we have Figure 3–14(c). The elimination of the loop containing H_2/G_1 gives Figure 3–14(d). Finally, eliminating the feedback loop results in Figure 3–14(e).

Notice that the numerator of the closed-loop transfer function $C(s)/R(s)$ is the product of the transfer functions of the feedforward path. The denominator of $C(s)/R(s)$ is equal to

$$1 - \sum \text{(product of the transfer functions around each loop)}$$
$$= 1 - (G_1G_2H_1 - G_2G_3H_2 - G_1G_2G_3)$$
$$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$$

(The positive feedback loop yields a negative term in the denominator.)

3–4 MODELING IN STATE SPACE

In this section we shall present introductory material on state-space analysis of control systems.

Modern Control Theory. The modern trend in engineering systems is toward greater complexity, due mainly to the requirements of complex tasks and good accuracy. Complex systems may have multiple inputs and multiple outputs and may be time varying. Because of the necessity of meeting increasingly stringent requirements on the performance of control systems, the increase in system complexity, and easy access to large scale computers, modern control theory, which is a new approach to the analysis and design of complex control systems, has been developed since around 1960. This new approach is based on the concept of state. The concept of state by itself is not new since it has been in existence for a long time in the field of classical dynamics and other fields.

Modern Control Theory Versus Conventional Control Theory. Modern control theory is contrasted with conventional control theory in that the former is applicable to multiple-input-multiple-output systems, which may be linear or nonlinear, time invariant or time varying, while the latter is applicable only to linear time-invariant single-input-single-output systems. Also, modern control theory is essentially a time-domain approach, while conventional control theory is a complex frequency-domain approach. Before we proceed further, we must define state, state variables, state vector, and state space.

State. The state of a dynamic system is the smallest set of variables (called *state variables*) such that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

Note that the concept of state is by no means limited to physical systems. It is applicable to biological systems, economic systems, social systems, and others.

State Variables. The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system. If at least n variables $x_1, x_2, \ldots, x_n$ are needed to completely describe the behavior of a dynamic system (so that once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then such n variables are a set of state variables.

Note that state variables need not be physically measurable or observable quantities. Variables that do not represent physical quantities and those that are neither measurable nor observable can be chosen as state variables. Such freedom in choosing state variables is an advantage of the state-space methods. Practically, however, it is convenient to choose easily measurable quantities for the state variables, if this is possible at all, because optimal control laws will require the feedback of all state variables with suitable weighting.

State Vector. If n state variables are needed to completely describe the behavior of a given system, then these n state variables can be considered the n components of a vector $\mathbf{x}$. Such a vector is called a *state vector*. A state vector is thus a vector that determines uniquely the system state $\mathbf{x}(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $u(t)$ for $t \geq t_0$ is specified.

State Space. The n-dimensional space whose coordinate axes consist of the x_1 axis, x_2 axis, $\ldots$, x_n axis, where $x_1, x_2, \ldots, x_n$ are state variables; is called a *state space*. Any state can be represented by a point in the state space.

State-Space Equations. In state-space analysis we are concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, output variables, and state variables. As we shall see in Section 3–5, the state-space representation for a given system is not unique, except that the number of state variables is the same for any of the different state-space representations of the same system.

The dynamic system must involve elements that memorize the values of the input for $t \geq t_1$. Since integrators in a continuous-time control system serve as memory devices, the outputs of such integrators can be considered as the variables that define the internal state of the dynamic system. Thus the outputs of integrators serve as state variables. The number of state variables to completely define the dynamics of the system is equal to the number of integrators involved in the system.

Assume that a multiple-input–multiple-output system involves n integrators. Assume also that there are r inputs $u_1(t), u_2(t), \ldots, u_r(t)$ and m outputs $y_1(t), y_2(t), \ldots, y_m(t)$. Define n outputs of the integrators as state variables: $x_1(t), x_2(t), \ldots, x_n(t)$ Then the system may be described by

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \dot{x}_2(t) &= f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ &\ \ \vdots \\ \dot{x}_n(t) &= f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{aligned} \tag{3–8}$$

The outputs $y_1(t), y_2(t), \ldots, y_m(t)$ of the system may be given by

$$\begin{aligned} y_1(t) &= g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ y_2(t) &= g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ &\cdot \\ &\cdot \\ &\cdot \\ y_m(t) &= g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{aligned} \tag{3–9}$$

If we define

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ f_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix},$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t) \end{bmatrix}, \quad \mathbf{g}(\mathbf{x}, \mathbf{u}, t) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ g_2(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_r; t) \end{bmatrix}, \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(t) \end{bmatrix}$$

then Equations (3–8) and (3–9) become

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{3–10}$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \tag{3–11}$$

where Equation (3–10) is the state equation and Equation (3–11) is the output equation. If vector functions $\mathbf{f}$ and/or $\mathbf{g}$ involve time t explicitly, then the system is called a time-varying system.

If Equations (3–10) and (3–11) are linearized about the operating state, then we have the following linearized state equation and output equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{3–12}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \tag{3–13}$$

where $\mathbf{A}(t)$ is called the state matrix, $\mathbf{B}(t)$ the input matrix, $\mathbf{C}(t)$ the output matrix, and $\mathbf{D}(t)$ the direct transmission matrix. (Details of linearization of nonlinear systems about the operating state are discussed in Section 3–10.) A block diagram representation of Equations (3–12) and (3–13) is shown in Figure 3–15.

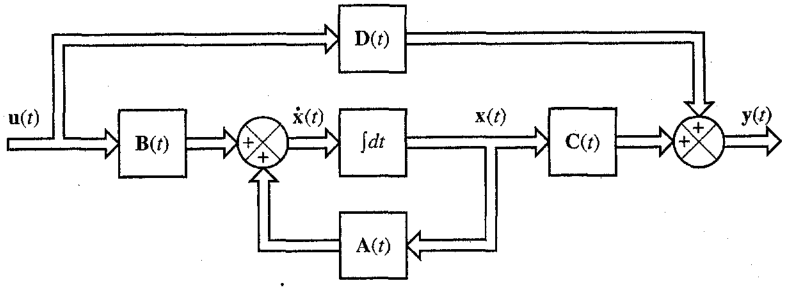


Figure 3–15
Block diagram of the linear, continuous-time control system represented in state space.

If vector functions **f** and **g** do not involve time t explicitly then the system is called a time-invariant system. In this case, Equations (3–12) and (3–13) can be simplified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3–14}$$

$$\dot{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{3–15}$$

Equation (3–14) is the state equation of the linear, time-invariant system.

Equation (3–15) is the output equation for the same system. In this book we shall be concerned mostly with systems described by Equations (3–14) and (3–15).

In what follows we shall present an example for deriving a state equation and output equation.

EXAMPLE 3–3 Consider the mechanical system shown in Figure 3–16. We assume that the system is linear. The external force $u(t)$ is the input to the system, and the displacement $y(t)$ of the mass is the output. The displacement $y(t)$ is measured from the equilibrium position in the absence of the external force. This system is a single-input–single-output system.

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u \tag{3–16}$$

This system is of second order. This means that the system involves two integrators. Let us define state variables $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

or

$$\dot{x}_1 = x_2 \tag{3–17}$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u \tag{3–18}$$

The output equation is

$$y = x_1 \tag{3–19}$$

In a vector-matrix form, Equations (3–17) and (3–18) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u \tag{3–20}$$

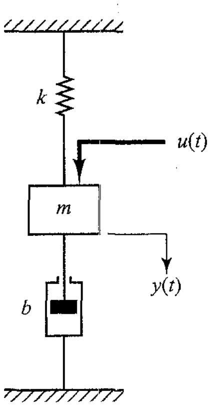


Figure 3–16
Mechanical system.

The output equation, Equation (3–19), can be written as

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3–21}$$

Equation (3–20) is a state equation and Equation (3–21) is an output equation for the system. Equations (3–20) and (3–21) are in the standard form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0], \qquad D = 0$$

Figure 3–17 is a block diagram for the system. Notice that the outputs of the integrators are state variables.

Correlation Between Transfer Functions and State-Space Equations. In what follows we shall show how to derive the transfer function of a single-input–single-output system from the state-space equations.

Let us consider the system whose transfer function is given by

$$\frac{Y(s)}{U(s)} = G(s) \tag{3–22}$$

This system may be represented in state space by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{3–23}$$

$$y = \mathbf{C}\mathbf{x} + Du \tag{3–24}$$

where $\mathbf{x}$ is the state vector, u is the input, and y is the output. The Laplace transforms of Equations (3–23) and (3–24) are given by

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \tag{3–25}$$

$$Y(s) = \mathbf{C}\mathbf{X}(s) + DU(s) \tag{3–26}$$

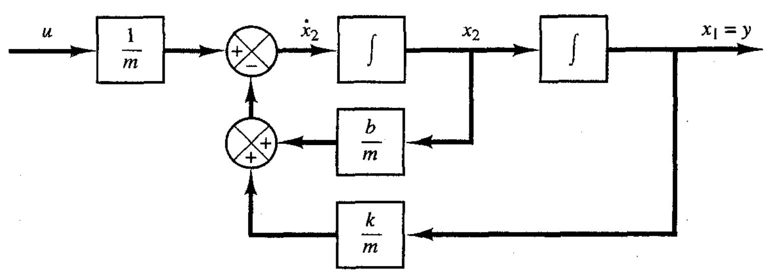


Figure 3–17 Block diagram of the mechanical system shown in Figure 3–16.

Since the transfer function was previously defined as the ratio of the Laplace transform of the output to the Laplace transform of the input when the initial conditions were zero, we set $\mathbf{x}(0)$ in Equation (3–25) to be zero. Then we have

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = \mathbf{B}U(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$

By premultiplying $(s\mathbf{I} - \mathbf{A})^{-1}$ to both sides of this last equation, we obtain

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) \tag{3–27}$$

By substituting Equation (3–27) into Equation (3–26), we get

$$Y(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\right]U(s) \tag{3–28}$$

Upon comparing Equation (3–28) with Equation (3–22), we see that

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \tag{3–29}$$

This is the transfer-function expression of the system in terms of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D.

Note that the right-hand side of Equation (3–29) involves $(s\mathbf{I} - \mathbf{A})^{-1}$. Hence $G(s)$ can be written as

$$G(s) = \frac{Q(s)}{|s\mathbf{I} - \mathbf{A}|}$$

where $Q(s)$ is a polynomial in s. Therefore, $|s\mathbf{I} - \mathbf{A}|$ is equal to the characteristic polynomial of $G(s)$. In other words, the eigenvalues of $\mathbf{A}$ are identical to the poles of $G(s)$.

EXAMPLE 3–4 Consider again the mechanical system shown in Figure 3–16. State-space equations for the system are given by Equations (3–20) and (3–21). We shall obtain the transfer function for the system from the state-space equations.

By substituting $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D into Equation (3–29), we obtain

$$\begin{aligned}
G(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \\
&= [1 \quad 0]\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0 \\
&= [1 \quad 0]\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}
\end{aligned}$$

Since

$$\begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

we have

$$G(s) = [1 \quad 0] \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$= \frac{1}{ms^2 + bs + k}$$

which is the transfer function of the system. The same transfer function can be obtained from Equation (3–16).

Transfer Matrix. Next, consider a multiple-input–multiple-output system. Assume that there are r inputs $u_1, u_2, \ldots, u_r$, and m outputs $y_1, y_2, \ldots, y_m$. Define

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_r \end{bmatrix}$$

The transfer matrix $\mathbf{G}(s)$ relates the output $\mathbf{Y}(s)$ to the input $\mathbf{U}(s)$, or

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

where $\mathbf{G}(s)$ is given by

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

[The derivation for this equation is the same as that for Equation (3–29).] Since the input vector $\mathbf{u}$ is r dimensional and the output vector $\mathbf{y}$ is m dimensional, the transfer matrix $\mathbf{G}(s)$ is an $m \times r$ matrix.

3–5 STATE-SPACE REPRESENTATION OF DYNAMIC SYSTEMS

A dynamic system consisting of a finite number of lumped elements may be described by ordinary differential equations in which time is the independent variable. By use of vector-matrix notation, an nth-order differential equation may be expressed by a first-order vector-matrix differential equation. If n elements of the vector are a set of state variables, then the vector-matrix differential equation is a *state* equation. In this section we shall present methods for obtaining state-space representations of continuous-time systems.

State-Space Representation of nth-Order Systems of Linear Differential Equations in which the Forcing Function Does Not Involve Derivative Terms. Consider the following nth-order system:

$$\overset{(n)}{y} + a_1\overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u \qquad (3\text{–}30)$$

Noting that the knowledge of $y(0), \dot{y}(0), \ldots, \overset{(n-1)}{y}(0)$, together with the input $u(t)$ for $t \geq 0$, determines completely the future behavior of the system, we may take

$y(t), \dot{y}(t), \ldots, \overset{(n-1)}{y}(t)$ as a set of n state variables. (Mathematically, such a choice of state variables is quite convenient. Practically, however, because higher-order derivative terms are inaccurate, due to the noise effects inherent in any practical situations, such a choice of the state variables may not be desirable.)

Let us define

$$
\begin{aligned}
x_1 &= y \\
x_2 &= \dot{y} \\
&\;\cdot \\
&\;\cdot \\
&\;\cdot \\
x_n &= \overset{(n-1)}{y}
\end{aligned}
$$

Then Equation (3–30) can be written as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\;\cdot \\
&\;\cdot \\
&\;\cdot \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= a_n x_1 - \cdots - a_1 x_n + u
\end{aligned}
$$

or

$$
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{3–31}
$$

where

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad
\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad
\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}
$$

The output can be given by

$$
y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}
$$

or

$$y = \mathbf{C}\mathbf{x} \tag{3–32}$$

where

$$\mathbf{C} = [1 \quad 0 \quad \cdots \quad 0]$$

[Note that D in Equation (3–24) is zero.] The first-order differential equation, Equation (3–31), is the state equation, and the algebraic equation, Equation (3–32), is the output equation.

Note that the state-space representation for the transfer function system

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is given also by Equation (3–31) and (3–32).

State-Space Representation of *n*th Order Systems of Linear Differential Equations in which the Forcing Function Involves Derivative Terms. Consider the differential equation system that involves derivatives of the forcing function, such as

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u \tag{3–33}$$

The main problem in defining the state variables for this case lies in the derivative terms. The state variables must be such that they will eliminate the derivatives of u in the state equation.

One way to obtain a state equation and output equation is to define the following n variables as a set of n state variables:

$$\begin{aligned}
x_1 &= y - \beta_0 u \\
x_2 &= \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \\
x_3 &= \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u \\
&\;\;\vdots \\
x_n &= \overset{(n-1)}{y} - \beta_0 \overset{(n-1)}{u} - \beta_1 \overset{(n-2)}{u} - \cdots - \beta_{n-2}\dot{u} - \beta_{n-1}u = \dot{x}_{n-1} - \beta_{n-1}u
\end{aligned} \tag{3–34}$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ are determined from

$$\begin{aligned}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1\beta_0 \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\
\beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \\
&\;\;\vdots \\
\beta_n &= b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_n\beta_0
\end{aligned} \tag{3–35}$$

With this choice of state variables the existence and uniqueness of the solution of the state equation is guaranteed. (Note that this is not the only choice of a set of state variables.) With the present choice of state variables, we obtain

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \beta_1 u \\
\dot{x}_2 &= x_3 + \beta_2 u \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{x}_{n-1} &= x_n + \beta_{n-1} u \\
\dot{x} &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + \beta_n u
\end{aligned}
\tag{3–36}
$$

[To derive Equation (3–36), see Problem **A–3–6**.] In terms of vector-matrix equations, Equation (3–36) and the output equation can be written as

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
0 & 0 & 0 & \cdots & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}
+
\begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u
$$

$$
y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \beta_0 u
$$

or

$$
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{3–37}
$$

$$
y = \mathbf{C}\mathbf{x} + Du \tag{3–38}
$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{n-1} \\ \beta_n \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad \cdots \quad 0], \quad D = \beta_0 = b_0$$

In this state-space representation, matrices **A** and **C** are exactly the same as those for the system of Equation (3–30). The derivatives on the right-hand side of Equation (3–33) affect only the elements of the **B** matrix.

Note that the state-space representation for the transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is given also by Equations (3–37) and (3–38).

There are many ways to obtain state-space representations of systems. Some of them are presented in this chapter. Methods for obtaining canonical representations of systems in state space (such as controllable canonical form, observable canonical form, diagonal canonical form, and Jordan canonical form) are presented in Chapter 11.

MATLAB can be used to obtain state-space representations of systems from transfer function representations, and vice versa. This subject is presented in Section 3-6.

EXAMPLE 3–5 Consider the spring-mass-dashpot system mounted on a massless cart as shown in Figure 3–18. A dashpot is a device that provides viscous friction, or damping. It consists of a piston and oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by the oil because the oil must flow around the piston (or through orifices provided in the piston) from one side of the piston to the other. The dashpot essentially absorbs energy. This absorbed energy is dissipated as heat, and the dashpot does not store any kinetic or potential energy. The dashpot is also called a *damper*.

Let us obtain mathematical models of this system by assuming that the cart is standing still for $t < 0$ and the spring-mass-dashpot system on the cart is also standing still for $t < 0$. In this system, $u(t)$ is the displacement of the cart and is the input to the system. At $t = 0$, the cart is moved at a constant speed, or $\dot{u} =$ constant. The displacement $y(t)$ of the mass is the output. (The displacement is relative to the ground.) In this system, m denotes the mass, b denotes the viscous friction coefficient, and k denotes the spring constant. We assume that the friction force of the dashpot

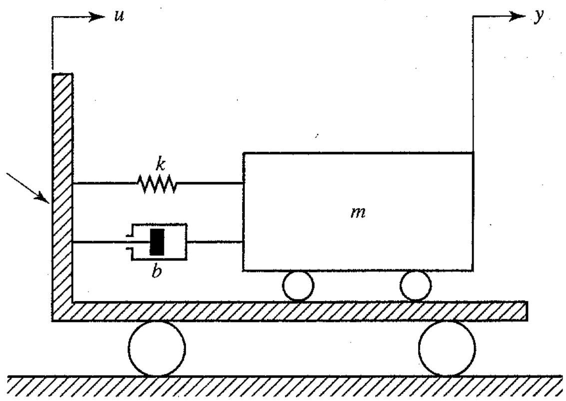


Figure 3–18 Spring-mass-dashpot system mounted on a cart.

is proportional to $\dot{y} - \dot{u}$ and that the spring is a linear spring; that is, the spring force is proportional to $y - u$.

For translational systems, Newton's second law states that

$$ma = \sum F$$

where m is a mass, a is the acceleration of the mass, and $\sum F$ is the sum of the forces acting on the mass in the direction of the acceleration a. Applying Newton's second law to the present system and noting that the cart is massless, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

This equation represents a mathematical model of the system considered. Taking the Laplace transform of this last equation, assuming zero initial condition, gives

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

Taking the ratio of $Y(s)$ to $U(s)$, we find the transfer function of the system to be

$$\text{Transfer function} = G(s) = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Such a transfer function representation of a mathematical model is used very frequently in control engineering.

Next we shall obtain a state-space model of this system. We shall first compare the differential equation for this system

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

with the standard form

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

and identify a_1, a_2, b_0, b_1, and b_2 as follows:

$$a_1 = \frac{b}{m}, \qquad a_2 = \frac{k}{m}, \qquad b_0 = 0, \qquad b_1 = \frac{b}{m}, \qquad b_2 = \frac{k}{m}$$

Referring to Equation (3–35), we have

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m}$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

Then, referring to Equation (3–34), define

$$x_1 = y - \beta_0 u = y$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u$$

From Equation (3–36) we have

$$\dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{b}{m}u$$

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u$$

and the output equation becomes

$$y = x_1$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ \frac{k}{m} - \left(\frac{b}{m}\right)^2 \end{bmatrix} u \tag{3–39}$$

and

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3–40}$$

Equations (3–39) and (3–40) give a state-space representation of the system. (Note that this is not the only state-space representation. There are infinitely many state-space representations for the system.)

3–6 TRANSFORMATION OF MATHEMATICAL MODELS WITH MATLAB

MATLAB is quite useful to transform the system model from transfer function to state space, and vice versa. We shall begin our discussion with transformation from transfer function to state space.

Let us write the closed-loop transfer function as

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s} = \frac{\text{num}}{\text{den}}$$

Once we have this transfer-function expression, the MATLAB command

```
[A, B, C, D] = tf2ss(num,den)
```

will give a state-space representation. It is important to note that the state-space representation for any system is not unique. There are many (infinitely many) state-space representations for the same system. The MATLAB command gives one possible such state-space representation.

Transformation From Transfer Function to State Space. Consider the transfer function system

$$\frac{Y(s)}{U(s)} = \frac{s}{(s+10)(s^2+4s+16)}$$

$$= \frac{s}{s^3+14s^2+56s+160} \tag{3–41}$$

There are many (infinitely many) possible state-space representations for this system. One possible state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Another possible state-space representation (among infinitely many alternatives) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{3–42}$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u \tag{3–43}$$

MATLAB transforms the transfer function given by Equation (3–41) into the state-space representation given by Equations (3–42) and (3–43). For the example system considered here, MATLAB Program 3–2 will produce matrices **A**, **B**, **C**, and D.

MATLAB Program 3–2

```
num = [0   0   1   0];
den = [1   14   56   160];
[A,B,C,D] = tf2ss(num,den)

A =

  -14  -56 -160
    1    0    0
    0    1    0

B =

    1
    0
    0

C =

    0    1    0

D =

    0
```

Transformation From State Space to Transfer Function. To obtain the transfer function from state-space equations, use the following command:

```
[num,den] = ss2tf(A,B,C,D,iu)
```

iu must be specified for systems with more than one input. For example, if the system has three inputs ($u1$, $u2$, $u3$), then iu must be either 1, 2, or 3, where 1 implies $u1$, 2 implies $u2$, and 3 implies $u3$.

If the system has only one input, then either

```
[num,den] = ss2tf(A,B,C,D)
```

or

```
[num,den] = ss2tf(A,B,C,D,1)
```

may be used. For the case where the system has multiple inputs and multiple outputs, see Problem **A–3–13**.

EXAMPLE 3–6 Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -25 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \\ -120 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

MATLAB Program 3-3 will produce the transfer function for the given system. The transfer function obtained is given by

$$\frac{Y(s)}{U(s)} = \frac{25s + 5}{s^3 + 5s^2 + 25s + 5}$$

MATLAB Program 3–3

```
A = [0  1  0;  0  0  1;  -5  -25  -5];
B = [0; 25; -120];
C = [1  0    0];
D = [0];
[num,den] = ss2tf(A,B,C,D)

num =

   0  0.0000   25.0000  5.0000

den

   1.0000  5.0000   25.0000  5.0000

% ***** The same result can be obtained by entering the following command: *****

[num,den] = ss2tf(A,B,C,D,1)

num =

   0  0.0000   25.0000  5.0000

den =

   1.0000  5.0000   25.0000  5.0000
```

3–7 MECHANICAL SYSTEMS

In this section we shall discuss mathematical modeling of mechanical systems. The fundamental law governing mechanical systems is Newton's second law. It can be applied to any mechanical system. In this section we shall derive mathematical models of three mechanical systems. (Mathematical models of additional mechanical systems will be derived and analyzed throughout the remaining chapters.)

EXAMPLE 3–7 Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system shown in Figure 3–19.

The equations of motion for the system shown in Figure 3–19 are

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Simplifying, we obtain

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = b\dot{x}_2 + k_2x_2 + u$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2x_1$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$\left[m_1s^2 + bs + (k_1 + k_2)\right]X_1(s) = (bs + k_2)X_2(s) + U(s) \tag{3–44}$$

$$\left[m_2s^2 + bs + (k_2 + k_3)\right]X_2(s) = (bs + k_2)X_1(s) \tag{3–45}$$

Solving Equation (3–45), for $X_2(s)$ and substituting it into Equation (3–44) and simplifying, we get

$$\left[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2\right]X_1(s)$$
$$= (m_2s^2 + bs + k_2 + k_3)U(s)$$

from which we obtain

$$\frac{X_1(s)}{U(s)} = \frac{m_2s^2 + bs + k_2 + k_3}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \tag{3–46}$$

From Equations (3–45) and (3–46) we have

$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \tag{3–47}$$

Equations (3–46) and (3–47) are the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$, respectively.

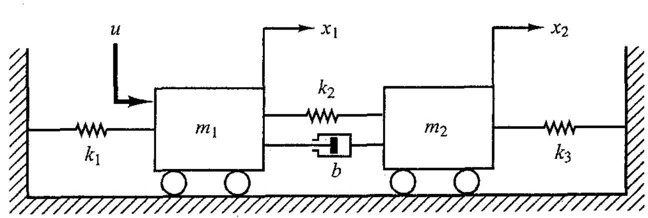


Figure 3–19 Mechanical system.

EXAMPLE 3–8 An inverted pendulum mounted on a motor-driven cart is shown in Figure 3–20(a). This is a model of the attitude control of a space booster on takeoff. (The objective of the attitude control problem is to keep the space booster in a vertical position.) The inverted pendulum is unstable in that it may fall over any time in any direction unless a suitable control force is applied. Here we consider

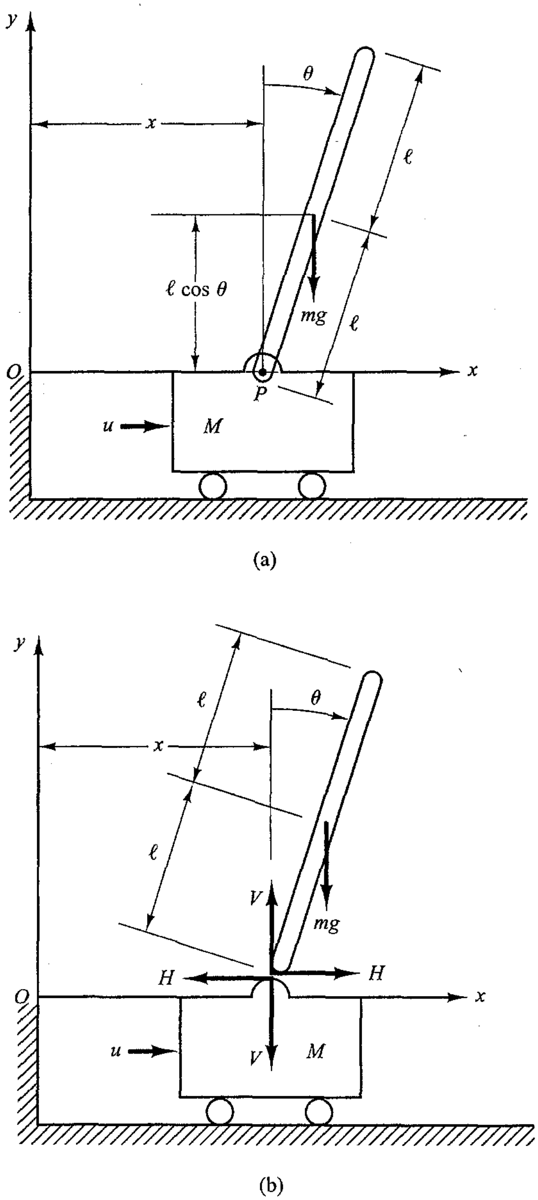


Figure 3–20
(a) Inverted pendulum system; (b) free-body diagram.

only a two-dimensional problem in which the pendulum moves only in the plane of the page. The control force u is applied to the cart. Assume that the center of gravity of the pendulum rod is at its geometric center. Obtain a mathematical model for the system.

Define the angle of the rod from the vertical line as θ. Define also the (x, y) coordinates of the center of gravity of the pendulum rod as (x_G, y_G). Then

$$x_G = x + l \sin\theta$$

$$y_G = l \cos\theta$$

To derive the equations of motion for the system, consider the free-body diagram shown in Figure 3–20(b). The rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta \tag{3–48}$$

where I is the moment of inertia of the rod about its center of gravity.

The horizontal motion of center of gravity of pendulum rod is given by

$$m\frac{d^2}{dt^2}(x + l\sin\theta) = H \tag{3–49}$$

The vertical motion of center of gravity of pendulum rod is

$$m\frac{d^2}{dt^2}(l\cos\theta) = V - mg \tag{3–50}$$

The horizontal motion of cart is described by

$$M\frac{d^2x}{dt^2} = u - H \tag{3–51}$$

Since we must keep the inverted pendulum vertical, we can assume that $\theta(t)$ and $\dot{\theta}(t)$ are small quantities such that $\sin\theta \doteq 0$, $\cos\theta = 1$, and $\theta\dot{\theta}^2 = 0$. Then, Equations (3–48) through (3–50) can be linearized. The linearized equations are

$$I\ddot{\theta} = Vl\theta - Hl \tag{3–52}$$

$$m(\ddot{x} + l\ddot{\theta}) = H \tag{3–53}$$

$$0 = V - mg \tag{3–54}$$

From Equations (3–51) and (3–53), we obtain

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \tag{3–55}$$

From Equations (3–52), (3–53), and (3–54), we have

$$\begin{aligned} I\ddot{\theta} &= mgl\theta - Hl \\ &= mgl\theta - l(m\ddot{x} + ml\ddot{\theta}) \end{aligned}$$

or

$$(I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{3–56}$$

Equations (3–55) and (3–56) describe the motion of the inverted-pendulum-on-the-cart system. They constitute a mathematical model of the system.

EXAMPLE 3–9 Consider the inverted pendulum system shown in Figure 3–21. Since in this system the mass is concentrated at the top of the rod, the center of gravity is the center of the pendulum ball. For this case, the moment of inertia of the pendulum about its center of gravity is small, and we assume $I = 0$ in Equation (3–56). Then the mathematical model for this system becomes as follows:

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \tag{3–57}$$

$$ml^2\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{3–58}$$

Equations (3–57) and (3–58) can be modified to

$$Ml\ddot{\theta} = (M + m)g\theta - u \tag{3–59}$$

$$M\ddot{x} = u - mg\theta \tag{3–60}$$

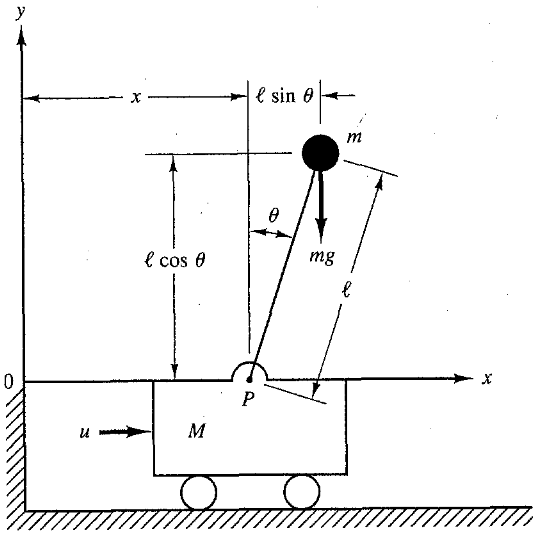


Figure 3–21 Inverted-pendulum system.

Equation (3–59) was obtained by eliminating $\ddot{x}$ from Equations (3–57) and (3–58). Equation (3–60) was obtained by eliminating $\ddot{\theta}$ from Equations (3–57) and (3–58). From Equation (3–59) we obtain the plant transfer function to be

$$\frac{\Theta(s)}{-U(s)} = \frac{1}{Mls^2 - (M+m)g}$$

$$= \frac{1}{Ml\left(s + \sqrt{\frac{M+m}{Ml}g}\right)\left(s - \sqrt{\frac{M+m}{Ml}g}\right)}$$

The inverted pendulum plant has one pole on the negative real axis $\left[s = -(\sqrt{M+m}/\sqrt{Ml})\sqrt{g}\right]$ and another on the positive real axis $\left[s = (\sqrt{M+m}/\sqrt{Ml})\sqrt{g}\right]$. Hence, the plant is open-loop unstable.

Define state variables x_1, x_2, x_3, and x_4 by

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$
$$x_3 = x$$
$$x_4 = \dot{x}$$

Note that angle θ indicates the rotation of the pendulum rod about point P, and x is the location of the cart. If we consider θ and x as the outputs of the system, then

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

(Notice that both θ and x are easily measurable quantities). Then, from the definition of the state variables and Equations (3–59) and (3–60), we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{M+m}{Ml} g x_1 - \frac{1}{Ml} u$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -\frac{m}{M} g x_1 + \frac{1}{M} u$$

In terms of vector-matrix equations, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M+m}{Ml}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M}g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} u \tag{3–61}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{3–62}$$

Equations (3–61) and (3–62) give a state-space representation of the inverted pendulum system. (Note that state-space representation of the system is not unique. There are infinitely many such representations for this system.

3–8 ELECTRICAL AND ELECTRONIC SYSTEMS

Basic laws governing electrical circuits are Kirchhoff's current law and voltage law. Kirchhoff's current law (node law) states that the algebraic sum of all currents entering and leaving a node is zero. (This law can also be stated as follows: The sum of currents entering a node is equal to the sum of currents leaving the same node.) Kirchhoff's voltage law (loop law) states that at any given instant the algebraic sum of the voltages around any loop in an electrical circuit is zero. (This law can also be stated as follows: The sum of the voltage drops is equal to the sum of the voltage rises around a loop.) A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's laws to it.

This section first deals with simple electrical circuits and then treats mathematical modeling of operational amplifier systems.

LRC Circuit. Consider the electrical circuit shown in Figure 3–22. The circuit consists of an inductance L (henry), a resistance R (ohm), and a capacitance C (farad). Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \tag{3–63}$$

$$\frac{1}{C}\int i\,dt = e_o \tag{3–64}$$

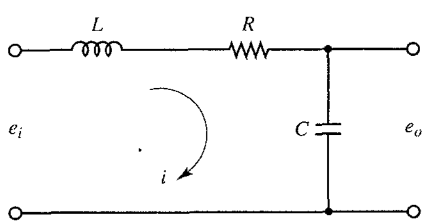


Figure 3–22
Electrical circuit.

Equations (3–63) and (3–64) give a mathematical model of the circuit.

A transfer function model of the circuit can also be obtained as follows: Taking the Laplace transforms of Equations (3–63) and (3–64), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

If e_i is assumed to be the input and e_o the output, then the transfer function of this system is found to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \tag{3–65}$$

State-Space Representation. A state-space model of the system shown in Figure 3–22 may be obtained as follows: First, note that the differential equation for the system can be obtained from Equation (3–65) as

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Then by defining state variables by

$$x_1 = e_o$$
$$x_2 = \dot{e}_o$$

and the input and output variables by

$$u = e_i$$
$$y = e_o = x_1$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{LC} \end{bmatrix} u$$

and

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a mathematical model of the system in state space.

Transfer Functions of Cascaded Elements. Many feedback systems have components that load each other. Consider the system shown in Figure 3–23. Assume that

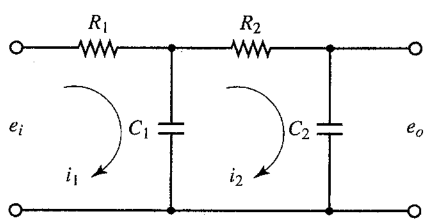


Figure 3–23
Electrical system.

e_i is the input and e_o is the output. The capacitances C_1 and C_2 are not charged initially. It will be shown that the second stage of the circuit (R_2C_2 portion) produces a loading effect on the first stage (R_1C_1 portion). The equations for this system are

$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i \tag{3–66}$$

and

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0 \tag{3–67}$$

$$\frac{1}{C_2}\int i_2\,dt = e_o \tag{3–68}$$

Taking the Laplace transforms of Equations (3–66) through (3–68), respectively, using zero initial conditions, we obtain

$$\frac{1}{C_1 s}\left[I_1(s) - I_2(s)\right] + R_1 I_1(s) = E_i(s) \tag{3–69}$$

$$\frac{1}{C_1 s}\left[I_2(s) - I_1(s)\right] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0 \tag{3–70}$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s) \tag{3–71}$$

Eliminating $I_1(s)$ from Equations (3–69) and (3–70) and writing $E_i(s)$ in terms of $I_2(s)$, we find the transfer function between $E_o(s)$ and $E_i(s)$ to be

$$\begin{aligned}\frac{E_o(s)}{E_i(s)} &= \frac{1}{(R_1C_1s + 1)(R_2C_2s + 1) + R_1C_2s} \\ &= \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}\end{aligned} \tag{3–72}$$

The term R_1C_2s in the denominator of the transfer function represents the interaction of two simple RC circuits. Since $(R_1C_1 + R_2C_2 + R_1C_2)^2 > 4R_1C_1R_2C_2$, the two roots of the denominator of Equation (3–72) are real.

The present analysis shows that, if two RC circuits are connected in cascade so that the output from the first circuit is the input to the second, the overall transfer function is not the product of $1/(R_1C_1s + 1)$ and $1/(R_2C_2s + 1)$. The reason for this is that, when we derive the transfer function for an isolated circuit, we implicitly assume that the output is unloaded. In other words, the load impedance is assumed to be infinite, which means that no power is being withdrawn at the output. When the second circuit is connected to the output of the first, however, a certain amount of power is withdrawn, and thus the assumption of no loading is violated. Therefore, if the transfer function of this system is obtained under the assumption of no loading, then it is not valid. The degree of the loading effect determines the amount of modification of the transfer function.

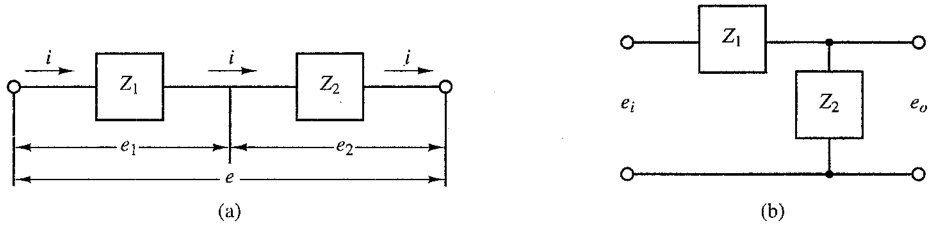


Figure 3–24
Electrical circuits.

Complex Impedances. In deriving transfer functions for electrical circuits, we frequently find it convenient to write the Laplace-transformed equations directly, without writing the differential equations. Consider the system shown in Figure 3–24(a). In this system, Z_1 and Z_2 represent complex impedances. The complex impedance $Z(s)$ of a two-terminal circuit is the ratio of $E(s)$, the Laplace transform of the voltage across the terminals, to $I(s)$, the Laplace transform of the current through the element, under the assumption that the initial conditions are zero, so that $Z(s) = E(s)/I(s)$. If the two-terminal elements is a resistance R, capacitance C, or inductance L, then the complex impedance is given by R, $1/Cs$, or Ls, respectively. If complex impedances are connected in series, the total impedance is the sum of the individual complex impedances.

Remember that the impedance approach is valid only if the initial conditions involved are all zeros. Since the transfer function requires zero initial conditions, the impedance approach can be applied to obtain the transfer function of the electrical circuit. This approach greatly simplifies the derivation of transfer functions of electrical circuits.

Consider the circuit shown in Figure 3–24(b). Assume that the voltages e_i and e_o are the input and output of the circuit, respectively. Then the transfer function of this circuit is

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

For the system shown in Figure 3–22,

$$Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs}$$

Hence the transfer function $E_o(s)/E_i(s)$ can be found as follows:

$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{1}{Cs}}{Ls + R + \dfrac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

which is, of course, identical to Equation (3–65).

EXAMPLE 3–10 Consider again the system shown in Figure 3–23. Obtain the transfer function $E_o(s)/E_i(s)$ by use of the complex impedance approach. (Capacitors C_1 and C_2 are not charged initially.)

The circuit shown in Figure 3–23 can be redrawn as that shown in Figure 3–25(a), which can be further modified to Figure 3–25(b).

In the system shown in Figure 3–25(b) the current I is divided into two currents I_1 and I_2. Noting that

$$Z_2 I_1 = (Z_3 + Z_4) I_2, \qquad I_1 + I_2 = I$$

we obtain

$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

Noting that

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} \right] I$$

$$E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$

we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}$$

Substituting $Z_1 = R_1$, $Z_2 = 1/(C_1 s)$, $Z_3 = R_2$, and $Z_4 = 1/(C_2 s)$ into this last equation, we get

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{C_1 s}\frac{1}{C_2 s}}{R_1\left(\frac{1}{C_1 s} + R_2 + \frac{1}{C_2 s}\right) + \frac{1}{C_1 s}\left(R_2 + \frac{1}{C_2 s}\right)}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

which is the same as that given by Equation (3–72).

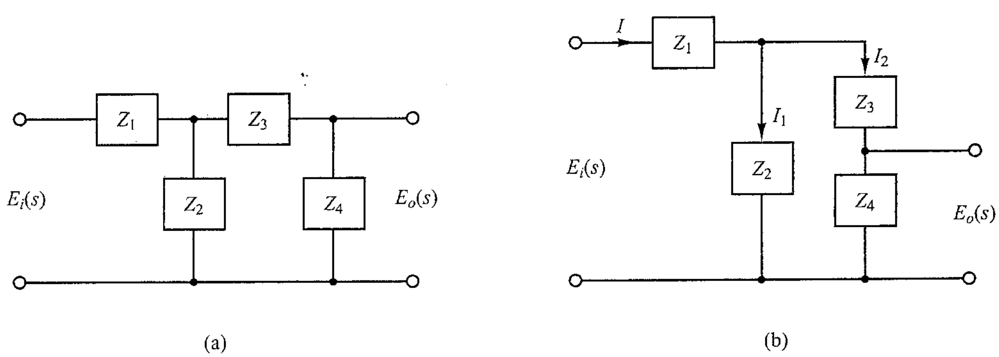


Figure 3–25
(a) The circuit of Figure 3–23 shown in terms of impedances; (b) equivalent circuit diagram.

Figure 3–26
(a) System consisting of two nonloading cascaded elements; (b) an equivalent system.

Transfer Functions of Nonloading Cascaded Elements. The transfer function of a system consisting of two nonloading cascaded elements can be obtained by eliminating the intermediate input and output. For example, consider the system shown in Figure 3–26(a). The transfer functions of the elements are

$$G_1(s) = \frac{X_2(s)}{X_1(s)} \quad \text{and} \quad G_2(s) = \frac{X_3(s)}{X_2(s)}$$

If the input impedance of the second element is infinite, the output of the first element is not affected by connecting it to the second element. Then the transfer function of the whole system becomes

$$G(s) = \frac{X_3(s)}{X_1(s)} = \frac{X_2(s)X_3(s)}{X_1(s)X_2(s)} = G_1(s)G_2(s)$$

The transfer function of the whole system is thus the product of the transfer functions of the individual elements. This is shown in Figure 3–26(b).

As an example, consider the system shown in Figure 3–27. The insertion of an isolating amplifier between the circuits to obtain nonloading characteristics is frequently used in combining circuits. Since amplifiers have very high input impedances, an isolation amplifier inserted between the two circuits justifies the nonloading assumption.

The two simple *RC* circuits, isolated by an amplifier as shown in Figure 3–27, have negligible loading effects, and the transfer function of the entire circuit equals the product of the individual transfer functions. Thus, in this case,

$$\begin{aligned}\frac{E_o(s)}{E_i(s)} &= \left(\frac{1}{R_1C_1s + 1}\right)(K)\left(\frac{1}{R_2C_2s + 1}\right)\\ &= \frac{K}{(R_1C_1s + 1)(R_2C_2s + 1)}\end{aligned}$$

Electronic Controllers. In what follows we shall discuss electronic controllers using operational amplifiers. We begin by deriving the transfer functions of simple operational-amplifier circuits. Then we derive the transfer functions of some of the operational-amplifier controllers. Finally, we give operational-amplifier controllers and their transfer functions in the form of a table.

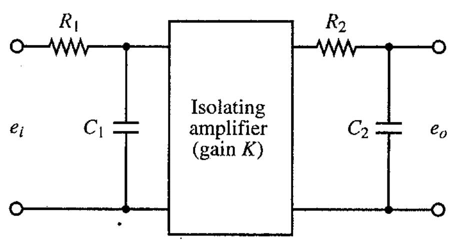


Figure 3–27
Electrical system.

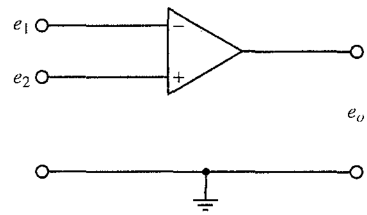


Figure 3–28 Operational amplifier.

Operational Amplifiers. Operational amplifiers, often called op amps, are frequently used to amplify signals in sensor circuits. Op amps are also frequently used in filters used for compensation purposes. Figure 3–28 shows an op amp. It is a common practice to choose the ground as 0 volt and measure the input voltages e_1 and e_2 relative to the ground. The input e_1 to the minus terminal of the amplifier is inverted, and the input e_2 to the plus terminal is not inverted. The total input to the amplifier thus becomes $e_2 - e_1$. Hence, for the circuit shown in Figure 3–28, we have

$$e_o = K(e_2 - e_1) = -K(e_1 - e_2)$$

where the inputs e_1 and e_2 may be dc or ac signals and K is the differential gain (voltage gain). The magnitude of K is approximately $10^5 \sim 10^6$ for dc signals and ac signals with frequencies less than approximately 10 Hz. (The differential gain K decreases with the signal frequency and becomes about unity for frequencies of 1 MHz $\sim$ 50 MHz.) Note that the op amp amplifies the difference in voltages e_1 and e_2. Such an amplifier is commonly called a differential amplifier. Since the gain of the op amp is very high, it is necessary to have a negative feedback from the output to the input to make the amplifier stable. (The feedback is made from the output to the inverted input so that the feedback is a negative feedback.)

In the ideal op amp, no current flows into the input terminals, and the output voltage is not affected by the load connected to the output terminal. In other words, the input impedance is infinity and the output impedance is zero. In an actual op amp, a very small (almost negligible) current flows into an input terminal and the output cannot be loaded too much. In our analysis here, we make the assumption that the op amps are ideal.

Inverting Amplifier. Consider the operational amplifier circuit shown in Figure 3–29. Let us obtain the output voltage e_o.

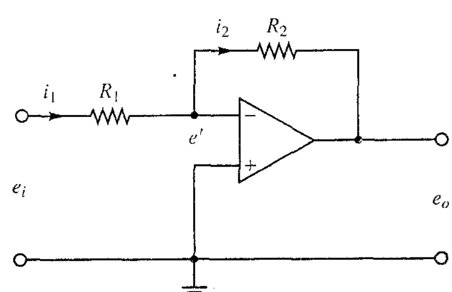


Figure 3–29 Inverting amplifier.

The equation for this circuit can be obtained as follows: Define

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = \frac{e' - e_o}{R_2}$$

Since only a negligible current flows into the amplifier, the current i_1 must be equal to current i_2. Thus

$$\frac{e_i - e'}{R_1} = \frac{e' - e_o}{R_2}$$

Since $K(0 - e') = e_0$ and $K \gg 1$, e' must be almost zero, or $e' \doteqdot 0$. Hence we have

$$\frac{e_i}{R_1} = \frac{-e_o}{R_2}$$

or

$$e_o = -\frac{R_2}{R_1} e_i$$

Thus the circuit shown is an inverting amplifier. If $R_1 = R_2$, then the op-amp circuit shown acts as a sign inverter.

Noninverting Amplifier. Figure 3–30(a) shows a noninverting amplifier. A circuit equivalent to this one is shown in Figure 3–30(b). For the circuit of Figure 3–30(b), we have

$$e_o = K\left(e_i - \frac{R_1}{R_1 + R_2} e_o\right)$$

where K is the differential gain of the amplifier. From this last equation, we get

$$e_i = \left(\frac{R_1}{R_1 + R_2} + \frac{1}{K}\right) e_o$$

Since $K \gg 1$, if $R_1/(R_1 + R_2) \gg 1/K$, then

$$e_o = \left(1 + \frac{R_2}{R_1}\right) e_i$$

This equation gives the output voltage e_o. Since e_o and e_i have the same signs, the op-amp circuit shown in Figure 3–30(a) is noninverting.

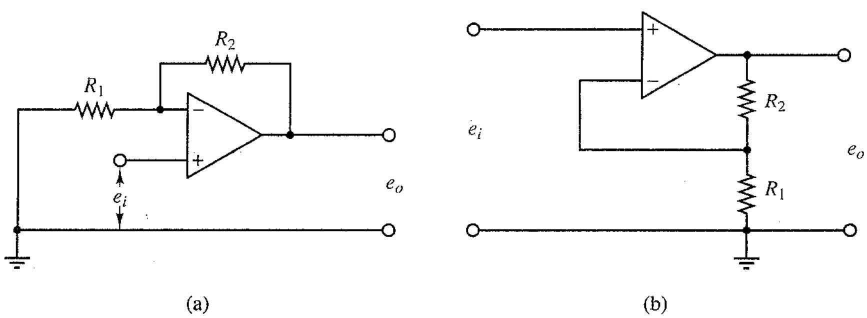


Figure 3–30 (a) Noninverting operational amplifier; (b) equivalent circuit.

EXAMPLE 3–11 Figure 3-31 shows an electrical circuit involving an operational amplifier. Obtain the output e_o.

Let us define

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = C\frac{d(e' - e_o)}{dt}, \qquad i_3 = \frac{e' - e_o}{R_2}$$

Noting that the current flowing into the amplifier is negligible, we have

$$i_1 = i_2 + i_3$$

Hence

$$\frac{e_i - e'}{R_1} = C\frac{d(e' - e_o)}{dt} + \frac{e' - e_o}{R_2}$$

Since $e' \doteqdot 0$, we have

$$\frac{e_i}{R_1} = -C\frac{de_o}{dt} - \frac{e_o}{R_2}$$

Taking the Laplace transform of this last equation, assuming the zero initial condition, we have

$$\frac{E_i(s)}{R_1} = -\frac{R_2Cs + 1}{R_2}E_o(s)$$

which can be written as

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}$$

The op-amp circuit shown in Figure 3–31 is a first-order lag circuit. (Several other circuits involving op amps are shown in Table 3–1 together with their transfer functions.)

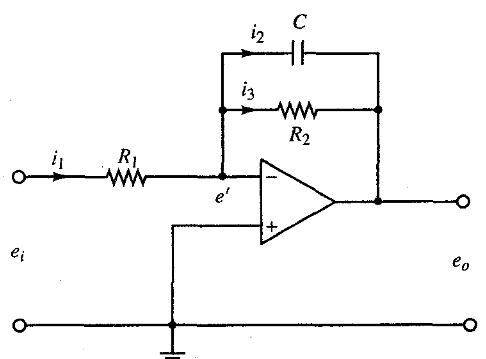


Figure 3–31
First-order lag circuit using operational amplifier.

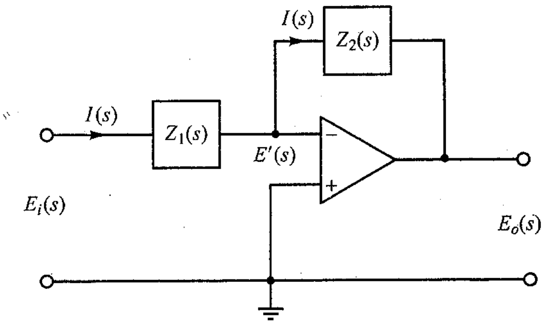


Figure 3–32 Operational amplifier circuit.

Impedance Approach to Obtaining Transfer Functions. Consider the op-amp circuit shown in Figure 3–32. Similar to the case of electrical circuits we discussed earlier, the impedance approach can be applied to op-amp circuits to obtain their transfer functions. For the circuit shown in Figure 3–32, we have

$$\frac{E_i(s) - E'(s)}{Z_1} = \frac{E'(s) - E_o(s)}{Z_2}$$

Since $E'(s) \doteqdot 0$, we have

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} \tag{3–73}$$

EXAMPLE 3–12 Referring to the op-amp circuit shown in Figure 3–31, obtain the transfer function $E_o(s)/E_i(s)$ by use of the impedance approach.

The complex impedances $Z_1(s)$ and $Z_2(s)$ for this circuit are

$$Z_1(s) = R_1 \quad \text{and} \quad Z_2(s) = \frac{1}{Cs + \frac{1}{R_2}} = \frac{R_2}{R_2Cs + 1}$$

The transfer function $E_o(s)/E_i(s)$ is, therefore, obtained as

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}$$

which is, of course, the same as that obtained in Example 3-11.

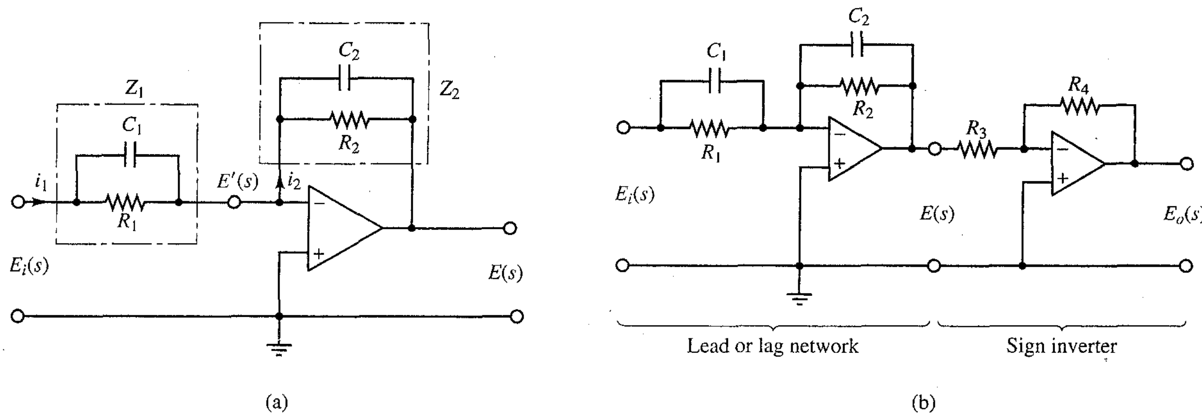


Figure 3–33
(a) Operational-amplifier circuit; (b) operational-amplifier circuit used as a lead or lag compensator.

Lead or Lag Networks Using Operational Amplifiers. Figure 3–33(a) shows an electronic circuit using an operational amplifier. The transfer function for this circuit can be obtained as follows: Define the input impedance and feedback impedance as Z_1 and Z_2, respectively. Then

$$Z_1 = \frac{R_1}{R_1 C_1 s + 1}, \qquad Z_2 = \frac{R_2}{R_2 C_2 s + 1}$$

Hence, referring to Equation (3–73), we have

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1} = -\frac{R_2}{R_1}\frac{R_1 C_1 s + 1}{R_2 C_2 s + 1} = -\frac{C_1}{C_2}\frac{s + \dfrac{1}{R_1 C_1}}{s + \dfrac{1}{R_2 C_2}} \tag{3–74}$$

Notice that the transfer function in Equation (3–74) contains a minus sign. Thus, this circuit is sign inverting. If such a sign inversion is not convenient in the actual application, a sign inverter may be connected to either the input or the output of the circuit of Figure 3–33(a). An example is shown in Figure 3–33(b). The sign inverter has the transfer function of

$$\frac{E_o(s)}{E(s)} = -\frac{R_4}{R_3}$$

The sign inverter has the gain of $-R_4/R_3$. Hence the network shown in Figure 3–33(b) has the following transfer function:

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 R_4}{R_1 R_3}\frac{R_1 C_1 s + 1}{R_2 C_2 s + 1} = \frac{R_4 C_1}{R_3 C_2}\frac{s + \dfrac{1}{R_1 C_1}}{s + \dfrac{1}{R_2 C_2}}$$

$$= K_c \alpha \frac{Ts + 1}{\alpha T s + 1} = K_c \frac{s + \dfrac{1}{T}}{s + \dfrac{1}{\alpha T}} \tag{3–75}$$

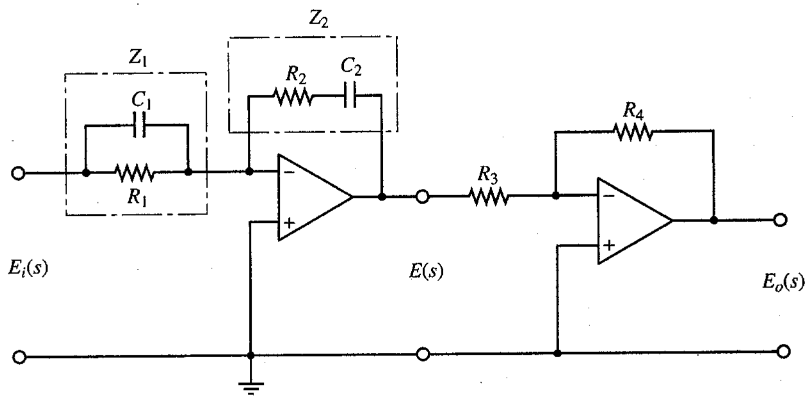


Figure 3–34 Electronic PID controller.

where

$$T = R_1C_1, \qquad \alpha T = R_2C_2, \qquad K_c = \frac{R_4C_1}{R_3C_2}$$

Notice that

$$K_c\alpha = \frac{R_4C_1}{R_3C_2}\frac{R_2C_2}{R_1C_1} = \frac{R_2R_4}{R_1R_3}, \qquad \alpha = \frac{R_2C_2}{R_1C_1}$$

This network has a dc gain of $K_c\alpha = R_2R_4/(R_1R_3)$.

Note that this network is a lead network if $R_1C_1 > R_2C_2$, or $\alpha < 1$. It is a lag network if $R_1C_1 < R_2C_2$.

PID Controller Using Operational Amplifiers. Figure 3–34 shows an electronic proportional-plus-integral-plus-derivative controller (a PID controller) using operational amplifiers. The transfer function $E(s)/E_i(s)$ is given by

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1}$$

where

$$Z_1 = \frac{R_1}{R_1C_1s + 1}, \qquad Z_2 = \frac{R_2C_2s + 1}{C_2s}$$

Thus

$$\frac{E(s)}{E_i(s)} = -\left(\frac{R_2C_2s + 1}{C_2s}\right)\left(\frac{R_1C_1s + 1}{R_1}\right)$$

Noting that

$$\frac{E_o(s)}{E(s)} = -\frac{R_4}{R_3}$$

we have

$$\frac{E_o(s)}{E_i(s)} = \frac{E_o(s)}{E(s)}\frac{E(s)}{E_i(s)} = \frac{R_4R_2}{R_3R_1}\frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_2C_2s}$$

$$= \frac{R_4R_2}{R_3R_1}\left(\frac{R_1C_1 + R_2C_2}{R_2C_2} + \frac{1}{R_2C_2s} + R_1C_1s\right)$$

$$= \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}\left[1 + \frac{1}{(R_1C_1 + R_2C_2)s} + \frac{R_1C_1R_2C_2}{R_1C_1 + R_2C_2}s\right] \quad (3\text{–}76)$$

Notice that the second operational-amplifier circuit acts as a sign inverter as well as a gain adjuster.

When a PID controller is expressed as

$$\frac{E_o(s)}{E_i(s)} = K_p\left(1 + \frac{T_i}{s} + T_d s\right)$$

K_p is called the proportional gain, T_i is called the integral time, and T_d is called the derivative time. From Equation (3–76) we obtain the proportional gain K_p, integral time T_i, and derivative time T_d to be

$$K_p = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}$$

$$T_i = \frac{1}{R_1C_1 + R_2C_2}$$

$$T_d = \frac{R_1C_1R_2C_2}{R_1C_1 + R_2C_2}$$

When a PID controller is expressed as

$$\frac{E_o(s)}{E_i(s)} = K_p + \frac{K_i}{s} + K_d s$$

K_p is called the proportional gain, K_i is called the integral gain, and K_d is called the derivative gain. For this controller

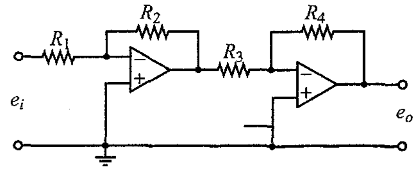
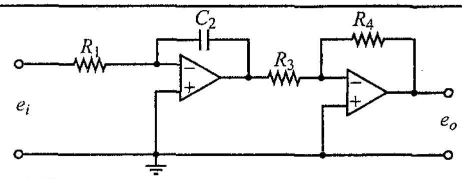
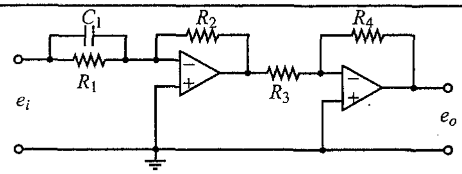
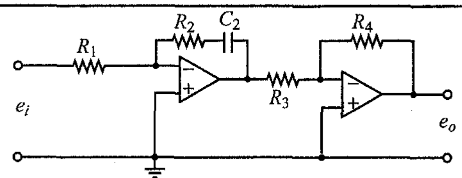
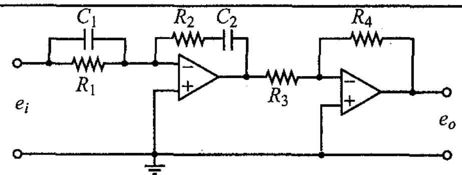
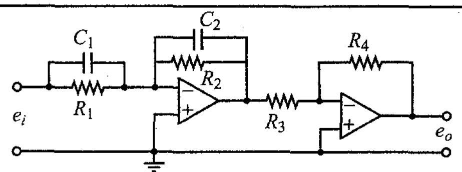
$$K_p = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2}$$

$$K_i = \frac{R_4}{R_3R_1C_2}$$

$$K_d = \frac{R_4R_2C_1}{R_3}$$

Table 3-1 shows a list of operational-amplifier circuits that may be used as controllers or compensators.

Table 3–1 Operational-Amplifier Circuits That May Be Used as Compensators

	Control Action	$G(s) = \frac{E_o(s)}{E_i(s)}$	Operational Amplifier Circuits
1	P	$\frac{R_4}{R_3}\frac{R_2}{R_1}$	
2	I	$\frac{R_4}{R_3}\frac{1}{R_1C_2s}$	
3	PD	$\frac{R_4}{R_3}\frac{R_2}{R_1}(R_1C_1s+1)$	
4	PI	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_2C_2s+1}{R_2C_2s}$	
5	PID	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{(R_1C_1s+1)(R_2C_2s+1)}{R_2C_2s}$	
6	Lead or lag	$\frac{R_4}{R_3}\frac{R_2}{R_1}\frac{R_1C_1s+1}{R_2C_2s+1}$	
7	Lag–lead	$\frac{R_6}{R_5}\frac{R_4}{R_3}\frac{[(R_1+R_3)C_1s+1](R_2C_2s+1)}{(R_1C_1s+1)[(R_2+R_4)C_2s+1]}$