Sight seeing through the world of Banach Spaces

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Abstract

The Math Mentorship Program is conducted by the Department of Mathematics at the University of Toronto pairing high school students interested in the field of mathematics study and research with graduate students, post-doctoral fellows and faculty in the field. Following are the notes and the thought process that has gone behind this document, as part of my individual research, in preparing this document with the guidance of my mentor Luke Volke.

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§1 Overview of our Journey

This is a document that I have made in order to capture and explain a complex topic through an example that may seem easy to begin with but leads us down many rabbit holes as many things go "wrong". I have also written this in a way I enjoy learning math, a more guided and self contained way of learning. The various next sections of this document will go through all the necessary steps and details.

§2 Banach Spaces

What is a Banach space? That is the main question I hope to answer in this section. However, detailed explanation will follow in the later sections.

Definition 2.0.1 (Banach Space)**.** A Banach space is a complete normed vector space $(A, \|\cdot\|)$. Where A is the complete normed vector space over a certain field $\mathbb{F} \subseteq \{\mathbb{R}, \mathbb{C}\}$, usually $\mathbb{F}$ is used to denote $\mathbb{R}$ or $\mathbb{C}$ but we can use it for any field. $\|\cdot\| : A \to \mathbb{R}$ is the norm in the pair.

We will be working with the following example to make life a bit easier. We will also define what **vector space, normed vector space and norm** are later on. We will also define what it means for these spaces to be **complete**. For now all we need to now is that there exist some mathematical object with these names.

Example 2.0.1. Let V be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\| := \sqrt{\int_{\mathbb{R}} |f(x)|^2 \, dx} < \infty$$

Here $\|f\|$ is the norm of f.

Before we jump into the nitty-gritty of the document, in our definiton for the norm we might notice some redundancies. That being the $|f|^2$, honestly $|f|^2 = f^2$ in our case but I left that in there just so that it becomes easier to generalize this to higher dimensions.

§2.1 Prove V is a vector space

We will prove that this set V is a Banach space. To prove this we will go through multiple steps. First we will be proving the fact that V is a **vector space**, after which we will be proving that it is a **normed vector space** then we will show that it is a **metric space** and finally we will be able to show that it is a Banach space. The purpose of going through such a lengthy process is it help us grasp the idea of what a Banach space is, without getting very general. These words of **vector space**, **normed vector space** and **metric spaces** will make more sense the more we move through the paper.

Proposition 2.1.1. V is a vector space.

To be more exact we want to show that $(V, +, \cdot)$ is a vector space but for connivance we say that V is a vector space. It will be made clear as to why it is $(V, +, \cdot)$, as you read through this paper.

Proof. Let us start on this long journey by first proving the stated proposition. What does it mean for something to be a vector space?

Definition 2.1.1 (Vector Space)**.** A vector space is a mathematical object that has elements over a field $\mathbb{F} \subseteq \{\mathbb{R}, \mathbb{C}\}$, along with two main operations(2.1.2) of addition and multiplication such that they satisfy a set of axioms(2.1.3).

We will start the proof by proving that V follows all the axioms of vector spaces. Consider the following $f, g, h \in V$, then we can notice that $f(x), g(x), h(x) \in \mathbb{R}$. We can say this because in this case f, g, h are elements of the set, while $f(x), g(x), h(x)$ talk about the function evaluated at some $x \in \mathbb{R}$ and spits out a value $f(x), g(x), h(x) \in \mathbb{R}$. This notation however might confuse functions for scalars but f, g, h will never be used as scalars. As I stated earlier, for V to be a vector space its elements must follow certain axioms and we will show that they in fact do follow them. Vector spaces also allow us to work with two main operations **addition** and **scalar multiplication.**

Definition 2.1.2 (Simple operations in vector spaces)**.** Addition is defined as follows $+ : A \times A \to A$ which basically takes 2 elements $a, b \in A$ and assigns a value $a + b \in A$. Scalar multiplication is defined as follows $\times : \mathbb{F} \times A \to A$ which takes any scalar $\mu \in \mathbb{F}$ and an $a \in A$ and gives us $\mu a \in A$

We will now proceed to prove that these operations yield the desired outcome that is also in our set V. We will do this in 2 different ways! We basically want to show that $\forall f, g \in V$ we get that $f + g \in V$. This will show that the addition operation works in our set. To show that $f + g \in V$ we have to somehow prove that $\|f + g\| < \infty$. We can do this simply as follows, let us consider $(\|f + g\|)^2$ and let us see what we can make of it

$$
\begin{aligned}
(\|f+g\|)^2 &= \left(\sqrt{\int_{\mathbb{R}} |f+g|^2 \, dx}\right)^2 \\
&\leq \left(\sqrt{\int_{\mathbb{R}} |f|^2 + 2|fg| + |g|^2 \, dx}\right)^2 \\
&= \int_{\mathbb{R}} |f|^2 + 2|fg| + |g|^2 \, dx \\
&= \int_{\mathbb{R}} |f|^2 \, dx + 2\int_{\mathbb{R}} |fg| \, dx + \int_{\mathbb{R}} |g|^2 \, dx
\end{aligned}
$$

We can already start to see that in our $(\|f + g\|)^2$ we have the $\|f\|$ and $\|g\|$ terms, but with them we also have this problematic $\int_{\mathbb{R}} |fg| \, dx$ term. For this we will now consider the following $(|f| - |g|)^2$. We can now notice the following

$$
\begin{gathered}
(|f| - |g|)^2 = |f|^2 - 2|fg| + |g|^2 \geq 0 \\
f^2 + g^2 \geq 2|fg|
\end{gathered}
$$

Lemma 2.1.1. For $f(x) \leq g(x), \forall x \in \mathbb{R}$ we get that

$$\int_{\mathbb{R}} f(x) \, dx \leq \int_{\mathbb{R}} g(x) \, dx$$

Proof. We can do the following

$$\begin{aligned}
&f(x) \leq g(x)\\
&f(x) - g(x) \leq 0\\
&\int_{\mathbb{R}} f(x) - g(x)\, dx \leq \int_{\mathbb{R}} 0\, dx\\
&\int_{\mathbb{R}} f(x) - g(x)\, dx \leq 0\\
&\int_{\mathbb{R}} f(x)\, dx - \int_{\mathbb{R}} g(x)\, dx \leq 0\\
&\int_{\mathbb{R}} f(x)\, dx \leq \int_{\mathbb{R}} g(x)\, dx
\end{aligned}$$

This concludes our proof of the proposed lemma. □

Now since we have that $f^2 + g^2 \geq 2|fg|$ we can safely say that

$$\int_{\mathbb{R}} f^2 + g^2\, dx = \int_{\mathbb{R}} f^2 + \int_{\mathbb{R}} g^2\, dx \geq 2\int_{\mathbb{R}} |fg|\, dx$$

And we can also see that

$$\infty > \int_{\mathbb{R}} f^2\, dx + \int_{\mathbb{R}} g^2\, dx \geq 2\int_{\mathbb{R}} |fg|\, dx$$

And we can safely conclude that

$$2\int_{\mathbb{R}} |fg|\, dx < \infty$$

Finally we can easily conclude that

$$(\|f+g\|)^2 \leq \int_{\mathbb{R}} |f|^2\, dx + 2\int_{\mathbb{R}} |fg|\, dx + \int_{\mathbb{R}} |g|^2\, dx < \infty$$

Then $f + g \in V$.
We can also get rid of the $2\int_{\mathbb{R}} |fg|$ by using the so called *Cauchy-Schwarz inequality*.

Theorem 2.1.1 (Cauchy-Schwarz inequality)**.** Suppose $f(x)$ and $g(x)$ and $f(x)g(x)$ are integrable, the Cauchy-Schwarz inequality states that

$$\int_{\mathbb{R}} f(x)g(x)\, dx \leq \sqrt{\int_{\mathbb{R}} [f(x)]^2\, dx}\sqrt{\int_{\mathbb{R}} [g(x)]^2\, dx}$$

Proof. We can start by noticing the following for functions f and g that are integrable and real number $t \in \mathbb{R}$

$$(tf(x) + g(x))^2 \geq 0$$

This works because a square is always non-negative. Moving on

$$\int_{\mathbb{R}} (tf(x) + g(x))^2\, dx \geq \int_{\mathbb{R}} 0\, dx$$

This is possible by Lemma 2.1.1. This leads us to

$$\int_{\mathbb{R}} t^2[f(x)]^2 + 2tf(x)g(x) + [g(x)]^2\,dx \geq 0$$
$$\int_{\mathbb{R}} t^2[f(x)]^2\,dx + \int_{\mathbb{R}} 2tf(x)g(x)\,dx + \int_{\mathbb{R}} [g(x)]^2\,dx \geq 0$$
$$t^2\int_{\mathbb{R}} [f(x)]^2\,dx + 2t\int_{\mathbb{R}} f(x)g(x)\,dx + \int_{\mathbb{R}} [g(x)]^2\,dx \geq 0$$

We can now define a few more variables

$$\alpha = \int_{\mathbb{R}} [f(x)]^2\,dx$$
$$\beta = \int_{\mathbb{R}} f(x)g(x)\,dx$$
$$\gamma = \int_{\mathbb{R}} [g(x)]^2\,dx$$

This transforms our equation into

$$\alpha t^2 + 2\beta t + \gamma \geq 0$$

This is now a more familiar object, a quadratic equation. Lets call this $h(t)$. Also we shall notice that $h(t)$ non-negative, meaning it has at most one root. Or in other words the discriminant is non-positive and as well know $D = b^2 - 4ac$, finding the discriminant for $h(t)$ we get

$$D = (2\beta)^2 - 4(\alpha)(\gamma) \leq 0$$
$$4\beta^2 \leq 4\alpha\gamma$$
$$\beta^2 \leq \alpha\gamma$$
$$\beta \leq \sqrt{\alpha\gamma}$$
$$\int_{\mathbb{R}} f(x)g(x)\,dx \leq \sqrt{\int_{\mathbb{R}} [f(x)]^2\,dx}\sqrt{\int_{\mathbb{R}} [g(x)]^2\,dx}$$

This shows us what we wanted. □

Then using this inequality we can see that

$$\int_{\mathbb{R}} |fg|\,dx \leq \sqrt{\int_{\mathbb{R}} |f|^2\,dx}\sqrt{\int_{\mathbb{R}} |g|^2\,dx} < \infty$$

Notice that in this step we have "replaced" f and g with $|f|$ and $|g|$ respectively. Then it follows that

$$(\|f+g\|)^2 \leq \int_{\mathbb{R}} |f|^2\,dx + 2\int_{\mathbb{R}} |fg|\,dx + \int_{\mathbb{R}} |g|^2\,dx < \infty$$

Finally $\|f+g\| < \infty \implies f+g \in V$. We now show that for $f \in V$ and $c \in \mathbb{R}$ we get that $cf \in V$. This is considerably easier to show than the addition operator. Consider $\|cf\|$, we can work with it and notice the following

$$\begin{aligned}\|cf\| &= \sqrt{\int_{\mathbb{R}} |cf|^2\,dx}\\ &= \sqrt{\int_{\mathbb{R}} |c|^2|f|^2\,dx}\\ &= \sqrt{|c|^2\int_{\mathbb{R}} |f|^2\,dx}\\ &= |c|\sqrt{\int_{\mathbb{R}} |f|^2\,dx}\end{aligned}$$

We can now see that $\|cf\| = |c|\|f\| < \infty$ therefore it is clear to see that $cf \in V$. We have now established the basic operations in over set V. Now we will go on to prove the axioms of a vector space.

You may now ask, what are these axioms? Using the operations defined above we can come up with a set of axioms as follows

Definition 2.1.3 (Axioms of a vector space)**.** For $f, g, h \in A$ and $a, b \in \mathbb{F}$

1. Commutativity of addition: $f + g = g + f$
2. Associativity of addition: $(f + g) + h = f + (g + h)$
3. Identity element of addition: There exists a $\mathbf{0} \in A$ such that $\mathbf{0} + f = f$, we will call $\mathbf{0}$, $o(x)$ or just o
4. Inverse elements of addition: For every element $f \in A$ there exists a $-f \in A$ such that $f + (-f) = o$
5. Zero multiplication: $0 \times f = 0$ such that $0 \in \mathbb{F}$
6. Identity element of scalar multiplication: $1f = f$ such that $1 \in \mathbb{F}$
7. Compatibility of scalar multiplication with field multiplication: $(ab)f = a(bf)$
8. Distributivity of scalar multiplication with respect to element addition: $a(f + g) = af + ag$
9. Distributivity of scalar multiplication with respect to field addition: $(a + b)f = af + bf$

We will just go through each and prove them in order. Most of these proof are one liners and in my opinion that's what makes it so unique.

Proof. Let us start with the first one, Commutativity of addition notice that

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$$

This works because $f, g \in V$ but $f(x), g(x) \in \mathbb{R}$.

□

Proof. The next one is Associativity of addition we can follow a rather similar approach and see that

$$((f + g) + h)(x) = (f + g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + (g + h)(x) = (f + (g + h))(x)$$

This works again because of the fact that $f, g.h \in V$ but $f(x), g(x), h(x) \in \mathbb{R}$.

□

We will now define a very important function, we will call it the "*Zero Function*".

Definition 2.1.4 (Zero Function)**.** Let $o(x)$ be the zero function and it is defined as follows, $o(x) = 0$.

Note that $o(x) : \mathbb{R} \to \mathbb{R}$ and also $\|o\| = \sqrt{\int_{\mathbb{R}} |0|^2 \, dx} = 0 < \infty$, therefore we can say that $o \in V$.

Proof. Let us continue with proving the axioms and let us move on to the next one, which is the Identity element of addition. Notice that

$$(o + f)(x) = o(x) + f(x) = 0 + f(x) = f(x)$$

□

Definition 2.1.5 (Addictive Inverse)**.** $\forall x \in A$ an additive inverse is an element in the vector space A usually denoted by $-x$ and has the property that $x + (-x) = o$.

Below we will prove that such an element exists $\forall f \in V$. Moving on we have the Inverse elements of addition axiom. We will now show that $f \in V$ has an inverse $-f \in V$. Consider the following

$$\|-f\| = \sqrt{\int_{\mathbb{R}} |-f(x)|^2 \, dx} = \sqrt{\int_{\mathbb{R}} |f(x)|^2 \, dx} < \infty$$

Proof. We can now see that if $f \in V$ there is an inverse $-f \in V$, now we will prove the remaining. We will see that

$$(f + (-f))(x) = f(x) + (-f(x)) = f(x) - f(x) = 0 = o(x)$$

□

Proof. The next axiom we wish to prove is the Zero multiplication axiom and we can do it as follows

$$(0 \times f)(x) = 0 \times f(x) = 0$$

□

Proof. Identity element of scalar multiplication, this one can be proven by doing the following

$$(1 \times f)(x) = 1 \times f(x) = f(x)$$

□

Proof. Compatibility of scalar multiplication with field multiplication, we can see this by doing the following

$$((ab)f)(x) = (ab)f(x) = abf(x) = a(bf(x)) = (a(bf))(x)$$

□

Proof. Distributivity of scalar multiplication with respect to element addition, we can motivate it with the following

$$a(f + g)(x) = a(f(x) + g(x)) = af(x) + ag(x) = (af + ag)(x)$$

□

Proof. Distributively of scalar multiplication with respect to field addition, we can show this by

$$(a+b)f(x) = af(x) + bf(x) = (af + bf)(x)$$

□

We have successfully shown all the axioms to be true. We can now conclude that our set V is a vector space over $\mathbb{R}$. □

An additional consequence of these axioms is that the zero function is unique.

Theorem 2.1.2 (Uniqueness of the 0)**.** $\exists! x \in V$ such that $x = 0$

Proof. Let us prove this first by considering two functions o and o', these **both** are the zero function. What we need to show is that both of these are in fact equal to each other. Consider $f \in V$, then

$$\begin{aligned} o + f &= o' + f \\ o + f + (-f) &= o' + f + (-f) \\ o + o &= o' + o \\ o &= o' \end{aligned}$$

We have shown that o and o' are equal, therefore the function is unique. □

The next step we will take is to prove that it is a normed vector space. We can do this very straightforwardly but we will see that we run into a problem.

§3 Prove V is a normed vector space

Like stated earlier, to prove that V is a Banach space we have to go through a lot of proving. In this section we will prove that V is a normed vector space, however we will have to make some changes on how we view our set.

Proposition 3.0.1. V is a normed vector space.

Proof. Let us start this journey of proving that our set is a normed vector space. However, we need to know what a normed vector space is exactly.

Definition 3.0.1 (Normed Vector Space)**.** A normed vector space is a vector space $(A, \|\cdot\|)$ over a field $\mathbb{F} \subseteq \{\mathbb{R}, \mathbb{C}\}$ on which a norm is defined. In this A is our vector space and $\|\cdot\| : A \to \mathbb{R}$ is our norm

Now in order to completely grasp this we must define what a norm really is.

Definition 3.0.2 (Norm)**.** A norm usually denoted by $\|\cdot\|$ is a real valued function, $\|\cdot\| : A \to \mathbb{R}^+ \cup \{0\}$ over our vector space and it must satisfy the following axioms. Let $f, g \in A$, $o \in A$ is the zero function in A and $a \in \mathbb{F}$

1. Non-negative: $\|f\| \geq 0 \; \forall f \in V$
2. Triangle Inequality: $\|f + g\| \leq \|f\| + \|g\|$
3. Absolute homogeneity: $\|af\| = |a|\|f\|$
4. Positive definiteness: $\|f\| = 0 \implies f = o$

We will now prove that our set V is a normed vector space. We will again go in chronological order and the non-negative property of for our norm first.

Proof. To prove this we will start with proving the following lemma, which will help us take care of this property relatively easily.

Lemma 3.0.1. If $f(x)$ is integrable and $f(x) \geq 0 \, \forall x \in \mathbb{R} \implies \int_{\mathbb{R}} f(x)\, dx \geq 0$

Proof. Since we know that $f(x)$ is integrable (We will use the Riemann definition of the integral) we have $U(f,P) = L(f,P) \forall P \subseteq \mathbb{R}$ it is sufficient to prove that if $L(f,P) \geq 0 \implies \int_{\mathbb{R}} f(x)\, dx \geq 0$. From the definition of the Riemman integral we have that

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

where we define

$$m_k = \inf_{[x_{k-1},x_k]} f$$

Since the function is always non-negative we can say that the $m_k = \inf_{\forall P} f \geq 0$. From the convention of writing out intervals we always say that for $[a,b]$ we have $a < b$, this implies that $0 < b - a$, why is this important you may ask? Well if we see out $L(f,P)$ sum we have $m_k \geq 0$ and $x_k - x_{k-1} > 0$ that are multiplied and the overall resultant is added all together. We can then notice that $m_k(x_k - x_{k-1}) \geq 0$, so the overall sum $L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \geq 0$. Now since $U(f,P) = L(f,P) \geq 0 \implies \int_{\mathbb{R}} f(x)\, dx \geq 0$ □

This now proves the lemma we were going for.

Now for our norm, let us notice that our norm is

$$\|f\| = \sqrt{\int_{\mathbb{R}} |f(x)|^2\, dx}$$

Notice that $|f(x)|^2 \geq 0 \; \forall x \in \mathbb{R}$, then by lemma 3.0.1 we have $\int_{\mathbb{R}} |f(x)|^2\, dx \geq 0$, this gives is that $\|f\| = \sqrt{\int_{\mathbb{R}} |f(x)|^2\, dx} \geq 0$. □

We will now prove the triangle inequality of our norm.

Proof. To begin off the proof I will go over some things that first came into my mind. Since norms are always non-negative, the first thing that came to mind was that proving $\|f+g\|^2 \leq (\|f\| + \|g\|)^2$ is not only easier but is sufficient. Let us consider $(\|f\| + \|g\|)^2$, I thought of this first because we are trying to prove that $\|f+g\| \leq \|f\| + \|g\|$ and what I have asked us to consider has a lot of this. This gives the following

$$\begin{aligned}
(\|f\| + \|g\|)^2 &= \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| \\
&= \left(\sqrt{\int_{\mathbb{R}} |f|^2\, dx}\right)^2 + \left(\sqrt{\int_{\mathbb{R}} |g|^2\, dx}\right)^2 + 2\sqrt{\int_{\mathbb{R}} |f|\, dx}\sqrt{\int_{\mathbb{R}} |g|\, dx} \\
&= \int_{\mathbb{R}} |f|^2\, dx + \int_{\mathbb{R}} |g|^2\, dx + 2\sqrt{\int_{\mathbb{R}} |f|\, dx}\sqrt{\int_{\mathbb{R}} |g|\, dx}
\end{aligned}$$

Keep this in mind as we go through other things. Next just consider the following $\|f+g\|^2$, let us go through what this leads us to

$$\|f+g\|^2 = \left(\sqrt{\int_{\mathbb{R}} |f+g|^2\, dx}\right)^2 = \int_R |f|^2 + 2|fg| + |g|^2\, dx$$

The following, as we know very well by this point, breaks down into the following

$$\|f+g\|^2 = \int_{\mathbb{R}} |f|^2\, dx + \int_{\mathbb{R}} 2|fg|\, dx + \int_{\mathbb{R}} |g|^2\, dx$$

Now one thing that should have caught your eye is that $\|f+g\|^2$ and $(\|f\| + \|g\|)^2$ have a bit too much in common and the only thing that is different from both is the $\int_{\mathbb{R}} 2|fg|\, dx$ and $2\sqrt{\int_{\mathbb{R}} |f|\, dx}\sqrt{\int_{\mathbb{R}} |g|\, dx}$ terms. However notice that we have proved an inequality for this before, using the Cauchy-Schwarz inequality 2.1.1. We proved earlier that

$$\int_{\mathbb{R}} |fg|\, dx \leq \sqrt{\int_{\mathbb{R}} |f|\, dx}\sqrt{\int_{\mathbb{R}} |g|\, dx}$$

This leads to the following

$$\int_{\mathbb{R}} 2|fg|\, dx \leq 2\sqrt{\int_{\mathbb{R}} |f|\, dx}\sqrt{\int_{\mathbb{R}} |g|\, dx}$$

we can now say that

$$\begin{gathered}
\int_{\mathbb{R}} |f|^2\, dx + \int_{\mathbb{R}} 2|fg|\, dx + \int_{\mathbb{R}} |g|^2\, dx \leq \int_{\mathbb{R}} |f|^2\, dx + \int_{\mathbb{R}} |g|^2\, dx + 2\sqrt{\int_{\mathbb{R}} |f|\, dx}\sqrt{\int_{\mathbb{R}} |g|\, dx} \\
\|f+g\|^2 \leq (\|f\| + \|g\|)^2 \\
\|f+g\| \leq \|f\| + \|g\|
\end{gathered}$$

This concludes our proof for the triangle inequality. □

An indirect consequence of proving that V is a vector space is that we have actually proven the absolute homogeneity of the norm, but just to reiterate the proof we will proving it again.

Proof. Consider $\|cf\|$ for $f \in V$ and $c \in \mathbb{R}$

$$\begin{aligned}
\|cf\| &= \sqrt{\int_{\mathbb{R}} |cf|^2\, dx} \\
&= \sqrt{\int_{\mathbb{R}} |c|^2 |f|^2\, dx} \\
&= \sqrt{|c|^2 \int_{\mathbb{R}} |f|^2\, dx} \\
&= |c| \sqrt{\int_{\mathbb{R}} |f|^2\, dx} \\
&= |c| \|f\|
\end{aligned}$$

This proves this point. □

We will now prove the final axiom to ensure that $(V, \|\cdot\|)$ is a normed vector space. This is the axiom of positive definiteness.

Proof. We wish to show that $\|f\| = 0 \implies f = o$, we however, rather quickly actually run into a very interesting problem. We can find functions $f \in V \neq 0$ but still give a 0 norm. I will you, the reader, find more examples of such functions but for now I will give and an example of one. Notice that even one such function is enough to disregard this pair $(V, \|\cdot\|)$ as a normed vector space. We will, of course, fix this issue later on.[1]. Notice that we do not have requirement for continuity of the function for the function to be in our vector space, so notice the function

$$f(x) = \begin{cases} 0 & x > 0 \\ 1 & x = 0 \\ 0 & 0 < x \end{cases}$$

Notice that $\|f\| = 0$ and also notice that $f(x) \neq o(x)$. We have just found a function that gave us 0 norm even though the function itself is not the 0 function. Before moving to the solution of such a problem, I will show that this function has 0 norm. □

□

[1] If I didn't present a solution to this problem, this "guide" would be rather redundant.