Problem Sheet 5 - Geometry of Physics

Febuary 2026

Question 1 (Projective Spaces).

1. *Take $\mathbb{R}^{n+1} \setminus \{0\}$, show that this is a manifold. What is its dimension?*
2. *Note that there is an action of $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ given by the map $A : \mathbb{R}^* \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ given by $A(\lambda, x) = \lambda x$. Show that the action is smooth. What are the orbits? What are the stabilizers?*
3. *Show that the preimage of any compact set under the action defined above is compact.*
4. *Look up **Quotient Manifold Theorem** and conclude that $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$ is a manifold. What is the dimension?*
5. *Show that $\mathbb{C}^n$ is a smooth manifold. What is its dimension?*
6. *Note that there is an action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ in the same way as above. Show that $\mathbb{C}P^n$ is a manifold. What is its dimension?*
7. *Show that $\mathbb{R}P^1 \cong S^1$ as manifolds.*
8. *Show that $\mathbb{C}P^1 \cong S^2$ as manifolds.*

Question 2 (Vector Fields).

1. *Show that there is a vector field $X \in \mathfrak{X}(S^2)$ such that $X_p = 0$ at only one point.*
2. *Let $f : M \to N$ be a diffeomorphism. Suppose that $X \in \mathfrak{X}(M)$. Define a vector field $Y \in \mathfrak{X}(M)$ as*

$$Y_p = df_{f^{-1}(p)}(X_{f^{-1}(p)})$$

   We say here that X and Y are f-related. Show that X and Y are f-related if and only if $X(g \circ f) = Y(g) \circ f$ for all $g \in C^\infty(N)$.
3. *Suppose X and Y are f-related vector fields on M and N respectively. Show that one has the following commutative diagram*

$$\begin{tikzcd} C^\infty(M) \arrow[r, "X"] \arrow[d, "f"'] & C^\infty(M) \\ C^\infty(N) \arrow[r, "Y"] & C^\infty(N) \arrow[u, "f^{-1}"'] \end{tikzcd}$$

4. *Show that if X_1, Y_1 and X_2, Y_2 are f-related. Show that $[X_1, X_2]$ and $[Y_1, Y_2]$ are f-related.*
5. *Show that a curve γ is an integral curve of X if and only if $\dfrac{\mathrm{d}}{\mathrm{d}t}$ and X are γ-related. Here what we mean by f-related is that $\mathrm{d}f_p(X_p) = Y_{f(p)}$.*
6. $X = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$ *on $\mathbb{R}^2$. What is the flow of X? Is it complete? Let $f(x, y) = x^2 + y^2$ and $Y = xy\dfrac{\partial}{\partial x} + y^2\dfrac{\partial}{\partial y}$.*
   (a) *Compute $\mathcal{L}_X f$.*
   (b) *Compute $\mathcal{L}_X Y$ using the flow definition.*
   (c) *Compute $\mathcal{L}_X Y$ using the fact that $\mathcal{L}_X Y = [X, Y]$. Verify that you get the same thing.*
7. *Let X, Y be a left invariant vector field on a Lie group G, show that $[X, Y]$ is also left invariant. Hence conclude that $[X, Y]_e$ induces a Lie algebra structure on $\mathfrak{g}$.*

Question 3 (Differential Forms). *1. Give an example of a one form $\alpha \in \Omega^1(\mathbb{R}^3)$ such that*

$$\alpha \wedge \mathrm{d}\alpha = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

2. *let $f(x) = |x|^2$ let ∇f be the gradient of f thought of as a vector field on $\mathbb{R}^n$. Show that $\iota_{\nabla f}\omega$ does not vanish on S^{n-1} where $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.*

3. *Let $F(u, v) = (\sin u, \cos v, \sin^2 u + \cos^2 v)$ and let $\omega = xz\,\mathrm{d}x$, compute $F^*\omega$. Compute $\mathrm{d}\omega$ and verify that $F^*\mathrm{d}\omega = \mathrm{d}F^*\omega$. Now let $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ and verify that $F^*\omega = 0$.*

4. *Using $X = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$ on $\mathbb{R}^2$ and $\omega = (x^2+y^2)\,\mathrm{d}x\wedge\mathrm{d}y$ verify Cartan's magic formula $\mathcal{L}_X\omega = \mathrm{d}\iota_X\omega + \iota_X\mathrm{d}\omega$.*

Question 4 (Integration and Cohomology). *1. Let $\omega = \dfrac{x}{x^2+y^2}\,\mathrm{d}y - \dfrac{y}{x^2+y^2}\mathrm{d}x$, show that $\mathrm{d}\omega = 0$. Is there a function f on $\mathbb{R}^2 \setminus \{0\}$ such that $\mathrm{d}f = \omega$? The condition that $\mathrm{d}\omega = 0$ means that ω is **closed** and the condition that $\omega = \mathrm{d}f$ means that ω is **exact**.*

2. *Compute the integral of*

$$\int_{S^1} i^*\omega$$

of the form above and use this to show that S^1 is not the boundary of any manifold in $\mathbb{R}^2 \setminus \{0\}$.

3. *Let M be an orientable compact n dimensional manifold and let $\omega \in \Omega^n(M)$ be nowhere vanishing. Show that if M doesn't have boundary then ω cannot be exact. Is this true if M has boundary?*

4. *Read a bit about the cohomology of S^n for various n and then show that S^2 is the only one that can have a symplectic form. In other words, show that S^2 is the only unit sphere for which there exists a two form ω such that ω_p is nondegenerate at every point and $\mathrm{d}\omega = 0$.*

5. *Suppose M is an orientable compact n dimensional manifold without boundary. Let $f_0, f_1 : M \to M$ be smooth such that there is a smooth map $F : [0,1] \times M \to M$ with the property that $F(0, x) = f_0(x)$ and $F(1, x) = f_1(x)$. Let $\omega \in \Omega^n(M)$ then show that*

$$\int_M f_0^*\omega = \int_M f_1^*\omega$$

Question 5 (Densities). *In the case that the manifold is not orientable, we can still define integration. This is because, at the end of the day, integration is done over measures. Differential forms just give us a nice way to define a measure on the space. Recall the the change of variable formula is given by*

$$\int_{g(U)} f = \int_U (f \circ g)|\det Dg|$$

and so we should look at objects that transform using the absolute value of a determinant. To this end, define an α-density on a vector space to be a map $\mu : V^{\dim V} \to \mathbb{R}$ such that

$$A^*\mu = |\det A|^\alpha \mu$$

for any linear map $A : V \to V$.

1. *Let $\omega \in \Lambda^n V^*$, show that $|\omega|$ defined by*

$$|\omega|(v_1, \ldots, v_n) = |\omega(v_1, \ldots, v_n)|$$

is a density on V. Use this to show that the space of α densities over V is one dimensional.

2. *Show however that densities are not tensors.*

3. *What representation of $\mathrm{GL}(n, \mathbb{R})$ leads to the definition of a density bundle over a manifold? Recall what we did for the tangent bundle and its various tensor bundles.*

4. *Define the pullback in a way similar to what it is for a differential form. Let $F : M \to N$ and $f : N \to \mathbb{R}$ be smooth. Let μ be an α-density over N*

$$F^*(f\mu) = (f \circ F)\mu$$

and show that if $\omega \in \Omega^n(N)$ then

$$F^*|\omega| = |F^*\omega|$$

Locally we may view a density on a manifold as an expression $\mu = f\,|\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n|$ and hence we may define the integral of this local density over a chart that it is defined in as

$$\int_U \mu = \int_{\varphi(U)} f$$

and we patch them together using partitions of unity as always.

5. *Assume without proof that for any non-orientable manifold M there exists a manifold $\tilde{M}$ with a projection map $\pi : \tilde{M} \to M$ such that $\tilde{M}$ is connected and $|\pi^{-1}(p)| = 2$. Assume further that π is a local diffeomorphism. Show that if μ is a density on M then*

$$\int_{\tilde{M}} \pi^* \mu = 2 \int_M \mu$$