(1) $\int_0^1 w\left(AE\frac{d^2u}{dx^2}+b(x)\right)dx=0 \qquad u(0)=1 \qquad u_{,x}(1)=e \qquad w(0)=0$

$$AE\int_0^1 w\frac{d^2u}{dx^2}\,dx+\int_0^1 w\,b(x)\,dx=0$$

$$AE\left[w\cdot u_{,x}\Big|_0^1-\int_0^1\frac{dw}{dx}\frac{du}{dx}dx\right]+\int_0^1 w\cdot b(x)\,dx=0$$

$$\int_0^1\frac{dw}{dx}AE\frac{du}{dx}\,dx=\int_0^1 w\cdot b(x)\,dx+w(1)\,AE\cdot e$$

Summary of the weak form:

Given $b(x)=-e^x$, $u(0)=1$, $\frac{du}{dx}\Big|_{x=1}=e$, find $u\in S$, such that for all $w\in V$, $\int_0^1\frac{dw}{dx}AE\frac{du}{dx}\,dx=\int_0^1 w\cdot b(x)\,dx+w(1)\,AE\cdot e$

Solution Space $S=\{u \mid u\in H^1,\ u(0)=1\}$

Test Space $V=\{u \mid u\in H^1,\ u(0)=0\}$

Formulation of Galerkin Method:

$$\int_0^1\frac{dw}{dx}AE\frac{du}{dx}\,dx=\sum_{e=1}^{n}\int_{\Omega^e}\frac{dw^h}{dx}AE\frac{du^h}{dx}\,dx=\sum_{e=1}^{n}\int_{\Omega^e}\underset{\sim}{w}^{eT}\underset{\sim}{B}^T AE\,\underset{\sim}{B}\,\underset{\sim}{u}^e\,dx=\sum_{e=1}^{n}\underset{\sim}{w}^{eT}\int_{\Omega^e}\underset{\sim}{B}^T AE\,\underset{\sim}{B}\,dx\,\underset{\sim}{u}^e$$

$$\underset{\sim}{K}^e=\int_{\Omega^e}\underset{\sim}{B}^T AE\,\underset{\sim}{B}\,dx=\frac{AE}{(l^e)^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\int_{\Omega^e}dx=\frac{AE}{l^e}\begin{bmatrix}1&-1\\-1&1\end{bmatrix},\quad \text{with } AE=1$$

Thus, $\boxed{\underset{\sim}{K}^e=\frac{1}{l^e}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}}$

$$\int_0^1 w\cdot b(x)\,dx=\sum_{e=1}^{n}\int_{\Omega^e}\underset{\sim}{w}^{eT}\underset{\sim}{N}^T b(x)\,dx=\sum_{e=1}^{n}\underset{\sim}{w}^{eT}\int_{\Omega^e}\underset{\sim}{N}^T b(x)\,dx$$

$$f_Q^e=\int_{\Omega^e}\underset{\sim}{N}^T b(x)\,dx=\int_{\Omega^e}\begin{bmatrix}\frac12(1-\xi)\\ \frac12(1+\xi)\end{bmatrix}(-e^x)\frac{dx}{d\xi}\,d\xi=\frac12 l^e\int_{\Omega^e}\begin{bmatrix}\frac12(1-\xi)\\ \frac12(1+\xi)\end{bmatrix}(-e^x)\,d\xi$$

$$x^e(\xi)=\frac12(1-\xi)x_L^e+\frac12(1+\xi)x_R^e,\quad \text{where } x_R^e-x_L^e=l^e$$

$$\text{So, } f_Q^e=\frac{l^e}{2}\int_{\Omega^e}\begin{bmatrix}\frac12(\xi-1)\\ \frac12(-\xi-1)\end{bmatrix}e^{\frac12x_L^e+\frac12x_R^e+\frac12x_R^e\xi-\frac12x_L^e\xi}\,d\xi$$

$$f_Q^e=\frac{l^e}{2}\cdot e^{\frac12(x_L^e+x_R^e)}\int_{\Omega^e}\begin{bmatrix}\frac12(\xi-1)\\ \frac12(-\xi-1)\end{bmatrix}e^{\frac{l^e}{2}\xi}\,d\xi$$

After I take the integral, I get the f_Q^e as following:

$$f_Q^e = \frac{l^e}{2}\, e^{\frac{1}{2}(x_L^e + x_R^e)} \begin{bmatrix} \frac{-2}{(l_e)^2} e^{\frac{l^e}{2}} + \frac{2}{l^e} e^{-\frac{l^e}{2}} + \frac{2}{(l^e)^2} e^{-\frac{l^e}{2}} \\ -\frac{2}{l^e} e^{\frac{l^e}{2}} + \frac{2}{(l^e)^2} e^{\frac{l^e}{2}} - \frac{2}{(l^e)^2} e^{-\frac{l^e}{2}} \end{bmatrix}$$

$$f_Q^e = \exp\left(\frac{1}{2}(x_L^e + x_R^e)\right) \begin{bmatrix} -\frac{1}{l^e} \cdot \exp\left(\frac{l^e}{2}\right) + \exp\left(-\frac{l^e}{2}\right) + \frac{1}{l^e} \exp\left(-\frac{l^e}{2}\right) \\ -\exp\left(\frac{l^e}{2}\right) + \frac{1}{l^e} \exp\left(\frac{l^e}{2}\right) - \frac{1}{l^e} \exp\left(-\frac{l^e}{2}\right) \end{bmatrix}$$

$$f_n = \begin{bmatrix} 0 \\ e \end{bmatrix}$$

(2)

Approximation of U with 5 Linear Elements:

```
U1 =

    1.0000
    1.2214
    1.4918
    1.8221
    2.2255
    2.7183
```

Approximation of U,x with 5 Linear Elements:

```
dU1 =

    1.1070
    1.3521
    1.6515
    2.0171
    2.4637
```

Approximation of U with 10 Linear Elements:

```
U2 =

    1.0000
    1.1052
    1.2214
    1.3499
    1.4918
    1.6487
    1.8221
    2.0138
    2.2255
    2.4596
    2.7183
```

Approximation of U,x with 10 Linear Elements:

```
dU2 =

    1.0517
    1.1623
    1.2846
    1.4197
    1.5690
    1.7340
    1.9163
    2.1179
    2.3406
    2.5868
```

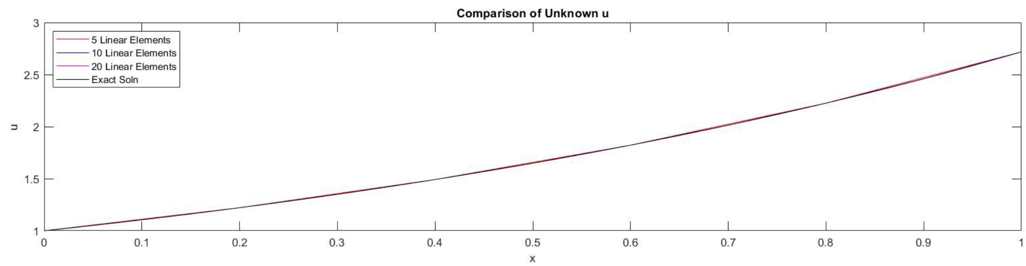
Approximation of U with 20 Linear Elements:

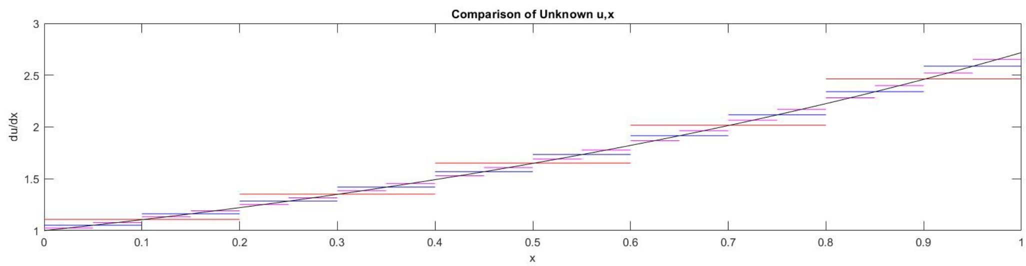
```
U3 =

    1.0000
    1.0513
    1.1052
    1.1618
    1.2214
    1.2840
    1.3499
    1.4191
    1.4918
    1.5683
    1.6487
    1.7333
    1.8221
    1.9155
    2.0138
    2.1170
    2.2255
    2.3396
    2.4596
    2.5857
    2.7183
```

Approximation of U,x with 20 Linear Elements:

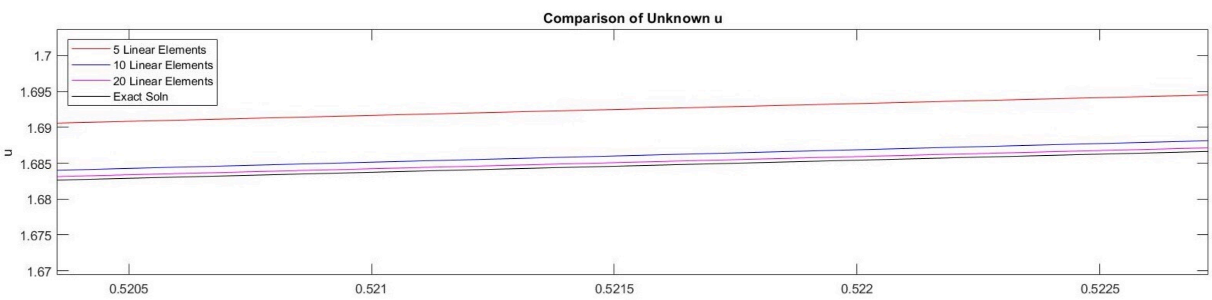
```
dU3 =

    1.0254
    1.0780
    1.1333
    1.1914
    1.2525
    1.3167
    1.3842
    1.4551
    1.5297
    1.6082
    1.6906
    1.7773
    1.8684
    1.9642
    2.0649
    2.1708
    2.2821
    2.3991
    2.5221
    2.6514

>>
```

(3)





(4)



If we take a closer look at the ‘Comparison of Unknown u’ graph, it is noticeable that the approximation is moving closer to the curve of exact solution in black, as more elements are involved in the approximation.

From the ‘Comparison of Unknown u,x’, we noticed that the approximated du/dx is discrete and discontinuous. However, as we are using more elements to perform the approximation, such discrete and discontinuous curves tend to be ‘smoother’ and more similar to the the exact solution. Moreover, regardless how much elements are used, the approximated du/dx is always exact at the midpoint of each element.

Thus, the element refinement by increasing the number of elements can lead to an approximation for u or du/dx closer to the exact solution.