

$$u\text{-sub: } \int f(g(x))g'(x) dx = \int f(u) du$$

$u = g(x)$

$$\int_{B(0;R)} x^2 + y^2 d^2 \mu \text{ how to integrate this?}$$

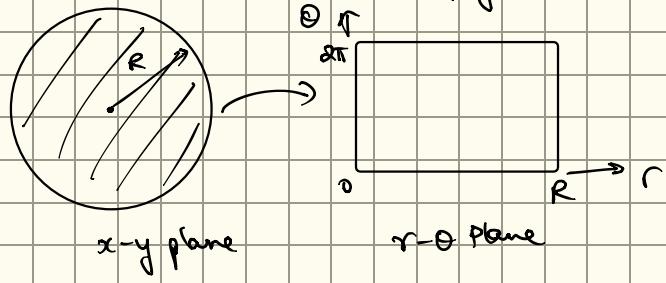
Can we deform $\overline{B(0;R)}$ into a rectangle? Almost, we can do continuously but not smoothly.

The singularities are measure 0, so they don't really matter for the integral.

So $F: V \rightarrow U$ is called a diffeomorphism if F is C^∞ and bijective so that F^{-1} is also C^∞ .
 (For what we will do, we allow F to be non-diffeomorphic up to a set of measure 0)

Example: $x = r \cos \theta$ where $r \in [0, \infty)$ so this fails only on $\{x > 0\} \subset \mathbb{R}^2$
 $y = r \sin \theta$ $\theta \in [0, 2\pi]$

So a ball is a rectangle in polar coordinates. $\overline{B(0;R)}$ we have $(x,y) \in B(0;R) \Leftrightarrow x^2 + y^2 \leq R^2$
 so we have that $x = r \cos \theta, y = r \sin \theta$ where $r \in [0, R]$ and $\theta \in [0, 2\pi]$



\mathbb{R}^n
U

Theorem (Change of Variables): If we have a function $f: U \rightarrow \mathbb{R}$ and we compute $\int_U f d^n \mu$
 and we have a diffeomorphism $F: V \rightarrow U$ then $\int_U f d^n \mu = \int_V (f \circ F) |\det DF| d^n \mu$
 (up to measure 0 sets)

Thm/

Def: Suppose X is a topological manifold and $\{U_\alpha\}$ is an open cover of X st.
 For each $x \in X$ there is only a finite no. of open sets U_α st. $x \in U_\alpha$. Then there is a
 collection $\{p_\alpha: U_\alpha \rightarrow \mathbb{R}\}$ of continuous maps st

(i) For each $x \in X$ we have $0 < p_\alpha(x) \leq 1$

(ii) $\forall x \in X$ we have $\sum_\alpha p_\alpha(x) = 1$

(iii) Each p_α vanishes outside an open set containing U_α .

Then $\{U_\alpha, p_\alpha\}$ is called a partition of unity.

$\int_{B(0;R)} x^2 + y^2 \, d^2\mu$ how to integrate this?

Step 1: Find diffeo from a simpler space into $B(O, R)$

$$F: [0, R] \times [0, 2\pi] \rightarrow B(0; R)$$

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta)$$

$$\text{Step 2: Find } DF : DF(r, \theta) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

$$\text{Step 3: Find } |\det D\mathbf{F}| = |r \cos^2 \theta + r \sin^2 \theta| = r$$

$$\text{Step 4: } \int_{B(0;R)} x^2 + y^2 \, d^2\mu = \int_{[0,R] \times [0,2\pi]} r^2 \, r \, dr \, d\theta = \int_{[0,R]} \left(\int_{[0,2\pi]} r^3 \, d\theta \right) dr = 2\pi \int_{[0,R]} r^3 \, dr = \frac{\pi R^4}{2}$$

We can compute area of circle : $\int_{B(0,R)} 1 d^2\mu = \pi R^2$

Consider the Gaussian integral : $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (If you want to try on your own check problem sheet 3, if not check meeting plan)

In \mathbb{R}^3 we have spherical coordinates: $x = r \cos \varphi \cos \theta$ $\Rightarrow DF = r^2 \sin \theta$
 $y = r \sin \varphi \cos \theta$
 $z = r \sin \theta$ $r \in [0, \infty), \varphi \in [0, \pi], \theta \in [0, \pi]$

so we can compute volume of sphere by

$$\int_{B(0;R)} 1 d^3\mu = \int_0^R \int_0^{2\pi} \int_0^\pi r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \frac{4}{3}\pi R^3$$

Higher dimensional spherical coordinates: $r, \theta_1, \theta_2, \dots, \theta_{n-1}$ where $r \in [0, \infty)$
 $\theta_i \in [0, 2\pi]$ for $i \neq n-1$
 $\theta_{n-1} \in [0, \pi]$

$$x_1 = r \cos \theta, \quad \underline{y_1}$$

$$x_p = \sum_{n=1}^N r \cos \theta_p \sum_{m=1}^M \sin \theta_m$$

$$x_n = r \prod_{m=1}^n \sin \theta_m$$

Def: (i) If $u: \mathbb{R}^n \rightarrow \mathbb{R}$ then $\nabla u = Du = \text{grad } u$

(ii) If $u: \mathbb{R}^m \rightarrow \mathbb{R}^n$ then $\nabla \cdot u = \text{tr}(\nabla u) = \text{div } u$

(iii) If $u: \mathbb{R}^n \rightarrow \mathbb{R}$ then $\Delta u = \nabla^2 u = \operatorname{div}(\operatorname{grad} u) = \operatorname{tr}(D(\operatorname{grad} u)) = \operatorname{tr}(D^2 u)$

line integrals: $\int_C \vec{F} \cdot d\vec{r} = \int_C \sum_i F^i dx^i$ these look "1-dimensional" so is there a FTC type theorem? The answer is yes!! So suppose $F: \mathbb{R}^n \rightarrow \mathbb{R} \Rightarrow \nabla F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ so then

$$\int_C \nabla F \cdot d\vec{r} = F(\text{end point}) - F(\text{start point})$$

1 dimensional integral

"0 dimensional integral"

if C is closed loop then $\int_C \nabla F \cdot d\vec{r} = 0$

if $\int_C \vec{F} \cdot d\vec{r} = 0$ is $\vec{F} = \nabla g$? we will answer this using deRham Cohomology.

then this is true

simply, piecewise smooth, closed

So suppose we have $\int_C P dx + Q dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$ where $C = \partial D$.

This is called green's theorem.

let us say we have a 2 dimensional surface A in \mathbb{R}^3 . We locally parametrize A by $r(s,t) = (x(s,t), y(s,t), z(s,t))$, $(s,t) \in D$

the surface area is given $\int_D \left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right| dA$

let \vec{F} and \vec{v} be vector (vector fields) then $\vec{F} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_x & F_y & F_z \\ v_x & v_y & v_z \end{vmatrix}$

If $f: A \rightarrow \mathbb{R}$ then $\iint_A f dA = \iint_D f(r(s,t)) \left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right| d^2s dt$

Note: $\left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right| = |\det((Dr)^T(Dr))|$

The normal of a surface is defined by $\hat{n} = \frac{\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}}{\left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right|}$

If \vec{F} is a vector field then $\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n} dS$

If $\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ then $\text{curl}(F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r} \quad \leftarrow \text{Stokes' theorem}$$

↑
1d integral
2d integral

$$\iiint_V \text{div } F \, d^3v = \iint_{\partial V} \vec{F} \cdot d\vec{S} \quad \leftarrow \text{Divergence theorem}$$

↑
3d integral 2d integral

Generalized Stokes' theorem: $\int_M d\omega = \int_{\partial M} \omega$