Problem Sheet 2 - Geometry of Physics

September 2025

Question 1 (Recap on Derivatives). *1. Argue that any $T \in \mathcal{T}_0^k(\mathbb{R}^n)$ is smooth. Moreover show that the derivative at $(a_1, \ldots, a_k) \in \mathbb{R}^{n\times k}$ is given by*

$$DT(a_1, \ldots, a_k)(x_1, \ldots, x_k) = \sum_{i=1}^{n} T(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_k)$$

Conclude that $\det(id_{n\times n})(A) = \mathrm{tr}(A)$.

2. *Show that if $f : \mathbb{R} \to S^n \subseteq \mathbb{R}^{n+1}$, show that f' and f are orthogonal.*

3. *Find the derivative of the map $A \to AA^T - id_n$ where $A \in \mathbb{R}^{n^2}$. Show that this maps into the space of symmetric matrices and show that the derivative is surjective at id_n.*

4. *Let $\mu : \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ be given by $\mu(A, B) = AB$. Show that $D\mu(id_{n\times n}, id_{n\times n})(A, B) = A + B$. Using this to then show that the derivative of the map $i : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ given by $i(A) = A^{-1}$ is $Di(id_{n\times n})(A) = -A$. Extend this to show that $D\mu(X, Y)(A, B) = AY + XB$. Hint: Consider the curve $\varphi(t) = (A + tX, B)$ so $\varphi(0) = (A, B)$ and $\varphi'(0) = (X, 0)$ then compute*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mu(\varphi(t))$$

Then use linearity and the fact that if the Gâteaux derivative exists for all points and is continuous then the derivative exists.

5. *Let $C_g(A) = gAg^{-1}$ use the above to compute $DC_g(A)(B)$.*

6. *Let $m : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be defined $m(v, w) = v \times w$. Show that m is differentiable and compute $Dm(x, y)(a, b)$ where $x, y, a, b \in \mathbb{R}^3$ are arbitrary. Does this look familiar?*

Question 2 (Banach Fixed Point and Uniqueness Theorem). *For the following let (X, d) be a complete metric space.*

1. *Let $f : X \to X$ be such that there is a $C \in [0, 1)$ such that $d(f(x), f(y)) \leq Cd(x, y)$ for every $x, y \in X$. Show that f must have a fixed point. Moreover show that this fixed point is unique.*

Now consider the following initial value problem on a closed rectangle $R \subseteq \mathbb{R} \times \mathbb{R}^n$

$$\begin{cases} y'(t) &= f(t, y) \\ y(t_0) &= y_0 \end{cases}$$

where $f : R \to \mathbb{R}^n$ is so that there is a number $M > 0$ such that

$$|f(t, y_1) - f(s, y_2)| \leq K|y_1 - y_2|$$

for every y_1, y_2. Such functions are called Lipschitz continuous in y. Our goal will be to prove that this initial value problem has a unique solution on some interval of the form $[t_0 - \epsilon, t_0 + \epsilon]$.

2. *You may assume that that space*

$$U(I, K) = \{\gamma : I \to K \mid \gamma(t_0) = y_0, \gamma \text{ is continuous}\}$$

along with the supremum norm is a complete metric space where I is a closed interval containing t_0 and K is a compact set containing y_0. Now let $0 < \epsilon < \dfrac{D}{M}$ where M is the maximum of f on $I \times K$ and D is the radius of a ball contained in K centered at y_0. With this show that the map

$$P(\gamma)(t) = y_0 + \int_{t_0}^{t} f(s, \gamma(s))\,\mathrm{d}s$$

defines a map $P : U((t_0 - \epsilon, t_0 + \epsilon), \overline{B(x, D)}) \to U((t_0 - \epsilon, t_0 + \epsilon), \overline{B(x, D)})$. *That is you must show that this map takes* $\gamma \in U((t_0 - \epsilon, t_0 + \epsilon), \overline{B(x, D)})$ *and gives* $P(\gamma) \in U((t_0 - \epsilon, t_0 + \epsilon), \overline{B(x, D)})$.

3. *Let* $\epsilon = \dfrac{1}{2} \min \left\{ \dfrac{1}{K}, \dfrac{D}{M} \right\}$. *Show that* P *is now a contraction.*

This now shows that there is a unique solution on this domain. Moreover the proof of the Banach fixed point theorem then shows that this solution also exists.

4. *Think about how we know the solution exists? Hint: It follows using the trick used in the first part of this question.*

Question 3 (Practice on Integration). 1. *Think of* $\mathbb{C}$ *as* $\mathbb{R}^2$ *and think of the integral of* $f = u + iv$ *as*

$$\int f = \int u + i \int v$$

now compute the integral of $f(z) = \dfrac{1}{z}$ *over a circle that goes around the origin once and has radius* R. *What about* $f(z) = z^n$ *for* $n > -1$? *What about* $f(z) = z^n$ *for* $n < -1$? *Use this to then conclude that if* f *as a power series expansion of the form (that uniformly converges to* f*)*

$$f(z) = \sum_{i \in \mathbb{Z}} a_i z^i$$

then

$$\int_{B(0;R)} f(z) = 2\pi i a_{-1}$$

2. *In this question we will find the center of mass and the moment of inertia tensor for a uniformly distributed ellipsoid with radii* a, b, c. *Moreover, we will see that the moment of inertia is a choice of inner product on* $\mathfrak{so}(3)$.

   (a) *First find the change of variables from a ball of radius* 1 *to a the ellipsoid and find its volume.*

   (b) *Find the density,* ρ, *of the ellipsoid given that the total mass is* M.

   (c) *Note that if* $\vec{R}$ *is the center of mass then*

   $$\int_V \rho(\vec{r})(\vec{r} - \vec{R}) \,\mathrm{d}V = 0$$

   use this to then find the center of mass.

   (d) *Now use the formula that*

   $$I_{jk} = \int_V \rho(\vec{r})(r^2 \delta_{jk} - x_j x_k) \,\mathrm{d}V$$

   here x_i *denotes* x, y, z *for* $i = 1, 2, 3$, *to compute the moments of inertia of the ellipsoid.*

Question 4 (Complexification and Symplectic Spaces). *This question will just be random facts about complexifications.*

1. *Show that*

$$J = \begin{pmatrix} 0 & id_{n\times n} \\ -id_{n \times n} & 0 \end{pmatrix}$$

is an almost complex structure on $\mathbb{R}^{2n}$. *Conclude that every even dimensional real vector space admits an almost complex structure.*

2. *On* $\mathbb{R}^{2n}$ *let* $e_1, \ldots, e_{2n}$ *be the standard basis, then note that the map above is* $e_i \to e_{i+n}$ *and* $e_{i+n} \to -e_i$ *for* $1 \le i \le n$. *Now define the action of* $\mathbb{C}$ *on* $\mathbb{R}^{2n}$ *by* $(a + bi)e_j = ae_j + bJ(e_j)$. *Show that this make* $\mathbb{R}^{2n}$ *into a* n*-dimensional complex vector space.*

3. *Show that the isomorphism given above makes the spaces* $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$ *and* $\mathbb{C}^n$ *naturally isomorphic.*

4. *Let* $z_1, \ldots, z_n$ *be the standard basis on* $\mathbb{C}^n$ *then we can treat* $\mathbb{C}^n$ *as a* $2n$*-dimensional real vector space with basis* $x_1, \ldots, x_n, y_1, \ldots, y_n$ *with* $z_j = x_j + iy_j$. *Let* $\mathrm{d}x^1, \ldots, \mathrm{d}x^n, \mathrm{d}y^1, \ldots, \mathrm{d}y^n$ *be the dual basis. Let* $\omega \in \bigwedge^2((\mathbb{R}^{2n})^*)$ *be given by*

$$\omega = \sum_i \mathrm{d}x^i \wedge \mathrm{d}y^i$$

find a suitable expression for $\mathrm{d}z^i$ *and* $\mathrm{d}\overline{z}^i$. *Moreover write* ω *in terms of these forms.*

5. *Show that* $\dfrac{\omega^n}{n!} = \det$.