Lectures on Resurgence and Trans-Series

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4. (**More $\lambda\phi^4$ integral**)
This time we want to calculate the "quartic partition function"
$$Z(\hbar) = \frac{1}{2\pi} \int_\Gamma e^{-\frac{1}{\hbar} V(z)}, \tag{1}$$
with the potential
$$V(z) = \frac{1}{2} z^2 - \frac{\lambda}{4!} z^4 \,, \tag{2}$$
and a contour Γ that starts at some "point" at (complex) infinity and ends at another "point" at (complex) infinity. Thus, for the integral to converge, one needs $\Re(V(z)) > 0$ as $z \to \infty$ along Γ.

   (a) (Warmup: **Change of variables**)
   Right now we have two small parameters in the exponent, $\hbar$ and λ. Find a change of variables so that
   $$Z(x) = \frac{\sqrt{\hbar}}{2\pi} \int_\Gamma \mathrm{d}z \, e^{-\frac{1}{2}z^2 + \frac{x}{4!} z^4} \,, \quad x = \lambda\hbar \,. \tag{3}$$
   This way we only have to care about x, which later will be complex.

   (b) (**Admissible directions for Γ**)
   Later we will have $x \in \mathbb{C}$, but right now assume $x \in \mathbb{R}^+$. Find the directions α along which $V(z) > 0$ for $z \to \infty$, by setting
   $$z = re^{i\alpha} \,. \tag{4}$$
   You should find four intervals for α.

   (c) (**Conversion to differential equation**)
   Show that the integral (3) can be written as an ordinary linear differential equation
   $$16x^2 Z''(x) + (32x - 24) Z'(x) + 3Z(x) = 0 \,. \tag{5}$$
   This can be done by plugging the definition of $Z(x)$, eq. (3), into the differential equation and showing that the integrand vanishes. Use the fact that $\Re(V(x)) \to +\infty$ for the start and end point of Γ. Start by partially integrating the last summand and then continue partially integrating some terms.
   Bonus points if you can tell me how to find the differential equation (5).

   (d) (**Series ansatz**) Make the ansatz
   $$Z(x) = e^{-\frac{A}{x}} \Phi(x) \,, \tag{6}$$
   $$\Phi(x) = \sum_{n=0}^{\infty} Z_n x^n \,, \tag{7}$$
   where $\Phi(x)$ is a formal power series (will be asymptotic). Find the two "solutions" for A, let's call them A_0 and A_1.

(e) (**Asymptotic series $\Phi(x)$**)
Corresponding to A_0 and A_1 we get two different asymptotic series $\Phi_0(x)$ and $\Phi_1(x)$,

$$\Phi_0(x) \simeq \sum_{n=0}^{\infty} Z_n^{(0)} x^n \,, \quad Z_n^{(0)} = \left(\frac{2}{3}\right)^n \frac{(4n)!}{2^{6n}(2n)!n!} Z_0^{(0)} \,, \tag{8}$$

$$\Phi_1(x) \simeq \sum_{n=0}^{\infty} Z_n^{(1)} x^n \,, \quad Z_n^{(1)} = (-1)^n \left(\frac{2}{3}\right)^n \frac{(4n)!}{2^{6n}(2n)!n!} Z_0^{(0)} \,, \tag{9}$$

with free parameters $Z_0^{(0)}$ and $Z_0^{(1)}$. You can check that Φ_0 and Φ_1 are correct if you want.
Calculate the Gevrey order $\frac{1}{m}$ of Φ_0 and Φ_1 as well as the instanton action A.
Reminders:

- A formal power series $\phi(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$ is of *Gevrey order* $\frac{1}{m}$ if the large order coefficients are bounded by
$$|c_n| \leq \alpha A^{-n} (n!)^m \,, \tag{10}$$
for some constants α and A, where A is called "instanton action".
- Stirling: $k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$.
- $n!$ scales just like $(n-1)!$ for large n.

(f) (**Non-perturbative saddle points and instantons**)
Find the saddle points of the exponent of the integrand of the original integral for $Z(x)$, eq. (3). Calculate the *instanton action* A of those saddle points.
Reminders:

$$\begin{aligned} I &= \int_{\mathcal{C}} \mathrm{d}z \, e^{-S(z)} \,, \\ S'(z_*) &= 0 \,, \quad \text{defines the saddle point } z_* \,, \\ S(z_*) &= 0 \,, \quad \text{instanton action} \,. \end{aligned}$$

Note: There is a factor $\frac{1}{x}$ that is sometimes in the instanton action A and sometimes not. I don't know if this is not clear in the literature or if I just don't have my definitions consistent.