Hilbert Space
Banach Space

Symmetric tensors
Lie Algebra

vector - valued forms

Symmetric tensor

let V be real vector space. Give V an inner product g note that g is symmetric

$\otimes$ We used $\mathcal{T}^k_\ell(V) = (V)^{\otimes k} \otimes (V^*)^{\otimes \ell}$ This is the physics convention

$\mathcal{T}^k_\ell(V) = (V^*)^{\otimes k} \otimes V^{\otimes \ell}$

$T \in \mathcal{T}^k_\ell(V)$ maps from $V \times \cdots \times V \times V^* \times \cdots \times V^* \to \mathbb{R}$

Let $T \in \mathcal{T}^0_k(V)$ or $T \in \mathcal{T}^k_0(V)$ we define an action of S_k on these spaces by : $\sigma \in S_k$

$$(\sigma \cdot T)(v_1, \ldots, v_n) = T(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

T is symmetric if $\sigma \cdot T = T \quad \forall \sigma \in S_k$

$$\operatorname{Sym} T = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \cdot T$$

So to check this symmetric : $\gamma \in S_k$ and apply to $\operatorname{Sym} T$

$$\gamma \cdot \operatorname{Sym} T = \frac{1}{k!} \sum_{\sigma \in S_k} (\gamma\sigma) \cdot T$$

Say I want the permutation $\alpha \in S_k$, choose $\sigma = \gamma^{-1}\alpha$

$$= \frac{1}{k!} \sum_{\alpha \in S_k} \alpha \cdot T$$

$$= \operatorname{Sym} T$$

$l_g(ab) \overset{?}{=} l_g(a)\, l_g(b)$ So left multiplication is not a group homomorphism

$$\begin{array}{ccc} l_g(ab) & & l_g(a)\,l_g(b) \\ \parallel & & \parallel \\ g(ab) & & ga\,gb \\ \parallel & & \\ gab & & \end{array}$$

Image of Sym is the symmetric k-tensors. Note $\mathrm{Sym}\,T = T$ for symmetric tensors.

So the space of symmetric tensor of type $(k,0)$, $\Sigma^k V^* := \mathrm{Sym}(\mathcal{T}^k{}_0(V))$

of type $(0,k)$, $\Sigma^k V := \mathrm{Sym}(\mathcal{T}^0{}_k(V))$

If $T \in \Sigma^k V^*$ and $S \in \Sigma^l V^*$ we define $T \odot S = \mathrm{Sym}(T \otimes S)$

Note $g \in \Sigma^2 V^*$.

On $\mathbb{R}^n$ we have standard basis $e_1, \dots, e_n$ and corresponding dual basis $dx^1, \dots, dx^n$

$g = g_{ij}\, dx^i \otimes dx^j$ note $g_{ij} = g_{ji}$ since $g(e_i, e_j) = g_{\alpha\beta}(dx^\alpha \otimes dx^\beta)(e_i, e_j)$

$= \frac{1}{2}\left(g_{ij}\, dx^i \otimes dx^j + \underset{g_{ij}}{g_{ji}}\, dx^j \otimes dx^i\right)$ $\qquad = g_{\alpha\beta}\, dx^\alpha(e_i)\, dx^\beta(e_j)$

$= g_{ij}\left[\frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right)\right]$ $\qquad = g_{\alpha\beta}\, \delta^\alpha{}_i\, \delta^\beta{}_j$

need $\alpha = i$, $\beta = j$ $\qquad = g_{ij}$

$= g_{ij}\, dx^i \odot dx^j = g_{ij}\, dx^i dx^j$

Similarly $g(e_j, e_i) = g_{ji}$ but symmetry gives $g_{ij} = g(e_i, e_j) = g(e_j, e_i) = g_{ji}$

$xy = yx$

Homogenous of degree k: $P(tx) = t^k P(x)$

$dx^i dx^j \leftarrow$ Homogenous polynomial of degree 2.

$g \in V^* \otimes V^*$ identified with a map $\hat{g}: V \to V^*$ since g is nondegenerate is an isomorphism and so its inverse $\hat{g}^{-1}: V^* \to V$
Hence $g^{-1} \in V \otimes V$ moreover $g^{-1} \in \Sigma^2 V$

We will denote the inner product by $[g_{ij}]$ ← inverses
We will denote the "inverse" inner product by $[g^{ij}]$ ←

Hilbert Spaces: let V be a complex vector space, then $g: V \times V \to \mathbb{C}$ is a hermitian inner product if

(i) $g(v,w) = \overline{g(w,v)}$

(ii) $g(u+v,w) = g(u,w) + g(v,w)$

(iii) $g(u, v+w) = g(u,v) + g(u,w)$

(iv) $g(\lambda v, w) = \bar{\lambda} g(v,w)$

(v) $g(v, \lambda w) = \lambda g(v,w)$

(vi) $g(v,v) \geq 0 \quad \forall v$ and $g(v,v) = 0 \Leftrightarrow v = 0$

Ex: (i) $\mathbb{C}^n$ with usual inner product

$$g((z_1, \dots, z_n), (w_1, \dots, w_n)) = \sum_{i=1}^{n} \bar{z}_i w_i$$

(ii) Consider particle in an infinite square well $V = \begin{cases} 0 & ; -\ell \leq x \leq \ell \\ \infty & ; \text{else} \end{cases}$

$L^2([-\ell,\ell])$ with the inner product: $\langle \varphi | \psi \rangle = \int_{-\ell}^{\ell} \varphi^* \psi \, dx$

An inner product induces a norm: $|x|^2 = g(x,x) \Rightarrow |x| = \sqrt{g(x,x)}$

We can define a metric $d(x,y) = |x-y|$

A metric on a set X is a function $d: X \times X \to \mathbb{R}$

(i) $d(x,y) \geq 0$ $\forall x,y \in X$ and $d(x,x) = 0$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$

We can define the metric topology on X by choosing the smallest topology so that sets of form $B(x,\varepsilon) = \{ y \in X \mid d(x,y) < \varepsilon \}$ are open

A metric space (X,d) is complete if every Cauchy sequence converges

A sequence $\{a_n\}$ is Cauchy if $\forall \varepsilon > 0$ $\exists N > 0$ s.t. $n,m > N \Rightarrow d(a_n, a_m) < \varepsilon$

Ex: $\mathbb{R}, \mathbb{C}$ are complete metric spaces

$L^2([-l,l])$ with the lebesgue integral

A Hilbert space is an inner product space (V,g) so that the metric topology is complete.

So if (V,g) is a Hilbert space we define $V^* = \{ \lambda: V \to \mathbb{F} \mid \lambda \text{ linear, continous} \}$

The riesz representation theorem: $V \cong V^*$ as long as V^* is the continous dual

We can give V^* a norm: $|\varphi| = \sup_{|v|=1} |\varphi(v)| \leftarrow$ operator norm

$T \in \mathcal{L}(V,W)$ we define $\|T\| = \sup_{\underset{\text{in } V}{|v|=1}} \underset{W}{|Tv|}$

Norms and inner products are different and a norm comes from an inner product $\iff |v+w|^2 + |v-w|^2 = 2(|v|^2+|w|^2)$

A Banach space is a vector space with a norm that is a complete metric space with induced metric talked about above.

Vector valued forms: We can think of (multilinear) maps $V\times\cdots\times V\times V^*\times\cdots\times V^* \to W$ as elements of $\mathcal{T}^k_\ell(V)\otimes W$ [W-valued (k,ℓ) tensor]

The alternating W-valued maps are in $\Lambda^k(V^*)\otimes W$, $\Lambda^k(V)\otimes W$ (W-valued forms)

Note if $T\in\mathcal{T}^k_\ell(V)\otimes W_1$ and $S\in\mathcal{T}^m_n(V)\otimes W_2$ then

$$T\otimes S\in\mathcal{T}^k_\ell(V)\otimes W_1\otimes\mathcal{T}^m_n(V)\otimes W_2 \cong \mathcal{T}^{k+m}_{\ell+n}(V)\otimes(W_1\otimes W_2)$$

$$(T\otimes S)(v_1,\dots,v_k,v_{k+1},\dots,v_{k+m},\varphi^1,\dots,\varphi^\ell,\varphi^{\ell+1},\dots,\varphi^{\ell+n})$$

$$= \underbrace{T(v_1,\dots,v_k,\varphi^1,\dots,\varphi^\ell)}_{W_1}\otimes\underbrace{S(v_{k+1},\dots,v_{k+m},\varphi^{\ell+1},\dots,\varphi^{\ell+n})}_{W_2}$$

$$\in (V^*)^{\otimes k+m}\otimes V^{\otimes \ell+n}\otimes W_1\otimes W_2 = \mathcal{T}^{k+m}_{\ell+n}(V)\otimes(W_1\otimes W_2)$$

let $\omega\in\Lambda^k(V^*)\otimes W_1$ and $\eta\in\Lambda^\ell(V^*)\otimes W_2$ then

$$(\omega\wedge\eta)(v_1,\dots,v_k,v_{k+1},\dots,v_{k+\ell}) = \frac{(k+\ell)!}{k!\,\ell!}\sum_{\sigma\in S_{k+\ell}}(\operatorname{sgn}\sigma)\,\omega(v_{\sigma(1)},\dots,v_{\sigma(k)})\otimes\eta(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)})$$

$$\omega\wedge\eta\in\Lambda^{k+\ell}(V^*)\otimes(W_1\otimes W_2)$$

Lie Algebra: let V be a vector space and let $[\cdot,\cdot]$ be a V-valued 2 form on V s.t. (lie bracket)

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \quad \forall x,y,z\in V$$

Exercise: Check $(\mathbb{R}^3, \times)$ is a lie algebra
cross product

let V be any vector space and consider the space $\mathcal{L}(V,V)$
this is a lie algebra using $[A,B] = A \circ B - B \circ A$

So $(\mathcal{L}(V,V), [\cdot,\cdot])$ is a lie algebra

If $e_1, \dots, e_n$ is a basis for $(V, [\cdot,\cdot])$ then

$[e_i, e_j] = C^k_{ij} e_k$

structure coefficients

So $C^k_{ij} = -C^k_{ji}$

Exercise: Show that the Jacobi identity is equivalent to:
$C^l_{ij} C^m_{kl} + C^l_{jk} C^m_{il} + C^l_{ki} C^m_{jl} = 0$. Moreover any n^3 constants that satisfy $C^k_{ij} = -C^k_{ji}$ and $C^l_{ij} C^m_{kl} + C^l_{jk} C^m_{il} + C^l_{ki} C^m_{jl} = 0$, then $[e_i, e_j] = C^k_{ij} e_k$ is a lie bracket.