Solution to a Nonzero Matrix

Benny Chen

April 19, 2023

1 Introduction

In this paper, we will be proving that a linear system of equations with real coefficients has a unique solution if and only if the determinant of its associated matrix A is nonzero. We will do this by proving the following theorem:

If the determinant of a matrix A is nonzero, then the system of equations $Ax = b$ has a unique solution.

2 Proof

Theorem: Let A be a $n \times n$ matrix with nonzero determinant. Then, the system of equations $Ax = b$ has a unique solution.

Let

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix} \tag{1}$$

be a system of equations with real coefficients.

Let A be a $n \times n$ matrix with nonzero determinant.

Let x be a vector in $\mathbb{R}^n$ such that $Ax = b$. We will prove that x is the unique solution to the system of equations $Ax = b$.

We find if $det(A)$ is invertible.

Suppose that M is invertible and $det(M) = 0$

Then there exists a matrix B such that $B = M^{-1}$ Which means that:

$$MB = I \tag{2}$$

Where I is the identity matrix. We can then solve for:

$$det(MB) = det(I) \tag{3}$$

Due to the fact that $det(I) = 1$, we can then solve for $det(MB)$:

$$det(B)0 = 0 \tag{4}$$

We can now see a that a contradiction has been made

$$0 = 1 \tag{5}$$

Therefore a invertible matrix cannot have an determinant of 0.

Since we know that A must be invertible now, we solve for $Ax = b$ by left multipling both sides by A^{-1} to get:

$$A^{-1}Ax = A^{-1}b \tag{6}$$

This means that:

$$x = A^{-1}b \tag{7}$$

We can now the original equation $Ax = b$ to get:

$$A(A^{-1}b) = b \tag{8}$$

With $AA^{-1} = I$, we can then solve for b:

$$I * b = b \tag{9}$$

So,

$$b = b \tag{10}$$

So, $x = A^{-1}b$ is a solution to the system of equations $Ax = b$.

We can now prove that x is the unique solution to the system of equations $Ax = b$.

Let x be a solution to the system of equations. Then,

$$Ax = b \tag{11}$$

So then,

$$A^{-1}Ax = A^{-1}b \tag{12}$$

Which shows that:

$$x = A^{-1}b \tag{13}$$

Therefor, this shows that whenever there is a solution to the system of equations, then x is the only unique solution to the system of equations.