

Lectures on Resurgence and Trans-Series

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1. (Borel Summation of the $\lambda\phi^4$ Integral)

The integral

$$Z(\lambda) = \int_{-\infty}^{\infty} dx e^{-x^2 - \lambda x^4} \quad (1)$$

has the analytic solution

$$Z(\lambda)_{\text{analytic}} = \frac{e^{\frac{1}{8\lambda}} K_{\frac{1}{4}}\left(\frac{1}{8\lambda}\right)}{2\sqrt{\lambda}} \quad (2)$$

and, as we have checked on the last exercise sheet, the asymptotic series

$$Z(\lambda)_{\text{asympt.}} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(4k)!(-1)^k}{2^{4k}(2k)!k!} \lambda^k. \quad (3)$$

(a) (MATHEMATICA: Asymptotic Expansion of the Analytic Solution)

Hankel came up with an asymptotic expansion for the *modified Hankel function*¹ $K_{\alpha}(z)$,

$$K_{\alpha}(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\alpha^2 - 1}{8z} + \frac{(4\alpha^2 - 1)(4\alpha^2 - 9)}{2!(8z)^2} + \frac{(4\alpha^2 - 1)(4\alpha^2 - 9)(4\alpha^2 - 25)}{3!(8z)^3} + \dots \right), \quad (4)$$

for $|\arg z| < \frac{3\pi}{2}$.² Check that the asymptotic expansion of $Z(\lambda)_{\text{analytic}}$ is exactly $Z(\lambda)_{\text{asympt.}}$. The easiest way is to let Mathematica expand $Z(\lambda)_{\text{analytic}}$ with `Series[]` and compare the first few terms. Note that Mathematica knows about the asymptotic expansion (4).

(b) (Borel Transform)

The *Borel transform* \mathcal{B} is defined as

$$B : z^{-1}\mathbb{C}[[z^{-1}]] \rightarrow \mathbb{C}[[\zeta]], \quad (5)$$

$$B : \tilde{\phi}(z) = \sum_{n=0}^{\infty} c_n z^{-n-1} \mapsto \hat{\phi}(\zeta) = \sum_{n=0}^{\infty} c_n \frac{\zeta^n}{n!}. \quad (6)$$

Note that we are working with $\lambda = \frac{1}{z}$ here. The Borel transform is not defined for the constant ($k = 0$) term in (3). Take out the constant term $\sqrt{\pi}$ and write

$$Z(\lambda)_{\text{asympt}} = \sqrt{\pi} + \tilde{Z}(\lambda)_{\text{asympt}}. \quad (7)$$

Perform the borel transform $\mathcal{B}[\tilde{Z}_{\text{asympt}}]$ by simply substituting

$$\lambda^k \rightarrow \frac{\zeta^{k-1}}{(k-1)!}. \quad (8)$$

By the ratio test³, figure out for what λ $\mathcal{B}[\tilde{Z}_{\text{asympt}}](\zeta)$ converges.

¹also called *Basset function*, *modified Bessel function of the third kind* and *Macdonald function*

²https://en.wikipedia.org/wiki/Bessel_function#Asymptotic_forms

³https://en.wikipedia.org/wiki/Ratio_test

- (c) (MATHEMATICA: **Analytic Continuation**) Type $\mathcal{B}[\tilde{Z}_{\text{asympt.}}]$ as an infinite sum into Mathematica, it should be able to write it as a Hypergeometric function.

$$\mathcal{B}[\tilde{Z}_{\text{asympt}}](\zeta) = -\frac{3\sqrt{\pi}}{4} {}_2F_1\left(\frac{5}{4}, \frac{7}{4}, 2 \middle| -4\zeta\right). \quad (9)$$

The Hypergeometric function ${}_2F_1(a, b, c|z)$ has a singularity at $z = 1$ and a branch cut from $z = 1$ to infinity. Note that (9) is the full analytic continuation of our Borel transform, since it works on all of $\mathbb{C} \setminus \{-\frac{1}{4}\}$. Is the singularity at $z = 1$ a problem for the Laplace transform?

- (d) (MATHEMATICA: **Inverse Borel Transform / Laplace Transform**)
Perform the Laplace transform

$$\mathcal{L}^\theta[\hat{\phi}](z) = \int_0^{e^{i\theta}\infty} d\zeta e^{-z\zeta} \hat{\phi}(\zeta), \quad (10)$$

with $\theta = 0$ in Mathematica and compare the result with $Z(\lambda)_{\text{analytic}}$. Don't forget to add the constant term again that we subtracted in (7).