Problem Sheet 6 - Geometry of Physics

Febuary 2026

Question 1 (Properties of the Metric)**.** *Let (M, g) be a Riemannian manifold. Let $S \subseteq M$ be a regular submanifold of M. Let $i : S \hookrightarrow M$ be the inclusion. The induced metric on S is the metric given by i^*g.*

1. *Let $\mathbb{R}^2$ be equipped with the usual metric $g = \mathrm{d}x^2 + \mathrm{d}y^2$. In coordinates $\theta \to (\cos\theta, \sin\theta)$ compute the induced metric on S^1.*
2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Let $\Gamma(f) = \{(x, f(x)) \mid x \in \mathbb{R}\}$ be the graph as a subset of $\mathbb{R}^2$. Show that this is always a submanifold and using coordinates $x \to (x, f(x))$ compute the induced metric.*
3. *Let (M, g) be a Riemannian manifold. Let ${\Gamma^i}_{jk}$ be the Christoffel symbols in some chart. Show that there is a sort of symmetry in these Christoffel symbols given by ${\Gamma^i}_{jk} = {\Gamma^i}_{kj}$. Moreover show that*

$$ {\Gamma^l}_{ki} g_{lj} + {\Gamma^l}_{kj} g_{il} = \partial_k g_{ij} $$

4. *Let (M, g) be a Riemannian manifold. Show that on any chart, there are vector fields $\{E_i\}$ such that $g(E_i, E_j) = \delta_{ij}$.*

Question 2 (Existence of Metric)**.** *Let M be any manifold. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to the atlas. Show that*

$$ g_M = \sum_\alpha \psi_\alpha(\varphi_\alpha^* g) $$

is a Riemannian metric on M where g is the standard metric on $\mathbb{R}^n$.

Question 3 (Raising and Lowering Indices)**.** *Recall that if (M, g) is a Riemannian manifold then g induces a bundle map $\hat{g} : TM \to T^*M$ given by $\hat{g}(v) = \iota_v g$.*

1. *Show that this is actually a bundle map, this means that show that the following diagram commutes*

$$
\begin{tikzcd}
TM \arrow[rr, "\hat{g}"] \arrow[dr, "\pi_{TM}"'] & & T^*M \arrow[dl, "\pi_{T^*M}"] \\
& M &
\end{tikzcd}
$$

2. *Let (U, φ) be a chart on M in which $g = g_{ij}\, \mathrm{d}x^i \mathrm{d}x^j$. Let $V = V^i \partial_i$ be a local vector field, what are the components of $\hat{g}(V)$?*
3. *Let $f : M \to \mathbb{R}$ be a smooth function, define grad f to be the vector field so that*

$$ \mathrm{d}f = \hat{g}(\mathit{grad}\, f) $$

what are the components of this vector field? Show moreover that if grad $f \neq 0$ then $(\mathit{grad} f)_p$ is orthogonal to $f^{-1}(0)$ for all $p \in f^{-1}(0)$

Question 4 (Curvature Computations)**.**

1. *Let S^1 be given by the parametrization $(\cos\theta, \sin\theta)$, what is the signed curvature κ of this.*
2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Compute the signed curvature of the graph.*

3. *Let S^2 be the unit sphere and let $N = x^i\partial_i$ be the unit normal vector field. Given the parametrization*

$$x = \sin\phi\cos\theta$$
$$y = \sin\phi\sin\theta$$
$$z = \cos\phi$$

verify that $\partial_\theta, \partial_\phi$ form an orthonormal basis for T_pS^2 whenever defined. Moreover Compute $\partial_\theta N$ and $\partial_\phi N$, write it as a matrix and find the determinant K of that matrix. Compute

$$\int_{S^2} K \, \mathrm{d}S$$

Question 5 (Spherical Pendulum without Inertia)**.** 1. *Given that kinetic energy is given by*

$$T = \frac{1}{2} g_{ij} v^i v^j$$

what metric on $\mathbb{R}^3$ gives the usual kinetic energy for a particle of mass m?

2. *Given a particle of mass m connected to a massless rod of length l, find the x, y, z positions of the particle in an arbitrary configuration. What is the configuration space manifold?*
3. *What is the kinetic energy of this particle? What is the potential energy due to gravity? What is the Lagrangian? What are the dynamics of the particle?*
4. *Using the fact that the azimuthal angle ϕ is not present in the Lagrangian, find a conserved quantity. Call this quantity M.*
5. *What is the total energy of the system? Is it conserved? Rewrite E in terms of θ and M.*
6. *What is the "potential" part of the energy? This is called the effective potential of the system.*
7. *What are the extrema of the effective potential? Are they stable? Are they unstable?*
8. *Using the equations of motion for θ, Taylor expanding and ignoring terms of order high enough, what is the period of small oscillations (if they exist) around the critical points?*

Question 6 (The Lie Algebra $\mathfrak{so}(n)$)**.** 1. *Recall that $\mathfrak{so}(n)$ is the space of traceless antisymmetric matrices. Verify that the collection T_{ab} given by $(T_{ab})^c{}_d = \frac{1}{2}(\delta_{ad}\delta^c_b - \delta^c_a\delta_{bd})$ where $a < b$ is a basis.*

2. *Restrict to the space $\mathfrak{so}(3)$, given the usual Lie bracket structure. What are the structure constants? Show that this is isomorphic to $(\mathbb{R}^3, \times)$.*
3. *Define on $\mathfrak{so}(n)$ a function $B : \mathfrak{so}(n) \times \mathfrak{so}(n) \to \mathbb{R}$ given by $B(x, y) = \mathrm{tr}(ad(x) \circ ad(y))$. Show that B is bilinear. Starting with the basis given in (a) find the structure constants of the Lie algebra and find a matrix representation of B with that. Conclude that $B(X, Y) = (n - 2)\,\mathrm{tr}(XY)$ for $n \geq 2$. Hint: $ad(x)(y) = [x, y]$. In $\mathbb{R}^3$ what is B?*
4. *Show that B is negative semi-definite, meaning that $B(X, X) \leq 0$.*