To those physics students who asked why q and $\dot{q}$ are independent in Lagrangian Mechanics

Niklas Zorbach[1] and Marco Knipfer[1, 2]

[1] *Institute for Theoretical Physics, Goethe-University Frankfurt, Germany*
[2] *Institute for Physics and Astronomy, The University of Alabama, USA*

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1 Manifolds

This chapter is basically taken from (Schuller, 2015) with our remarks to it.

We start with a set $\mathcal{M}$ which is supposed to be the space where physics happens. The weakest structure we need in order to talk about continuity (of curves or fields) is called a topology.

Definition 1.1 (Power set $\mathcal{P}$)
The set of all subsets of $\mathcal{M}$.

Definition 1.2 (Topology)
A Topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(\mathcal{M})$ satisfying:

1. $\emptyset \in \mathcal{O}$, $\mathcal{M} \in \mathcal{O}$,
2. $U \in \mathcal{O}, \quad V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$
3. $U_\alpha \in \mathcal{O}, \quad \alpha \in A \Rightarrow \left(\bigcup_{\alpha \in A} U_\alpha \right) \in \mathcal{O}$

Every set has the *chaotic topology*

$$\mathcal{O}_{\text{chaotic}} := \{\emptyset, \mathcal{M}\} \,, \tag{1}$$

and the *discrete topology*

$$\mathcal{O}_{\text{discrete}} := \mathcal{P}(M) \,, \tag{2}$$

which are both useless.

The special case $\mathcal{M} = \mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ has a standard topology for which we need the definition of a soft ball.

Definition 1.3 (Soft Ball in $\mathcal{R}^d$)

$$B_r(p) := \left\{ (q_1, \cdots, q_d) | \sum_{i=1}^{d} (p_i - p_i) < r \right\} \,, \tag{3}$$

with $r \in \mathcal{R}^+$, $p \in \mathcal{R}^d$. Note: This does not need a norm or vector space structure on $\mathbb{R}^d$.

Definition 1.4 ($\mathcal{O}_{\text{standard}}$ on $\mathbb{R}^d$)

$$U \in \mathcal{O}_{\text{standard}} :\Leftrightarrow \forall p \in U : \exists r \in \mathcal{R}^+ : B_r(p) \subseteq U \tag{4}$$

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Bibliography

Schuller, Frederic (Feb. 2015). *International Winter School on Gravity and Light 2015*. https://www.youtube.com/watch?v=7G4SqIboeig&list=PLFeEvEPtX_0S6vxxiiNPrJbLu9aK1UVC_.