Groups are important because we can use them to "act" on various types of object.

We have seen this with how invertible matrices "act" by matrix multiplication to vectors in $\mathbb{R}^n$.

But before we can talk about those we first talk about maps between group. A function $\varphi: G \to H$ is called a (group) homomorphism if $\varphi(gh) = \varphi(g)\varphi(h)$
Basically they preserve the group operation.

Now given a set M and a group G we can make G "act" on M by bijections of M. (Show that this is infact a group with composition)

We will call this group $\mathrm{Aut}(M)$ [In field theory and such if M is an algebraic object then $\mathrm{Aut}(M)$ is usually the isomorphism group]

Now take a homorphism $G \to \mathrm{Aut}(M)$ then $\varphi(g): M \to M$ is a bijection.
$g \mapsto \varphi(g)$

We see this in physics with spin of quantum particles.

An important class of group actions are groups acting on vector spaces.
We call a group homomorphism $\varphi: G \to GL(V)$ a representation of G on V. We have see that $\varphi: GL(\dim V, \mathbb{R}) \to GL(V)$ is a representation of the matrix group on the isomorphism group of V.

Representations are very important in study of Quantum systems and we will see more when we talk about lie groups.

let now talk about representations on a function space. Say we take a group G and it has a representation π on V. Say now we have some function space on $V = W$ (common examples are $L^2(V)$) then $\pi : G \to GL(V)$ can also act on W.

let $g \in G$ and $f \in W$ then define $(\pi^*(g) f)(v) = f(\pi(g^{-1})v)$. Now see that we needed to invert the group element. Check that π^* is a representation on W and check why we need to invert g.

(doesn't have to be a subspace)
Suppose now that there is a subset $U \subseteq V$ s.t. $\forall g \in G$ $\pi(g) U \subseteq U$. Then we can define a sort of restriction representation on U. Since U doesn't have to be a subspace this map may not map into a general linear group but some function group on U. So we will still call these representations.

Suppose now that U was a vector subspace then $\pi(g)|_U \in GL(U)$ and hence π restricts to a representation on U. We call such representations reducible.

Now one might ask when are two representations are the same. The idea here is that we want to be able to move "freely" with the representations so Say $\pi_1 : G \to GL(V_1)$ and $\pi_2 : G \to GL(V_2)$ are representations. Then π_1 and π_2 are equivalent if $T : V_1 \xrightarrow[\text{linear}]{\cong} V_2$ s.t.

$$\begin{tikzcd} V_1 \arrow[r, "\pi_1(g)"] \arrow[d, "T"'] \arrow[dr, phantom, "\circlearrowright"] & V_1 \arrow[d, "T"] \\ V_2 \arrow[r, "\pi_2(g)"'] & V_2 \end{tikzcd}$$

or in other words $T\pi_1(g) = \pi_2(g) T$

Equivalence is a good word to use here since the action of $\pi_1(g)$ and $\pi_2(g)$ are the same up to certain isomorphism of the underlying spaces. We write $\pi_1 \cong \pi_2$

Let's now see a technical result regarding finite dimensional representation over $\mathbb{C}$ [I think this works over arb. algebraically closed fields]

This theorem is called Schur's lemma.

Let $\pi_1 : G \to GL(V_1)$ and $\pi_2 : G \to GL(V_2)$ be irreducible representations on complex v.s. V_1 and V_2

Now suppose $S : V_1 \to V_2$ is linear s.t.

$$\begin{tikzcd} V_1 \arrow[r, "\pi_1(g)"] \arrow[d, "S"'] \arrow[dr, phantom, "\circlearrowright"] & V_1 \arrow[d, "S"] \\ V_2 \arrow[r, "\pi_2(g)"'] & V_2 \end{tikzcd}$$

(i) If $\pi_1 \not\cong \pi_2$ then $S \equiv 0$

(ii) If $\pi_1 = \pi_2$ then $S = \lambda I$ for some $\lambda \in \mathbb{C}$

Got this proof from article by Noah Miller

Proof: (i) let $g \in G$. Let $v_1 \in \ker S \Rightarrow S(\pi_1(g)v_1) = \pi_2(g)(Sv_1) = \pi_2(g)(0) = 0 \Rightarrow \pi_1(g)v_1 \in \ker S$

let $v_2 \in \operatorname{Im} S \Rightarrow \exists w \in V_1$ s.t. $Sw = v_2 \Rightarrow \pi_2(g)(w) = \pi_2(g)(Sw) = S(\pi_2(g)w) \in \operatorname{Im} S$

Hence $\ker S$ and $\operatorname{Im} S$ are invariant subspaces and hence $\ker S$ is a subrep of π_1 and $\operatorname{Im} S$ is a subrep of π_2. But they are irreducible hence $\ker S = \{0\}$ or V_1

If $\ker S = \{0\}$ then S is injective hence $\operatorname{Im} S \neq 0 \Rightarrow \operatorname{Im} S = V_2$ hence S is an isomorphism meaning $\pi_1 \cong \pi_2$. Which is a contradiction

Hence $\ker S = V_1 \Rightarrow S \equiv 0$.

exists by algebraic closure

(ii) the statement says $\pi(g) S = S \pi(g)$. let λ be an eigenvalue of S. Then

$$\begin{aligned}(S-\lambda I)(\pi(g)v) = S\pi(g)v - \lambda\pi(g)v &= \pi(g)Sv - \pi(g)\lambda v \\ &= \pi(g)(Sv - \lambda v) \\ &= \pi(g)(S-\lambda I)v\end{aligned}$$

Now if $Sv = \lambda v$ then
$$\begin{aligned}(S-\lambda I)(\pi(g)v) &= \pi(g)(Sv - \lambda v) \\ &= \pi(g)(\lambda v - \lambda v) \\ &= 0\end{aligned}$$

and so $\pi(g)(v)$ is also an eigenvalue meaning $\ker(S - \lambda I)$ is a subrep. Since π_1 is irreducible $\Rightarrow S = \lambda I$.

Corollary 1 and 2 in the document are nice and you should check them out.

taken from Gauge fields, knots, gravity

Examples: $U(1)$ has representations π_n on $\mathbb{C}$ by $\pi_n(\theta)v = e^{in\theta}v$

$SO(n)$ has a representation on $\mathbb{R}^n$ by rotation

Poincaré group has a representation on $\mathbb{R}^4$ by Lorentz transformations

Now we discuss, what I think, is the most important representation in mathematical physics. Those are the $\mathrm{Spin}(n)$ representations. These describe the spin of particles and give rise to the dirac equation on curved spacetimes. But to define these in general we require some more machinery. So let us look at $\mathrm{Spin}(3) = SU(2)$

So let use treat $\mathbb{R}^3 \subseteq U(2)$ by $(x,y,z) \xmapsto{f} x\sigma_x + y\sigma_y + z\sigma_z$

$$= x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -yi \\ yi & 0 \end{pmatrix} + \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}$$

$$= \begin{pmatrix} z & x-yi \\ x+yi & -z \end{pmatrix}$$

Note that $\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}^\dagger = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$. More f is actually a bijection between $\mathbb{R}^3$ and 2×2 hermitian matrices with trace 0.

Now take $T \in \mathbb{R}^3$ and $g \in SU(2)$ then $tr(gTg^{-1}) = 0$ by change of basis.
$(gTg^{-1})^\dagger = (g^{-1})^\dagger T^\dagger g^\dagger = gTg^{-1}$ hen gTg^{-1} is hermitian and hence $gTg^{-1} \in \mathbb{R}^3$.

Now we check $\rho(g)(T) = gTg^{-1}$ is a representation of $SU(2)$ on $\mathbb{R}^3$.

1) $\rho(1)(T) = 1T1^{-1} = T$

2) $\rho(gh)(T) = (gh)T(gh)^{-1} = ghTh^{-1}g^{-1} = g(hTh^{-1})g^{-1} = g(\rho(h)T)g^{-1}$
$$= \rho(g)\rho(h)T$$

Now we note if $T = (T^1, T^2, T^3) \cong T^1\sigma_1 + T^2\sigma_2 + T^3\sigma_3$ then

$$\det(T^1\sigma_1 + T^2\sigma_2 + T^3\sigma_3) = \det\begin{pmatrix} T^3 & T^1 - iT^2 \\ T^1 + iT^2 & -T^3 \end{pmatrix}$$

$$= -(T^3)^2 - (T^1 - iT^2)(T^1 + iT^2)$$

$$= -\left[(T^1)^2 + (T^2)^2 + (T^3)^2\right]$$

$$= -\|T\|^2$$

Now we check that $\|T\| = \|\rho(g)T\|$. Note $-\|\rho(g)T\|^2 = \det(\rho(g)T)$

$$= \det(gTg^{-1})$$

$$= \det(T)$$

$$= -\|T\|^2$$

hence $\|T\| = \|\rho(g)T\|$. So $\rho(g) \in U(2)$. When we do topology we will prove that $\rho(g)$ has determinant 1. Hence $\rho(g) \in SO(3)$. This means we have a group homomorphism $SU(2) \longrightarrow SO(3)$. Now is this map an isomorphism? No. it is not but it is 2:1. Note that $\rho(-g)T = (-g)T(-g)^{-1} = gTg^{-1} = \rho(g)T \Rightarrow \rho(g) = \rho(-g)$.

Also if $\rho(g) = \rho(h)$ then $\rho(gh^{-1}) = \rho(g)\rho(h^{-1}) = 1$. So now $\rho(gh^{-1})$ must commute with every $S: \mathbb{R}^3 \to \mathbb{R}^3$. But then gh^{-1} commute with everything. And so $gh^{-1} = \lambda I$ but $\lambda I \in SU(2) \Rightarrow \lambda = \pm 1$ hence

$gh^{-1} = \pm 1 \Rightarrow g = \pm h$. The surjectivity will be proven later.