

# Problem Sheet 4 - Geometry of Physics

January 2026

**Question 1** (Category Theory). (i) Show that  $\text{Top}_*$  and  $\text{Grp}$  are categories and show that  $\pi_1 : \text{Top}_* \rightarrow \text{Grp}$  is a covariant functor.

(ii) Given a small category  $C$  a (set valued) presheaf is a contravariant functor from  $C$  to  $\text{Set}$ , in other words it is a functor  $C^{\text{op}} \rightarrow \text{Set}$ . Show that this forms a category with natural transformations.

**Question 2** (Manifolds). (i) Show that  $S^n$  can be made into a smooth manifold with the following atlas: Define the set  $U_{i,\pm} \subseteq S^n$  so that  $x \in U_{i,\pm}$  if and only if the  $i$ th component of  $x$  is strictly positive or negative respectively. Let the chart maps be  $\varphi_{i,\pm} : U_{i,\pm} \rightarrow B(0,1)$  given by  $\varphi(x) = (x_1, \dots, \hat{x}_i, \dots, x_n)$ .

(ii) Show that an open set of a manifold is naturally a manifold. Use this to show that  $\text{GL}(n, \mathbb{R})$  is a manifold. What is its dimension?

**Question 3** (Adjoint Representation of Lie Groups and their Lie Algebra). Suppose  $G$  is a Lie group.

1. Using the fact that  $C_g(h) = ghg^{-1}$  is smooth and fixes the identity, show that the connected component of  $G$  containing  $e$  is open and normal. Conclude that it is the maximal proper normal lie subgroup of  $G$ .
2. Note that for each  $g \in G$  we have  $(dC_g)_e : T_e G \rightarrow T_e G$ . Show that  $g \rightarrow (dC_g)_e$  is a smooth action of  $G$  on its lie algebra  $\mathfrak{g}$ . This is called the adjoint representation of  $G$  and will be denoted  $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ .
3. For  $G = \text{GL}(n, \mathbb{R})$  compute its adjoint representation under the identification that  $\mathfrak{gl}(n, \mathbb{R}) = \text{Mat}(n, \mathbb{R})$ .
4. Now we can take the derivative of  $\text{Ad}$  at identity to get a map  $\text{ad} : \mathfrak{g} \rightarrow \text{GL}(\mathfrak{g})$  given by  $\text{ad}(X) = d(\text{Ad})_e(X)$ . Compute this for  $\text{GL}(n, \mathbb{R})$ .

**Question 4** (Lie Groups and tangent bundles). Let  $G$  be a Lie group.

1. Let  $X \in \mathfrak{X}(G)$  such that for all  $g, h \in G$  we have  $(d\ell_g)_h(X_h) = X_{gh}$ , then  $X$  is called a left invariant. Show that any  $X_e \in \mathfrak{g}$  can be extended to a smooth left invariant vector field  $X_g = (d\ell_g)_e(X_e)$ .
2. Let  $X_e \in \mathfrak{g}$  be nonzero. Show that the induced left invariant vector field of  $X_e$  is a nonzero vector field on  $G$ .
3. Show that if  $X_{1e}, \dots, X_{ne}$  is a basis for  $\mathfrak{g}$  then the extended vector fields  $X_1, \dots, X_n$  form a basis for the tangent space at each point.
4. Use the above to conclude that  $TG = G \times \mathbb{R}^n$  by noting that the map  $G \times \mathbb{R}^n \rightarrow TG$  given by  $(g, a^1, \dots, a^n) \rightarrow a^i X_{ig}$  is a diffeomorphism.
5. Using the hairy ball theorem conclude that  $S^2$  is not a Lie group in any way with the standard structure. What about the other spheres?
6. Let  $X \in \mathfrak{X}(G)$  be left invariant. Show that if  $\gamma : (-\epsilon, \epsilon) \rightarrow G$  is a smooth curve with  $\gamma(0) = e$  and  $\gamma'(0) = X_e$  then show that  $(h\gamma)(t) = h\gamma(t)$  is the flow through  $h$ . Use this to conclude that the flow of left invariant vector fields exist for all time.