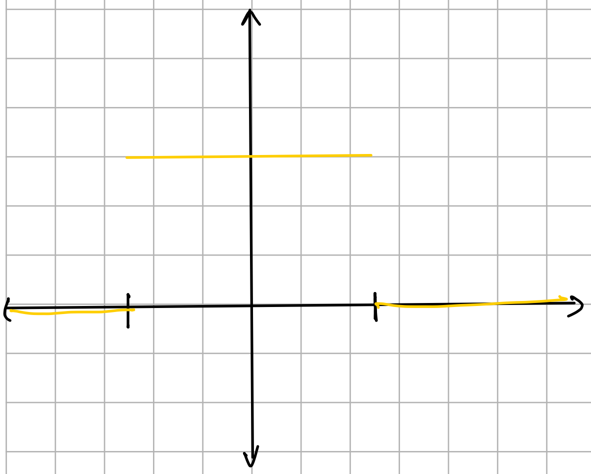
Why define calculus on the reals and not $\mathbb{Q}$?

consider $f: \mathbb{Q} \to \mathbb{Q}$ by $f(x) = \begin{cases} 1 & ; x^2 < 2 \\ 0 & ; x^2 > 2 \end{cases}$



this is continous over $\mathbb{Q}$

this differentiable over $\mathbb{Q}$

This give motivation to restrict to complete spaces with a V.S. structure and norm $\Rightarrow$ Banach spaces (real)

So suppose we have $\gamma: (-\varepsilon, \varepsilon) \to X$ ($X$ is Banach space)

$$\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t}$$

What happens when $X = \mathbb{R}^n$: $\gamma(t) = \gamma^i(t) e_i \Rightarrow \gamma'(0) = (\gamma^i)'(0) e_i$

So how to generalize to functions between arb. Banach spaces?

Note $X$ is path connected, $a, h \in X$ are connected by $(1-t)a + th$

Now let $f: X \to Y$ then $\gamma(t) = a + th \Rightarrow f \circ \gamma : (-\varepsilon, \varepsilon) \to Y$

we can differentiate this to get $(f \circ \gamma)'(0) = \delta f(a; h)$

Gâteaux variation of $f$ along $h$

We say $f: X \to Y$ is Gateaux differentiable at $a \in X$ if there is a map $\delta f(a): X \to Y$ s.t. $\delta f(a)(h) = \underbrace{\delta f(a; h)}$

$\delta f(a)$ — Gâteaux Derivative of $f$ at $a$

require this to exist for all $h \in X$.

Gâteaux Derivative is not strong enough:

Consider $f:\mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & ; (x,y) \neq 0 \\ 0 & ; (x,y) = 0 \end{cases}$

let $h = (a,b)$ then

$$\delta f(0;h) = \lim_{t\to 0} \frac{f(0+th) - f(0)}{t}$$

$$= \lim_{t\to 0} \frac{1}{t}\left(\frac{(t^2a^2)(tb)}{t^4a^4+t^2b^2}\right)$$

$$= \lim_{t\to 0} \frac{1}{\cancel{t^3}}\left(\frac{\cancel{t^3}a^2b}{t^2a^4+b^2}\right)$$

$$= \frac{a^2}{b}$$

$\delta f(0;h)$ is not linear in $h$.

Consider $\lim_{(a,b)\to(0,0)} \frac{a^2}{b}$ but if $b = a^2 \Rightarrow \lim_{(a,b)\to(0,0)} \frac{a^2}{a^2} = 1 \neq 0$

Note $f$ is not continous:

$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ along $(x,y) = (t,t^2) \Rightarrow \lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2} \neq 0$

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$f:\mathbb{R}\to\mathbb{R}$ then $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$$\Rightarrow 0 = \lim_{h\to 0} \frac{f(a+h) - f(a) - f'(a)\cdot h}{h}$$

$$\Rightarrow 0 = \lim_{h\to 0} \frac{|f(a+h) - f(a) - f'(a)\cdot h|}{|h|}$$

real Banach spaces

$f: X \to Y$ we say $f$ is Frechét Differentiable at $a \in X$ if there is a linear map $Df(a): X \to Y$ s.t. $\lim_{h \to 0} \frac{\|f(a+h) - f(a) - Df(a)(h)\|_Y}{\|h\|_X} = 0$

Note that if $f$ is Frechét differentiable at $a \in X$ then $f$ is Gâteaux differentiable at $a$.

Note also that Gâteaux differentiable may not imply Frechét differentiable

↳ If $\delta f: X \times X \to Y$ is continuous map and $\delta f(a): X \to Y$ is linear then $f$ is Frechét differentiable (only always true in finite dimensions, it may be false in infinite dimensions)

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(Frechét differentiable at every point in $X$)

let $f: X \to Y$ be Frechét differentiable then we note that

$Df: X \to \mathrm{Lin}(X, Y)$ in case we have $Df: X \to \mathcal{L}(X, Y)$ ← Bounded linear maps $X \to Y$

in other words $Df(a): X \to Y$ is bounded $\forall a \in X$.

we may ask if $Df$ is differentiable.

(*) $\mathcal{L}(X, Y)$ is a Banach space under sup norm.

If $Df$ exists and is bounded then $Df$ is automatically continous in $a$. $a \mapsto Df(a)$ is continous.

We say $f: X \to Y$ is of class $C^1$ ($f \in C^1(X, Y)$) if $Df(a)$ exists $\forall a$ and $Df$ is continous.

Now we define $D^2 f(a): X \times X \to Y$ is bilinear what this means is that $D^2 f(a) \in \mathcal{T}^2{}_0(X) \otimes Y$

$$\lim_{h \to 0} \frac{\|Df(a+h) - Df(a) - D^2 f(a)(h)\|_{\mathcal{L}(X,Y)}}{\|h\|_X} = 0$$ 

this has to be a map on $X \to Y$

Then we define $D^3 f(a): X \times X \times X \to Y$ similarly.

Hence $D^n f(a) \in \mathcal{T}^n{}_0(X) \otimes Y$ that satisfies the limit difition.

We now say that $f$ is smooth at $a \in X$ if $D^n f(a)$ exists for all $n \in \mathbb{N}$.

Example: $f: \mathbb{R}^n \to \mathbb{R}^m$ is linear. We check that $Df(a) = f$.

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - Df(a)(h) \|_{\mathbb{R}^m}}{\|h\|_{\mathbb{R}^n}} = \lim_{h \to 0} \frac{\| f(a) + f(h) - f(a) - f(h) \|_{\mathbb{R}^m}}{\|h\|_{\mathbb{R}^n}}$$

$$= 0$$

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Say we have $X = X_1 \oplus \cdots \oplus X_m$ and we have $f: X \to Y$

a point $a \in X$ can be written as $a = (a_1, \ldots, a_m)$ so for each $1 \leq i \leq m$ we have a map $f_{i,a}: X_i \to Y$ given by

$f_{i,a}(x) = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m)$ is this differentiable?

then we define $D_i f(a)(h) = Df_{i,a}(h)$

In finite dimensions this is nice since $\mathbb{R}^n = \overbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}^{n \text{ times}}$

and so the matrix representation of $Df(a) = \begin{bmatrix} D_1 f(a) & \cdots & D_n f(a) \end{bmatrix}$

⊗ One can check that $D^2 f$ exists and is continuous then $D^2 f(a) \in \Sigma^2 \mathbb{R}^n$

Example: $f(x,y) = x^2 + y^2 + 1$ then $f$ is smooth moreover

$Df = \begin{bmatrix} 2x & 2y \end{bmatrix}$, $D^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \ldots$

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Three main theorems

1) Chain rule: $f: X \to Y$ that is differentiable at $a \in X$ and $g: Y \to Z$ that is differentiable at $f(a)$ then $g \circ f: X \to Z$ is differentiable at $a \in X$. $D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$

2) Inverse function Theorem: If $f: X \to Y$ is $C^1$ at $a \in X$ and $Df(a): X \to Y$ is invertible then $f$ is invertible in some neighbourhood $V$ of $a$ and moreover $f^{-1}$ is differentiable at $f(a)$ with

open

$Df^{-1}(a) = [Df(a)]^{-1}$

(*) $V$ may not be Banach but derivative is still defined this is because limits are local and $V$ is locally convex

(*) Using the fact that open sets are locally convex we can define derivatives for maps $f: U \to Y$ where $U$ is open in $X$, and derivative is defined exactly the same way, so $Df(a) \in Lin(X, Y)$

3) Implicit function Theorem: If $f: X \oplus Y \to Z$ is $C^1$. Suppose further that $f(x_0, y_0) = 0$ and $Df(x_0, y_0)(0, y) =: g(y)$ is an isomorphism from $Y \xrightarrow{g} Z$. Then there is a neighbourhood $U$ of $x_0$ and $V$ of $y_0$ and a Frechét differentiable function $h: U \to V$ s.t.

$$f(x, h(x)) = 0 \quad \text{and} \quad f(x,y) = 0 \iff y = h(x) \quad \forall (x,y) \in U \times V$$

Example: $f(x,y) = x^2 + y^2 - 1$ then $f$ satisfies the conditions at each point $(x,y) \in S^1$, $U = (-1,1)$ and $V = (0,1)$ then

$h(x) = \sqrt{1 - x^2}$ this is for $(x_0, y_0) = (0,1)$