A set $E \subseteq \overline{\mathbb{R}}$ is measurable $\iff E \cap \mathbb{R}$ is measurable

Integration of function on measure space

$\mathbb{R} \cup \{-\infty, \infty\} = \overline{\mathbb{R}}$

let (X, Σ, μ) be a measure space and let $f: X \to \mathbb{R} \cup \{-\infty, \infty\}$ we say f is μ-measurable if for every lebesgue measurable set $E \subseteq \overline{\mathbb{R}}$ we have $f^{-1}(E) \in \Sigma$

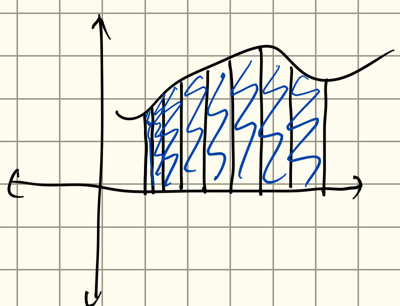
⊛ All the theory we will develope is for σ-finite measures. (X, Σ, μ) measure space then μ is called σ-finite if there is a countable collection of measurable sets with finite measure that cover X.

example : (1) $\mathbb{R}^n$ with lebesgue

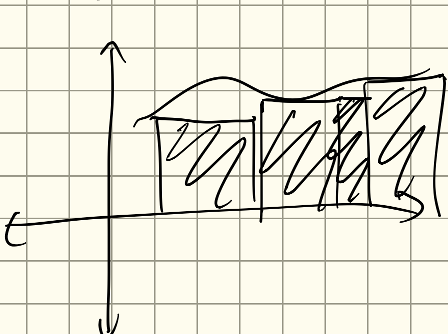
(2) Any probability space

Note: The Borel σ-Algebra of $\mathbb{R}$ can be constructed using sets of the form (a, ∞)
The σ-algebra generated by these sets is the same as the Borel σ-Algebra.
In all situations it suffices to check that $f^{-1}((a,\infty))$ is measurable.

$$f^{-1}((a,\infty)) = \{x \in X \mid f(x) > a\}$$



← we take a limit of area underneath



We now define a characteristic function of a set.
let $E \subseteq X$

$$\chi_E(x) = \begin{cases} 1 & ; x \in E \\ 0 & ; x \notin E \end{cases}$$

χ_E is μ-measurable $\iff$ E is measurable

$\chi_E^{-1}(\mathbb{R}) = E$ hence χ_E measurable $\iff E$ is.

Note : Every continuous function $f: \underset{\substack{\uparrow \\ \text{topological space}}}{X} \to \overline{\mathbb{R}}$ is Borel (in X) measurable

Theorem: If $\{f_n\}$ is a sequence of μ-measurable functions then $f(x) = \lim_{n \to \infty} f_n(x)$ is also measurable.

Def: A function $f: X \to \overline{\mathbb{R}}$ is called simple if it takes on finitely many values.

Example : χ_E is simple
$f(x) = x^2$ is <u>not</u> simple

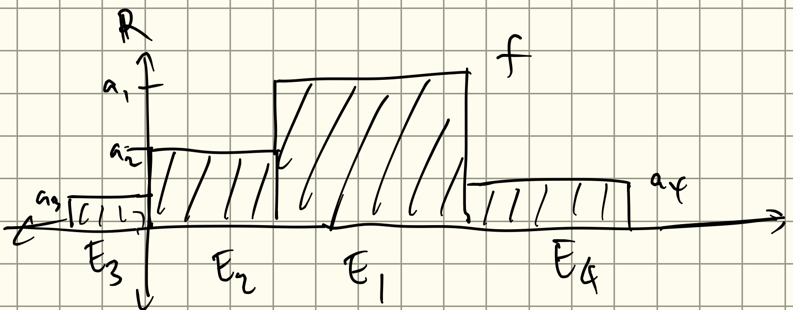
Every simple function is a <u>finite</u> linear combination of characteristic functions.

To see this suppose f takes on $a_1, \dots, a_n$ then define $E_i = f^{-1}(\{a_i\})$ then $f = \sum_{i=1}^{n} a_i \chi_{E_i}$

Theorem: Suppose $f: X \to \overline{\mathbb{R}}$ is μ-measurable then there is a sequence $\{f_n\}$ of μ-measurable simple functions that non-decreasing and bounded above by f.

non-decreasing: $f_n(x) \leq f_{n+1}(x)$

Moreover if f is bounded then $f_n \to f$ uniformly



here define the integral for non-negative simple functions

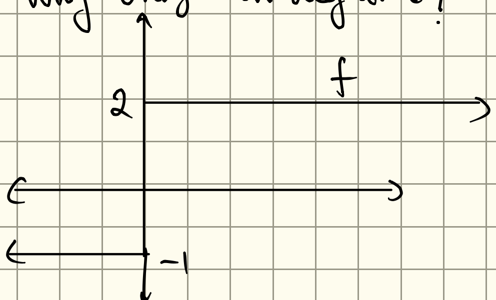
$$\boxed{\int_X f \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i)} \quad \text{here } f = \sum_{i=1}^{n} a_i \chi_{E_i}$$

Note: we allow the value to be infinite

We don't generalize directly since if f is allowed to be both positive and negative we might end up with situations where we have an expression $\infty - \infty$

So we now let $f: X \to \overline{\mathbb{R}}$ measurable and non-negative then define $\int_X f \, d\mu = \sup\left\{ \int_X s \, d\mu \,\middle|\, 0 \leq s \leq f(x) \text{ (} s \text{ is simple and measurable)} \right\}$

Why only non-negative?



$$\int_{\mathbb{R}} f \, d\mu = 2 \cdot \mu((0,\infty)) + (-1) \cdot \mu((-\infty, 0])$$
$$= 2 \cdot \infty - \infty$$

$f(x) = \begin{cases} 2 & ; x > 0 \\ -1 & ; x \leq 0 \end{cases}$ $\quad f^+(x) = \begin{cases} 2 & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$ $\quad f^-(x) = \begin{cases} 1 & ; x \leq 0 \\ 0 & ; \text{else} \end{cases}$

How to deal with functions that are both +ve and -ve?

$f: X \to \overline{\mathbb{R}}$ we extract the positive and negative parts.

$f^+(x) = \begin{cases} f(x) & ; f(x) > 0 \\ 0 & ; \text{else} \end{cases}$ $\qquad f^-(x) = \begin{cases} -f(x) & ; f(x) < 0 \\ 0 & ; \text{else} \end{cases}$

$f = f^+ - f^-$ and $|f| = f^+ + f^-$

Now f^+ and f^- are both measurable and non-negative. So we can look at $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ but if one is finite we can define

$$\boxed{\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu} \leftarrow \text{lebesgue integral}$$

$$\int_X |f|\,d\mu = \int_X f^+\,d\mu + \int_X f^-\,d\mu$$

we say f is lebesgue integrable $\Leftrightarrow \int_X |f|\,d\mu < \infty$

Note: The integral is linear!!!

let $f: X \to \mathbb{C}$ then $f(x) = u(x) + iv(x)$ if f is measurable u and v are also measurable and so we define $\int_X f\,d\mu = \int_X u\,d\mu + i\int_X v\,d\mu$

$\mathbb{R}^2$ with the lebesgue measure

In Quantum we require $\int_X |\psi|^2\,d\mu < \infty$

here $p \in [1,\infty)$

We now define the spaces $\mathcal{L}^p(X,\mu) = \{f: X \to \bar{\mathbb{R}} \mid \int_X |f|^p\,d\mu < \infty\}$ it has $\|f\|_p := \left(\int_X |f|^p\,d\mu\right)^{1/p}$

THERE IS AN ISSUE!!! There are many functions that have 0 norm!!!

So to rectify this we define an equivalence relation on $\mathcal{L}^p(X,\mu)$ by

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Say we have some property A and the set $\{x \in X \mid x \text{ doesn't satisfy } A\}$ is μ-measure 0 then A is satisfied <u>almost everywhere</u>.

$f \sim g \Leftrightarrow f =_{a.e} g$ (f and g are the same everywhere except a set of measure 0)

Example: $f(x) = x^2$ $\quad g(x) = \begin{cases} 0 & ; x = 1 \\ x^2 & ; \text{else} \end{cases}$ the set $\{x \in \mathbb{R} \mid f(x) \neq g(x)\} = \{1\}$

So $f =_{a.e} g$.

$L^p(X,\mu) = \{[f] \mid f \in \mathcal{L}^p(X,\mu)\} \leftarrow$ These are Banach spaces !!!

Hölders inequality $\|fg\|_1 \leq \|f\|_p \|g\|_q$

$ev_f : L^q(X,\mu) \to \mathbb{R}$ defined by $ev_f(g) = \int_X fg \, d\mu$ here $f \in L^p(X,\mu)$ with $\frac{1}{p} + \frac{1}{q} = 1$

continuous dual

$ev_f \in (L^q(X,\mu))^*$ moreover $f \mapsto ev_f$ is an isomorphism (and is distance preserving)

We can then say that $(L^p(X,\mu))^* = L^q(X,\mu)$

Note: $L^2(X,\mu)$ is isometrically isomorphic to its own dual space.

Two Big Theorems:

1) Monotone Convergence Theorem: Suppose $\{f_n\}$ is a sequence of (measurable) (non-negative) functions and $f_n \leq f_{n+1}$ then $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu$

2) Dominated Convergence Theorem: Suppose $\{f_n\}$ is a sequence (measurable) functions with $f = \lim_{n\to\infty} f_n$. If there is some $g \in L^1(X,\mu)$ s.t. $|f_k| \leq g$ $\forall k \in \mathbb{N}$ then f is lebesgue integrable and $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$

Example: $\int_{[0,1]} \chi_{\mathbb{Q}} \, d\mu = 1 \cdot \mu([0,1] \cap \mathbb{Q}) = 0$

Downside: No notion of orientation. To fix this we define $\int_{[a,b]} f \, d\mu = \int_a^b f(x) \mu(dx)$
$= -\int_b^a f(x) \mu(dx)$

Note: Lebesgue Integral and Riemann Integral are the same for Riemann integrable functions