Category Theory:

In math we study objects and maps between objects. We also note that we can use different areas of math to study a particular area.

let V and W be vector spaces then the tensor product $V \otimes W$ is the unique vector space (up to isomorphism) s.t. for any vector space U and bilinear map $f: V \times W \to U$ there is a unique linear map $\hat{f}: V \otimes W \to U$ s.t. $\hat{f} \circ i = f$. Where $i: V \times W \to V \otimes W$ is an inclusion.

$$\begin{tikzcd} V \times W \arrow[r, "i"] \arrow[dr, "f"'] & V \otimes W \arrow[d, "\exists! \hat{f}"] \\ & U \end{tikzcd}$$

This "commutes" which means $\hat{f} \circ i = f$

Another example: let A and B be sets then $A \times B$ is the unique set s.t. we have maps $\pi_A: A \times B \to A$ and $\pi_B: A \times B \to B$ s.t. for any set Z and maps $f_A: Z \to A$ and $f_B: Z \to B$ there is a unique map $f: Z \to A \times B$ s.t.

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & Z \arrow[l, "f_A"] \arrow[r, "f_B"'] \arrow[u, dashed, "\exists! f"] & B \end{tikzcd}$$

Exercise: Check this!

But we can reverse arrows and get "dual" structures

$$\begin{tikzcd} & A \amalg B & \\ A \arrow[ur, "i_A"] \arrow[r, "f_A"'] & Z \arrow[u, dashed, "\exists! f"] & B \arrow[ul, "i_B"'] \arrow[l, "f_B"] \end{tikzcd}$$

in case of sets we get the disjoint union.

Exercise: Check this!

Is the collection of all sets a set?

Russel's Paradox: Consider $S = \{A \text{ is a set} \mid A \notin A\}$. Is $S \in S$?

if $S \in S \Rightarrow S \notin S$ so this is a contradiction
if $S \notin S \Rightarrow S \in S$ so this is a contradiction!!

S is therefore not a set. Hence the collection of all sets is not a set.

S is a <u>class</u>

↳ Any thing that can be written as a formula using first order logic.

Def: A category C has 2 things (i) Class of objects $Ob(C)$
(ii) For every $A, B \in Ob(C)$ there is a class $Hom(A,B)$ of arrows from $A \to B$

such that there is a composition rule between arrows: $\circ : Hom(A,B) \times Hom(B,C) \to Hom(A,C)$

with the following properties:

(i) For every object A, there is a arrow called $id_A : A \to A$
↳ If $f: A \to B$ then $f \circ id_A = id_B \circ f = f$

(ii) For every $f: A \to B$, $g: B \to C$, $h: C \to D$ we have $h \circ (g \circ f) = (h \circ g) \circ f$

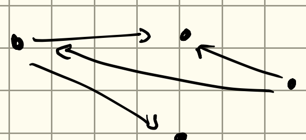
A category C is called small if $Ob(C)$ and $Hom(A,B)$ are sets
A category C is called locally small if $Hom(A,B)$ is a set.

Examples: (i) Category of sets with set functions (called **Set**)

(ii) Vector spaces + linear functions ($Vect_K$, K is the base field)

(iii) Groups + Group homomorphisms (Grp)

(iv) Top. spaces + continous functions (Top)

(v)  (directed) graphs are categories

(vi) let G be a group. Then G is a category!! It is a category with 1 object and the arrows are elements of G.

(vii) let C be any category we have a category called C^{op} which has the same objects as C but for every arrow $f: A \to B$ there is an arrow $f^{op}: B \to A$ ($\mathrm{Hom}_{C^{op}}(A,B) = \mathrm{Hom}_C(B,A)$)

(viii) Pointed Top. spaces + cont maps preserving base point (Top_*)
(X, x_0) $\quad f: (X, x_0) \to (Y, y_0)$
$f(x_0) = y_0$

A (covariant) functor from a category C to a category D is an arrow $F: C \to D$ s.t.

(i) We have an arrow from $\mathrm{Ob}(C) \to \mathrm{Ob}(D)$
$A \mapsto F(A)$

(ii) For each $A, B \in \mathrm{Ob}(C)$ an arrow $\mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(F(A), F(B))$
$f: A \to B \longmapsto F(f): F(A) \to F(B)$

s.t.

a) $F(g \circ f) = F(g) \circ F(f)$ whenever $A \xrightarrow{f} B \xrightarrow{g} C$

b) $F(id_A) = id_{F(A)}$

A (contravariant) functor is a (covariant) functor $F: C^{op} \to D$.

Examples: (i) The fundamental group can be seen as a functor $\pi_1: \mathrm{Top}_* \to \mathrm{Grp}$ to each $(X, x_0) \mapsto \pi_1(X, x_0)$ and to each $f: (X, x_0) \to (Y, y_0)$ we assign $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$

(ii) Group representations are (covariant) functors $\mathrm{Grp} \to \mathrm{Vect}_k$. More specifically group actions are functors from $\mathrm{Grp} \to \mathrm{Set}$.

Now we've seen categories and arrows between them. So is there a category of categories!

But we may define Cat: Ob(Cat): small categories and whose morphisms are functors

Are there arrows between functors? These are called natural transformations: suppose we have functors $F: C \to D$ and $G: C \to D$ then $\alpha: F \to G$ is a natural transformation if: α is a family of arrows in D ($F(A) \xrightarrow{\alpha_A} G(A)$) s.t for every $f: A \to B$ the following commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\circlearrowright"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

we have a natural category defined as follows: let C and D be categories then the category $[C, D]$ has objects $F: C \to D$ functors and the morphisms are natural transformations.

Note we have an identify natural transformation $1_F: F \to F$ given by $(1_F)_A = id_{F(A)}$.

Moreover given $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ natural transformations we have $F \xrightarrow[\beta\alpha]{} H$ s.t. $(\beta\alpha)_A = \beta_A \alpha_A$ $\quad (\beta\alpha)_A : F(A) \to H(A)$

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[dr, dashed, "\beta_A\alpha_A"'] & G(A) \arrow[d, "\beta_A"] \\ & H(A) \end{tikzcd}$$

we can also write natural transformations as
$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$$

C, D are categories

$F, G: C \to D$ functors then α is a natural transformation.