Mathematical Notes on Manifolds in Physics

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Version: June 25, 2019

We are writing these notes in order to learn Riemannian Geometry and better understand Lagrangian Mechanics and General Relativity. As a physicist one usually learns all of this in a rather practical way without understanding the basic mathematical concepts. For example a physicist usually does not learn that the Lagrangian lives on the tangent bundle, because one implicitly always identifies some spaces (here the space and the tangent space), which is possible on a flat manifold.

1 Manifolds

1.1 Topology

This chapter is basically taken from (Schuller, 2015) with our remarks to it.

We start with a set M which is supposed to be the space where physics happens. The weakest structure we need in order to talk about continuity (of curves or fields) is called a topology.

***Definition* 1.1** (Power set $\mathcal{P}$)
The set of all subsets of M.

***Definition* 1.2** (Topology)
A Topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$ satisfying:

1. $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$,
2. $U \in \mathcal{O}, \quad V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$
3. $U_\alpha \in \mathcal{O}, \quad \alpha \in A \Rightarrow \left(\bigcup_{\alpha \in A} U_\alpha \right) \in \mathcal{O}$

Every set has the *chaotic topology*

$$\mathcal{O}_{\text{chaotic}} := \{\emptyset, M\} \,, \tag{1}$$

and the *discrete topology*

$$\mathcal{O}_{\text{discrete}} := \mathcal{P}(M) \,, \tag{2}$$

which are both useless.

The special case $M = \mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ has a standard topology for which we need the definition of a soft ball.

***Definition* 1.3** (Soft Ball in $\mathbb{R}^d$)

$$B_r(p) := \left\{ (q_1, \cdots, q_d) | \sum_{i=1}^{d} (p_i - p_i) < r \right\} \,, \tag{3}$$

with $r \in \mathbb{R}^+$, $p \in \mathbb{R}^d$. Note: This does not need a norm or vector space structure on $\mathbb{R}^d$.

***Definition* 1.4** ($\mathcal{O}_{\text{standard}}$ on $\mathbb{R}^d$)

$$U \in \mathcal{O}_{\text{standard}} :\Leftrightarrow \forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U \tag{4}$$

Some terminology: Let M be a set with a topology $\mathcal{O}$ =: set of open sets. We call $(M, \mathcal{O}$ a *topological space* and:

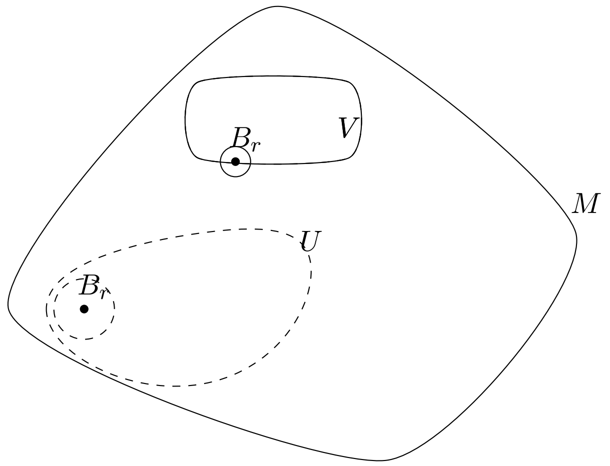

Figure 1: *The set U is in the standard topology, V not.*

- $U \in \mathcal{O} \Leftrightarrow$: call $U \subseteq M$ an *open set*
- $M \backslash A \in \mathcal{O} \Leftrightarrow$: call $U \subseteq M$ a *closed set*

Note: The empty set is open and closed. If a set is open we cannot directly follow that it is not closed or vise versa. For $M = \{1, 2\}$ and $\mathcal{O}_M = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ the set $\{2\}$ is open and closed.

1.2 Continuous Maps

A map

$$f : M \to N\,, \tag{5}$$

takes every point from the domain M (a set) to the target N (a set). If one point $p \in N$ is not reached, the map is not *surjective*. If a point is hit twice, the map is not *injective*. A map that is injective and surjective is called *surjective*.

***Definition* 1.5** (Preimage)

$$\begin{aligned} f : M \to N \supseteq V \\ \mathrm{preim}_f(V) := \{m \in M \mid f(f) \in V\} \end{aligned} \tag{6}$$

***Definition* 1.6** (Continuity)
$(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ topological spaces. Then a map $f : M \to N$ is called *continuous with respect to $\mathcal{O}_M$ and $\mathcal{O}_N$* if

$$\forall V \in \mathcal{O}_N : \mathrm{preim}_f(V) \in \mathcal{O}_M\,. \tag{7}$$

"A map is open iff the preimages of all open sets are open sets."

Note: If a map is not surjective there are sets with preimage $\emptyset$, thus we need to have $\emptyset$ in $\mathcal{O}$, otherwise only surjective maps could be continuous.

Note: The inverse of a continuous function does not need to be continuous.

***Definition* 1.7** (Composition of maps)
For f and g

$$f : M \to N, \quad g : N \to P\,,$$

we define the *composition* as

$$\begin{aligned} g \circ f : M \to P \\ m \mapsto (g \circ f)(m) := g(f(m)) \end{aligned} \tag{8}$$

***Theorem* 1.8** (Composition of continuos maps)
For f, g continuos also $g \circ f$ is continuous (if spaces match).

***Definition* 1.9** (Subset topology, Inherited topology)
A set M with topology $\mathcal{O}_M$. Given any subset $S \subseteq M$ we can construct the inherited topology $\mathcal{O}\big|_S \subseteq \mathcal{P}(S)$

$$\mathcal{O}\big|_S := \{U \cap S \mid U \in \mathcal{O}M\}\,. \tag{9}$$

Note: For $S \subseteq M$, if f is continuous then $f\big|_S$ is also continuous if $\mathcal{O}\big|_S$ is chosen. This is for example important if you are on a trajectory γ through $\mathbb{R}^n$ and measure the temperature $T\big|_\gamma$.

***Definition* 1.10** (Topological manifold)
A topological space $(\mathcal{M}, \mathcal{O})$ is called a *d-dimensional topological manifold* if

$$\forall p \in \mathcal{M} : \exists U \in \mathcal{O},\ p \in U : \exists x : U \to x(U) \subseteq \mathbb{R}^d\,, \tag{10}$$

with the following properties (wrt. $\mathcal{O}_{\text{std}}$ on $\mathbb{R}^d$):

1. x intervitble: $x^{-1} : x(U) \to U$,
2. x continuous,
3. x^{-1} continuous.

"Invertible, in both directions continuous map to $\mathbb{R}^n$."

Note: Thus in the above definition $x(U)$ is also open (from the definition of continuity).

Terminology:
- (U, x) is a *chart* of $\mathcal{M}, \mathcal{O}$,
- $\mathcal{A} = \left\{(U_{(\alpha)}, x_{(\alpha)} | \alpha \in A\right\}$ is an *atlas* of $(\mathcal{M}, \mathcal{O})$ if $\bigcup_{\alpha \in A} U_{(\alpha)}$ covers the whole manifold $\mathcal{M}$,
- $x : U \to x(U) \subseteq \mathbb{R}^d$ is a *chart map* $x(p) = (x^1(p), \ldots, x^d(p))$, where the *component maps* $x^i : U \to \mathbb{R}$ are called *coordinate maps*,
- $p \in U$, then $x^1(p)$ is the first coordinate of the point p wrt. the chosen chart (U, x).

Note: The choice of the chart (choice of coordinates) has nothing to do with the physics. Physics is chart independent. $\mathcal{M}$ is "the real world".

1.3 Chart Transition Maps

Given (U, x) and (V, y) charts, on $U \cup V$ one can transition from one chart to the other by (see figure 2)

$$y \circ x^{-1} : \mathbb{R}^d \supseteq x(U \cap V) \to y(U \cap V) \subseteq \mathbb{R}^d \,, \quad (11)$$

which is called the *chart transition map*.

Note: As a physicist one talks about a “change in coordinates”.

1.4 Manifold Philosophy

The idea is to define properties of some object in the real world $\mathcal{M}$ by at a chart-representative of it. For example the continuity of a curve $\gamma : [0,1] \to \mathcal{M}$ can be judged by looking at $x \circ \gamma : [0,1] \to \mathbb{R}^d$, because x is invertible and in both directions continuous and the composition of two continuous maps is also continuous.

Note: One needs to make sure that the property of the object on $\mathcal{M}$ does not depend on the map x or y. For continuity this is the case, since $y \circ \gamma = (y \circ x^{-1}) \circ x \circ \gamma$ and the chart transition map $y \circ x^{-1}$ is also continuous.

Other properties like “differentiability” are not even defined on $\mathcal{M}$ a priori, so one can only talk about the chart representative. Here the definition that γ is differentiable iff $x \circ \gamma : [0,1] \to \mathbb{R}^d$ is differentiable has the problem that x and y only need to be continuous and so the chart transition map $y \circ x^{-1}$ does not need to be differentiable unless one restricts oneself to only differentiable charts.

2 Vector Spaces

2.1 Vectors and Linear Maps

In order to understand the tangent space we need to understand vector spaces.

***Definition* 2.1** (Vector space $(V, +, \cdot)$)
A vector space $(V, +, \cdot)$ is a set V with

- an “addition” $+ : V \times V\ toV$,
- an “S-multiplication” $\cdot : \mathbb{R} \times V\ toV$

and the properties CANI ADDU:
$\forall v, w, u \in V, \lambda, \mu \in \mathbb{R}$

C⁺: $v + w = w + v \,,$
A⁺: $(u + v) + w = u + (v + w) \,,$
N⁺: $\exists 0 \in V : \forall v \in V : v + 0 = v \,,$
I⁺: $\forall v \in V : \exists(-v) \in V : v + (-v) = 0 \,,$
A: $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v \,,$
D: $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v \,,$
D: $\lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w) \,,$
U: $1 \cdot v = v \,.$

An element of a vector space is called a *vector*.

Note: The addition $+$ in definition 2.1 sometimes is between vectors and sometimes between scalars. It is important to know the difference.

***Definition* 2.2** (Linear maps)
(Structure respecting maps between vector spaces)
$(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ vector spaces. A map

$$\phi : V \to W \quad (12)$$

is called *linear* if $\forall v, \tilde{v} \in V, \forall \lambda \in \mathbb{R}$

1. $\phi(v +_V + \tilde{v}) = \phi(v) +_W \phi(\tilde{v})$
2. $\phi(\lambda \cdot_V v) = \lambda \cdot_W \phi(v)$

We write:

$$\phi : V \to W \text{ linear } \Leftrightarrow: \phi : V \xrightarrow{\sim} W \,. \quad (13)$$

***Theorem* 2.3** (Transitivity of linearity of maps)
V, W, U vector spaces, $\psi : V \xrightarrow{\sim} W$, $\phi : W \xrightarrow{\sim} U$ then $\phi \circ \psi$ is also linear: $\phi \circ \psi : V \xrightarrow{\sim} U$.

***Definition* 2.4** (Homomorphisms $\mathrm{Hom}(V, W)$)

$$\mathrm{Hom}(V, W) := \left\{ \phi : V \xrightarrow{\sim} W \right\} \,. \quad (14)$$

Note: $\mathrm{Hom}(V, W)$ can be made into a vector space by defining an addition and a multiplication

- $(\phi + \psi)(v) := \phi(v) + \psi(v) \,,$
- $(\lambda \psi)(v) := \lambda(\psi(v)) \,.$

***Definition* 2.5** (Dual vector space $V^\star$)

$$V^\star := \left\{ \phi : V \xrightarrow{\sim} \mathbb{R} \right\} = \mathrm{Hom}(V, \mathbb{R}) \,. \quad (15)$$

The vector space $(V^\star, +, \cdot)$ is the *dual vector space* to V. $\phi \in V^\star$ is informally called a *covector*.

***Definition* 2.6** ((r, s) - Tensors)
$(V, +, \cdot)$ vector space, $r, s \in \mathbb{N}_0$. An (r, s)-tensor T over V is a multi-linar map

$$T : \overbrace{V^\star \times \cdots \times V^\star}^{r} \times \overbrace{V \times \cdots \times V}^{s} \xrightarrow{\sim} \mathbb{R} \quad (16)$$

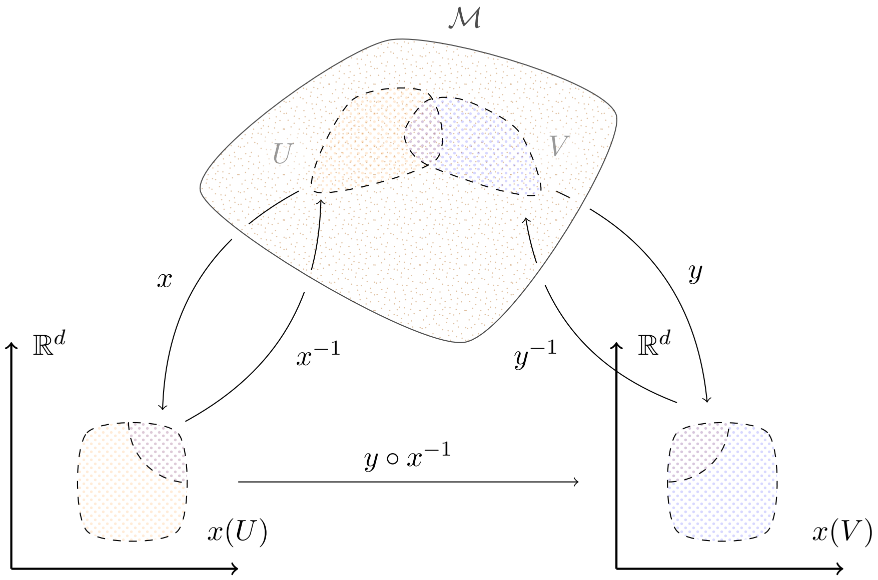

Figure 2: *Visualization of chart transition maps. "How to glue together the charts of an atlas." Plot modified from (*Drawing manifolds in tikz *n.d.)*

***Theorem* 2.7** (Covector is (0,1)-tensor)

$$\phi \in V^\star \Leftrightarrow \phi : V \xrightarrow{\sim} \mathbb{R} \Leftrightarrow \phi\,(0,1)\,\text{tensor}\,. \tag{17}$$

***Theorem* 2.8** (Vector is (1,0)-tensor)
If $\dim V < \infty$

$$v \in V = (V^\star)^\star \Leftrightarrow v : V^\star \xrightarrow{\sim} \mathbb{R} \Leftrightarrow v\,\text{is}\,(1,0)\text{−tensor}\,. \tag{18}$$

2.2 Bases

***Definition* 2.9** (Hamel-basis)
$(V, +, \cdot))$ vector space. A subset $B \subset V$ is called a Hamel-basis if

$$\forall v \in V\, \exists!\,\text{finite}\, F = \{f_1, \ldots, f_n\} \subset B : \exists!\, \underbrace{v^1, \cdots, v^n}_{\in \mathbb{R}}\,, \tag{19}$$

such that

$$v = v^1 f_1 + \cdots + v^n f_n\,. \tag{20}$$

(and all f_i linearly independent).

***Definition* 2.10** (Dimension of a vector space)
If a basis B with $d < \infty$ many elements, then we call $d =: \dim V$.

If we have chosen a basis $\{e_1, \ldots, e_n\}$ of $(V, +, \cdot)$ then $(v^1, \ldots, v^n)$ are called the *components of* V w.r.t. the chosen basis if

$$v = v^1 e_1 + \cdots v^n e_n\,. \tag{21}$$

***Definition* 2.11** (Dual basis)
Choose basis $\{e_1, \cdots, e_n\}$ for V. The basis $\{\epsilon^1, \cdots, \epsilon^n\}$ for $V^\star$ can be chosen that

$$\epsilon^a(e_b) = \delta^a_b \quad \forall a, b = 1, \ldots, n\,. \tag{22}$$

$\{\epsilon^1, \ldots, \epsilon^n\}$ is then called *the dual basis* of the dual space.

2.3 Components of a tensor

T an (r, s)-tensor. Then the real numbers

$$T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} = T\left(\epsilon^{i_1}, \ldots, \epsilon^{i_r}, e_{j_1}, \ldots e_{j_s}\right) \tag{23}$$

are the components of T with respect to the chosen basis. From the components and the basis one can reconstruct the entire tensor: Example for (1, 1)-tensor:

$$T(\varphi, v) = T^i_{\;j}\,\varphi_i v^j \tag{24}$$

where φ_i are the components of $\varphi \in V^\star$ and v^j the components of $v \in V$ with respect to the chosen basis. In equation (24) the *Einstein summation convention* is used, *i.e.* an index that appears up and down in an expression is summed over.

Note: The Einstein summation convention is only useful because we are working with *linear maps*, otherwise the expression

$$\varphi\left(\sum_i v^i e_i\right) = \sum_i \varphi(v^i e_i)\,, \tag{25}$$

would not hold and with the summation index we would not know where the sum sign goes.

$$\begin{tikzcd}
& y(U\cup V)\subseteq\mathbb{R}^d \\
\gamma:\mathbb{R} \arrow[ur, "y\circ\gamma"] \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma"'] & U\cup V\neq\emptyset \arrow[u, "y"'] \arrow[d, "x"] \\
& x(U\cup V)\subseteq\mathbb{R}^d \arrow[uu, bend right=60, "y\circ x^{-1}"']
\end{tikzcd}$$

Figure 3: *Curve γ in chart.*

3 Differentiable Manifolds

So far we only had topological manifolds. We also want to be able to talk about the velocity of curves. The problem is that the notion of a topological manifold is not enough to define differentiability of curves. In this section we will find out what additional structure we need to be able to talk about the differentiability of

- curves: $\mathbb{R}\to\mathcal{M}$
- functions: $\mathcal{M}\to\mathbb{R}$
- maps: $\mathcal{M}\to\mathcal{N}$

Strategy: Choose a chart (U,x) and consider portion of the curve in the domain of the chart: $\gamma:\mathbb{R}\to U$ (see figure 3). Since $x\circ\gamma:\mathbb{R}\to\mathbb{R}^d$ we can try to "lift" the notion of differentiability of a curve on $\mathbb{R}^d$ to that of a curve on $\mathcal{M}$. The problem is to make this independent of the chart.

$$y\circ\gamma=\underbrace{(y\circ x^{-1})}_{\mathbb{R}^d\to\mathbb{R}^d}\circ\underbrace{(x\circ\gamma)}_{\substack{\mathbb{R}\to\mathbb{R}^d\\ \text{differentiable}}}\quad, \qquad (26)$$

but we only know that the *chart transition map* $y\circ x^{-1}$ is continuous (because of the definition of a top. Manifold). Thus it is not guaranteed that $y\circ\gamma$ is continuons, not differentiable. Reminder: The composition of continuous maps is continuous, same for differentiable. The above definition of differentiability of γ by checking the differentiability of $x\circ\gamma$ with some chart x is not independent of the chart.

***Definition* 3.1** ($\star$ - compatibility of charts)
Two charts (U,x) and (V,y) of a topological manifold are called $\star$-compatible if either

1. $U\cup V=\emptyset$ or
2. $U\cup V\neq\emptyset$ and the chart transition maps

$$y\circ x^{-1}:\mathbb{R}^d\supseteq x(U\cup V)\to y(U\cup V)\subseteq\mathbb{R}^d$$
$$x\circ y^{-1}:\mathbb{R}^d\supseteq y(U\cup V)\to x(U\cup V)\subseteq\mathbb{R}^d$$

have the $\star$-property in the $\mathbb{R}^d$-sense.

***Definition* 3.2** ($\star$-compatible atlas)
An atlas $\mathcal{A}_\star$ is a $\star$-compatible atlas if any two charts in $\mathcal{A}_\star$ are $\star$-compatible.

***Definition* 3.3** ($\star$-manifold)
A $\star$-manifold is a triple $(\underbrace{\mathcal{M},\mathcal{O}}_{\text{top. manif.}},\underbrace{\mathcal{A}_\star}_{\in\mathcal{A}_{\max}})$.

$\star$	$\star$ property in $\mathbb{R}^d$-sense
C^0	$C^0(\mathbb{R}^d\to\mathbb{R}^d)$ continuous maps
C^1	$C^1(\mathbb{R}^d\to\mathbb{R}^d)$ differentiable and result is cont.
C^k	$C^k(\mathbb{R}^d\to\mathbb{R}^d)$ k times diffble and result is cont.
D^k	$D^k(\mathbb{R}^d\to\mathbb{R}^d)$ k times differentiable
C^∞	$C^\infty(\mathbb{R}^d\to\mathbb{R}^d)$ smooth functions
C^ω	$\exists$ multidim. Taylor expansion, $C^\omega\subset C^\infty$

Note: The more fancy properties one wants for the objects on the manifold, the more restrictive one has to be for the atlas.

***Theorem* 3.4** ($C^1\to C^\infty$)
Any $C^{k\leq 1}$-manifold atlas $\mathcal{A}_{C^{k\leq 1}}$ of a topological manifold contains a C^∞-atlas.

Thus we may without loss of generality always consider C^∞-manifolds. "smooth" manifolds, unless we wish to define Taylor expandibility or complex differentiability, ...

***Definition* 3.5** (Smooth manifold)
$(\mathcal{M},\mathcal{O},\mathcal{A})$, where $(\mathcal{M},\mathcal{O})$ is a topological manifold and $\mathcal{A}$ is a C^∞-atlas.

4 Diffeomorphisms

$$M\xrightarrow{\ \phi\ }N$$

M, N naked sets, then the structure-preserving maps are bijections (invertionable maps).

***Definition* 4.1** (Set-theoretically isomorphic)
Two sets M, N are said to be *set-thoretically isomorphic* $M\cong_{\text{st}}N$ if $\exists$ a bijection $\phi:M\to N$ between them.

Note: Then they are "of the same size". Examples: $\mathbb{N}\cong_{\text{st}}\mathbb{Z}$, $\mathbb{N}\cong_{\text{st}}\mathbb{Q}$, $\mathbb{N}\not\cong_{\text{st}}\mathbb{R}$

Now $(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ and $(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$.

$$\mathcal{M} \xrightarrow{\phi} \mathcal{N}$$

***Definition* 4.2** (Topologically isomorphic (homeomorphic))
$(\mathcal{M}, \mathcal{O}_{\mathcal{M}}) \cong_{\text{top}} (\mathcal{N}, \mathcal{O}_{\mathcal{N}})$ topologically isomorphic = "homeomorphic" if $\exists \phi : \mathcal{M} \to \mathcal{N}$ and ϕ, ϕ^{-1} are *continuous*.

Note: Continuity is the important property here. This is a stronger notion. If two spaces are hoemomorphic then they are also set-theoretically isomorphic.

***Definition* 4.3** (Isomorphic vector spaces)
$(V, +_V, \cdot_V) \cong_{\text{vec}} (W, +_W, \cdot_W)$ if $\exists$ a bijection $\phi : V \to W$ that is linear in both directions.

***Definition* 4.4** (diffeomorphic)
Two C^∞ manifolds $(\mathcal{M}, \mathcal{O}_{\mathcal{M}}, \mathcal{A}_{\mathcal{M}})$ and $(\mathcal{N}, \mathcal{O}_{\mathcal{N}}, \mathcal{A}_{\mathcal{N}})$ are said to be *diffeomorphic* if $\exists$ a bijeciton $\phi : \mathcal{M} \to \mathcal{N}$ such that ϕ, ϕ^{-1} are both C^∞-maps, where by C^∞ we mean that $y \circ \phi \circ x^{-1}$ is in C^∞ in the $\mathbb{R}^d$-sense, see figure 4.

$$\begin{tikzcd}
\mathbb{R}^d \arrow[rr, "\tilde{y} \circ \phi \circ \tilde{x}^{-1}"'] & & \mathbb{R}^l \\
\mathcal{M} \supseteq U \arrow[u, "\tilde{x}"] \arrow[rr, "\phi"] \arrow[d, "x"'] & & V \subseteq N \arrow[u, "\tilde{y}"'] \arrow[d, "y"] \\
\mathbb{R}^d \arrow[rr, "y \circ \phi \circ x^{-1}"] \arrow[uu, bend left=60, "C^\infty"] & & \mathbb{R}^l \arrow[uu, bend right=60, "C^\infty"']
\end{tikzcd}$$

Figure 4: *In the definition of diffeomorphic 4.4 ϕ, ϕ^{-1} have to be C^∞, which is defined such that $y \circ \phi \circ x^{-1}$ has to be C^∞ in the $\mathbb{R}^d$-sense, which is chart-independent here.*

Note: Since we started with C^∞-manifolds, the chart transition maps are C^∞ and thus the notion of differentiability in the definition 4.4 is independent of the choice of charts, *i.e.* $\tilde{y} \circ \phi \tilde{x}^{-1}$ is also C^∞, see figure 4.

***Theorem* 4.5**
= number of C^∞-manifolds one can make of a given C^∞-manifold (if any) — up to diffeomorphisms —.

dim$M$	#	
1	1	Moise-Radon theorem
2	2	
3	3	
4	uncountable infinitely many	
5	finite	surgery theory
6	finite	
$\vdots$	finite	

5 Tangent Spaces

"What is the velocity of a curve γ at a point p?"

5.1 Velocities, Tangent Spaces

***Definition* 5.1** (Velocity)
$(\mathcal{M}, \mathcal{O}, \mathcal{A})$ smooth manifold, curve $\gamma : \mathbb{R} \to \mathcal{M}$ at least C^1. Suppose $\gamma(\lambda_0) = p$. The velocity v of γ at p is the linear map

$$v_{\gamma,p} : C^\infty(\mathcal{M}) \xrightarrow{\sim} \mathbb{R}\,, \tag{27}$$

with

$$f \mapsto v_{\gamma,p}(f) := (f \circ \gamma)'(\underbrace{\gamma^{-1}(p)}_{\lambda_0})\,, \tag{28}$$

i.e., the directional derivative of f along γ at the point p

Note: Remember

$$C^\infty(\mathcal{M}) = \{f : \mathcal{M} \to \mathbb{R} | f \text{ smooth}, \tag{29}$$
$$(f+g)(p) := f(p) + g(p) \tag{30}$$
$$(\lambda \cdot g)(p) := \lambda \cdot g(p), \lambda \in \mathbb{R}\} \tag{31}$$

is a vector space.

Since $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ we can simply take the normal derivative.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "f \circ \gamma"'] & M \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

Note: In differential calculus one had the directional derivative as $v^i(\partial_i f)$. The shift in philosophy is now to see $v^i \partial_i$, *i.e.*, the operator that acts on f, as the vector.

***Definition* 5.2** (Tangent space)
$\forall p \in \mathcal{M}$ the "tangent space to $\mathcal{M}$ at p" consists of all the velocities of curves at that point :

$$T_p\mathcal{M} := \{v_{\gamma,p} | \gamma \text{ smooth curves}\} \ . \tag{32}$$

Note: There is no reference to any external space or embedding in definition 5.2, see also figure 5!

$T_p\mathcal{M}$ can be made into a vector space. The proof for this is so important and contains so many important things, that one should go through it in detail.

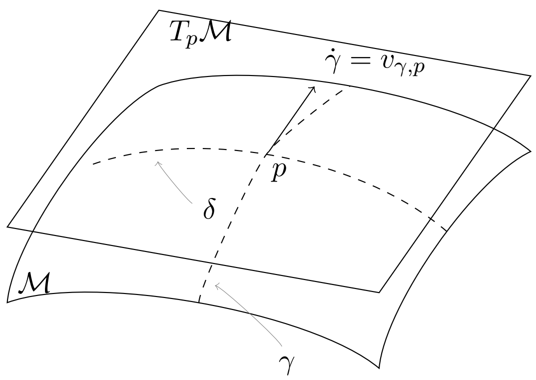

Figure 5: *Picture for immagining the tangent space. Keep in mind that there is no embedding needed like in this picture. Also in one can only think of $v_{\gamma,p}$ as an arrow in the chart, but not at manifold level. Also it is usefull to think of the arrow as the directional derivative ∂_v in the direciton of this arrow.*

***Definition* 5.3** (Addition and multiplication for tangent space)
For $p \in \mathcal{M}$, γ smooth curve on $\mathcal{M}$, $\alpha \in \mathbb{R}$:

$$\begin{aligned}
+ &: T_p\mathcal{M} \times T_p\mathcal{M} \to \mathrm{Hom}(C^\infty(\mathcal{M}), \mathbb{R}) \\
&(v_{\gamma,p} + v_{\delta,p})(f) := v_{\gamma,p}(f) + v_{\delta,p}(f) \qquad (33) \\
&f \in C^\infty(M) \\
\cdot &: \mathbb{R} \times T_p\mathcal{M} \to \mathrm{Hom}(C^\infty(\mathcal{M}), \mathbb{R}) \\
&(\alpha \cdot v_{\gamma,p})(f) = \alpha \cdot v_{\gamma,p}(f) \qquad (34)
\end{aligned}$$

But do they close, *i.e.*, is the tangent space a vector space? It remains to be shown that

1. $\exists$ curve $\sigma : v_{\gamma,p} + v_{\delta,p} = v_{\sigma,p}$
2. $\exists$ curve $\tau : \alpha \cdot v_{\gamma,p} = v_{\tau,p}$

The problem for 1 is that one cannot define $v_{\gamma,p} + v_{\delta,p}$ as just adding the points of the curves, since there is no such thing as adding two points on a manifold (what would be Paris + Berlin?).

Proof: (Tangent space is a vector space)

2. Construct $\tau : \mathbb{R} \to \mathcal{M}$:

$$\tau(\lambda) := \gamma(\alpha \cdot \lambda + \lambda_0) = (\gamma \circ \mu_\alpha)(\lambda) \tag{35}$$

with $\mu_\alpha : \mathbb{R} \to \mathbb{R}, r \mapsto \alpha r + \lambda_0$. Then $\tau(0) = \gamma(\lambda_0) = p$.

$$\begin{aligned}
v_{\tau,p} &:= (f \circ \tau)'(0) = (f \circ \gamma \circ \mu_\alpha)'(0) \\
&= (f \circ \gamma)(\lambda_0)\alpha = \alpha v_{\gamma,p} \,. \qquad (36)
\end{aligned}$$

1. Two curves $\gamma(\lambda), \delta(\lambda)$ with $\gamma(\lambda_0) = p$ and $\delta(\lambda_1) = p$. Make a choice of chart (U, x) with $p \in U$, later show independence of chart. Define

$$\begin{aligned}
\sigma_x &: \mathbb{R} \to \mathcal{M} \,, \\
\sigma_x(\lambda) &:= x^{-1}\left[(x \circ \gamma)(\lambda_0 + \lambda) + \right. \qquad (37) \\
&\left. + (x \circ \delta)(\lambda_1 + \lambda) - (x \circ \gamma)(\lambda_0)\right] \,.
\end{aligned}$$

$$\begin{aligned}
\sigma_x(0) &= \delta(\lambda_1) = p \,, \qquad (38) \\
v_{\sigma_x,p}(f) &= (f \circ \sigma_x)'(0) \qquad (39) \\
&= \left[\underbrace{(f \circ x^{-1})}_{\mathbb{R}^d \to \mathbb{R}} \circ \underbrace{(x \circ \sigma_x)}_{\mathbb{R}^d \to \mathbb{R}}\right]'(0) \,,
\end{aligned}$$

where now we use the multidimensional chain rule that a physicist would rather know as $\frac{\mathrm{d}}{\mathrm{d}\lambda} f(\vec{y}(\lambda)) = (\vec{\nabla}_y f) \cdot \frac{\mathrm{d}\vec{y}}{\mathrm{d}\lambda}$.

$$v_{\sigma_x,p}(f) = (x^i \circ \sigma_x)'(0) \left[\partial_i (f \circ x^{-1})\right] (\underbrace{x(\sigma_x(0))}_{x(p)})$$

$$\begin{aligned}
(x^i \circ \sigma_x)'(0) &= [(x^i \circ \gamma(\lambda_0 + \lambda) + (x^i \circ \delta)(\lambda_1 + \lambda) \\
&\quad - (x^i \circ \gamma)(\lambda_0)]' \\
&= (x^i \circ \gamma)'(\lambda_0) + (x^i \circ \delta)'(\lambda_1) = v^i_{\gamma,p} + v^i_{\delta,p} \,.
\end{aligned}$$

Plugging this into equation (39) and doing the same step backwards we get

$$v_{\sigma_x,p}(f) = v_{\gamma,p}(f) + v_{\delta,p}(f) \quad \forall f \in C^\infty(\mathcal{M}) \,, \tag{40}$$

independent of the chart.

Note: It turns out that the sum of curves like this depends on the map (U, x), but the derivative at the point p does not.

□

Bibliography

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