Different Definitions of the Borel Transform

Resurgence Lecture 7

Def: (Borel 1) $z \to \infty$ (Dorigoni, Aniceto)

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} c_n \frac{1}{z^{n+1}} \longrightarrow \hat{\phi}(\zeta) = \mathcal{B}[\tilde{\phi}](\zeta) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \zeta^n$$

Def (Laplace 1)

$$\mathcal{L}^{\theta}[\hat{\phi}](z) = \int_0^{\infty e^{i\theta}} d\zeta \, e^{-z\zeta} \hat{\phi}(\zeta)$$

Def (Borel 2) (Hatsuda, others)

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} c_n \frac{1}{z^n} \longrightarrow \hat{\phi}(\zeta) = \mathcal{B}[\tilde{\phi}](\zeta) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \zeta^n$$

includes constant term

Def (Laplace 2)

$$\mathcal{L}^{\theta}[\hat{\phi}](z) = \int_0^{\infty e^{i\theta}} d\zeta \, e^{-\zeta} \hat{\phi}(z\zeta)$$

Def (Variant of B2) (Hatsuda) $g \to 0$, $g = \frac{1}{z}$

$$\tilde{\phi}(g) = \sum_{n=0}^{\infty} c_n g^n \longrightarrow \mathcal{B}[\tilde{\phi}](\zeta) = \hat{\phi}(\zeta) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \zeta^n$$

$$\mathcal{L}(\hat{\phi})(g) = \int_0^{\infty e^{i\theta}} d\zeta \, e^{-\zeta} \hat{\phi}(g\zeta)$$

B1	B2	B3
$z \to \infty$	$z \to \infty$	$g \to 0$
$c_n \frac{1}{z^{n+1}}$	$c_n \frac{1}{z^n}$	$c_n g$
$\frac{c_n}{n!} \zeta^n$	$\frac{c_n}{n!} \zeta^n$	$\frac{c_n}{n!} \zeta^n$
$e^{-z\zeta} \hat{\phi}(\zeta)$	$e^{-\zeta} \hat{\phi}(\frac{1}{z}\zeta)$	$e^{-\zeta} \hat{\phi}(z\zeta)$

Why do they work?

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-z} \, dz$$

$\boxed{B1}$, convergent series $\tilde{\Phi} = \sum_{n=0}^{\infty} c_n \frac{1}{z^{n+1}}$

$$\mathcal{L}^0[\tilde{\Phi}](z) = \int_0^\infty d\zeta \, e^{-z\zeta} \sum_{n=0}^{\infty} \frac{c_n}{n!} \zeta^n \qquad \Big| \text{ convergent} \to \text{can } \int \leftrightarrow \sum$$

$$= \sum_{n=0}^{\infty} \frac{c_n}{n!} \int_0^\infty d\zeta \, \zeta^n e^{-z\zeta} \qquad \Big| \; \xi = z\zeta \,, \; d\zeta = \frac{1}{z} d\xi$$

$$= \sum_{n=0}^{\infty} \frac{c_n}{n!} \frac{1}{z^{n+1}} \underbrace{\int_0^\infty d\xi \, \xi^n e^{-\xi}}_{\Gamma(n+1) = n!} = \tilde{\Phi}(z)$$

if $\tilde{\Phi}$ is asympt.
$\tilde{\Phi}$ has finite R
$\Rightarrow$ anal. cont.
+ right contour,
then can $\int \leftrightarrow \sum$

Constant term: (what would be)

$$\mathcal{L}^0[1](z) = \int_0^\infty d\zeta \, e^{-z\zeta} \frac{C_{-1}}{(-1)!} \zeta^{-1} \overset{?}{=} \frac{C_{-1}}{\Gamma(0)} \int_0^\infty d\zeta \, e^{-z\zeta} \frac{1}{\zeta} \qquad \Big| \; \xi = z\zeta$$

$\Gamma(0) \to \infty$

$$= \frac{\overset{=1}{C_{-1}}}{\Gamma(0)} \underbrace{\int_0^\infty d\xi \, \frac{1}{\cancel{z}} \cancel{z} \frac{1}{\xi} e^{-\xi}}_{\Gamma(0)} \overset{?}{=} 1$$

More problems with the convolution property

$\boxed{B2}$

$$\tilde{\Phi} = \int_0^\infty d\zeta \, e^{-\zeta} \sum_{n=0}^{\infty} \frac{c_n}{n!} \zeta^n = \sum_{n=0}^{\infty} \frac{c_n}{n!} \underbrace{\int_0^\infty d\zeta \, e^{-\zeta} (\xi^n}_{\Gamma(n+1) = n!} z^{-n}) = \sum_{n=0}^{\infty} c_n \frac{1}{z^n}$$

Constant term has no problem here

Convolutive Property $\boxed{B1}$

$$\tilde{\Phi}(z)\,\tilde{\Psi}(z) = \sum_{n=0}^{\infty} \phi_n \frac{1}{z^{n+1}} \sum_{m=0}^{\infty} \psi_m \frac{1}{z^{m+1}}$$

$$= \sum_{\substack{n=0\\m=0}}^{\infty} \phi_n \psi_m \frac{1}{z^{n+m+2}} \qquad \Big| \; \ell = n+m+1$$

$$= \sum_{\ell=1}^{\infty} c_n \frac{1}{z^{\ell+1}} \quad , \quad c_\ell = \sum_{p+q=\ell+1} \phi_p \psi_q$$

$$\mathcal{B}\left[\tilde{\Phi}(z)\tilde{\Psi}(z)\right](\zeta) = \sum_{\ell=1}^{\infty} \frac{c_n}{n!} \zeta^n$$

What if we work with $\hat{\Phi}, \hat{\Psi}$?

$$\boxed{\hat{\Phi}(\zeta) * \hat{\Psi}(\zeta)} = \int_0^{\zeta} d\zeta' \, \hat{\Phi}(\zeta') \, \hat{\Psi}(\zeta-\zeta') \qquad \text{convolution}$$

$$= \int_0^{\zeta} d\zeta' \, \hat{\Psi}(\zeta') \, \hat{\Phi}(\zeta-\zeta') \qquad \text{plug in } \hat{\Phi}, \hat{\Psi}$$

$$= \int_0^{\zeta} d\zeta' \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \zeta'^n \sum_{m=0}^{\infty} \frac{\psi_m}{m!} (\zeta-\zeta')^m$$

$$= \sum_{\substack{n=0\\m=0}}^{\infty} \frac{\phi_n}{n!} \frac{\psi_m}{m!} \underbrace{\int_0^{\zeta} d\zeta' \, \zeta'^n (\zeta-\zeta')^m}_{\text{Euler Beta function}}$$

$$= \zeta^{m+n+1} \frac{n!\,m!}{(n+m+1)!}$$

$$\mathcal{B}(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt$$

$$\mathcal{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!\,(y-1)!}{(x+y-1)!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\phi_n \psi_m}{(n+m+1)!} \zeta^{m+n+1} \qquad \Big| \; \ell = n+m+1$$

$$= \sum_{\ell=1}^{\infty} \frac{c_\ell}{\ell!} \zeta^\ell = \mathcal{B}\left[\tilde{\Phi}\tilde{\Psi}\right](\zeta) \quad \checkmark$$

And for a constant?

Works with tricks $\to$ Later $\hat{1} = \delta$ symbol with exactly this prescription

$$\mathcal{B}[1](\zeta) = \frac{1}{\zeta} \underbrace{\frac{1}{(-1)!}}_{\text{not good}} \quad , \quad 1\tilde{\Phi} = \tilde{\Phi}, \quad \mathcal{B}[1\tilde{\Phi}](\zeta) = \hat{\Phi}(\zeta) = \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \zeta^n$$

(that's actually zero)

$$\hat{1} * \hat{\Phi} = \int_0^{\zeta} d\zeta' \frac{1}{(-1)!} \frac{1}{\zeta'} \sum_{n=0}^{\infty} \frac{\phi_n}{n!} (\zeta-\zeta')^n = \frac{1}{\underbrace{(-1)!}_{=\delta(\zeta)}} \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \overbrace{\int_0^{\zeta} d\zeta' (\zeta')^{-1} (\zeta-\zeta')^n}^{\zeta^{n+1-1} \frac{n!\,(-1)!}{(n-\cancel{1}+\cancel{1})!}}$$

And in the easier def.?

$\boxed{B2}$ $\phi(z)\psi(z) = \sum_{n=0}^{\infty} \phi_n \frac{1}{z^n} \sum_{m=0}^{\infty} \psi_m \frac{1}{z^m}$

no $\sim$, keep that for $z^{-1}\mathbb{C}[[z^{-1}]]$

$\phi = \text{const.} + \tilde{\phi}$

$$= \sum_{n,m} \phi_n \psi_m \frac{1}{z^{n+m}} \quad \Big| \ l = n+m$$

$$= \sum_{l=0}^{\infty} c_l \frac{1}{z^l}, \quad c_l = \sum_{p+q=l} \phi_p \psi_q \longrightarrow \mathcal{B}[\phi\psi](\zeta) = \sum_{l=0}^{\infty} \frac{c_l}{l!} \zeta^l$$

$$\hat{\phi}(\zeta) * \hat{\psi}(\zeta) = \int_0^{\zeta} d\zeta' \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \zeta^n \sum_{m=0}^{\infty} \frac{\psi_m}{m!} \zeta^m = \sum_{l=1}^{\infty} \frac{c_l}{l!} \zeta^l$$

what happened to $l=0$?

$$1 \to \mathcal{B}[1](\zeta) = \frac{1}{0!} 1 \zeta^0 = 1$$

$$\underset{\overset{\shortparallel}{1}}{\hat{1}} * \hat{\phi}(\zeta) = \int_0^{\zeta} d\zeta' \, 1 \, \hat{\phi}(\zeta - \zeta') \neq \mathcal{B}[1\phi] = \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \zeta^n$$

$$\boxed{\int_0^{\zeta} d\zeta' \hat{\phi}(\zeta-\zeta') \Big| \ \xi = \zeta - \zeta' \\ = \int_{\zeta}^{0} -d\xi \hat{\phi}(\xi) = \int_0^{\zeta} d\xi \hat{\phi}(\xi)}$$

$$= \int_0^{\zeta} d\zeta' \sum_{n=0}^{\infty} \frac{\phi_n}{n!} (\zeta - \zeta')^n$$

$$= \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \underbrace{\int_0^{\zeta} d\zeta' \zeta'^0 (\zeta - \zeta')^n}_{\frac{0!\,n!}{(n+0+1)!} \zeta^{n+0+1}} = \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \frac{n!\,0!\,\zeta^{n+1}}{(n+0+1)!} = \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \frac{1}{n+1} \zeta^{n+1}$$

$$\hookrightarrow \underbrace{1 \cdot 1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1}_{k} \phi(z) = \phi(z)$$

$$\mathcal{B}[1 \cdots 1 \phi](\zeta) = \hat{\phi}(\zeta)$$

$$\underbrace{\mathcal{B}[1] * \ldots * \mathcal{B}[1] * \mathcal{B}[\phi]}_{k} = \underbrace{\mathcal{B}[1] * \ldots * \mathcal{B}[1]}_{k-1} \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \frac{1}{n+1} \zeta^{n+1}$$

$$= \underbrace{\mathcal{B}[1] * \cdots * \mathcal{B}[1]}_{k-2} \sum_{n=0}^{\infty} \frac{\phi_n}{n!} \frac{1}{(n+1)} \frac{1}{(n+2)} \zeta^{n+2}$$

$$= \sum_{n=0}^{\infty} \frac{\phi_n}{(n+k)!} \zeta^{n+k}$$

I wouldn't call this well defined

Solution

$\boxed{B1}$ Just define the right property

$$\delta * \hat{\phi} = \hat{\phi} \ \forall \hat{\phi} \in \mathbb{C}[[\zeta]]$$

$$\mathbb{C} \oplus z^{-1}\mathbb{C}[[z^{-1}]] \xrightarrow{\mathcal{B}} \delta\mathbb{C} \oplus \mathbb{C}[[\xi]], \quad \mathcal{B}[a](\zeta) = a\delta(\zeta) \text{ for } a \in \mathbb{C}$$

Resurgence Lecture

An interesting DEQ

$$y'(x) = \cos[\pi x\, y(x)]$$

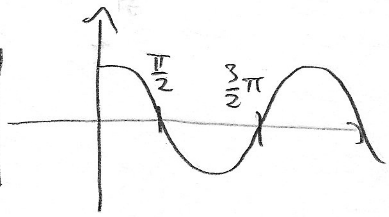
Boundary condition: $y(x\to\infty) = 0$

Ansatz: $y(x) = \frac{a_0}{x} + \frac{a_1}{x^2} + \frac{a_2}{x^3}$

$$\Rightarrow y'(x\to\infty) = 0$$

$$y'(x) = \cos\Big[\pi a_0 + \underbrace{\pi\frac{a_1}{x} + \pi\frac{a_2}{x^2} + \dots}_{\to 0}\Big]$$

$$0 = \cos[\pi a_0] \Rightarrow \boxed{a_0 = n + \frac{1}{2}}$$



$$y'(x) = \cos\left[\pi\left(n+\tfrac{1}{2}\right) + \pi\frac{a_1}{x} + \pi\frac{a_2}{x^2} + \dots\right] \quad , n \in \mathbb{Z}$$

$$= (-1)^{n+1} \sin\left[\pi\frac{a_1}{x} + \pi\frac{a_2}{x^2} + \dots\right] \qquad \Big|\ \sin(x) = x - \frac{x^3}{3} + \dots$$

$$\boxed{-\frac{a_0}{x^2}} - \frac{2a_1}{x^3} - \frac{3a_2}{x^4} = (-1)^{n+1}\left[\pi\frac{a_1}{x} + \boxed{\pi\frac{a_2}{x^2}} - \pi^3\frac{a_1^3}{x^3} + \dots\right]$$

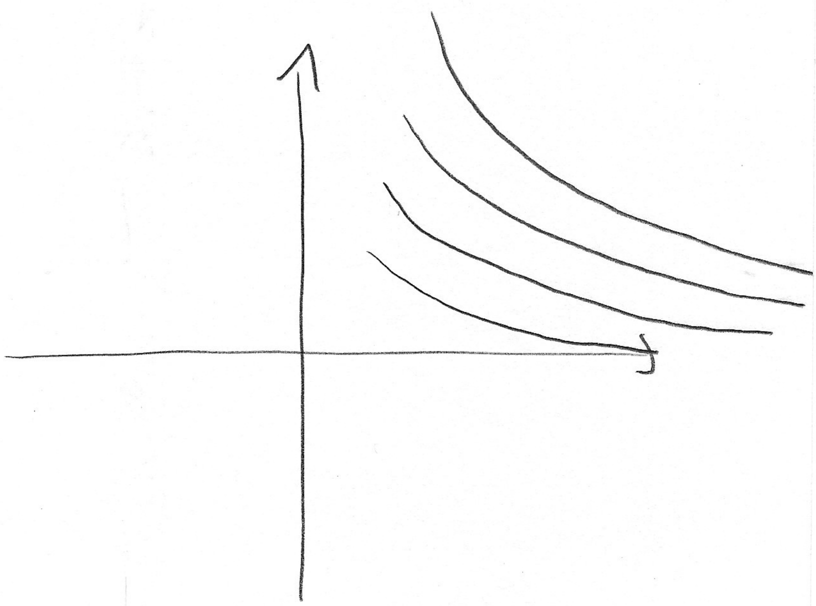
$$\Rightarrow a_1 = 0$$

$$(-1)^{n+1}\pi\frac{a_2}{x^2} = -\frac{a_0}{x^2} = -\frac{1}{x^2}\left(n+\tfrac{1}{2}\right)$$

$$\boxed{a_2 = \frac{1}{\pi}(-1)^{n}\left(n+\tfrac{1}{2}\right)}$$

for large x: $y(x) = \frac{n+\frac{1}{2}}{x} + \frac{(-1)^n}{\pi x^2}\left(n+\frac{1}{2}\right) + \dots$

$$\sim \frac{n+\frac{1}{2}}{x}$$



$$\frac{1}{2x}, \frac{3}{2x}, \frac{5}{2x}, \dots$$

Only "quantized" solutions, but we have one free b.c. because of 1st. order deq

Show numerical plot

↳ Solutions asymptote to "quantized" solutions,
but only those with even n !

Turns out odd n are "unstable" and need precise initial conditions

Take $y_1(x)$, $y_2(x)$ of same $a_0 = n + \frac{1}{2}$, $n \in \mathbb{Z}$, but different $y(0)$

↳ $y_1 - y_2$ is asymptotically smaller than any $\frac{1}{x^\alpha}$

non-perturbative

$u(x) = y_1(x) - y_2(x)$, since $y_1(0) \neq y_2(0)$: $u \not\equiv 0$

$$u'(x) = y_1'(x) - y_2'(x)$$

$$= \cos\left[\pi x y_1(x)\right] - \cos\left[\pi x y_2(x)\right]$$

$$\cos x - \cos y = -2 \sin\left[\frac{x+y}{2}\right] \sin\left[\frac{x-y}{2}\right]$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$= -2 \sin\left[\pi x \frac{y_1(x) + y_2(x)}{2}\right] \sin\left[\frac{u(x)}{2} \pi x\right]$$

for large x: $y_1 \sim y_2 \sim (n + \frac{1}{2}) \frac{1}{x} \Rightarrow y_1 + y_2 \sim \frac{2(n+\frac{1}{2})}{x}$

$$\sim -2 \underbrace{\sin\left[\pi x \frac{2(n+\frac{1}{2})}{x}\right]}_{(-1)^{n}} \sin\left[\frac{u(x)}{2} \pi x\right]$$

$$u'(x) \sim (-1)^{n+1} 2 \sin\left(\frac{\pi}{2} x u(x)\right) \approx (-1)^{n+1} \cancel{2} \frac{\pi}{\cancel{2}} x\, u(x)$$

$$u'(x) \sim (-1)^{n+1} \pi x\, u(x)$$

$$u(x) = A e^{(-1)^{n+1} \frac{\pi}{2} x^2}$$

→ n odd: Grows : Can't trust $u(x)$ here, since $\to \infty$, but we see that y_1, y_2 go away from each other

→ n even: The difference goes away as $x \to \infty$