The Erdős-Gyárfás problem and color energy graphs

Bob Krueger
rak5@illinois.edu

Joint work with József Balogh, Sean English, Emily Heath

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Erdős-Gyárfás Problem

Definition

A (p, q)-*coloring* of K_n is an edge-coloring where every p vertices span at least q colors. Let

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$f(n, p, 2)$ is equivalent to multi-color Ramsey problem for K_p:

$$f(5, 3, 2) = 2, f(6, 3, 2) = 3 \Leftrightarrow R(3, 3) = 6$$

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Aim: asymptotics in n, think of p and q as constants.
Note: $f(n,p,q)$ increasing in q.

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Note: $f(n,p,q)$ increasing in q.

Ex: $f\left(n,p,\binom{p}{2}-p+3\right) = \Omega(n)$.
Proof: each vertex meets less than $p-1$ edges of the same color.

Upper Bounds

Local lemma bound (Erdős-Gyárfás '97):

$$f(n,p,q) = O\left(n^{\frac{p-2}{\binom{p}{2}-q+1}}\right).$$

Thus

$$f\left(n,p,\binom{p}{2}-p+3\right) = \Theta(n)$$

$$f\left(n,p,\binom{p}{2}-p+2\right) = o(n)$$

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Mubayi '98:	$f(n,4,3) = n^{o(1)}$ (explicit)
Eichhorn, Mubayi '00:	$f(n,5,4) = n^{o(1)}$
Mubayi '04:	$f(n,4,4) = n^{1/2+o(1)}$ (+ explicit algebraic coloring)
Conlon, Fox, Lee, Sudakov '15:	$f(n,p,p-1) = n^{o(1)}$
Heath, Cameron '18:	$f(n,5,5) = n^{1/3+o(1)}$
Heath, Cameron '22:	$f(n,6,6), f(n,8,8)$ improvements
Bennett, Cushman, Dudek, Prałat '22+:	$f(n,4,5) = \frac{5}{6}n + o(n)$

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Question (Erdős, Gyárfás '97)

Is $f(n,5,9)$ linear in n? (Axenovich '00: $n^{1+o(1)}$ with Behrend construction.)

Lower Bounds

Recall

$$f\left(n, p, \binom{p}{2} - p + 3\right) = \Theta(n)$$

$$f\left(n, p, \binom{p}{2} - p + 2\right) = o(n)$$

Question (Erdős-Gyárfás '97)

Is $f(n, p, \binom{p}{2} - p + 4) = O(n)$*?*

Theorem (Sárközy, Selkow '00)

$f\left(n, p, \binom{p}{2} - p + \log_2 p + 3\right) = \Omega\left(n^{1+\frac{1}{2p-1}}\right)$.

Lower Bounds

Our range of interest: $q = \binom{p}{2} - \beta p$. Local Lemma:

$$f\left(n, p, \binom{p}{2} - \beta p\right) = O\left(n^{\frac{p-2}{\beta p+1}}\right).$$

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Theorem (Pohoata, Sheffer '19)

For $k, m \in \mathbb{Z}^+$,

$$f\left(n, k(m+1), \binom{k(m+1)}{2} - m(k+1) + 1\right) = \Omega\left(n^{\frac{m+1}{m}}\right).$$

Proof via color energy: given $c : E(K_n) \to C$, the color energy is

$$\left|\{(w, x, y, z) : c(wx) = c(yz)\}\right|.$$

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Theorem (Pohoata, Sheffer, Fish '20)

$$f\left(n, 4k, \binom{4k}{2} - 2k + 1\right) = \Omega\left(n^{2-2/k}\right)$$

$$f\left(n, 3\cdot 8, \binom{3\cdot 8}{2} - 2\cdot 8 + 1\right) = \Omega\left(n^{\frac{3}{4}\cdot\frac{3}{2}}\right)$$

via color energy graphs.

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"It seems that similar new bounds could be obtained using the same methods. . . so far we were not able to derive a family of such bounds each bound requires a separate technical proof. We thus leave the further exploration of this technique for future works."

Lower Bounds

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$$f\left(n, k(m+1), \binom{k(m+1)}{2} - m(k+1) + 1\right) = \Omega\left(n^{\frac{m+1}{m}}\right) \leftarrow F = K_{1,k}$$

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$$f\left(n, 4k, \binom{4k}{2} - 2k + 1\right) = \Omega\left(n^{2-2/k}\right) \quad \leftarrow F = C_{2k}, r = 2$$

$$f\left(n, 3\cdot 8, \binom{3\cdot 8}{2} - 2\cdot 8 + 1\right) = \Omega\left(n^{\frac{3}{4}\cdot\frac{3}{2}}\right) \quad \leftarrow F = C_8, r = 3$$

General Framework: try to leverage $\mathrm{ex}(n, F) = O\left(n^{2-\alpha}\right)$ into

$$f\left(n, r|V(F)|, \binom{r|V(F)|}{2} - (r-1)|E(F)| + 1\right) = \Omega(n^{\alpha\frac{r}{r-1}})$$

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Theorem (Balogh, English, Heath, K. '22)

The above holds for all r when F is a 1-subdivision of K_t, $\Theta(a, b)$, or subdivisions of $K_{a,b}$ (for some r).

O. Janzer independently proved this for even cycles.

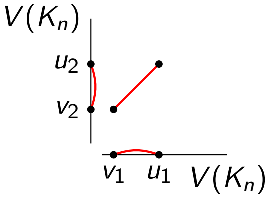
Color Energy

Fix $c : V(K_n) \to C$.

Definition (Pohoata, Sheffer, Fish '20)

The *color energy graph* $\vec{G}$ on $\vec{V} = V(K_n) \times V(K_n)$ is

$$\vec{E} = \{(v_1, v_2)(u_1, u_2) : c(v_1 u_1) = c(v_2 u_2)\}.$$



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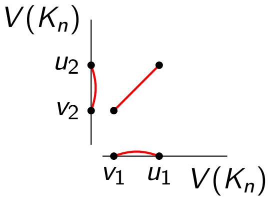

Let $m_i = |\{(w, x) : c(wx) = i\}|$. By Cauchy-Schwarz,

$$|\vec{E}| = \sum_{i \in C} m_i^2 \geq \frac{1}{|C|} \left(\sum_{i \in C} m_i \right)^2 = \frac{1}{|C|} \binom{n}{2}^2.$$

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The r^{th} *color energy graph* $\vec{G}$ on $\vec{V} = V(K_n) \times \cdots \times V(K_n)$ is

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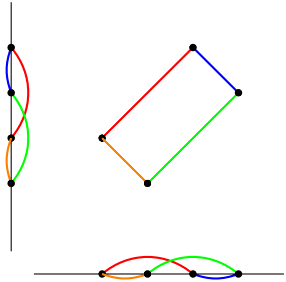
Project to coordinates to get r color-isomorphic copies of F:



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Proof Idea: Find F in $\vec{G}$ by $\mathrm{ex}(n, F) = O\left(n^{2-\alpha}\right)$:

$$|\vec{E}| \geq \frac{1}{|C|^{r-1}} \binom{n}{2}^r = \Omega\left(n^{2r-\alpha r}\right).$$

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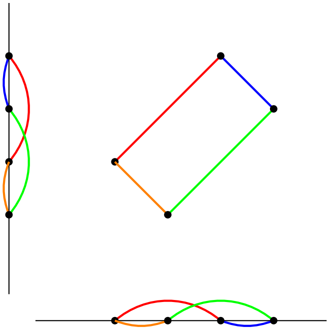
Dealing with Homomorphisms

Problem: projections not isomorphic to F, only homomorphic, and not necessarily disjoint.



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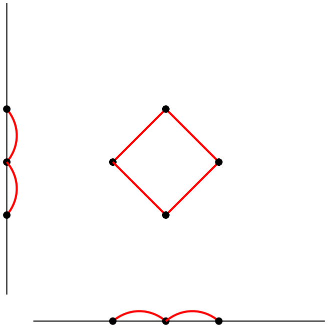

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- Randomly partition V_i into V_i', V_i'' and define $\vec{G}$ as bipartite between $V_1' \times \cdots \times V_r'$ and $V_1'' \times \cdots \times V_r''$: **homomorphic copies are bipartite**.

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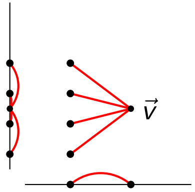
- Prune so P_3-preserving homomorphisms: $\vec{u}\vec{v}, \vec{v}\vec{w} \in \vec{E}$, then $u_i \neq w_i$ for all $i \in [r]$. (Can do if number of colors is superlinear.)



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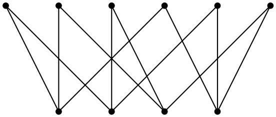

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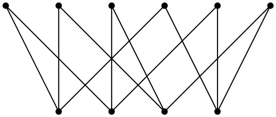

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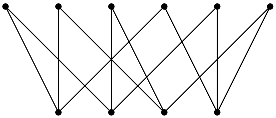

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Side note: O. Janzer's approach to homomorphisms: instead of making sure homomorphisms are nice enough, just count number of non-isomorphism homomorphisms.

Conlon-Tyomkyn Problem

Let $f_r(n, F)$ be the minimum number of colors in a **proper** edge-coloring of K_n which does not contain r disjoint color-isomorphic copies of F.

Question

Is $f_2(n, C_4) = O(n)$*?*

Theorem (Xu, Zhang, Jing, Ge '20)

$f_3(n, C_4) = O(n)$.

Erdős-Gyárfás for other graphs

Let $f(n, H, q)$ be the minimum number of colors in an edge-coloring of K_n where every copy of H receives at least q colors. Let P_v be a path on v vertices.

Theorem (K '20)

$f(n, P_v, q)$ is either linear in n or quadratic in n, unless v is odd and $q = \frac{v+1}{2}$.

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What is $f(n, P_7, 4)$? It's somewhere between $\tilde{\Omega}(n^{4/3})$ and $O(n^{5/3})$.

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Thanks for Listening