one direction — General Relativity
↳ Derivation of Einstein Equations
↳ Study Black hole solutions
↳ Possible Higher dimensional Theory (Kaluza-Klein)

Other direction — Quantum Mechanics on Arb. spaces
↳ Dirac Equation in 3 space
↳ Spinors and spinor bundles
↳ Dirac equation on Riemannian spin manifolds

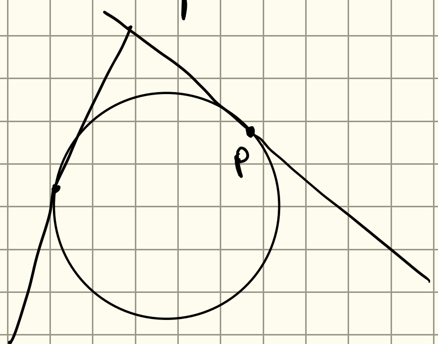
Defined smooth manifolds and defined smooth maps.

Recall a map $f: M^m \to N^n$ is smooth if for any point $p \in M$ and any chart (U, φ) around p and (V, ψ) around $f(p)$ the map $\psi \circ f \circ \varphi^{-1}$ is smooth.

What is the "derivative" of f?

The derivative (at $p \in M$) must be a linear map. Note M may not be a linear space and so we need some vector space $\underbrace{T_pM}_{}$ at the point p.

↑ tangent space of M at p.



Assume $S^1 \subseteq \mathbb{R}^2$ and $\gamma: (-\varepsilon, \varepsilon) \to S^1$ we hope that γ and γ' be tangent at at each $t \in (-\varepsilon, \varepsilon)$. WTS that $\langle \gamma', \gamma \rangle = 0$. $1 = \langle \gamma, \gamma \rangle \Rightarrow 0 = \frac{d}{dt} \langle \gamma, \gamma \rangle$

$$= \langle \gamma, \gamma' \rangle + \langle \gamma', \gamma \rangle$$
$$= 2 \langle \gamma, \gamma' \rangle$$

$$\Rightarrow \langle \gamma, \gamma' \rangle = 0$$

$f: M \to \mathbb{R}$ say this is smooth. suppose I have a smooth curve $\gamma: (-\varepsilon, \varepsilon) \to M$

then $\underset{\substack{\uparrow \\ \gamma^* f}}{f \circ \gamma}: (-\varepsilon, \varepsilon) \to \mathbb{R}$ we can look at $\frac{d}{dt}(f \circ \gamma)(t)$ this is precicely the directional derivative of f in the direction γ.

Now we can define tangent vectors at $p \in M$ as functions $v_p : C^\infty(M) \to \mathbb{R}$ with the property that it is linear and $v_p(fg) = f(p)v_p(g) + g(p)v_p(f) =: T_pM$

No reason to believe this is finite dimensional. Remarkably these are indeed finite dimensional.

In $\mathbb{R}^n$ only linear combinations $\sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}\Big|_p$ are tangent vectors at $p \in \mathbb{R}^n$.

$$\Updownarrow$$

$$(\alpha^1, \dots, \alpha^n) \in \mathbb{R}^n$$

$f : \mathbb{R}^n \to \mathbb{R} \Rightarrow Df(p) : \mathbb{R}^n \to \mathbb{R}$ the directional derivative of f along $v \in \mathbb{R}$

standard basis $\downarrow$ $\qquad \nabla_v f = Df(p)(v)$

$$\left\{ \frac{\partial f}{\partial x^1} \quad \frac{\partial f}{\partial x^2} \quad \cdots \quad \frac{\partial f}{\partial x^n} \right\}$$

$(1,0,0,\dots,0)$, $(0,1,0,\dots,0)$, $\dots$

let $f : M^m \to N^n$ then $df_p : T_pM \to T_{f(p)}N$ defined as let $v_p \in T_pM$

then $df_p(v_p) : C^\infty(N) \to \mathbb{R}$, let $g \in C^\infty(N)$ then define $df_p(v_p)(g) = v_p(g \circ f)$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, dashed, "g \circ f"'] & N \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

If v_p is the velocity vector of some $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ i.e. $v_p(f) = \frac{d}{dt}\Big|_{t=0} (f \circ \gamma)(t)$ then $df_p(v_p)(g)$ is the directional derivative of g in the direction of $f \circ \gamma = \gamma^* f$

Now the usual theorems of calculus apply, $f : M \to N$ and $g : N \to S$ is smooth then $g \circ f : M \to S$ is smooth and that $d(g \circ f)_p = dg_{f(p)} \circ df_p$ hence chain rule holds.

Moreover by the inverse function theorem if $f : M \to N$ has non zero derivative and $\dim M = \dim N$ then f is locally invertible and f^{-1} is differentiable whenever defined.

$\Rightarrow$ If f is a local diffeomorphism then $df_p : T_pM \to T_{f(p)}N$ is a isomorphism.

Recall that for a chart (U, φ) we have U is diffeomorphic to $\varphi(U)$ hence

$$\begin{array}{l} T_pU = T_{\varphi(p)}\varphi(U) \cong T_{\varphi(p)}\mathbb{R}^m \cong \mathbb{R}^m \\ \ \ \| \wr \\ T_pM \end{array}$$

Hence T_pM is isomorphic to $T_{\varphi(p)}\mathbb{R}^m$. Hence T_pM is finite dimensional.

We also have a basis for T_pM given by $d\varphi^{-1}_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\right)$

$$d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^m$$

$$\Downarrow$$

$$d\varphi^{-1}_p : \underbrace{T_{\varphi(p)}\mathbb{R}^m}_{= \operatorname{span}\left\{\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right\}} \to T_pM$$

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let $f : M \to N$ we want to see how df_p looks like in this basis $d\varphi^{-1}_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\right)$ and $d\psi^{-1}_{\psi(f(p))}\left(\frac{\partial}{\partial y^i}\right)$ but it suffices to look at maps

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

let $g \in C^\infty(\mathbb{R}^n)$ then
$$df_p\left(\frac{\partial}{\partial x^i}\right)(g) = \left.\frac{\partial}{\partial x^i}\right|_p (g \circ f)$$
$$= \left.\frac{\partial g}{\partial y^j}\right|_{f(p)} \left.\frac{\partial f^j}{\partial x^i}\right|_p$$

hence df_p is just the jacobian.

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let G be a group and also be a smooth manifold. We then suppose that the maps
$$\mu : G \times G \to G, \quad (g,h) \to gh$$
and
$$\imath : G \to G, \quad g \mapsto g^{-1}$$
be smooth.

What is $d\mu_{(e,e)}$ and $d\iota_e$?

We want to compute $d\mu_{(e,e)}(X_e, Y_e)(f)$ where $(X_e, Y_e) \in T_{(e,e)}(G\times G) \cong T_eG \oplus T_eG$

let $X_e \in T_eG$ and let $\gamma : (-\varepsilon, \varepsilon) \to G$ s.t. $\gamma(0) = e$ and $\gamma'(0) = X_e$ $\left(X_e f = \frac{d}{dt}\Big|_{t=0} (f\circ\gamma)(t) \right)$

then $\sigma : (-\varepsilon,\varepsilon) \to G\times G$ given $\sigma(t) = (\gamma(t), e)$ then $\sigma(0) = (e,e)$ and $\sigma'(0) = (X_e, 0)$

then $f \in C^\infty(G)$ then $d\mu_{(e,e)}(X_e, 0)(f) = (X_e, 0)(f\circ\mu)$

$$= \frac{d}{dt}\Big|_{t=0} (f\circ\mu)\circ\sigma \qquad\qquad (\mu\circ\sigma)(t) = \gamma(t)$$

$$= \frac{d}{dt}\Big|_{t=0} (f\circ\gamma)(t)$$

$$= X_e f$$

Similarly $d\mu_{(e,e)}(0, Y_e) = Y_e f \Rightarrow d\mu_{(e,e)}(X_e, Y_e)(f) = X_e f + Y_e f$

Now we compute $d\iota_e$. Note that $(\mu\circ(id_M \times \iota))(g) = \mu(g, g^{-1}) = gg^{-1} = e$
this is constant. Hence $d(\mu\circ(id_M\times\iota))_e(X_e) = 0$

$$\Downarrow$$

$$d\mu_{(e,e^{-1})}(X_e, d\iota_e(X_e)) = 0$$

$$\Rightarrow X_e + d\iota_e(X_e) = 0$$

$$\Rightarrow d\iota_e(X_e) = -X_e$$

We placed importance on $c_g : G \to G$, $h \mapsto ghg^{-1}$ and note that $c_g(e) = geg^{-1} = gg^{-1} = e$

hence we can differentiate this $(dc_g)_e : \underset{\mathfrak{g}}{T_eG} \to \underset{\mathfrak{g}}{T_eG}$ [T_eG is called the lie algebra of G, note lie algebra need a lie bracket, which we will define now]

$$\begin{aligned} G &\longmapsto GL(\mathfrak{g}) \\ g &\longmapsto (dc_g)_e \end{aligned}$$ hence we have a representation of G on $\mathfrak{g}$. This is called the adjoint representation $Ad(g) = (dc_g)_e$

$Ad(e) = (dc_e)_e = id_{\mathfrak{g}}$

$$\begin{aligned} \mathfrak{g} &\longmapsto GL(\mathfrak{g}) \\ X &\longmapsto d(Ad)_e(X) \end{aligned}$$

$ad(X,Y) = d(Ad(X))_e(Y)$

$ad : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is bilinear moreover one can check that it satisfies the conditions for a lie bracket.