Tensor Products:

Finite-dimensional vectors over $\mathbb{R}$.

V is a vector space $V^* = \{ f: V \to \mathbb{R} \mid f \text{ linear} \}$ is the "dual space".

$$V \xrightarrow{T} W$$

$$V^* \xleftarrow[T^{tr}]{} W^*$$

$f \in W^*$

$T^{tr}(f) \in V^*$ so its action on $v \in V$ is defined as

$$T^{tr}(f)(v) = f(Tv)$$

In finite dimension $V \cong V^*$ but isomorphism is not canonical.

Fix basis $e_1, \dots, e_n$ for V. Then define $e_i^* \in V^*$ to be the linear map s.t. $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Kronecker delta

Now the linear map $e_i \to e_i^*$ is an isomorphism.

However it turns out that $V \cong V^{**}$ is canonical.

Consider the map $v \mapsto ev_v$ now $ev_v(\varphi) = \varphi(v)$

$$\begin{array}{ccc} v & \longmapsto & ev_v \ (\eta_v) \\ V & \longrightarrow & V^{**} \\ & & \cup \\ & & \varphi \end{array}$$

$\varphi : V^* \longrightarrow \mathbb{R}$

$\eta_v \varphi = \varphi(v)$ so from now on $v(\varphi) := \varphi(v)$

(For now V and W may not be finite dim) $\to F(V,W)$

Take vector space V and W. let the free vector space over V and W be the space s.t. every element in $V \times W$ is a basis element.

ex: $F(\mathbb{R},\mathbb{R}) = \text{span}\{(a,b) \mid a \in \mathbb{R},\ b \in \mathbb{R}\}$

$(1,1)$ is linearly independent to $(2,2)$

In particular $2 \cdot (1,1) \neq (2,2)$

$$V \otimes W = F(V,W) \Big/ \left\{ \begin{matrix} (v,\lambda w) - \lambda(v,w) & (v_1+v_2, w) - (v_1,w) - (v_2,w) \\ (\lambda v, w) - \lambda(v,w) & (v, w_1+w_2) - (v,w_1) - (v,w_2) \end{matrix} \right\}$$

we can think of vectors in $V \otimes W$ as finite linear combinations of elements of the form $v \otimes w$

$\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$

$(v_1+v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

Ex: In Quantum mechanics, for multiparticle systems a general state is a linear combination of tensor product states

$|+\rangle, |-\rangle \leftarrow$ eigenbasis

$\alpha |+\rangle|+\rangle + \beta|-\rangle|+\rangle + \gamma|+\rangle|-\rangle + \delta|-\rangle|-\rangle$

$\uparrow$

tensor product

Usefulness: If V and W are finite dimensional turns out

$V^* \otimes W \cong \mathcal{L}(V,W)$ let $\varphi \in V^*$ and let $w \in W$

$(\varphi \otimes w)$ has a natural action on V that gives a vector in W

$(\varphi \otimes w)(v) = \varphi(v) w \in W$

$\nearrow$

$\mathbb{R}$ or $\mathbb{C}$

This action extends linearly to $V^* \otimes W$

$$\sum_{i=1}^{n} \alpha_i (\varphi_i \otimes w) \in V^* \otimes W$$

$$\left(\sum_{i=1}^{n} \alpha_i (\varphi_i \otimes w_i) \right)(v) = \sum_{i=1}^{n} \alpha_i \underbrace{\varphi_i(v)}_{\in \mathbb{F}} w_i$$

Exercise: Pick basis $\varphi_1, \dots, \varphi_n \in V^*$ and $w_1, \dots, w_m$ of W and construct an isomorphism $V^* \otimes W \to \mathcal{L}(V, W)$ by the action above.

An algebra over $\mathbb{R}$ (or $\mathbb{C}$) is a vector space with a multiplication (associative) that bilinear.

Ex: $(\mathbb{R}^3, \times)$ is an algebra over $\mathbb{R}$.

Define $V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}} = \bigotimes_{i=1}^{n} V$

$V \otimes (U \otimes W) \cong (V \otimes U) \otimes W$

$V \otimes W \cong W \otimes V$

check this.

these behave like homogenous polynomials. $P(\lambda z) = \lambda^n P(z)$

then P is homogenous of degree n.

tensor algebra over V

$$T(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i} = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

Exterior Algebra and Symmetric Algebra and Clifford Algebra

$V^* \otimes V \cong \mathcal{L}(V, V)$

$V^* \otimes V^* \cong \mathcal{L}(V, V^*)$

if $T \in \mathcal{L}(V, V^*) \Rightarrow \hat{T} : V \times V \to \mathbb{R}$ bilinear

$\hat{T}(v, w) = T(v)(w)$

So a multilinear map $\overbrace{V\times V\times V\times\cdots\times V}^{n}\to\mathbb{R}$ can be thought of as a member of $(V^*)^{\otimes n}$

Take a basis $\{e_i\}$ of V we get $\{e_i^*\}$ of V^*, $e^i := e_i^*$

$T: V\times V\to\mathbb{R}$ is $\sum_{i=1}^{n}\sum_{j=1}^{n} T^i{}_j\, e_i\otimes e^j \Rightarrow T^i{}_j\, e_i\otimes e^j$

summed over

summed over

Einstein summation convention.

Using $V^{**}\cong V$ we look at multilinear maps over V^* into $\mathbb{R}$. as members of $V\otimes\cdots\otimes V$

So in physics the tensors we use will be from the following space: $\mathcal{T}^k{}_l(V) = \underbrace{V\otimes\cdots\otimes V}_{k\text{-times}}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{l\text{-times}}$

we call $T\in\mathcal{T}^k{}_l(V)$ a tensor of type (k,l). This means $\mathcal{L}(V,V)$ are of type $(1,1)$

dual vectors are of type $(1,0)$

vectors are of type $(0,1)$ ← follows from $V\cong V^{**}$

bilinear maps are of type $(0,2)$

---

basis $\{a_i\}$ of V and $\{a^i\}$ of V^*

basis $\{b_i\}$ of V and $\{b^i\}$ of V^*

$T: V\to V$

$a_i\mapsto b_i$

We want to figure out how do the components of a vector in basis a_i transform to basis b_i

$\hat{T} \in V \otimes V^* \Rightarrow T = T^i{}_j a_i \otimes a^j$ moreover $b_i = T^j{}_i a_j$

$v = v^i a_i = \check{v}^i b_i$

$\check{v}^i b_i = \check{v}^i T^j{}_i a_j \Rightarrow v^j = \check{v}^i T^j{}_i$

$$\Rightarrow A^i{}_j v^j = \check{v}^i \underbrace{T^j{}_i A^i{}_j}_{\delta^j{}_j} = \check{v}^i$$

So the components of v transform by inverse of T.

How does the dual basis transform?

$b^i = C_j{}^i a^j$ where $C_i{}^j$ is some transformation $V^* \to V^*$

$$\delta^i{}_j = b^i(b_j) = C_k{}^i a^k(T^l{}_j a_l)$$

$$= C_k{}^i T^l{}_j a^k(a_l)$$

$$= C_k{}^i T^l{}_j \delta^k{}_l$$

$$= C_k{}^i T^k{}_j$$

$[C_k{}^i]$ is inverse matrix of $[T^k{}_j]$

Take a general tensor $T \in \mathcal{T}^k{}_l(V)$ we have the following

$$T = T^{i_1 \cdots i_k}{}_{j_1 \cdots j_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l}$$

$A : e_i \mapsto e'_i$ how does $e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l}$ transform?

$$e'_{i_1}\otimes\cdots\otimes e'_{i_k}\otimes e'^{j_1}\otimes\cdots\otimes e'^{j_\ell}$$

$$=(A^{m_1}{}_{i_1}e_{m_1})\otimes\cdots\otimes(A^{m_k}{}_{i_k}e_{m_k})\otimes\left((A^{-1})^{j_1}{}_{n_1}e^{n_1}\right)\otimes\cdots\otimes\left((A^{-1})^{j_\ell}{}_{n_\ell}e^{n_\ell}\right)$$

$$=A^{m_1}{}_{i_1}\cdots A^{m_k}{}_{i_k}(A^{-1})^{j_1}{}_{n_1}\cdots(A^{-1})^{j_\ell}{}_{n_\ell}\,e_{m_1}\otimes\cdots\otimes e_{m_k}\otimes e^{n_1}\otimes\cdots\otimes e^{n_\ell}$$

Now components must transform by inverse:

$$T=T'^{i_1\cdots i_k}{}_{j_1\cdots j_\ell}\,e'_{i_1}\otimes\cdots\otimes e'_{i_k}\otimes e'^{j_1}\otimes\cdots\otimes e'^{j_\ell}$$

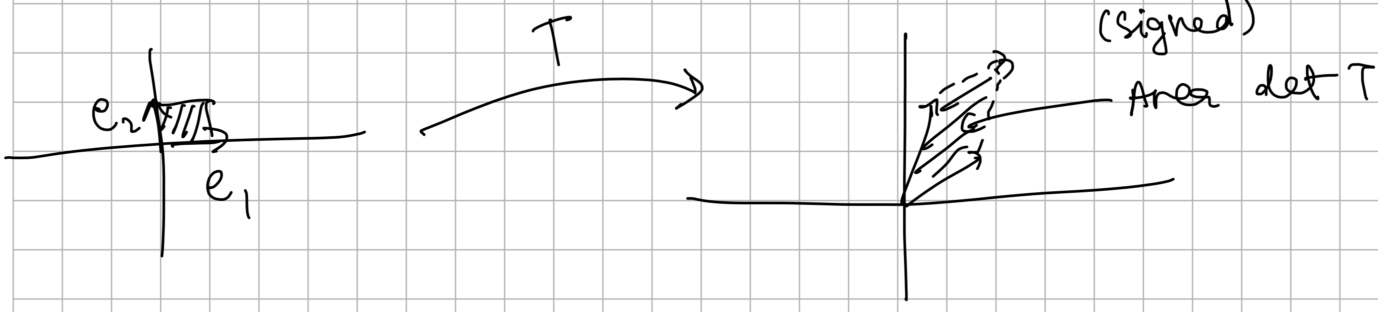
$$=T'^{i_1\cdots i_k}{}_{j_1\cdots j_\ell}A^{m_1}{}_{i_1}\cdots A^{m_k}{}_{i_k}(A^{-1})^{j_1}{}_{n_1}\cdots(A^{-1})^{j_\ell}{}_{n_\ell}\,e_{m_1}\otimes\cdots\otimes e_{m_k}\otimes e^{n_1}\otimes\cdots\otimes e^{n_\ell}$$

Hence

$$T^{m_1\cdots m_k}{}_{n_1\cdots n_k}=T'^{i_1\cdots i_k}{}_{j_1\cdots j_\ell}A^{m_1}{}_{i_1}\cdots A^{m_k}{}_{i_k}(A^{-1})^{j_1}{}_{n_1}\cdots(A^{-1})^{j_\ell}{}_{n_\ell}$$

$$T'^{i_1\cdots i_k}{}_{j_1\cdots j_\ell}=(A^{-1})^{i_1}{}_{m_1}\cdots(A^{-1})^{i_k}{}_{m_k}A^{n_1}{}_{j_1}\cdots A^{n_\ell}{}_{j_\ell}T^{m_1\cdots m_k}{}_{n_1,\cdots,n_\ell}$$

Idea: determinat tells you the "hyper" area of the parallelogram spanned by $e_1,\dots,e_n$ after the transformation



gives us an notion of orientation.

Another property: determinant is multilinear in the coloums of a matrix.

$\det\in\mathcal{T}^0{}_n(V)$

n is the dimension of V. the determinant is alternating.