(1) $\underbrace{\int_\Omega w(u_{,xx}+u_{,yy})\,d\Omega}_{①} - \int_\Omega w\,(x^2+y^2)\,e^{xy}\,d\Omega = 0 \qquad (1)$

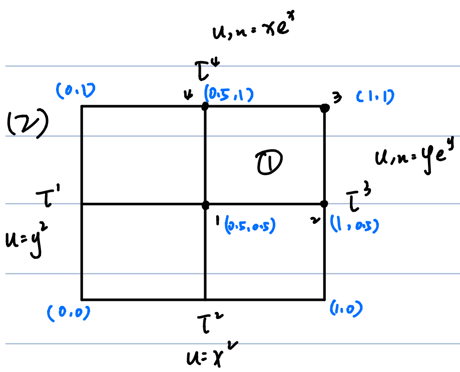
$$
\begin{aligned}
① = \int_\Omega w\,u_{,ii}\,d\Omega &= \int_\Omega (w u_{,i})_{,i}\,d\Omega - \int_\Omega w_{,i}\,u_{,i}\,d\Omega \\
&= \int_\Gamma w\,u_{,i}\,n_i\,d\Gamma - \int_\Omega w_{,i}\,u_{,i}\,d\Omega \\
&= \int_{\Gamma^3} w\,(y e^y)\,d\Gamma + \int_{\Gamma^4} w\,(x e^x)\,d\Gamma - \int_\Omega w_{,i}\,u_{,i}\,d\Omega
\end{aligned}
$$

Substitute ① $\longrightarrow$ (1)

$$\int_\Omega w_{,i}\,u_{,i}\,d\Omega = -\int_\Omega w(x^2+y^2)\,e^{xy}\,d\Omega + \int_{\Gamma^3} w\,(y e^y)\,d\Gamma + \int_{\Gamma^4} w\,(x e^x)\,d\Gamma$$

Summary of the weak form: Find $u \in S$, $S = \{u \mid u \in H^1,\ u = y^2 \text{ on } \Gamma^1,\ u = x^2 \text{ on } \Gamma^2\}$ such that for all $w \in \mathcal{V}$, $\mathcal{V} = \{u \mid u \in H^1,\ u = 0 \text{ on } \Gamma^1 \cup \Gamma^2\}$, there exists equation $\int_\Omega w_{,i}\,u_{,i}\,d\Omega = -\int_\Omega w(x^2+y^2)\,e^{xy}\,d\Omega + \int_{\Gamma^3} w\,(y e^y)\,d\Gamma + \int_{\Gamma^4} w\,(x e^x)\,d\Gamma$

(2)



$$\underbrace{\int_\Omega w_{,i}\,u_{,i}\,d\Omega}_{①} = \underbrace{-\int_\Omega w(x^2+y^2)\,e^{xy}\,d\Omega}_{②} + \underbrace{\int_{\Gamma^3} w\,(y e^y)\,d\Gamma + \int_{\Gamma^4} w\,(x e^x)\,d\Gamma}_{③}$$

$$\left.\begin{aligned} u^h_{,i} &= \underset{\sim}{B}\,\underset{\sim}{u}^e \\ w^h_{,i} &= \underset{\sim}{B}\,\underset{\sim}{w}^e \end{aligned}\right\} \longrightarrow ① = \sum_e \underset{\sim}{w}^e \int_{\Omega^e} \underset{\sim}{B}\,\underset{\sim}{B}^T\,d\Omega\,\underset{\sim}{u}^e$$

$$\underset{\sim}{k}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{B}\,d\Omega = \int_{-1}^{1}\int_{-1}^{1} \underset{\sim}{B}^T \underset{\sim}{B}\,\underset{\sim}{J}\,d\xi\,d\eta, \text{ where } \underset{\sim}{B} = \begin{bmatrix} N_{1,x} & N_{2,x} & N_{3,x} & N_{4,x} \\ N_{1,y} & N_{2,y} & N_{3,y} & N_{4,y} \end{bmatrix}$$

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \longrightarrow N_{1,\xi} = \frac{1}{4}(\eta-1) \,,\; N_{1,\eta} = \frac{1}{4}(\xi-1) \qquad X_1 = 0.5 \quad Y_1 = 0.5$$

$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta) \longrightarrow N_{2,\xi} = \frac{1}{4}(1-\eta) \quad N_{2,\eta} = -\frac{1}{4}(\xi+1) \qquad X_2 = 1 \quad Y_2 = 0.5$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta) \longrightarrow N_{3,\xi} = \frac{1}{4}(1+\eta) \quad N_{3,\eta} = \frac{1}{4}(1+\xi) \qquad X_3 = 1 \quad Y_3 = 1$$

$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta) \longrightarrow N_{4,\xi} = -\frac{1}{4}(1+\eta) \quad N_{4,\eta} = \frac{1}{4}(1-\xi) \qquad X_4 = 0.5 \quad Y_4 = 1$$

$$\frac{\partial x}{\partial \xi} = \sum_{I=1}^{4} N_{I,\xi} X_I = \frac{1}{4}(\eta-1)\times\frac{1}{2} + \frac{1}{4}(1-\eta)\times 1 + \frac{1}{4}(1+\eta)\times 1 - \frac{1}{4}(1+\eta)\times\frac{1}{2} = \frac{1}{4}$$

$$\frac{\partial y}{\partial \xi} = \sum_{I=1}^{4} N_{I,\xi} Y_I = \frac{1}{4}(\eta-1)\times\frac{1}{2} + \frac{1}{4}(1-\eta)\times\frac{1}{2} + \frac{1}{4}(1+\eta)\times 1 - \frac{1}{4}(1+\eta)\times 1 = 0$$

$$\frac{\partial x}{\partial \eta} = \sum_{I=1}^{4} N_{I,\eta} X_I = \frac{1}{4}(\xi-1)\times\frac{1}{2} - \frac{1}{4}(\xi+1)\times 1 + \frac{1}{4}(1+\xi)\times 1 + \frac{1}{4}(1-\xi)\times\frac{1}{2} = 0$$

$$\frac{\partial y}{\partial \eta} = \sum_{I=1}^{4} N_{I,\eta} Y_I = \frac{1}{4}(\xi-1)\times\frac{1}{2} - \frac{1}{4}(\xi+1)\times\frac{1}{2} + \frac{1}{4}(1+\xi)\times 1 + \frac{1}{4}(1-\xi)\times 1 = \frac{1}{4}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \longrightarrow \hat{J} = \det(J) = \frac{1}{16}$$

$$\underset{\sim}{B_I} = \begin{bmatrix} N_{I,x} \\ N_{I,y} \end{bmatrix} = J^{-1} \begin{bmatrix} N_{I,\xi} \\ N_{I,\eta} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} N_{I,\xi} \\ N_{I,\eta} \end{bmatrix}$$

$$\boxed{\begin{gathered} \text{Thus, } \underset{\sim}{B} = \begin{bmatrix} (\eta-1) & (1-\eta) & (1+\eta) & -(1+\eta) \\ (\xi-1) & -(\xi+1) & (1+\xi) & (1-\xi) \end{bmatrix} \\ \underset{\sim}{K}^{①} = \frac{1}{16} \int_{-1}^{+1}\int_{-1}^{+1} \underset{\sim}{B}^T \underset{\sim}{B} \, d\xi \, d\eta \end{gathered}}$$

$$② = -\int_\Omega w(x^2+y^2)\, e^{xy}\, d\Omega \qquad I = 1 \sim 4$$

$$= \sum_e \underset{\sim}{w}^{e^T} -\int_{-1}^{+1}\int_{-1}^{+1} \underset{\sim}{N}^T \left[\left(\sum X_I N_I\right)^2 + \left(\sum Y_I N_I\right)^2\right] \exp\left[\left(\sum X_I N_I\right)\cdot\left(\sum Y_I N_I\right)\right] \tilde{J}\, d\xi\, d\eta$$

$$\boxed{\longrightarrow f_Q^{①} = -\frac{1}{16}\int_{-1}^{+1}\int_{-1}^{+1} \underset{\sim}{N}^T \left[\left(\sum X_I N_I\right)^2 + \left(\sum Y_I N_I\right)^2\right] \exp\left[\left(\sum X_I N_I\right)\cdot\left(\sum Y_I N_I\right)\right] d\xi\, d\eta}$$

$$③ = \int_{\Gamma^3} w\,(y e^{y})\, d\Gamma + \int_{\Gamma^4} w\,(x e^{x})\, d\Gamma$$

$$= \sum_e \underset{\sim}{w}^{e^T} \int_{-1}^{+1} \underset{\sim}{N}^T \sum N_I Y_I \cdot \exp\left(\sum N_I Y_I\right)\cdot J_1^S \Big|_{\xi=1} d\eta + \sum_e \underset{\sim}{w}^{e^T} \int_{-1}^{+1} \underset{\sim}{N}^T \sum N_I X_I \cdot \exp\left(\sum N_I X_I\right)\cdot J_2^S \Big|_{\eta=1} d\xi$$

$$J_1^S = \sqrt{(x_{,\eta})^2 + (y_{,\eta})^2} = \frac{1}{4}$$

$$J_2^S = \sqrt{(x_{,\xi})^2 + (y_{,\xi})^2} = \frac{1}{4}$$

$$\boxed{\text{Thus, } f_h^{①} = \frac{1}{4}\int_{-1}^{+1} \underset{\sim}{N}^T \sum N_I Y_I \cdot \exp\left(\sum N_I Y_I\right)\Big|_{\xi=1} d\eta + \frac{1}{4}\int_{-1}^{+1} \underset{\sim}{N}^T \sum N_I X_I \cdot \exp\left(\sum N_I X_I\right)\Big|_{\eta=1} d\xi}$$

```
>> Computer_Assignment3_BoXiao

K_local =

    0.6667   -0.1667   -0.3333   -0.1667
   -0.1667    0.6667   -0.1667   -0.3333
   -0.3333   -0.1667    0.6667   -0.1667
   -0.1667   -0.3333   -0.1667    0.6667


f =

   -0.0929
    0.2027
    0.7967
    0.2027

>>
```