Painleve I

Following the paper by Costin and Dunne 1904.11593.

ToDo

- Create the Series for the Painleve I
- Borel Transform

Creating the Series

$$y''(x) - 6y^2(x) + x = 0$$

- want smooth real solution for large x
- Ecalle time $t = \frac{(24x)^{5/4}}{30}$
- $y(x) = -\sqrt{\frac{x}{6}}(1 + h(t))$
- now for $h(t)$ we have

$$\ddot{h}(t) + \frac{1}{t}\dot{h}(t) + h(t)\left(1 + \frac{1}{2}h(t)\right) - \frac{4}{25t^2}\left(1 + h(t)\right) = 0$$

- expand series to lowest needed order such that we have all terms to solve for the lowest order
- solve for lowest order
- expand again for next order
- not sure what it's good for, but I found that one can also write it analytically as

$$\sum_{n=1}^{\infty}\left[\left(2a_n(2n(n+1)-1) + a_{n+1}\right)\frac{1}{t^{2n+1}} + \frac{1}{2}\sum_{m=1}^{n}\frac{a_n a_{m-n+1}}{t^{2m+2}}\right] - \frac{4}{25t^2} + \frac{a_1}{t^2} = 0$$

From this one can immediately see $a_1 = 4/25$.

The algorithm is as follows:

- Start with an empty array `aArr` that will contain all known coefficients
- Loop over `order` starting from 1
  - Expand the DEQ to the needed order (so that everything to solve for `a[order]` is there). This might need some work with pencil and paper to figure out to which order one has to go.
  - Apply all coefficients of `aArr`.
  - Solve for `a[order]` and add it to `aArr`.

Possible improvements could be:

- maybe better to have the expansion step directly use the known coefficients and not applying them later.

Richardson's Extrapolation

The theory is described in Bender, Orszag on p. 375. One assumes that the error we make when evaluating a series $A_\infty = \sum_{k=0}^{\infty} a_k$ only to order n goes like

$$A_n = \sum_{k=0}^{n} a_k \sim A_\infty + Q_1 n^{-1} + Q_2 n^{-2} + \ldots \,, \quad n \to \infty \,,$$

where we want to extract A_∞ from only finitely many terms. Using different lengths n one can extract some of the Q_i and thus improve the order. For example if we use A_n and A_{n+1} we can extract Q_1 and thus improve convergence by subtracting Q_1/n from our result.

The general formula if we have $n + N$ terms and want to do a Richardson's extrapolation of order N is

$$A_\infty = \sum_{k=0}^{N} \frac{A_{n+k}(n+k)^N(-1)^{k+n}}{k!(N-k)!}$$

In my code I called the length of the series n and thus I replaced $n \to n - N$ everywhere.