Examples of manifolds:

let $f: M \to N$ be smooth. Then f is called an immersion if $\forall p \in M$ $df_p: T_pM \to T_{f(p)}N$ is injective
f is called a submersion if $\forall p \in M$ $df_p: T_pM \to T_{f(p)}N$ is surjective
f is called a local diffeomorphism if $\forall p \in M$ $df_p: T_pM \to T_{f(p)}N$ is bijective

Let $q \in f(M) \subseteq N$ then q is a regular value if $\forall p \in f^{-1}(\{q\})$, $df_p: T_pM \to T_qN$ is surjective.

let $S \subseteq M$ then S is called a k-dimensional submanifold of M if $\forall p \in S$ $\exists (U, \varphi)$ a chart s.t. $\varphi(U \cap S) = \varphi(U) \cap \mathbb{R}^k$

Implicit function theorem $\Rightarrow$ If q is a regular for $f: M \to N \Rightarrow f^{-1}(\{q\})$ is a submanifold of dimension $\dim M - \dim N$.

The above is called regular level set theorem.

Now lets go into examples:

i) $\mathbb{R}^n$ any real / complex finite dimensional vector.

ii) let $f: \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(x) = |x|^2$. let $S^n = f^{-1}(\{1\}) = \{x \in \mathbb{R}^{n+1} \mid |x|^2 = 1\}$

$df_p: \underbrace{T_p\mathbb{R}^{n+1}}_{\mathbb{R}^{n+1}} \to \underbrace{T_1\mathbb{R}}_{\mathbb{R}}$ is surjective. $df_p = [\underset{\frac{\partial f}{\partial x^1}}{2p^1} \; \cdots \; \underset{\frac{\partial f}{\partial x^{n+1}}}{2p^{n+1}}]$ since $|p|^2 = 1$ $\Rightarrow p^i \neq 0$ for some i

$$f(x) = \sum_{i=1}^{n+1} (x^i)^2 \Rightarrow \frac{\partial f}{\partial x^j} = \sum_{i=1}^{n+1} \delta_{ij} (2x^i) = 2x^j \Rightarrow \left.\frac{\partial f}{\partial x^j}\right|_p = 2p^j$$

Hence S^n is a manifold of dimension $(n+1) - 1 = n$.

iii) $GL(n,\mathbb{R}) = \{A: \mathbb{R}^n \to \mathbb{R}^n \mid A \text{ is invertible}\} \subseteq \mathbb{R}^{n^2}$ is open. $GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$
Hence $GL(n,\mathbb{R})$ is a submanifold of $\mathbb{R}^{n^2}$ and has dimension n^2.

iv) $SL(n,\mathbb{R}) \subseteq GL(n,\mathbb{R})$ s.t. $A \in SL(n,\mathbb{R}) \Leftrightarrow \det A = 1$. Now we need to show that for all $A \in \det^{-1}(\{1\}) = SL(n,\mathbb{R})$ $d(\det)_A: \underset{\mathbb{R}^{n^2}}{T_A GL(n,\mathbb{R})} \to \underset{\mathbb{R}}{T_1\mathbb{R}}$

consider $\gamma(t) = A + tA \Rightarrow \gamma'(0) = A$: $d(\det)_A(A) = \left.\frac{d}{dt}\right|_{t=0} \det(A + tA) = \left.\frac{d}{dt}\right|_{t=0} \det((1+t)A))$

$$= \left.\frac{d}{dt}\right|_{t=0} (1+t)^n \underbrace{\det(A)}_{1}$$

$$= n \neq 0$$

Hence 1 is a regular value for $\det \Rightarrow SL(n,\mathbb{R})$ is a submanifold of dimension $n^2 - 1$.

v) let $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a nondegenerate, symmetric, bilinear form : $G(x,y) = x^T g y$. Now G has signature (p,q)

symmetric matrix

⊛ Look at Sylvester's law of inertia

$O(p,q) = \{A \in GL(n,\mathbb{R}) \mid G(x,y) = G(Ax,Ay)\}$. $A \in O(p,q) \iff A^T g A = g$

$$\iff \underbrace{x^T A^T g A y}_{= G(Ax, Ay)} = x^T g y = G(x,y)$$

show this is a vector space of dimension $\frac{n(n+1)}{2}$

Consider the map $f : \mathbb{R}^{n^2} \to \mathrm{Sym}(n,\mathbb{R})$ by $f(A) = A^T g A$. $O(p,q) = f^{-1}(\{g\})$

we need to show that $\forall A \in O(p,q)$, $df_A : \mathbb{R}^{n^2} \to \mathrm{Sym}(n,\mathbb{R})$ is surjective.

$$df_A(X) = \left.\frac{d}{dt}\right|_{t=0} f(A+tX)$$

$$= \left.\frac{d}{dt}\right|_{t=0} A^T g A + t A^T g X + t X^T g A + t^2 X^T g X$$

$$= A^T g X + X^T g A$$

$$g g^{-1} = Id$$
$$\Downarrow$$
$$(g^{-1})^T g^T = Id$$
$$\Rightarrow (g^{-1})^T = (g^T)^{-1} = g^{-1}$$

Now let $Y \in \mathrm{Sym}(n,\mathbb{R})$ then let $X = \frac{1}{2} A g^{-1} Y \Rightarrow df_A(X) = A^T g \left(\frac{1}{2} A g^{-1} Y\right) + \frac{1}{2} Y^T (g^{-1})^T A^T g A$

$$= \frac{1}{2}\Big(\underbrace{A^T g A}_{g} g^{-1} Y + Y g^{-1} \underbrace{A^T g A}_{g}\Big)$$

$$= \frac{1}{2}\left(g g^{-1} Y + Y g^{-1} g\right)$$

$$= Y$$

Hence df_A is surjective $\Rightarrow O(p,q)$ are manifolds of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$

vi) $SO(p,q) \leq O(p,q)$ given by $A \in SO(p,q) \iff \det A = 1$ and $A \in O(p,q)$

$f : \mathbb{R}^{n^2} \to \mathrm{Sym}(n,\mathbb{R}) \times \mathbb{R}$

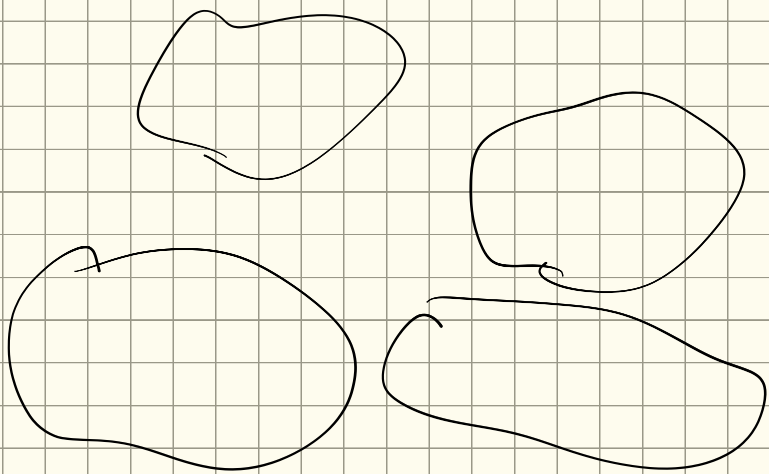
Hence $f(A) = (A^T g A, \det A)$ is a submersion on $SO(p,q) \Rightarrow SO(p,q)$ is a manifold of dimension $\frac{n(n-1)}{2} - 1$

In special relativity we have inner product $\eta(v,w) = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$

$$= \begin{bmatrix} v^0 & v^1 & v^2 & v^3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \end{bmatrix}$$

Lorentz group is the group $O(3,1)$. Note that $\dim O(3,1) = \frac{4(3)}{2} = 6$

The restricted Lorentz group $SO^+(3,1)$ = identity component of $O(3,1)$



Exercise 1: Show $O(3,1)$ has 4 connected components

Exercise 2: let G be a Lie group and N be the connected component of e. Then $N \trianglelefteq G$ and N is submanifold.

We have $Spin(3,1)$ which is the double cover of $SO(3,1)$

$$\begin{array}{c} Spin(3,1) \\ \text{\rotatebox{90}{$\cong$}} \\ SL(2,\mathbb{C}) \\ \text{\rotatebox{90}{$\cong$}} \\ Sp(2,\mathbb{C}) \end{array}$$

$S^n \subseteq \mathbb{R}^{n+1}$ is a manifold. Moreover $i: S^n \hookrightarrow \mathbb{R}^{n+1}$, $x \mapsto x$ is smooth. $T_pS^n \cong di_p(T_pS^n) \subseteq T_p\mathbb{R}^{n+1}$

$\Rightarrow T_pS^n \subseteq \mathbb{R}^{n+1}$

Exercise: If $f: M \to N$ is smooth and $\overset{S}{\overset{\shortparallel}{f^{-1}(\{q\})}}$ is a regular level set then $T_pS = \ker(df_p)$

$T_pS^n = \ker(df_p)$ where $df_p = [2p^1 \cdots 2p^{n+1}]$ but $\overset{v \in \mathbb{R}^{n+1}}{df_p(v)} = [2p^1 \cdots 2p^{n+1}]\begin{bmatrix} v^1 \\ \vdots \\ v^{n+1} \end{bmatrix}$

Hence $v \in T_pS^n \Leftrightarrow p \cdot v = 0$

$= 2p \cdot v$
$= 0$

Next time we compute $T_A O(p,q)$, $T_{Id} SL(n,\mathbb{R}) \Longrightarrow T_A SL(n,\mathbb{R})$