

Problem Sheet 4 - Geometry of Physics

January 2026

Question 1 (Category Theory). (i) Show that Top_* and Grp are categories and show that $\pi_1 : \text{Top}_* \rightarrow \text{Grp}$ is a covariant functor.

(ii) Given a small category C a (set valued) presheaf is a contravariant functor from C to Set , in other words it is a functor $C^{\text{op}} \rightarrow \text{Set}$. Show that this forms a category with natural transformations.

Question 2 (Manifolds). (i) Show that S^n can be made into a smooth manifold with the following atlas: Define the set $U_{i,\pm} \subseteq S^n$ so that $x \in U_{i,\pm}$ if and only if the i th component of x is strictly positive or negative respectively. Let the chart maps be $\varphi_{i,\pm} : U_{i,\pm} \rightarrow B(0, 1)$ given by $\varphi(x) = (x_1, \dots, \hat{x}_i, \dots, x_n)$.

(ii) Show that an open set of a manifold is naturally a manifold. Use this to show that $\text{GL}(n, \mathbb{R})$ is a manifold. What is its dimension?

Question 3 (Adjoint Representation of Lie Groups and their Lie Algebra). Suppose G is a Lie group.

1. Using the fact that $C_g(h) = ghg^{-1}$ is smooth and fixes the identity, show that the connected component of G containing e is open and normal. Conclude that it is the maximal proper normal lie subgroup of G .
2. Note that for each $g \in G$ we have $(dC_g)_e : T_e G \rightarrow T_e G$. Show that $g \rightarrow (dC_g)_e$ is a smooth action of G on its lie algebra \mathfrak{g} . This is called the adjoint representation of G and will be denoted $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$.
3. For $G = \text{GL}(n, \mathbb{R})$ compute its adjoint representation under the identification that $\mathfrak{gl}(n, \mathbb{R}) = \text{Mat}(n, \mathbb{R})$.
4. Now we can take the derivative of Ad at identity to get a map $\text{ad} : \mathfrak{g} \rightarrow \text{GL}(\mathfrak{g})$ given by $\text{ad}(X) = d(\text{Ad})_e(X)$. Compute this for $\text{GL}(n, \mathbb{R})$.

Question 4 (Lie Groups and tangent bundles). Let G be a Lie group.

1. Let $X \in \mathfrak{X}(G)$ such that for all $g, h \in G$ we have $(d\ell_g)_h(X_h) = X_{gh}$, then X is called a left invariant. Show that any $X_e \in \mathfrak{g}$ can be extended to a smooth left invariant vector field $X_g = (d\ell_g)_e(X_e)$.
2. Let $X_e \in \mathfrak{g}$ be nonzero. Show that the induced left invariant vector field of X_e is a nonzero vector field on G .
3. Show that if X_{1e}, \dots, X_{ne} is a basis for \mathfrak{g} then the extended vector fields X_1, \dots, X_n form a basis for the tangent space at each point.
4. Use the above to conclude that $TG = G \times \mathbb{R}^n$ by noting that the map $G \times \mathbb{R}^n \rightarrow TG$ given by $(g, a^1, \dots, a^n) \mapsto a^i X_{ig}$ is a diffeomorphism.
5. Using the hairy ball theorem conclude that S^2 is not a Lie group in any way with the standard structure. What about the other spheres?
6. Let $X \in \mathfrak{X}(G)$ be left invariant. Show that if $\gamma : (-\epsilon, \epsilon) \rightarrow G$ is a smooth curve with $\gamma(0) = e$ and $\gamma'(0) = X_e$ then show that $(h\gamma)(t) = h\gamma(t)$ is the flow through h . Use this to conclude that the flow of left invariant vector fields exist for all time.