Recall: Manifold is a $\underbrace{2^{nd}\text{ countable}}_{\Downarrow\ \text{paracompactness}}$, Hausdorff locally Euclidean topological space

last time: we took the tangent bundle TM we dualized it to get the cotangent bundle T^*M.

Sections: $\pi : E \to M$ then a section σ of this surjection is a right inverse: $\pi \circ \sigma = id_M$

$E_p = \pi^{-1}(\{p\})$ and $v_p \in E_p$

so sections of TM are vector fields
sections of T^*M are called covector fields
sections of $M \times N$ are maps from $M \to N$

We denote by $\mathfrak{X}(M) = \{X \text{ section of } TM \to M \text{ and smooth}\}$
$\Omega^1(M) = \{\omega \text{ sections of } T^*M \to M \text{ and smooth}\} = \{\text{space of 1-forms}\}$ (differential)

Exercise: Show sections of $M \times \mathbb{R}$ is the same as the space $C^\infty(M)$

John Baez - Gauge Fields, Knots and Gravity.

Smooth sections of a vector bundle $\pi : E \to M$ will be denoted $\Gamma(E)$

Take TM we used some representation of $GL(\mathbb{R}^n)$ to get cocycles $g_{\alpha\beta}$ for TM.

So now we take tensor representations of $GL(\mathbb{R}^n)$. $\rho : GL(\mathbb{R}^n) \to GL(T^k{}_\ell \mathbb{R}^n)_\ell$
$A \mapsto \bigotimes_{i=1}^{k} (A^*)^{-1} \otimes \bigotimes_{i=1}^{\ell} A$

then we define tensor bundles $T^k{}_\ell M = T^{(k,\ell)}M$.

In local coordinate (U, φ) with coordinate vector fields $\frac{\partial}{\partial x^i}$ and coordinate 1-forms dx^i $\left[(dx^i)_p : \underset{\substack{\| \\ T_pM}}{T_pU} \to \mathbb{R} \text{ so that } dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i{}_j \right]$

$\omega_p \in T_p^{(k,\ell)}M$, $\omega_p = \omega_{i_1 \cdots i_k}{}^{j_1 \cdots j_\ell}(p)\, dx^{i_1}_p \otimes dx^{i_2}_p \otimes \cdots \otimes dx^{i_k}_p \otimes \left.\frac{\partial}{\partial x^{j_1}}\right|_p \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{j_\ell}}\right|_p$

Exercise: Suppose (V, ψ) is another coordinate system. How do the components transform?

Now $\omega \in \Gamma(T^{(k,l)}M) \Leftrightarrow \forall p \in M$ and charts (U,φ) around p. The component functions are smooth.

we can look at $\omega \in \Gamma(T^{(k,l)}M)$ as a $C^\infty(M)$ - linear function

$$\omega : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{l} \longrightarrow C^\infty(M)$$

Examples of smooth tensor fields:

1) Riemannian metric: $g \in \Gamma(T^{(2,0)}M)$ that is symmetric, positive definite and non-degenerate.

$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ that is an inner product

locally $g = g_{ij}\, dx^i \otimes dx^j$ where $g_{ij} = g_{ji}$

2) Symplectic form: $\omega \in \Gamma(T^{(2,0)}M)$ that is antisymmetric, non-degenerate and is closed. $\omega = \omega_{ij}\, dx^i \wedge dx^j$.

$d\omega = 0$

wedge from linear algebra