Lectures on Resurgence and Trans-Series

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1. (**The Airy Equation: Borel Summation**)
We have that

$$Z_{\mathrm{Ai}}(x) = \frac{1}{2\sqrt{\pi}x^{1/4}} e^{\frac{2}{3}x^{3/2}} \sum_{n=0}^{\infty} (-1)^n a_n x^{-\frac{3}{2}n} \,, \tag{1}$$

with

$$a_n = \frac{1}{2\pi}\left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6}))}{n!} \,, \tag{2}$$

(a) (**Asymptotic behavior of a_n**)
Calculate the asymptotic behavior of a_n as $n \to \infty$ using Stirling's formula. Determine the instanton action A from it. Where will the pole of the Borel transform be? You can read it off from the large order behavior of a_n.

(b) (**Borel Transform**)
Let's focus on the sum that is multiplying the exponential. With the transformation $x = z^{\frac{2}{3}}$ we write it as

$$\phi_{\mathrm{Ai}}(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n} \,. \tag{3}$$

Show that the Borel transform

$$\hat{\phi}(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n \,, \tag{4}$$

can be analytically continued to

$$\hat{\phi}_{\mathrm{Ai}} = -\frac{5}{48}\, {}_2F_1\left(\frac{7}{6}, \frac{11}{6}; 2; -\frac{3}{4}\zeta\right) \,. \tag{5}$$

(c) (**Laplace Transform**)
The Hypergeometric Function ${}_2F_1(a, b; c; z)$ has a branch point at $z = 1$. Check that the pole is really where you thought it would be based on a. With Mathematica you can check the *Borel resummation* (Borel+Laplace)

$$S[\phi_{\mathrm{Ai}}](z) = a_0 + \int_0^{\infty} \mathrm{d}\zeta\, \hat{\phi}_{\mathrm{Ai}}(\zeta) e^{-z\zeta} \tag{6}$$

is really the full Airy function for $\Re(z) \geq 0$. What does this mean for $\arg(x)$?

(d) (Branch cut of the Hypergeometric Function)
At some point we will have to "jump" over the singularity at $\arg(z) = \pi$ and the branch cut. Using the discontinuity of the Hypergeometric function

$${}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{3}{4}x + i\epsilon\right) - {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \frac{3}{4}x - i\epsilon\right) = i\, {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; 1 - \frac{3}{4}x\right) \tag{7}$$

show that the difference in the lateral Borel resummations is

$$S_{\pi^+}[\pi_{\mathrm{Ai}}](z) - S_{\pi^-}[\pi_{\mathrm{Ai}}](z) = -ie^{\frac{4}{3}z} S_{\pi}[\phi_{\mathrm{Bi}}](z)\,, \tag{8}$$

where $\phi_{\mathrm{Bi}}(z)$ has the same coefficients as $\phi_{\mathrm{Ai}}(z)$, but non-alternating.

NOTE: You have showed that the expansion around the one saddle point has a branch cut in the Borel transform with discontinuity proportional to the full expansion around the other saddle point.