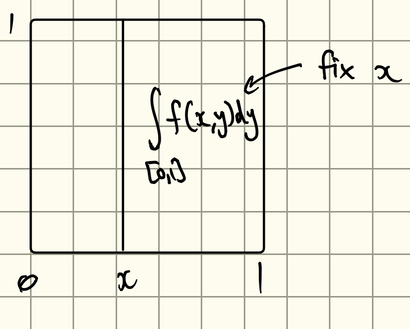
October 16, 2025: We have some measure space (X, Σ, μ) we looked at how to integrate functions.

$(\mathbb{R}^n, \lambda^n) \leftarrow$ we mainly worry about this.

$$\int_{[0,1]^2} 1 \, d^2\lambda = 1 \cdot \lambda^2([0,1]^2) = 1$$

Say we wanted to integrate $f(x,y) = x^2 + y^3$ over $[0,1]^2$



we now look at how to decompose an integral in "higher dimension" into "lower dimensional" integrals.

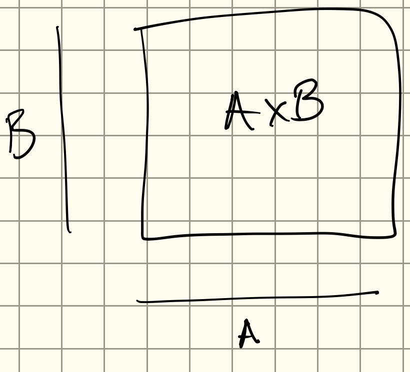
$$\int_{[0,1]^2} f \, d^2\lambda = \int_{[0,1]} \Bigg(\int_{[0,1]} f(x,y) \underbrace{d\lambda(y)}_{\text{integrate wrt } y} \Bigg) \underbrace{d\lambda(x)}_{\text{integrate wrt } x.} \; ?$$

Idea: Start with measurable spaces (X, Σ_X) and (Y, Σ_Y) and try to define a reasonable σ-algebra on $X \times Y$. Then define a reasonable measure on $X \times Y$ if X, Y are measure spaces.

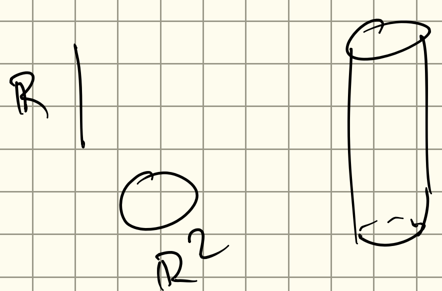
"Reasonable" here means we hope that the theory allows the lebesgue measure on $\mathbb{R}^n$ to decompose into the lebesgue measure on $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$: $\mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$

$$\mathbb{R}^n \cong \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$$

let us take (X, Σ_X) and (Y, Σ_Y) then note we define the σ-algebra $\Sigma_X \otimes \Sigma_Y$ to be the smallest σ-algebra generated by $\{A \times B \mid A \in \Sigma_X \text{ and } B \in \Sigma_Y\}$



we will call $A \times B$ a rectangle (geometrically this might always be a rectangle



let $E \subseteq X \times Y$ then the cross-section of E over $x \in X$ is defined as $[E]_x = \{(x,y) \in E\}$
$y \in Y$ is defined as $[E]^y = \{(x,y) \in E\}$

let $f: X \times Y \to A$ then the cross-section of f over $x \in X$ is defined as $[f]_x(y) = f(x,y)$
$y \in Y$ is defined as $[f]^y(x) = f(x,y)$

If $f: X \times Y \to \overline{\mathbb{R}}$ is measurable we have $[f]_x$ is measurable in Y
$[f]^y$ is measurable in X

(σ-finite)

let (X, Σ_X, μ) and (Y, Σ_Y, ν) be measure spaces. How can we define a measure on $(X \times Y, \Sigma_X \otimes \Sigma_Y)$?

If $f: X \to \overline{\mathbb{R}}$ then we write $\int_X g \, d\mu = \int_X g(x) \, d\mu(x)$

Now note that if $E \in \Sigma_X$ then $\int_X \chi_E \, d\mu = \mu(E)$. Then suppose $E \in \Sigma_X \otimes \Sigma_Y$

then note first that $\forall x \in X$ $[E]_x$ is measurable $\Leftrightarrow$ $[\chi_E]_x$ is also measurable
$\forall y \in Y$ $[E]^y$ is measurable $\Leftrightarrow$ $[\chi_E]^y$ is also measurable

using this define $(\mu \times \nu)(E) = \int_X \left(\int_Y \underbrace{\chi_E(x,y)}_{[\chi_E]_x : Y \to \overline{\mathbb{R}}} \, d\nu(y) \right) d\mu(x)$ another way to look at

this is that define $g(x) = \int_Y [\chi_E]_x(y) \, d\nu(y)$ then $(\mu \times \nu)(E) = \int_X g(x) \, d\mu(x)$

So then we can define the iterated integral of $f: X \times Y \to \overline{\mathbb{R}}$ as $\int_X \int_Y f(x,y) \, d\nu(y) \, d\mu(x)$

$$= \int_X \left(\int_Y [f]_x(y) \, d\nu(y) \right) d\mu(x)$$

We could have also defined $(\mu \times \nu)(E) = \int_Y \int_X \chi_E \, d\mu(x) \, d\nu(y)$ but we will see that it will not matter.

Example: $\int_{[0,1]} \int_{[0,1]} x^2 + y^3 \, d\lambda(y) \, d\lambda(x) = \int_{[0,1]} \left(\int_{[0,1]} x^2 + y^3 \, d\lambda(y) \right) d\lambda(x) = \int_{[0,1]} x^2 + \frac{1}{4} \, d\lambda(x) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

$\int_{[0,1]} \left(\int_{[0,1]} x^2 + y^3 \, d\lambda(x) \right) d\lambda(y) = \int_{[0,1]} \frac{1}{3} + y^3 \, d\lambda(y) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

We would now <u>hope</u> that $\int_{[0,1]^2} f \, d^2\lambda = \frac{7}{12}$

So three main questions: (i) Is $\mu \times \nu$ even a measure ?? - Yes it is :)

(ii) Is the lebesgue measure on $\mathbb{R}^n$ a product measure of the lebesgue measure on $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$? Yes it is a product of the lebesgue measures.

(iii) Is the integral over the product measure equal to the iterated integrals ?

Theorem 1 (Tonelli's Theorem): Suppose (X, Σ_X, μ) and (Y, Σ_Y, ν) are (σ-finite) measure spaces. Suppose also that $f : X \times Y \to \overline{\mathbb{R}}$ is non-negative and is also $(\mu \times \nu)$-measurable.

Then consider $g(x) = \int_Y f(x,y)\, d\nu(y)$ and $h(y) = \int_X f(x,y)\, d\mu(x)$, g and h are both measurable (in their respective spaces) and

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y)$$

Example: $$\int_{[0,1]^2} xy \, d^2\lambda = \int_{[0,1]} \left(\int_{[0,1]} xy \, d\lambda(x) \right) d\lambda(y) = \int_{[0,1]} \frac{y}{2} \, d\lambda(y) = \frac{y^3}{6} \Big|_0^1 = \frac{1}{6}$$

Theorem 2 (Fubini's Theorem): Suppose (X, Σ_X, μ) and (Y, Σ_Y, ν) are (σ-finite) measure spaces. Suppose $f : X \times Y \to \overline{\mathbb{R}}$ is $(\mu \times \nu)$-measurable with $\int_{X \times Y} |f| \, d(\mu \times \nu) < \infty$ then

$$\int_{X \times Y} |f| \, d(\mu \times \nu) = \int_{X \times Y} f^+ \, d(\mu \times \nu) + \int_{X \times Y} f^- \, d(\mu \times \nu)$$

(i) $\int_Y f(x,y)\, d\nu(y) < \infty$ for almost every $x \in X$

(ii) $\int_X f(x,y)\, d\mu(x) < \infty$ for almost every $y \in X$

Then consider $g(x) = \int_Y f(x,y)\, d\nu(y)$ and $h(y) = \int_X f(x,y)\, d\mu(x)$, g and h are both measurable (in their respective spaces) and

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y)$$

Example: $f : [0,1] \times [0,1] \to \overline{\mathbb{R}}$ by $f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$ then

$$\int_{[0,1]} \int_{[0,1]} f(x,y)\, d\lambda(y)\, d\lambda(x) = \frac{\pi}{4}$$

$$\int_{[0,1]} \int_{[0,1]} f(x,y)\, d\lambda(x)\, d\lambda(y) = -\frac{\pi}{4}$$

$$f^+ = \begin{cases} \frac{x^2 - y^2}{(x^2+y^2)^2} & ; x > y \\ 0 & ; \text{else} \end{cases} \qquad f^- = \begin{cases} \frac{y^2 - x^2}{(x^2+y^2)^2} & ; y < x \\ 0 & ; \text{else} \end{cases}$$

$\int_{[0,1]^2} f^+ \, d^2\lambda$ and $\int_{[0,1]} f^- \, d^2\lambda$ diverges