Resurgence Exercise Sheet 1

Exercise 2

$$f(z) = \sum_{n=0}^{\infty} z^n \quad , \quad z \in \mathbb{C}$$

a) $$\lim_{n\to\infty} \left| \frac{z^{n+1}}{z^n} \right| = \lim_{n\to\infty} |z| \overset{!}{<} 1 \Rightarrow |z| < 1$$

$\boxed{R = 1}$ radius of convergence

b) Expansion around $z = -\frac{1}{2}$

$$f(z) = \frac{1}{1-z} = \frac{1}{1+\frac{1}{2}-(z+\frac{1}{2})}$$

$$= \frac{2}{3} \frac{1}{1-\frac{2}{3}(z+\frac{1}{2})} = \frac{2}{3} \sum_{n=0}^{\infty} \underbrace{\left(\frac{2}{3}\right)^n \left(z+\frac{1}{2}\right)^n}_{a_n}$$

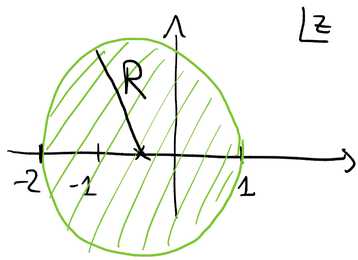
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^{n+1} \left(z+\frac{1}{2}\right)^{n+1}}{\left(\frac{2}{3}\right)^n \left(z+\frac{1}{2}\right)^n}$$

$$= \frac{2}{3}\left(z+\frac{1}{2}\right)$$

$$\left| \frac{2}{3}\left(z+\frac{1}{2}\right) \right| < 1$$

$$\left| z+\frac{1}{2} \right| < \frac{3}{2}$$

$\boxed{R = \frac{3}{2}}$



c) Without master representation

$$f(z) = \sum_{n=0}^{\infty} z^n$$

expansion around $z = -\frac{1}{2}$

$$f(-\tfrac{1}{2}) = \sum_{n=0}^{\infty} \left(-\tfrac{1}{2}\right)^n$$

$$f'(z) = \sum_{n=1}^{\infty} n\, z^{n-1} = \sum_{n=0}^{\infty} (n+1)\, z^n$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1)\, z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)\, z^n$$

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!}\, z^n$$

actually could have already used it here: $\sum_{n=0}^{\infty} \frac{(n+k)!}{n!} = \frac{1}{(1-z)^{1+k}}\, k!$.

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(-\tfrac{1}{2})\, \frac{1}{k!}\, (z+\tfrac{1}{2})^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{\cancel{k!}}\, (z+\tfrac{1}{2})^k \underbrace{\sum_{n=0}^{\infty} \frac{(n+k)!}{n!} (-\tfrac{1}{2})^n}_{\left(\frac{2}{3}\right)^{k+1} \cancel{k!}}$$

Mathematica

Okay, this is kind of cheating, since it is like using a master representation

$$= \boxed{\frac{2}{3} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k \left(z+\frac{1}{2}\right)^k}$$

d) $f(2) = \sum_{n=0}^{\infty} 2^n = \boxed{1 + 2 + 2^2 + 2^3 + \ldots}$

$f_{master}(2) = \frac{1}{1-2} = \boxed{-1}$

for $z=2$ the representation $\sum_{n=0}^{\infty} 2^n$ is not valid, since it diverges.

Because the analytic continuation is unique, one could just define $1+2+2^2+2^3+\ldots := -1$.

to find $f(2)$ one could do the following chain of Taylor expansions: