Lectures on Resurgence and Trans-Series

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1. ($\lambda\phi^4$ **- integral**)
   Let's look at the integral

$$Z(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-x^2-\lambda x^4} \, . \tag{1}$$

   (a) (**Asymptotic expansion**) Expand the $\exp(-\lambda x^4)$ part in a power series around $\lambda = 0$, add a spurious factor a to the x^2 part (in the end $a \to 1$) and change $x^4 \to \partial_a^2$. Finally, do the Gaussian integral to get

$$Z(\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \partial_a^{2n} \sqrt{\frac{\pi}{a}}\bigg|_{a\to 1} \, . \tag{2}$$

   (b) (**Performing the derivative**) Show that the above eq.(2) can be written in the form of an oscillating sum,

$$Z(\lambda) = \sum_{k=0}^{\infty} \sqrt{\pi} \frac{(4k)!}{2^{4k}(2k)!k!} (-\lambda)^k = \sum_{k=0}^{\infty} c_k (-\lambda)^k \, . \tag{3}$$

   (c) (**Large order Behavior**) Use Stirling's approximation

$$k! \approx \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \tag{4}$$

   to show that

$$\sqrt{\pi} \frac{(4k)!}{2^{4k}(2k)!k!} \approx \frac{1}{\sqrt{2\pi}} 4^k k! \, . \tag{5}$$

   Why does this mean that the expansion eq. (3) diverges?

   (d) (**Optional: Alternative derivation of the series**) Don't use the "derivative trick" from part a, but just expand the exponential $e^{-\lambda x^4}$, exchange sum and integral and solve the integral by substitution and use of the definition of the Γ-function/factorial or just type it into Mathematica.

   (e) (**Comparison with exact result and optimal truncation**) The integral can actually be performed analytically,

$$Z(\lambda) = \frac{e^{\frac{1}{8\lambda}}}{2\sqrt{\lambda}} K_{1/4}\left(\frac{1}{8\lambda}\right) \, , \tag{6}$$

   with the modified Bessel function of the second kind $K_n(x)$. Take eq. (3) and $\lambda = \frac{1}{50}$. Truncate the sum (3) at N and let's call this truncated sum $Z(\lambda; N)$. Plot $Z(\frac{1}{50}, N)$ against N for $N = 1 \ldots 40$ and also show the exact value $Z(\frac{1}{50})$. For $c_n \sim A^{-n} n!$ the optimal truncation is around[1]

$$N_* = \left|\frac{A}{\lambda}\right| \, . \tag{7}$$

   Do you find this here?

[1] take some closest integer, I don't think it matters if you round up or down, since it's only an estimate anyways.

(f) (**Perturbative ambiguity**) The closest we can get to the exact result by optimal truncation scales like

$$\epsilon(\lambda) \sim e^{-|A/\lambda|} . \tag{8}$$

Why can't we just add a term like this in the power series (3) to get rid of the errors?

(g) (**Scaling of λ**) What happens to the instanton action A if you scale λ in eq. (1) to $\alpha\lambda$, $\alpha > 0$? What happens to the perturbative ambiguity?

2. (**Analytic Continuation**) In this question we will look at the function

$$f(z) = \sum_{n=0}^{\infty} z^n , \quad z \in \mathbb{C} . \tag{9}$$

(a) (**Radius of Convergence**) Using the ratio test find the radius of convergence for $f(z)$.

(b) (**Expansion around $z = -1/2$**) The function $f(z)$ can actually be represented as

$$f_{\text{master}}(z) = \frac{1}{1-z} , \tag{10}$$

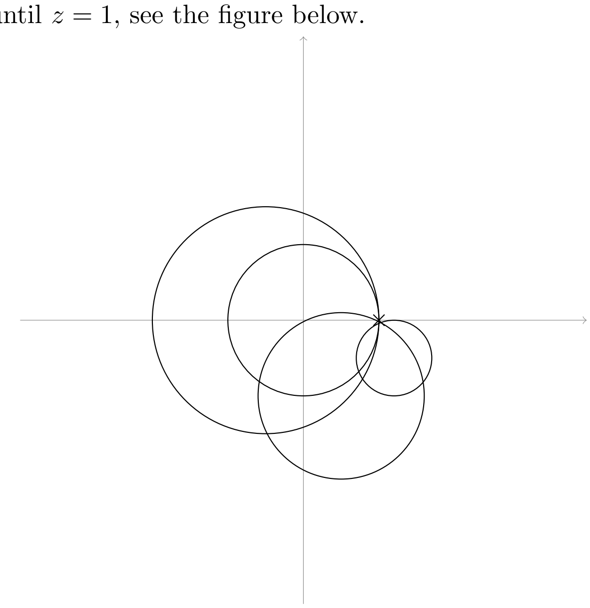
which has a singularity at $z = 1$. This representation is called *master representation*[2], because it works everywhere except at $z = 1$. It is the analytic continuation of $f(z)$ to all of $\mathbb{C}\backslash\{1\}$. Eq. (9) is the expansion of $f_{\text{master}}(z)$ around $z = 0$. Expand f_{master} around $z = -\frac{1}{2}$ and calculate the radius of convergence using the ratio test.

(c) (**Without Master Representation**) Without using f_{master}, calculate the Taylor series of $f(z)$ around $z = -\frac{1}{2}$ by taking the derivatives of the series. You should get the same as above.

Tipp: Plug the series you get for $f^{(n)}(-\frac{1}{2})$ into Mathematica.

(d) (**Calculating $2 + 4 + 8 + \ldots$**) In principle, how would you analytically continue $f(z)$ to $z = 2$ if you did not know the master representation? Calculate $f_{\text{master}}(2)$. Does this make sense in terms of the original function (9)?

Tipp: There is only the singularity at $z = 1$ and every expansion around any point $z_* \neq 1$ will have radius of convergence until $z = 1$, see the figure below.



[2] at least I think it is. If it is not, it is now.