Prerequiste

- Group Theory
  - Representations
    - Lie Algebras (Groups)
- Topology
- Linear and Multilinear Algebra
- Calculus
- ODE and PDE
- Category Theory

Main Content

- (real) Smooth finite dimensional manifolds
- Bundles
  - Principle bundles
  - Associated bundles
  - Fiber bundles
    - vector bundles
  - deRham Complex (differential forms)
- (psudo) Riemannian metric
  - Affine connection
  - Geodesics
- Symplectic Geometry

Side Quests

- Simulations
- Poisson Geometry

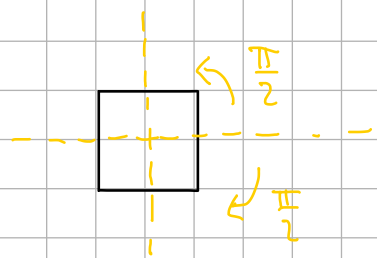
Group is a set G and a binary operation $\bullet : G \times G \to G$

(i) $\exists e \in G$ s.t. $\bullet(e,g) = \bullet(g,e) = g$ $(\forall g \in G)$

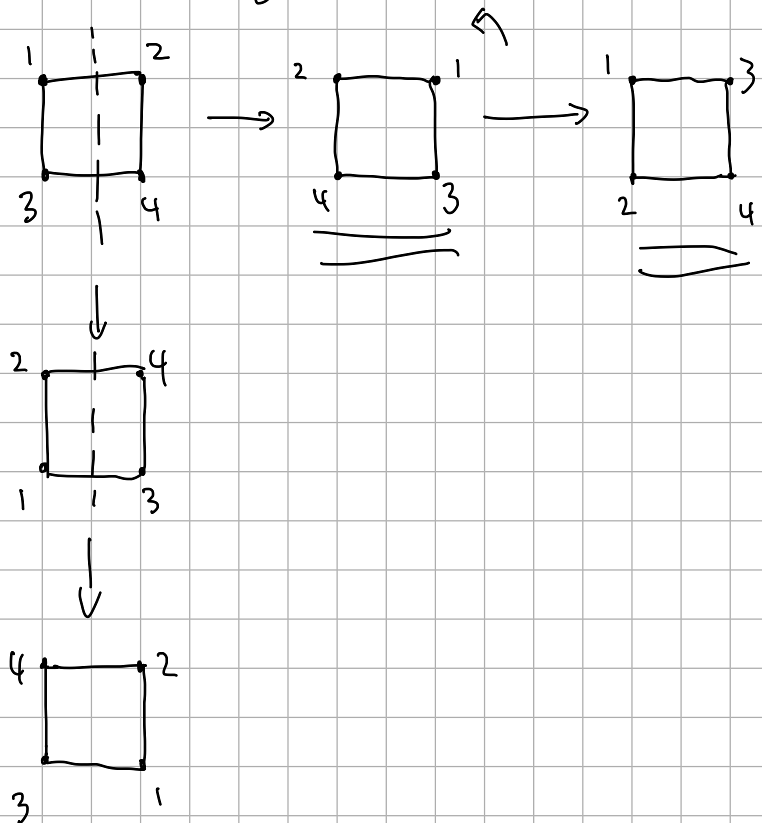
(ii) $\forall x \in G \ \exists x^{-1} \in G$ s.t. $\bullet(x, x^{-1}) = \bullet(x^{-1}, x) = e$

(iii) $\forall x,y,z \in G$ $\bullet(x, \bullet(y,z)) = \bullet(\bullet(x,y), z)$

$\bullet(x,y) := xy$ or $x \cdot y$



Due to Emmy Noether we see that symmetries of our dynamical systems are closly linked to conservation laws.



- $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{Q}, \mathbb{Z}/p\mathbb{Z}$ ← addition
- These are not group under multiplication

↳ $\mathbb{R}^{\times}$

↑ remove additive identity

- $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

  ↳ $e^{i\theta}$, $e^{i\theta} \cdot e^{i\gamma} = e^{i(\theta+\gamma)}$

$\psi, \varphi \rightarrow |\langle\varphi|\psi\rangle|^2$

$\psi \rightarrow e^{i\theta}\psi \rightarrow |\langle\varphi|e^{i\theta}\psi\rangle|^2 \equiv |e^{i\theta}|^2 |\langle\varphi|\psi\rangle| = |\langle\varphi|\psi\rangle|$

↑

measurements
dont care about phase

- Poincaré Group : $\mathbb{R}^{1,3} \rtimes O(1,3)$ [See below for $O(1,3)$]

  ↳ Group of Lorentz transformations.

Classical Group:

→ $GL(n,\mathbb{R})$: Group invertible linear transformations
→ $\underline{S}L(n,\mathbb{R}) \subseteq GL(n,\mathbb{R})$: Group of det $= 1$ L.T.

→ $U(n)$: Group of complex linear transformations on n space $UU^\dagger = U^\dagger U = Id$

  ↳ $U(1) := S^1$

  unitary group

$(re^{i\theta})(re^{-i\theta}) = 1 \Rightarrow r^2 = 1$
$\Rightarrow r = \pm 1$

→ $\underline{S}U(n)$: det 1 $U(n)$ maps

"special" unitary group

(real)

Bilinear forms

V is a vector space. $\varphi : V \times V \rightarrow \mathbb{R}$ if φ is bilinear.

$\varphi(v+\alpha w, \beta) = \varphi(v,\beta) + \alpha\varphi(w,\beta)$
$\varphi(\beta, v+\alpha w) = \varphi(\beta,v) + \alpha\varphi(\beta,w)$

$\mathbb{R}^n$, $x \cdot y = \sum_i x^i y^i \leftarrow (n,0)$

$x \cdot y = x^0y^0 - x^1y^1 - \cdots - x^ny^n \leftarrow$ lorentz $(1, n-1)$

Signature of φ : (p,q) s.t. there is a basis $\{e_i\}$ where $\varphi(e_i,e_i) = \pm 1$ (i)
$\varphi(e_i,e_j) = 0$ if $i \neq j$

\# of + in (i) is p
\# of − in (i) is q

→ $O(p,q)$ = Group of linear transformations T s.t. $\langle Tv, Tw \rangle = \langle v, w \rangle$
$\langle v, w \rangle = v^1w^1 + \dots + v^p w^p - v^{p+1}w^{p+1} - \dots - v^{p+q}w^{p+q}$

Fun exercise: Show $O(n) := O(n,0)$ are rotations + reflections

→ $SO(p,q) = T \in O(p,q)$ with $\det T = 1$

On the Poincaré Group: $\mathbb{R}^{1,3} \rtimes O(1,3)$ the set $\mathbb{R}^4 \times O(1,3)$ with
$(\alpha, f) \cdot (\beta, g) = (\alpha + f\beta, fg)$

Group Homomorphism: $\varphi : G \to H$ then φ is a group homomorphism

$\varphi(gh) = \varphi(g)\varphi(h)$

Group Actions: let X be a set $\underset{\text{group homomorphism}}{\varphi : G \to \mathrm{Bij}(X)}$

Representation: A group G and vector space V along group homorphism from $G \to GL(V) := \mathrm{End}(V)$

$SO(2)$ representation on $\mathbb{C}^2 \longleftrightarrow \frac{1}{2}$-Spin states

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Normal subgroups
Examples of representations

Meetings will be on Tuesdays – 5 EST
Saturdays – 6 EST