

u-sub: $\int f(g(x))g'(x) dx = \int f(u) du$
 $u = g(x)$

$\int_{B(0;R)} x^2 + y^2 d^2\mu$ how to integrate this?

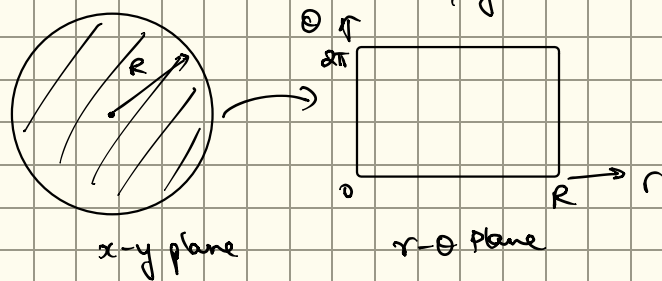
Can we deform $\overline{B(0;R)}$ into a rectangle? Almost, we can do continuously but not smoothly.

The singularities are measure 0, so they don't really matter for the integral.

So $F: V \rightarrow U$ is called a diffeomorphism if F is C^∞ and bijective so that F^{-1} is also C^∞ .
 (For what we will do, we allow F to be nondiffeomorphic upto a set of measure 0)

Example: $x = r \cos \theta$ where $r \in [0, \infty)$ so this fails only on $\{x, 0\} \subseteq \mathbb{R}^2$
 $y = r \sin \theta$ $\theta \in [0, 2\pi]$

So a ball is a rectangle in polar coordinates $\overline{B(0;R)}$ we have $(x,y) \in B(0;R) \Leftrightarrow x^2 + y^2 \leq R^2$
 so we have that $x = r \cos \theta$, $y = r \sin \theta$ where $r \in [0, R]$ and $\theta \in [0, 2\pi]$



Theorem (Change of variables): If we have a function $f: U \rightarrow \mathbb{R}$ and we compute $\int_U f d^2\mu$
 and we diffeomorphism $F: V \rightarrow U$ then $\int_U f d^2\mu = \int_V (f \circ F) |\det DF| d^2\mu$
 (up to measure 0 sets) \uparrow
 \mathbb{R}^n

Thm/

Def: Suppose X is a topological manifold and $\{U_\alpha\}$ is an open cover of X s.t.
 For each $x \in X$ there is only a finite no. of open sets U_α s.t. $x \in U_\alpha$. Then there is a collection $\{\rho_\alpha: U_\alpha \rightarrow \mathbb{R}\}$ of continuous maps s.t.

(i) For each $x \in X$ we have $0 \leq \rho_\alpha(x) \leq 1$

(ii) $\forall x \in X$ we have $\sum_\alpha \rho_\alpha(x) = 1$

(iii) Each ρ_α vanishes outside an open set containing U_α .

Then $\{U_\alpha, \rho_\alpha\}$ is called a partition of unity.

$\int_{B(0;R)} x^2 + y^2 d^2\mu$ how to integrate this?

Step 1: Find diffeo from a simpler space into $B(0;R)$

$$F: [0, R] \times [0, 2\pi] \rightarrow B(0;R)$$

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta)$$

Step 2: Find DF: $DF(r, \theta) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$

Step 3: Find $|\det DF| = |r \cos^2 \theta + r \sin^2 \theta| = r$

Step 4: $\int_{B(0;R)} x^2 + y^2 d^2\mu = \int_{[0, R] \times [0, 2\pi]} r^2 r dr d\theta = \int_{[0, R]} \left(\int_{[0, 2\pi]} r^3 d\theta \right) dr = 2\pi \int_{[0, R]} r^3 dr = \frac{\pi R^4}{2}$

We can compute area of circle: $\int_{B(0;R)} 1 d^2\mu = \pi R^2$

Consider the Gaussian integral: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (If you want to try on your own check problem sheet 3, if not check meeting plan)

In \mathbb{R}^3 we have spherical coordinates: $x = r \cos \varphi \cos \theta$
 $y = r \sin \varphi \cos \theta$
 $z = r \sin \theta$ $\Rightarrow DF = r^2 \sin \theta$
 $r \in [0, \infty), \varphi \in [0, 2\pi], \theta \in [0, \pi]$

so we can compute volume of sphere by $\int_{B(0;R)} 1 d^3\mu = \int_0^R \int_0^{2\pi} \int_0^\pi r^2 \sin \theta d\theta d\varphi dr$
 $= \frac{4}{3} \pi R^3$

Higher dimensional spherical coordinates: $r, \theta_1, \theta_2, \dots, \theta_{n-1}$ where $r \in [0, \infty)$
 $\theta_i \in [0, 2\pi)$ for $i \neq n-1$
 $\theta_{n-1} \in [0, \pi]$

$$x_1 = r \cos \theta_1$$

$$x_p = r \cos \theta_p \prod_{m=1}^{p-1} \sin \theta_m$$

$$x_n = r \prod_{m=1}^{n-1} \sin \theta_m$$

Def: (i) If $u: \mathbb{R}^n \rightarrow \mathbb{R}$ then $\nabla u = Du = \text{grad } u$
(ii) If $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$ then $\nabla \cdot u = \text{tr}(Du) = \text{div } u$
(iii) If $u: \mathbb{R}^n \rightarrow \mathbb{R}$ then $\Delta u = \nabla^2 u = \text{div}(\text{grad } u) = \text{tr}(D(\text{grad } u)) = \text{tr}(D^2 u)$

line integrals: $\int_C \vec{F} \cdot d\vec{r} = \int_C \sum_i F^i dx^i$ these look "1-dimensional" so is there a FTC type theorem? The answer is yes!! So suppose $F: \mathbb{R}^n \rightarrow \mathbb{R} \Rightarrow \nabla F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ so then

$$\int_C \nabla F \cdot d\vec{r} = F(\text{end point}) - F(\text{start point})$$

1 dimensional integral
"0 dimensional integral"

if C is closed loop then $\int_C \nabla F \cdot d\vec{r} = 0$

if $\int_C \vec{F} \cdot d\vec{r} = 0$ is $\vec{F} = \nabla g$? We will answer this using deRham cohomology. then this is true

So suppose we have $\int_C P dx + Q dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$ where $C = \partial D$. This is called Green's theorem.

simple, piecewise smooth, closed

let us say we have a 2 dimensional surface A in \mathbb{R}^3 . We locally parametrize A by $r(s,t) = (x(s,t), y(s,t), z(s,t))$, $(s,t) \in D$

the surface area is given $\int_D \left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right| dA$

let \vec{F} and \vec{V} be vector (vector fields) then $\vec{F} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_x & F_y & F_z \\ V_x & V_y & V_z \end{vmatrix}$

If $f: A \rightarrow \mathbb{R}$ then $\iint_A f dA = \iint_D f(r(s,t)) \left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right| d^2 \mu$

Note: $\left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right| = |\det((Dr)^T(Dr))|$

The normal of a surface is defined by $\hat{n} = \frac{\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}}{\left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right|}$

If \vec{F} is a vector field then $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} dS$

If $\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ the $\text{curl}(F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$

$\iint_S \nabla \times \vec{F} \, dS = \int_{\partial S} \vec{F} \cdot d\vec{r} \leftarrow \text{Stokes' theorem}$

\uparrow 2d integral \uparrow 1d integral

$\iiint_V \text{div } F \, d^3\mu = \iint_{\partial V} \vec{F} \cdot d\vec{S} \leftarrow \text{Divergence theorem}$

\uparrow 3d integral \uparrow 2d integral

Generalized Stokes theorem: $\int_M d\omega = \int_{\partial M} \omega$