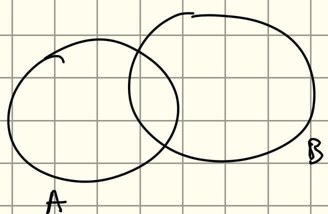
Fiber Bundles, we will spend time today on vector bundles.

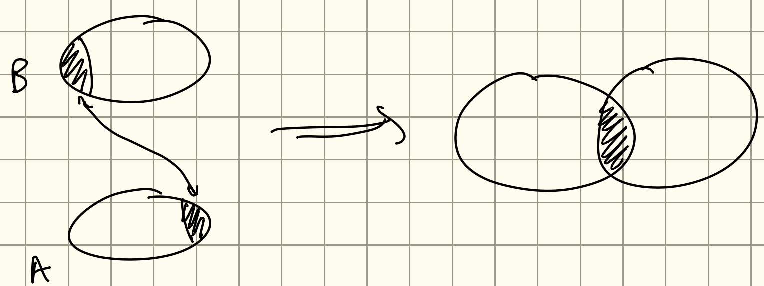
let M be a smooth manifold and let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ be an atlas for M.

We start by $Y = \underbrace{\bigsqcup_{\alpha \in \mathcal{A}} \varphi_\alpha(U_\alpha)}$ $\overset{\text{as a set}}{=}$ $\{(\alpha, x) \mid \alpha \in \mathcal{A} \text{ and } x \in \varphi_\alpha(U_\alpha)\}$ so we want to patch the U_α together in such a way so that we recover M. To do this we define an equivalence relation on Y: $V_\alpha = \varphi_\alpha(U_\alpha)$, $V_{\beta\alpha} = \varphi_\alpha(U_\alpha \cap U_\beta)$ and $V_{\alpha\beta} = \varphi_\beta(U_\beta \cap U_\alpha)$ then the transition functions $\psi_{\beta\alpha} : V_{\beta\alpha} \longrightarrow V_{\alpha\beta}$ by $\psi_{\beta\alpha} = \varphi_\beta \circ \varphi_\alpha^{-1}$

We say that $x \in V_{\beta\alpha}$ is equivalent to $y \in V_{\alpha\beta}$ if $y = \psi_{\beta\alpha}(x) = (\varphi_\beta \circ \varphi_\alpha^{-1})(x)$



but we have



we need to show that (i) $x \sim x$
(ii) $x \sim y \Leftrightarrow y \sim x$
(iii) $x \sim y$ and $y \sim z$ $\Rightarrow x \sim z$

(i) If $x \in V_{\alpha\alpha}$ then $\psi_{\alpha\alpha} = \varphi_\alpha \circ \varphi_\alpha^{-1} = id_{\mathbb{R}^m}$ but then $x = id_{\mathbb{R}^m}(x) = (\varphi_\alpha \circ \varphi_\alpha^{-1})(x) = \psi_{\alpha\alpha}(x)$. Hence $x \sim x$

(ii) If $x \in V_{\beta\alpha}$ and $y \in V_{\alpha\beta}$ with $x \sim y$ this means that $y = \psi_{\beta\alpha}(x) = (\varphi_\beta \circ \varphi_\alpha^{-1})(x)$ hence note that
$$\psi_{\alpha\beta}(y) = (\varphi_\alpha \circ \varphi_\beta^{-1})(\psi_{\beta\alpha}(x)) = \varphi_\alpha(\varphi_\beta^{-1}(\varphi_\beta(\varphi_\alpha^{-1}(x)))) = x$$
Hence $y \sim x$.

(iii) $x \in V_{\beta\alpha}$, $y \in V_{\gamma\beta}$ and $z \in V_{\beta\gamma}$ and $x \sim y$ and $y \sim z \Rightarrow y \in \varphi_\beta(U_\alpha \cap U_\beta)$ and $y \in \varphi_\beta(U_\beta \cap U_\gamma)$
$\Rightarrow y \in \varphi_\beta(U_\alpha \cap U_\beta \cap U_\gamma)$

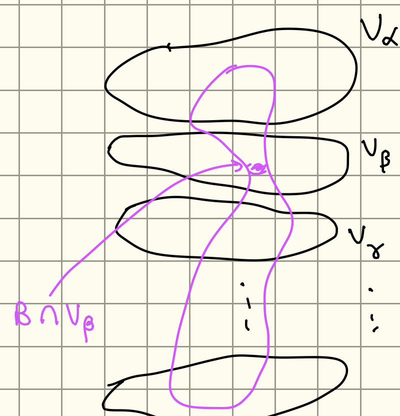
$y = \psi_{\beta\alpha}(x)$ and $z = \psi_{\gamma\beta}(y) = \psi_{\gamma\beta}(\psi_{\beta\alpha}(x)) = \varphi_\gamma(\varphi_\beta^{-1}(\varphi_\beta(\varphi_\alpha^{-1}(x)))) = (\varphi_\gamma \circ \varphi_\alpha^{-1})(x) = \psi_{\gamma\alpha}(x)$. Hence $x \sim z$.

This means that $\sim$ is an equivalence relation on Y.

let $X = Y/\sim = \{[x] \mid x \in Y\}$

Quick Aside: The topology of disjoint unions and quotients.

Suppose $A = \bigsqcup_{\alpha \in I} V_\alpha$ where V_α are top. spaces. Then we may visualize A as



Now if $B \subseteq A$ then B is open in the disjoint union topology $\Leftrightarrow$ $B \cap V_\alpha$ must be open in V_α.

Another way to look at this: we have maps $\iota_\alpha : V_\alpha \longrightarrow A$, $x \longmapsto (\alpha, x)$
then A has the "largest" topology that makes all maps ι_α continuous.

let A be a topological space and let $\sim$ be an equivalence relation of A.
let $B = A/\sim$. We have a map $q: A \to B$, $x \mapsto [x]$ then B has the largest topology so that q is continous.

Moreover one can show that as topological spaces M and X are homeomorphic.

let $f: M \to X$ be the homeomorphism and let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be an atlas of M.
Note that $f(U_\alpha) \subseteq X$ is open then consider the map $\varphi_\alpha \circ f^{-1}: f(U_\alpha) \to \hat{U}_\alpha = \varphi_\alpha(U_\alpha)$
this is a homeomorphism. Now the transition functions are given by $(\varphi_\beta \circ f^{-1}) \circ (\varphi_\alpha \circ f^{-1})$

$$= \varphi_\beta(f^{-1}(f(\varphi_\alpha^{-1}))))$$

$$= \varphi_\beta \circ \varphi_\alpha^{-1}$$

then $\{(f(U_\alpha), \varphi_\alpha \circ f^{-1})\}_{\alpha \in A}$ is a smooth atlas for X. Extending to a smooth structure on X we get M and X are diffeomorphic smooth manifolds.

Tangent Bundle:

First what is the tangent bundle of open subsets of $\mathbb{R}^n$? $TU = \bigsqcup_{p \in U} T_pU = U \times \mathbb{R}^n$

Note this works since U is an open subset of $\mathbb{R}^n$ and the transition functions are quite nice.
As we will see later $TS^2 \neq S^2 \times \mathbb{R}^2$. Also we will see that if M is a $\underbrace{\text{Lie group}}_{\substack{\text{manifold + group} \\ \text{s.t. multiplication} \\ \text{and inversion is} \\ \text{smooth.}}}$ then
$TM = M \times \mathbb{R}^m$

M is said to be parallelizable

---

let M^m be a (smooth) manifold with an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ then we take as our sets $T\varphi_\alpha(U_\alpha)$
so now the transition functions between overlapping tangent bundles has to be the derivative of the transition function $\psi_{\beta\alpha}: \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\beta \cap U_\alpha)$

$$T\psi_{\beta\alpha}: T\varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow T\varphi_\beta(U_\beta \cap U_\alpha)$$

$$\underset{\underset{\mathbb{R}^{2m}}{\cap}}{\varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^m} \ni (x, y) \longmapsto (\psi_{\beta\alpha}(x), D\psi_{\beta\alpha}(x)(y))$$

we say $x \in T\varphi_\alpha(U_\alpha \cap U_\beta)$ and $y \in T\varphi_\beta(U_\beta \cap U_\alpha)$ are equivalent $\iff$ $y = T\psi_{\beta\alpha}(x)$

Now we define the tangent bundle $TM = \left(\bigsqcup_{\alpha \in A} T\varphi_\alpha(U_\alpha) \right) \Big/ \sim$

$(\varphi_\alpha(x), v) \in T\varphi_\alpha(U_\alpha) = \varphi_\alpha(U_\alpha) \times \mathbb{R}^m$ then this equivalent to $T\psi_{\beta\alpha}(\varphi_\alpha(x), v) = (\psi_{\beta\alpha}(\varphi_\alpha(x)), D\psi_{\beta\alpha}(\varphi_\alpha(x))(v))$
which commutes out to $(\varphi_\alpha(x), v) \sim (\varphi_\beta(x), D(\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(x))(v))$

Aside: $F_{\mu\nu}\, dx^\mu \wedge dx^\nu$ and $F = B + E \wedge dt$

We have a collection of set U_α and maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow G$ s.t. (some lie group)

(i) $g_{\alpha\alpha} = e$

(ii) $g_{\beta\alpha}^{-1} = g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$

(iii) $g_{\gamma\alpha} = g_{\gamma\beta}\, g_{\beta\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$

These maps are called cocycles. If $G = GL(V)$ then we would have a vector bundle.
If $G = \mathrm{Diff}(F)$ then we would get fiber bundle.

Def: let $\pi : E \to M$ be a smooth surjection (π is submersion if $d\pi_p : T_pE \to T_{\pi(p)}M$ is surjective) then E is said to be a rank k-vector bundle of M if
(i) For each $p \in M$ the fiber $\pi^{-1}(p) = E_p$ is a k-dimensional real vector space
(ii) For each $p \in M$ the is a open set U containing p and diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\pi_U \circ \Phi = \pi$ where $\pi_U : U \times \mathbb{R}^m \longrightarrow U$ and $\Phi|_{E_p} : E_p \longrightarrow \mathbb{R}^k$ is a vector space isomorphism.

Note: we may take a representation $\rho : G \to GL(\mathbb{R}^n)$ to get: $E = \left(\bigsqcup_{\alpha \in A} U_\alpha \times \mathbb{R}^n \right) / \sim$
where $(\varphi_\alpha(x), v) \sim g_{\beta\alpha}(\varphi_\alpha(x), v) = (\varphi_\beta(x), \rho(g_{\beta\alpha}(x))v)$

Note that $X_p \in T_pM$ then $X_p = X_p^i\, d\varphi^{-1}\left(\frac{\partial}{\partial x^i}\right) = X_p^i \frac{\partial}{\partial x^i}\bigg|_p$

A vector field on M is a map $\sigma : M \longrightarrow TM$ st. $\pi \circ \sigma = id_M$

So if we have a vector bundle $\pi : E \to M$ then a map $\sigma : M \to E$ is a section if $\pi \circ \sigma = id_M$.