Lectures on Resurgence and Trans-Series

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1. (**Borel Summation of the $\lambda\phi^4$ Integral**)
The integral
$$Z(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-x^2 - \lambda x^4} \tag{1}$$
has the analytic solution
$$Z(\lambda)_{\text{analytic}} = \frac{e^{\frac{1}{8\lambda}} K_{\frac{1}{4}}\left(\frac{1}{8\lambda}\right)}{2\sqrt{\lambda}} \tag{2}$$
and, as we have checked on the last exercise sheet, the asymptotic series
$$Z(\lambda)_{\text{asympt.}} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(4k)!(-1)^k}{2^{4k}(2k)!\,k!} \lambda^k \,. \tag{3}$$

   (a) (Mathematica: **Asymptotic Expansion of the Analytic Solution**)
   Hankel came up with an asymptotic expansion for the *modified Hankel function*[1] $K_\alpha(z)$,
$$K_\alpha(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\alpha^2 - 1}{8z} + \frac{\left(4\alpha^2 - 1\right)\left(4\alpha^2 - 9\right)}{2!(8z)^2} + \frac{\left(4\alpha^2 - 1\right)\left(4\alpha^2 - 9\right)\left(4\alpha^2 - 25\right)}{3!(8z)^3} + \cdots \right), \tag{4}$$
   for $|\arg z| < \frac{3\pi}{2}$.[2] Check that the asymptotic expansion of $Z(\lambda)_{\text{analytic}}$ is exactly $Z(\lambda)_{\text{asympt.}}$. The easiest way is to let Mathematica expand $Z(\lambda)_{\text{analytic}}$ with `Series[ ]` and compare the first few terms. Note that Mathematica knows about the asymptotic expansion (4).

   (b) (**Borel Transform**)
   The *Borel transform* $\mathcal{B}$ is defined as
$$B : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[\zeta]]\,, \tag{5}$$
$$B : \tilde{\phi}(z) = \sum_{n=0}^{\infty} c_n z^{-n-1} \mapsto \hat{\phi}(\zeta) = \sum_{n=0}^{\infty} c_n \frac{\zeta^n}{n!}\,. \tag{6}$$
   Note that we are working with $\lambda = \frac{1}{z}$ here. The Borel transform is not defined for the constant ($k = 0$) term in (3). Take out the constant term $\sqrt{\pi}$ and write
$$Z(\lambda)_{\text{asympt.}} = \sqrt{\pi} + \tilde{Z}(\lambda)_{\text{asympt.}}\,. \tag{7}$$
   Perform the borel transform $\mathcal{B}[\tilde{Z}_{\text{asympt.}}]$ by simply substituting
$$\lambda^k \to \frac{\zeta^{k-1}}{(k-1)!}\,. \tag{8}$$
   By the ratio test[3], figure out for what ζ $\mathcal{B}[\tilde{Z}_{\text{asympt.}}](\zeta)$ converges.

[1] also called *Basset function, modified Bessel function of the second kind and Macdonald function*
[2] `https://en.wikipedia.org/wiki/Bessel_function#Asymptotic_forms`
[3] `https://en.wikipedia.org/wiki/Ratio_test`

(c) (MATHEMATICA: **Analytic Continuation**)
Type $\mathcal{B}[\tilde{Z}_{\text{asympt.}}]$ as an infinite sum into Mathematica, it should be able to write it as a Hypergeometric function.

$$\mathcal{B}[\tilde{Z}_{\text{asympt.}}](\zeta) = -\frac{3\sqrt{\pi}}{4}\, {}_2F_1\left(\frac{5}{4}, \frac{7}{4}, 2 \middle| -4\zeta\right) . \tag{9}$$

The Hypergeometric function ${}_2F_1(a, b, c|z)$ has a singularity at $z = 1$ and a branch cut from $z = 1$ to infinity. Note that (9) is the full analytic continuation of our Borel transform, since it works on all of $\mathbb{C}\backslash\left\{-\frac{1}{4}\right\}$. Is the singularity at $z = 1$ a problem for the Laplace transform?

(d) (MATHEMATICA: **Inverse Borel Transform / Laplace Transform**)
Perform the Laplace transform

$$\mathcal{L}^{\theta}[\hat{\phi}](z) = \int_0^{e^{i\theta}\infty} \mathrm{d}\zeta\, e^{-z\zeta}\hat{\phi}(\zeta)\,, \tag{10}$$

with $\theta = 0$ in Mathematica and compare the result with $Z(\lambda)_{\text{analytic}}$. Don't forget to add the constant term again that we subtracted in (7).