Lectures on Resurgence and Non-Perturbative Effects

$\boxed{03/04/2020}$

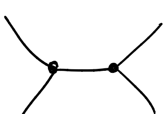
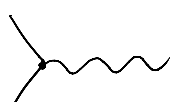
1) Introduction

- Dyson paper
- Mariño lectures : 1) + 2.1) + 2.2) partly
  - Asymptotic series
  - Instantons

  ⋮

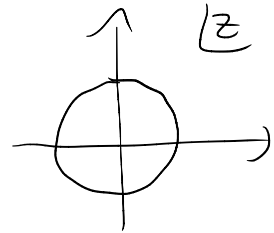
1.1) Divergence of Pert. Theory in QED

$$F(e^2) = \sum_{n=0}^{\infty} a_{2n} e^{2n} \qquad \text{around } e=0$$

Power Series : $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$

- Converges $\forall\, |z| < R$
- Diverges $\forall\, |z| > R$
- Complicated $|z| = R$

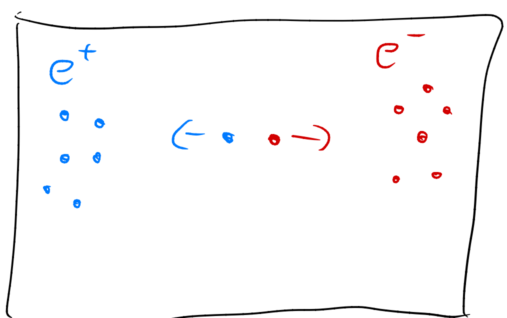


$(z-1)\ln z$, $z = 1$

$F(e^2)$, $\boxed{e^2 < 0}$, like charges attract

ie_1^+ ie_2^+

$$\vec{F}(\vec{r}) = \frac{ie_1\, ie_2}{r^3}\,\vec{r}$$



$\Rightarrow$ unstable

$\Rightarrow F(-e^2)$ not analytic , Σ cannot converge

Dyson: 2 alternatives for the future

A) New method w/o power series

B) All info we can get is in a_n

$\hookrightarrow$ need new physical theory

1.2) Intro to Non-Pert. Effects

$$\underbrace{\sum_{n=0}^{\infty} a_n g^n}_{\substack{\text{perturbative} \\ \text{series} \\ \text{(formal)}}} + \underbrace{\boxed{e^{-A/g}} \sum_n a_n^{(1)} g^n}_{\text{1-instanton contribution}} + \mathcal{O}\left(e^{-2A/g}\right) \qquad \text{transseries}$$

More generally

$$\sum_n a_n^{(0)} g^n + \sum_{i=1} e^{-A_i/g} \left[\sum_{n=}^{\infty} a_n^{(i)} g^n \right]$$

$f = e^{-A/g}$: $\quad f'(g) = \frac{A}{g^2} e^{-A/g}$, $\quad f'(0) = 0$

$\quad f'' = \dots$, $\quad f''(0) = 0$

$\quad \vdots$

power series $a_n = 0$

1.3 Instantons in QM

- solutions to the EoMs in euclidean spacetime

$\Updownarrow$

- critical points of the action
- Action $\neq 0$
- connected to tunneling in QM

Example $\quad$ QM : $-\frac{\hbar}{2m} \partial_x^2 \psi(x) + V(x)\, \psi(x) = E\, \psi(x)$

$$V(x) = \frac{1}{4}\left(x^2 - 1\right)^2$$

A) WKB

$$\psi'' = \frac{2m}{\hbar^2}\left[V(x) - E\right]\psi \quad , \hbar \text{ small}$$

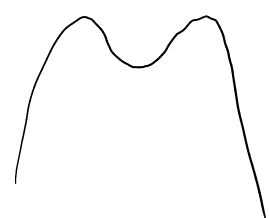
$$\psi = e^{-ikx} \quad , E < V$$

$\hookrightarrow$ tunneling amplitude $e^{-\frac{1}{\hbar}\int_a^b \sqrt{2m(V(x)-E)}\,dx}$ $\qquad e^{-A/g}$

B) Path Integral → Instantons

$$K(a,b;t) = \int \mathcal{D}x(t)\, e^{i\frac{S[x(t)]}{\hbar}} \xrightarrow{it \to \tau} \int \mathcal{D}x(\tau)\, e^{-\frac{1}{\hbar}S_E[x(\tau)]}$$

$$S_E = \int_{\tau_a}^{\tau_b} d\tau \left[\frac{1}{2}m\dot{x}^2 + V(x)\right]$$



V flips sign

$$V(x) = \frac{1}{4}(x^2-1)^2 \quad , m=1$$

$$S_E = \int_{\tau_a}^{\tau_b} d\tau \; \frac{1}{2}\left[\dot{x} - \sqrt{2V(x)}\right]^2 + \sqrt{2}\int_{\tau_a}^{\tau_b} \cancel{d\tau}\, \frac{dx}{\cancel{d\tau}} \sqrt{V(x)}$$

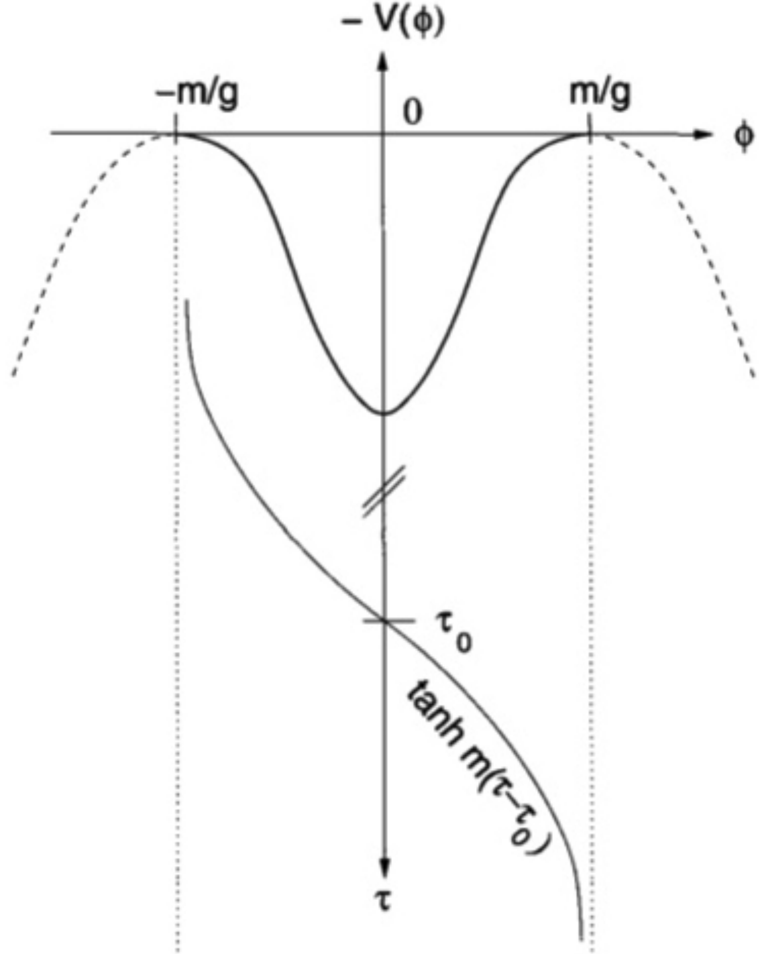
$$= \underbrace{\int_{\tau_a}^{\tau_b} d\tau \; \frac{1}{2}\Big[\underbrace{\dot{x} - \sqrt{2V(x)}}_{0}\Big]^2}_{\geq 0} + \sqrt{2}\int_{-1}^{1} dx \sqrt{V(x)}$$

minimum for $\dot{x} = \sqrt{2V(x)}$, $S_E = \frac{2\sqrt{2}}{3}$

$$\frac{dx}{d\tau} = \frac{1}{\sqrt{2}}(x^2-1), \quad x(\tau_a) = -1, \quad x(\tau_b) = 1$$

$$\tau_a \to -\infty, \; \tau_b \to \infty \quad \Rightarrow \quad x(\tau) = \tanh\left(\frac{1}{\sqrt{2}}(\tau - \tau_0)\right)$$

$$K(-1, 1, \infty) \stackrel{?}{=} e^{-\frac{1}{\hbar}\int_a^b \sqrt{2m(V(x)-E)}\, dx}, \quad x(\tau)$$



03/04/20

WKB $$e^{-\frac{1}{\hbar}\int_a^b \sqrt{2m\left(V(x)-E\right)}\,dx}$$

same if $E=0$?

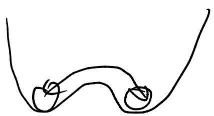
Path integral $$\int \mathcal{D}x(\tau)\, e^{-\frac{1}{\hbar}S_E[x(\tau)]} = e^{-\frac{1}{\hbar}\sqrt{2}\int_0^1 dx\,\sqrt{V(x)}}$$

Resurgence Lecture 2

$\boxed{04/10/20}$

- Instanton: Extra saddle point in the path integral

~~renormalons~~



QM tunneling

- Transseries

two small parameters: $e^{-A/g}$, g

should be treated independently

$$\sum_{n=0}^{\infty} a_n g^n + \boxed{e^{-A/g}} \sum_n a_n^{(1)} g^n + \mathcal{O}\left(e^{-2A/g}\right)$$

3 important steps

1) Calculating the formal series

- $\sum a_n g^n$ perturbative series
- instanton corrections

ODE + irregular singular points

↳ asympt. series → transseries can be calculated recursively

– Steepest descent

2) Clasical asymptotics

perturbative series $S(x)$ / $f(x)$

3) Beyond Classical Asymptotics

transseries → $f(x)$

Transser. + Borel = Resurgence (Jean Écalle)

Def: (Asymptotic)

Series $S(\omega) = \sum_{n=0}^{\infty} a_n \omega^n$ is asympt. to $f(\omega)$ if $\forall N > 0$: $\lim_{\omega \to 0} \omega^{-N} \left[f(\omega) - \sum_{n=0}^{N} a_n \omega^n \right] = 0$

$e^{-A/\omega} \; \Big| \; f(\omega) \to \sum_{n=0}^{\infty} a_n \omega^n \; \Big| \; f(\omega) + e^{-A/\omega}$

Every function can only have one asymptotic expansion, but an asymptotic series can be asymptotic to many functions

$\Downarrow$

Knowing the asymptotic expansion does not tell us much about the f, just how f approaches $f(0)$ as $z \searrow 0$

Note: Remainder does not have to go to zero as $N \to \infty$, ω fixed

Example: (e^x)

$$\lim_{x \to 0} \Big[\underbrace{e^x - \sum_{n=0}^{N} \frac{x^n}{n!}}_{x^{N+1}} \Big] x^{-N} \simeq \lim_{x \to 0} x = 0$$

Example: (Stirling)

$$\sqrt{\frac{z}{2\pi}} \left(\frac{z}{e}\right)^{-z} \Gamma(z) = 1 + \frac{1}{12z} + \frac{1}{180 z^2} + \dots \quad , z \to \infty \quad , \omega = \frac{1}{z}$$

Exponential Integral

$$u = \omega/t$$

$$\int_0^\infty \frac{e^{-\omega/t}}{1-\omega}\, d\omega \stackrel{!}{=} \int_0^\infty du\, t\, e^{-u} \frac{1}{1-ut} = e^{-1/t} \int_{-1/t}^\infty \frac{e^{-u}}{-u} = e^{-\frac{1}{t}}\, \mathcal{E}i\left(\frac{1}{t}\right)$$

$$\mathcal{E}i(x) = -\int_{-x}^\infty \frac{e^{-u}}{u}\, du$$

$$e^{-\frac{1}{t}}\, \mathcal{E}i\left(\frac{1}{t}\right) = \int_0^\infty \frac{e^{-\omega/t}}{1-\omega}\, d\omega \quad \Bigg| \quad \frac{1}{1-\omega} = \sum_{n=0}^\infty \omega^n$$

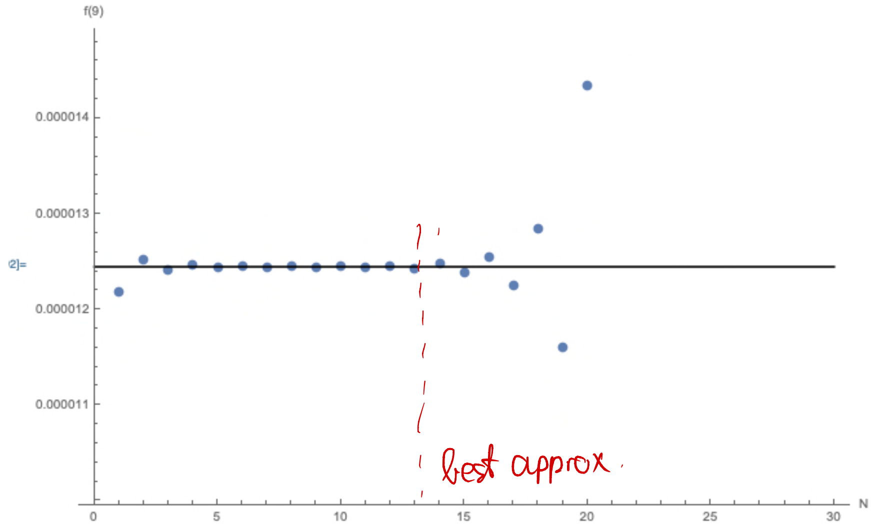
$$\stackrel{?}{=} \sum_{n=0}^\infty \int_0^\infty \omega^n e^{-\omega/t}\, d\omega = \dots = \sum_{n=0}^\infty n!\, t^{n+1}$$

diverges for $t > 0$

$R = 0$

$$e^{-\frac{1}{t}}\, \mathcal{E}i\left(\frac{1}{t}\right) = \sum_{n=0}^\infty n!\, t^{n+1} \quad \Big| \quad x = \frac{1}{-t}$$

$$x e^{x} \underbrace{\mathcal{E}i(-x)}_{\mathcal{E}_1(x)} \sim \sum_{n=0}^\infty \frac{(-1)^n n!}{x^n} \quad (x \to \infty) \quad \Bigg| \quad e^{-x}$$



More generally: $\sum_{n=1}^{\infty} a_n x^n$, $(x \to 0)$, if $a_n \sim A^{-n} n!$ (large n)

(sum diverges)

then the best approx is at $N_* = \left| \frac{A}{x} \right|$ optimal truncation

minimal error: $\mathcal{E}(x) \sim \underbrace{e^{-|A/x|}}_{\text{non-pert.}}$ "perturbative ambiguity"

first sign of resurgence:

In the large order behav. of the div. series we found info about the non-pert. part

Example ($\lambda\phi^4$)

$$I(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, e^{-a\frac{z^2}{2} - g\frac{z^4}{4}} \Big|_{a=1}, \quad g \text{ small}$$

∂_a^2

A) like pert. theory (c.f. my talk)

B) $$e^{-g\frac{z^4}{4}} = \sum_{n=0}^{\infty} \left(-\frac{z^4}{4} g\right)^n \frac{1}{n!} \qquad \int z^4 e^{-z^2} dz$$

$$\rightarrow I(g) = \sum_{n=0}^{\infty} a_n g^n, \quad a_n = (-4)^n \frac{(4n-1)!}{n!} \sim (-4)^n n!$$

$\hookrightarrow R = 0$ radius of convergence

$$A^{-n} n!$$

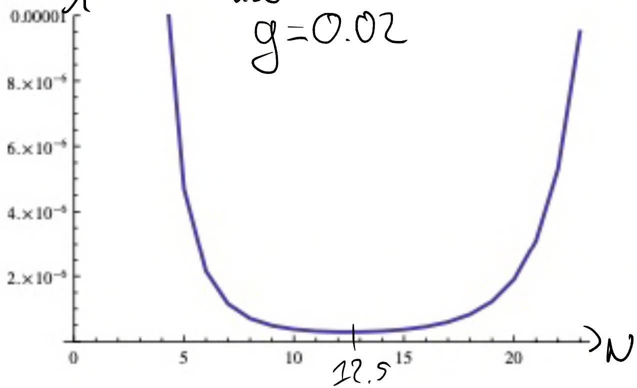
$$A = -\frac{1}{4}$$

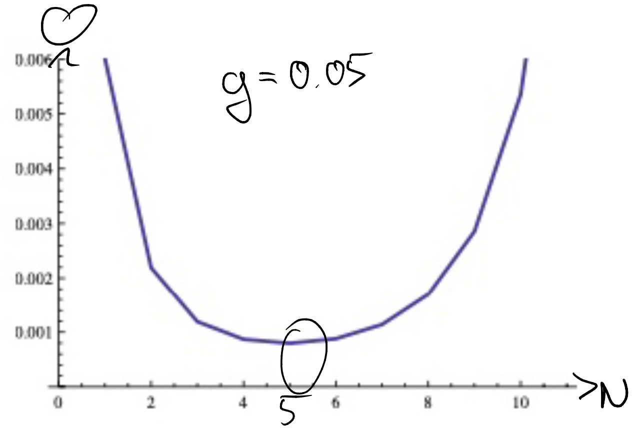
optimal truncation: $N_* = \left|\frac{1}{4}\frac{1}{g}\right|$

$0.02, \quad \frac{1}{4} 50 = 12.5$

$0.05, \quad \frac{1}{4} 20 = 5$

$$\varepsilon = I(g) - \sum_{n=0}^{N} a_n g^n$$





optimal truncation

Transseries

Example: (Painlevé II) (Mariño)

- 2D Yang-Mills
- All genus free energy 2D-SUGRA

$$u''(k) - 2u^3(k) + 2ku(k) = 0$$

Transseries solution: $u(k) = \sum_{\ell=0}^{\infty} C^{\ell} u^{(\ell)}(k)$

$$= \sqrt{k} \sum_{\ell=0}^{\infty} C^{\ell} k^{-\frac{3}{4}\ell} e^{-\ell A k^{3/2}} \varepsilon^{(\ell)}(k) \quad , \quad k \to \infty$$

$A = \frac{4}{3}$ instanton action

$\ell = 0$: ~~non~~ - pert. $\quad u^{(0)}(k) = \sqrt{k} - \frac{1}{16k^{5/2}} + \ldots \qquad a_n \sim A^{-n} n!$

$\ell > 0$: instantons

$\ell = 1$: one-instanton-solution : $\mathcal{E}^{(1)}(k) = 1 - \frac{17}{96} k^{-3/2} + \frac{1513}{18432} k^{-3} - \ldots$

10.04.20

17.04.20

Mathematical Methods

Analytic Functions

Def: (Analytic function on $\mathbb{C}$)

$f: \mathbb{C} \to \mathbb{C}$ is analytic on the open set $D \subset \mathbb{C}$

if $\forall z_0 \in D$ one can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad , \; a_n \in \mathbb{C} \quad \text{and the series converges}$$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

$\hookrightarrow$ inf. often differentiable

$f(z)$ analytic $\Leftrightarrow$ holomorphic

Def (Holomorphic)

$f: \mathbb{C} \to \mathbb{C}$ is holomorphic at $z_0 \in \mathbb{C}$ if it is compl. differentiable on some neighborhood of z_0.

f is hol. on $U \subset \mathbb{C}$ open if it is hol. $\forall z \in U$

if $U = \mathbb{C}$: "entire function"

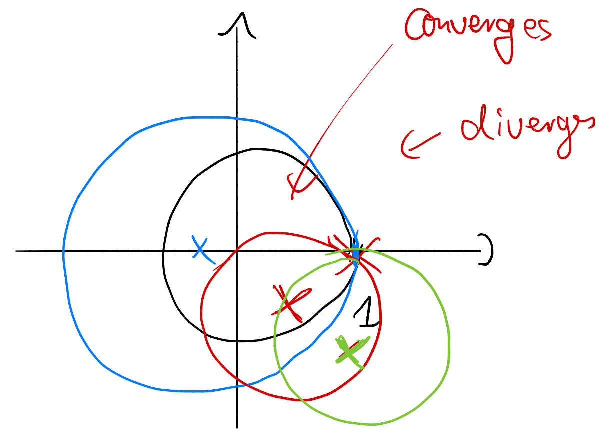
Theorem: (Looman - Menchoff)

$f: \overset{\text{open}}{U} \subset \mathbb{C} \to \mathbb{C}$ continuous is holomorphic iff it satisfies the Cauchy Riemann eqs.

$$\frac{\partial f}{\partial \bar{z}} = 0 = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right], \qquad z = x + iy \qquad f(z, \cancel{\bar{z}})$$

$$f = u + iv \quad \to \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \wedge \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Analytic Continuation



Example: (Geometric Series)

$$f(z) = \sum_{n=0}^{\infty} z^n$$ diverges for $z = R = 1$

$$f(z) = \frac{1}{1-z}$$ master representation

↳ only diverges at $z=1$

↳ Power series around $z=0$: radius of convergence $R=1$

Series around $z = -\frac{1}{2}$

$$f(z) = \frac{1}{1+\frac{1}{2}-(z+\frac{1}{2})} = \frac{2}{3}\frac{1}{1-\frac{2}{3}(z+\frac{1}{2})} = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n\left(z+\frac{1}{2}\right)^n$$

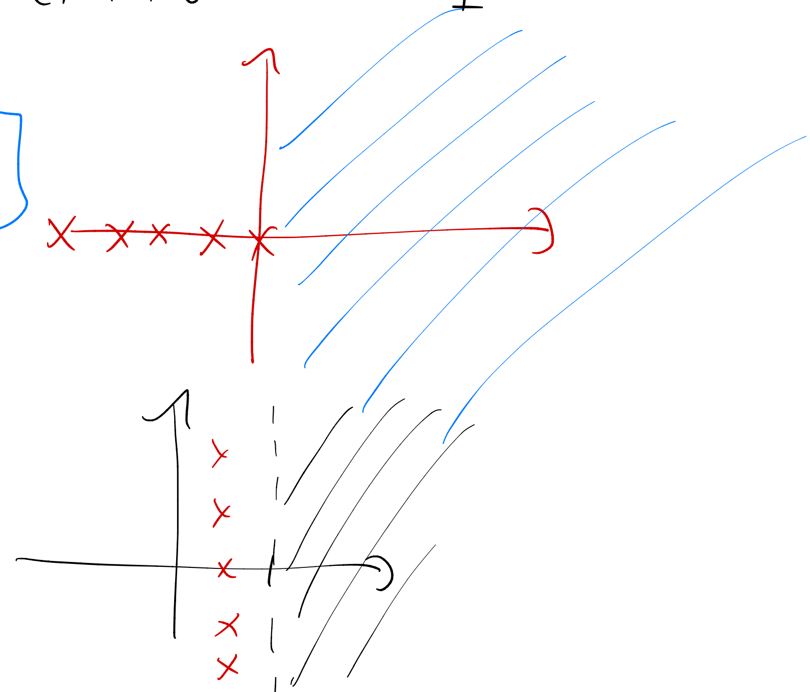
has $R = \frac{3}{2}$

This way all of $\mathbb{C}$ can be reached

$f(z) = \frac{1}{1-z}$ is defined on $\mathbb{C} \setminus \{1\}$

$f(2) = \frac{1}{1-2} = -1$, $1 + 2 + 4 + 8 + \dots = -1$

$$\Gamma = \frac{-1}{1 - e^{2\pi i z}} \boxed{\int^{\infty} dt \; t^{z-1} e^{-t}}$$



$\zeta(z) = \sum_{s} \frac{1}{s^z}$, $\mathrm{Re}\, z > 1$

$$\langle T^{00} T^{00} \rangle \sim \frac{1}{\omega(k) - \omega_n} \,, \quad \begin{array}{l} \omega \in \mathbb{C} \\ k \in \mathbb{C} \end{array}$$

$\downarrow$ 17.04.20