Problem Sheet 1 - Geometry of Physics

September 2025

Question 1 (Topological Groups). *We define a topological group to be a group (G, m) such that the multiplication map $m : G \times G \to G$ is continuous and the map $v \to v^{-1}$ is continuous.*

1. *Show that $\mathbb{R}$ is a topological group under addition.*

2. *Show that $\mathrm{GL}(n, \mathbb{R})$ is a topological group under multiplication. This will show that every subgroup is a topological group. In particular the spaces $\mathrm{SL}(n, \mathbb{R})$ is a topological group, we will see later that it is infact a Lie Group.*

3. *Recall that a subgroup $A \le G$ is normal when it is invariant under conjugation i.e. $gAg^{-1} = A$. Show that the connected component E of G that contains e is normal.*

4. *Suppose further that G is a topological manifold then show that the quotient space G/E is discrete[1]. Is this a topological manifold?*

Topological groups (with additional structure) will be of great interest to us later on as everything in physics has an action by a *Lie group* on it.

Question 2 (Tensor and Other Products).   1. *Given two vector spaces U, V show that the tensor product $U \otimes V$ has the following property: Let W be any vector space and let $f : U \times V \to W$ be bilinear. Show that there is a unique linear map $\hat{f} : U \otimes V \to W$ such that $\hat{f} \circ i = f$ where $i : U \times V \to U \otimes V$ given by $i(u, v) = u \otimes v$. This introduces the idea of a commutative diagram, the following property can be written as a diagram like this:*

$$
\begin{array}{ccc}
U \times V & \xrightarrow{\quad i \quad} & U \otimes V \\
& {\scriptstyle f} \searrow & \Big\downarrow {\scriptstyle \exists! \hat{f}} \\
& & W
\end{array}
$$

     This is called a commutative diagram since going along the arrows in any way is all the same. We will see when we talk about category theory that a lot of the constructions we talk about regularly actually all have the same structure of satisfying a universal property.

2. *Show that the tensor product is unique up to isomorphism. Hint: Show that i is a bilinear map. Then suppose W was another tensor product what can you say about the induced map given by using f above as one of the inclusions? Drawing a diagram like the one above may help.*

3. *Show that the space $\mathcal{T}_l^k(V) = (V^*)^{\otimes k} \otimes V^{\otimes l}$ is naturally identified with the space of multilinear maps $V^k \times (V^*)^l \to \mathbb{R}$ with the tensor product given below. Moreover show that $\mathcal{T}_l^k(V) \otimes W$ is the space of multilinear maps $V^k \times (V^*)^l \to W$.*

4. *The tensor product of multilinear maps (since they are tensors under the identification we talked about) is given as follows: Let $T \in \mathcal{T}_k^l(V)$ and $S \in \mathcal{T}_n^m(V)$ then $T \otimes S \in \mathcal{T}_{k+n}^{l+m}(V)$ has the following action, let $v_1, \ldots, v_l, v_{l+1}, \ldots, v_{l+m} \in V$ and $\varphi^1, \ldots, \varphi^k, \varphi^{k+1}, \ldots, \varphi^{k+n} \in V^*$ then*

$$(T \otimes S)(v_1, \ldots, v_l, v_{l+1}, \ldots, v_{l+m}, \varphi^1, \ldots, \varphi^k, \varphi^{k+1}, \ldots, \varphi^{k+n})$$
$$= T(v_1, \ldots, v_l, \varphi^1, \ldots, \varphi^k) S(v_{l+1}, \ldots, v_{l+m}, \varphi^{k+1}, \ldots, \varphi^{k+n})$$

     Now take an invertible linear map $f : V \to W$ and take a tensor $T \in \mathcal{T}_k^l(W)$ and define a tensor $f^*T \in \mathcal{T}_k^l(V)$ by taking $\varphi^1, \ldots, \varphi^k \in V^*$ and $v_1, \ldots, v_n \in V$ and defining

$$(f^*T)(v_1, \ldots, v_n, \varphi^1, \ldots, \varphi^k) = T(f(v_1), \ldots, f(v_n), \varphi^1 \circ f^{-1}, \ldots, \varphi^k \circ f^{-1})$$

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[1] Has the topology every every subset is open.

Show that this is well defined and is actually a tensor on V. Why do we require f to be invertible? Note that on tensors of type $(k,0)$ we do not require f to be an isomorphism and in some sense we can see that these tensors are different to tensors of type $(0,k)$. In fact these are duals and we will see this more when we talk about duality in category theory. Finally show that $f^*(T \otimes S) = f^*T \otimes f^*S$.

5. *Finally conclude that $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$ and that $f^*(\alpha\beta) = f^*(\alpha \odot \beta) = f^*\alpha \odot f^*\beta$. You will need to show that $f^* : \mathcal{T}_k^l(W) \to \mathcal{T}_k^l(V)$ is linear.*

Question 3 (Trace and Contraction). *Note that $\mathcal{L}(V,V) \cong V \otimes V^*$ and hence $\mathcal{L}(V,V)$ are tensors of type $(1,1)$. We will define the trace operator in a way that will allow us to define a map that moves us down dimensions and will justify the name contraction.*

1. *Let $T : V \to V$ be linear such that $T = v \otimes \varphi \in V \otimes V^*$ and define $\operatorname{tr}(T) = \varphi(v)$. Extend this map to $\mathcal{L}(V,V)$ by linearity. Show that this agrees with the usual notion of the trace of an operator. To do this think about what the coefficient of $e_i \otimes \varphi^j$ says about the matrix representation in the basis $\{e_i\}$ and dual basis $\{\varphi^i\}$.*

From here we see that the trace operator contracts a tensor of type $(1,1)$ to tensors of type $(0,0)$. So we can use this to a contraction operator $\mathcal{T}_l^k(V) \to \mathcal{T}_{l-1}^{k-1}(V)$.

2. *Let $T \in \mathcal{T}_l^k(V)$ be of the form $T = \varphi^1 \otimes \cdots \otimes \varphi^k \otimes v_1 \otimes \cdots \otimes v_l$ then we can contract the i-th vector index and the j-th covector index to define $C_i^j(T) = \varphi^j(v_i)\varphi^i \otimes \cdots \otimes \widehat{\varphi^j} \otimes \cdots \otimes \varphi^k \otimes v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_l$. Extend this linearly. Show that the coefficients of $C_i^j(T)$ in some basis are given as*

$$C_i^j(T) = T_{\beta_1 \cdots \beta_{j-1} \gamma \beta_{j+1} \cdots \beta_l}^{\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\alpha_1 \cdots \alpha_{i-1} \gamma \alpha_{i+1} \cdots \alpha_k} \varphi^{\beta_1} \otimes \cdots \otimes \varphi^{\beta_l} \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}$$

When we define inner products spaces we will talk about contractions with respect to the inner products and those are used everywhere in GR and other related fields. There are other operations we have on tensor spaces and we will see those soon.

Question 4 (Spin(3)). *We start by identifying $\mathbb{R}^3$ with a matrix space via $(x,y,z) \to x\sigma_x + y\sigma_y + z\sigma_z = T_{(x,y,z)}$. From here on $\mathbb{R}^3$ is this matrix space.*

1. *Show that $T_{(x,y,z)}$ is hermitian.*

2. *Show that $T_{(x,y,z)}$ is traceless.*

3. *Show that if T is any 2×2 complex matrix then $T \in \mathbb{R}^3$ if and only if it is hermitian and traceless.*

4. *Show that $\det T_{(x,y,z)} = -|(x,y,z)|^2$.*

5. *Show that $SU(2)$ contains 2×2 complex matrix and hence acts on $\mathbb{R}^3$. Show that $\rho(g)T = gTg^{-1}$ is a representation on $\mathbb{R}^3$.*

6. *Show that $|\rho(g)T| = |T|$ for every $T \in \mathbb{R}^3$.*

7. *Show that ρ is continuous.*

8. *Note that $\rho(I) \in SO(3)$ conclude that $\rho(g) \in SO(3)$ for all $g \in SU(2)$.*

9. *Show that $\rho(g) = 1$ then $g = \alpha I$ by using the fact that $g\sigma_i = \sigma_i g$ and deriving equations for the entries of g. Conclude that $g = \pm I$.*

We will explicitly talk about this being surjective when we talk about Lie Algebras. So we will call $\mathrm{Spin}(3)$ the group $SU(2)$. Moreover generally we will show that for each n there is a group $\mathrm{Spin}(n)$ that has a group homomorphism $\rho : \mathrm{Spin}(n) \to SO(n)$ with $\ker \rho = \{I, -I\}$ and is onto. These play important roles when discussing the dynamics and evolution of particles that have spin.