Lectures on Resurgence and Trans-Series

Marco Knipfer, University of Alabama

Week 7, Version: May 15, 2020

6. (Warmup: **Euler's Equation**)
The differential equation

$$\phi'(z) - \phi(z) = -\frac{1}{z}\,, \tag{1}$$

is not really called Euler's equation, but he studied it. Here we are working in the $z \to \infty$ limit.

(a) (**Perturbative Solution**)
Show that the alternating (asymptotic) series

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} (-1)^n n!\, z^{-n-1}\,, \tag{2}$$

solves eq. (1).

(b) (**Borel Transformation and Analytic Continuation**)
Perform the Borel transform

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} a_n z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]]\,, \tag{3}$$

$$\hat{\phi}(z) = \mathcal{B}[\tilde{\phi}][\zeta] = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n\,. \tag{4}$$

on the perturbative solution (2). Perform an analytic continuation of the Borel transform by sharply looking at it and realizing that it's a geometric series. Where are the poles of the analytic continuation of the Borel transform?

(c) (Optional: **Laplace transform**)
Do the Laplace transform and find

$$\mathcal{L}^0[\hat{\phi}](z) = e^z\Gamma(0; z)\,. \tag{5}$$

The most general solution is $e^z\Gamma(0; z) + ce^z$, which you can find by plugging the differential equation into Mathematica's `DSolve`. Why can't we find the ce^z term? Keep in mind that we are approximating around $z = \infty$.

7. (**Modification of Euler's Equation**)
Let's change a sign,

$$\phi'(z) + \phi(z) = +\frac{1}{z}\,. \tag{6}$$

Then the asymptotic series changes to

$$\tilde{\phi}(z) = \sum_{n=0}^{\infty} n!\, z^{-n-1}\,, \tag{7}$$

which is no longer alternating.

(a) (**Borel Transform and Analytic Continuation**)
Perform the Borel transform as in eq. (4) and use your magic powers to find the analytic continuation $\hat{\phi}(\zeta) = \frac{1}{1-\zeta}$. Where is the pole and will it matter for the Laplace transform?

(b) (**Lateral Laplace Transform**)
Perform a lateral Laplace transform

$$\mathcal{L}^\theta[\hat{\phi}](z) = \int_0^{\infty e^{i\theta}} \mathrm{d}\zeta \, e^{-z\zeta} \hat{\phi}(\zeta) \,, \tag{8}$$

by first making a change of variables, $\zeta = e^{i\theta}\xi$, and then asking Mathematica.

(c) (**Ambiguity is Purely Imaginary**)
By *e.g.* using `ReImPlot` in Mathematica, show that for $z \in \mathbb{R}$ the lateral Laplace transforms for $\frac{\pi}{2} > \theta > 0$ and $0 > \theta > -\frac{\pi}{2}$ have the same real part, but differ in the imaginary part. This is by the way a general feature of this Borel-Laplace stuff.

Derive this ambiguity by the residue theorem. The two directions enclose the singularity and the part at infinity vanishes, thus with the notation

$$S_\theta \tilde{\phi} \sim \mathcal{L}^\theta[\mathcal{B}[\tilde{\phi}]] \,, \tag{9}$$

one just has to calculate

$$(S_{0^+} - S_{0^-})\tilde{\phi}(z) = -2\pi i \operatorname*{Res}_{\zeta \to 1} \left(\frac{e^{-z\zeta}}{1-\zeta} \right) \tag{10}$$

You can see that this ambiguity is non-analytic and can't be "touched" by the perturbative expansion.

Can you find the instanton action of the series (7) in the exponent of the ambiguity? Reminder: $c_n \sim A^{-n} n!$, A instanton action.

(d) (**Median Summation**)
Simply perform a *median resummation* (for $\theta = 0$) defined as

$$S_\theta^{\text{med}} \sim \frac{1}{2} \left(S_{\theta^+} + S_{\theta^-} \right) \,, \tag{11}$$

in order to get rid of the imaginary part. Plot the Borel-Laplace (median resummed) solution, a truncation of the asymptotic series (7) and the analytic solution from `DSolve`.

8. (**Simple Singularity**)
Our Borel transform in question 7 had the form $\frac{1}{1-\zeta}$. Generally a function is said to have a *simple singularity* at ω if close to the singularity it has the form

$$\hat{\phi}(\zeta) = \underbrace{\frac{\alpha}{2\pi i(\zeta - \omega)}}_{\text{simple pole}} + \frac{1}{2\pi i} \underbrace{\hat{\Phi}(\zeta)}_{\text{minor}} \ln(\zeta - \omega) + \operatorname{reg}(\zeta - \omega) \,, \tag{12}$$

where $\alpha \in \mathbb{C}$ is called the *residue*, $\hat{\Phi}(\zeta)$ is a holomorphic function called the *minor* and reg is a regular term close to ω.

For $\arg(\omega) = 0$, *i.e.* $\omega \in \mathbb{R}^+$, calculate

$$(S_{0^+} - S_{0^-})\tilde{\phi}(\zeta) \tag{13}$$

by using definitions (9), (8) and integrating around the simple singularity. Remember that $\ln(\zeta - \omega)$ has a branch cut and just above the real line the argument of the log is $(\zeta - \omega)$, whereas just below the real line it is $(\zeta - \omega)e^{2\pi i}$. You should recover something like (10) plus a term containing the minor.