let M be a smooth manifold. A Riemannian metric $g \in \Gamma(\Sigma^2 TM)$ that is $\underbrace{\text{positive definite}}_{g_p(v_p, v_p) \geq 0 \text{ and } g_p(v_p,v_p) = 0 \iff v_p = 0}$ and nondegenerate at all $p \in M$. $g_p : T_pM \times T_pM \to \mathbb{R}$ is an inner product

A manifold M with a Riemannian metric is called a Riemannian manifold
locally we have $g = g_{ij}\, dx^i \otimes dx^j = g_{ij}\, dx^i dx^j$, $g_{ij} = g_{ji}$

Example: In $\mathbb{R}^2$: $g = dx \otimes dx + dy \otimes dy$ this is the standard metric.
$= dx^2 + dy^2$

On $\operatorname{Int} \mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ we have $g = \frac{1}{y^2}(dx^2 + dy^2)$ ← Poincaré half plane.

looking back in $\mathbb{R}^n$ when we have $\gamma : [a,b] \to \mathbb{R}^n$ the length of γ
$L(\gamma) = \int_a^b |\gamma'(t)|\, dt$

Now if $\gamma : [a,b] \to (M,g)$ then $L(\gamma) = \int_a^b \left(g(\gamma'(t), \gamma'(t))\right)^{1/2} dt$

Theorem: let M be a Riemannian manifold then there exists a Riemannian metric

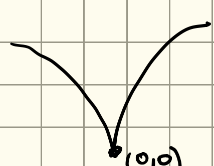
let $\gamma : [a,b] \to M$ be smooth.

i) γ is regular if $\dot\gamma \neq 0$ $\forall t \in [a,b]$
ii) γ is simple if γ is injective $[a,b)$
iii) γ is closed if $\gamma(a) = \gamma(b)$

Example: i) Going around S^1 once is closed and if $\gamma(t) = (\cos t, \sin t)$ $t \in [0, 2\pi]$
the γ is regular and simple

ii) $\gamma(t) = (t^2, t^3)$ is smooth by the image is not.

$\gamma'(t) = (2t, 3t^2)$
$\gamma'(0) = 0$



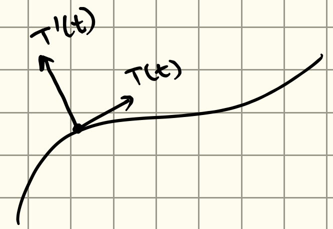
Hence we restrict ourselves to regular curves.

If $\eta : [c,d] \to [a,b]$ is a diffeomorphism then $\gamma \circ \eta$ is called a reparametrizat
of γ.

Theorem: A curve γ is regular $\iff$ it has unit speed reparametrization.

If γ is a curve $T(t) = \gamma'(t) \Rightarrow |T| = 1 \Rightarrow T \cdot T' = 0$. let $n(t)$ be unit norm, perpendicular to $T(t)$ and $[T, n] = [\hat{x}, \hat{y}] \Rightarrow T'(t) = k\, n(t)$

this is the signed curvature.



What about 3d?

for unit speed γ

we define $k(t) = |\gamma''(t)|$

let $T(t) = \gamma'(t)$ then $|T| = 1$. Define $n(t) = \frac{1}{k(t)} T'(t)$ we can complete T, n to a basis: $B = T \times n$

Binormal

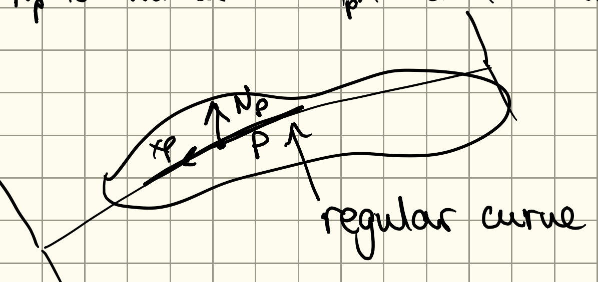
Note $|B| = |T||n| \sin\theta = 1$. At each point $\gamma(t)$ we have an ordered orthonormal basis $(T(t), n(t), B(t))$ this Frenet-Serret Basis.

Note $\dot{B} = \underbrace{\dot{T} \times n}_{0} + T \times \dot{n} = T \times \dot{n} \Rightarrow \dot{B} \perp T$ but since $|B| = 1 \Rightarrow \dot{B} \perp B \Rightarrow \dot{B} = -\tau n(t)$ we call τ the torsion of the curve.

We get the Frenet-Serret Equations:
$$\frac{d}{dt}\begin{bmatrix} T \\ n \\ B \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ B \end{bmatrix}$$

---

let M be a surface in $\mathbb{R}^3$. let $p \in M$ then choose $N_p \in T_p\mathbb{R}^3 \setminus T_pM$ s.t. N_p is normal to T_pM and unit length. let $X_p \in T_pM$ let γ define X_p then



we can define the "normal" curvature
$k_{N_p}(X_p) = \langle \gamma''(0), N_p \rangle$
this induces a function $k_{N_p} : S^1 \to \mathbb{R}$.

The extrema of k_{N_p} are called the principle curvatures, $k_{N_p,min}$, $k_{N_p,max}$.

The mean curvature $H_{N_p} = \frac{k_{p,min} + k_{p,max}}{2}$

The Gaussian curvature $K_{N_p} = k_{p,min}\, k_{p,max}$

Theorem (Gauss Bonnet): $\int_M K \, dS = 2\pi \chi(M)$