# Matsuda Method (Overview)

Resurgence Lecture 5

- Metric (Schwarzschild) $\rightarrow$ $\Box \phi = 0$ scalar DEQ
- Transform to Schrödinger form (tortoise coordinates, $\varepsilon \to i\hbar$)
- Potential $V(r_*)$ $\rightarrow$ shift so that minimum at $\delta_* = 0$

  $\hookrightarrow$ Bender-Wu package solves for $\mathcal{E}_n^{pert}(\hbar)$ (energy)

  as a series in $\hbar$.

  (asymptotic series)
- Borel transform the series
- Padé approximant of the Borel transform

  $\hookrightarrow$ analytic continuation
- Laplace transform
- $\hbar \to i$, $\omega_n^2 = -(V_0 + 2i\,\varepsilon_n^{pert}(i))$

$\rightarrow$ Show Notebooks

# Hatsuda's Method (Detail)

Schwarzschild black hole

$$ds^2 = -\underbrace{\left(1-\frac{2M}{r}\right)}_{f(r)} dt^2 + \frac{1}{1-\frac{2M}{r}} dr^2 + r^2 d\Omega^2$$

$$\Box \Phi = 0 \;, \quad \Phi(x) = \Phi(\vec{x}, t) = \int d\omega\, e^{i\omega t} \phi(\vec{x}, \omega)$$

$\downarrow$

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r_*)\right]\phi(r_*) = 0$$

, $r_*$ tortoise coordinate

$r_*$: $\to r(r_*)$

$$V(r) = \left(1-\frac{1}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{1-s^2}{r^3}\right] \;, \quad s = 0, 1, 2$$

Regge-Wheeler potential

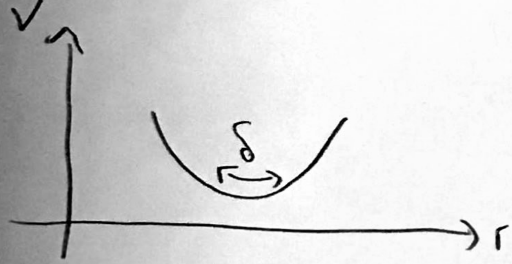
scalar ($s=0$), vector ($s=1$), grav. ($s=2$)

also from linearized Einstein equations

Tortoise coordinate: $r_*(r) = \int^r dr' \frac{1}{f(r')}$

here: $r_*(r) = r + \ln(r-1)$

but: We need $\tilde{V}(r)$ around the minimum of $V(r)$



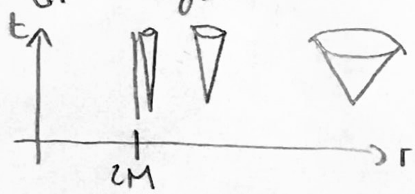

$V(r_0) = 0$

- $\delta_*(r) = r_*(r) - r_*(r_0) = r_*(\delta + r_0) - r_*(r_0)$
  expand in $\delta$
- Take inverse series $\to \delta(\delta_*)$ series in $\delta_*$
- $V(r(r_*)) = V(r_0 + \delta(\delta_*))$ : – function of $\delta_*$
  – minimum at $\delta_* = 0$
- Then: Bender-Wu package

## Tortoise Coordinate

- infalling light ray: $ds^2 = 0$
  $d\theta = d\phi = 0$

$$\frac{dt}{dr} = \pm \frac{1}{f(r)} \to \infty \text{ as } r \to 2M$$




light seems to never get to 2M in these coordinates
$\to$ because coords are singular there

$\to ds^2 = f(r)\left[-dt^2 + dr_*^2\right] + r^2 d\Omega^2$

everywhere, $2GM \to r_* = -\infty$

$$\delta_*(\delta) = r_*(r_0+\delta) - r_*(r_0)$$
$$= \int_{r_\infty}^{r_0+\delta} dr \, \frac{1}{f(r)} - \int_{r_\infty}^{r_0} dr \, \frac{1}{f(r)} = \int_0^{\delta} d\delta' \, \frac{1}{f(r_0+\delta')}$$

just Taylor expand this, no need to solve integral for $r_*(r)$

$$\delta_*(0) = 0$$
$$\delta_*'(0) = \frac{1}{f(r_0)}$$
$$\vdots$$

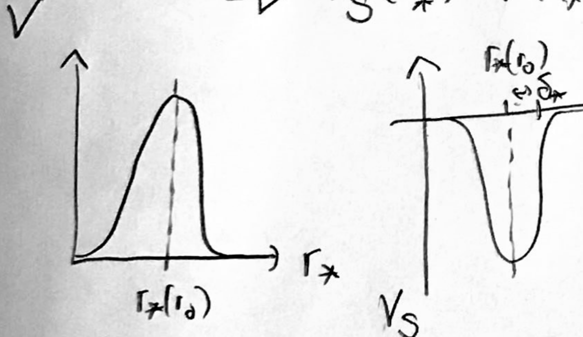
(Hatsuda expanded the explicit solution of $r_*(r) = \int^r dr' \frac{1}{f(r)}$, but mine is better I think)

Back to $\left[\varepsilon^2 \frac{d^2}{dr_*^2} + \omega^2 - V(r_*)\right] \phi(r_*) = 0 \qquad \varepsilon^2 = 1$

$$\Big\downarrow \; \hbar = i\varepsilon, \; \varepsilon = -i\hbar, \; E = -\omega^2$$

$$\left[-\hbar^2 \frac{d^2}{dr_*^2} - V(r_*)\right] \phi(r_*) = E\phi \qquad \Bigg\| \qquad \left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + V_S(r,t)\right] \psi = E\psi$$

$\Rightarrow V_S(r_*) = -V(r_*)$ inverted potential




Expand potential:
$$-V(r_*) = V_0 + \sum_{k=2}^{\infty} V_k \big[\underbrace{r_* - r_*(r_0)}_{\delta_*^k}\big]^k$$

The Bender-Wu package takes an eq. of the form

$$-\frac{1}{2}\psi''(x) + \frac{1}{g^2} V_0(gx)\psi(x) + V_2(gx)\psi(x) = \mathcal{E}\psi(x)$$

we get this by $r_* - r_*(r_0) = \sqrt{\hbar}\, x$, $g = \hbar$, $\mathcal{E} = \frac{E - V_0}{2\hbar}$, $V_{int} = \frac{1}{2}\sum_{k=2}^{\infty} \hbar^{k/2-1} V_k x^k$

$\rightarrow$ Plug into Bender-Wu

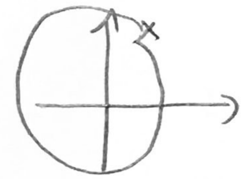
Bender-Wu $\rightarrow \mathcal{E}_n^{pert}(\hbar) = \sum_{k=0}^{N} \mathcal{E}_n^{(k)} \hbar^k$ asymptotic series

Check poles → no poles on $\mathbb{R}^+$

$\boxed{\text{Borel:}}$ $\mathcal{B}[\mathcal{E}_n^{pert}](\zeta) := \sum_{k=0}^{\infty} \frac{\mathcal{E}_n^{(k)}}{k!} \zeta^k$ other definition than e.g. Dorigoni

$\boxed{\text{Padé}}$ Need to analytically continue $\mathcal{E}_n^{pert}$.

It has no pole on the real axis, but still a finite radius of convergence.



Padé approximant works for this:

$$f^{[M/N]}(z) = \frac{\sum_{j=0}^{M} a_j z^j}{1 + \sum_{k=1}^{N} b_k z^k}$$

convention

→ Taylor series of order N+M of $f^{[M/N]}$ agrees with same order expansion of the original $f$.

→ Can be shown that Padé is best approx. at given order.

→ Captures pole structure of $f$ quite well. Taylor series of finite order has no poles

→ Usually use diagonal (M=N) Padé approximant

$$\mathcal{B}^{[M/N]}[\mathcal{E}_n^{pert}](\zeta)$$

$\boxed{\text{Laplace}}$ first check poles

$$\mathcal{E}_n^{pert,[M/N]}(\hbar) = \int_0^{\infty} d\zeta \, e^{-\zeta} \mathcal{B}^{[M/N]}[\mathcal{E}_n^{pert}](\hbar\zeta)$$

also different from Dorigoni b/c of different Borel transf. def.

remember: $\varepsilon^2 = 1$

$\varepsilon = \frac{\hbar}{i} \Rightarrow \hbar = i$

finally: $\omega_n^2 = -(V_0 + 2i\mathcal{E}_n^{pert}(i))$

Next: Physical Resurgent Extrapolation by Costin & Dunne?
Or formal definitions of transseries etc?