let $X \in \mathfrak{X}(M)$. let us start a point $p \in M$ now we want to find a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(t) = X_{\gamma(t)} \iff d\gamma\left(\frac{d}{dt}\Big|_{t=t_0}\right) = X_{\gamma(t_0)}$

$$\iff \forall f \in C^\infty(M) : \frac{d}{dt}\Big|_{t=t_0} (f \circ \gamma)(t) = X_{\gamma(t_0)} f$$

So such curve is called an integral curve of X through p.

let (U, φ) be coordinates around p. In these coordinates $X = X^i \frac{\partial}{\partial x^i}$ and $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$ and $\gamma^i : (-\varepsilon, \varepsilon) \to \mathbb{R}$. $\gamma'(t) = \gamma^{i\prime}(t) \frac{\partial}{\partial x^i}$ hence we get a system of ODEs $\gamma^{i\prime}(t) = X^i(\gamma(t))$

$\Rightarrow$ solution exists for some small neighbourhood of p.

Hence we a map $\varphi_X : (-\varepsilon, \varepsilon) \times V \to M$ s.t. for any fixed q the map $\varphi_X^q : (-\varepsilon, \varepsilon) \to M$ is the integral curve of X through q.

where the solution exists around p

we say X is a complete vector field if the flow at all points exists for all time.

let $f : X \to Y$ be a function then $\operatorname{supp} f = \overline{\{p \in X \mid f(p) \neq 0\}}$

If $X \in \mathfrak{X}(M)$ and $\operatorname{supp} X$ is compact then X is complete.

$\Rightarrow$ If M is compact then every vector field has complete flow.

Examples: (i) let $X = x^2 \frac{d}{dx}$ on $\mathbb{R}$ then an integral curve must satisfy $\gamma' = \gamma^2$

$\Rightarrow \gamma(t) = \dfrac{1}{c - t}$ where $c = \dfrac{1}{\gamma(0)}$. Note: at $x = 0 \Rightarrow X\big|_0 = 0$ and so at 0 it has complete flow.

ii) let $X = x \frac{d}{dx}$ then $\gamma_X^p(t) = pe^t \Rightarrow \varphi_X(t, p) = pe^t$.

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let $X \in \mathfrak{X}(M)$ with flow defined on $\varphi_X : J \to M$ where $J = \{(t, p) \mid$ the integral curve through p exists at $t\}$

Then note that $(t, \varphi_X(s, p)) \in J \iff (t+s, p) \in J$ moreover $\underbrace{\varphi_X(t+s, p) = \varphi_X(t, \varphi_X(s, p))}_{\text{flow property}}$.

Proof: Let $f \in C^\infty(M)$ then $\frac{d}{dt} f(\varphi_X(t+s,p)) = \frac{d}{d\tau}\Big|_{\tau = t+s} f(\varphi_X(\tau,p)) = X_{\varphi_X(t+s,p)}(f)$

Hence by uniqueness $\varphi_X(t+s,p) = \varphi_X(t, \varphi_X(s,p))$.

Note: $\varphi_X(0,p) = p \Rightarrow \varphi_X^0 = \mathrm{id}_M$ hence φ_X^t is a diffeomorphism of its domain. Since φ_X^{-t} is its inverse.

If X is complete we get smooth action of $\mathbb{R}$ on M. $t \mapsto \varphi_X^t$

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let X and $Y \in \mathfrak{X}(M)$. Is $X \circ Y$ a vector field?

let $f \in C^\infty(M)$ then
$$\begin{aligned} X(Yf) &= X^i \partial_i (Y^j \partial_j f) \\ &= X^i \partial_i Y^j \partial_j f + X^i Y^j \partial_i \partial_j f \end{aligned}$$

$$\begin{aligned} Y(Xf) &= Y^i \partial_i (X^j \partial_j f) \\ &= Y^i \partial_i X^j \partial_j f + X^j Y^i \partial_i \partial_j f \end{aligned}$$

Hence $[X,Y]f = X(Yf) - Y(Xf)$ is a vector field. This is the Lie bracket of X and Y.

Suppose $F: M \to N$ is a diffeomorphism $\Rightarrow F^*f : M \to \mathbb{R}$
$p \mapsto f(F(p))$

$$\begin{tikzcd} C^\infty(M) \arrow[r, dashed, "F^*Y"] & C^\infty(M) \\ C^\infty(N) \arrow[u, "F^*"] \arrow[r, "Y"'] \arrow[ur, phantom, "\circlearrowright"] & C^\infty(N) \arrow[u, "F^*"'] \end{tikzcd}$$

$(F^*Y)(F^*g) = F^*(Y(g)) \Leftrightarrow (F^*Y)_p = dF^{-1}_{F(p)}(Y_{F(p)})$

$F : p \mapsto F(p)$
$T_pM \mapsto T_{F(p)}N$

$F^{-1} : F(p) \mapsto p$
$T_{F(p)}N \mapsto T_pM$

$F^*[X,Y] = [F^*X, F^*Y]$

let $X \in \mathfrak{X}(M)$ be complete $(\varphi_X^t)^* : \mathfrak{X}(M) \to \mathfrak{X}(M)$
$Y \mapsto (\varphi_X^t)^* Y$

$\mathcal{L}_X f = \frac{d}{dt}\Big|_{t=0} (\varphi_X^t)^* f = \frac{d}{dt}\Big|_{t=0} f \circ \varphi_X^t \leftarrow$ Lie derivative of f wrt X.

$\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} (\varphi_X^t)^* Y$

Theorem: let $X, Y \in \mathfrak{X}(M)$ then $\mathcal{L}_X Y = [X,Y]$

Proof: $(\varphi_X^t)^*(Y(f)) = ((\varphi_X^t)^* Y)((\varphi_X^t)^* f)$ take derivative at $t=0$

$$X(Y(f)) = \left(\frac{d}{dt}\Big|_{t=0} (\varphi_X^t)^* Y\right)(f) + Y\left(\frac{d}{dt}\Big|_{t=0} (\varphi_X^t)^* f\right)$$

$$= (\mathcal{L}_X Y)(f) + Y(Xf)$$

$\Rightarrow \mathcal{L}_X Y = [X,Y]$

Note $\mathcal{L}_X Y = -\mathcal{L}_Y X$