# 1) Introduction

## 1.1 Divergence of Perturbation Theory in QED (Dyson 1952)

QED perturbation theory:

not sure why $e^2$

$$F(e^2) = \sum_{n=0}^{\infty} a_{2n} e^{2n}$$ power series in $e$

finite coeffs after renormalization (those are the Feynm. diagrams)

typically one can only solve the free theory in QFT and then one does free theory + small perturbation ($e$ here)

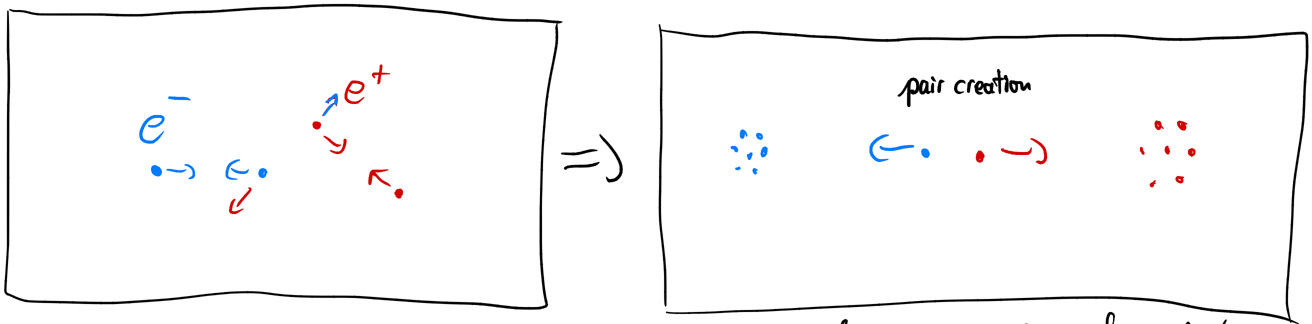
If $F(e^2)$ converges for $\varepsilon > e^2 > 0$, then it has to converge for $e^2 = 0$, but also for $-\varepsilon < e^2 < 0$!

expansion around $e^2 = 0$

Power series: $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$

- Converges $\forall |z| < R$ radius of convergence
- Diverges $\forall |z| > R$
- Complicated for $|z| = R$. Will have at least one singularity on the radius of convergence!

Theory for $e^2 < 0$: like charges attract, Coulomb potential with flipped sign.




can create more pairs of particles

"Every physical state is unstable against the spontaneous creation of many particles"

"In these circumstances it is impossible that the integration of the equations of motion of the theory over any finite or infinite time interval, starting from a given state of the fictitious world, should lead to well defined analytic functions"

$\Rightarrow$ $F(e^2)$ not analytic $\Rightarrow$ $\sum$ cannot converge

Dyson: 2 alternatives for the future

A) New method w/o power series (new mathematical methods)

$F(e^2)$, not analytic at $e^2=0$

$\hookrightarrow$ Power series of $F$ should reproduce pert. theory, but not sufficient to determine $F$ uniquely (hint at transseries)

B) All info we can get (in principle) is already in $a_n$.

Asympt. series only good for approximation

New physical theory needed.

Right now: More like A, but not quite.

Resurgence: Info about transseries is contained in the large order behaviour of the asympt. series

## 1.2 Intro to Non-Perturbative Effects

Mariño lecture notes 1), 1.1), part of 1.2)

Most perturbative series in QFT are asymptotic

↳ add non-perturbative effects somehow.

g: Coupling const.

$$\underbrace{\sum_n a_n g^n}_{\substack{\text{perturbative} \\ \text{(asympt. series)}}} + \underbrace{\boxed{e^{-A/g}} \sum_n a_n^{(1)} g^n}_{\text{one-instanton contribution}} + \mathcal{O}(e^{-2A/g}) \quad \boxed{\text{transseries}}$$

often people use instanton for anything that is non-perturb.

More generally:

$$\boxed{\sum_n a_n^{(0)} g^n + \sum_{i=1} e^{-A_i/g} \left( \sum_{n=0}^{\infty} a_n^{(i)} g^n \right)}$$

$$f = e^{-A/g}: \quad f' = \frac{A}{g^2} e^{-A/g}, \quad f'(0) = 0$$

$$f'' = \dots, \quad f''(0) = 0$$

$$\vdots$$

Power series $\equiv 0$

"non-perturbative"

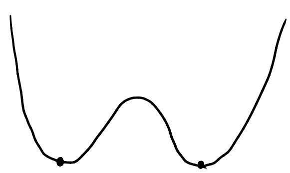
Can be expanded around different point, but we can only solve the theory for the free case, $g=0$.

<u>Instanton</u>: - Solution to the EoM in euclidean spacetime
- critical points of the action
- Nonzero action $A$
- Connected to tunneling, but I have not really understood that yet.

**Example**: (Wikipedia: Instanton)

QM: Schrödinger Equation: $-\frac{\hbar^2}{2m}\partial_x^2 \psi + V(x)\psi(x) = E\psi(x)$

$$V(x) = \frac{1}{4}(x^2-1)$$



- two classical minima
- ground state is only one state because of tunneling

## A) <u>WKB</u>

$$\psi'' = \frac{2m}{\hbar^2}\left[V(x)-E\right]\psi \quad , \quad \hbar \text{ small}$$

$$\psi = e^{-ikx} \quad , \quad E < V(x)$$

$\rightarrow$ tunneling amplitude $e^{-\frac{1}{\hbar}\int_a^b \sqrt{2m[V(x)-E]}\,dx}$

non-perturb.!

$a$: start point of tunneling
$b$: end point

## B) <u>Path integral, instantons</u> $\rightarrow$ Same result

$$K(a,b,t) = \int \mathcal{D}x(t)\, e^{i\frac{S[x(t)]}{\hbar}} \xrightarrow{it \to \tau} \int \mathcal{D}x(\tau)\, e^{-\frac{1}{\hbar}S_E[x(\tau)]}$$

$$S_E = \int_{\tau_a}^{\tau_b} d\tau \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]$$


V flips sign when doing Wick rotation

$$V(x) = \frac{1}{4}(x^2-1)^2 \quad \text{set } m=1$$

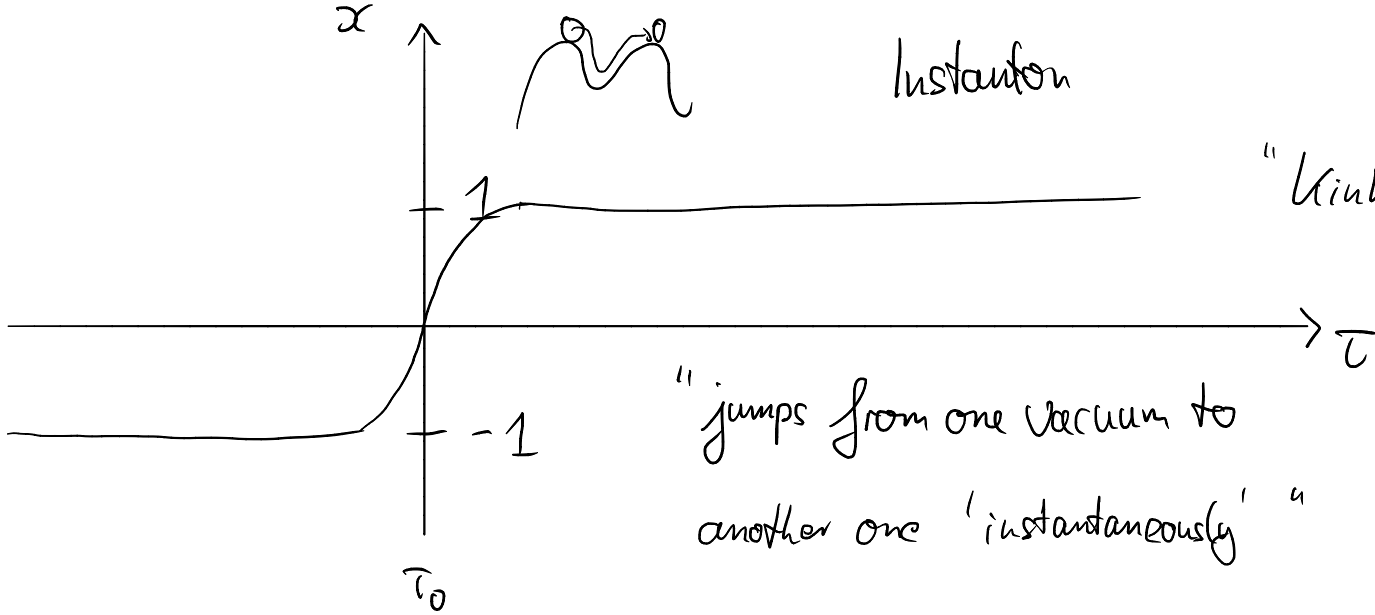
$$S_E = \int_{\tau_a}^{\tau_b} d\tau \, \frac{1}{2}\left[\dot{x} - \sqrt{2V(x)}\right]^2 + \sqrt{2}\int_{\tau_a}^{\tau_b} d\tau \, \frac{dx}{d\tau}\sqrt{V(x)}$$

$$= \underbrace{\int_{\tau_a}^{\tau_b} d\tau \, \frac{1}{2}\left[\dot{x} - \sqrt{2V(x)}\right]^2}_{\geq 0} + \underbrace{\sqrt{2}\int_{-1}^{1} dx \sqrt{V(x)}}_{\int_{-1}^{1} dx \frac{1}{\sqrt{2}}(1-x^2) \, = \, \frac{2\sqrt{2}}{3}}$$

minimal for $\dot{x} = \sqrt{2V(x)}$, $S_E = \frac{2\sqrt{2}}{3}$ instanton action

$$\frac{dx}{d\tau} = \frac{1}{\sqrt{2}}(x^2-1), \quad x(\tau_a) = -1, \quad x(\tau_b) = +1$$

for $\tau_a \to -\infty$, $\tau_b \to +\infty$: $\boxed{x(\tau) = \tanh\left(\frac{1}{\sqrt{2}}(\tau - \tau_0)\right)}$



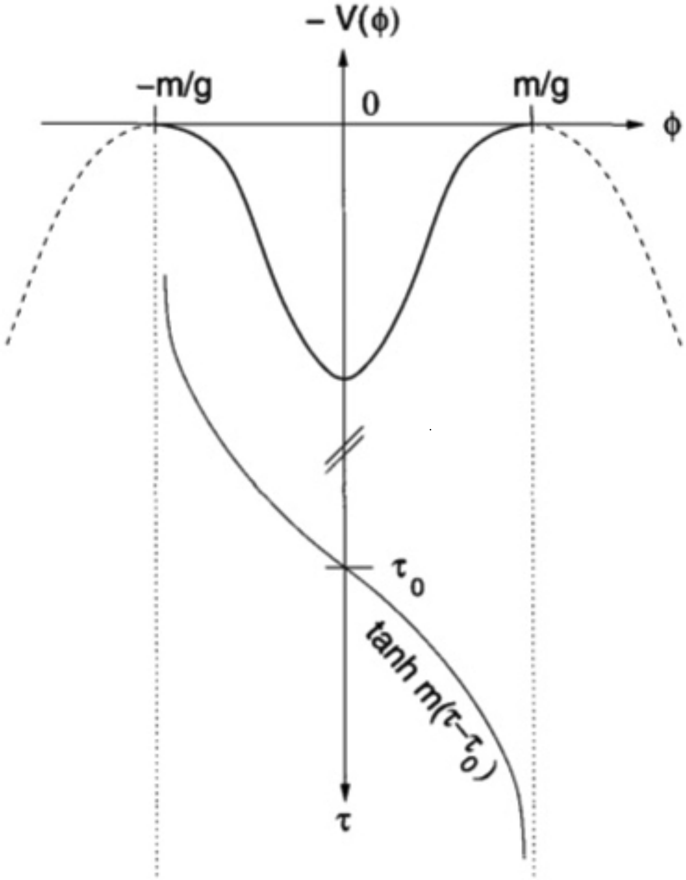

Instanton

"Kink"

"jumps from one vacuum to another one 'instantaneously' "

(naive) perturbation theory around one of the vacua $x = \pm 1$ would not show this tunneling (non-perturb.)

Haven't calculated it, but $K(-1, 1, \underset{\infty?}{t})$ should give same result as WKB.




from Müller-Kirsten "Intro to QM ..."

4.3.20

## 1.3 Overview over Transseries and Resurgence

Back to $\sum_n a_n g^n + e^{-A/g} \sum_n a_n^{(1)} g^n + \mathcal{O}(e^{-2A/g})$

3 Steps to understand transseries and resurgence

$\boxed{\text{two small parameter} : g, e^{-A/g}}$

1) Formal series:
   Get perturbative series + instanton corrections

   transseries "formal": already perturb. series diverges

2) Classical asymptotics

   See the perturb series as perturb expansion of a well defined function

   original function $\xrightarrow{\text{"classical asymptotics}}$ perturbative series, no $e^{-A/g}$ corrections