

Manifolds (Smoller)

- ↳ Fiber Bundles (Things like tangent bundle, cotangent bundle, tensor bundles etc)
- ↳ Lie Groups
- ↳ Principle G -bundle

(Pseudo) Riemannian manifolds

- ↳ Connections
- ↳ Curvature
- ↳ Torsion

Symplectic Manifold

- ↳ Hamiltonian Mechanics
- ↳ Fibration ★

Lagrangian Mechanics

* Spinors and Spin Geometry

★ Poisson Geometry

A top. space (M, τ) is a ^(real) topological manifold if

- (i) Hausdorff (T_2)
- (ii) 2nd countable
- (iii) For every $p \in M$ there is an open neighborhood $U \subseteq M$ and a homeomorphism $\varphi: U \rightarrow \hat{U} \subseteq \mathbb{R}^n$

The pair (U, φ) is called a chart.

Ex: \mathbb{R}^n is a top. manifold
 $S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^n x_i^2 = 1\}$

- $\mathbb{R}^{n^2} \leftarrow$ matrices in \mathbb{R}^n
 - $\hookrightarrow GL(n, \mathbb{R})$ is a manifold
 - $\hookrightarrow SL(n, \mathbb{R})$ is a manifold
 - $\hookrightarrow U(n)$ is a manifold
 - $\hookrightarrow SU(n)$ is a manifold
 - $\hookrightarrow O(p, q)$ is a manifold
 - $\hookrightarrow Sol(p, q)$ is a manifold

An atlas for a manifold R the collection of all charts on that manifold.

Given charts (U, φ) and (V, ψ) the maps $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are called transition maps

$$\begin{array}{ccc} U & \xrightleftharpoons[\varphi^{-1}]{\varphi} & \hat{U} \\ V & \xrightleftharpoons[\varphi]{\varphi^{-1}} & \hat{V} \end{array}$$

We could have charts that maps us into \mathbb{C}^n , X where X is a Banach space.

If $\varphi: U \rightarrow \hat{U} \subseteq \mathbb{C}^n$ then M is complex manifold

If $\varphi: U \rightarrow \hat{U} \subseteq X$ then M is a Banach manifold

Smooth manifolds:

Transition maps are maps from open subsets of \mathbb{R}^n into open subsets of \mathbb{R}^n . We can now ask if these are smooth.

If (U, φ) and (V, ψ) are charts with $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ both smooth. Then the charts are said to be C^∞ -compatible.

We define a smooth structure on a top. manifold M to be a maximal C^∞ -compatible atlas on M .

Ex: \mathbb{R} . Note that $\text{id}: \mathbb{R} \rightarrow \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ are homeomorphisms. But they are not C^∞ compatible. $f \circ \text{id}^{-1} = f$ but $f(x) = x^{1/3}$ is not smooth at 0.

Note however $\text{id}: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}$ then id and g are C^∞ -compatible.
 $x \mapsto x^{1/3}$

A smooth manifold is a top. manifold M with a smooth structure.

Ex: \mathbb{R}^n with the chart $(\mathbb{R}^n, \text{id})$

\mathbb{R} with the chart (\mathbb{R}, f)

S^n is a smooth manifold (determined by stereographic projection)

All the matrix groups above are smooth manifolds with the standard charts.

\mathbb{C}^n is a smooth manifold determined by $\varphi(z_1, \dots, z_n) = (x_1, y_1, \dots, x_n, y_n)$
 $z_j = x_j + iy_j$

Let M be a smooth manifold then M is said to be n dimensional if all charts have codomain in \mathbb{R}^n .

Note: $\mathbb{R}^n \cong \mathbb{R}^m \Leftrightarrow n=m$ (Homology)



maps in the category of smooth manifolds should "smooth".

Suppose $f: M^{\overset{\text{dimension}}{m}} \rightarrow N^n$, this is smooth at $p \in M$ if there is a chart (U, φ) around p and a chart (V, ψ) around $f(p)$ s.t. the map $\psi \circ f \circ \varphi^{-1}$ is smooth.

f is smooth if f is smooth at all points $p \in M$.

Note: If f is smooth then choice of chart does not matter: $(U, \varphi), (U', \varphi')$ and $(V, \psi), (V', \psi')$ charts around p and $f(p)$ respectively. We have $\psi \circ f \circ \varphi^{-1}$ is smooth, $\psi' \circ \psi^{-1}$ and $\varphi \circ \varphi'^{-1}$ are smooth then note that the map $(\psi' \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \varphi'^{-1})$ is smooth.
" $\psi' \circ f \circ \varphi'^{-1}$

Projective space: let V be a vector space. Then we define an equivalence of V as follows: we say $v \sim u$ if there is a $\lambda \in \mathbb{F}^\times$ s.t. $v = \lambda u$. Then $PV = \{[v] \mid v \in V\}$

We show a vector space V is a manifold. So a chart here will be a choice of basis. This is a chart since a choice of basis gives a linear map $T: V \rightarrow \mathbb{R}^n$ by map $v \mapsto (a_1, \dots, a_n)$ s.t. $a_1 v_1 + \dots + a_n v_n = v$ (v_1, \dots, v_n is the choice of basis)