

# Problem Sheet 3 - Geometry of Physics

October 2025

**Question 1** (Differential Equations). *In the last problem sheet we showed that the solution to the following ODE  $y' = f(x, y)$  exists and is unique for given initial data when  $f$  is Lipschitz in  $y$ . Now consider the ODE  $y' = Ay$  where  $A$  is a constant matrix, then we know that the solution to any initial value problem is unique and must exist. Define by  $\exp(tA)y_0$  be the solution to this ODE subject to  $y(0) = y_0$ . We will derive some important facts about the exponential map now.*

1. Show that, using a power series, that

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{n!} A^n$$

2. Show moreover that

$$\exp(tA)\exp(sB) - \exp(sB)\exp(tA) = st[A, B] + \text{higher order terms in } s \text{ and } t$$

3. Show that the map  $t \mapsto \exp(tA)$  is a group homomorphism.

4. Let  $J : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  be the standard almost complex structure on  $\mathbb{R}^{2n}$ . Show that for any  $t \in \mathbb{R}$  we have  $\exp(tJ) = \cos(t)I + \sin(t)J$ , this is somewhat of a generalization of Euler's identity.

We now talk a little about PDEs.

5. Consider the wave equation with friction

$$u_{tt} - c^2 \Delta u + mu_t = 0$$

for  $(t, x) \in (0, \infty) \times \mathbb{R}^n$ . Let

$$E(t) = \int_{\mathbb{R}^n} \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 d^n \mu$$

is this conserved? What about if we remove the  $mu_t$  above.

6. Consider now the Klein Gordon equation

$$u_{tt} - c^2 \Delta u + m^2 u = 0$$

now show that  $u \rightarrow e^{i\theta}u$  leaves the equation invariant.

**Question 2** (Integration). *This question mainly exists because integrals are so central and I think its good to know how to do them.*

1. In this first question we will derive the formula for the volume of a ball of radius  $R$ . To do this, we will first derive the equation for the surface area of a sphere in  $n$  dimensions of radius  $R$ . Then we will use this to find a formula for the volume of the radius  $R$ .

- (a) We will first consider the integral

$$\int_{\mathbb{R}} e^{-x^2} d\mu$$

to do this we will consider the integral

$$\int_{\mathbb{R}^2} e^{-r^2} d^2 \mu$$

where  $r^2 = x^2 + y^2$ . Show that

$$\int_{\mathbb{R}^2} e^{-r^2} d^2 \mu = \left( \int_{\mathbb{R}} e^{-x^2} d\mu \right)^2$$

now using a change of variable, compute

$$\int_{\mathbb{R}^2} e^{-r^2} d^2\mu$$

directly. Use this to show that

$$\int_{\mathbb{R}} e^{-x^2} d\mu = \sqrt{\pi}$$

(b) Now that we have this, convince yourself that  $\text{vol}(S^{n-1}(R)) = R^{n-1} \text{vol}(S^{n-1})$ .

(c) Now let

$$I_n = \int_{\mathbb{R}^n} e^{-r^2} d^n\mu$$

Show that  $I_n = \pi^{\frac{n}{2}}$ .

(d) Compute  $I_n$  another way by integrating over spheres of radius  $r$ . Note that the function is constant on these sphere and hence show that

$$I_n = \frac{1}{2} \text{vol}(S^{n-1}) \Gamma\left(\frac{n}{2}\right)$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^x dt$$

also note that  $x\Gamma(x) = \Gamma(1+x)$  when  $\text{Re}(x) > 0$ .

(e) Conclude that

$$\text{vol}(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

(f) Now integrate the function 1 over a ball of radius  $R$  to show that

$$\text{vol}(D^n) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(1 + \frac{n}{2}\right)} R^n$$

2. Suppose now that you have a smooth closed and simple curve  $\gamma : (a, b) \rightarrow \mathbb{R}^2$ . Let  $A$  be the region bounded by  $\gamma$ . Using Green's theorem show that

$$\text{vol}(A) = \frac{1}{2} \int -\gamma^2(t)(\gamma^1(t))' + \gamma^1(t)(\gamma^2(t))' dt$$

use this to find the area of a circle of radius  $R$ .