

# Lectures on Resurgence and Trans-Series

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Week 4, Version: April 28, 2020

## 4. (More $\lambda\phi^4$ integral)

This time we want to calculate the “quartic partition function”

$$Z(\hbar) = \frac{1}{2\pi} \int_{\Gamma} e^{-\frac{1}{\hbar} V(z)}, \quad (1)$$

with the potential

$$V(z) = \frac{1}{2}z^2 - \frac{\lambda}{4!}z^4, \quad (2)$$

and a contour  $\Gamma$  that starts at some “point” at (complex) infinity and ends at another “point” at (complex) infinity. Thus, for the integral to converge, one needs  $\Re(V(z)) > 0$  as  $z \rightarrow \infty$  along  $\Gamma$ .

### (a) (WARMUP: **Change of variables**)

Right now we have two small parameters in the exponent,  $\hbar$  and  $\lambda$ . Find a change of variables so that

$$Z(x) = \frac{\sqrt{\hbar}}{2\pi} \int_{\Gamma} dz e^{-\frac{1}{2}z^2 + \frac{x}{4!}z^4}, \quad x = \lambda\hbar. \quad (3)$$

This way we only have to care about  $x$ , which later will be complex.

### (b) (**Admissible directions for $\Gamma$** )

Later we will have  $x \in \mathbb{C}$ , but right now assume  $x \in \mathbb{R}^+$ . Find the directions  $\alpha$  along which  $V(z) > 0$  for  $z \rightarrow \infty$ , by setting

$$z = re^{i\alpha}. \quad (4)$$

You should find four intervals for  $\alpha$ .

### (c) (**Conversion to differential equation**)

Show that the integral (3) can be written as an ordinary linear differential equation

$$16x^2 Z''(x) + (32x - 24)Z'(x) + 3Z(x) = 0. \quad (5)$$

This can be done by plugging the definition of  $Z(x)$ , eq. (3), into the differential equation and showing that the integrand vanishes. Use the fact that  $\Re(V(x)) \rightarrow +\infty$  for the start and end point of  $\Gamma$ . Start by partially integrating the last summand and then continue partially integrating some terms.

Bonus points if you can tell me how to find the differential equation (5).

### (d) (**Series ansatz**) Make the ansatz

$$Z(x) = e^{-\frac{A}{x}} \Phi(x), \quad (6)$$

$$\Phi(x) = \sum_{n=0}^{\infty} Z_n x^n, \quad (7)$$

where  $\Phi(x)$  is a formal power series (will be asymptotic). Find the two “solutions” for  $A$ , let’s call them  $A_0$  and  $A_1$ .

(e) **(Asymptotic series  $\Phi(x)$ )**

Corresponding to  $A_0$  and  $A_1$  we get two different asymptotic series  $\Phi_0(x)$  and  $\Phi_1(x)$ ,

$$\Phi_0(x) \simeq \sum_{n=0}^{\infty} Z_n^{(0)} x^n, \quad Z_n^{(0)} = \left(\frac{2}{3}\right)^n \frac{(4n)!}{2^{6n}(2n)!n!} Z_0^{(0)}, \quad (8)$$

$$\Phi_1(x) \simeq \sum_{n=0}^{\infty} Z_n^{(1)} x^n, \quad Z_n^{(1)} = (-1)^n \left(\frac{2}{3}\right)^n \frac{(4n)!}{2^{6n}(2n)!n!} Z_0^{(0)}, \quad (9)$$

with free parameters  $Z_0^{(0)}$  and  $Z_0^{(1)}$ . You can check that  $\Phi_0$  and  $\Phi_1$  are correct if you want. Calculate the Gevrey order  $\frac{1}{m}$  of  $\Phi_0$  and  $\Phi_1$  as well as the instanton action  $A$ .

REMINDERS:

- A formal power series  $\phi(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$  is of *Gevrey order*  $\frac{1}{m}$  if the large order coefficients are bounded by

$$|c_n| \leq \alpha A^{-n} (n!)^m, \quad (10)$$

for some constants  $\alpha$  and  $A$ , where  $A$  is called “instanton action”.

- Stirling:  $k! \sim \sqrt{2\pi k} \left(\frac{k}{e}\right)^k$ .
- $n!$  scales just like  $(n-1)!$  for large  $n$ .

(f) **(Non-perturbative saddle points and instantons)**

Find the saddle points of the exponent of the integrand of the original integral for  $Z(x)$ , eq. (3). Calculate the *instanton action*  $A$  of those saddle points.

REMINDERS:

$$I = \int_{\mathcal{C}} dz e^{-S(z)},$$

$$S'(z_*) = 0, \quad \text{defines the saddle point } z_*,$$

$$S(z_*) = 0, \quad \text{instanton action.}$$

NOTE: There is a factor  $\frac{1}{x}$  that is sometimes in the instanton action  $A$  and sometimes not. I don't know if this is not clear in the literature or if I just don't have my definitions consistent.