Borel Summation of Asympt Series

Idea: – Describe procedure
– Apply to Hatsuda's Method

Def: (Set of all formal power series in $\frac{1}{z}$)

$$\mathbb{C}[[z^{-1}]] = \left\{ \sum_{n=0}^{\infty} c_n \frac{1}{z^n} \ , \ c_n \in \mathbb{C} \right\}$$

Def: (Algebra of Formal Power Series in $\frac{1}{z}$ for $z \sim \infty$)

$$z^{-1}\mathbb{C}[[z^{-1}]] = \left\{ \sum_{n=0}^{\infty} c_n \frac{1}{z^{n+1}} \ , \ c_n \in \mathbb{C} \right\}$$

Algebra over a field:
A vector space equipped with a bilinear product.
- Set
- Mult., Add., Scalar Mult. by elements of the underlying field
- Vector space axioms
- Mult. is bilinear

No constant term z^0:
– there are two models for the algebra of formal power series
- Multiplicative model $\tilde{\Phi}$
  $\downarrow$ Borel
- Convolutive model: Has no unity
  $\hat{\phi}$ $\quad$ ↳ will just introduce it as a "symbol" δ

Def: (Gevrey order $\frac{1}{m}$)

$$\tilde{\Phi}(z) = \sum_{n=0}^{\infty} c_n \frac{1}{z^{n+1}} \in z^{-1}\mathbb{C}[[z^{-1}]]$$

is of Gevrey order $\frac{1}{m}$ if the large order asymptotics of $|c_n|$ are bounded by

$$|c_n| \leq \alpha\, C^n (n!)^m \quad , \quad \alpha, C \in \mathbb{R}$$

Def: (Borel Transform $\mathcal{B}$)

$$\mathcal{B} : z^{-1}\mathbb{C}[[z^{-1}]] \xrightarrow[\simeq]{\text{linear}} \mathbb{C}[[\zeta]]$$

$$\mathcal{B} : \tilde{\Phi}(z) = \sum_{n=0}^{\infty} c_n z^{-n-1} \longmapsto \hat{\phi}(\zeta) = \sum_{n=0}^{\infty} c_n \frac{\zeta^n}{n!}$$

$$\left\{ \begin{array}{l} c_n \mapsto \frac{c_n}{n!} \\ z^{-n-1} \mapsto \zeta^n \end{array} \right\}$$

(invert, subtract one power)

write $\mathcal{B}[\tilde{\Phi}](\zeta) = \hat{\phi}(\zeta)$

note that a constant term in $\tilde{\Phi}$ would give a $\frac{1}{\zeta}$ term.

I don't know why we don't define the Borel transform as

$c_n z^{-n} \to \frac{c_n}{n!}\zeta^n$

Proposition (Borel Transform of Gevrey-1)

$$\hat{\Phi}(\zeta) = \mathcal{B}[\tilde{\Phi}](\zeta)$$

$$\tilde{\Phi}(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$$

- exactly what we had last time
- typical for QFT

has finite radius of convergence iff $\tilde{\Phi}$ is of Gevrey-1 type: $|c_n| = \mathcal{O}(C^n n!)$

$\rightarrow$ from now on assume Gevrey-1

Note:

$\tilde{\Phi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ Multiplicative Model

$\hat{\Phi}(\zeta) = \mathcal{B}[\tilde{\Phi}](\zeta)$ Convolutive Model

because $\mathcal{B}[\tilde{\Phi}_1(z)\tilde{\Phi}_2(z)] = \int_0^{\zeta} d\zeta' \, \hat{\Phi}_1(\zeta')\hat{\Phi}_2(\zeta - \zeta')$ (symmetric $\tilde{\Phi}_1 \leftrightarrow \tilde{\Phi}_2$)

(also works out if you look at the coefficients in each Model)

Def (Directional Laplace Transform)

$$\mathcal{L}^{\theta}[\hat{\Phi}](z) = \int_0^{e^{i\theta}\infty} d\zeta \, e^{-z\zeta} \hat{\Phi}(\zeta)$$

Convolutive Model, Borel transformed, has finite R iff Gevrey-1

Analytic functions $\hat{\Phi}$ on the line $e^{i\theta}\mathbb{R}$ which don't grow faster than $e^{r|\zeta|}$ to $\mathcal{L}^{\theta}[\hat{\Phi}]$ (analytic function in the half plane $\mathrm{Re}(ze^{i\theta}) > r$.

Note (Directional Laplace Transform)

We use the directional Laplace Transform to transform the Laplace transformed function $\mathcal{B}[\tilde{\Phi}](\zeta)$ "back".
Under some circumstances we will recover the original function, under others there will be ambiguities because of singularities in the Borel plane (ζ-plane).

...ency:

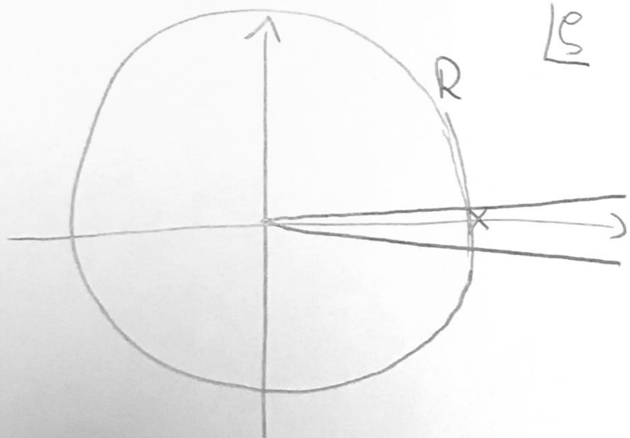
take a convergent function $\tilde{\phi}(z)=\sum_{n=0}^{\infty} c_n z^{-n-1}$

$\hookrightarrow$ $\mathcal{B}[\tilde{\phi}](\zeta)=\sum_{n=0}^{\infty}\frac{c_n}{n!}\zeta^n$ also converges and has $R=\infty$

$$\mathcal{L}^0\left[\mathcal{B}[\tilde{\phi}](\zeta)\right](z)=\sum_{n=0}^{\infty}\frac{c_n}{n!}\int_0^{\infty}d\zeta\, e^{-z\zeta}\zeta^n \quad \Big| t=z\zeta$$

$$=\sum_{n=0}^{\infty}\frac{c_n}{n!}z^{-n-1}\underbrace{\int_0^{\infty}dt\, t^n e^{-t}}_{=\Gamma(n+1)=n!}=\sum_{n=0}^{\infty}\frac{c_n}{\cancel{n!}}n!\,z^{-n-1}=\tilde{\phi}(z)$$

Note: (Analytic Continuation)



Usually the Borel Transform ~~has~~ of a Gevrey-1 series has a pole somewhere.

Thus ~~the~~ $\hat{\phi}$ won't converge outside R

$\hookrightarrow$ Analytic continuation to get $\mathcal{L}^0$

Example: (Euler Equation)

$$\phi'(z)-\phi(z)=-\frac{1}{z}$$

$$\tilde{\phi}(z)=\sum_{n=0}^{\infty}(-1)^n n!\, z^{-n-1}$$ Gevrey type 1, diverges $\forall z<\infty$

test: $\tilde{\phi}'=\sum_{n=0}^{\infty}(-1)^n n!(-n-1)\,z^{-n-2}$

$$\sum_{n=0}^{\infty}(-1)^n n!(-n-1)\,z^{-n-2}-\sum_{n=0}^{\infty}(-1)^n n!\,z^{-n-1}=$$

$$=\sum_{n=1}^{\infty}(-1)^{n+1}(n+1)!\,z^{-(n+1)-1}-\sum_{n=0}^{\infty}(-1)^n n!\,z^{-n-1}$$

$$=\cancel{\sum_{n=0}^{\infty}(-1)^n n!\,z^{-n-1}}-(-1)^0\,0!\,z^{-0-1}-\cancel{\sum_{n=0}^{\infty}(-1)^n n!\,z^{-n-1}}=-\frac{1}{z}\ \checkmark$$

$$\mathcal{B}[\tilde{\Phi}(z)](\zeta) = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{n!} \zeta^n = \sum_{n=0}^{\infty} (-\zeta)^n = \hat{\Phi}(\zeta)$$

radius of convergence $R=1$

$\hookrightarrow$ analytic continuation $\hat{\Phi}(\zeta) = \frac{1}{1+\zeta}$, singul. at $\zeta = -1$.

$$\mathcal{L}^0[\hat{\Phi}](z) = \int_0^\infty d\zeta\, e^{-z\zeta} \frac{1}{1+\zeta} = e^z \Gamma(0;z)$$

(we don't hit the singul.)

$$\Gamma(a,z) = \int_z^\infty dt\, t^{a-1} e^{-t} \quad \text{upper incomplete Gamma function}$$

$$\mathrm{Ei}(z) = -\int_{-z}^{\infty} dt\, t^{-1} e^{-t} \Rightarrow \Gamma(0,z) = -\mathrm{Ei}(-z)$$

e^z for $z \to \infty$ non-analytic

Also: $c e^z$ is a homogene solution $\phi' - \phi = 0$

$\hookrightarrow$ can be added

<u>But</u> 1) Is non-analytic

2) $e^z \to \infty$ for $z \to \infty$, but $e^z \Gamma(0;z) \to 0$

$\hookrightarrow$ we have the unique solution with $\tilde{\Phi}(z\to\infty) = 0$

$$\tilde{\Phi}(z) = \sum_{n=0}^{\infty} c_n z^{-n-1} \xrightarrow{\mathcal{B}[\tilde{\Phi}]} \hat{\Phi}(\zeta) = \mathcal{B}[\tilde{\Phi}](\zeta) = \sum_{n=0}^{\infty} c_n \frac{\zeta^n}{n!}$$

Germ of analytic f at the origin

$\mathcal{L}$ $\swarrow$

$$\mathcal{L}^0[\hat{\Phi}](z) = \int_0^\infty d\zeta\, e^{-z\zeta} \hat{\Phi}(\zeta)$$

analytic function for $\mathrm{Re}(z) > 0$

$\uparrow$

why?

$\nwarrow$ Asympt. expansion $z \to \infty$