Mathematical Notes on Manifolds in Physics

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1 Manifolds

1.1 Topology

This chapter is basically taken from (Schuller, 2015) with our remarks to it.

We start with a set M which is supposed to be the space where physics happens. The weakest structure we need in order to talk about continuity (of curves or fields) is called a topology.

***Definition* 1.1** (Power set $\mathcal{P}$)
The set of all subsets of M.

***Definition* 1.2** (Topology)
A Topology $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$ satisfying:

1. $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$,
2. $U \in \mathcal{O}, \quad V \in \mathcal{O} \Rightarrow U \cap V \in \mathcal{O}$
3. $U_\alpha \in \mathcal{O}, \quad \alpha \in A \Rightarrow \left(\bigcup_{\alpha \in A} U_\alpha \right) \in \mathcal{O}$

Every set has the *chaotic topology*

$$\mathcal{O}_{\text{chaotic}} := \{\emptyset, M\} \ , \tag{1}$$

and the *discrete topology*

$$\mathcal{O}_{\text{discrete}} := \mathcal{P}(M) \, , \tag{2}$$

which are both useless.

The special case $M = \mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R}$ has a standard topology for which we need the definition of a soft ball.

***Definition* 1.3** (Soft Ball in $\mathbb{R}^d$)

$$B_r(p) := \left\{ (q_1, \cdots, q_d) | \sum_{i=1}^{d} (p_i - p_i) < r \right\} \ , \tag{3}$$

with $r \in \mathbb{R}^+$, $p \in \mathbb{R}^d$. Note: This does not need a norm or vector space structure on $\mathbb{R}^d$.

***Definition* 1.4** ($\mathcal{O}_{\text{standard}}$ on $\mathbb{R}^d$)

$$U \in \mathcal{O}_{\text{standard}} :\Leftrightarrow \forall p \in U : \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U \tag{4}$$

Some terminology: Let M be a set with a topology $\mathcal{O}$ =: set of open sets. We call $(M, \mathcal{O}$ a *topological space* and:

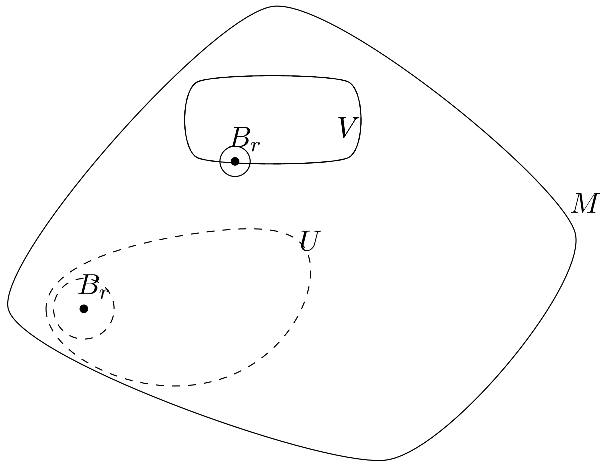

Figure 1: *The set U is in the standard topology, V not.*

- $U \in \mathcal{O} \Leftrightarrow$: call $U \subseteq M$ an *open set*
- $M \backslash A \in \mathcal{O} \Leftrightarrow$: call $U \subseteq M$ a *closed set*

Note: The empty set is open and closed. If a set is open we cannot directly follow that it is not closed or vise versa. For $M = \{1, 2\}$ and $\mathcal{O}_M = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ the set $\{2\}$ is open and closed.

1.2 Continuous Maps

A map

$$f : M \to N\,, \tag{5}$$

takes every point from the domain M (a set) to the target N (a set). If one point $p \in N$ is not reached, the map is not *surjective*. If a point is hit twice, the map is not *injective*. A map that is injective and surjective is called *surjective*.

***Definition* 1.5** (Preimage)

$$\begin{aligned} f : M \to N \supseteq V \\ \text{preim}_f(V) := \{m \in M \mid f(f) \in V\} \end{aligned} \tag{6}$$

***Definition* 1.6** (Continuity)
$(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ topological spaces. Then a map $f : M \to N$ is called *continuous with respect to $\mathcal{O}_M$ and $\mathcal{O}_N$* if

$$\forall V \in \mathcal{O}_N : \text{preim}_f(V) \in \mathcal{O}_M\,. \tag{7}$$

"A map is open iff the preimages of all open sets are open sets."

Note: If a map is not surjective there are sets with preimage $\emptyset$, thus we need to have $\emptyset$ in $\mathcal{O}$, otherwise only surjective maps could be continuous.

Note: The inverse of a continuous function does not need to be continuous.

***Definition* 1.7** (Composition of maps)
For f and g

$$f : M \to N, \quad g : N \to P\,,$$

we define the *composition* as

$$\begin{aligned} g \circ f : M \to P \\ m \mapsto (g \circ f)(m) := g(f(m)) \end{aligned} \tag{8}$$

***Theorem* 1.8** (Composition of continuos maps)
For f, g continuos also $g \circ f$ is continuous (if spaces match).

***Definition* 1.9** (Subset topology, Inherited topology)
A set M with topology $\mathcal{O}_M$. Given any subset $S \subseteq M$ we can construct the inherited topology $\mathcal{O}\big|_S \subseteq \mathcal{P}(S)$

$$\mathcal{O}\big|_S := \{U \cap S \mid U \in \mathcal{O}M\}\,. \tag{9}$$

Note: For $S \subseteq M$, if f is continuous then $f\big|_S$ is also continuous if $\mathcal{O}\big|_S$ is chosen. This is for example important if you are on a trajectory γ through $\mathbb{R}^n$ and measure the temperature $T\big|_\gamma$.

***Definition* 1.10** (Topological manifold)
A topological space $(\mathcal{M}, \mathcal{O})$ is called a *d-dimensional topological manifold* if

$$\forall p \in \mathcal{M} : \exists U \in \mathcal{O},\ p \in U : \exists x : U \to x(U) \subseteq \mathbb{R}^d\,, \tag{10}$$

with the following properties (wrt. $\mathcal{O}_{\text{std}}$ on $\mathbb{R}^d$):

1. x intervitble: $x^{-1} : x(U) \to U$,
2. x continuous,
3. x^{-1} continuous.

"Invertible, in both directions continuous map to $\mathbb{R}^n$."

Note: Thus in the above definition $x(U)$ is also open (from the definition of continuity).

Terminology:
- (U, x) is a *chart* of $\mathcal{M}, \mathcal{O}$,
- $\mathcal{A} = \left\{(U_{(\alpha)}, x_{(\alpha)} | \alpha \in A\right\}$ is an *atlas* of $(\mathcal{M}, \mathcal{O})$ if $\bigcup_{\alpha \in A} U_{(\alpha)}$ covers the whole manifold $\mathcal{M}$,
- $x : U \to x(U) \subseteq \mathbb{R}^d$ is a *chart map* $x(p) = (x^1(p), \ldots, x^d(p))$, where the *component maps* $x^i : U \to \mathbb{R}$ are called *coordinate maps*,
- $p \in U$, then $x^1(p)$ is the first coordinate of the point p wrt. the chosen chart (U, x).

Note: The choice of the chart (choice of coordinates) has nothing to do with the physics. Physics is chart independent. $\mathcal{M}$ is "the real world".

1.3 Chart Transition Maps

Given (U, x) and (V, y) charts, on $U \cup V$ one can transition from one chart to the other by (see figure 2)

$$y \circ x^{-1} : \mathbb{R}^d \supseteq x(U \cap V) \to y(U \cap V) \subseteq \mathbb{R}^d\,, \quad (11)$$

which is called the *chart transition map*.

Note: As a physicist one talks about a "change in coordinates".

1.4 Manifold Philosophy

The idea is to define properties of some object in the real world $\mathcal{M}$ by at a chart-representative of it. For example the continuity of a curve $\gamma : [0,1] \to \mathcal{M}$ can be judged by looking at $x \circ \gamma : [0,1] \to \mathbb{R}^d$, because x is invertible and in both directions continuous and the composition of two continuous maps is also continuous.

Note: One needs to make sure that the property of the object on $\mathcal{M}$ does not depend on the map x or y. For continuity this is the case, since $y \circ \gamma = (y \circ x^{-1}) \circ x \circ \gamma$ and the chart transition map $y \circ x^{-1}$ is also continuous.

Other properties like "differentiability" are not even defined on $\mathcal{M}$ a priori, so one can only talk about the chart representative. Here the definition that γ is differentiable iff $x \circ \gamma : [0,1] \to \mathbb{R}^d$ is differentiable has the problem that x and y only need to be continuous and so the chart transition map $y \circ x^{-1}$ does not need to be differentiable unless one restricts oneself to only differentiable charts.

2 Vector Spaces

2.1 Vectors and Linear Maps

In order to understand the tangent space we need to understand vector spaces.

***Definition* 2.1** (Vector space $(V, +, \cdot)$)
A vector space $(V, +, \cdot)$ is a set V with

- an "addition" $+ : V \times V\ toV$,
- an "S-multiplication" $\cdot : \mathbb{R} \times V\ toV$

and the properties CANI ADDU:
$\forall v, w, u \in V, \lambda, \mu \in \mathbb{R}$

C$^+$: $v + w = w + v\,,$
A$^+$: $(u + v) + w = u + (v + w)\,,$
N$^+$: $\exists 0 \in V : \forall v \in V : v + 0 = v\,,$
I$^+$: $\forall v \in V : \exists(-v) \in V : v + (-v) = 0\,,$
A: $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) \cdot v\,,$
D: $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v\,,$
D: $\lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w)\,,$
U: $1 \cdot v = v\,.$

An element of a vector space is called a *vector*.

Note: The addition $+$ in definition 2.1 sometimes is between vectors and sometimes between scalars. It is important to know the difference.

***Definition* 2.2** (Linear maps)
(Structure respecting maps between vector spaces)
$(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ vector spaces. A map

$$\phi : V \to W \quad (12)$$

is called *linear* if $\forall v, \tilde{v} \in V, \forall \lambda \in \mathbb{R}$

1. $\phi(v +_V + \tilde{v}) = \phi(v) +_W \phi(\tilde{v})$
2. $\phi(\lambda \cdot_V v) = \lambda \cdot_W \phi(v)$

We write:

$$\phi : V \to W \text{ linear } \Leftrightarrow: \phi : V \xrightarrow{\sim} W\,. \quad (13)$$

***Theorem* 2.3** (Transitivity of linearity of maps)
V, W, U vector spaces, $\psi : V \xrightarrow{\sim} W$, $\phi : W \xrightarrow{\sim} U$ then $\phi \circ \psi$ is also linear: $\phi \circ \psi : V \xrightarrow{\sim} U$.

***Definition* 2.4** (Homomorphisms $\mathrm{Hom}(V, W)$)

$$\mathrm{Hom}(V, W) := \left\{ \phi : V \xrightarrow{\sim} W \right\}\,. \quad (14)$$

Note: $\mathrm{Hom}(V, W)$ can be made into a vector space by defining an addition and a multiplication

- $(\phi + \psi)(v) := \phi(v) + \psi(v)\,,$
- $(\lambda\psi)(v) := \lambda(\psi(v))\,.$

***Definition* 2.5** (Dual vector space $V^\star$)

$$V^\star := \left\{ \phi : V \xrightarrow{\sim} \mathbb{R} \right\} = \mathrm{Hom}(V, \mathbb{R})\,. \quad (15)$$

The vector space $(V^\star, +, \cdot)$ is the *dual vector space* to V. $\phi \in V^\star$ is informally called a *covector*.

***Definition* 2.6** ((r, s) - Tensors)
$(V, +, \cdot)$ vector space, $r, s \in \mathbb{N}_0$. An (r, s)-tensor T over V is a multi-linar map

$$T : \overbrace{V^\star \times \cdots \times V^\star}^{r} \times \overbrace{V \times \cdots \times V}^{s} \overset{\sim}{\underset{\vdots}{\longrightarrow}} \mathbb{R} \quad (16)$$

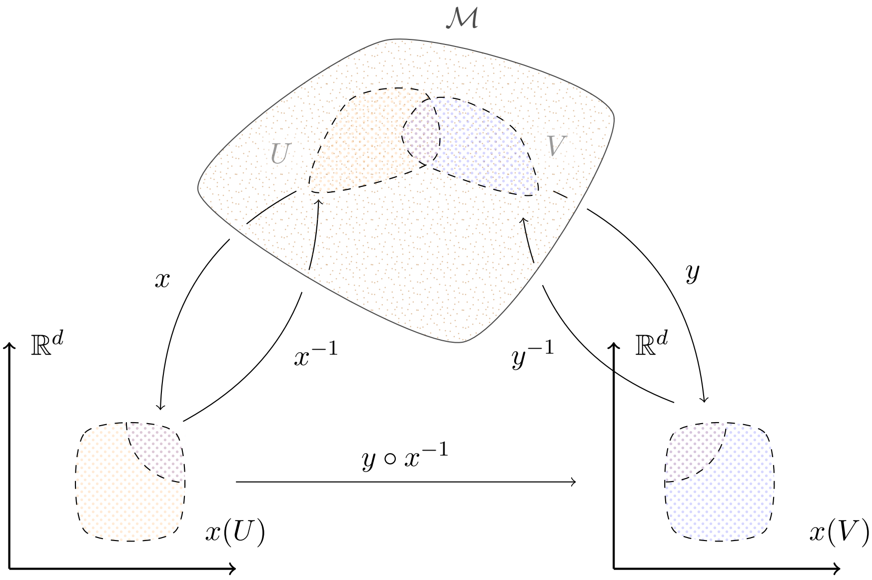

Figure 2: *Visualization of chart transition maps. "How to glue together the charts of an atlas." Plot modified from (*Drawing manifolds in tikz *n.d.)*

***Theorem* 2.7** (Covector is (0,1)-tensor)

$$\phi \in V^\star \Leftrightarrow \phi : V \xrightarrow{\sim} \mathbb{R} \Leftrightarrow \phi\,(0,1)\,\text{tensor}\,. \tag{17}$$

***Theorem* 2.8** (Vector is (1,0)-tensor)
If dim$V < \infty$

$$v \in V = (V^\star)^\star \Leftrightarrow v : V^\star \xrightarrow{\sim} \mathbb{R} \Leftrightarrow v\,\text{is}\,(1,0)\text{−tensor}\,. \tag{18}$$

2.2 Bases

***Definition* 2.9** (Hamel-basis)
$(V, +, \cdot))$ vector space. A subset $B \subset V$ is called a Hamel-basis if

$$\forall v \in V\ \exists!\,\text{finite}\ F = \{f_1, \dots, f_n\} \subset B : \exists!\, \underbrace{v^1, \cdots, v^n}_{\in\mathbb{R}}, \tag{19}$$

such that

$$v = v^1 f_1 + \cdots + v^n f_n\,. \tag{20}$$

(and all f_i linearly independent).

***Definition* 2.10** (Dimension of a vector space)
If a basis B with $d < \infty$ many elements, then we call $d =:$ dim$V$.

If we have chosen a basis $\{e_1, \dots, e_n\}$ of $(V, +, \cdot)$ then $(v^1, \dots, v^n)$ are called the *components of* V w.r.t. the chosen basis if

$$v = v^1 e_1 + \cdots v^n e_n\,. \tag{21}$$

***Definition* 2.11** (Dual basis)
Choose basis $\{e_1, \cdots, e_n\}$ for V. The basis $\{\epsilon^1, \cdots, \epsilon^n\}$ for $V^\star$ can be chosen that

$$\epsilon^a(e_b) = \delta^a_b \quad \forall a, b = 1, \dots, n\,. \tag{22}$$

$\{\epsilon^1, \dots, \epsilon^n\}$ is then called *the dual basis* of the dual space.

2.3 Components of a tensor

T an (r, s)-tensor. Then the real numbers

$$T^{i_1\dots i_r}{}_{j_1\dots j_s} = T\left(\epsilon^{i_1}, \dots, \epsilon^{i_r}, e_{j_1}, \dots e_{j_s}\right) \tag{23}$$

are the components of T with respect to the chosen basis. From the components and the basis one can reconstruct the entire tensor: Example for (1, 1)-tensor:

$$T(\varphi, v) = T^i{}_j\,\varphi_i v^j \tag{24}$$

where φ_i are the components of $\varphi \in V^\star$ and v^j the components of $v \in V$ with respect to the chosen basis. In equation (24) the *Einstein summation convention* is used, *i.e.* an index that appears up and down in an expression is summed over.

Note: The Einstein summation convention is only useful because we are working wit *linear maps*, otherwise the expression

$$\varphi\left(\sum_i v^i e_i\right) = \sum_i \varphi(v^i e_i)\,, \tag{25}$$

would not hold and with the summation index we would not know where the sum sign goes.

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