Toric Code: A Foray Into Topological Quantum

From Topological Matter to Topological Computing

Bosons and Fermions

Fermions are particles which obey the Pauli Exclusion Principle: no two particles may occupy the same state.

We find that fermionic particles such as electrons, protons, and neutrons have exclusively half-integer spin! Examples include 3/2, 5/2,...but not 2/2, or 4/2.

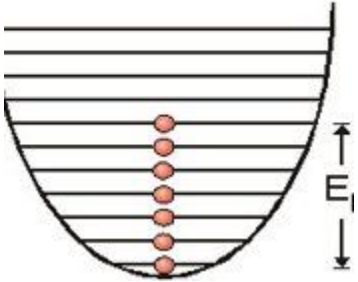
This is in stark contrast to the behaviour of Bosons, which have whole integer spins like 0, 1, 2... This class of particles include the mesons, photons, and gluons.

Besides these differences in spin, the statistics each corresponding class of particles follows also differs. That is, they have different 'quantum rules'.

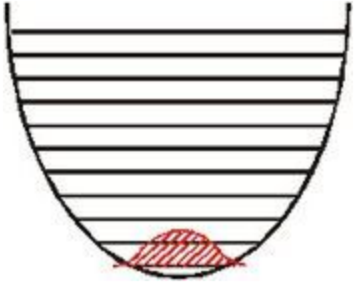
Relations Between Spins & Statistics

The most profound observation of these so called differing properties occurs when we cool our system of particles down to reach the quantum degeneracy regime.

In this regime, fermions actually stack up as shown on the left, while the bosons collect around the ground state as shown on the right.



Fermi Level



Bose-Einstein Condensate

Relations Between Spins & Statistics

In fact, the distribution functions for fermions and bosons follow something like this:

$$f(E) = \frac{1}{e^{(E-\mu)/kT} \pm 1}$$

We use -1 for bosons and +1 for fermions, making the distributions distinct for the two types of particles.

The relationship between spin and statistics is profound and almost emergent in a way. One finds that a self-consistent quantum field theory cannot be constructed in any other way.

Spin-Statistics Theorem

How do we know what the spin of these class of particles should be? Because the spin-statistics theorem says so.

Spin-statistics theorem: The fields of integral spins commute (and therefore must be quantized as bosons) while the fields of half-integral spin anticommute (and therefore must be quantized as fermions).

Now what could this mean?

Relativistic causality requires quantum fields at two spacetime points x and y separated by a space-like interval to either commute or anticommute with each other.

Spin-Statistics Theorem

Suppose we have two field operators. Then what the spin-statistics theorem says is that the following relations must hold:

$$\widehat{\phi}_A(x)\phi_B^\dagger(y) = +\widehat{\phi}_B(y)\phi_A^\dagger(y)$$
$$\widehat{\phi}_A(x)\phi_B^\dagger(y) = -\widehat{\phi}_B(y)\phi_A^\dagger(y)$$

Then from this we can gather (after a lot of QFT machinery):

$$\left[\widehat{\phi}_A(x),\ \phi_B^\dagger(y)\right] \propto 1 - (-1)^{2j}$$
$$\left\{\widehat{\phi}_A(x),\ \phi_B^\dagger(y)\right\} \propto 1 - (-1)^{2j}$$

We then must have that for these relations to hold, bosons must have integer spin only, while fermions must have half-integer spins.

Spin-Statistics and 2-dimensional failure

The spin-statistics theorem applies to all quantum field theories which have:

1. Special relativity, i.e. Lorentz invariance and relativistic causality.

2. Positive energies of all particles.

3. Hilbert space with positive norms of all states*.

The theorem is valid for both free or interacting quantum field theories and in any spacetime dimension $d > 2$.

That’s a peculiar restriction on the spacetime dimension...

Not Bosons, nor Fermions, but a Secret Third Thing

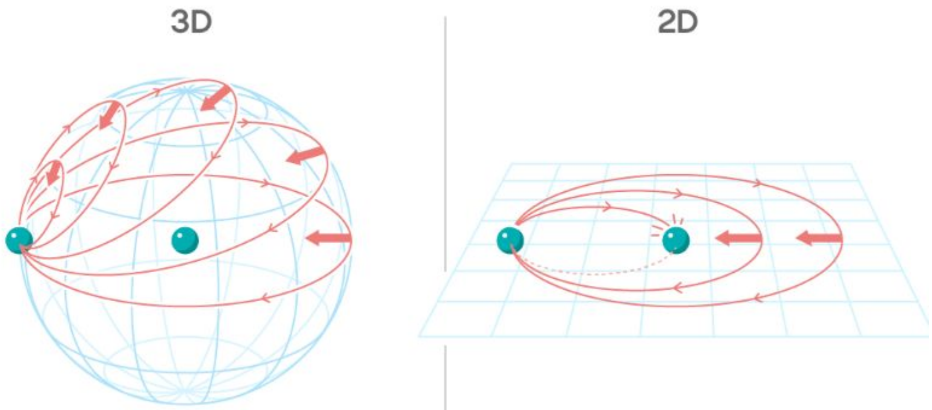
The spin-statistics theorem should hold for all spacetime dimensions greater than 2. But why the restriction?

In 2D, things get *weird*. Researchers have thought for a long time that everything we know of follows the statistics of either fermions or bosons.

‘Exchange’ of particles gives you a +1 as a phase factor for your wavefunction if your particle is a boson, -1 if it’s a fermion.

However in 2D you can get fractional statistics (phase factors) → the third kind of particles: anyons!

Not Bosons, nor Fermions, but a Secret Third Thing



In 3D, a loop can shrink to a point.

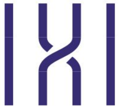
In 2D, the loop gets caught on the other particle.

Braids

Braids can be thought of as strings that cross over.

We use braids to show the motion of particles through spacetime and the crossings indicates the interactions

In (2 + 1)d we can have nontrivial braids, which ties into why we need anyons to live in 2d. In higher spatial dimensions we do not have such non trivial braids

$\sigma_1 =$  $\sigma_2 =$  $\sigma_3 =$ 

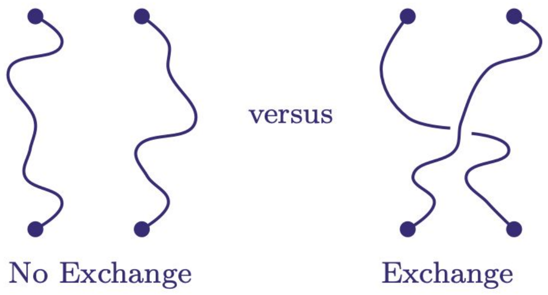
$$\sigma_n \sigma_{n+1} \sigma_n = \sigma_{n+1} \sigma_n \sigma_{n+1}$$

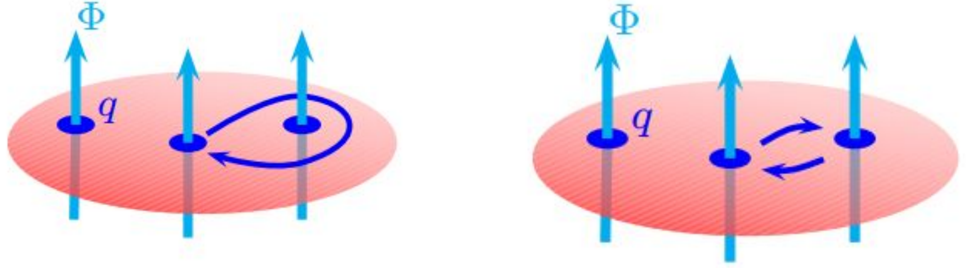
Anyons

Anyons are particles that are confined to a 2d surface so that there is a number θ such that when we exchange two anyons we get a relative phase shift of

$$e^{i\theta}$$

In this way fermions and bosons are anyons with $\theta = \pi$ and $\theta = 0$

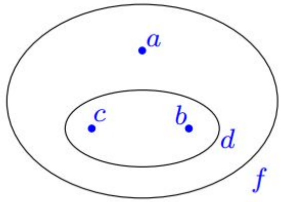




Fusion of Anyons

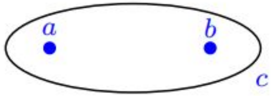
Before we move forward to our main topic of error correction, we need to model multi-anyon systems. The way we do this is through fusion channels.

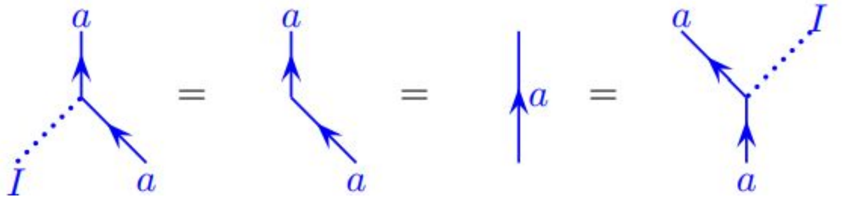
Suppose we have some finite species of particles **a, b, c, ...**



Then we have rules for such combinations of particles as such:

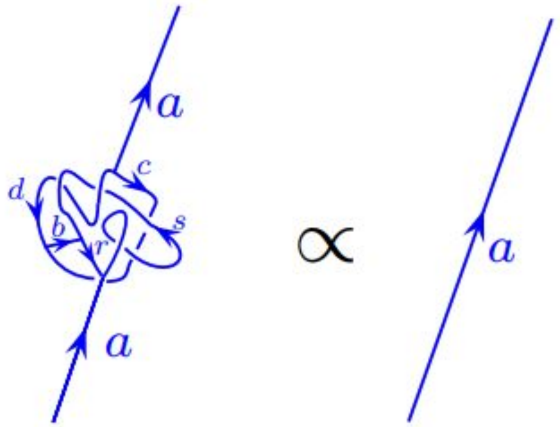
$$a \times b = c$$
$$a \times I = a$$
$$a \times \bar{a} = I$$

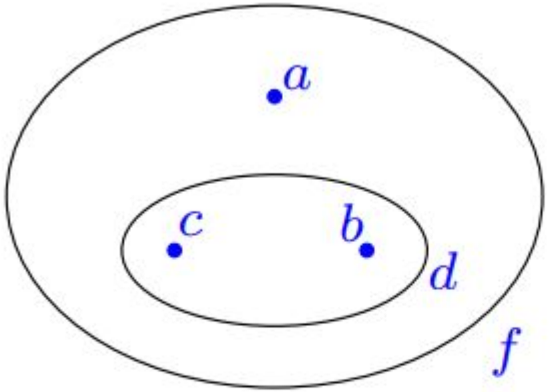




Fusion of Anyons

Principle of Locality: Locality is an extremely ubiquitous in all of physics. This applies to our anyon systems as well. Locality presents itself in our case through local *fusion channels*.





Ising Anyons

While the abelian cases for anyon systems are simple, the non-abelian cases are more useful for our purposes. The general fusion rule follows something like:

$$a \times b = c + d...$$

A prominent example are **Ising anyons**:

$$\text{Particle types} = \{I, \sigma, \psi\}$$

The fusion rules are:

$$\begin{aligned} \psi \times \psi &= I \\ \psi \times \sigma &= \sigma \\ \sigma \times \sigma &= I + \psi \end{aligned}$$

Error Correction(Classical)

In classical error correction we can duplicate bits and so we take full advantage of this and duplicate bits.

logical bit	physical bits
0	000
1	111

No cloning theorem

It states that there is no unitary operator that can clone a qubit

Theorem: Given a qubit in an arbitrary unknown state $|\phi_1\rangle$ and another qubit in a known or unknown initial state $|\phi_2\rangle$, there does not exist any unitary operator U (i.e. any quantum-mechanical evolution) such that

$$U(\,|\phi_1\rangle \otimes |\phi_2\rangle\,) = e^{i\chi(\phi_1)}\,|\phi_1\rangle \otimes |\phi_1\rangle \tag{26.2}$$

for all possible inputs $|\phi_1\rangle$.

This has profound implications when it comes to quantum error correction as we cannot simply make copies to store the information

Error Correction (Intro Quantum)

There are many types of errors that can occur but we focus on qubit flip errors (associated to σ_x application) and sign errors (associated to σ_z application)

It takes a logical qubit $$|\psi\rangle_L = \alpha|0\rangle_L + \beta|1\rangle_L$$

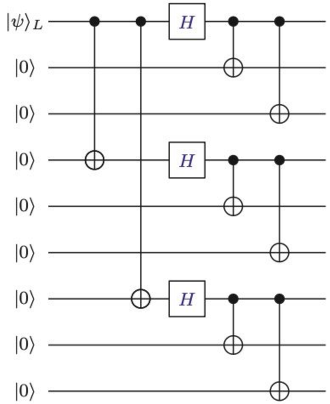


Fig. 26.2 A quantum circuit to prepare the nine-qubit Shor code. Here, H is a Hadamard gate, which can be written as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

And then it is put through the circuit on the right, and it spits out the physical qubit

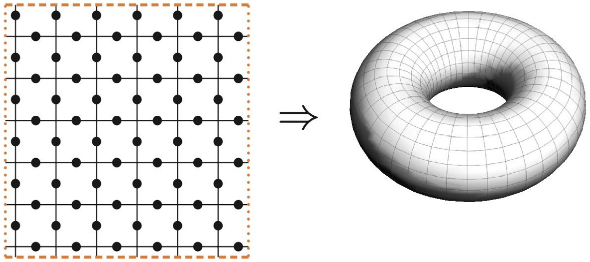
$$\begin{aligned} |\psi_9\rangle &= \frac{\alpha}{2\sqrt{2}}[(|000\rangle + |111\rangle)\otimes(|000\rangle + |111\rangle)\otimes(|000\rangle + |111\rangle)] \\ &+ \frac{\beta}{2\sqrt{2}}[(|000\rangle - |111\rangle)\otimes(|000\rangle - |111\rangle)\otimes(|000\rangle - |111\rangle)] \end{aligned}$$

The method for which we need this physical qubit is called

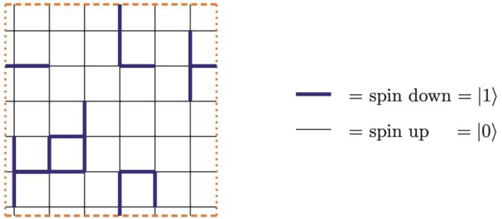
The Shor Code

Toric Code

The Hilbert Space we work with is the a Discrete Torus made by taking an n by m square and giving periodic boundary conditions, like Pac Man! Each lattice is a qubit. This means that there are 2mn = a qubits.



The Hilbert Space is exp(aln(2)) dimensional.



Operators on the Torus

Vertex operator is defined as $V_\alpha = \prod_{i \in \text{vertex } \alpha} \sigma_z^{(i)}$ for every vertex α

Plaquette operator is defined as $P_\beta = \prod_{i \in \text{plaquette } \beta} \sigma_x^{(i)}$ for every plaquette β

They are both involution meaning they square to identity.

Vertex operators commute with other vertex operators. Same with Plaquette operators.

Nontrivially Vertex and Plaquette operators commute for every edge and plaquette.

Code Space and Stabilizers

Code space just refers to the space of all possible configurations of qubits that we allow.

Stabilizers are operators of the code space that leave the code space unchanged. They are like projections.

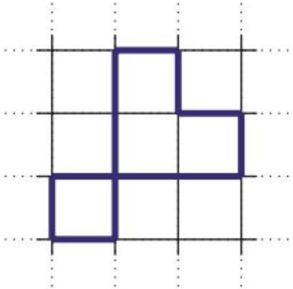
The stabilizers of Shor code are

$$
\begin{array}{ll}
\hat{O}_{12} = \sigma_z^1 \sigma_z^2 & \hat{O}_{23} = \sigma_z^2 \sigma_z^3 \\
\hat{O}_{45} = \sigma_z^4 \sigma_z^5 & \hat{O}_{56} = \sigma_z^5 \sigma_z^6 \\
\hat{O}_{78} = \sigma_z^7 \sigma_z^8 & \hat{O}_{89} = \sigma_z^8 \sigma_z^9 \\
\hat{O}_{1-6} = (\sigma_x^1 \sigma_x^2 \sigma_x^3)(\sigma_x^4 \sigma_x^5 \sigma_x^6) & \\
\hat{O}_{4-9} = (\sigma_x^4 \sigma_x^5 \sigma_x^6)(\sigma_x^7 \sigma_x^8 \sigma_x^9) & .
\end{array}
$$

The stabilizers for the toric code are the vertex and plaquette operator

Toric Code Space

We specify that for each vertex α that the $V_\alpha = 1$



We also specify that for each plaquette β that $P_\beta = 1$

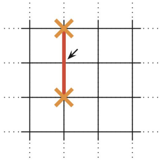
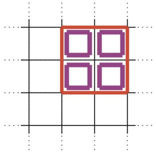
Then a wave function(Just one of four basis functions) is given by

$$|\psi\rangle = \mathcal{N}^{-1/2} \sum_{\substack{\text{all loop configs that can}\\\text{be obtained by flipping pla-}\\\text{quettes from a reference}\\\text{loop config}}} |\text{loop config}\rangle$$

So the general wave function is $|\psi\rangle = \alpha|\psi_{ee}\rangle + \beta|\psi_{eo}\rangle + \gamma|\psi_{oe}\rangle + \delta|\psi_{oo}\rangle$

σ_x Errors

Single σ_x applications make it so that on 2 vertices joined by the qubit to give $V_\alpha = -1$ this means that we can fix these by scanning through with vertex operators to find eigenvalues -1 and when we find it we flip it back.

For multiple errors, if the errors are far away enough we do the same thing as above. If they are touching  then the black arrow shows $V_\alpha = 1$ So we start on a vertex in orange and spam the σ_x till we either go back to the original state or we form a loop  , this is sufficient because of:

$$\prod_{i \text{ around loop}} \sigma_x^{(i)} = \prod_{\beta \text{ enclosed in loop}} P_\beta$$

σ_z Errors

Single σ_z applications makes it so that on 2 plaquettes joined by the qubit to give $P_\alpha = -1$ this means that we can fix these by scanning through with vertex operators to find eigenvalues -1 and when we find it we flip it back.

For multiple errors, if the errors are far away enough we do the same thing as above. If not, then we do the same thing as we did with σ_x so keep applying the operator on affected plaquettes till we either get back to the original or we form a loop

Mixed Errors

For this we note that σ_x and σ_z anti-commute, so the order of correction matters!

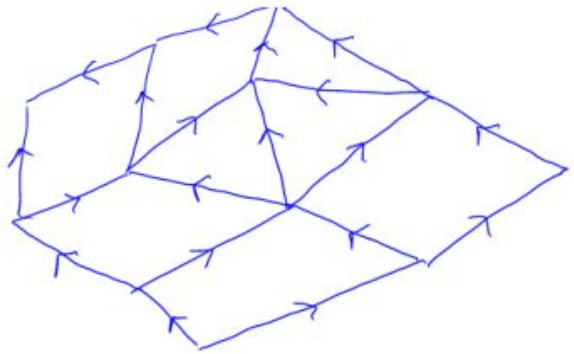
The idea is to just take one of the operators and try to correct all the associated errors. If that doesn't work, form a loop and make it cancel out.

Generalization of Toric Code

We can generalize the concept of the toric code to a full lattice gauge theory.

So what does that mean?

Alexei Kitaev, a Russian-American theoretical physicist developed the generalization of the toric code method by using groups on a lattice:



$\mathbb{Z}_2$ Toric Code!

Continuing from the last slide, what the group on the lattice allows us to do is define $\mathbb{Z}_2$ group of {1, -1}, on the lattice by identifying the group elements to be spin up or spin down.

$$P_\beta(1)\ identity\ operator$$
$$P_\beta(-1)\ multiplies\ edges\ by\ -1\ (flips\ it)$$

The full plaquette operator is then:

$$P_\beta = P_\beta(1) + P_\beta(-1)$$

Consequently the vertex operator will be:

$$V_\alpha = \begin{cases} 1 & if\ even\ number\ of\ edges\ are\ spin\ down \\ 0 & if\ an\ odd\ number \end{cases}$$

Thank you!