To those physics students who asked why q and $\dot{q}$ are independent in Lagrangian Mechanics

Niklas Zorbach[1] **and Marco Knipfer**[1, 2]

[1] *Institute for Theoretical Physics, Goethe-University Frankfurt, Germany*
[2] *Institute for Physics and Astronomy, The University of Alabama, USA*

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1 Manifolds

1.1 Topology

This chapter is basically taken from (Schuller, 2015) with our remarks to it.

We start with a set M which is supposed to be the space where physics happens. The weakest structure we need in order to talk about continuity (of curves or fields) is called a topology.

Definition 1.1 (Power set $\mathcal{P}$)
The set of all subsets of M.

Definition 1.2 (Topology)
A Topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$ satisfying:

1. $\emptyset \in \mathcal{O}, M \in \mathcal{O}$,
2. $U \in \mathcal{O}, \quad V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$
3. $U_\alpha \in \mathcal{O}, \quad \alpha \in A \Rightarrow \left(\bigcup_{\alpha \in A} U_\alpha \right) \in \mathcal{O}$

Every set has the *chaotic topology*

$$\mathcal{O}_{\text{chaotic}} := \{\emptyset, M\}, \tag{1}$$

and the *discrete topology*

$$\mathcal{O}_{\text{discrete}} := \mathcal{P}(M), \tag{2}$$

which are both useless.

The special case $M = \mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ has a standard topology for which we need the definition of a soft ball.

Definition 1.3 (Soft Ball in $\mathcal{R}^d$)

$$B_r(p) := \left\{ (q_1, \cdots, q_d) | \sum_{i=1}^{d} (p_i - p_i) < r \right\}, \tag{3}$$

with $r \in \mathcal{R}^+$, $p \in \mathcal{R}^d$. Note: This does not need a norm or vector space structure on $\mathbb{R}^d$.

Definition 1.4 ($\mathcal{O}_{\text{standard}}$ on $\mathbb{R}^d$)

$$U \in \mathcal{O}_{\text{standard}} :\Leftrightarrow \forall p \in U : \exists r \in \mathcal{R}^+ : B_r(p) \subseteq U \tag{4}$$
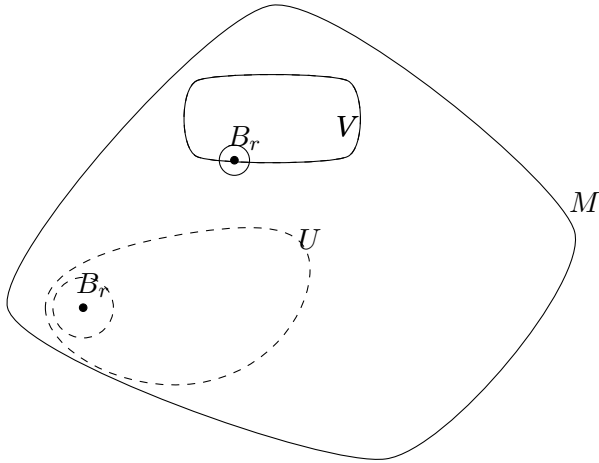
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Figure 1: *The set U is in the standard topology, V not.*

Some terminology: Let M be a set with a topology $\mathcal{O} =:$ set of open sets. We call $(M, \mathcal{O}$ a *topological space* and:

- $U \in \mathcal{O} \Leftrightarrow:$ call $U \subseteq M$ an *open set*
- $M \backslash A \in \mathcal{O} \Leftrightarrow:$ call $U \subseteq M$ a *closed set*

Note: The empty set is open and closed. If a set is open we cannot directly follow that it is not closed or vise versa.

1.2 Continuous maps

A map
$$f : M \to N \,, \tag{5}$$

takes every point from the domain M (a set) to the target N (a set). If one point $p \in N$ is not reached, the map is not *surjective*. If a point is hit twice, the map is not *injective*. A map that is injective and surjective is called *surjective*.

***Definition* 1.5** (Preimage)

$$f : M \to N \supseteq V$$
$$\operatorname{preim}_f(V) := \{m \in M \mid f(f) \in V\} \tag{6}$$

***Definition* 1.6** (Continuity)
$(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ topological spaces. Then a map $f : M \to N$ is called *continuous with respect to $\mathcal{O}_M$ and $\mathcal{O}_N$* if

$$\forall V \in \mathcal{O}_N : \operatorname{preim}_f(V) \in \mathcal{O}_M \,. \tag{7}$$

"A map is open iff the preimages of all open sets are open sets."

Note: If a map is not surjective there are sets with preimage $\emptyset$, thus we need to have $\emptyset$ in $\mathcal{O}$, otherwise only surjective maps could be continuous.

Bibliography

Schuller, Frederic (Feb. 2015). *International Winter School on Gravity and Light 2015*. `https : / / www . youtube . com / watch ? v = 7G4SqIboeig & list = PLFeEvEPtX _ 0S6vxxiiNPrJbLu9aK1UVC_`.