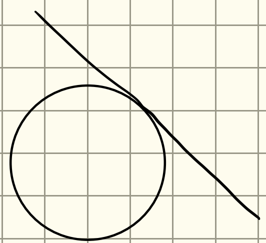
Frobenius' Theorem.

$TS \neq TM|_S$



We say a vector field $X \in \mathfrak{X}(M)$ is tangent to a submanifold $S \subseteq M$ if $X_p \in T_pS \ \forall p \in S$.

Suppose $X, Y \in \mathfrak{X}(M)$ that are tangent to S then we claim $[X,Y]$ is also tangent to S.

Proof: let $i : S \hookrightarrow M$ be inclusion then $X|_S \sim_i X$ and $Y|_S \sim_i Y \Rightarrow [X|_S, Y|_S] \sim_i [X,Y]$

$\Rightarrow [X,Y]$ is tangent to S.

let $X_1, \dots, X_n \in \mathfrak{X}(M)$ tangent to S if these are a basis then

$[X_i, X_j] = c_{ij}^k X_k$

This condition is called the Frobenius condition.

Frobenius' Theorem: let $X_1, \dots, X_n \in \mathfrak{X}(M)$ s.t. they are linearly independent $\forall p \in M$. Then the following are equivalent:

(i) For all $p \in M$ there $\exists S \subseteq M$ n-dim submanifold with $p \in S$ and $X_1, \dots, X_n$ every tangent to S.

(ii) $X_1, \dots, X_n$ satisfy the Frobenius condition.

Bundle	Section
TM	Vector Fields
T^*M	Differential 1-forms
$T^{(k,l)}M$	

$\omega = x^2 dx + y^2 dy$

$\omega_{(1,0)} = dx : T_{(1,0)}\mathbb{R}^2 \to \mathbb{R}$

$\pi_1 : \mathbb{R}^2 \to \mathbb{R}$

$\omega_{(0,1)} = dy$

π_2

If $\alpha \in \Omega^1(M)$ locally $\alpha_p = \alpha_i(p)\, dx^i$

$\alpha : M \to T^*M$

$\alpha_p : T_pM \to \mathbb{R}$

$\omega \in T^{(k,0)}M$ then locally $\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$ this naturally lets us think of ω as a multilinear map on T_pM.

Sections of $T^{(k,0)}M$: At each point I get a multilinear map.

At each $p \in M \rightsquigarrow T_p^{(k,0)}M = \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{k\text{-times}} \supseteq \Lambda^k T_p^* M$

$V^* \otimes \cdots \otimes V^* \supseteq \Lambda^k V^*$

$\rightsquigarrow \Lambda^k T^* M$, sections of $\Lambda^k T^*M = \Gamma(\Lambda^k T^* M) = \Omega^k(M) \leftarrow$ these are differential k-forms.

let $\omega \in \Omega^k(M)$ then locally $\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

We can define the wedge product of 2 forms $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ by
$(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p$, $\alpha_p \wedge \beta_p = \frac{(k+l)!}{k!\,l!} \mathrm{Alt}(\alpha_p \otimes \beta_p)$

$$(\mathrm{Alt}(\omega))(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \frac{1}{k!} \mathrm{sgn}(\sigma) \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

$f: \mathbb{R}^3 \to \mathbb{R}$ then $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

$$X = \frac{\partial}{\partial x} \Rightarrow df(X) = \frac{\partial f}{\partial x} \underbrace{dx\left(\frac{\partial}{\partial x}\right)}_{1} + \cancel{\frac{\partial f}{\partial y} dy \left(\frac{\partial}{\partial x}\right)}^{0} + \cancel{\frac{\partial f}{\partial z} dz \left(\frac{\partial}{\partial x}\right)}^{0}$$

$$= \frac{\partial f}{\partial x}$$

$d^0: \Omega^0(M) \to \Omega^1(M)$

$d^1: \Omega^1(M) \to \Omega^2(M)$

$d(fg) = g\,df + f\,dg = df \wedge g + f \wedge dg$

$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$

$d(df) = 0 \Leftrightarrow d \circ d = 0$

Theorem: There is a unique collection of maps $d: \Omega^k(M) \to \Omega^{k+1}(M)$ s.t.

(i) linear over $\mathbb{R}$
(ii) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
(iii) $d(d\omega) = 0$
(iv) If $f \in C^\infty(M) = \Omega^0(M)$ then $df(X) = Xf$

Exercise: Show locally that if $\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ then $d\omega = \partial_j \omega_{i_1 \dots i_k} dx^j \wedge dx^{i_1} \dots \wedge dx^{i_k}$

If $T: V \times V \to \mathbb{R}$ and $w \in V$ then $\iota_w T : V \to \mathbb{R}$ interior multiplication
$v \mapsto T(w, v)$

$$d\omega(X_1, \dots, X_k, \underbrace{X_{k+1}}_{\uparrow}) = \sum_{1 \le i \le k+1} (-1)^{i-1} X_i(\omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1})) + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1})$$

extra compared to ω

$\mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega$ $\leftarrow$ Cartan's magic Formula

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$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

$\leftarrow$ dim M

Hence we get a exact sequence of vector spaces.

deRham cohomology (degree k): $H^k_{dR}(M) = \ker(d: \Omega^k(M) \to \Omega^{k+1}(M)) / \operatorname{im}(d: \Omega^{k-1}(M) \to \Omega^k(M))$

$\omega, \eta \in \Omega^k(M)$ are equivalent if $\omega - \eta = d\alpha$ for some $\alpha \in \Omega^{k-1}(M)$

ω is exact if $\omega = d\alpha$
ω is closed if $d\omega = 0$

If ω is exact then $d\omega = dd\eta = 0$

$$\chi(M) = \sum_{k=0}^{n} \dim H^k_{dR}(M)$$

Example: $\omega = \frac{-x}{x^2+y^2} dy + \frac{y}{x^2+y^2} dy$ is closed but not exact $(\mathbb{R}^2 \setminus \{0\})$

$f: V \to W$ this induces a map $f^*: W^* \otimes \cdots \otimes W^* \to V^* \otimes \cdots \otimes V^*$

$(f^*T)(v_1, \ldots, v_n) = T(f(v_1), \ldots, f(v_n))$ where $T \in W^* \otimes \cdots \otimes W^*$

If $f: M \to N$ smooth this induces a map $f^*: \Omega^k(N) \to \Omega^k(M)$

$(f^*\omega)_p(X_1, \ldots, X_k) = \omega_{f(p)}(df_p(X_1), \ldots, df_p(X_k))$

Exercise: show $dF^*\omega = F^*d\omega$

M smooth manifold and let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas on M.

$\varphi_\alpha: U_\alpha \to \hat{U}_\alpha \subseteq \mathbb{R}^m$

$\varphi_\alpha^{-1}: \hat{U}_\alpha \to U_\alpha$ $\qquad \omega \in \Omega^m(M)$ say $\int_M \omega$ but $\int_{U_\alpha} \omega = \int_{\hat{U}_\alpha} \overbrace{(\varphi_\alpha^{-1})^*\omega}^{\Omega^m(\hat{U}_\alpha)}$

⊛ Read up on partitions of unity