$\pi : TM \to M$. let ∇ be a connection and let X_i be a local frame. let α_i be its dual frame.

$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

or

$\nabla : \mathfrak{X}(M) \to \Gamma(T^*M \otimes TM)$

$\nabla_{X_i} X_j = \Gamma^k_{ij} X_k$. we ~~can~~ can also look at ∇X_j is a $(1,1)$ tensor field. let $Y \in \mathfrak{X}(M)$

$Y = Y^i X_i$ let us compute the components of $\nabla Y = (\nabla Y)_i{}^j \alpha^i \otimes X_j$

Note $(\nabla Y)_i{}^j = (\nabla Y)(X_i, \alpha^j)$

$$
\begin{aligned}
&= (\nabla_{X_i} Y)(\alpha^j) \\
&= (\nabla_{X_i}(Y^k X_k))(\alpha^j) \\
&= \alpha^j((X_i Y^k) X_k + Y^k \nabla_{X_i} X_k) \\
&= \alpha^j((X_i Y^k) X_k) + \alpha^j(Y^k \Gamma^l_{ik} X_l) \\
&= (X_i Y^k)\delta^j{}_k + Y^k \Gamma^l_{il} \delta^j_l \\
&= X_i Y^j + Y^k \Gamma^j_{ik}
\end{aligned}
$$

$\nabla : \Gamma(T^{(k,l)}M) \to \Gamma(T^*M \otimes T^{(k,l)}M) \cong \Gamma(T^{(k+1,l)}M) \Rightarrow$ if $\omega \in \Omega^1(M)$ then $\nabla\omega \in \Gamma(T^{(2,0)}M)$

Exercise: Is $\nabla\omega$ a 2 form?

Note that $\omega = \omega_i \alpha^i$. Recall $\nabla_{X_i} \alpha^j = -\Gamma^j_{ik} \alpha^k$

$$
\begin{aligned}
(\nabla\omega)_{ij} &= (\nabla\omega)(X_i, X_j) \\
&= (\nabla_{X_i}\omega)(X_j) \\
&= (\nabla_{X_i}(\omega_k \alpha^k))(X_j) \\
&= ((X_i\omega_k)\alpha^k + \omega_k \nabla_{X_i}\alpha^k)(X_j) \\
&= (X_i\omega_k)\delta^k_j - \omega_k \Gamma^k_{il} \alpha^l(X_j) \\
&= X_i\omega_j - \omega_k \Gamma^k_{ij}
\end{aligned}
$$

If $F \in \Gamma(T^{(k,l)}M) \Rightarrow \nabla F \in \Gamma(T^{(k+1,l)}M)$ we can "differentiate" again to get

$$\Rightarrow \nabla^2 F \in \Gamma(T^{(k+2,l)}M)$$
$$\cong \Gamma(T^*M \otimes T^*M \times T^{(k,l)}M)$$

Let (M,g) be a Riemannian manifold. Then $\hat{g}: TM \to T^*M$ is a bundle isomorphism. Hence we have an inverse $\hat{g}^{-1}: T^*M \to TM$.

If $f \in C^\infty(M)$ then $df \in \Omega^1(M) \Rightarrow df_p \in T_p^*M$

Define $\mathrm{grad}(f) \in \mathfrak{X}(M)$ so that $\hat{g}(\mathrm{grad} f) = df$ or in other words $\mathrm{grad} f = \hat{g}^{-1}(df)$.

If $\{\partial_i\}$ are coordinate vector fields then $g_{ij} = g(\partial_i, \partial_j)$ and $df = \partial_i f\, dx^i$

Note that $\hat{g}: \partial_i \mapsto g_{ij} dx^j$ and so $\hat{g}^{-1}: dx^i \mapsto g^{ij}\partial_j$

$$\hat{g}^{-1}(df) = (\partial_i f)\, \hat{g}^{-1}(dx^i)$$
$$= (\partial_i f)\, g^{ij}\partial_j$$

Examples: If $g_{ij} = \delta_{ij} \Rightarrow (\mathrm{grad} f)_i = \partial_i f$

If $g = dr^2 + r^2 d\theta^2 \Rightarrow \mathrm{grad} f = \partial_r f\, g^{rr}\partial_r + \partial_\theta f\, g^{\theta\theta}\partial_\theta$

g_{rr} $\quad$ $g_{\theta\theta}$

$$= (\partial_r f)\partial_r + \frac{1}{r^2}(\partial_\theta f)\partial_\theta$$

$$[g] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \Rightarrow [g^{-1}] = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

Let (M,g) be an oriented Riemannian manifold. Given an orthonormal oriented frame α^i we can define $dV_g = \alpha^1 \wedge \dots \wedge \alpha^n$.

Exercise: show that this well defined. Hint: If $T_p: \alpha_p^i \to \beta_p^i$ where β_p^i is also orthonormal the $\det T_p = \pm 1$

In a coordinate frame, one can check that $dV_g = \sqrt{g}\, dx^1 \wedge \dots \wedge dx^n$ where $\sqrt{g} = \sqrt{\det g}$

Exercise: Check this.

Note If $X \in \mathfrak{X}(M)$ then $\iota_X dV_g \in \Omega^{n-1}(M) \Rightarrow d(\iota_X dV_g) \in \Omega^n(M)$

$$d(\iota_X dV_g) = (\operatorname{div} X)\, dV_g$$

We can now define $\Delta f = \operatorname{div}(\operatorname{grad} f)$. We can define $\Delta \omega$ for $\omega \in \Omega^k(M)$ by defining the "Hodge star" operator.

If $X = X^i \partial_i$ in a coordinate frame then

$$\iota_X dV_g = (-1)^{i-1} \sqrt{g}\, X^i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

then
$$\begin{aligned} d(\iota_X dV_g) &= (-1)^{i-1} \partial_j(\sqrt{g}\, X^i)\, dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \partial_i(\sqrt{g}\, X^i)\, dx^1 \wedge \cdots \wedge dx^n \\ &= \left(\frac{1}{\sqrt{g}} \partial_i(\sqrt{g}\, X^i)\right) \underbrace{\left(\sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n\right)}_{dV_g} \end{aligned}$$

Hence $\operatorname{div} X = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g} X^i)$ moreover $\Delta f = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g}\, g^{ij} \partial_j f)$

Examples: If $g = (dx^1)^2 + \cdots + (dx^n)^2$ then $\operatorname{div} X = \partial_i X^i$

If $g = dr^2 + r^2 d\theta^2$ and so $\operatorname{div} X = \frac{1}{r}\left(\partial_r(r X^r) + \partial_\theta(r X^\theta)\right)$

$$[g] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad = \frac{1}{r}\left(X^r + r\partial_r X^r + r\partial_\theta X^\theta\right)$$

$\det g = r^2$ $\qquad = \frac{1}{r} X^r + \partial_r X^r + \partial_\theta X^\theta$

$\sqrt{\det g} = r$

Let $\pi : E \to M$ be a vector bundle and let ∇ be a connection

We define the map $\mathrm{Riem} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$ defined by

$$\mathrm{Riem}_\nabla(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \in \Gamma(E)$$

Exercise: Show that this is multilinear over $C^\infty(M)$. Hence it is a tensor

If $E = TM$ we can ask a stronger question: Is $\nabla_X Y - \nabla_Y X = [X,Y]$?

The failure of this is the torsion: $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

$T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

Theorem: If (M,g) is a Riemannian manifold then there exists a unique connection ∇ on TM s.t. $T_\nabla = 0$ and $\nabla g = 0$.

This connection is called the Levi-Civita connection.

$\pi : E \to M$ vector bundle with connection ∇.

let $\{s_i\}$ be a local frame for E and let α^i be the dual
let $\{X_i\}$ be a local frame for TM and let β^i be its dual.

$\mathrm{Riem}_\nabla = R_{ijk}{}^{l} \beta^i \otimes \beta^j \otimes \alpha^i \otimes s_l$ where $R_{ijk}{}^{l} = \alpha^l(R(X_i,X_j)s_k)$

Note $\nabla_{X_i} s_k = \Gamma^\sigma_{ik} s_\sigma$ and so

$[X_i,X_j] = \Lambda^\gamma_{ij} X_\gamma$

$$\mathrm{Riem}_\nabla(X_i,X_j)s_k = \nabla_{X_i}\nabla_{X_j} s_k - \nabla_{X_j}\nabla_{X_i} s_k - \nabla_{[X_i,X_j]} s_k$$

$$= \nabla_{X_i}(\Gamma^\sigma_{jk} s_\sigma) - \nabla_{X_j}(\Gamma^\sigma_{ik} s_\sigma) - \nabla_{\Lambda^\gamma_{ij}X_\gamma} s_k$$

$$= (X_i \Gamma^\sigma_{ij}) s_\sigma + \Gamma^\sigma_{jk}\Gamma^\gamma_{i\sigma} s_\gamma - (X_j \Gamma^\sigma_{ik}) s_\sigma - \Gamma^\sigma_{ik}\Gamma^\gamma_{j\sigma} s_\gamma - \Lambda^\gamma_{ij}\Gamma^\sigma_{\gamma k} s_\sigma$$

Hence $$R_{ijk}{}^{l} = \alpha^l\left((X_i \Gamma^\sigma_{ij}) s_\sigma + \Gamma^\sigma_{jk}\Gamma^\gamma_{i\sigma} s_\gamma - (X_j \Gamma^\sigma_{ik}) s_\sigma - \Gamma^\sigma_{ik}\Gamma^\gamma_{j\sigma} s_\gamma - \Lambda^\gamma_{ij}\Gamma^\sigma_{\gamma k} s_\sigma\right)$$

$$= (X_i \Gamma^l_{ij}) - (X_j \Gamma^l_{ik}) + \Gamma^\sigma_{jk}\Gamma^l_{i\sigma} - \Gamma^\sigma_{ik}\Gamma^l_{j\sigma} - \Lambda^\gamma_{ij}\Gamma^l_{\gamma k}$$

In a coordinate frame $\Lambda^\gamma_{ij} = 0$ hence

$$R_{ijk}{}^{l} = \partial_i \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^\sigma_{jk}\Gamma^l_{i\sigma} - \Gamma^\sigma_{ik}\Gamma^l_{j\sigma}$$

on TM with the levi-Civita connection we can define the Ricci tensor $\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y)$ and this has components $R_{ij} = R_{kij}{}^{k}$

Exercise: Check this formula.

We can define the scalar curvature as $R = R_i{}^i$

We say a metric g is an Einstein metric if $\mathrm{Ric} = \lambda g$.
(vacuum, no cosmological constant)

Hilbert realised that the Einstein field equations are the "Euler-Lagrange equations" corresponding to $S(g) = \int_M R \, dV_g$