Lectures on Resurgence and Trans-Series

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11. (**Integral Definition of the Borel Transform**[1])

For an asymptotic expansion

$$f(z) \sim \phi(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}\, z \to +\infty\,, \tag{1}$$

we had the *Borel transform* defined as

$$\mathcal{B}[\phi](\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n\,. \tag{2}$$

In this exercise we want to prove that

$$\mathcal{B}[f](\zeta) = \frac{1}{2\pi i}\int_{\mathcal{C}_a} \mathrm{d}z\, e^{z\zeta} f(z)\,, \tag{3}$$

where $\mathcal{C}_a$ is a path $a + iy$, $y \in \mathbb{R}$ with a constant a such that the path is to the right of all singularities of f.

Check that the integral definition (3) gives the same Borel transform as the usual definition (2).

Tipp: Use the expansion (1) and integrate using the residue theorem.

12. (Cauchy's Integral Formula and Branch Cuts)

Say we have a function $f(z)$ that has a branch cut along the negative real axis with discontinuity $\operatorname{Disc} f(z)$ and no other singularities (except the branch point at $z = 0$).

(a) (Example: log)
What is $\operatorname{Disc} f(z)$ for $f(z) = \log(z)$?

(b) (Cauchy's Theorem)
Show that the generic function $f(z)$ can be represented as

$$f(z) = -\frac{1}{2\pi i}\int_{\delta}^{R} \mathrm{d}w\, \frac{\operatorname{Disc} f(-w)}{w+z} + I_\delta(z) + \mathcal{J}_R(z)\,. \tag{4}$$

Use Cauchy's integral formula[2]

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,\mathrm{d}\,z\,. \tag{5}$$

and the contour given in figure 1

[1] Following [1]
[2] See *e.g.* [2].

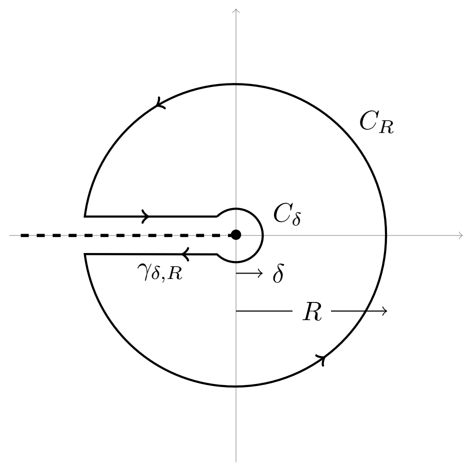

Figure 1: Contour for part 12 b, figure from [1].

References

[1] R. M. Mas, "Resurgence, a problem of missing exponential corrections in asymptotic expansions," [arXiv:1904.07217 [hep-th]].

[2] Wikipedia contributors. Cauchy's integral formula. Wikipedia, The Free Encyclopedia. May 7, 2020, 09:44 UTC. Available at: `https://en.wikipedia.org/w/index.php?title=Cauchy%27s_integral_formula&oldid=955351355`. Accessed July 30, 2020.