Mathematical Notes on Manifolds in Physics

Niklas Zorbach[1] and Marco Knipfer[1, 2]

[1] *Institute for Theoretical Physics, Goethe-University Frankfurt, Germany*
[2] *Institute for Physics and Astronomy, The University of Alabama, USA*

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1 Manifolds

1.1 Topology

This chapter is basically taken from (Schuller, 2015) with our remarks to it.

We start with a set M which is supposed to be the space where physics happens. The weakest structure we need in order to talk about continuity (of curves or fields) is called a topology.

***Definition* 1.1** (Power set $\mathcal{P}$)
The set of all subsets of M.

***Definition* 1.2** (Topology)
A Topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$ satisfying:

1. $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$,
2. $U \in \mathcal{O}, \quad V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$
3. $U_\alpha \in \mathcal{O}, \quad \alpha \in A \Rightarrow \left(\bigcup_{\alpha \in A} U_\alpha \right) \in \mathcal{O}$

Every set has the *chaotic topology*

$$\mathcal{O}_{\text{chaotic}} := \{\emptyset, M\} \,, \tag{1}$$

and the *discrete topology*

$$\mathcal{O}_{\text{discrete}} := \mathcal{P}(M) \,, \tag{2}$$

which are both useless.

The special case $M = \mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ has a standard topology for which we need the definition of a soft ball.

***Definition* 1.3** (Soft Ball in $\mathbb{R}^d$)

$$B_r(p) := \left\{ (q_1, \cdots, q_d) | \sum_{i=1}^{d} (p_i - p_i) < r \right\} \,, \tag{3}$$

with $r \in \mathbb{R}^+$, $p \in \mathbb{R}^d$. Note: This does not need a norm or vector space structure on $\mathbb{R}^d$.

***Definition* 1.4** ($\mathcal{O}_{\text{standard}}$ on $\mathbb{R}^d$)

$$U \in \mathcal{O}_{\text{standard}} :\Leftrightarrow \forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U \tag{4}$$

Some terminology: Let M be a set with a topology $\mathcal{O} =:$ set of open sets. We call $(M, \mathcal{O}$ a *topological space* and:

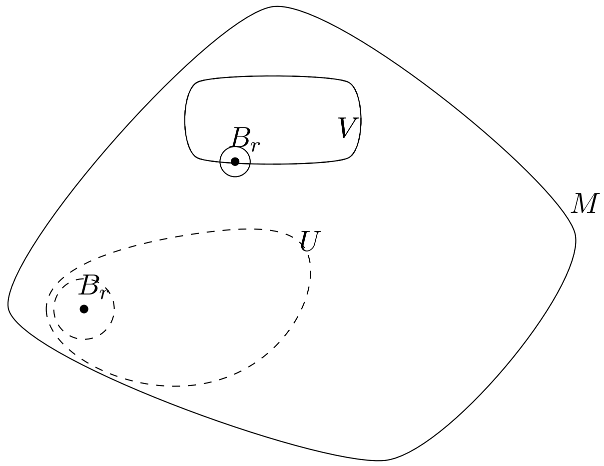

Figure 1: *The set U is in the standard topology, V not.*

- $U \in \mathcal{O} \Leftrightarrow$: call $U \subseteq M$ an *open set*
- $M \backslash A \in \mathcal{O} \Leftrightarrow$: call $U \subseteq M$ a *closed set*

Note: The empty set is open and closed. If a set is open we cannot directly follow that it is not closed or vise versa. For $M = \{1, 2\}$ and $\mathcal{O}_M = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ the set $\{2\}$ is open and closed.

1.2 Continuous Maps

A map

$$f : M \to N \,, \tag{5}$$

takes every point from the domain M (a set) to the target N (a set). If one point $p \in N$ is not reached, the map is not *surjective*. If a point is hit twice, the map is not *injective*. A map that is injective and surjective is called *surjective*.

***Definition* 1.5** (Preimage)

$$\begin{aligned} f : M \to N \supseteq V \\ \mathrm{preim}_f(V) := \{m \in M \mid f(f) \in V\} \end{aligned} \tag{6}$$

***Definition* 1.6** (Continuity)
$(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ topological spaces. Then a map $f : M \to N$ is called *continuous with respect to $\mathcal{O}_M$ and $\mathcal{O}_N$* if

$$\forall V \in \mathcal{O}_N : \mathrm{preim}_f(V) \in \mathcal{O}_M \,. \tag{7}$$

"A map is open iff the preimages of all open sets are open sets."

Note: If a map is not surjective there are sets with preimage $\emptyset$, thus we need to have $\emptyset$ in $\mathcal{O}$, otherwise only surjective maps could be continuous.

Note: The inverse of a continuous function does not need to be continuous.

***Definition* 1.7** (Composition of maps)
For f and g

$$f : M \to N, \quad g : N \to P \,,$$

we define the *composition* as

$$\begin{aligned} g \circ f : M \to P \\ m \mapsto (g \circ f)(m) := g(f(m)) \end{aligned} \tag{8}$$

***Theorem* 1.8** (Composition of continuos maps)
For f, g continuos also $g \circ f$ is continuous (if spaces match).

***Definition* 1.9** (Subset topology, Inherited topology)
A set M with topology $\mathcal{O}_M$. Given any subset $S \subseteq M$ we can construct the inherited topology $\mathcal{O}\big|_S \subseteq \mathcal{P}(S)$

$$\mathcal{O}\big|_S := \{U \cap S \mid U \in \mathcal{O}M\} \,. \tag{9}$$

Note: For $S \subseteq M$, if f is continuous then $f\big|_S$ is also continuous if $\mathcal{O}\big|_S$ is chosen. This is for example important if you are on a trajectory γ through $\mathbb{R}^n$ and measure the temperature $T\big|_\gamma$.

***Definition* 1.10** (Topological manifold)
A topological space $(\mathcal{M}, \mathcal{O})$ is called a *d-dimensional topological manifold* if

$$\forall p \in \mathcal{M} : \exists U \in \mathcal{O},\ p \in U : \exists x : U \to x(U) \subseteq \mathbb{R}^d \,, \tag{10}$$

with the following properties (wrt. $\mathcal{O}_{\text{std}}$ on $\mathbb{R}^d$):

1. x intervitble: $x^{-1} : x(U) \to U$,
2. x continuous,
3. x^{-1} continuous.

"Invertible, in both directions continuous map to $\mathbb{R}^n$."

Note: Thus in the above definition $x(U)$ is also open (from the definition of continuity).

Terminology:
- (U, x) is a *chart* of $\mathcal{M}, \mathcal{O}$,
- $\mathcal{A} = \left\{(U_{(\alpha)}, x_{(\alpha)} | \alpha \in A\right\}$ is an *atlas* of $(\mathcal{M}, \mathcal{O})$ if $\bigcup\limits_{\alpha \in A} U_{(\alpha)}$ covers the whole manifold $\mathcal{M}$,
- $x : U \to x(U) \subseteq \mathbb{R}^d$ is a *chart map* $x(p) = (x^1(p), \ldots, x^d(p))$, where the *component maps* $x^i : U \to \mathbb{R}$ are called *coordinate maps*,
- $p \in U$, then $x^1(p)$ is the first coordinate of the point p wrt. the chosen chart (U, x).

Note: The choice of the chart (choice of coordinates) has nothing to do with the physics. Physics is chart independent. $\mathcal{M}$ is "the real world".

1.3 Chart Transition Maps

Given (U,x) and (V,y) charts, on $U \cup V$ one can transition from one chart to the other by (see figure 2)

$$y \circ x^{-1} : \mathbb{R}^d \supseteq x(U \cap V) \to y(U \cap V) \subseteq \mathbb{R}^d \,, \quad (11)$$

which is called the *chart transition map*.

Note: As a physicist one talks about a “change in coordinates”.

1.4 Manifold Philosophy

The idea is to define properties of some object in the real world $\mathcal{M}$ by at a chart-representative of it. For example the continuity of a curve $\gamma : [0,1] \to \mathcal{M}$ can be judged by looking at $x \circ \gamma : [0,1] \to \mathbb{R}^d$, because x is invertible and in both directions continuous and the composition of two continuous maps is also continuous.

Note: One needs to make sure that the property of the object on $\mathcal{M}$ does not depend on the map x or y. For continuity this is the case, since $y \circ \gamma = (y \circ x^{-1}) \circ x \circ \gamma$ and the chart transition map $y \circ x^{-1}$ is also continuous.

Other properties like “differentiability” are not even defined on $\mathcal{M}$ a priori, so one can only talk about the chart representative. Here the definition that γ is differentiable iff $x \circ \gamma : [0,1] \to \mathbb{R}^d$ is differentiable has the problem that x and y only need to be continuous and so the chart transition map $y \circ x^{-1}$ does not need to be differentiable unless one restricts oneself to only differentiable charts.

Bibliography

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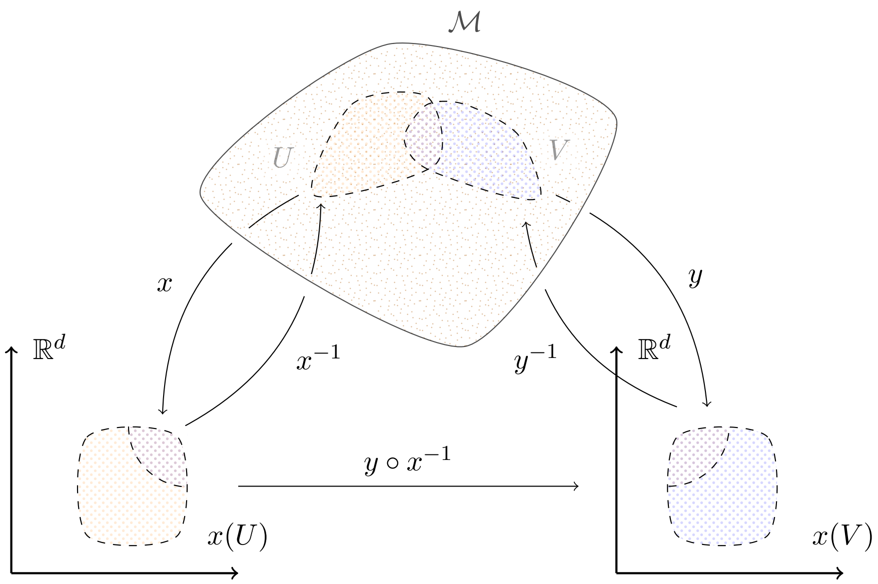

Figure 2: *Visualization of chart transition maps. "How to glue together the charts of an atlas." Plot modified from (*Drawing manifolds in tikz *n.d.)*