Notes on: SIGNALS & SYSTEMS.

⊛ Differential Equation of Systems:

Mathematical models of most physical systems are characterized by differential equations.

A mathematical model is "Linear", if the differential equation describing it has co-efficients which are either functions only of independent variables or are constants.

If the coefficients are function of time (independent variable) the mathematical model is "Linear-time-varying".

If the coefficients are constant then the model is "Linear time-invariant".

The differential equations describing a LTI system can be reshaped into different forms for convenience for analysis - for example - Transfer function representation for SISO LTI & vector-matrix notation for MIMO LTI systems.

The mathematical model of a system having been obtained the available mathematical tools can then be used for analysis & synthesis of the same.

* Differential equations of physical systems

Presents the method of obtaining differential equation models of physical systems by utilizing the physical laws of the process.

In the first step one should build up the physical model of the system as an interconnection of idealized system elements & describe them in the form of elemental laws.

An ideal element results from two basic assumptions:

(i) spatial distribution of the element is ignored. It is regarded as a point phenomenon. For example, mass, which has physical dimensions, is considered concentrated at a point. Temperature in a room, which is distributed over the entire room is replaced by a representative one as if of a single point in the room.

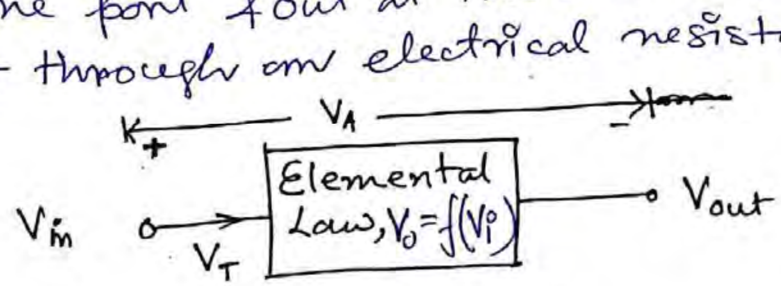
This process is called "lumping" & the model is called lumped-parameter model.

(ii) It is assumed that the variables associated with the elements lie in the range so that the element can be described by simple linear law of (a) a constant of proportionality,
(b) a first-order derivative,
(c) a first-order integration.

(b) & (c) are in fact inter-converted by a simple differentiation or integration.

— To start with let us consider ideal elements which have a single-port or two-terminal representation. So we have two variables associated with it, which are identified as

1. Through Variable (V_T): which sort of passes through the element & so has the same value in at one port & out at the other. For example, current through an electrical resistance.



Ideal Element

2. Across Variable (V_A): which appears across the two terminals of the element. For example, voltage across an electrical resistance.

Another classification of the element variables:

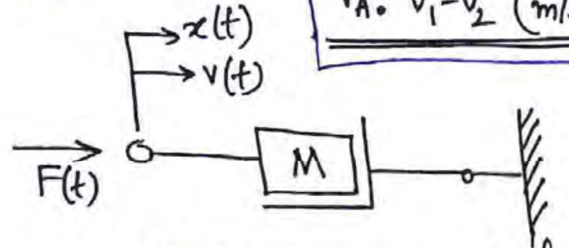
A. Input Variables (V_i) or independent variables.
B. Output Variables (V_o) or dependent (response) ".

V_i could be V_T or V_A & correspondingly V_o would be V_A or V_T.

Mechanical Systems:

Mechanical systems & devices can be modelled by means of 3 (plus gear train) ideal translatory & 3 ideal rotary elements.

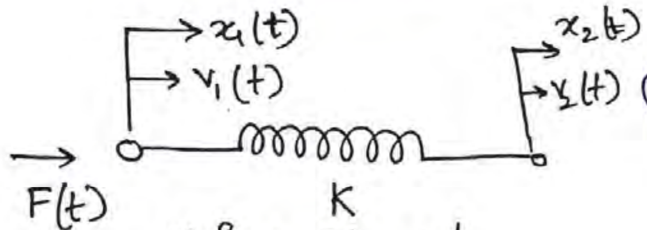
V_T : F (in $N = kg\,m/sec^2$). V_A : $v_1 - v_2$ (m/sec)



Mass Element

(1) Mass Element:

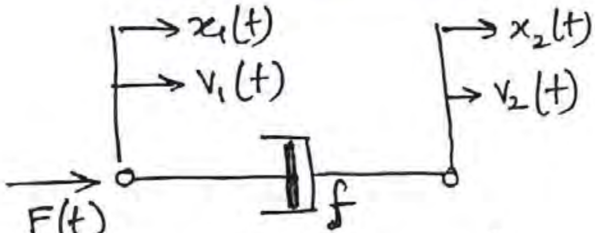
$$F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$$



Spring Element

(2) Spring Element:

$$F = K(x_1 - x_2) = Kx = K\int_{-\infty}^{t}(v_1 - v_2)\,dt = K\int_{-\infty}^{t} v\,dt.$$



Damper Element

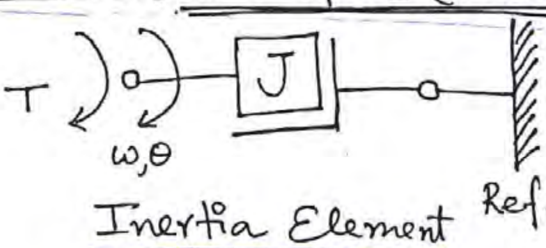
(3) Damper Element:

$$F = f(v_1 - v_2) = f_v = f(x_1 - x_2) = f_x$$

V_T : T (in N-m), V_A : $\omega_1 - \omega_2$ (in rad/s)

(1) Inertia Element:

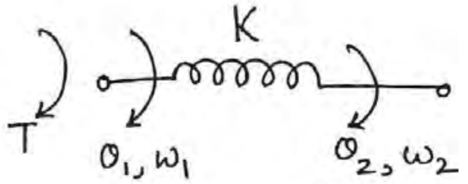
$$T = J\cdot\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2}$$



Inertia Element

(2) Torsional Spring Element:

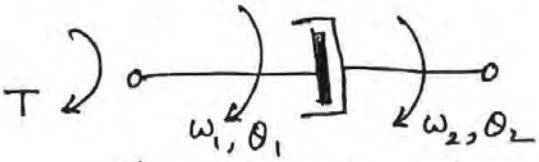
$$T = K(\theta_1 - \theta_2) = K\theta = K\int_{-\infty}^{t}(\omega_1 - \omega_2)\,dt = K\int_{-\infty}^{t}\omega\,dt$$



Torsional Spring Element

(3) Damper Element:

$$T = f(\omega_1 - \omega_2) = f_\omega = f(\theta_1 - \theta_2) = f_\theta$$



Damper Element

- Mass/Inertia & the two kinds of springs are the energy storage elements, where energy can be stored & retrieved without loss — these are called "Conservative elements"

The energy stored in Mass: $E = \left(\frac{1}{2}\right) M v^2$ = Kinetic energy (J) (motional energy)

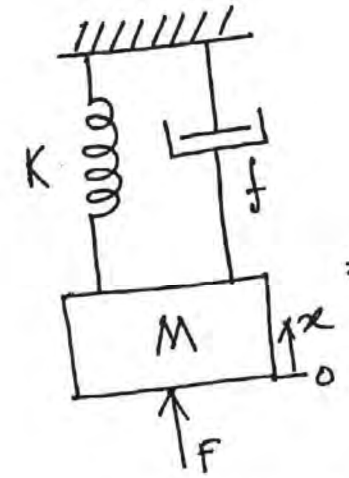
" " " " Inertia: $E = \frac{1}{2} J \omega^2$ = Kinetic (/motional) energy.

" " " " Spring (translatory): $E = \frac{1}{2} k x^2$ = potential (/deformation) energy.

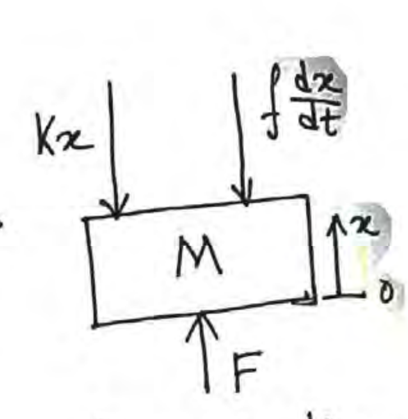
" " " " Spring (torsional): $E = \frac{1}{2} k \theta^2$ = potential (/deformation) energy.

- Damper is a dissipative element & power it consumes (lost in the form of heat) is given as: $P = f v^2 (= f \omega^2)$ (watt)

⊛Translational System:



mass-spring-dashpot system.



free-body diagram

A mass (M) attached to a spring (stiffness K) & a dashpot (viscous friction coefficient of f) on which the force F acts. Displacement (x) is +ve in the direction shown. The zero position is taken to be at the point where spring & mass are in static equilibrium (note that the gravitational effect is eliminated by this choice of zero position).

Now, applying Newton's law of motion to the free-body diagram the force equation can be written as,

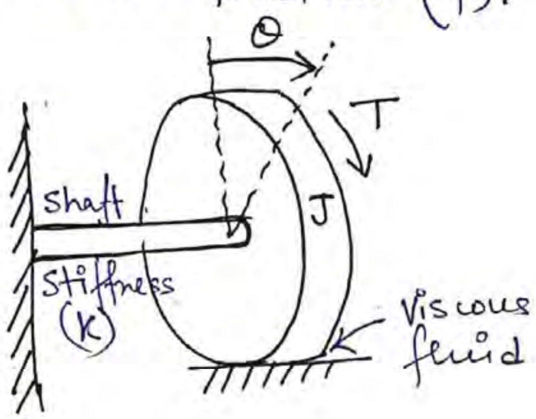
$$F - f\frac{dx}{dt} - Kx = M\frac{d^2x}{dt^2}$$

$$\Rightarrow F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx.$$

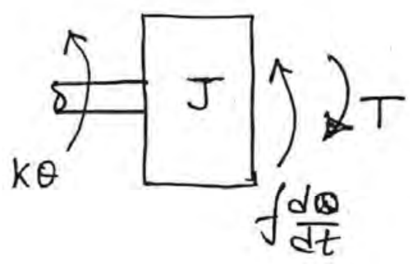
* It is a <u>linear</u>, <u>constant co-efficient</u> differential equation of <u>2^{nd} order</u>.
* Note that the system has 2 storage elements, the mass M & spring K.

⊛ Rotational Systems: Mechanical systems involving fixed-axis rotation occur in study of rotating machinery.

- The modelling is very close to that used in translation.
- The variables of interest are: Torque (T) & angular velocity (or displacement)
- Three basic components: moment of inertia (J), torsional spring (K) & viscous friction (f).
- Consider the rotational mechanical system consisting of a rotatable disc of moment of inertia J & a shaft of stiffness K.



- The disc rotates in a viscous medium with viscous friction co-efficient f.
- T be the torque applied, which tends to rotate the disc.



- The torque equation from the free-body diagram

$$T - f\frac{d\theta}{dt} - K\theta = J\frac{d^2\theta}{dt^2}$$

$$\text{or, } T = J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} + K\theta.$$

↳ linear, constant co-efficient differential equation describing the dynamics of the system.

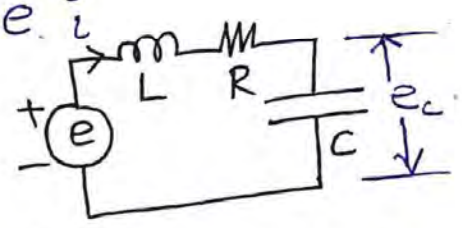
- It has 2 storage elements — inertia J & shaft of stiffness K.

Electrical Systems:

- The basic elements are R, L & C.
- Analysis is done based on KCL & KVL.

R-L-C series circuit:

Applying KVL: $L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt = e$



let, $\frac{1}{C}\int_{-\infty}^{t} i\,dt = e_o$ & $e_o = e_c$

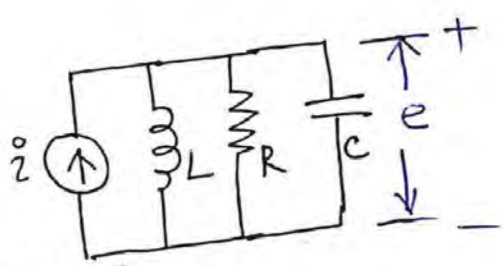
- Inductor & Capacitor — storage (conservative) element
- Resistor — dissipative element.
- In terms of electric charge, $q = \int i\,dt$,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e$$ 2nd order differential equation

R-L-C parallel circuit:

Applying KCL:

$$C\frac{de}{dt} + \frac{1}{L}\int_{-\infty}^{t} e\,dt + \frac{e}{R} = i$$



- In terms of flux linkage, $\phi = \int e\,dt$

$$C\frac{d^2\phi}{dt^2} + \frac{1}{R}\cdot\frac{d\phi}{dt} + \frac{1}{L}\phi = i$$ 2nd order differential equation.

Analogous System:

Analogous systems are those whose differential equations are identical.

- The concept of analogous system is useful technique for study of various systems — mechanical, electrical, thermal, liquid-level, etc..
- If the solution is known for one system, then the same can be extended to its analogous ones.
- It is convenient to study electrical analogous systems since they are easily amenable to experimental study.

Analogous quantities in FORCE (or TORQUE) - VOLTAGE A[nalogy]:

Mechanical System: Translational	Mechanical System: Rotational	Electrical System
Force (F)	Torque (T)	Voltage (e)
Mass (M)	Moment of Inertia (J)	Inductance (L)
Viscous friction Co-efficient (f)	Viscous friction co-efficient (f)	Resistance (R)
Spring stiffness (K)	Torsional Spring Stiffness (K)	Reciprocal of capacitance $\left(\frac{1}{C}\right)$
Displacement (x)	Angular Displacement (θ)	charge (q)
Velocity ($\dot{x}$)	Angular Velocity ($\dot{\theta}=\omega$)	Current (i)

Analogous quantities in FORCE (TORQUE) - CURRENT Analogy:

Mechanical System: Translational	Mechanical System: Rotational	Electrical System
F	T	Current (i)
M	J	Capacitor (C)
f	f	$1/R$
K	K	$1/L$
x	θ	Magnetic flux linkage (ϕ)
$\dot{x}(v)$	$\dot{\theta}(\omega)$	voltage (v).

State Equations: In general an n-th order differential equation can be decomposed into n first-order ones.

- Since first order differential equations are easier to solve, they are used in the analytical studies.
- Let us consider series R-L-C circuit represented by differential equation: $R\,i(t) + L\dfrac{di(t)}{dt} + \dfrac{1}{C}\int i(t)\,dt = e(t)$.
- let us assume, $x_1(t) = \int i(t)\,dt$ & $x_2(t) = \dfrac{dx_1(t)}{dt} = i(t)$.
- Therefore, one obtains two first-order differential equati-ons:

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\& \quad \frac{dx_2(t)}{dt} = -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{L}e(t)$$

- In general, the differential equation of an n-th order system given as

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = f(t)$$

can be decomposed into n- first order differential equations as:

$$\frac{dx_1(t)}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = x_3(t)$$

$$\vdots$$

$$\frac{dx_n(t)}{dt} = -a_0x_1(t) - a_1x_2(t) - \cdots - a_{n-2}x_{n-1}(t) - a_{n-1}x_n(t) + f(t)$$

with, $x_1(t) = y(t)$; $x_2(t) = \dfrac{dy}{dt}$; $\cdots$, $x_n(t) = \dfrac{d^{n-1}y(t)}{dt^{n-1}}$.

- This set of first order differential equations is called the State Equations & $x_1, x_2, \cdots, x_n$ are called the State Variables.

Example: R-L-C series circuit

- Differential equation: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e$

Let, $x_1(t) = q(t)$ & $x_2(t) = \dot{x}_1(t) = \dot{q}(t) = i(t)$

$\therefore \quad \dot{x}_1 = x_2; \quad \dot{x}_2 = -\frac{R}{L}x_2(t) - \frac{1}{LC}x_1(t) + \frac{e}{L}$.

& output equation: $i(t) = x_2(t)$.

$\therefore$ In a vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} e$$

& $$i(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$\Rightarrow \left.\begin{aligned} \dot{X} &= AX + Bu \\ \& \quad Y &= CX + Du \end{aligned}\right\} \text{State-variable representation}$$

- State Variables: A minimal set of variables such that the knowledge of them at any time t_0 & the information on the input at time t_0 will be sufficient to determine the state of the system at any time $t > t_0$.
- The value of the state-variables at any initial time t_0 defines the initial states of the system.
- n^{th} order system —— n-state variables (n-states).