we are looking for "Riemann normal coordinates"

$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha}(\partial_j g_{\alpha k} + \partial_k g_{\alpha j} - \partial_\alpha g_{jk})$

In Riemann normal coordinates $\Gamma^i_{jk}(p) = 0$

If (U, φ) is a chart on M then the geodesic equation may be written as $\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$ this is a system of ODES. we have existence of a geodesic through a point $p \in M$ with initial velocity $X_p \in T_pM$.

Define $\exp_p(v) = \gamma(1)$ where γ is the geodesic through p with velocity $v \in T_pM$.

$\exp_p : U \subseteq T_pM \longrightarrow M$ is only defined on some $\underbrace{\text{open subset}}_{\text{topology given by } g_p.}$ of T_pM.

Lemma: For any $p \in M$ we have $(d\exp_p)_0 = id_{T_pM}$ (identify $T_0(T_pM) \cong T_pM$)

Proof: let $X \in T_pM$ then $(d\exp_p)_0(X) = \frac{d}{dt}\Big|_{t=0} \exp(tX) = \frac{d}{dt}\Big|_{t=0} \gamma(tX) = \dot{\gamma}(0) = X$.

This now shows that T_pM is a local diffeomorphism around $0 \in T_pM$ with some open subset containing p.

With this let $g = g_{ij}dx^i dx^j$ (around p) note that $g_{ij} : U \to \mathbb{R}$

$g_{ij} = g_{ij}(p) + \frac{\partial g_{ij}}{\partial x^k}(p)\, x^k + \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\frac{\partial g_{ij}}{\partial x^l} x^k x^l + \dots$

we want $g_{ij}(p) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$ and so $\Gamma^i_{jk}(p) = 0$

let $V \subseteq T_pM$ be an open subset s.t. $\exp_p$ is a diffeomorphism and $\exp_p(V) \subseteq U$.

Let $X_1, \dots, X_n$ be an orthonormal basis of T_pM hence we get an isomorphism $E : T_pM \to \mathbb{R}^n$, $E(a^i X_i) = (a^1, \dots, a^n)$ then $\psi = E \circ \exp_p^{-1}$ is

$\Psi(p) = E(\exp_p^{-1}(p)) = E(0) = 0$

a chart on $\hat{V} = \exp_p(V)$.

$d\Psi_p^{-1} = d(\exp_p)_0 \circ dE^{-1} = E^{-1}$ hence $\left.\frac{\partial}{\partial x^i}\right|_p = d\Psi_p^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_0\right) = E^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_0\right) = X^i$

the coordinate frame is an orthonormal basis of T_pM.

$$g_{ij}(p) = g_p\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta_{ij}$$

Note that $\exp \circ c$ is a geodesic starting at p with initial velocity $v = c^i X_i$.

Hence for geodesics in these coordinates $\ddot{x}^k = 0$ hence the geodesic equation reduces to $\Gamma^i_{jk}\dot{x}^i\dot{x}^j = 0$ for any initial conditions $\Rightarrow \Gamma^i_{jk}(p) = 0$ at p.

Define $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ s.t. it is bilinear over $\mathbb{R}$ and s.t.

(i) $\nabla(fX + gY, Z) = f\nabla(X,Z) + g\nabla(Y,Z)$ where $f,g \in C^\infty(M)$

(ii) $\nabla(X, fY) = (Xf)Y + f\nabla(X,Y)$ (Leibniz rule)

$\nabla(X,Y) := \nabla_X Y$

let $\pi : E \to M$ be a vector bundle then we can define a map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ similarly as above.

We will call maps like this <u>connections</u>.

If ∇ is a connection on $\pi: E \to M$ then ∇ extends to a connection on each tensor bundle s.t. $\nabla_X(\mathrm{tr}\, F) = \mathrm{tr}\, \nabla_X F$ and $\nabla_X(F \otimes G) = \nabla_X F \otimes G + F \otimes \nabla_X F$

and $\nabla_X f = Xf$ for $f \in C^\infty(M)$ for f $E = TM$

let $\pi : E \to M$ be a vector bundle then $g \in \Gamma(\Sigma^2 E)$ that is positive definite and non-degenerate on each fiber. Then we call (E, M, π, g) a Riemannian vector bundle.

let (M,g) be a Riemannian manifold we get Γ^i_{jk} as above.

These do not form a tensor. However we can define a connection out of these:

let ∂_i be the coordinate frame define $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$.

Next time: we will show that this defines a connection on TM.

let dx^i be the dual to ∂_i. Then $\underbrace{dx^i(\partial_j)}_{\delta^i_j} = tr(dx^i \otimes \partial_j)$

which means that $0 = X\delta^i_j = \nabla_X tr(dx^i \otimes \partial_j) = tr(\nabla_X(dx^i \otimes \partial_j))$

$$= tr(\nabla_X dx^i \otimes \partial_j + dx^i \otimes \nabla_X \partial_j)$$

$$= (\nabla_X dx^i)(\partial_j) + dx^i(\nabla_X \partial_j)$$

hence $\underbrace{(\nabla_X dx^i)(\partial_j)}_{\alpha^i_j} = -dx^i(\nabla_X \partial_j)$

$$\nabla_X dx^i \in \Omega^1(M) \Rightarrow \nabla_X dx^i = \alpha^i_j dx^j$$

$$\Rightarrow (\nabla_X dx^i)(\partial_k) = \alpha^i_j dx^j(\partial_k)$$

$$= \alpha^i_k$$

if $\partial_k = X$ then $(\nabla_{\partial_k} dx^i)(\partial_j) = -dx^i(\nabla_{\partial_k}\partial_j)$

$$= -dx^i(\Gamma^\alpha_{kj}\partial_\alpha)$$

$$= -\Gamma^i_{kj}$$

if $X = X^i\partial_i$ we get $\nabla_X dx^i = \nabla_{X^j\partial_j} dx^i$

$$= X^j \nabla_{\partial_j} dx^i$$

$$= -X^j \Gamma^i_{jk} dx^k$$

$$\nabla_X g = \frac{1}{2}\nabla_X(g_{ij}\, dx^i \otimes dx^j)$$

$$= \frac{1}{2}\left((\nabla_X g_{ij})\, dx^i \otimes dx^j + g_{ij} \nabla_X dx^i \otimes dx^j + g_{ij}\, dx^i \otimes \nabla_X dx^j\right)$$

$$= \frac{1}{2}\left(X^k \partial_k g_{ij}\, dx^i \otimes dx^j - g_{ij} X^\alpha \Gamma^i_{\alpha k}\, dx^k \otimes dx^j - g_{ij} X^\alpha \Gamma^j_{\alpha k}\, dx^i \otimes dx^k\right)$$

$$\downarrow k \to \alpha \qquad \downarrow i \leftrightarrow k \qquad \downarrow j \leftrightarrow k$$

$$= \frac{1}{2} X^\alpha \left(\partial_\alpha g_{ij} - g_{jk} \Gamma^k_{\alpha i} - g_{ik} \Gamma^k_{\alpha j}\right) dx^i \otimes dx^j$$

$$= \frac{1}{2} X^\alpha \left(\partial_\alpha g_{ij} - \frac{1}{2}\partial_\alpha g_{ij} - \frac{1}{2}\partial_i g_{\alpha j} + \frac{1}{2}\partial_j g_{\alpha i} - \frac{1}{2}\partial_\alpha g_{ji} - \frac{1}{2}\partial_j g_{\alpha i} + \frac{1}{2}\partial_i g_{\alpha j}\right) dx^i \otimes dx^j$$

$$= 0$$

Aside: Every Riemannian manifold admits a unique "torsion free", metric compatible connection called the Levi-Civita connection.

If $\nabla_X F = 0$ $\forall X \in \mathfrak{X}(M)$ we say F is "flat" or "parallel"

Corollary: If $A, B \in \mathfrak{X}(M)$ then $\nabla_X(g(A,B)) = 0$.

Proof: $g(A,B) = tr(tr(g \otimes A \otimes B))$

$$\nabla_X(g(A,B)) = tr(tr(\nabla_X(g \otimes A \otimes B)))$$
$$= tr(tr(\underbrace{(\nabla_X g)}_{0} \otimes A \otimes B + g \otimes \nabla_X A \otimes B + g \otimes A \otimes \nabla_X B))$$
$$= tr(tr(g \otimes \nabla_X A \otimes B + g \otimes A \otimes \nabla_X B))$$
$$= g(\nabla_X A, B) + g(A, \nabla_X B)$$

Given a connection ∇ we can define a map

$$\hat{\nabla} : \Gamma(E) \to \Gamma(\underbrace{T^*M \otimes E}_{})$$

this bundle can be thought of as the bundle whose fibers are maps $T_pM \to E_p$

$\hat{\nabla} F$ so that $(\hat{\nabla} F)(X) = \nabla_X F$

$\uparrow$
$X \in \mathfrak{X}(M)$