous (because of the definition of a top. Manifold). Thus it is not guaranteed that $y \circ \gamma$ is continuons, not differentiable. Reminder: The composition of continuous maps is continuous, same for differentiable. The above definition of differentiability of γ by checking the differentiability of $x \circ \gamma$ with some chart x is not independent of the chart.

***Definition* 3.1** ($\star$ - compatibility of charts)
Two charts (U, x) and (V, y) of a topological manifold are called $\star$-compatible if either

1. $U \cup V = \emptyset$ or
2. $U \cup V \neq \emptyset$ and the chart transition maps

$$y \circ x^{-1} : \mathbb{R}^d \supseteq x(U \cup V) \to y(U \cup V) \subseteq \mathbb{R}^d$$
$$x \circ y^{-1} : \mathbb{R}^d \supseteq y(U \cup V) \to x(U \cup V) \subseteq \mathbb{R}^d$$

have the $\star$-property in the $\mathbb{R}^d$-sense.

***Definition* 3.2** ($\star$-compatible atlas)
An atlas $\mathcal{A}_\star$ is a $\star$-compatible atlas if any two charts in $\mathcal{A}_\star$ are $\star$-compatible.

***Definition* 3.3** ($\star$-manifold)
A $\star$-manifold is a triple $(\underbrace{\mathcal{M}, \mathcal{O}}_{\text{top. manif.}}, \underbrace{\mathcal{A}_\star}_{\in \mathcal{A}_{\text{max}}})$.

$\star$	$\star$ property in $\mathbb{R}^d$-sense
C^0	$C^0(\mathbb{R}^d \to \mathbb{R}^d)$ continuous maps
C^1	$C^1(\mathbb{R}^d \to \mathbb{R}^d)$ differentiable and result is cont.
C^k	$C^k(\mathbb{R}^d \to \mathbb{R}^d)$ k times diffble and result is cont.
D^k	$D^k(\mathbb{R}^d \to \mathbb{R}^d)$ k times differentiable
C^∞	$C^\infty(\mathbb{R}^d \to \mathbb{R}^d)$ smooth functions
C^ω	$\exists$ multidim. Taylor expansion, $C^\omega \subset C^\infty$

Note: The more fancy properties one wants for the objects on the manifold, the more restrictive one has to be for the atlas.

***Theorem* 3.4** ($C^1 \to C^\infty$)
Any $C^{k \leq 1}$-manifold atlas $\mathcal{A}_{C^{k \leq 1}}$ of a topological manifold contains a C^∞-atlas.

Thus we may without loss of generality always consider C^∞-manifolds. "smooth" manifolds, unless we wish to define Taylor expandibility or complex differentiability, ...

***Definition* 3.5** (Smooth manifold)
$(\mathcal{M}, \mathcal{O}, \mathcal{A})$, where $(\mathcal{M}, \mathcal{O})$ is a topological manifold and $\mathcal{A}$ is a C^∞-atlas.

4 Diffeomorphisms

$$M \xrightarrow{\phi} N$$

M, N naked sets, then the structure-preserving maps are bijections (invertionable maps).

***Definition* 4.1** (Set-theoretically isomorphic)
Two sets M, N are said to be *set-thoretically isomorphic* $M \cong_{\text{st}} N$ if $\exists$ a bijection $\phi : M \to N$ between them.

Note: Then they are "of the same size". Examples: $\mathbb{N} \cong_{\text{st}} \mathbb{Z}$, $\mathbb{N} \cong_{\text{st}} \mathbb{Q}$, $\mathbb{N} \ncong_{\text{st}} \mathbb{R}$

Now $(\mathcal{M}, \mathcal{O}_\mathcal{M})$ and $(\mathcal{N}, \mathcal{O}_\mathcal{N})$.

$$\mathcal{M} \xrightarrow{\phi} \mathcal{N}$$

***Definition* 4.2** (Topologically isomorphic (homeomorphic))
$(\mathcal{M}, \mathcal{O}_\mathcal{M}) \cong_{\text{top}} (\mathcal{N}, \mathcal{O}_\mathcal{N})$ topologically isomorphic = "homeomorphic" if $\exists \phi : \mathcal{M} \to \mathcal{N}$ and ϕ, ϕ^{-1} are *continuous*.

Note: Continuity is the important property here. This is a stronger notion. If two spaces are hoemomorphic then they are also set-theoretically isomorphic.

***Definition* 4.3** (Isomorphic vector spaces)
$(V, +_V, \cdot_V) \cong_{\text{vec}} (W, +_W, \cdot_W)$ if $\exists$ a bijection $\phi : V \to W$ that is linear in both directions.

***Definition* 4.4** (diffeomorphic)
Two C^∞ manifolds $(\mathcal{M}, \mathcal{O}_\mathcal{M}, \mathcal{A}_\mathcal{M})$ and $(\mathcal{N}, \mathcal{O}_\mathcal{N}, \mathcal{A}_\mathcal{N})$ are said to be *diffeomorphic* if $\exists$ a bijeciton $\phi : \mathcal{M} \to \mathcal{N}$ such that ϕ, ϕ^{-1} are both C^∞-maps, where by C^∞ we mean that $y \circ \phi \circ x^{-1}$ is in C^∞ in the $\mathbb{R}^d$-sense, see figure 4.

$$\begin{tikzcd}
\mathbb{R}^d \arrow[rr, "\tilde{y} \circ \phi \circ \tilde{x}^{-1}"'] & & \mathbb{R}^l \\
\mathcal{M} \supseteq U \arrow[u, "\tilde{x}"] \arrow[d, "x"'] \arrow[rr, "\phi"] & & V \subseteq N \arrow[u, "\tilde{y}"'] \arrow[d, "y"] \\
\mathbb{R}^d \arrow[rr, "y \circ \phi \circ x^{-1}"] \arrow[uu, bend left=60, "C^\infty"] & & \mathbb{R}^l \arrow[uu, bend right=60, "C^\infty"']
\end{tikzcd}$$

Figure 4: *In the definition of diffeomorphic 4.4 ϕ, ϕ^{-1} have to be C^∞, which is defined such that $y \circ \phi \circ x^{-1}$ has to be C^∞ in the $\mathbb{R}^d$-sense, which is chart-independent here.*

Note: Since we started with C^∞-manifolds, the chart transition maps are C^∞ and thus the notion of differentiability in the definition 4.4 is independent of the choice of charts, *i.e.* $\tilde{y} \circ \phi \tilde{x}^{-1}$ is also C^∞, see figure 4.

***Theorem* 4.5**
\# = number of C^∞-manifolds one can make of a given C^∞-manifold (if any) — up to diffeomorphisms —.

dim$M$	#	
1	1	Moise-Radon theorem
2	2	
3	3	
4	uncountable infinitely many	
5	finite	surgery theory
6	finite	
$\vdots$	finite	

5 Tangent Spaces

"What is the velocity of a curve γ at a point p?"

5.1 Velocities, Tangent Spaces

***Definition* 5.1** (Velocity)
$(\mathcal{M}, \mathcal{O}, \mathcal{A})$ smooth manifold, curve $\gamma : \mathbb{R} \to \mathcal{M}$ at least C^1. Suppose $\gamma(\lambda_0) = p$. The velocity v of γ at p is the linear map

$$\begin{aligned} v_{\gamma,p} : C^\infty(\mathcal{M}) &\xrightarrow{\sim} \mathbb{R}\,, \\ f &\mapsto v_{\gamma,p}(f)\,, \end{aligned}$$

with

$$\boxed{v_{\gamma,p}(f) := (f \circ \gamma)'(\underbrace{\gamma^{-1}(p)}_{\lambda_0})}\,, \tag{27}$$

i.e., the directional derivative of f along γ at the point p

Note: Remember

$$\begin{aligned} C^\infty(\mathcal{M}) = \{f : \mathcal{M} \to \mathbb{R} | f \text{ smooth}, \\ (f+g)(p) := f(p) + g(p) \\ (\lambda \cdot g)(p) := \lambda \cdot g(p), \lambda \in \mathbb{R}\} \end{aligned} \tag{28}$$

is a vector space.

Since $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ we can simply take the normal derivative.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "f \circ \gamma"'] & M \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

Note: In differential calculus one had the directional derivative as $v^i(\partial_i f)$. The shift in philosophy is now to see $v^i \partial_i$, *i.e.*, the operator that acts on f, as the vector.

***Definition* 5.2** (Tangent space)
$\forall p \in \mathcal{M}$ the "tangent space to $\mathcal{M}$ at p" consists of all the velocities of curves at that point :

$$\mathrm{T}_p\mathcal{M} := \{v_{\gamma,p} | \gamma \text{ smooth curves}\}\ . \tag{29}$$

Note: There is no reference to any external space or embedding in definition 5.2, see also figure 5!

$\mathrm{T}_p\mathcal{M}$ can be made into a vector space. The proof for this is so important and contains so many important things, that one should go through it in detail.

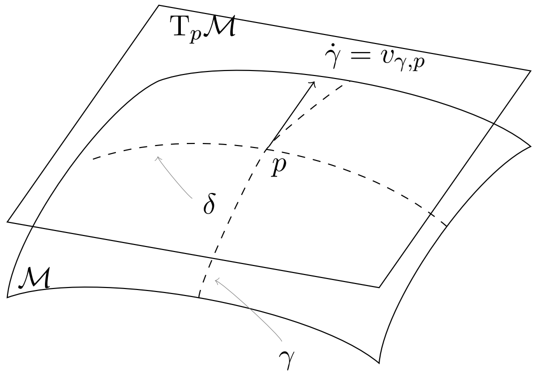


Figure 5: *Picture for immagining the tangent space. Keep in mind that there is no embedding needed like in this picture. Also in one can only think of $v_{\gamma,p}$ as an arrow in the chart, but not at manifold level. Also it is usefull to think of the arrow as the directional derivative ∂_v in the direciton of this arrow. Picture adapted from Menke, 2015*

***Definition* 5.3** (Addition and multiplication for tangent space)
For $p \in \mathcal{M}$, γ smooth curve on $\mathcal{M}$, $\alpha \in \mathbb{R}$:

$$\begin{aligned} + :\ & \mathrm{T}_p\mathcal{M} \times \mathrm{T}_p\mathcal{M} \to \mathrm{Hom}(C^\infty(\mathcal{M}), \mathbb{R}) \\ & (v_{\gamma,p} + v_{\delta,p})(f) := v_{\gamma,p}(f) + v_{\delta,p}(f) \qquad (30) \\ & f \in C^\infty(M) \\ \cdot :\ & \mathbb{R} \times \mathrm{T}_p\mathcal{M} \to \mathrm{Hom}(C^\infty(\mathcal{M}), \mathbb{R}) \\ & (\alpha \cdot v_{\gamma,p})(f) = \alpha \cdot v_{\gamma,p}(f) \qquad (31) \end{aligned}$$

But do they close, *i.e.*, is the tangent space a vector space? It remains to be shown that

1. $\exists$ curve $\sigma : v_{\gamma,p} + v_{\delta,p} = v_{\sigma,p}$
2. $\exists$ curve $\tau : \alpha \cdot v_{\gamma,p} = v_{\tau,p}$

The problem for 1 is that one cannot define $v_{\gamma,p} + v_{\delta,p}$ as just adding the points of the curves, since there is no such thing as adding two points on a manifold (what would be Paris + Berlin?).
Proof: (Tangent space is a vector space)

2. Construct $\tau : \mathbb{R} \to \mathcal{M}$:

$$\tau(\lambda) := \gamma(\alpha \cdot \lambda + \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda) \qquad (32)$$

with $\mu_\alpha : \mathbb{R} \to \mathbb{R}, r \mapsto \alpha r + \lambda_0$. Then $\tau(0) = \gamma(\lambda_0) = p$.

$$\begin{aligned} v_{\tau,p} &:= (f \circ \tau)'(0) = (f \circ \gamma \circ \mu_\alpha)'(0) \\ &= (f \circ \gamma)(\lambda_0)\alpha = \alpha v_{\gamma,p} \,. \end{aligned} \qquad (33)$$

1. Two curves $\gamma(\lambda), \delta(\lambda)$ with $\gamma(\lambda_0) = p$ and $\delta(\lambda_1) = p$. Make a choice of chart (U, x) with $p \in U$, later show independence of chart. Define

$$\begin{aligned} &\sigma_x : \mathbb{R} \to \mathcal{M} \,, \\ &\sigma_x(\lambda) := x^{-1}\left[(x \circ \gamma)(\lambda_0 + \lambda) + \right. \\ &\quad \left. + (x \circ \delta)(\lambda_1 + \lambda) - (x \circ \gamma)(\lambda_0)\right] \,. \end{aligned} \qquad (34)$$

$$\sigma_x(0) = \delta(\lambda_1) = p \,, \qquad (35)$$

$$\begin{aligned} v_{\sigma_x,p}(f) &= (f \circ \sigma_x)'(0) \qquad (36) \\ &= \left[\underbrace{(f \circ x^{-1})}_{\mathbb{R}^d \to \mathbb{R}} \circ \underbrace{(x \circ \sigma_x)}_{\mathbb{R}^d \to \mathbb{R}}\right]'(0) \,, \end{aligned}$$

where now we use the multidimensional chain rule that a physicist would rather know as $\frac{\mathrm{d}}{\mathrm{d}\lambda} f(\vec{y}(\lambda)) = (\vec{\nabla}_y f) \cdot \frac{\mathrm{d}\vec{y}}{\mathrm{d}\lambda}$.

$$v_{\sigma_x,p}(f) = (x^i \circ \sigma_x)'(0)\left[\partial_i(f \circ x^{-1})\right](\underbrace{x(\sigma_x(0))}_{x(p)})$$

$$\begin{aligned} (x^i \circ \sigma_x)'(0) &= [(x^i \circ \gamma(\lambda_0 + \lambda) + (x^i \circ \delta)(\lambda_1 + \lambda) \\ &\quad - (x^i \circ \gamma)(\lambda_0)]' \\ &= (x^i \circ \gamma)'(\lambda_0) + (x^i \circ \delta)'(\lambda_1) = v^i_{\gamma,p} + v^i_{\delta,p} \,. \end{aligned}$$

Plugging this into equation (36) and doing the same step backwards we get

$$v_{\sigma_x,p}(f) = v_{\gamma,p}(f) + v_{\delta,p}(f) \quad \forall f \in C^\infty(\mathcal{M}) \,, \qquad (37)$$

independent of the chart.

Note: It turns out that the sum of curves like this depends on the chart (U, x), but the derivative of the sum of the curves at the point p does not.

□

5.2 Components of vectors

Let $(U, x) \in \mathcal{A}_{\text{smooth}}$,

$$\begin{aligned} v_{\gamma,p}(f) : &= (f \circ \gamma)'(0) = \left[(f \circ x^{-1}) \circ (x \circ \gamma)\right]'(0) \\ &= (x^i \circ \gamma)'(0) \cdot \left(\partial_i(f \circ x^{-1})\right)(x(p)) \\ &=: \dot\gamma^i_x(0)\left(\frac{\partial f}{\partial x^i}\right)_p \end{aligned} \qquad (38)$$

Note: The last step in equation (38) is pure notation. If one sees $(\partial f/\partial x^i)_p$, one should first think of $\left(\partial_i(f\circ x^{-1})\right)(x(p))$ and also for $\dot\gamma_x^i(x)$.

$$\boxed{\left(\frac{\partial f}{\partial x^i}\right)_p := \left(\partial_i(f\circ x^{-1})\,(x(p))\right)}\,, \tag{39}$$

$$\boxed{\dot\gamma_x^i(0) := (x^i\circ\gamma)'(0)}\,. \tag{40}$$

Nevertheless one can show that what we write like a partial derivative here, behaves (and maybe tastes) like a partial derivative.

Thus we write

$$v_{\gamma,p}(f) = \dot\gamma_x^i(0)\left(\frac{\partial}{\partial x^i}\right)_p f\,, \quad \forall f\in C^\infty\,, \tag{41}$$

which leads to

$$v_{\gamma,p} = \underbrace{\dot\gamma_x^i(0)}_{\text{components}}\underbrace{\left(\frac{\partial}{\partial x^i}\right)_p}_{\substack{\text{chart induced}\\ \text{basis of } \mathrm{T}_pU}}\,. \tag{42}$$

***Theorem* 5.4** (Chart induced basis of T_pU)
Let $(U,x)\in\mathcal{A}_{\text{smooth}}$. The

$$\left(\frac{\partial}{\partial x^1}\right)_p,\ldots,\left(\frac{\partial}{\partial x^d}\right)_p \quad \in \mathrm{T}_p\mathcal{M} \tag{43}$$

constitute a *basis* of T_pU.

Proof: Show linear independence

$$\lambda^i\left(\frac{\partial}{\partial x^i}\right)_p \overset{!}{=} 0 \tag{44}$$

$$\begin{aligned}\lambda^i\left(\frac{\partial}{\partial x^i}\right)_p(x^j) &= \lambda^i\,\overbrace{\partial_i(x^j\circ x^{-1})}^{\delta_i^j}(x(p))\\ &= \lambda^j\,,\quad \forall j=1,\ldots,d\,,\end{aligned} \tag{45}$$

□

***Corollary* 5.5**

$$\underbrace{\dim\mathrm{T}_p\mathcal{M}}_{\substack{\text{vector space}\\ \text{dimension}}} = d = \underbrace{\dim\mathcal{M}}_{\substack{\text{top. manif.}\\ \text{dimension}}}\,. \tag{46}$$

5.3 Change of vector components under a change of charts

Note: A vector is an abstract object and does not change under a change of charts. How should it? It does not depend on the chart. If it would, we could change objects by looking at them differently (different coordinates), so we could do telekinesis.

What does change are the components of a vector.

Terminology: $X\in\mathrm{T}_p\mathcal{M}$ always means that

- $\exists\,\gamma:\mathbb{R}\to\mathcal{M}$ s.t. $X=v_{\gamma,p}$.
- $\exists\,X^1,\ldots,X^d\in\mathbb{R}: X = X^i\left(\partial/\partial x^i\right)_p$.

Let (U,x) and (V,y) be overlapping charts and $p\in U\cup V$. Let $X\in\mathrm{T}_p\mathcal{M}$.

$$\begin{aligned}X &= X^i_{(x)}\left(\frac{\partial}{\partial x^i}\right)_p\\ &= X^i_{(y)}\left(\frac{\partial}{\partial y^i}\right)_p\end{aligned} \tag{47}$$

Again we have to translate the "partial derivative"-notation:

$$\left(\frac{\partial}{\partial x^i}\right)_p f = \partial_i\left(f\circ x^{-1}\right)(x(p)) \tag{48}$$

$$= \partial_i\left(\underbrace{(f\circ y^{-1})}_{\mathbb{R}^d\to\mathbb{R}}\circ\underbrace{(y\circ x^{-1})}_{\mathbb{R}^d\to\mathbb{R}^d}\right)(x(p))\,,$$

and with the multidimensional chain rule, which a physicist would write as

$$\frac{\partial}{\partial x^i}\left(f(\vec y(\vec x))\right) = \left(\vec\nabla_y f(\vec y)\right)\cdot\frac{\partial\vec y}{\partial x^i}\,, \tag{49}$$

equation (48) becomes

$$\begin{aligned}\left(\frac{\partial}{\partial x^i}\right)_p f &= \left(\partial_i(y\circ x^{-1})^j\right)(x(p))\cdot\left(\partial_j(f\circ y^{-1})\right)(y(p))\\ &= \left(\frac{\partial y^j}{\partial x^i}\right)_p\cdot\left(\frac{\partial}{\partial y^j}\right)_p f\,.\end{aligned} \tag{50}$$

Comparison with equation (47) yields the transformation for vector components. Namely the components of X in the y-chart, $X^j_{(y)}$ can be calculated from the components in the x-chart, $X^i_{(x)}$ through

$$\boxed{X^j_{(y)} = \left(\frac{\partial y^j}{\partial x^i}\right)_p X^i_{(x)}}\,. \tag{51}$$

5.4 Cotangent spaces

The cotangent space of $\mathrm{T}_p\mathcal{M}$ is

$$(\mathrm{T}_p\mathcal{M})^* := \left\{\phi:\mathrm{T}_p\mathcal{M}\xrightarrow{\sim}\mathbb{R}\right\}\,. \tag{52}$$

One often just writes $\mathrm{T}_p\mathcal{M}^*$.

***Example* 5.6** (Gradient)
For $f \in C^\infty(\mathcal{M})$, define

$$\begin{aligned}(\mathrm{d}f)_p : \mathrm{T}_p\mathcal{M} &\xrightarrow{\sim} \mathbb{R} \\ X &\mapsto (\mathrm{d}f)_p(X) := Xf\,, \end{aligned} \tag{53}$$

then $(\mathrm{d}f)_p(X) \in \mathbb{R}$, thus $(\mathrm{d}f)_p \in \mathrm{T}_p\mathcal{M}^*$ and we call $(\mathrm{d}f)_p$ the *gradient* of f at $p \in M$. Note that the gradient is a $(0, 1)$-tensor, a covector, and not a $(1, 0)$-tensor, which would be a vector.

The components of the gradient with respect to the chart induced basis are (choosing a chart (U, x))

$$\begin{aligned}((\mathrm{d}f)_p)_j := (\mathrm{d}f)_p\left(\frac{\partial}{\partial X^j}\right) &= \left(\frac{\partial f}{\partial x^j}\right)_p \\ &= \partial_j(f \circ x^{-1})(x(p))\,. \end{aligned} \tag{54}$$

***Theorem* 5.7** (Chart induced basis for the cotangent space)
Consider chart (U, x), $x^i : U \to \mathbb{R}$. Then

$$\left(\mathrm{d}x^1\right), \left(\mathrm{d}x^2\right), \ldots, \left(\mathrm{d}x^d\right), , \tag{55}$$

is a basis of $\mathrm{T}_p\mathcal{M}^*$. In fact, if one chooses the chart induced basis in $\mathrm{T}_p\mathcal{M}$, it is the dual basis of the dual space $\mathrm{T}_p\mathcal{M}^*$, *i.e.*,

$$(\mathrm{d}x^a)_p\left(\frac{\partial}{\partial x^b}\right) = \delta^a_b\,. \tag{56}$$

Again, one can calculate how the components of a covector behave under a *change of chart*. Of a covector $\omega \in \mathrm{T}_p\mathcal{M}^\star$ the components are given by

$$\omega = \omega_{(x)i}(\mathrm{d}x^i)_p = \omega_{(y)i}(\mathrm{d}y^i)_p\,, \tag{57}$$

Since ω is a 1-form it always acts on a vector $v \in \mathrm{T}_p\mathcal{M}$,

$$\omega(v) = \omega_{(x)i}\left(\mathrm{d}x^i\right)_p(v) = \omega_{(x)i}v(x^i)\,. \tag{58}$$

Take a curve $\gamma(\lambda)$ for which v is the tangent vector at p, then

$$\begin{aligned} v(x^i) = (x^i \circ \gamma)'(0) &= \left((x^i \circ y^{-1}) \circ (y \circ \gamma)\right)'(0) \\ &= \left[\partial_j(x^i \circ y^{-1}(y(p))\right] \frac{\mathrm{d}(y^j \circ \gamma)}{\mathrm{d}\lambda}(0) \\ &= \left(\frac{\partial x^i}{\partial y^j}\right)_p (y^j \circ \gamma)'(0) \\ &= \left(\frac{\partial x^i}{\partial y^j}\right)_p v(y^j)\,. \end{aligned} \tag{59}$$

Plugging this in equation (57)

$$\begin{aligned} w(v) = w_{(x)i}\left(\mathrm{d}x^i\right)_p &= \omega_{(x)i}\left(\frac{\partial x^i}{\partial y^j}\right)_p \left(\mathrm{d}y^i\right)_p \\ &= w_{(y)i}\left(\mathrm{d}y^i\right)_p\,. \end{aligned} \tag{60}$$

So the transformation for the components of a 1-form is

$$\boxed{\omega_{(y)i} = \left(\frac{\partial x^j}{\partial y^i}\right)_p \omega_{(x)j}}\,. \tag{61}$$

5.5 Bundles and Vector Fields

Up until now we were always working at a single point $p \in \mathcal{M}$. The vectors and tensors we defined live in the tangent space at that point, $\mathrm{T}_p\mathcal{M}$.

***Definition* 5.8** (Bundle)
A *bundle* is a triple

$$\mathcal{E} \xrightarrow{\quad \pi \quad} \mathcal{M}\,, \tag{62}$$

where

- $\mathcal{E}$: smooth manifold ("total space"),
- π: surjective smooth map ("projection map"),
- $\mathcal{M}$: smooth manifold ("base space").

***Definition* 5.9** (Fibre)
$\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ a bundle, $p \in \mathcal{M}$, then

$$\mathrm{preim}_\pi(p) \tag{63}$$

is called the *fibre over* p.

***Definition* 5.10** (Section of a bundle)
A *section* σ of a bundle is a map $\mathcal{M} \to \mathcal{E}$ with $\pi \circ \sigma = \mathrm{Id}_\mathcal{M}$, *i.e.* a section projects a point from the base space to a point in the total space that is in the same fibre.

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ \mathcal{M} \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

Note: In figure 6 a picture for a specific bundle is given. We choose $\mathcal{M}$ to be a circle and $\mathcal{E}$ to be a cylinder with the same radius. Then π can be any map $\mathcal{E} \to \mathcal{M}$ as long as it is surjective and smooth (it can also be $\mathcal{E} \supset U \to \mathcal{M}$).

One possibility for π would be to just project the point down the cylinder to the circle. Indeed, in this picture we can see every fibre (red line) as the tangent space $T_p\mathcal{M}$ at the point p on the circle $\mathcal{M}$. Then $\mathcal{E}$ consists of all $T_p\mathcal{M}$ and π takes a vector $v \in \mathrm{T}_p\mathcal{M}$ and returns p.

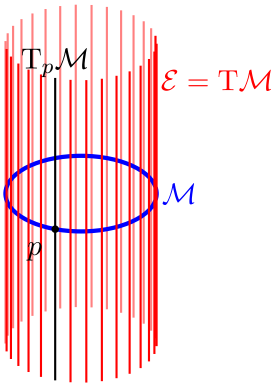


Figure 6: *Picture for a bundle, where $\mathcal{M}$ is a circle and $\mathcal{E}$ a cylinder. We have chosen π such that every point on the cylinder ($\mathcal{E}$) is projected down to the circle. This way every vertical line is a fibre. For example the black vertical line is the fibre over the point p. A way to think of this specific example for the tangent space of the circle is that the tangent at every point of the circle is rotated so to give this picture. Picture modified from Arun Debray, 2016.*

Note: A section is a field and what kind of field it is depends on the choice of total space. In figure 6 every fibre is a vector space, namely the tangent space to the circle at that point. The base space $\mathcal{M}$ and the total space $\mathcal{E}$ don't have to lie in the same space like in the picture, they don't need to have anything to do with each other.

Note: The wave function $\psi : \mathcal{M} \to \mathbb{C}$ is actually a scalar field and not a function.

***Definition* 5.11** (Tangent Bundle of a Smooth Manifold)
$(\mathcal{M}, \mathcal{O}, \mathcal{A})$ a smooth manifold.

(a) The tangent bundle is the *disjoint union* of all the tangent spaces,

$$\mathrm{T}\mathcal{M} := \dot{\bigcup_{p\in\mathcal{M}}} \mathrm{T}_p\mathcal{M} . \tag{64}$$

(b) The projection map π projects down to the base point of the tangent space $T_p\mathcal{M}$ that the vector X is in,

$$\pi : \mathrm{T}\mathcal{M} \to \mathcal{M} , \tag{65}$$
$$X \mapsto p, \quad p \in \mathcal{M} : X \in \mathrm{T}_p\mathcal{M} . \tag{66}$$

(c) Construct the *coarsest* topology on $\mathrm{T}\mathcal{M}$ such that π is (just) continuous ("the initial topology with respect to π"), which here is given by

$$\mathcal{O}_{\mathrm{T}\mathcal{M}} := \{\mathrm{preim}_\pi(\mathcal{U}) \,|\, \mathcal{U} \in \mathcal{O}\} . \tag{67}$$

Note: An element X of the tangent bundle $X \in \mathrm{T}\mathcal{M}$ is of course an element of the tangent space at its point in the manifold:

$$X \in \mathrm{T}_{\pi(X)}\mathcal{M} . \tag{68}$$

The tangent bundle itself can be made to be a smooth manifold. Construct a C^∞-atlas on $\mathrm{T}\mathcal{M}$ from the C^∞-atlas $\mathcal{A}$ on $\mathcal{M}$,

$$\mathcal{A}_{\mathrm{T}\mathcal{M}} := \{(\mathrm{T}\mathcal{U}, \xi_x) \,|\, (\mathcal{U}, x) \in \mathcal{A}\} , \tag{69}$$

where

$$\xi_x : \mathrm{T}\mathcal{U} \to \mathbb{R}^{2\dim\mathcal{M}} , \tag{70}$$
$$X \mapsto \Big((x^1 \circ \pi)(X), \ldots, (x^d \circ \pi)(X), (\mathrm{d}x^i)_{\pi(X)}(X), \ldots, (\mathrm{d}x^d)_{\pi(X)}(X)\Big) . \tag{71}$$

Note: x is the chart that we have chosen. $X \in \mathrm{T}\mathcal{M}$ is a vector in $\mathrm{T}_p\mathcal{M}$ at a base point p. We can get the base point p through the map π, $p = \pi(X)$. The first part of ξ_x are the coordinates of the base point in the chart x, *i.e.* $(x \circ \pi)(X)$ and the second part are the components of the vector which we can get by acting with $(\mathrm{d}x)_{\pi(X)}$ on X, since

$$X =: X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_{\pi(X)} , \tag{72}$$

and

$$(\mathrm{d}x^j)_{\pi(X)} \left(\frac{\partial}{\partial_i}\right)_{\pi(X)} = \delta^j_i . \tag{73}$$

Let's write the components of $\xi_x X$ as

$$(\underbrace{\alpha^1, \ldots, \alpha^d}_{\text{coordinates in chart } x} , \underbrace{\beta^1, \ldots, \beta^d}_{\text{compoenents of vector in } \mathrm{T}_{\pi(X)}\mathcal{M}}) \tag{74}$$

The inverse of ξ_x

$$\xi_x^{-1} : \mathbb{R}^{2\dim\mathcal{M}} \ni \xi_x(\mathrm{T}\mathcal{U}) \to \mathrm{T}\mathcal{U} , \tag{75}$$
$$(\alpha^1, \ldots, \alpha^d, \beta^1, \ldots, \beta^d) \mapsto \beta^i \left(\frac{\partial}{\partial x^i}\right)_{x^{-1}(\alpha^1,\ldots,\alpha^d)} , \tag{76}$$

and of course $x^{-1}(\alpha^1, \ldots, \alpha^d)$ is just $\pi(X)$ explicitly written out.

The atlas is *smooth* if the chart transition map $\xi_y \circ \xi_x^{-1}$ is smooth. We can check that this is true by explicitly acting with ξ_y on eq. (76). Then we see that the α^i transform like the coordinates in a map (because they are),

$$(y^j \circ x^{-1})(\alpha^1, \ldots, \alpha^d)\,, \tag{77}$$

and the β^i transform like the components of a tangent vector (because they are),

$$\beta^m \left(\frac{\partial y^j}{\partial x^m} \right) = \beta^m \partial_m (y^i \circ x^{-1})(\alpha^1, \ldots, \alpha^d)\,, \tag{78}$$

The chart transition map on the manifold level, $y^i \circ x^{-1}$, is smooth by assumption, and the derivative $\partial_m (y^i \circ x^{-1})$ is also smooth, since the derivative of C^∞ is still C^∞.

We can summarize all of that in one picture:

$$\begin{tikzcd} \mathrm{T}\mathcal{M} \arrow[rr, "\pi"] & & \mathcal{M} \\ C^\infty \text{ mfd.} & C^\infty \text{ map} & C^\infty \text{ mfd.} \end{tikzcd}$$

***Definition* 5.12** (Vector Field χ)
A *smooth vector field* χ is a *smooth map*

$$\begin{tikzcd} \mathrm{T}\mathcal{M} \arrow[d, "\pi"'] \\ \mathcal{M} \arrow[u, bend right, "\chi"'] \end{tikzcd} \qquad \pi \circ \chi = \mathrm{id}_{\mathcal{M}}$$

Note: We needed all the sommersaults with bundles and fibres for the word SMOOTH in above definition.

5.6 The $C^\infty(\mathcal{M})$ Module $\Gamma(\mathrm{T}\mathcal{M})$

Remember: C^∞ is the collection of smooth functions and also a vector space $(C^\infty(\mathcal{M}), +)$ (can add functions). One can also multiply functions, but for the multiplication fg there exists only an inverse g^{-1} if the function g has no zeros (And in the definition of vector space only the multiplication with the function that is zero everywhere is excluded). Thus $(C^\infty(\mathcal{M}), +, \cdot)$ is a *ring*.

- Field: Fulfills[1] (C$^+$, A$^+$, N$^+$, I$^+$, C$^\cdot$, A$^\cdot$, N$^\cdot$, I$^\cdot$, D$^+$)
- Ring Fulfills (C$^+$, A$^+$, N$^+$, I$^+$, C$^\cdot$, A$^\cdot$, N$^\cdot$, ~~I$^\cdot$~~, D$^+$)

$$\Gamma(\mathrm{T}\mathcal{M}) := \{\chi : \mathrm{T}\mathcal{M} \to \mathcal{M} \,|\, \text{smooth section}\}\,, \tag{79}$$

which means all smooth vector fields on $\mathcal{M}$; a section with total space $\mathrm{T}\mathcal{M}$ and base space $\mathcal{M}$ is a vector field, see definitions 5.12 and 5.10.

***Definition* 5.13** (Set of smooth vector fields $\Gamma(\mathrm{T}\mathcal{M})$)

$$(\chi + \tilde{\chi})(f) := \chi(f) + \tilde{\chi}(f)\,. \tag{80}$$

Watch out for the $+$ and $\cdot$ and on what spaces they operate!

One can make a vector field to an $\mathbb{R}$-vector space, by allowing multiplication with real numbers,

$$(\alpha \cdot \chi)(f) := \alpha \cdot \chi(f)\,, \tag{81}$$

but actually we can even make more! We can allow $C^\infty(\mathcal{M})$ functions instead of $\mathbb{R}$,

$$(g \cdot \chi)(f) := g \cdot \chi(f)\,, \tag{82}$$

where $g \in C^\infty(\mathcal{M})$. The point is, that $C^\infty(\mathcal{M})$ is only a RING and so we don't call it a $C^\infty(\mathcal{M})$-vector space, but $C^\infty(\mathcal{M})$-module[2]!

The set of all smooth vector fields $\Gamma(\mathrm{T}\mathcal{M})$ can be made into a C^∞-module

A module does not have all the properties of vector spaces. A module is *not guaranteed* to always have a basis! Thus we are *not* in general able to write every vector field χ as

$$\chi = f^i \chi_{(i)}\,, \tag{83}$$

$$\chi_{(1)}, \ldots, \chi_{(d)} \in \Gamma(\mathrm{T}\mathcal{M})\,, \quad \text{global basis} \tag{84}$$

Example: Every vector field on the sphere has to vanish somewhere, but that means at this point it cannot be used as a basis. We can do it locally, though, *i.e.* for subsets $\mathcal{U} \in \mathcal{M}$.

[1] **C**ommutative, **A**ssociative, **N**eutral element, **I**nverse element, **D**istributive

[2] So a vector space over a ring is a module.

5.7 Tensor Fields

Since $\mathrm{T}^*\mathcal{M}$ is also a vector space, we can define $\Gamma(\mathrm{T}^*\mathcal{M})$ as all smooth covector fields, which is again a $C^\infty(\mathcal{M})$-module.

***Definition* 5.14** ((r,s)-tensor field T)
An (r,s)-tensor field T is a C^∞, in every element multi-linear[3], map

$$T : \underbrace{\Gamma(\mathrm{T}^*\mathcal{M}) \times \cdots \times \Gamma(\mathrm{T}^*\mathcal{M})}_{r} \times \\ \times \underbrace{\Gamma(\mathrm{T}\mathcal{M}) \times \cdots \times \Gamma(\mathrm{T}\mathcal{M})}_{s} \xrightarrow{\sim} C^\infty(M) \quad (85)$$

Note: So T is a map from $C^\infty(\mathcal{M})$ modules to the $C^\infty(\mathcal{M})$ module $C^\infty(\mathcal{M})$. Yes, $C^\infty(\mathcal{M})$ itself is a $C^\infty(\mathcal{M})$ module, just like $\mathbb{R}$ is an $\mathbb{R}$-vector space. I just want to write $C^\infty(\mathcal{M})$ once more: $C^\infty(\mathcal{M})$.

Example: $f \in C^\infty(\mathcal{M})$

$$\mathrm{d}f : \Gamma(\mathrm{T}\mathcal{M}) \xrightarrow{\sim} C^\infty(\mathcal{M}) \quad (86)$$
$$\chi \mapsto \mathrm{d}f(\chi) := \chi f\,, \quad (87)$$

where χf is defined by its action on $p \in \mathcal{M}$,

$$(\chi f)(p) := \chi(p) f\,, \quad (88)$$

which works since $\chi(p) \in \mathrm{T}p\mathcal{M}$ can act on a function f. Thus $\mathrm{d}f$ is a gradient co-vector field.

6 Connections/Covariant Derivatives

Most of the time it is enough to think about a *vector field* X just as a vector in each point. Remember: X gives a *directional derivative* Xf. We define new notation

$$\nabla_X f := Xf = (\mathrm{d}f)(X)\,, \quad f \in C^\infty(M)\,, \quad (89)$$

because ∇_X can be generalized to tensors.

6.1 Directional Derivatives of Tensor Fields

We make a wishlist for properties of ∇_X acting on a tensor field. There will remain a freedom in the definition of ∇_X which we need to fix by providing additional structure.

[3] Addition is like always, but s-multiplication means multiplying with a C^∞ function.

***Definition* 6.1** (Covariant Derivative/Affine Connection)
A *connection* ∇ on a smooth manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ is a map that takes a pair consisting of a vector (field) X and a (p,q)-tensor *field* T and sends them to a (p,q)-tensor (field) $\nabla_X T$, satisfying

1. *Extension of normal derivative*:
$$\nabla_X f = Xf\,, \quad (90)$$
$\forall f \in C^\infty(\mathcal{M})$.
2. *Additivity*:
$$\nabla_X(T+S) = \nabla_X T + \nabla_X S\,, \quad (91)$$
for (p,q)-tensors T, S,
3. *Leibnitz rule*: For a $(1,1)$ tensor field T, already evaluated with a covector ω and a vector Y, so $T(\omega, Y) \in C^\infty(\mathcal{M})$,,
$$\nabla_X\left(T(\omega,Y)\right) = (\nabla_X T)(\omega, Y) + T(\nabla_X \omega, Y) \\ + T(\omega, \nabla_X Y)\,. \quad (92)$$
For (p,q) tensor fields analogously. Easier formulation (definition of the tensor product below):
$$\nabla_X(T \otimes S) = (\nabla_X T) \otimes S + T \otimes \nabla_X S\,. \quad (93)$$
4. *C^∞-linearity in lower argument*:
$$\nabla_{fX+Z} T = f \nabla_X T + \nabla_Z T\,, \quad (94)$$
$\forall f \in C^\infty(\mathcal{M})$.

A manifold with connection (or affine manifold) is a quadruple of structures, $(\mathcal{M}, \mathcal{O}, \mathcal{A}, \nabla)$.

***Definition* 6.2** (Tensor Product $T \otimes S$)
For a (p,q)-tensor T and a (l,m)-tensor S, the *tensor product* is defined as

$$(T \otimes S)(\omega_1, \ldots, \omega_{p+l}, X_1, \ldots, X_{q+m}) = \\ T(w, \ldots, \omega_p, X_1, \ldots, X_q) \cdot \\ \cdot S(\omega_{p+1}, \ldots, \omega_{p+l}, X_{q+1}, \ldots, X_{q+m})\,. \quad (95)$$

The first tensor eats as many covectors and vectors as it can followed by the second tensor who eats the rest.

Note:

- $\nabla_X \cdot$ is the extension of X,
- $\nabla_\cdot \cdot$ is the extension of d.

6.2 New Structure on $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ Required to Define ∇

How much freedom do we have in choosing such a structure? How fixed is ∇_X by the definition above?

We consider vector fields X, Y and choose a chart (U, x), using the rules above

$$\begin{aligned}\nabla_X Y &= \nabla_{X^i \frac{\partial}{\partial x^i}} \left(Y^m \frac{\partial}{\partial x^m} \right) \qquad (96)\\ &= X^i \left(\nabla_{\frac{\partial}{\partial x^i}} Y^m \right) \frac{\partial}{\partial x^m} + X^i Y^m \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^m} \right)\\ &= X^i \left(\frac{\partial}{\partial x^i} Y^m \right) \frac{\partial}{\partial x^m} + X^i Y^m \Gamma^q_{(x)mi} \frac{\partial}{\partial x^m} ,\end{aligned}$$

where in the last step we have expanded $\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^m} \right) = \Gamma^q_{(x)mi} \frac{\partial}{\partial x^q}$ with the *connection coefficient functions* ${\Gamma^q}_{mi}$ (on $\mathcal{M}$) of ∇ with respect to the chart (U, x). The (x) on $\Gamma^q_{(x)mi}$ denotes that it depends on the chart x.

***Definition* 6.3** (Connection coefficient functions Γ)
Given $(\mathcal{M}, \mathcal{O}, \mathcal{A}, \nabla)$ and a chart $(U, x) \in \mathcal{A}$ the *connection coefficient functions* (the "Γs") with respect to (U, x) are the $(\dim \mathcal{M})^3$ many chart dependent functions

$$\begin{aligned}\Gamma^i_{(x)jk} &: U \to \mathbb{R} \qquad (97)\\ p &\mapsto \left(\mathrm{d}x^i \left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right) \right) (p) . \qquad (98)\end{aligned}$$

Thus:

$$(\nabla_X Y)^i = X^m \left(\frac{\partial}{\partial x^m} Y^i \right) + {\Gamma^i}_{nm} Y^n X^m . \qquad (99)$$

Note: The new structure that we need to fix ∇ acting on a *vector field* are the $(\dim \mathcal{M})^3$ many functions ${\Gamma^i}_{jl}$. Actually we are lucky and they already fix ∇ acting on any tensor field of any rank as we will see.

For a dual vector field we arrive at one point at

$$\nabla_{\frac{\partial}{\partial x^m}} \left(\mathrm{d}x^i \right) = {\Sigma^i}_{qm} \, \mathrm{d}x^q , \qquad (100)$$

but now

$$\begin{aligned}&\nabla_{\frac{\partial}{\partial x^m}} \left(\mathrm{d}x^i \left(\frac{\partial}{\partial x^j} \right) \right) = 0\\ &= \left(\nabla_{\frac{\partial}{\partial x^m}} \mathrm{d}x^i \right) \left(\frac{\partial}{\partial x^j} \right) + \mathrm{d}x^i \nabla_{\frac{\partial}{\partial x^m}} \left(\frac{\partial}{\partial x^j} \right) =\\ &= {\Sigma^i}_{qm} \, \mathrm{d}x^q \left(\frac{\partial}{\partial x^j} \right) + {\Gamma^i}_{qm} \, \mathrm{d}x^q \left(\frac{\partial}{\partial x^q} \right) ,\end{aligned}$$

so $\Sigma = -\Gamma$ and we will just use Γ.

∇ comes with a $+$ for vectors and a $-$ for covectors. The last index of ${\Gamma^i}_{jm}$ goes always with the direction X of ∇_X.

$$\begin{aligned}(\nabla_X Y)^i &= X(Y^i) + {\Gamma^i}_{jm} Y^j X^m , \qquad (101)\\ (\nabla_X \omega)_i &= X(Y^i) - {\Gamma^j}_{im} \omega^j X^l . \qquad (102)\end{aligned}$$

For higher rank tensors every upper index comes with a $+\Gamma$ and every lower index with a $-\Gamma$, *e.g.* for a $(1, 2)$-tensor T:

$$\begin{aligned}(\nabla_X T)^i_{jk} = X(T^i_{jk}) &+ {\Gamma^i}_{sm} T^s_{jk} X^m\\ &- {\Gamma^s}_{jm} T^i_{sk} X^m - {\Gamma^s}_{km} T^i_{js} X^m . \qquad (103)\end{aligned}$$

Note: In Euclidean space (non-curved) ${\Gamma^i}_{lm} = 0$ for non-curvilinear coordinates. So in $\mathbb{R}$ for the standard basis the Γ vanish, but not for *e.g.* polar coordinates they are nonzero.

***Definition* 6.4** (Divergence)
Let X be a vector field on $(\mathcal{M}, \mathcal{O}, \mathcal{A}, \nabla)$. The *divergence* of X is the function

$$\mathrm{div}(X) := \left(\nabla_{\frac{\partial}{\partial x^i}} X \right)^i , \qquad (104)$$

where there is a sum over i. This definition is *chart independent*.

6.3 Change of Γs Under Change of Chart

Let (U, x), $(V, y) \in \mathcal{A}$ and $U \cap V \neq \emptyset$, then using the transformations of $\mathrm{d}x^q$ and $\partial / \partial y^q$,

$$\begin{aligned}\Gamma^i_{(y)jk} &:= \mathrm{d}y^i \left(\nabla_{\frac{\partial}{\partial y^l}} \frac{\partial}{\partial y^j} \right)\\ &= \frac{\partial y^i}{\partial x^q} \mathrm{d}x^q \left(\nabla_{\frac{\partial x^p}{\partial y^k} \frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y j} \frac{\partial}{\partial x^s} \right)\\ &= \frac{\partial y^i}{\partial x^q} \mathrm{d}x^q \frac{\partial x^p}{\partial y^k} \left\{ \left(\nabla_{\frac{\partial}{\partial x^p}} \frac{\partial x^s}{\partial y^i} \right) \frac{\partial}{\partial x^s} \right.\\ &\qquad \left. + \frac{\partial x^s}{\partial y^j} \left(\nabla_{\frac{\partial}{\partial x^p}} \frac{\partial}{\partial x^s} \right) \right\}\\ &= \frac{\partial y^i}{\partial x^q} \left(\frac{\partial}{\partial y^k} \frac{\partial x^s}{\partial y^j} \right) {\delta^q}_s + \frac{\partial y^i}{\partial x^q} \frac{\partial x^p}{\partial y^k} \frac{\partial x^s}{\partial y^j} \Gamma^q_{(x)sp} ,\end{aligned}$$

and in summary the transformation of the Γs is

$$\boxed{\begin{aligned}\Gamma^i_{(y)jk} = \frac{\partial y^i}{\partial x^q} \frac{\partial x^s}{\partial y^j} \frac{\partial x^p}{\partial y^k} \Gamma^q_{(x)sp} &\qquad (105)\\ + \frac{\partial y^i}{\partial x^q} \frac{\partial^2 x^q}{\partial y^k \partial y^j} .&\end{aligned}}$$

The first part is like the transformation of the components of a $(1,2)$-tensor. The second part only depends on x and y and even if Γ is zero in one chart, it does not have to be zero in another, depending on this term. If all componenets of a tensor are zero in one chart, then they are zero in all charts. We see that for linear transformations $x(y)$ this term is zero.

Note:

$$\frac{\partial}{\partial x^p}\frac{\partial}{\partial y^j} \neq \frac{\partial}{\partial y^j}\frac{\partial}{\partial x^p}\,, \tag{106}$$

no Schwartz rule in this case, but if we write it as only derivatives with respect to y, then there is.

6.4 Normal Coordinates

Let $p \in \mathcal{M}$ of $(\mathcal{M}, \mathcal{O}, \mathcal{A}, \nabla)$. Then one can construct a chart (U, x) with $p \in U$ such that

$$\Gamma^i_{(x)(jk)}(p) = 0\,, \tag{107}$$

at the point p, but not necessarily in a neighbourhood.

Note: In equation (107) (jk) means the symmetrized part of $\Gamma^i{}_{(x)jk}$.

PROOF: Let (V, y) be any chart, $p \in V$. Then in general $\Gamma^i_{(y)jk} \neq 0$. Consider a new chart (U, x) with the chart transition map $y \to x$:

$$\begin{aligned}(x \circ y^{-1})^i(\alpha^i, \dots, \alpha^d) &:= \alpha^i - \frac{1}{2}\Gamma^i_{(y)jk}(p)\alpha^j\alpha^k\,,\\ \frac{\partial x^i}{\partial y^k \partial y^j} &= -\Gamma^i_{(y)(kj)}(p)\,,\\ \Gamma^i_{(x)jk} &= \Gamma^i_{(y)jk}(p) - \Gamma^i_{(y)(jk)}(p) = \Gamma^i_{(y)[jk]}\,,\end{aligned}$$

and thus $\Gamma_{(x)}$ has vanishing symmetric part (lower two indices). $\Gamma^i_{[jk]}(p)$ is actually a tensor (the components transform like for a tensor) and is called the *torsion tensor*,

$$\Gamma^i_{[jk]} = T^i_{jk}\,. \tag{108}$$

We call this chart (U, x) a *normal coordinate chart* of ∇ at the point $p \in \mathcal{M}$.

Note: Nonzero curvature prevents us from extending this to a neighbourhood around that point, but we will be able to extend it to a curve in $\mathcal{M}$.

7 Parallel Transport & Curvature

Parallel transport of a vector Y along a curve γ means $\nabla_{v_\gamma} Y = 0$, where v_γ is the tangent vector along γ. As we see in figure 7 if there is curvature, then the vector we get actually depends on the path we take.

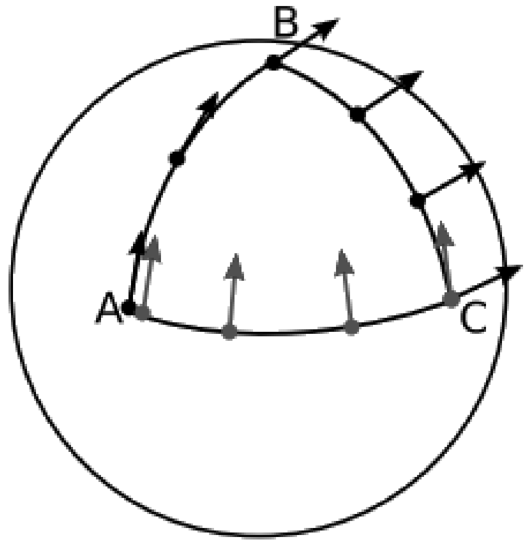


Figure 7: *Parallel Transport on a sphere. Parallel transporting the vector along ABC gives a different vector than along AC. Figure from Crowell, n.d.*

7.1 Parallelity of Vector Fields

Let $(\mathcal{M}, \mathcal{O}, \mathcal{A}, \nabla)$ a vector field with connection.

***Definition* 7.1** (Parallel Transport)
A vector field X on $\mathcal{M}$ is said to be *parallely transported* along a smooth curve $\gamma : \mathbb{R} \to \mathcal{M}$ if

$$\boxed{\nabla_{v_\gamma} X = 0\,.} \tag{109}$$

Another way of writing this is

$$\left(\nabla_{v_{\gamma,\gamma(\lambda)}} X\right)_{\gamma(\lambda)}\,, \quad \forall \lambda \tag{110}$$

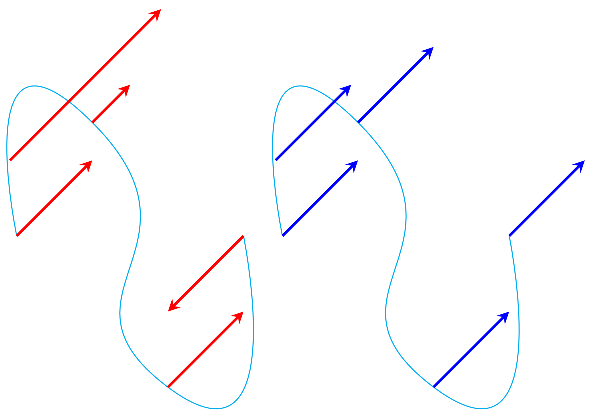
Note: v_γ is not a vector field, but a vector at each point of the curve. Here it is actually important that the derivative $\nabla_Y X$ only needs a vector field X and a vector Y at the point where the derivative is taken!

***Definition* 7.2** (Parallel)
A vector X is said to be parallel along the curve γ if

$$\left(\nabla_{\gamma,\gamma(\lambda)} X\right)_{\gamma(\lambda)} = \mu(\lambda) X_{\gamma(\lambda)}\,, \tag{111}$$

for $\mu : \mathbb{R} \to \mathbb{R}$. This is a weaker notion than *parallel transported*.

Example: *Euclidean plane* $(\mathbb{R}^2, \mathcal{O}, \mathcal{A}, \nabla_E)$ The red arrows (left picture) are *parallel* along the curve and the blue arrows (right picture) are *parallel transported* along the curve.



Note: Explanation by Schuller:

- *Parallel transport*: Pinocchio move along the curve and point your nose in the same direction always and DO NOT LIE.
- *Parallel*: Now you're allowed to lie.

7.2 Autoparallelly Transported Curves

***Definition* 7.3** (Autoparallelly transported)
A curve $\gamma : \mathbb{R} \to \mathcal{M}$ is called *autoparallelly transported* (or just *autoparallel*) if

$$\nabla_{v_\gamma} v_\gamma = 0\,. \tag{112}$$

or (this is the same)

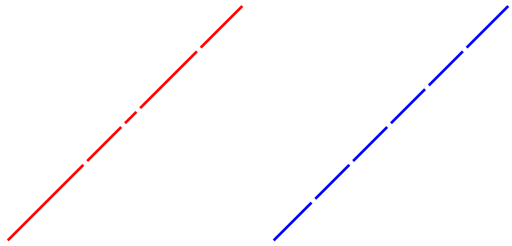
$$\left(\nabla_{v_{\gamma,\gamma(\lambda)}} v_\gamma\right)_{\gamma(\lambda)} = 0\,. \tag{113}$$

Note: An autoparallel curve is

$$\nabla_{v_\gamma} v_\gamma = \mu v_\gamma\,. \tag{114}$$

even though most of the time one also uses the notion "autoparallel" for an autoparallelly transported curve. An autoparallelly transported curve is the "straightest curve possible".

Example: *Euclidean plane* $(\mathbb{R}^2, \mathcal{O}, \mathcal{A}, \nabla_E)$ Red (left): autoparallel curve, blue (right): autoparallelly transported curve. Equal distances mean equal affine parameter λ.



7.3 Autoparallel Equation

Let γ be an autoparallelly transported curve. Consider that portion of the curve that lies in U, where $(U, x) \in \mathcal{A}$ (atlas). Express $\nabla_{v_\gamma} v_\gamma = 0$ (condition for the curve to be autoparallelly transported) in terms of chart representatives: Using $v_{\gamma,\gamma(\lambda)} = \dot\gamma^m_{(x)}(\lambda) \left(\frac{\partial}{\partial x^m}\right)_{\gamma(\lambda)}$ and $\gamma^m_{(x)} := x^m \circ \gamma$ we get

$$\begin{aligned}\nabla_{v_\gamma} v_\gamma &= \left(\nabla_{\dot\gamma^m_{(x)}\left(\frac{\partial}{\partial x^m}\right)} \dot\gamma^n_{(x)} \frac{\partial}{\partial x^n}\right) \\ &= \underbrace{\dot\gamma^m \frac{\partial \dot\gamma^q}{\partial x^m}}_{\ddot\gamma^m_{(x)}} \frac{\partial}{\partial x^q} + \dot\gamma^m \dot\gamma^n \Gamma^q{}_{nm} \frac{\partial}{\partial x^q}\,. \end{aligned} \tag{115}$$

In summary we have the chart expression of the condition that γ be autoparallelly transported:

$$\boxed{\ddot\gamma^m_{(x)}(\lambda) + \Gamma^m{}_{ab}(\gamma(\lambda))\dot\gamma^a(\lambda)\dot\gamma^b(\lambda) = 0}\,. \tag{116}$$

As we will see later if for Γ we choose the so called *Levi Civita connection* then this is the *geodesic equation*.

Examples

1. Euclidean plane:
   $U = \mathbb{R}^d$, $x = \mathrm{id}_{\mathbb{R}^d}$, $\Gamma_{(x)}{}^i{}_{jl} = 0$, $\Rightarrow$ $\ddot\gamma^m_{(x)} = 0$
   $\Rightarrow \gamma^m_{(x)}(\lambda) = a^m\lambda + b^m$, $a, b \in \mathbb{R}^d$
2. Round sphere $(S^2, \mathcal{O}, \mathcal{A}, \nabla_{\text{round}})$:
   The sphere S^2 as a manifold does not contain the notion of distances like we are used to from a sphere. Also a squished and stretched sphere is still a sphere. Only when we choose a specific connection ∇_{round} we get what we usually see as the sphere, but it's actually the *round sphere*. Consider a chart (polar coordinates)

$$\begin{aligned} x(p) &= (\theta, \phi)\,, \\ \theta \in (0, \pi)\,, &\quad \phi \in (0, 2\pi) \\ \Gamma^1_{(x)22}(x^{-1}(\theta,\phi)) &:= -\sin\theta\cos\theta\,, \quad (117) \\ \Gamma^2_{(x)21} = \Gamma^2_{(x)12} &:= \cot\theta\,. \quad (118) \end{aligned}$$

and all other Γs zero. Using sloppy notation

$$x^1(p) = \theta p \,, x^2(p) = \phi(p) \,, \tag{119}$$

the autoparallel equation becomes

$$\ddot\theta - \sin\theta\cos\theta\,\dot\phi\dot\phi = 0 \,, \tag{120}$$
$$\ddot\phi + 2\cot\theta\,\dot\theta\dot\phi = 0 \,. \tag{121}$$

For example a solution is

$$\theta(\lambda) = \pi/2 = \text{const} \,, \tag{122}$$
$$\phi(\lambda) = \omega\lambda + \phi_0 \,. \tag{123}$$

This is a curve around the equator with constant speed. Similarity other curves along great circles are solutions.

Note: Thus if someone gives you the connection ∇_{potato} on a potato, you can calculate the straightest curves on that potato. Still, the potato is a 2-sphere S^2 as a smooth manifold.

7.4 Torsion

***Definition* 7.4** (Commutator)
The commutator between two vector fields X and Y is defined as

$$[X, Y]f := X(Yf) - Y(Xf) \,. \tag{124}$$

***Definition* 7.5** (Torsion)
The *torsion* of a connection ∇ is the $(1,2)$-tensor field

$$T(\omega, X, Y) := \omega\left(\nabla_X Y - \nabla_Y X - [X, Y]\right) \,, \tag{125}$$

Proof that T is a tensor: Check T is C^∞-linear in each entry

1.

$$T(f\omega, X, Y) = f\omega(\ldots) = fT(\omega, X, Y) \,, \tag{126}$$

$$\begin{aligned} T(\omega + \psi, X, Y) &= (\omega + \psi)(\ldots) \\ &= T(\omega, X, Y) + T(\psi, X, Y) \,, \end{aligned} \tag{127}$$

2.

$$\begin{aligned} T(\omega, fX, Y) &= \omega\left(\nabla_{fX} Y - \nabla_Y (fX) - [fX, Y]\right) \\ &= \omega\,[f\nabla_X Y - f\nabla_Y X - \cancel{(Yf)X} - fXY \\ &\quad - fYX - \cancel{(Yf)X}] = fT(\omega, X, Y) \,, \end{aligned} \tag{128}$$

where we have used

$$\begin{aligned} [fX, Y]g &= fX(Yg) - Y(fXg) \\ &= fX(Yg) - (Yf)Xg - f(YXg) \,, \end{aligned} \tag{129}$$

and $\nabla_Y f = Yf$. Since $T(\omega, X, Y) = -T(\omega, Y, X)$ we don't have to check the scaling in the last argument and the additivity in the middle argument also is easy.

***Definition* 7.6** (Torsion-free connection)
$(\mathcal{M}, \mathcal{O}, \mathcal{A}, \nabla)$ is called *torsion-free* if $T = 0$. In a chart:

$$T^i{}_{ab} := T\left(\mathrm{d}x^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) = 2\Gamma^i_{[ab]} \,. \tag{130}$$

From now on we will be focusing on *torsion-free* connections.

7.5 Curvature

7.5.1 Riemann Curvature Tensor

***Definition* 7.7** (Riemann Curvature)
The *Riemann Curvature* of a connection ∇ is the (1,3)-tensor field

$$\begin{aligned} \mathrm{Riem}(\omega, Z, X, Y) := w\,(&\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \\ &- \nabla_{[X,Y]} Z) \,. \end{aligned} \tag{131}$$

Note: The Riemmann curvature tensor contains all information about the curvature. For a two-dimensional manifold the Ricci tensor is enough.

Note: Of course one has to show that Riem is C^∞-linear in each slot. The first slot is trivial, I will show it for the second:

$$\begin{aligned} &\mathrm{Riem}(\omega, fZ, X, Y) := \\ &= \omega\left(\nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ)\right) \,. \\ &= \omega\,[\nabla_X((Yf)Z) + f\nabla_Y Z) - \\ &\quad \nabla_Y((Xf)Z - f\nabla_X Z) - ([X, Y]f)Z - f\nabla_{[X,Y]}Z] \\ &= \omega\,[((XY - YX)f)Z + f(\nabla_X \nabla_Y - \nabla_Y \nabla_X)Z \\ &\quad -([X, Y]f)Z - f\nabla_{[X,Y]}Z] \\ &= f\,\mathrm{Riem}(\omega, Z, X, Y) \,. \end{aligned}$$

The third (and by symmetry also forth) argument works the same, one just has to use

$$\begin{aligned} \nabla_{[fX,Y]} Z &= \nabla_{f[X,Y]Z - (Yf)X} Z \\ &= f\nabla_{[X,Y]} Z - (Yf)\nabla_X Z \,. \end{aligned} \tag{132}$$

7.5.2 Algebraic Relevance of Riem

$$
\begin{aligned}
(\nabla_X \nabla_Y - \nabla_Y \nabla_X)\, Z = \\
\mathrm{Riem}(\cdot, Z, X, Y) + \nabla_{[X,Y]} Z
\end{aligned}
\tag{133}
$$

Let's look at a chart (U, x). We write $\nabla_{\frac{\partial}{\partial x^a}} = \nabla_a$, but be careful, because when writing it like this we throw away the information of the chart in ∇.

$$
\begin{aligned}
\boxed{(\nabla_a \nabla_b Z)^m - (\nabla_b \nabla_a Z)^m = R^m{}_{nab} Z^n} \\
+ \cancel{\nabla_{[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}]} Z}\,,
\end{aligned}
\tag{134}
$$

where with $R^m{}_{nab}$ we now denote the components of Riem in the basis. «««< HEAD

7.5.3 Geometric Relevance of Riem

PUT PICTURE HERE OF $[X, Y] = 0$ AND $[X, Y] \neq 0$. Schuller did a really god job explaining this at the end of lecture 8, but it's hard to write down.

Assuming a torsion-free connection, $T = 0$, then one can imagine curvature as follows. Parallel transporting a vector Z along two different paths from p to q changes the vector. Going infinitesimal and "along" X or Y (first along X and then along Y or the other way round) one can find (for $[X, Y] = 0$)

$$
(\delta Z)^m = R^m{}_{nab} X^a Y^b Z^n \, \delta s \delta t\,,
\tag{135}
$$

plus higher order terms in the "lengths of the curves" δs, δt.

8 Newtonian Spacetime is Curved!?

Let's review Newton's axioms:

1. A body on which *no force* acts moves uniformly along a straight line.
2. *Deviation* of a body's motion from such uniform straight motion is effected by a force, reduced by a factor of the body's reciprocal mass.

One might think that the first axiom is a special case of the second. The problem with that is that the second axiom needs to know what a straight line is. So it might be a better idea to interpret the first axiom as a measurement prescription for the geometry of space that tells us what a straight line is. The second problem is: Gravity acts universally on every particle, so how should the first axiom ever be applicable if there are at least two particles in the universe?

Maybe similar reasoning lead Laplace (1749-1827) to state the following question:

> Can gravity be encoded in a curvature of space, such that particles that are subject to no other force than gravity, move in straight lines in this curved space? In other words: Can we get rid of the gravitational "force" and put it into the geometry of space?[4]

Answer : No!

Proof : Newton's equation and Laplace's equation are

$$
\cancel{m}\ddot{x}^i(t) = \cancel{m} f^i(x(t)) = F^i\,,
\tag{136}
$$
$$
-\partial_i f^i = 4\pi G \rho\,,
\tag{137}
$$
$$
\tag{138}
$$

where F denotes the force, m the mass and ρ the density. If we could encode this in curvature then we should be able to write equation (136) as

$$
\ddot{x}^i(t) - f^i(x(t)) = 0
\tag{139}
$$
$$
= \ddot{x}^i(t) + \Gamma^i_{ab}\dot{x}^a(t)\dot{x}^b(t) = 0
\tag{140}
$$

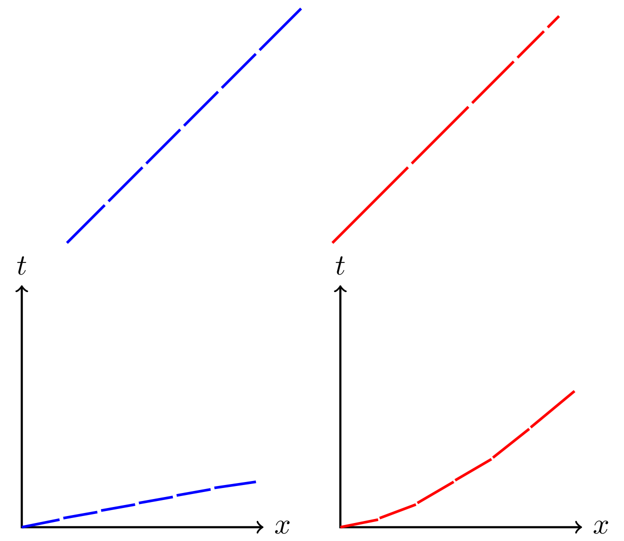
But that is just not possible, since f^i does not depend on the velocity $(\dot{x}^a)$.

But lo and behold: We have not used the word *uniformly* in the first axiom. That basically means that the equal distances are passed in equal times.

Note: A curve is more than the set of its points! It's the set of its points and its parametrization!

The trick is: When we have a curve $\gamma(t)$ we can plot it in a coordinate system with one dimension more that we take as time (just like s-t diagrams in school). Then we basically put the information of the parametrization in the form of the line in this space.

[4] Not sure if I wrote this correctly. The text in the lecture Schuller, 2015 is strangely formulated.



So now let us try not only in space, but in (Newtonian) spacetime:

Let $x : \mathbb{R} \to \mathbb{R}^3$ be the particle's trajectory in space fulfilling Newton's law $\ddot{x}^i = f^i(x(t))$.	$\leftrightarrow$	worldline (history) of the particle $X : \mathbb{R} \to \mathbb{R}^4$ $t \mapsto (t, x^1(t), x^2(t), x^3(t)) := (X^0(t), X^1(t), X^2(t), X^3(t))$

Trivial rewritings:

$$\dot{X}^0 = 1 \tag{141}$$

and

$$\ddot{X}^0 = 0\,, \tag{142}$$
$$\ddot{X}^i - f^i(X(t))\dot{X}^0\dot{X}^0 = 0\,, \quad i = 1,2,3\,, \tag{143}$$

which is equivalent to the autoparallel equation in spacetime

$$\ddot{X}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{X}^\beta\dot{X}^\gamma = 0\,, \quad \alpha = 0,1,2,3\,, \tag{144}$$

with

$$\Gamma^i{}_{00} = -f^i\,, \tag{145}$$

and all other components of Γ zero.

This is not a coordinate-choice artefact, since

$$R^a{}_{0b0} = -\frac{\partial}{\partial x^b} f^a \neq 0 \tag{146}$$

and

$$R_{00} = R^\mu{}_{0\mu 0} = -\partial f^a = 4\pi G\rho\,. \tag{147}$$

If you already know the solution (General Relativity) you can cheat and write $T_{00} = \rho/2$ to get

$$R_{00} = 8\pi G T_{00}\,. \tag{148}$$

Thus Newtonian spacetime is curved (only in time) even if we do not have relativity and the curvature is prescribed by the distribution of matter ρ. Uniformly straight in space $\to$ straight in spacetime.

In fact Einstein proposed an equation similar to (148) in 1912, namely

$$R_{\mu\nu} 8\pi G T_{\mu\nu}\,, \quad \mu,\nu = 0,1,2,3\,, \tag{149}$$

which is not entirely correct, but almost.

8.1 Foundations of the Geometric Formulation of Newton's Axioms

***Definition* 8.1** (Newtonian Spacetime)
A Netwonian spacetime (space+time) is a quintuple

$$(\mathcal{M}, \mathcal{O}, \mathcal{A}, \nabla, t)\,, \tag{150}$$

where $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ form a 4-dimensional smooth manifold and the *absolute time*

$$t : \mathcal{M} \to \mathbb{R}\,, \quad \text{smooth function}\,, \tag{151}$$

satisfies

1. There is an absolute space that follows from the existence of absolute time.

$$(\mathrm{d}t)_p \neq 0\,, \quad \forall p \in \mathcal{M}\,, \tag{152}$$

2. Absolute time flows uniformly,

$$\nabla\,\mathrm{d}t = 0\,, \quad \text{everywhere} \tag{153}$$

3. ∇ is torsion free.

***Definition* 8.2** (Absolute space S_τ at time τ)

$$S_\tau := \{p \in \mathcal{M} | t(p) = \tau\}\,, \tag{154}$$

and thus because of $(\mathrm{d}\tau)_p \neq 0$

$$\mathcal{M} = \dot{\bigcup} S_\tau\,, \tag{155}$$

where $\dot{\bigcup}$ means the disjoint union.

This means that S_τ *foliate* spacetime.

Note: By $\nabla\,\mathrm{d}t$ we mean that the argument that $\nabla.$ takes is open, so what comes out is a (0,2)-tensor.

Note: You can view Gravity as curvature of spacetime and already in Newtonian mechanics this is not just an alternative formulation, but if you look at the first axiom there is not really another possible choice. It's not relativity that forces us to use spacetime, it's gravity itself.

***Definition* 8.3**
A vector $X \in \mathrm{T}p\mathcal{M}$ is called

1. future-directed if
$$\mathrm{d}t(X) > 0\,, \tag{156}$$
2. spatial if
$$\mathrm{d}(X) = 0\,, \tag{157}$$
3. past-directed if
$$\mathrm{d}X < 0\,. \tag{158}$$

Newton 1: The worldline of a particle under the influence of no force (gravity isn't a force now) is a *future directed autoparallel,i.e.* everywhere

$$\nabla_{v_X} v_X = 0\,, \mathrm{d}t(v_X) > 0\,. \tag{159}$$

Newton 2: The acceleration of a worldline

$$\underbrace{\nabla_{v_X} v_X}_{a} = \frac{F}{m}\,, \tag{160}$$

where F is a spatial vector field: $\mathrm{d}t(F) = 0$, X is a future directed vector and a is the acceleration.

Convention: Restrict attention to *stratified atlases* $\mathcal{A}_{\text{stratified}}$ whose charts (U, x) have the property

$$x^0 = t\big|_U \tag{161}$$

In a stratified atlas the first axiom becomes

$$0 = \left(\nabla_{\frac{\partial}{\partial x^a}} \mathrm{d}x^0\right)_b = -\Gamma^0_{ba}\,. \tag{162}$$

8.1.1 Geometric Relevance of Riem

PUT PICTURE HERE OF $[X,Y]=0$ AND $[X,Y]\neq 0$. Schuller did a really god job explaining this at the end of lecture 8, but it's hard to write down.

Assuming a torsion-free connection, $T = 0$, then one can imagine curvature as follows. Parallel transporting a vector Z along two different paths from p to q changes the vector. Going infinitesimal and "along" X or Y (first along X and then along Y or the other way round) one can find (for $[X,Y]=0$)

$$(\delta Z)^m = R^m{}_{nab} X^a Y^b Z^n\, \delta s \delta t\,, \tag{163}$$

plus higher order terms in the "lengths of the curves" δs, δt. One contracts the first and the third index, because all others are either zero or equivalent.

Let's evaluate Newton 2 in a chart (U, x) of a stratified atlas $\mathcal{A}_{\text{stratified}}$:

$\boxed{\text{Finish this section}}$

9 Metric Manifolds

9.1 The metric g

We establish a new structure called a *metric* on a smooth manifold $\mathcal{M}$ that allows to assign a length to each vector X in each tangent space $X \in \mathrm{T}_p\mathcal{M}$ and an angle between vectors in the same tangent space.

Since the velocity $v_{\gamma,p}$ of a curve γ at the point $p \in \mathcal{M}$ (defined by equation (27)) is a vector, we can then integrate up the lengths of the velocities to get the length of a curve.

Then we require that the shortest curves are also the straightest curves, $\nabla_{v_\gamma} v_\gamma = 0$, which will result in the metric determining the connection ∇ if there is no torsion ($T = 0$) and thus also the curvature.

***Definition* 9.1** (Metric)
A metrig g on a smooth manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ is a (0,2)-tensor field satisfying

- *symmetry*:
$$g(X,Y) = g(Y,X)\,, \quad \forall X, Y \in \Gamma(\mathrm{T}\mathcal{M})\,, \tag{164}$$
- *non-degeneracy:* There are no non-zero vectors $X \in \mathrm{T}p\mathcal{M}$ with
$$g(X,Y) = 0 \quad \forall Y \in \mathrm{T}_p\mathcal{M}\,. \tag{165}$$

***Definition* 9.2** (Inverse Metric g^{-1})
The symmetric (2,0)-tensor field g^{-1} with respect to a metric g is

$$g^{-1} : \Gamma(\mathrm{T}^*\mathcal{M}) \times \Gamma(\mathrm{T}^*\mathcal{M}) \xrightarrow{\sim} C^\infty(\mathcal{M}) \tag{166}$$
$$(\omega, \sigma) \mapsto w\left(\flat^{-1}(\sigma)\right)\,. \tag{167}$$

***Definition* 9.3** (Musical map $\flat$)
The musical map ("flat")

$$\flat : \Gamma(\mathrm{T}\mathcal{M}) \to \Gamma(\mathrm{T}^*\mathcal{M}) \tag{168}$$
$$X \mapsto \flat(X)\,, \tag{169}$$

where
$$\flat(X)(Y) := g(X,Y)\,, \tag{170}$$

i.e. the musical map is like a partial evaluation of the metric, $\flat(X) = g(X, \cdot)$ and can also be written with indices and the so called raising and lowering of indices

$$X_a := g_{am} X^m := (\flat(X))_a \tag{171}$$
$$X^a := g^{am} X_m := \left(\flat^{-1}(X)\right)^a \tag{172}$$

Note: In a chart $g_{ab} = g_{ba}$ and

$$\left(g^{-1}\right)^{am} g_{mb} = \delta^a_b \,, \tag{173}$$

but the inverse metric is not really an inverse of the metric. We see this by looking at the spaces:

$$g : \Gamma(\mathrm{T}\mathcal{M}) \times \Gamma(\mathrm{T}\mathcal{M}) \to C^\infty(\mathcal{M}) \,, \tag{174}$$
$$g^{-1} : \Gamma(\mathrm{T}^*\mathcal{M}) \times \Gamma(\mathrm{T}^*\mathcal{M}) \to C^\infty(\mathcal{M}) \,, \tag{175}$$

so it is not really the inverse map, but the inverse matrix in the sense of equation (173).

Note: Pulling down or up indices is a dangerous business. It means we are suppressing the metric and hiding that the object depends on the metric. Actually it then is not clear if X_a are the components of a genuine one form or if it is constructed by pulling down the index of the index of a vector and hiding the metric.

Example: Sphere $(S^2, \mathcal{O}, \mathcal{A})$ with a chart (U, x), $\phi \in (0, 2\pi)$, $\theta \in (0, \pi)$.

$$g_{ij}\left(x^{-1}(\theta, \phi)\right) = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}_{ij} \,, \tag{176}$$

is the metric of the *round sphere* of radius $R \in \mathbb{R}^+$.

9.2 Signature

Remember *linear algebra*: Eigenvalues and eigenvectors

$$A^a{}_m v^m = \lambda v^a \,. \tag{177}$$

How does this translate the notion of eigenvectors to our case of a metric? $A^a{}_m$ is a (1,1)-tensor and eigenvectors are a good notion, but for a (0,2)-tensor eq. (177) does not work

$$g_{am} v^m \neq \lambda v^a \,. \tag{178}$$

Note: A (1,1) tensor cannot be symmetric on its own, it can only be symmetric with respect to a metric, *i.e.* one can pull down an index and then switch indices.

- A (1,1)-tensor has eigenvalues and can be transformed to look like

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \,, \tag{179}$$

  with eigenvalues λ_i.
- A (0,2)-tensor like the metric has a *signature* (p, q) and can be transformed to

$$\mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}, \underbrace{0, \ldots, 0}_{\dim V - p - q}) \,, \tag{180}$$

  which we can agree has way less information than the eigenvalues.

Note: The condition that the musical isomorphism $\flat$, eq. (168) is invertible means that there are no zeros in the signature. Basically a zero would mean that a whole subspace is mapped to zero and this is not invertible.

***Definition* 9.4** (Riemannianian and Lorentzian Metric)

- A metric is called *Riemannian* if its signature is $(+, \ldots, +)$ (or $(-, \ldots, -)$ is equivalent).
- A metric is called *Lorentzian* if its signature is $(+, -, \ldots, -)$ (or $(-, +, \ldots, +)$ is equivalent). We will chose $(+, -, \ldots, -)$ here. This is what we need for General Relativity.
- All other signature including Lorentzian metrics are called *pseudo Riemannian*.

Note: One might call a non-Riemannian metric a *pseudo metric*, since there are nonzero vectors that have zero length under such a metric. In a Lorentzian manifold one says they lie on the light cone.

Note: Generally the metric itself will change from point to point in spacetime $\mathcal{M}$, but the signature stays.

9.3 Length of a Curve

Let γ be a smooth curve. Then we know its velocity $v_{\gamma,\gamma(\lambda)}(f) := (f \circ \gamma)'(\lambda)$ at each $\gamma(\lambda) \in \mathcal{M}$ from definition 5.1.

***Definition* 9.5** (Speed of a curve)
On a *Riemannian metric manifold* $(\mathcal{M}, \mathcal{O}, \mathcal{A}, G)$ the *speed* of a curve γ at $\gamma(\lambda)$ is the number

$$s(\lambda) = \left(\sqrt{g(v_\gamma, v_\gamma)}\right)_{\gamma(\lambda)} \,. \tag{181}$$

Basically its just the magnitude of the velocity.

Note: I feel the need, the need for ~~speed~~ a metric to define speed.

Note: The physical dimensions are

$$[v^a] = \frac{1}{T}\,,$$
$$[g_{ab}] = L^2\,,$$
$$[\sqrt{g_{ab}v^a v^b}] = \frac{L}{T}\,.$$

The idea that coordinate distance has anything to do with real distance is just wrong. Going double as far in coordinates has nothing to do with going double as far in "reality" (the manifold $\mathcal{M}$).

***Definition* 9.6** (Length of a curve)
Let $\gamma : (0,1) \to \mathcal{M}$ be a smooth curve. Then the *length of* γ is the number

$$\begin{aligned} L[\gamma] &:= \int_0^1 \mathrm{d}\lambda\, s(\lambda) \qquad (182)\\ &= \int_0^1 \mathrm{d}\lambda\, \sqrt{(g(v_\gamma, v_\gamma)_{\gamma(\lambda)}}\,. \end{aligned}$$

It is a functional, *i.e.* a function is mapped to a number.

Note: It's exactly the other way than one usually thinks. Velocity is more fundamental than speed and speed is more fundamental than length.

Example: The round sphere of radius R. Its equator is a curve

$$\theta(\lambda) := (x^1 \circ \gamma)(\lambda) = \frac{\pi}{2}\,, \qquad (183)$$
$$\phi(\lambda) := (x^2 \circ \gamma)(\lambda) = 2\pi\lambda^3\,, \qquad (184)$$

where we have randomly chosen any parametrization that has $\phi(0) = 0$, $\phi(1) = 2\pi$. Then the components of the velocity are (eq. (40))

$$\begin{aligned} v^i &= \dot{\gamma}^i_x(0) := (x^i \circ \gamma)'(0)\,,\\ v^1 &= \left(\frac{\pi}{2}\right)' = 0\,,\\ v^2 &= \left(2\pi\lambda^3\right)' = 6\pi\lambda^2\,. \end{aligned}$$

and $g_{ij} = \operatorname{diag}(R^2, R^2\sin^2\theta)$ the length of the curve around the equator ($\theta = \pi/2$, $\sin(\theta) = 1$)

$$\begin{aligned} L[\gamma] &= \int_0^1 \mathrm{d}\lambda\, \sqrt{R^2 \cdot 0 + R^2 \sin^2(\theta(\lambda))(6\pi\lambda^2)^2}\\ &= \int_0^1 \mathrm{d}\lambda\, R 6\pi\lambda^2 = 2\pi R\,, \end{aligned}$$

or just to have it written down in a rigorous way

$$L[\gamma] = \int_0^1 \mathrm{d}\lambda\, \sqrt{g_{ij}(x^{-1}(\theta(\lambda), \phi(\lambda))(x^i \circ \gamma)'(\lambda)(x^j \circ \gamma)'(\lambda)}$$

***Theorem* 9.7** (Reparametrization invariance of the lengh of a curve)
Let $\gamma : (0,1) \to \mathcal{M}$ be a curve and $\sigma : (0,1) \to (0,1)$ a smooth bijection and increasing (don't drive back on the curve), then the reparametrized curve has the same length,

$$L[\lambda] = L[\gamma \circ \sigma]\,. \qquad (185)$$

9.4 Geodesics

***Definition* 9.8** (Geodesic)
A curve $\gamma : (0,1) \to \mathcal{M}$ is called a *geodesic* on a Riemannian manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A}, g)$ is a *stationary* curve with respect to the length functional $L[\gamma]$.

***Theorem* 9.9**
A curve γ is a *geodesic* iff it satisfies the Euler-Lagrange equations for the Lagrangian

$$\mathcal{L} : \mathrm{T}\mathcal{M} \to \mathbb{R}\,, \qquad (186)$$
$$X \mapsto \sqrt{g(X,X)}\,. \qquad (187)$$

In a chart the Euler-Lagrange equations take the form (chart dependent)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m}\right) - \frac{\partial \mathcal{L}}{\partial \gamma^m} = 0\,, \qquad (188)$$

where

$$\mathcal{L}(\gamma^i, \dot{\gamma}^i) = \sqrt{g_{ij}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda)}\,. \qquad (189)$$

Plugging the Lagrangian (189) into the Euler-Lagrange equations (188) and using the parametrization of γ such that $g(\dot{\gamma}, \dot{\gamma}) = 1$ (always driving at unit speed) we get after raising the index with $(g^{-1})^{qm}$

$$\ddot{\gamma}^q + \overbrace{\frac{1}{2}\left(g^{-1}\right)^{qm}\left(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}\right)}^{:= {}^{\text{L.C.}}\Gamma^q{}_{ij}(\gamma(\lambda))} \dot{\gamma}^i \dot{\gamma}^j = 0\,. \qquad (190)$$

Equation (190) is the *geodesic equation* for γ. We call ${}^{\text{L.C.}}\Gamma^q{}_{ij}$ the *Levi-Civita connection coefficient functions*.

***Definition* 9.10** (Levi-Civita connection)
The *Levi-Civita connection* coefficient functions ${}^{\text{L.C.}}\Gamma^{q}{}_{ij}(\gamma(\lambda))$ (also called *Christoffel symbols* or "Christ awful symbols" because of the labour needed to calculate them) of the so called *Levi-Civita connection* ${}^{\text{L.C.}}\nabla$ and they are

$$\boxed{{}^{\text{L.C.}}\Gamma^{q}{}_{ij} = \frac{1}{2}\left(g^{-1}\right)^{qm}\left(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}\right)} \tag{191}$$

Note: If we use the Levi-Civita connection as the connection on our manifold, then the geodesic equation

$$\ddot{\gamma}^{q} + \Gamma^{q}{}_{ij}\dot{\gamma}^{i}\dot{\gamma}^{j} = 0\,, \tag{192}$$

is exactly the equation (115) for an autoparallelly transported curve, *i.e.* for a curve that is as straight as possible. Choice of the connection as Christoffel conection thus means we identify autoparallelly transported curves (straight as possible) with the shortest curves.

Note: Thus a metric manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A}, g)$ implies a manifold with the Christoffel connection

$$(\mathcal{M}, \mathcal{O}, \mathcal{A}, g) \to (\mathcal{M}, \mathcal{O}, \mathcal{A}, g, {}^{\text{L.C.}}\nabla)\,, \tag{193}$$

and we usually make this choice of connection.

Note: If for a metric manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A}, g)$ one imposes

1. *Metric compatibility:* $\nabla g = 0$,
2. *Absence of torsion:* $T = 0$,

then the connection is already fixed to be the Levi Civita connection $\nabla = {}^{\text{L.C.}}\nabla$. This is the way many General Relativity textbooks go. They impose metric compatibility and write the equation $\nabla_i g_{ab}$ in three permutations, add them in some way and find an expression for ∇. Sadly they usually don't talk about implicitly identifying autoparallelly transported curves with the shortest curves by doing this.

***Definition* 9.11** (Riemann-Christoffel Curvature)
The *Riemann-Christoffel Curvature* R_{abcd} of a manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A}, g)$ is defined by (in coordinates)

$$R_{abcd} := g_{am}R^{m}{}_{bcd}\,, \tag{194}$$

where the connection used to calculate the Riemann tensor $R^{m}{}_{bcd}$ is the Levi-Civita connection.

Note: In contrast to the Riemann curvature (131), which only needs a connection, the Riemann-Christoffel cuvature also needs a metric.

***Definition* 9.12** (Ricci Tensor)
The *Ricci tensor* R_{ab} is defined by (in coordinates)

$$R_{ab} := R^{m}{}_{amb}\,, \tag{195}$$

where again for the connection to calculate $R^{c}{}_{amb}$ the Levi-Civita connection is used.

***Definition* 9.13** (Ricci Scalar Curvature)
The *Ricci scalar curvature* is

$$R := g^{ab}R_{ab}\,, \tag{196}$$

where we have introduced the *convention*

$$g^{ab} := (g^{-1})^{ab}\,. \tag{197}$$

***Definition* 9.14** (Einstein Curvature)
The *Einstein Curvature* G_{ab} is defined by

$$G_{ab} := R_{ab} - \frac{1}{2}g_{ab}R\,. \tag{198}$$

10 Symmetry

We have the feeling that the *round sphere*

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{round}}) \tag{199}$$

has rotational symmetry, while the potato

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{potato}}) \tag{200}$$

does not.

10.1 Push-Forward

***Definition* 10.1** (Push-forward map)
Let $\mathcal{N}$ and $\mathcal{M}$ be smooth manifolds. Let ϕ be a map $\phi : \mathcal{M} \to \mathcal{N}$. Then the *push-forward* ϕ_* is the map

$$X \mapsto \phi_*(X)\,, \tag{201}$$

where $\forall f \in C^\infty(\mathcal{N})$ and $X \in \mathrm{T}\mathcal{M}$

$$\phi_*(X)f := X(f \circ \phi)\,. \tag{202}$$

$$\begin{tikzcd}
\mathrm{T}\mathcal{M} \arrow[r, "\phi_*"] \arrow[d, "\pi_{\mathrm{T}\mathcal{M}}"] & \mathrm{T}\mathcal{N} \arrow[d, "\pi_{\mathrm{T}\mathcal{N}}"] & \\
\mathcal{M} \arrow[r, "\phi"] & \mathcal{N} \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

Vectors are pushed forward.

Note: Not much happens in the push forward. Take a vector X on $\mathcal{M}$ and the push-forward gives a vector on $\mathcal{M}$ that has the same effect on a function f as X would have on $(f \circ \phi)$. One can also take the push-forward of a whole fiber and then

$$\phi_*(\mathrm{T}_p\mathcal{M}) \subseteq T_{\phi(p)}\mathcal{N}\,, \tag{203}$$

Components ϕ^a_{*i} of ϕ_* wrt. two charts $(U, x) \in \mathcal{A}_\mathcal{M}$ and $(V, y) \in \mathcal{A}_\mathcal{N}$, where $i = 1, \dots, \dim \mathcal{M}$ and $a = 1, \dots, \dim \mathcal{N}$.

$$
\begin{tikzcd}
M \supset U \arrow[r, "\phi"] \arrow[d, "x"] \arrow[dr, "\hat{\phi}"] & V \supset \mathcal{N} \arrow[d, "y"] \\
x(U) \arrow[r, "y \circ \phi \circ x^{-1}"'] & y(U)
\end{tikzcd}
$$

Remember the definition $\mathrm{d}g(X) = X(g)$ in equation (53), then

$$
\begin{aligned}
\phi^a_{*i} &= \mathrm{d}y^a \left(\phi_* \left(\frac{\partial}{\partial^i} \right)_p \right) \\
&= \phi_* \left(\frac{\partial}{\partial x^i} \right)_p y^a \qquad (204) \\
&= \left(\frac{\partial}{\partial x^i} \right)_p (y \circ \phi)^a = \left(\frac{\partial \hat{\phi}^a}{\partial x^i} \right)_p .
\end{aligned}
$$

Note: To better understand the push-forward take a smooth curve $\gamma : \mathbb{R} \to \mathcal{M}$ with a tangent vector $v_{\gamma,p} : C^\infty(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}$. Then with ϕ one can map the whole curve to $\mathcal{N}$. So what happens to the curve γ is described by the map ϕ and what happens to the tangent vector $v_{\gamma,p}$ is described by the *push-forward* ϕ_*,

$$
\phi_*(v_{\gamma,p} li) = v_{\phi\circ\gamma,\phi(p)} . \qquad (205)
$$

> The push-forward pushes tangent vectors of curves forward to tangent vectors of the mapped (pushed forward) curves.

Proof: Let $\gamma(\lambda_0) = p$, then $\forall f \in C^\infty(\mathcal{N})$

$$
\begin{aligned}
\phi_*(v_{\gamma,p}) f &= v_{\gamma,p}(f \circ \phi) \\
&= ((f \circ \phi) \circ \gamma)' (\lambda_0) \\
&= (f \circ (\phi \circ \gamma))' (\lambda_0) \\
&= v_{\phi\circ\gamma,\phi(p)} f . \qquad (206)
\end{aligned}
$$

□

Note: If $\dim \mathcal{M} < \dim \mathcal{N}$ then ϕ is (or can be?) an embedding. Then ϕ_* converts intrinsic tangent vectors in $\mathcal{M}$ to extrinsic tangent vectors in $\mathcal{N}$, just like we often imagine tangent vectors to come out of the manifold into a higher space.

10.2 Pull-Back

***Definition* 10.2** (Pull-back)
Let $\phi : \mathcal{M} \to \mathcal{N}$ be a smooth map between two manifolds $\mathcal{M}$ and $\mathcal{N}$. Then the *pull-back* ϕ^* of ϕ is

$$
\begin{aligned}
\phi^* : \mathrm{T}^*\mathcal{N} &\to \mathrm{T}^*\mathcal{M} \qquad (207) \\
\omega &\mapsto \phi^*(\omega), \qquad (208)
\end{aligned}
$$

where

$$
\phi^*(\omega)(X) := \omega\left(\phi_*(X)\right) , \qquad (209)
$$

for $\omega \in \mathrm{T}^*\mathcal{N}$ and $X \in \mathrm{T}\mathcal{M}$. So a form ω is pulled back, $\phi_*(\omega)$, such that its action on a vector X is the same as the action of ω on the pushed forward vector $\phi_*(X)$ on ω.

> Forms are pulled-back.

Components of the pull-back wrt. charts are the same as for the pull-back, which can be seen by just plugging in definitions of the pull-back, differential $\mathrm{d}y$ and the push-forward,

$$
\begin{aligned}
\phi^{*a}_i &:= \phi^* \left((\mathrm{d}y^a)_{\phi(p)} \right) \left(\left(\frac{\partial}{\partial x^i} \right)_p \right) \qquad (210) \\
&= \dots = \phi^a_{*i} . \qquad (211)
\end{aligned}
$$

- push-forward:

$$
(\phi_*(X))^a := \phi^a_{*i} X^i , \qquad (212)
$$

- pull-back:

$$
(\phi^*(\omega))_i := \phi^{*a}_i \omega_a . \qquad (213)
$$

Note: The picture is that if we have a function $f : \mathbb{R} \to \mathcal{N}$ and take its gradient $\mathrm{d}f$, we can pull back the gradient to $\mathcal{M}$, which should be the same as taking the gradient after "pulling back" the function f with ϕ:

$$
\phi^*(\mathrm{d}f) = \mathrm{d}(f \circ \phi) . \qquad (214)
$$

$$
\boxed{
\begin{array}{c}
\mathrm{T}\mathcal{M} \xrightarrow[\text{push-forward}]{\phi_*} \mathrm{T}\mathcal{N} \\[2em]
\mathrm{T}^*\mathcal{M} \xleftarrow[\text{pull-back}]{\phi^*} \mathrm{T}^*\mathcal{N}
\end{array}
}
$$

Note: If ϕ is invertible of course one can also pull back a vector by pushing it forward with the inverse map, same for covectors/forms.

10.3 Induced Metric

An important application of the pull-back is when we have an embedding of one manifold $\mathcal{M}$ in another, higher dimensional one, $\mathcal{N}$,

$$\mathcal{M} \underset{\text{injective}}{\overset{\phi}{\hookrightarrow}} \mathcal{N}\,, \tag{215}$$

From the metric g on $\mathcal{N}$ and the inclusion map ϕ we can calculate the *induced metric* $g_{\mathcal{M}}$. For $X, Y \in \mathrm{T}_p\mathcal{M}$

$$g_{\mathcal{M}}(X,Y) := g\left(\phi_*(X), \phi_*(Y)\right)\,, \tag{216}$$

$$(g_{\mathcal{M}})_{ij,p} = (g_{ab})_{\phi(p)} \left(\frac{\partial \hat{\phi}^a}{\partial x^i}\right)_{\phi(p)} \left(\frac{\partial \hat{\phi}^b}{\partial x^j}\right)_{\phi(p)}\,. \tag{217}$$

where again $\hat{\phi}^a = (y \circ \phi)^a$ like in equation (204).

Note: This way we can for example get the induced metric of a 2-sphere S^2 in $\mathbb{R}^3$. This apparently is done in the tutorials of Schuller, 2015 and I should do it!

Note: Think of $\mathcal{M} = S^2$ embedded in $\mathcal{N} = \mathbb{R}^3$. The length of a curve γ is defined by eq. (183), so basically by the tangent vectors. Now one can map the whole curve to $\mathcal{N}$ with ϕ. That means the tangent vectors of γ in $\mathcal{N}$ are the pushed forward tangent vectors from $\mathcal{M}$. I think we basically require that the lenghts of both curves are the same.

11 Flow of a Complete Vector Field

Let $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ be a smooth manifold and X be a vector field on $\mathcal{M}$.

***Definition* 11.1** (Integral Curve of a Vector Field)
A curve $\gamma : \mathbb{R} \supseteq I \to \mathcal{M}$ is called an *integral curve* of X if

$$v_{\gamma,\gamma(\lambda)} = X_{\gamma(\lambda)}\,, \tag{218}$$

i.e. at every point the tangent vector of the curve is the vector of the vector field at that point, see figure 8.

Note: Think of X as the velocity of water molecules in a river and γ the trajectory of a paper ship that you throw into that river.

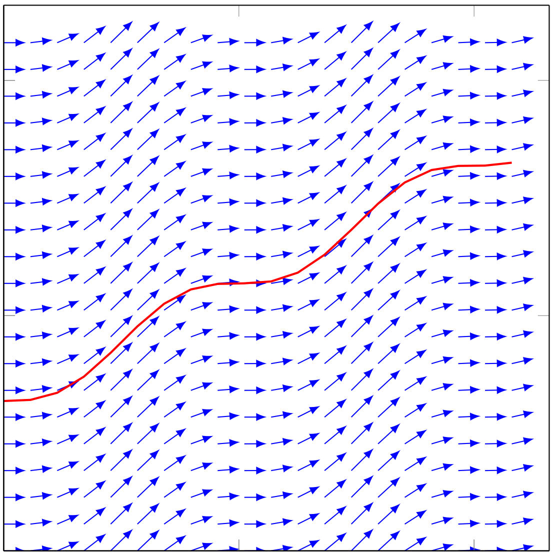


Figure 8: *One integral curve of a vector field. Plot adapted from BambOo, 2018*

***Definition* 11.2** (Complete Vector Field)
A vector field X is said to be *complete* if all integral curves have $I = \mathbb{R}$.

Note: You cannot just reparametrize the curve with something like an arctan, because then it is no longer an integral curve!

***Theorem* 11.3**
A compactly supported smooth vector field is complete. (Compact: Any open cover a finite subcover, we don't have to understand this at the current point.)

***Definition* 11.4** (Flow of a Vector Field)
The *flow of a vector field* X is a one-parameter family

$$h^X : \mathbb{R} \times \mathcal{M} \to \mathcal{M}\,, \tag{219}$$

$$(\lambda, p) \mapsto \gamma_p(\lambda)\,, \tag{220}$$

where $\gamma_p : \mathbb{R} \to \mathcal{M}$ is the integral curve of X with $\gamma(0) = p$.

Note: For a fixed $\lambda \in \mathbb{R}$

$$h^X_\lambda : \mathcal{M} \to \mathcal{M}, \quad \text{smooth}\,, \tag{221}$$

it takes points of the manifold and pushes them further along the curve γ a parameter distance λ. The same can be done for a set $S \in \mathcal{M}$ and in general

$$h^X_\lambda(S) \neq S\,, \quad (\text{if } X \neq 0)\,. \tag{222}$$

11.1 Lie Subalgebras of the Lie Algebra $(\Gamma(\mathrm{T}\mathcal{M}, [\cdot,\cdot])$ of Vector Fields

Lie algebra:

$$\Gamma(\mathrm{T}\mathcal{M}) = \{\text{set of all vector fields}\} \tag{223}$$

is a $C^\infty(\mathcal{M})$-module. We can also restrict ourselfs to the $\mathbb{R}$-vector space, *i.e.* only multiplication by numbers, not functions. Then the *Lie bracket* is $[X, Y] \in \Gamma(\mathrm{T}\mathcal{M})$, where again

$$[X, Y]f := X(Yf) - Y(Xf)\,, \tag{224}$$

with properties

1. $[X, Y] = -[Y, X]\,,$
2. $[\lambda X + Z, Y] = \lambda[X, Y] + [Z, Y]\,,$
3. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ (Jacobi identity).

Every structure that fulfills above items and so especially $(\Gamma(\mathrm{T}\mathcal{M}), [\cdot,\cdot])$ is called a *Lie algebra*.

Lie subalgebra: Let $X_1, \ldots, X_s$ be s many vector fields on $\mathcal{M}$, such that

$$[X_i, X_j]\; C^k{}_{ij} X_k\,, \quad \forall i, j = 1, \ldots, s\,, \tag{225}$$

with the *structure constants* $C^k{}_{ij} \in \mathbb{R}$. Then

***Definition* 11.5** (Lie subalgebra)

$$(\mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_s\}, [\cdot,\cdot])\,, \tag{226}$$

with eq. (225) is called a *Lie subalgebra*.

Example on S^2:

$$[X_1, X_2] = X_3\,, [X_2, X_3] = X_1\,, [X_3, X_1] = X_2\,,$$

$(\mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_s\}, [\cdot,\cdot]) = so(3)$. Note that we did not need any metric, only a smooth manifold $(S^2, \mathcal{O}_{\text{sd}}, \mathcal{A})$ was needed. The vectors are given in a chart (θ, ϕ) by

$$X_1(p) = -\sin(\phi(p))\frac{\partial}{\partial\theta} - \cot(\theta(p))\cos(\phi(p))\frac{\partial}{\partial\phi}\,,$$
$$X_2(p) = \cos(\phi(p))\frac{\partial}{\partial\theta} - \cot(\theta(p)\sin(\phi(p)))\frac{\partial}{\partial\phi}\,,$$
$$X_3(p) = \frac{\partial}{\partial\phi}\,.$$

11.2 Symmetry

***Definition* 11.6** (Symmetry of a Metric)
An s-dimensional Lie subalgebra $(L, [\cdot,\cdot])$ is said to be a *symmetry* of asymmetry metric tensor field g if $\forall X \in L$ (vector field), $\forall A, B \in \mathrm{T}_p\mathcal{M}$, $\forall\lambda \in \mathbb{R}$:

$$g\left((h^X_\lambda)_*(A), (h^X_\lambda)_*(B)\right) = g(A, B)\,, \tag{227}$$

or put differently:

$$(h^X_\lambda)^* g = g\,. \tag{228}$$

11.3 Lie Derivative

If $\forall X \in L$ the *Lie derivative* of the metric

$$\mathcal{L}_X g := \lim_{\lambda\to 0}\frac{(h^X_\lambda)^* g - g}{\lambda} = 0\,, \tag{229}$$

then L is a symmetry of g.

***Definition* 11.7** (The Lie Derivative $\mathcal{L}$)
On a smooth manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A})$, the *Lie derivative* $\mathcal{L}$ sends a pair of a *vector field* X and a (p, q)-*tensor field* T to a (p, q)-tensor field $\mathcal{L}_X T$ such that

1. $\mathcal{L}_X f = Xf\,,\; f \in C^\infty(\mathcal{M})\,,$
2. $(\mathcal{L}_X Yp)_p = [X, Y]_p\,,\; Y \in \Gamma(\mathrm{T}\mathcal{M})\,,$
3. $\mathcal{L}_X(T + S) = \mathcal{L}_X T + \mathcal{L}_X S\,,$
4. $\mathcal{L}_X(T(\omega, Y)) = (\mathcal{L}_X T)(\omega, Y) + T(\mathcal{L}_X\omega, Y) + T(\omega, \mathcal{L}_X Y)\,,$
5. $\mathcal{L}_{X+Y} T = \mathcal{L}_X T + \mathcal{L}_Y T\,.$

It is a good exercise to calculate the components of the Lie derivative in a chart:

$$\begin{aligned}
(\mathcal{L}_X Y)^i &= \mathrm{d}x^i[XY - YX] \\
&= \mathrm{d}x^i\left[X^m\frac{\partial}{\partial x^m}Y^n\frac{\partial}{\partial x^n} - Y^n\frac{\partial}{\partial x^n}X^m\frac{\partial}{\partial x^m}\right] \\
&= \mathrm{d}x^i\left[X^m\left(\frac{\partial}{\partial x^m}Y^n\right)\frac{\partial}{\partial x^n} - Y^n\left(\frac{\partial}{\partial x^n}X^m\right)\frac{\partial}{\partial x^n}\right] \\
&= X^m\frac{\partial}{\partial x^m}Y^i - Y^m\frac{\partial}{\partial x^m}X^i\,,
\end{aligned}$$

where we have used the product rule for derivatives and that the second derivatives commute. Summary:

$$(\mathcal{L}_X Y)^i = X^m\frac{\partial}{\partial x^m}Y^i - Y^m\frac{\partial}{\partial x^m}X^i\,, \tag{230}$$

$$(\nabla_X Y)^i = X^m\frac{\partial}{\partial x^m}Y^i + \Gamma^i{}_{ab}X^a Y^b\,, \tag{231}$$

and in general for the Lie derivative every index up comes with a "-" and every index down comes with a "+".

$$\boxed{(\mathcal{L}_X T)^i_j = X^m\frac{\partial}{\partial x^m}\left(T^i_j\right) - \frac{\partial X^i}{\partial x^m}T^m_j + \frac{\partial X^m}{\partial x^j}T^i_m} \tag{232}$$

∇_X: index up comes with $+\Gamma$, index down with $-\Gamma$.
$\mathcal{L}_X$: index up comes with $-Y^s\partial/\partial x^s$, index down with $+$.

Note: The *connection* ∇_X is $C^\infty(\mathcal{M})$-linear in X, whereas the *Lie derivative* $\mathcal{L}_X$ is not, it is just $\mathbb{R}$-linear. Also we only have to give a vector X to ∇_X in order to take the *derivative* of a tensor field. On the other side $\mathcal{L}_X$ needs a vector field X. For ∇_X we had to introduce a new structure, the Γs, the Lie derivative $\mathcal{L}_X$ needs no new structure.

Note: Use

$$0 = (\mathcal{L}_X g)_{ij} = \dots \tag{233}$$

to check whether a metric features a symmetry.

12 Integration on Manifolds

Now we do the completion of our "lift" of analysis on the charts to the manifold level. We want to define

$$\int_{\mathcal{M}} f\,, \tag{234}$$

and this requires a mild new structure called the *volume form* and we need to restrict the atlas a little bit ("orientation").

12.1 Review of Integration on $\mathbb{R}^d$

1. A function $F : \mathbb{R} \to \mathbb{R}$. Assume a notion of integration is known (Riemann, Lebesque):

$$\int_{(a,b)} F := \int_a^b \mathrm{d}x\, F(x)\,. \tag{235}$$

2. $F : \mathbb{R}^d \to \mathbb{R}$. On a box-shaped domain $I = (a,b) \times (c,d) \times \dots \times (u,v) \subseteq \mathbb{R}^d$,

$$\int_I \mathrm{d}^d x\, F(x) := \int_{(a,b)} \mathrm{d}x^1 \cdots \int_{(u,v)} \mathrm{d}x^d\,. \tag{236}$$

3. Other domains $G \subseteq \mathbb{R}^d$:

***Definition* 12.1** (Indicator function)

$$\mu_G : \mathbb{R}^d \to \mathbb{R}\,,$$
$$x \mapsto \begin{cases} 1\,, & \text{if } x \in G \\ 0\,, & \text{if } x \notin G \end{cases} \tag{237}$$

and then define

$$\int_G \mathrm{d}^d x F(x) := \int_{-\infty}^{\infty} \mathrm{d}x^1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x^d \mu_G(x) F(x)\,, \tag{238}$$

if it exists.

Change of variables

$$\begin{tikzcd} \mathbb{R}^d \supseteq \mathrm{preim}_\phi(G) \arrow[r, "\phi"] \arrow[dr, "F\circ\phi"'] & G \subseteq \mathbb{R}^d \arrow[d, "F"] \\ & \mathbb{R} \end{tikzcd}$$

$$\int_G \mathrm{d}^d x\, F(x) = \int\limits_{\mathrm{preim}_\phi(G)} \mathrm{d}^d\, y\, |\det(\partial.\phi^{\cdot})(y)|\, (F\circ\phi)(y)\,. \tag{239}$$

12.2 Integration on a Chart

Let $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ be a smooth manifold and $f : M \to \mathbb{R}$. Choose two charts $(U, x) \in \mathcal{A} \ni (U, y)$.

$$\begin{tikzcd} y(u) \subseteq \mathbb{R}^d \arrow[dr, "f_{(y)} := f\circ y^{-1}"] & \\ U \arrow[u, "y"'] \arrow[r, "f"] \arrow[d, "x"] & \mathbb{R} \\ x(U) \arrow[uu, bend left, "\phi = y\circ x^{-1}"] \arrow[ur, "f_{(x)} := f\circ x^{-1}"'] & \end{tikzcd}$$

The naive way of integration, namely just integrating $f_{(y)}$ over $y(U)$, *i.e.* just integrating on a chart, *does not work*, because it depends on the chart. We can see it by transforming from chart y to chart x.

$$\begin{aligned} &\int\limits_{y(U)} \mathrm{d}^d\beta f_{(y)}(\beta) = \\ &= \int\limits_{x(U)} \mathrm{d}^d\alpha \left|\det(\partial_a(y^b\circ x^{-1})(\alpha)\right| \left(f_{(y)}\circ(y\circ x^{-1}\right)(\alpha) \\ &= \int\limits_{x(U)} \mathrm{d}^d\alpha \left|\det\left(\frac{\partial y^b}{\partial x^a}\right)_{x^{-1}(\alpha)}\right| f_{(x)}(\alpha) \\ &\neq \int\limits_{x(U)} \mathrm{d}^d\alpha f_{(x)}(\alpha)\,. \end{aligned} \tag{240}$$

So we need to put something into the integral that transforms with the inverse of the factor that is too much.

12.3 Volume Forms

***Definition* 12.2** (Volume Form)
On a smooth manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A})$ a $(0, \dim \mathcal{M})$-tensor field Ω is called a *volume form* if

1. Ω vanishes nowhere,
2. Ω is totally antisymmetric:

$$\Omega(\ldots, X, \ldots, Y, \ldots) = -\Omega(\ldots, Y, \ldots, X, \ldots), \tag{241}$$

for any vectors X, Y in any of the positions of Ω. In a chart we write

$$\Omega_{i_1 \cdots i_d} = \Omega_{[i_1 \cdots i_d]} . \tag{242}$$

On a metric manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A}^\uparrow, g)$ one can construct a volume for Ω from the metric g. In any chart:

$$\Omega_{i_1 \cdots i_d} := \sqrt{\det(g_{(x)ij}} \epsilon_{i_1 \cdots i_d} , \tag{243}$$

where $\epsilon_{i_1 \cdots i_d}$ is the *Levi-Civita symbol* and defined as

$$\epsilon_{123 \cdots d} = 1 , \tag{244}$$
$$\epsilon_{i_1 \cdots i_d} = \epsilon_{[i_1 \cdots i_d]} . \tag{245}$$

It is not a tensor, just see it as a symbol!

For Ω to be well defined we actually need to assume the atlas to be an *oriented* atlas $\mathcal{A}^\uparrow$. The reason is the following: Transforming Ω that comes form g should give a factor that cancels the problematic factor we had in eq. (240), but

$$\begin{aligned}\Omega_{(y)i_1 \cdots i_d} &= \sqrt{\det(g_{(y)ij})}\, \epsilon_{i_1 \ldots i_d} \qquad (246)\\ &= \sqrt{\det\left(g_{(x)mn} \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j}\right)} \epsilon_{m_1 \cdots m_d}\end{aligned}$$

where now one has to be careful with the transformations. The transformation in the metric has $\partial x / \partial y$, which is just the way the metric transforms. On the other hand for the Levi-Civita symbol we employ a trick now: This is for example described in Carroll, 2004: The determinant $\det(A)$ of a matrix A^m_n is

$$\det(A^m_n) \epsilon_{i_1 \cdots i_d} = \epsilon_{m_1 \cdots m_d} A^{m_1}_{i_1} \cdots A^{m_d}_{i_d} , \tag{247}$$

Then setting $A^m_n = \frac{\partial x^m}{\partial y^n}$ we get

$$\epsilon_{i_1 \cdots i_d} = \det\left(\frac{\partial y^n}{\partial x^m}\right) \epsilon_{m_1 \cdots m_d} \frac{\partial x^{m_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{m_d}}{\partial y^{i_d}} , \tag{248}$$

which is of course just an identity and not really a transformation, since the Levi-Civita symbol is the same in every chart. It however looks like a transformation. Plugging this into equation (246)

$$\begin{aligned}&\sqrt{\det\left(g_{(x)mn} \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j}\right)} \frac{\partial y^{m_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{m_d}}{\partial x^{i_d}} \det\left(\frac{\partial y^a}{\partial x^b}\right) \epsilon_{m_1 \cdots m_d}\\ &= \sqrt{\det g_{(x)}} \left|\det\left(\frac{\partial x^c}{\partial y^d}\right)\right| \det\left(\frac{\partial y^a}{\partial x^b}\right) \frac{\partial x^{m_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{m_d}}{\partial y^{i_d}} \epsilon_{m_1 \cdots m_d}\\ &= \sqrt{\det g_{(x)}} \operatorname{sgn}\left(\det\frac{\partial x^c}{\partial y^d}\right) \det\left(\frac{\partial x^a}{\partial y^b}\right) \epsilon_{i_1 \cdots i_d}\\ &= \det\left(\frac{\partial x^a}{\partial y^b}\right) \operatorname{sgn}\left(\det\frac{\partial x^c}{\partial y^d}\right) \Omega_{(x)i_1 \cdots i_d} , \qquad (249)\end{aligned}$$

where we now see that the first term is exactly the inverse of the term that was too much in the naive integral eq. (240), but we have the sign of the determinant extra.[5] I actually don't see the problem, because in (240) we also have an absolute value sign which should make it okay again? Nevertheless, Schuller states that we want the sign to be 1, restrict ourselves to an *oriented atlas*, which we denote by $\mathcal{A}^\uparrow$.

***Definition* 12.3** (Oriented Atlas $\mathcal{A}^\uparrow$)
An atlas $\mathcal{A}$ such that any two charts (U, x), (V, y) have a chart transition map $y \circ x^{-1}$ with

$$\det\left(\frac{\partial y^a}{\partial x^b}\right) > 0 , \tag{250}$$

is called an oriented atlas and denoted with $\mathcal{A}^\uparrow$.

We introduce the *Levi-Civita symbol* with upper indices $\epsilon^{i_1 \cdots i_d}$ with exactly the same values as lower indices (no pulling up the indices with metric) and then define the scalar density

***Definition* 12.4** ($\omega_{(y)}$)
Let Ω be a volume form on $(\mathcal{M}, \mathcal{O}, \mathcal{A}^\uparrow)$ and a chart (U, x), then

$$\omega_{(x)}(p) := \Omega_{i_1 \cdots i_d}(p) \epsilon^{i_1 \cdots i_n} . \tag{251}$$

Looking at the transformation of Ω in equation (249) we directly see that (for an oriented atlas, which is already needed for the definition of Ω)

$$\boxed{\omega_{(y)} = \det\left(\frac{\partial x^a}{\partial y^b}\right) \omega_{(x)} .} \tag{252}$$

An object with such a transformation behaviour is called a *scalar density*.

[5] I am not sure about what I did here, but I think Schuller made a mistake in his lecture and the way I did it seems correct to me.

12.4 Integration on One Chart Domain

***Definition* 12.5** (Integration on one chart domain U)
On a chart (U, x)

$$\int_U f := \int\limits_{x(U)} \mathrm{d}^d\alpha \omega_{(x)} \left(x^{-1}(\alpha)\right) f_{(x)}(\alpha) \,. \tag{253}$$

From the transformation behaviour of ω and the integral (Jacobian) we see that this definition is independent of the chart. On an oriented metric manifold $(\mathcal{M}, \mathcal{O}, \mathcal{A}^\uparrow, g)$

$$\int_U f := \int\limits_{x(U)} \mathrm{d}^d\alpha \underbrace{\sqrt{\det(g_{(x)ij}(x^{-1}(\alpha))}}_{\sqrt{g}} f_{(x)}(\alpha) \,. \tag{254}$$

12.5 Integration on the Entire Manifold

We need to require that the manifold admits a so-called *partition of unity*. Roughly: For any finite subatlas $\mathcal{A}' \subseteq \mathcal{A}^\uparrow$ $\exists$ continuous functions

$$\rho_i : U_i \to \mathbb{R} \,, \tag{255}$$

such that

$$\sum_{p \,\in\, U_i} \rho_i(p) = 1 \,, \quad \forall p \in \mathcal{M} \,. \tag{256}$$

This is called a *partition of unity*. The point is that the subatlas needs to be finite in order to not have problems with convergence. Then we define

$$\int\limits_{\mathcal{M}} f = \sum_{i=1}^{\text{finite}} \int\limits_{U_i} \rho_i f \,. \tag{257}$$

An example of a partition of unity can be seen in figure 9

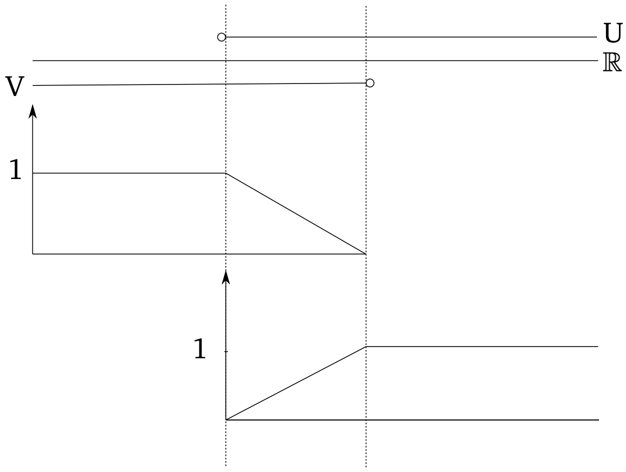


Figure 9: *A partition of unity for $\mathcal{M} = \mathbb{R}$*

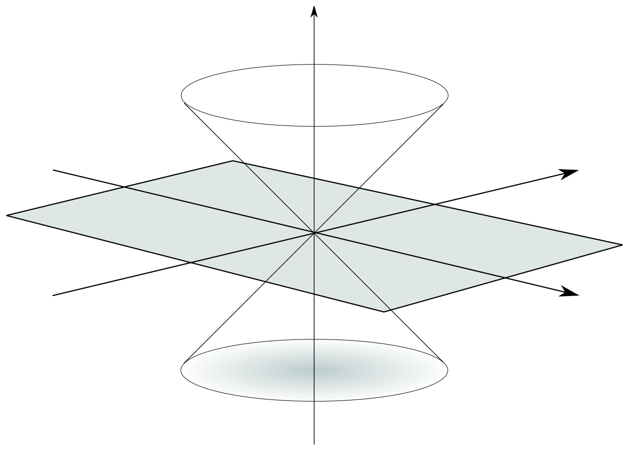


Figure 10: *Light cone. One should rather think of a light cone in the tangent space of every point.*

13 Relativistic Spacetime

On a Lorentzian metric, here we choose the sign $(+ - - -)$ in every tangent space there is a *light cone* defined by the metric g. In figure 10 such a light cone is drawn. Inside the cone there is $g(X, X) > 0$ (timelike, same sign as time in metric) , on the (surface of) the cone there is $g(X, X) = 0$ (lightlike or null) and outside $g(X, X) < 0$ (spacelike, same sign as space in metric).

We have to choose a time orientation, *i.e.* which side of the cone is future and which is past. We do this by introducing a vector T.

***Definition* 13.1** (Time-Orientation)
Let $(\mathcal{M}, \mathcal{O}, \mathcal{A}^\uparrow, g)$ be a Lorentzian manifold. Then a time-orientation is given by a *smooth* vector field T that

1. does *not vanish* anywhere,
2. $g(T, T) > 0$.

We made this definition of *spacetime* in order to enable the following physical postulates:

(P_1) The worldline γ of a *massive* particle satisfies
   (a) $g_{\gamma(\lambda)} \left(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}\right) > 0 \,\forall\lambda \in I$,
   (b) $g_{\gamma(\lambda)} \left(T, v_{\gamma,\gamma(\lambda)}\right) > 0 \,\forall\lambda \in I$,

(P_2) The worldlines of *massless* particles satisfy
   (a) $g_{\gamma(\lambda)} \left(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}\right) = 0 \,\forall\lambda \in I$,
   (b) $g_{\gamma(\lambda)} \left(T, v_{\gamma,\gamma(\lambda)}\right) > 0 \,\forall\lambda \in I$,

I am not quite sure about the index of g, but I think the thing is that g is a tensor field, so it takes two vector fields $\mathrm{T}\mathcal{M}\, \mathrm{T}\mathcal{M} \to \mathbb{R}$, whereas we plug in two vectors, so we have to take the tensor field at the point $p \in \mathcal{M}$ where the vectors are in the tangent space $\mathrm{T}p\mathcal{M}$

Now the picture is that at every point p in spacetime $\mathcal{M}$ there is a (double) cone like in figure 10. Through the time orientation T we decide to only take one side of that double cone where T lies in. A massive particle has its velocity always in the light cone and a massless particle has its velocity always on the boundary of the cone, see figure 11 *Claim:* 9/10 of a metric are determined by the cone(s).

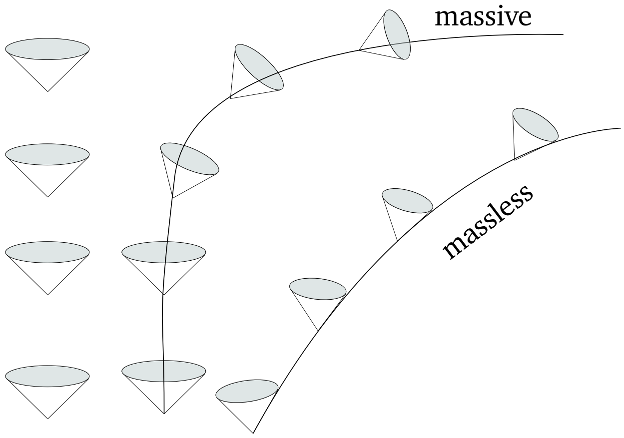


Figure 11: *Light cones in space time, massive and massless particle.*

Note: The difference to Newtonian physics is now, that in Newtonian physics every tangent space was divided in a future and a past. Now in General Relativity every tangent space is divided in the future light cone and what is not the future light cone (past light cone and something that is not in the future or past light cone).

13.1 Observers

Spacetime $(\mathcal{M}, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T)$.

***Definition* 13.2** (Observer)
An *observer* is a *worldline* γ with $g(v_\gamma, v_\gamma) = 1$, $g(T, v_\gamma) > 0$ together with a choice of basis

$$\left\{e_0(\lambda) = v_{\gamma,\gamma(\lambda)}, e_1(\lambda), e_2(\lambda), e_3(\lambda)\right\} , \quad (258)$$

of each $T_{\gamma(\lambda)}\mathcal{M}$ where the observer worldline passes, if

$$g\left(e_a(\lambda), e_b(\lambda)\right) = \eta_{ab} = \mathrm{diag}(1, -1, -1, -1) . \quad (259)$$

(More precisely: Observer = *smooth* curve in the frame bundle $\mathrm{L}\mathcal{M}$ over $\mathcal{M}$.)

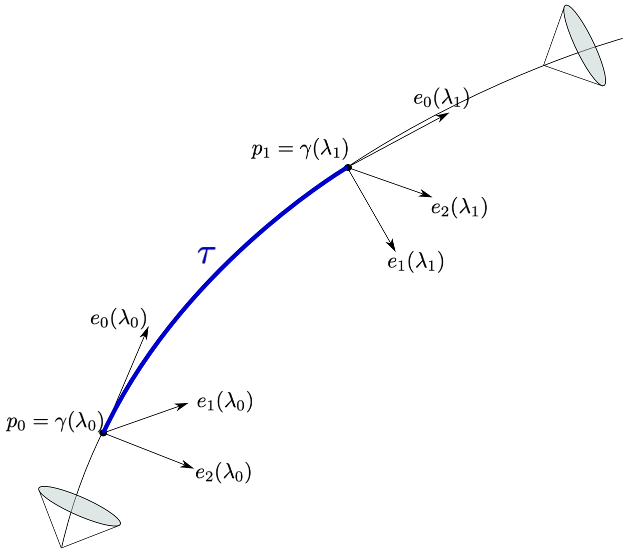


Figure 12: *Observers*

(P_3) A *clock* carried by a specific observer (γ, e) will measure a *time* called *proper time* or *eigentime*

$$\tau := \int_{\lambda_0}^{\lambda_1} \mathrm{d}\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})} \quad (260)$$

between the two *events* (see figure 12)

- $\gamma(\lambda_0)$: "start the clock" and
- $\gamma(\lambda_1)$: "stop the clock".

Example: (Twin paradoxon) Let $\mathcal{M} = \mathbb{R}^4$, $\mathcal{O} = \mathcal{O}_{\mathrm{st}}$, $\mathcal{A}^{\uparrow} \ni (\mathbb{R}^4, \mathrm{id}_{\mathbb{R}^4})$, $T^i_{(x)} = (1,0,0,0)^i$, $g_{(x)ij} = \eta_{ij} \Rightarrow \Gamma^i_{(x)jl} = 0$ everywhere, $\Rightarrow$ Riem $= 0$ everywhere. This spacetime is flat and this situation is called *Special Relativity*.

Note: Otherwise Special Relativity is not different from General Relativity.

Consider two observers

$$\gamma : (0,1) \to \mathcal{M} \quad (261)$$
$$\gamma^i_{(x)} = (\lambda, 0, 0, 0)^i \quad (262)$$

and

$$\delta : (0,1) \to \mathcal{M} \quad (263)$$
$$\delta^i_{(x)} = \begin{cases} (\lambda, \alpha\lambda, 0, 0)^i , & \lambda \leq \frac{1}{2} \\ (\lambda, (1-\lambda)\alpha, 0, 0)^i , & \lambda > \frac{1}{2} \end{cases} , \quad (264)$$

for $\alpha \in (0,1)$. Let's calculate

$$\tau_\gamma := \int_0^1 \mathrm{d}\lambda \sqrt{g_{(x)ij}\dot{\gamma}^i_{(x)}\dot{\gamma}^j_{(x)}} = 1\,, \tag{265}$$

$$\tau_\delta = \int_0^{\frac{1}{2}} \mathrm{d}\lambda \sqrt{1-\alpha^2} + \int_{\frac{1}{2}}^1 \sqrt{1-(-\alpha)^2} = \sqrt{1-\alpha^2}\,. \tag{266}$$

That means if we push α close to 1 we can make the time that the observer δ measures close to zero.

Taking the clock postulate (P_3) seriously, one should better come up with a realistic clock design that suports the postulate. Idea: Light clock, see figure 13.

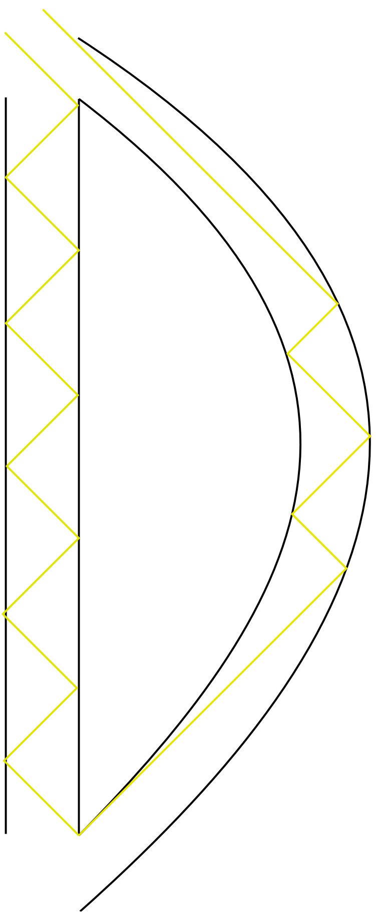


Figure 13: *A light clock. Left: 11 ticks, right: 6 ticks.*

(P_4) Let (γ, e) be an observer and δ be a *massive* particle worldline that is parametrized such that $g(v_\delta, v_\delta) = 1$. Suppose the observer and the particle *meet* somewhere in spacetime.

$$\delta(\tau_2) = p = \gamma(\tau_1) \tag{267}$$

This observer measures the 3-velocity (spatial velocity) of this particle as

$$v^{(3)}_{\delta,\delta(\tau_2)} := \epsilon^\alpha\left(v_{\delta,\delta(\tau_2)}\right) e_\alpha\,,\ \alpha = 1,2,3\,, \tag{268}$$

where $\{\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3\}$ is the unique dual basis of the basis $\{e_0, e_1, e_2, e_3\}$ of the observer γ and we only take the spatial vectors, see figure 14

Note: What happens here is that we project the 4-velocity v_δ onto the "spatial part" of the frame given by the observer γ. Of course v_δ is an objective thing, but $v_\delta^{(3)}$ depends on the observer, which is the whole point of doing it. $v_\delta^{(3)}$ is actually a 4-vector, but it just happens to "lie in the span of e_1, e_2, e_3". We will see that $|v_\delta^{(3)}| \leq 1$.

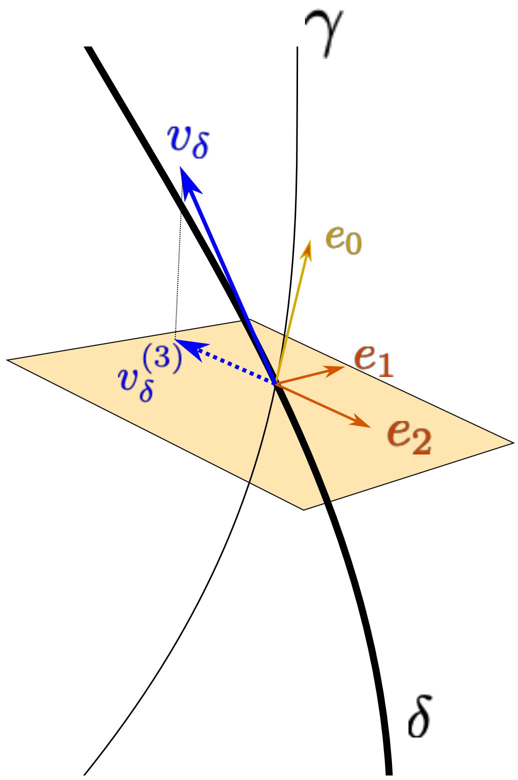


Figure 14: *Projection of the velocity v_δ to the 3-velocity $v_\delta^{(3)}$.*

Consequence: An observer (γ, e) will extract quantities measurable in his laboratory frame from objective quantities always like that. paragraphExample: The Faraday (0,2)-tensor F of electromagnetism is the objective quantity and

$$F(e_a, e_b) = F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{269}$$

is what the observer measures.

Note: I am not quite sure how to understand this. Don't we have the tensor field F that we can express as F_{ab} for some chart? As I see it, an observer gives something like a chart on a worldline, but F_{ab} can exist on the whole are where we defined a chart.

13.2 Role of the Lorentz Transformations

Lorentz transformations emerge as follows: Let (γ, e) and $(\tilde{\gamma}, \tilde{e})$ be observers with $\gamma(0) = \tilde{\gamma}(0) = p$. Now $\{e_0, \ldots, e_3\}$ and $\{\tilde{e}_0, \ldots, \tilde{e}_3\}$ at $\tau = 0$ are both bases for the same $\mathrm{T}_{\gamma(0)}\mathcal{M}$. Thus

$$\tilde{e}_a = \Lambda^b{}_a e_b\,, \quad \Lambda \in GL(4)\,. \tag{270}$$

Now, since both are observers,

$$\eta_{ab} = g(\tilde{e}_a, \tilde{e}_b) \tag{271}$$

$$= g\left(\lambda^m{}_a e_m, \Lambda^m{}_b e_n\right) = \Lambda^m{}_a \Lambda^n{}_b g(e_m, e_n) \tag{272}$$

$$= \Lambda^m{}_a \Lambda^n{}_b \eta_{mn}\,, \tag{273}$$

i.e. $\Lambda \in O(1,3)$, the transformation Λ is an element of the *Lorentz transformations*.

Result: Lorentz transformations relate the frames of any two observers *at the same point*. They do not transform spacetime! They just act on one tangent space of spacetime where the two observers meet!

Note: $\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ is utter nonsense. This is just abuse of flat space structure. In SRT as well as GR Lorentz transformations transform between tangent spaces between observers at the point where they meet.

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