Chapter 2 : Mathematical Methods

Resurgence Lecture 3

2.1 Analytic Functions

Def: (Analytic Function on $\mathbb{C}$) (Wiki: Analytic function

$f: \mathbb{C} \to \mathbb{C}$ is analytic on the open set $D \subset \mathbb{C}$

if $\forall z_0 \in D$ one can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad , \; a_n \in \mathbb{C}$$

and the series converges

An analytic function is infinitely often differentiable ("smooth")

and $a_n = \frac{f^{(n)}(z_0)}{n!}$. the Taylor series converges to $f(z)$ on a neighborhood of z_0 pointwise

a function $f(z)$ is analytic $\Leftrightarrow$ holomorphic (complex differentiable

Def: (Holomorphic Function) Wiki: Holomorphic

$f: \mathbb{C} \to \mathbb{C}$ is holomorphic at $z_0 \in \mathbb{C}$

if it is complex differentiable ~~at~~ on some neighborhood of z_0.

f is holomorphic on $\mathbb{C} \supset U$ open, if it is complex differentiable $\forall z \in U$.

f is "entire" if $U = \mathbb{C}$

Theorem: (Looman-Menchoff) Wiki: Looman-Menchoff theorem

$f: \overset{\text{open}}{U} \subset \mathbb{C} \to \mathbb{C}$ continuous is holomorphic iff it satisfies the Cauchy-Riemann equations $z = x + iy$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right] = 0$$

"only function of z, not $\bar{z}$"

$$\Leftrightarrow (f = u + iv) \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

2.2 Analytic Continuation

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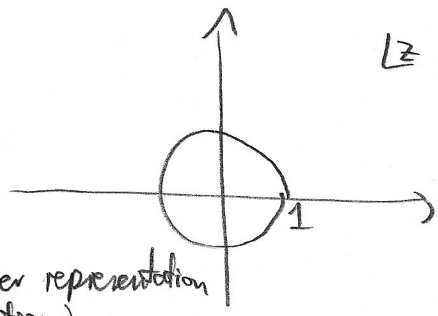
Example: (Geometric Series)

$$f(z)=\sum_{n=0}^{\infty} z^n$$



radius of convergence

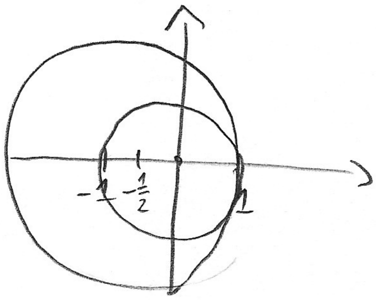
diverges for $z=R=1$

can write it as $\boxed{f(z)=\frac{1}{1-z}}$ master representation (equation)

- only diverges at $z=1$!
- The power series around $z=0$ has radius $R=1$
- Series around $z=-\frac{1}{2}$:

$$f(z)=\frac{1}{1+\frac{1}{2}-(z-\frac{1}{2})}=\frac{2}{3}\,\frac{1}{1-\frac{2}{3}(z+\frac{1}{2})}=\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n\left(z+\frac{1}{2}\right)^n$$

has $R=\frac{3}{2}$ around $z=-\frac{1}{2}$



- This way every $z\neq 1$ can be reached with a power series
- $\sum_{n=0}^{\infty} z^n$ is only the power series representation of $f(z)=\frac{1}{1-z}$

$$f(2)=\frac{1}{1-2}=-1 \quad \overset{?}{\neq}\sum_{n=0}^{\infty}2^n$$

not really, this representation is not good for $z=2$, but the analytic continuation is unique, so one could define

$$\sum_{n=0}^{\infty}2^n\overset{!}{=}-1$$

- Without master representation we would just calc. the derivatives at the new point and build the Taylor series, which would have radius until $z=1$.

↓ 17.04.