# To those physics students who asked why $q$ and $\dot{q}$ are independent in Lagrangian Mechanics

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## 1 Manifolds

### 1.1 Topology

This chapter is basically taken from (Schuller, 2015) with our remarks to it.

We start with a set $M$ which is supposed to be the space where physics happens. The weakest structure we need in order to talk about continuity (of curves or fields) is called a topology.

***Definition* 1.1** (Power set $\mathcal{P}$)
The set of all subsets of $M$.

***Definition* 1.2** (Topology)
A Topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$ satisfying:

1. $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$,
2. $U \in \mathcal{O}, \quad V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$
3. $U_\alpha \in \mathcal{O}, \quad \alpha \in A \Rightarrow \left( \bigcup_{\alpha \in A} U_\alpha \right) \in \mathcal{O}$

Every set has the *chaotic topology*

$$\mathcal{O}_{\text{chaotic}} := \{\emptyset, M\} \,, \tag{1}$$

and the *discrete topology*

$$\mathcal{O}_{\text{discrete}} := \mathcal{P}(M) \,, \tag{2}$$

which are both useless.

The special case $M = \mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ has a standard topology for which we need the definition of a soft ball.

***Definition* 1.3** (Soft Ball in $\mathbb{R}^d$)

$$B_r(p) := \left\{ (q_1, \cdots, q_d) | \sum_{i=1}^{d} (p_i - p_i) < r \right\} \,, \tag{3}$$

with $r \in \mathbb{R}^+$, $p \in \mathbb{R}^d$. Note: This does not need a norm or vector space structure on $\mathbb{R}^d$.

***Definition* 1.4** ($\mathcal{O}_{\text{standard}}$ on $\mathbb{R}^d$)

$$U \in \mathcal{O}_{\text{standard}} :\Leftrightarrow \forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U \tag{4}$$

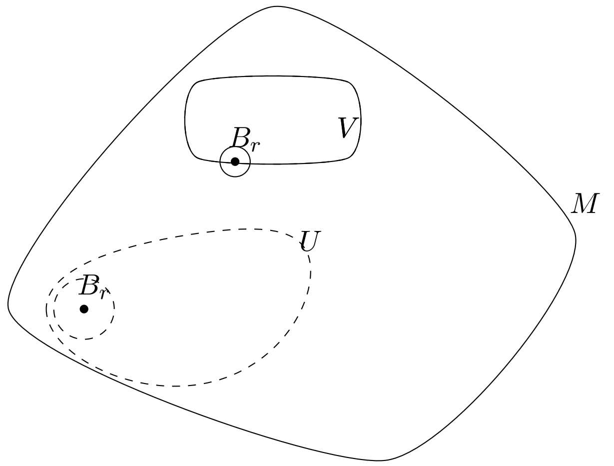


**Figure 1:** *The set $U$ is in the standard topology, $V$ not.*

Some terminology: Let $M$ be a set with a topology $\mathcal{O}$ =: set of open sets. We call $(M, \mathcal{O}$ a *topological space* and:

- $U \in \mathcal{O} \Leftrightarrow$: call $U \subseteq M$ an *open set*
- $M \backslash A \in \mathcal{O} \Leftrightarrow$: call $U \subseteq M$ a *closed set*

*Note:* The empty set is open and closed. If a set is open we cannot directly follow that it is not closed or vise versa. For $M = \{1, 2\}$ and $\mathcal{O}_M = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ the set $\{2\}$ is open and closed.

## 1.2 Continuous maps

A map

$$f : M \to N \,, \tag{5}$$

takes every point from the domain M (a set) to the target N (a set). If one point $p \in N$ is not reached, the map is not *surjective*. If a point is hit twice, the map is not *injective*. A map that is injective and surjective is called *surjective*.

***Definition* 1.5** (Preimage)

$$\begin{aligned} f : M \to N \supseteq V \\ \text{preim}_f(V) := \{m \in M \mid f(f) \in V\} \end{aligned} \tag{6}$$

***Definition* 1.6** (Continuity)
$(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ topological spaces. Then a map $f : M \to N$ is called *continuous with respect to $\mathcal{O}_M$ and $\mathcal{O}_N$* if

$$\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M \,. \tag{7}$$

*"A map is open iff the preimages of all open sets are open sets."*

*Note:* If a map is not surjective there are sets with preimage $\emptyset$, thus we need to have $\emptyset$ in $\mathcal{O}$, otherwise only surjective maps could be continuous.

*Note:* The inverse of a continuous function does not need to be continuous.

***Definition* 1.7** (Composition of maps)
For $f$ and $g$

$$f : M \to N, \quad g : N \to P \,,$$

we define the *composition* as

$$\begin{aligned} g \circ f : M \to P \\ m \mapsto (g \circ f)(m) := g(f(m)) \end{aligned} \tag{8}$$

***Theorem* 1.8** (Composition of continuos maps)
For $f, g$ continuos also $g \circ f$ is continuous (if spaces match).

***Definition* 1.9** (Subset topology, Inherited topology)
A set $M$ with topology $\mathcal{O}_M$. Given any subset $S \subseteq M$ we can construct the inherited topology $\mathcal{O}\big|_S \subseteq \mathcal{P}(S)$

$$\mathcal{O}\big|_S := \{U \cap S \mid U \in \mathcal{O}M\} \,. \tag{9}$$

*Note:* For $S \subseteq M$, if $f$ is continuous then $f\big|_S$ is also continuous if $\mathcal{O}\big|_S$ is chosen. This is for example important if you are on a trajectory $\gamma$ through $\mathbb{R}^n$ and measure the temperature $T\big|_\gamma$.

# Bibliography

Schuller, Frederic (Feb. 2015). *International Winter School on Gravity and Light 2015*. `https://www.youtube.com/watch?v=7G4SqIboeig&list=PLFeEvEPtX_0S6vxxiiNPrJbLu9aK1UVC_`.