$f: \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}$ given by $f(x) = |x|^2$ is smooth and is a submersion this means $\forall r \in \mathbb{R}_{>0}$ we have $S^n(r) = f^{-1}(r)$ is a smooth submanifold.

$T_p S^n(r) = \ker(df_p)$

$$\overset{\mathbb{R}^{n+1}}{\overset{\text{S||}}{}}$$

$df_p : \mathbb{R}^{n+1} \to \mathbb{R}$ note $df_p = [2p^1 \; \cdots \; 2p^{n+1}] \Rightarrow v \in T_p S^n(r) \Leftrightarrow df_p(v) = 0 \Leftrightarrow 2p\cdot v = 0 \Leftrightarrow p\cdot v = 0$

Now let $A \in O(p,q) \Leftrightarrow A^T g A = g$. Note $O(p,q) = f^{-1}(g)$ where $f(A) = A^T g A$ we showed that $df_A(X) = A^T g X + X^T g A$ (took the curve $\gamma(t) = A + tX$ note $\gamma(0) = A$, $\gamma'(0) = X$)

This means $T_A O(p,q) = \ker(df_A) = \{X \in \mathbb{R}^{n^2} \cong \mathrm{Mat}(n,\mathbb{R}) \mid A^T g X + X^T g A = 0\}$

Now let us compute $T_{Id} SL(n,\mathbb{R}) = \mathfrak{sl}(n)$. From the exercises note that $D(\det)(Id)(A) = tr(A)$
so $T_{Id} SL(n,\mathbb{R}) = \{X \in \mathbb{R}^{n^2} \mid tr(X) = 0\}$

$L_B : SL(n,\mathbb{R}) \to SL(n,\mathbb{R})$ is a diffeomorphism this means $T_B SL(n,\mathbb{R}) \cong dL_B(T_{Id} SL(n,\mathbb{R}))$
$\quad\quad A \mapsto BA$

$L_{B^{-1}} \circ L_B = L_{B^{-1}} \circ L_B = Id_{SL(n,\mathbb{R})}$

Let $\gamma : I \to SL(n,\mathbb{R})$ be a curve s.t. $\gamma(0) = Id$ and $\gamma'(0) = X \Leftrightarrow X = \frac{d}{dt}\Big|_{t=0} \gamma(t)$ then consider $\psi : I \to SL(n,\mathbb{R})$ given by $\psi(t) = B\gamma(t)$ then $\psi(0) = B$ and $\psi'(0) = \frac{d}{dt}\Big|_{t=0} B\gamma(t) = B\gamma'(0) = BX$

So this show that dL_B can be thought of as $L_B \Rightarrow T_B SL(n,\mathbb{R}) = \{BX \mid tr(X) = 0\}$

Exercise: Show that $T_B SL(n,\mathbb{R}) \cong \{XB \mid tr(X) = 0\}$.

Let us start with the cotangent bundle. So its fibres at each point will be the $T_p^* M = (T_p M)^*$.

Suppose $f: M \to \mathbb{R}$ smooth then $df_p : T_p M \to T_{f(p)}\mathbb{R} \cong \mathbb{R} \Rightarrow df_p \in T_p^* M$

If we have some configuration space M then we care about T^*M since it is the phase space and hence we can talk about Hamiltonian dynamics.

Start with a chart (U,φ) around $p \in M$. Then we have a basis for T_pM given by $\frac{\partial}{\partial x^i}\Big|_p = d\varphi^{-1}_{\varphi(p)}\left(\frac{\partial}{\partial y^i}\Big|_{\varphi(p)}\right)$ where $\frac{\partial}{\partial y^i}\Big|_{\varphi(p)}$ is the standard basis for $T_{\varphi(p)}\mathbb{R}^m$. Now we define the dual vectors $dx^i_p \in T_p^* M$ s.t. $dx^i_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \delta^i_j$, $dx^i_p = d(\pi^i \circ \varphi)_p$. Check this !!

Now let (V,ψ) be another chart around $p \in M$. With basis tangent vectors $\left.\frac{\partial}{\partial \tilde{x}^i}\right|_p = d\psi^{-1}_{\psi(p)}\left(\left.\frac{\partial}{\partial y^i}\right|_{\psi(p)}\right)$

$$\left.\frac{\partial}{\partial x^i}\right|_p = d\varphi^{-1}_{\varphi(p)}\left(\left.\frac{\partial}{\partial y^i}\right|_{\varphi(p)}\right)$$

$$= d(\psi^{-1}\circ\psi\circ\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial y^i}\right|_{\varphi(p)}\right)$$

$$= d\psi^{-1}_{\psi(p)}\left(d(\psi\circ\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial y^i}\right|_{\varphi(p)}\right)\right)$$

$$= \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p))\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p$$

Exercise: Fill in the steps.

This tells us that $\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j} \Leftrightarrow \frac{\partial}{\partial \tilde{x}^j} = \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial}{\partial x^i}$

let $\omega_p = \omega_i\, dx^i_p = \tilde{\omega}_i\, d\tilde{x}^i_p$ note $\omega_i = \omega_j\, dx^j_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \tilde{\omega}_j\, d\tilde{x}^j_p\left(\left.\frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial}{\partial \tilde{x}^k}\right|_p\right)$

$$= \tilde{\omega}_j\left(\frac{\partial \tilde{x}^k}{\partial x^i}\right)\delta^j{}_k$$

$$= \tilde{\omega}_j\,\frac{\partial \tilde{x}^j}{\partial x^i}$$

this means $dx^i_p = \frac{\partial x^i}{\partial \tilde{x}^j}\, d\tilde{x}^j$

Defining the cotangent Bundle:

Let $(U_\alpha, \varphi_\alpha)$ be an atlas on M. Then let $V_\alpha = \varphi_\alpha^{-1}(U_\alpha)$ and $\psi_{\beta\alpha} = \varphi_\alpha^{-1}(U_\alpha \cap \varphi_\beta)$

then the cocycle defining TM was $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(m,\mathbb{R})$ given by $g_{\beta\alpha}(x) = D\psi_{\beta\alpha}\circ\varphi_\alpha$.

We look for a representation $\rho : GL(n,\mathbb{R}) \to GL(l,\mathbb{R})$ then we cocycles $\rho\circ g_{\beta\alpha} = t_{\beta\alpha}$

Exercise: Check $t_{\beta\alpha}$ satisfies the cocycle condition.

So for the cotangent bundle $\rho(A) = (A^*)^{-1}$. Then doing a similar construction as TM we define T^*M.

In any chart (U, φ) we can write $\omega \in T^*M$ as $\omega_i dx^i_p$ where $\omega_i \in \mathbb{R}$ and dx^i_p are the duals of $\left.\frac{\partial}{\partial x^i}\right|_p$.

A vector field: the idea is that to each point associate a tangent vector.

Def: A section σ of a surjection $\pi: E \to M$ is a right inverse $\pi \circ \sigma = id_M$

Then vector fields are sections of TM.

A vector field $X: M \to TM$ is smooth if $\forall f \in C^\infty(M)$ $Xf: M \to \mathbb{R}$ is smooth.
$(Xf)(p) = X_p f$. We denote the space of smooth vector fields on M by $\mathfrak{X}(M)$

\mathfrak{X}

In general if we have a smooth vector bundle $\pi: E \to M$ then its space of sections is denoted by $\Gamma(E)$

$\mathfrak{X}(M) = \Gamma(TM)$

Smooth sections $\Gamma(T^*M) = \Omega^1(M)$ are called differential 1-forms

$M \times \mathbb{R} \to M$ is a vector bundle. Hence the space of smooth sections is

$$\begin{array}{l}\Gamma(M \times \mathbb{R}) = C^\infty(M) \\ \quad\; \| \\ \quad \Omega^0(M)\end{array}$$

Note: $\omega \in \Omega^1(M) \iff \forall X \in \mathfrak{X}(M)$ we have $\omega(X)$ defined by $\omega(X)(p) = \omega_p(X_p)$ is smooth