Before we continue with our discussion of alternating tensors lets talk about complexification of a real vector space V.

let V be a real vector space, $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ this a real vector space of dim. $2 \cdot \dim V$.

let $\lambda \in \mathbb{C}$ then we define $\lambda(v \otimes \alpha) = v \otimes (\lambda\alpha)$ extend this linearly. Then $V_{\mathbb{C}}$ is now a natural complex vector space

let $T: V \to W$ be linear (over $\mathbb{R}$) so we define $T_{\mathbb{C}}: V_{\mathbb{C}} \to W_{\mathbb{C}}$

$T_{\mathbb{C}}(v \otimes \alpha) = Tv \otimes \alpha$ extend this linearly.

Another way to define complexification:

$i^2 + 1 = 0$. In words its a root of x^2+1.

We define an <u>almost</u> complex structure ($J: V \to V$ linear) s.t.

$J^2 + Id = 0$. This means J will act as "i" for us.

$(a+bi)v = av + bJ(v)$ ⟵ this makes V into a complex vector space.

The minimal polynomial of J is x^2+1. So this construction only works for $\dim V$ is even.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

All the eigenvalues of J are $\pm i$.

$$\begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$(V, J) \to (V_{\mathbb{C}}, J_{\mathbb{C}})$

$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ where $V^{1,0} = \{v \in J_{\mathbb{C}} \mid J_{\mathbb{C}} v = iv\}$

$V^{0,1} = \{v \in J_{\mathbb{C}} \mid J_{\mathbb{C}} v = -iv\}$

$$v \mapsto \underbrace{\frac{1}{2}(v - iJ_{\mathbb{C}}(v))}_{\text{eigenvalue } -i} + \underbrace{\frac{1}{2}(v + iJ_{\mathbb{C}}(v))}_{\text{eigenvalue } i}$$

$$\frac{1}{2}\left(J_{\mathbb{C}}(v) + iJ^2_{\mathbb{C}}(v)\right) = \frac{1}{2}\left(J_{\mathbb{C}}(v) - iv\right) = -i\left(\frac{1}{2}\left(v - iJ_{\mathbb{C}}(v)\right)\right)$$

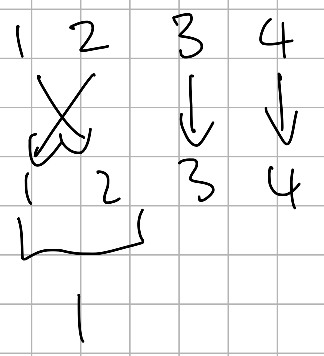
If we have $T \in \mathcal{T}^0{}_k(V)$ or $T \in \mathcal{T}^k{}_0(V)$ then we say

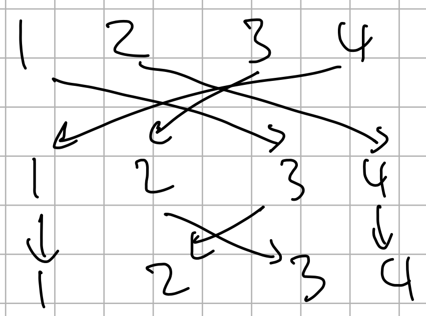
(i) T is symmetric if $\sigma \cdot T = T$

$\sigma \in S_k$, $(\sigma \cdot T)(v_1, \ldots, v_k) = T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$

(ii) T is alternating if $\sigma \cdot T = (\operatorname{sgn} \sigma) T$

$$\operatorname{sgn} \sigma = \begin{cases} 1 & ; \text{ \# of transpositions need to get } \sigma \to id \text{ is even} \\ -1 & ; \text{ \# " " " " " } \sigma \to id \text{ is odd} \end{cases}$$



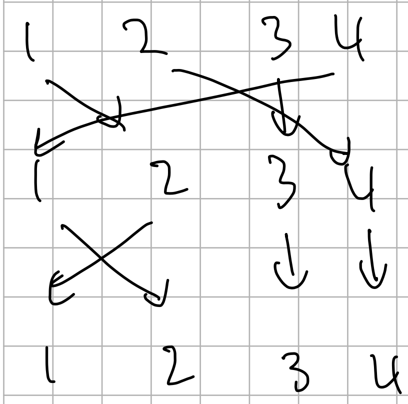


$1 \to 2$

$2 \to 4$

$3 \to 3$

$4 \to 1$

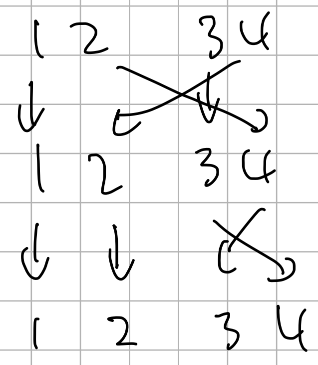


$1 \to 1$

$2 \to 4$

$3 \to 3$

$4 \to 2$

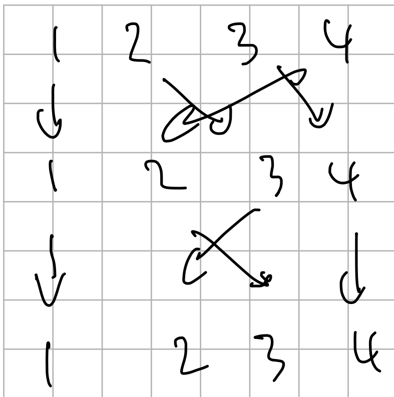


$1 \to 1$

$2 \to 3$

$3 \to 4$

$4 \to 2$



$1 \to 1$

$2 \to 2$

$3 \to 4$

$4 \to 3$

5 transpositions needed so sgn is -1.

So since det is alternating we look mainly at alternating tensors.

$$T \in \underbrace{\mathcal{T}^0{}_k(V)}_{V^* \otimes \cdots \otimes V^*} \xrightarrow{\text{Alt}} \text{Alt}(T) \text{ is alternating.}$$

$$\text{Alt}(T)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \frac{1}{k!} \operatorname{sgn}(\sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

Note that if T already alternating then $\text{Alt}(T) = T$.

We call the image of Alt : $\Lambda^k(V^*)$ the space of alternating $(0,k)$ tensors

$\Lambda^k(V)$ — space of alternating $(k,0)$ tensors

$\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^l(V^*)$

$$\alpha \wedge \beta := \frac{(k+l)!}{k!\,l!} \text{Alt}(\alpha \otimes \beta)$$

k-form on V

Now we define $\Lambda(V^*) := \bigoplus_{k=0}^{\infty} \Lambda^k(V^*)$

$$\dim \Lambda^k(V^*) = \binom{n}{k}$$

$\leftarrow \dim V$

$\Lambda(V)$ is finite dimensional algebra.

$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$

Ex: e_1, e_2, e_3 , $dx(e_1) = 1$ $\quad \frac{1}{2}(dx \otimes dy - dy \otimes dx)$

$\downarrow \quad \downarrow \quad \downarrow$ $\quad dx(e_2) = 0$

$dx \quad dy \quad dz$ $\quad dx(e_3) = 0$

$$(dx \wedge dy)(e_1, e_2) = dx(e_1) dy(e_2) - dy(e_1) dx(e_2)$$
$$= 1$$

$$(dx \wedge dy)(e_2, e_1) = -1$$

$$(dx \wedge dy \wedge dz)(e_1, e_2, e_3) = \begin{vmatrix} dx(e_1) & dy(e_1) & dz(e_1) \\ dx(e_2) & dy(e_2) & dz(e_2) \\ dx(e_3) & dy(e_3) & dz(e_3) \end{vmatrix}$$
$$= 1$$

$$(dx \wedge dy \wedge dz)(e_2, e_1, e_3) = \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1$$

$$T(V) / \langle \alpha \otimes \beta + \beta \otimes \alpha \mid \alpha, \beta \in V \rangle = \Lambda(V)$$

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det

let $e_1, \dots, e_n$ be the standard basis $(dx^1 \wedge \dots \wedge dx^n)(e_1, \dots, e_n) = 1$

Say we have $A: \mathbb{R}^n \to \mathbb{R}^n$ change of basis $e_i \to u_i$

$u_j = A^i{}_j e_i$

$$\det(u_1, \dots, u_n) = \det(Ae_1, \dots, Ae_n)$$

$$= \det(A^{i_1}{}_1 e_{i_1}, \dots, A^{i_n}{}_n e_{i_n})$$

$$= A^{i_1}{}_1 \cdots A^{i_n}{}_n \underbrace{\det(e_{i_1}, \dots, e_{i_n})}_{\text{permution in } S_n}$$

$$= \operatorname{sgn}(1 \to i_1, \dots, n \to i_n) A^{i_1}{}_1 \cdots A^{i_n}{}_n \underbrace{\det(e_1, \dots, e_n)}_{=1}$$

$$= \det A$$

if $\det A > 0$ we say $u_1, \dots, u_n$ has same orientation as $e_1, \dots, e_n$.

if $\det A < 0$ then it has the opposite orientation.

Pick some basis $e_1, \dots, e_n$ of V and we just call it positively oriented. This is an orientation on V.

Collection of all "frame" the orientation partitions into 2 subcollections.

$\neq 0$ highest k $\Lambda^k(V^*)$

Choice of orientation is equivalent to a nonzero top form $\omega \in \Lambda^{\dim(V)}(V^*)$

$e_1, \dots, e_n$ ← basis is some orientation then $e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V^*)$

and is nonzero $(e_1^* \wedge \cdots \wedge e_n^*)(e_1, \dots, e_n) = 1$.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$[a \longrightarrow \longrightarrow \longrightarrow]b$

(*) Frames over $\mathbb{R}$ are $\mathbb{R}^\times := \mathbb{R}\setminus\{0\}$

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let V be real vector space. Then an inner product g on V is a symmetric $(0,2)$ tensor on V that is non-degenerate

$\forall v \in V^\times\ \exists w \in V$ s.t. $g(v,w) \neq 0$

ex: On $\mathbb{R}^n$, $g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + \cdots + dx^n \otimes dx^n$ is an inner product on $\mathbb{R}^n$.

In Special relativity: $g = dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - dx^2 \otimes dx^2 - dx^3 \otimes dx^3$

Minkowski metric.

\# of minus signs in an "orthonormal" basis is signature.

non-degeneracy allows us to find a ON basis similar to G.S algorithm!

$(V,g) \longrightarrow V^*$ canonical isomorphism

$v \longmapsto \iota_v g = g(v,-)$ (lowering index)

interior multiplication by v

Inner product on $\mathcal{T}^k{}_\ell(V)$:

let $\varphi, \psi \in V^*$ there $\exists v, w \in V$ s.t. $\varphi = \iota_v g$ and $\psi = \iota_w g$

then define $g_{V^*}(\varphi, \psi) = g(v,w)$

g_{V^*} has the same signature as g.

Pick an ON basis $e_1, \dots, e_n$ of V, dx^i is dual to e_i

$$g = dx^1 \otimes dx^1 + \dots + dx^p \otimes dx^p - dx^{p+1} \otimes dx^{p+1} - \dots - dx^n \otimes dx^n$$

pick $v = v^i e_i$ consider

$$\imath_v g = (dx^1 \otimes dx^1 + \dots + dx^p \otimes dx^p - dx^{p+1} \otimes dx^{p+1} - \dots - dx^n \otimes dx^n)(v, -)$$
$$= dx^1(v)\, dx^1 + \dots + dx^p(v) \otimes dx^p - dx^{p+1}(v) \otimes dx^{p+1} - \dots - dx^n(v)\, dx^n$$
$$= v^1 dx^1 + \dots + v^p dx^p - v^{p+1} dx^{p+1} - \dots - v^n dx^n$$
$$= v_i dx^i$$

$$v_i = \begin{cases} v^i & \text{if } i \leq p \\ -v^i & \text{if } i > p \end{cases}$$

$$\imath_v g = \underbrace{g_{ij} v^i}_{v_j} \varphi^j \qquad \text{where } \{\varphi^i\} \text{ is a basis for } V^*$$

let $T = v_1 \otimes \dots \otimes v_k \otimes \varphi^1 \otimes \dots \otimes \varphi^l \in \mathcal{T}^k_l(V)$
$S = w_1 \otimes \dots \otimes w_k \otimes \psi^1 \otimes \dots \otimes \psi^l \in \mathcal{T}^k_l(V)$

$$g(T,S) = g(v_1, w_1) \cdots g(v_k, w_k)\, g(\varphi^1, \psi^1) \cdots g(\varphi^l, \psi^l)$$

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \binom{n}{n-k} = \dim \Lambda^{n-k}(V^*)$$

$$\star : \Lambda^k(V^*) \longrightarrow \Lambda^{n-k}(V^*)$$
$$\alpha \longmapsto \star\alpha$$

Pick an oriented basis $e_1, \dots, e_n$ then define $\star\alpha \in \Lambda^{n-k}(V^*)$ so that for all $\beta \in \Lambda^k(V^*)$ we have:

$$\beta \wedge \star\alpha = \langle \beta, \alpha \rangle\, e_1^* \wedge \cdots \wedge e_n^*$$

ex: $\mathbb{R}^2$ with $g = dx^2 + dy^2$ $\quad \star 1$

$\star 1 = dx \wedge dy$, $\quad x \cdot \star 1 = \underbrace{\langle x, 1 \rangle}_{x}\, dx \wedge dy$

$\star dx$: $dy \wedge \star dx = \langle dy, dx \rangle\, dx \wedge dy = 0$

$\star dx = \alpha\, dy$

$dx \wedge \star dx = \langle dx, dx \rangle\, dx \wedge dy = dx \wedge dy$

$\alpha\, dx \wedge dy$

$\star dx = dy$

$\star dy = -dx$