We study topology to be able to talk about "closeness".

Def: let X be a set. let $P(X)$ or 2^X be the power set of X. then a collection $\mathcal{T} \subseteq P(X)$ is called a topology on X if

(i) $\phi, X \in \mathcal{T}$

(ii) If $\{U_\alpha\}_{\alpha \in I}$ is an arbitrary collection of set in $\mathcal{T}$ then

$$\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$$

(iii) If $\{U_i\}_{i=1}^n$ is a finite collection of sets in $\mathcal{T}$ then $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

elements in $\mathcal{T}$ are called open set in $\underbrace{(X, \mathcal{T})}_{\text{topological space}}$

So conditions (ii) and (iii) mimic open intervals in $\mathbb{R}$, $\mathbb{R}^n$.

(a_α, b_α) indexed by $\alpha \in A$. Unions of open intervals is yet another open interval.

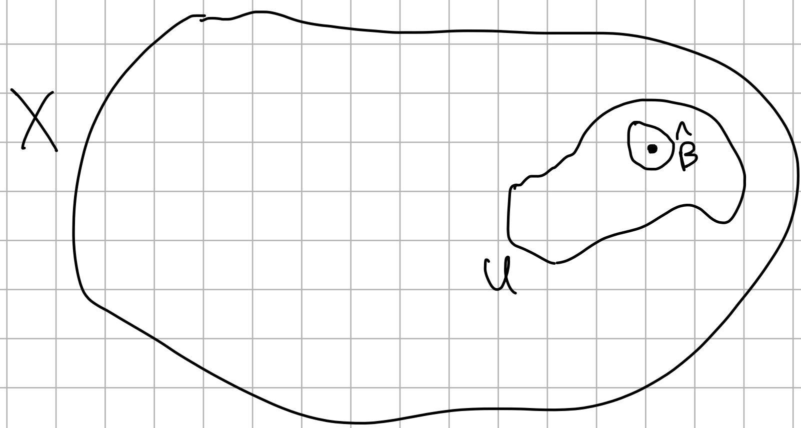
Consider $\bigcap_{i=1}^{\infty} \underbrace{\left(-\frac{1}{i}, \frac{1}{i}\right)}_{\text{for each } i \ \left(-\frac{1}{i}, \frac{1}{i}\right) \text{ is an open interval}} \ni x \Rightarrow \bigcap_{i=1}^{\infty} \left(-\frac{1}{i}, \frac{1}{i}\right) = \{0\}$ this is NOT an open interval

note that $-\frac{1}{i} < x < \frac{1}{i} \Rightarrow x = 0$

$$\begin{array}{ccc} -\frac{1}{i} & & \frac{1}{i} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Exercise: let $\hat{\mathcal{T}} \subseteq P(X)$ then show $\mathcal{T} = \bigcap_{\substack{\mathcal{A} \text{ is a topology on } X \\ \text{and } \hat{\mathcal{T}} \subseteq \mathcal{A}}} \mathcal{A}$ is a topology

Def: let (X, τ) be a top. space let $\mathcal{B} \subseteq \tau$ s.t.
If U is an open set and $x \in X$ then $\exists \hat{B} \in \mathcal{B}$ s.t. $x \in \hat{B} \subseteq U$,
then $\mathcal{B}$ is a called a basis for τ.

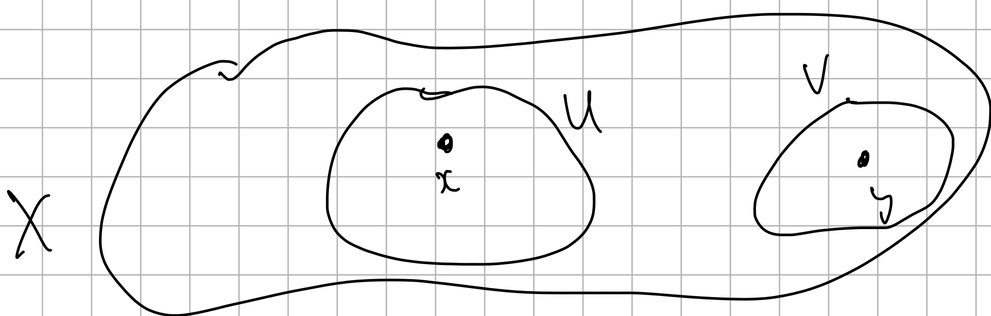


Def: A top. space X is called 2^{nd} countable if it has a countable basis.

Ex: $(\mathbb{R}, T_{std}) =: \mathbb{R}_{std}$, T_{std} is topology generated by the open intervals.

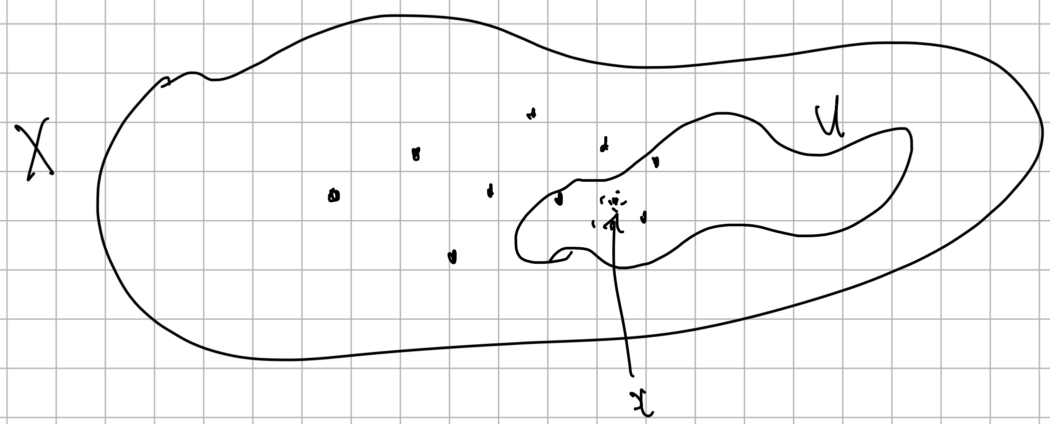
Note that $\mathcal{B} = \{(p-q, p+q) \mid q \in \mathbb{Q}_{>0} \text{ and } p \in \mathbb{Q}\}$ is a basis for $\mathbb{R}_{std}$. Hence $\mathbb{R}_{std}$ is 2^{nd} countable.

Def: A top. space X is T_2 or Hausdorff if $x \neq y$ in X then there are open sets U and V s.t. $U \ni x$ and $y \in V$ with $U \cap V = \emptyset$



Ex: $\mathbb{R}_{std}$ is T_2.

Say $x_1, \dots, x_n, \dots$ is a sequence in X. we say that this converges to $x \in X$ if for every open set U containing x there is $N > 0$ s.t. $\forall n > N$ $x_n \in U$.



In general top. spaces limits are not unique. But limits are unique in T_2 spaces.

Def: let $f: X_1 \to X_2$ be a map between top. spaces. Then we say f is continous if $\forall U$ open in X_2 the set $f^{-1}(U)$ is open in X_1.

"pulling back"

Ex of where $U \subseteq X_1$ is open but $f(U)$ is not open

$f(x) = 1$, $U = \mathbb{R}$ but $f(\mathbb{R}) = \{1\}$ which is not open.

Def: $f: X_1 \to X_2$ cont. and $f^{-1}: X_2 \to X_1$ exists and is cont. then f is called a homeomorphism.

Def: A top. space M is called a topological manifold if

(i) M is T_2

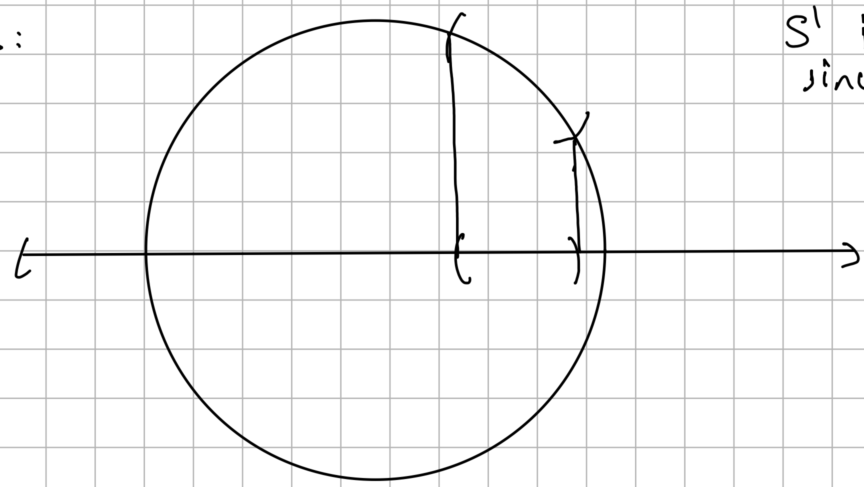
(ii) M is 2^{nd} countable

(iii) M is "locally Euclidean": For every $p \in M$ there is an open set U containing p and a map $\varphi: U \to \hat{U}$ s.t. φ is a

$\hat{U} \subseteq \mathbb{R}^n_{std}$ and is open homeomorphism. And the # n is called the dimension if it is unique.

Def: If $A \subseteq X$ then A has a natural topology where $U \subseteq A$ is open if there is an open set $V \subseteq X$ s.t. $U = A \cap V$

Ex:



S^1 is not homeo to $\mathbb{R}$ since S^1 is "compact".

Def: A top. space X is called compact if $\{U_i\}_{i \in I}$ arb. collection of open sets s.t. $X \subseteq \bigcup_{i \in I} U_i$ then I have a finite subcollection $\{U_j\}_{j=1}^n$ s.t. $\bigcup_{j=1}^n U_j \supseteq X$.

Def: A subset $A \subseteq X$ is closed if $X \setminus A$ is open.

Using deMorgan we equivalently define a topology in terms of closed set instead of open sets.

Fundamental group: To each topological space $X \to \pi_1(X, x)$ a group.

(this sort of assignment is called a Functor)

Consider loops in X

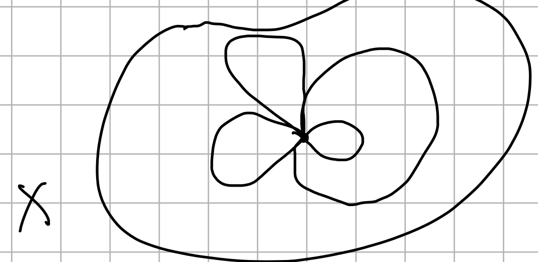
$\gamma : S^1 \to X$

s.t. γ is cont.

equivalently

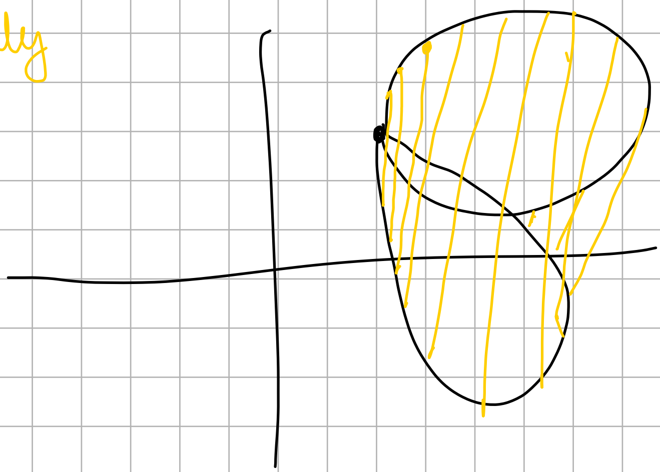
$\gamma : [0,1] \to X$

and $\gamma(0) = \gamma(1)$.



we say γ_1 and γ_2 are equivalent loops if we can deform one loop into the other

continously

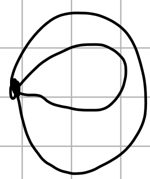


then consider the set of all equivalence classes of loops

$\pi_1(X, x) = \{ [\gamma] \mid \gamma \sim \beta$ is I can deform $\gamma \to \beta \}$

this becomes a group by concatonation of loops

First Fundamental group



$\pi_1(SO(3)) \neq 0$

So the topological space SO(3) is <u>not</u> simply connected

We can check $\pi_1(SO(3)) \cong \mathbb{Z}_2$ ← SL(2) representation that descends to a 2:1 homomorphism onto SO(3)

SL(2) is simply connected and is a "covering space" of SO(3). So it is up to homeomorphism the only top. space.

On Saturday we will use the fundamental group to talk about

↳ Existence problem for magnetic vector potentials

↳ Theorize the existence of non-elemantory particles called

Anyons.