Resurgence Lecture 2

Mostly Mariño lectures

- Most series in physics are asymptotic instead of converging } → what exactly is computable from pert. theory and what not?
  don't consider renormalons here, they are also non-pert.
- Instantons: Extra saddle points in the path integral
  ↳ non-perturbative contributions

  QM: Tunneling

tunneling between those two vacua ⇒ non-pert. contr. to the ground state energy

- Typical trans-series

$$\underbrace{\sum_{n=0}^{\infty} a_n g^n}_{\text{asympt series}} + \underbrace{e^{-A/g} \sum_n a_n^{(1)} g}_{\text{one-instanton}} \underbrace{+ \mathcal{O}(e^{-2A/g})}_{\text{higher instanton contribution}}$$

{ two small parameters $g, e^{-A/g}$ should be regarded as independent

3 important steps

Mariño

1) Calculate the formal series $\boxed{\text{formal calc. of trans-series}}$
   - $\sum_n a_n g^n$ perturb. (asympt.) series
   - instanton corrections

ODEs with irregular singular points: formal solutions are usually asympt. series

Transseries (incl. instantons) can be calc. recursively

Method of steepest descent also important tool

function → asymptotic expansion

2) $\boxed{\text{Classical Asymptotics}}$

Think of the perturb series of the asymptotic expansion of a well defined function

3) $\boxed{\text{Beyond Classical Asymptotics}}$

use full info of the transseries to reconstruct the function fully/exactly

Trans-series + Borel Resummation = Theory of Resurgence (Jean Écalle)

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Def: (Asymptotic)

Series $S(w) = \sum_{n=0}^{\infty} a_n w^n$

is asympt. to $f(w)$ (in the sense of Poincaré) if

$$\forall N > 0 \quad \lim_{w \to 0} w^{-N} \left[f(w) - \sum_{n=0}^{N} a_n w^n \right] = 0$$

Note: Remainder does not have to go to zero as $N \to \infty$, w fixed.

Example: (Stirling)

$$\sqrt{\frac{z}{2\pi}} \left(\frac{z}{e}\right)^{-z} \Gamma(z) = 1 + \frac{1}{12z} + \frac{1}{288 z^2} + \dots$$

$$w = \frac{1}{z} \quad \Big| \quad = 1 + \frac{1}{12} w + \frac{1}{288} w^2 + \dots \quad , \quad \Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^{z} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots\right]$$

$$= \sqrt{2\pi w} \left(\frac{1}{ew}\right)^{1/w} \underbrace{\left[1 + \frac{1}{12} w + \frac{1}{288} w^2 + \dots\right]}_{\text{diverges I think}}$$

$n=1$:

$$\Gamma(z) \to \Gamma\left(\frac{1}{w}\right)$$

$$\lim_{w \to 0} \frac{1}{w^1} \left\{ \Gamma\left(\frac{1}{w}\right) - \sqrt{2\pi w} \left(\frac{1}{ew}\right)^{1/w} \left[1 + \frac{1}{12} w\right] \right\} \stackrel{?}{=} 0$$

Example: (e^z)

$$\lim_{w \to 0} \left[e^w - \sum_{n=0}^{N} \frac{w^n}{n!} \right] w^{-N} = \lim_{w \to 0} \sum_{n=N+1}^{\infty} \frac{w^n}{n!} w^{-N} \overset{n \to n+N}{=} \lim_{w \to 0} \sum_{n=1}^{\infty} \frac{w^n}{n+N} = 0$$

convergent series is asymptotic to its func. in this sense, but asympt. is more general

Example: Exponential integral (Wiki: Asympt. expansion)

$$\int_0^{\infty} \frac{e^{-w/t}}{1-w} dw \overset{u = w/t,\ w = ut}{=} \int_0^{\infty} du\, t\, e^{-u} / (1-ut) = \int_0^{\infty} du \frac{e^{-u} e^{-\frac{1}{t}} e^{\frac{1}{t}}}{u(u - \frac{1}{t})} = e^{-\frac{1}{t}} \int_0^{\infty} \frac{e^{-(u-\frac{1}{t})}}{-(u - \frac{1}{t})} du$$

$$\boxed{Ei(x) = -\int_{-x}^{\infty} \frac{e^{-u}}{u} du}$$

$$= e^{-\frac{1}{t}} \int_{-1/t}^{\infty} \frac{e^{-u}}{-u} = e^{-\frac{1}{t}} Ei\left(\frac{1}{t}\right)$$

$$e^{-\frac{1}{t}} Ei\left(\frac{1}{t}\right) = \int_0^{\infty} \frac{e^{-w/t}}{1-w} dw \quad \Big| \quad \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n, \quad |w| < 1$$

actually should not be used here!

$$\stackrel{?}{=} \sum_{n=0}^{\infty} \int_0^{\infty} w^n e^{-w/t} dw \overset{u = w/t,\ dt = t du}{=} \sum_{n=0}^{\infty} t^{n+1} \underbrace{\int_0^{\infty} e^{-u} u^n}_{= \Gamma(n+1) = n!} = \sum_{n=0}^{\infty} n!\, t^{n+1}$$

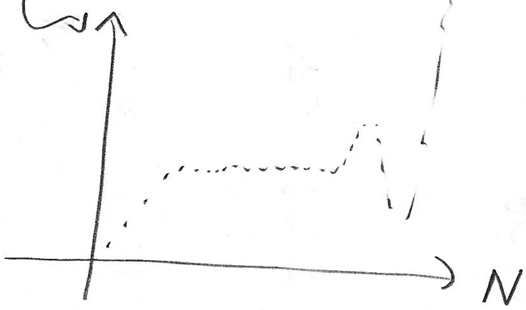
$$e^{-\frac{1}{t}}\, Ei\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} n!\, t^{n+1} \qquad \Big|_{x=\frac{1}{t}}$$

$$x\, e^{+x}\, Ei(-x) = \sum_{n=0}^{\infty} \frac{n!\,(-1)^{n+1}}{x^n} \quad , \quad Ei(-x) = -E_1(x)$$

$$\boxed{x\, e^{+x}\, E_1(x) \approx \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{x^n}}$$

truncation gives good approx for $\boxed{\text{large } x}$ / small t

→ $\boxed{\text{show plot}}$



Turns out: $\sum_{n=1}^{\infty} a_n x^n$ small x if $a_n \sim A^{-n}\, n!$ (diverges)

(often the case)

In QFT this $n!$ comes from the factorial growth in the number of diagrams

(Mariño)

$x \in \mathbb{C}$

c.f. Casey's talk

$$\phi(z) = \sum_{n=0}^{\infty} c_n \frac{1}{z^{n+1}} \quad \text{Gevrey order } \frac{1}{m} \text{ if} \quad (\text{large } z)$$

$$|c_n| \leq \alpha\, C^n (n!)^m$$

$$\sum_{n=1}^{\infty} \overset{a_n}{\overset{\shortparallel}{c_{n-1}}}\, x^n \qquad \Big|_{x=\frac{1}{z}}$$

$$|a_n| \leq \alpha\, C^{n-1} \left[(n-1)!\right]^m$$

Best approx at $N_* = \left|\frac{A}{\cancel{z}}\right|$

with minimal error $\varepsilon(x) \sim e^{-|A/x|}$

"best resolution"

"perturbative ambiguity"

nonperturbative in x!
Would not be able to put this part in a perturbative expansion

That's a first sign of resurgence: In the large order behaviour of the divergent series, $a_n \sim A^{-n}\, n!$ we found info about the non-pert. part, $e^{-A/x}$

Example:

$$I(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-\frac{z^2}{2} - g\frac{z^4}{4}} \quad , \; g \text{ small} \qquad \lambda\phi^4$$

A) like in perturbation theory (my Journal talk)

$\hookrightarrow$ won't do that here

B) $e^{-g\frac{z^4}{4}} = \sum_{n=0}^{\infty} \left(\frac{-z^4}{4}g\right)^n \frac{1}{n!}$

stirling

$$\Rightarrow I(g) = \sum_{n=0}^{\infty} a_n g^n \quad , \; a_n = (-4)^{-n} \frac{(4n-1)!!}{n!} \approx (-4)^n n!$$

$A^{-n}\, n!$

zero radius of convergence

$\Rightarrow |A| = \frac{1}{4}$

optimal at $\left|\frac{1}{4}\frac{1}{g}\right|$

$g = 0.02 \Rightarrow$ best at $\frac{1}{4} 50 \simeq 12.5$

$g = 0.05 \Rightarrow \quad \frac{1}{4} 20 \sim 5$

$\boxed{\text{Show plots}}$ (Mariño

Will look at this integral later

Transseries

Example: (Painleve II)

$$u''(k) - 2u^3(k) + 2k\,u(k) = 0$$

– 2D Yang-Mills,

– All-genus free energy of 2D Sugra

Transseries solution:

$$u(k) = \sum_{\ell=0}^{\infty} C^\ell u^{(\ell)}(k)$$

$$= \sqrt{k} \sum_{\ell=0}^{\infty} C^\ell k^{-\frac{3}{4}\ell} e^{-\ell A k^{3/2}} \, \varepsilon^{(\ell)}(k) \qquad , \; k \to \infty$$

$A = \frac{4}{3}$ instanton action

$$\varepsilon^{(\ell)}(k) = \sum_{n=0}^{\infty} u_{\ell,n} \, k^{-3n/2}$$

$\ell = 0$: perturbative solution : $u^{(0)}(k) = \sqrt{k} - \frac{1}{16k^{5/2}} - \ldots$

$\ell > 0$: instantons

one instanton solution: $\ell = 1$, $\varepsilon^{(1)} = 1 - \frac{17}{96} k^{-3/2} + \frac{1513}{18432} k^{-3} - \ldots$