Lectures on Resurgence and Trans-Series

Marco Knipfer, University of Alabama

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5. (**Consistency of Borel Summation**)
   As a reminder: We had a series

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} a_n z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]]\,, \tag{1}$$

and the *Borel transform* $\mathcal{B}$ was defined as

$$\hat{\phi}(z) = \mathcal{B}[\tilde{\phi}][\zeta] = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n\,. \tag{2}$$

If the series is asymptotic (and Gevrey type 1) the Borel transform will have finite radius of convergence. After analytic continuation of the Borel transform one uses the lateral Laplace transform

$$\mathcal{L}^\theta[\hat{\phi}](z) = \int_0^{e^{i\theta}\infty} \mathrm{d}\zeta\, e^{-z\zeta}\hat{\phi}(\zeta)\,, \tag{3}$$

which then (hopefully) gives a finite answer. The upper bound $e^{i\theta}\infty$ means that one integrates in a straight line from 0 to infinity in the complex plane at an angle θ with the real axis.

(a) (**Convergent series, $\theta = 0$**)
Let's start with a *convergent* series $\tilde{\phi}(z)$ of the form (1). Perform the Borel transform. The Borel transform of course is convergent on all $\mathbb{C}$, since already $\tilde{\phi}(z)$ was. Thus no analytic continuation is needed. Perform the Laplace transform $\mathcal{L}^0$, make a change of variables and use the definition of the Gamma function

$$\int_0^\infty \mathrm{d}t\, t^{n-1}e^{-t} = \Gamma(n) = (n-1)!\,, \tag{4}$$

to recover the original convergent series.

Tipp: Since the Borel transform converges you are allowed to exchange sum and integral.

(b) (**Convergent series, $\theta \neq 0$**)
Again start with a *convergent* series $\tilde{\phi}(z)$ of the form (1) and perform the Borel transform. This time perform the *lateral Laplace transform* $\mathcal{L}^\theta$ with $\theta \neq 0$. A change of variables, $\zeta = t/z$, $\alpha = \arg(z)$, will give

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} z^{-n-1} \int_0^{e^{i(\theta+\alpha)}\infty} \mathrm{d}t\, e^{-t}t^n\,. \tag{5}$$

Perform another change of variables such that the integral is from 0 to ∞. The resulting integral can be calculated *e.g.* with Mathematica. Find the solution and a condition on θ for the integral to converge. What is the condition for θ if $\alpha = \arg(z) = 0$? What if $z < 0$?

Note: If there is a singularity on the real line, we will have a jump when θ crosses it. We are going to investigate this on the next exercise sheet.