

Lectures on Resurgence and Trans-Series

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11. (Integral Definition of the Borel Transform¹)

For an asymptotic expansion

$$f(z) \sim \phi(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}} \quad z \rightarrow +\infty, \quad (1)$$

we had the *Borel transform* defined as

$$\mathcal{B}[\phi](\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n} \zeta^n. \quad (2)$$

In this exercise we want to prove that

$$\mathcal{B}[f](\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}_a} dz e^{z\zeta} f(z), \quad (3)$$

where \mathcal{C}_a is a path $a + iy$, $y \in \mathbb{R}$ with a constant a such that the path is to the right of all singularities of f .

Check that the integral definition (3) gives the same Borel transform as the usual definition (2).

TIPP: Use the expansion (1) and integrate using the residue theorem.

12. (Cauchy's Integral Formula and Branch Cuts)

Say we have a function $f(z)$ that has a branch cut along the negative real axis with discontinuity $\text{Disc } f(z)$ and no other singularities (except the branch point at $z = 0$).

(a) (Example: \log)

What is $\text{Disc } f(z)$ for $f(z) = \log(z)$?

(b) (Cauchy's Theorem)

Show that the generic function $f(z)$ can be represented as

$$f(z) = -\frac{1}{2\pi i} \int_{\delta}^R dw \frac{\text{Disc } f(-w)}{w + z} + I_{\delta}(z) + \mathcal{J}_R(z). \quad (4)$$

Use Cauchy's integral formula²

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - a} dz. \quad (5)$$

and the contour given in figure 1

¹Following Mas

²See e.g. [2].

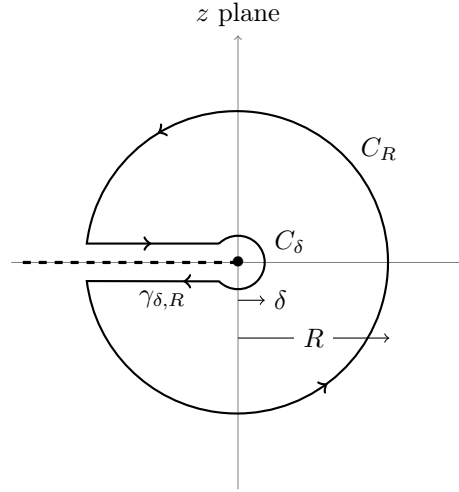


Figure 1: Contour for part 12 b, figure from [1].

References

- [1] R. M. Mas, “Resurgence, a problem of missing exponential corrections in asymptotic expansions,” [arXiv:1904.07217 [hep-th]].
- [2] Wikipedia contributors. Cauchy’s integral formula. Wikipedia, The Free Encyclopedia. May 7, 2020, 09:44 UTC. Available at: https://en.wikipedia.org/w/index.php?title=Cauchy%27s_integral_formula&oldid=955351355. Accessed July 30, 2020.