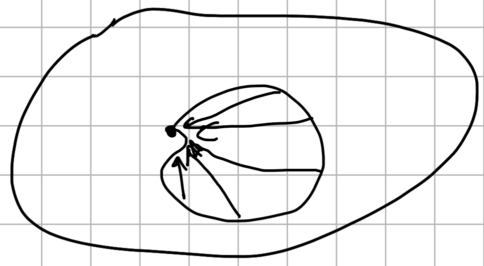
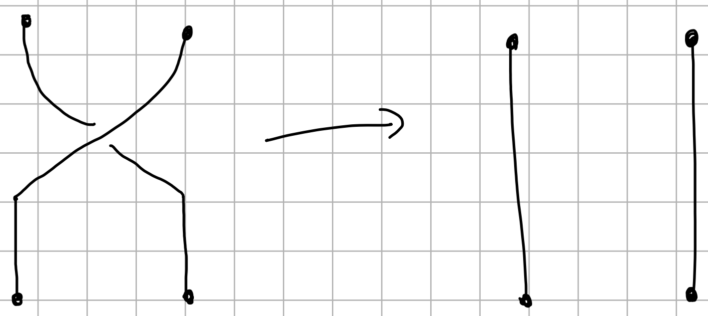


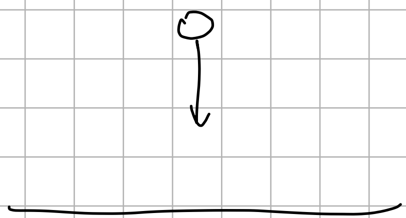
Fermions under exchange pick up -1 (phase factor)

Bosons don't pick up a phase factor



Side - Tangent: Degrees of freedom

Classical Mechanics:



One degree freedom — position

One degree freedom — velocity

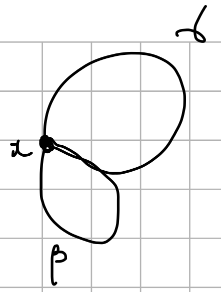
Degrees of freedom — dimension of configuration space x 2

If we are constrained to a smooth manifold M then

D.O.F $= \dim T^*M \leftarrow$ dimension of the cotangent bundle.

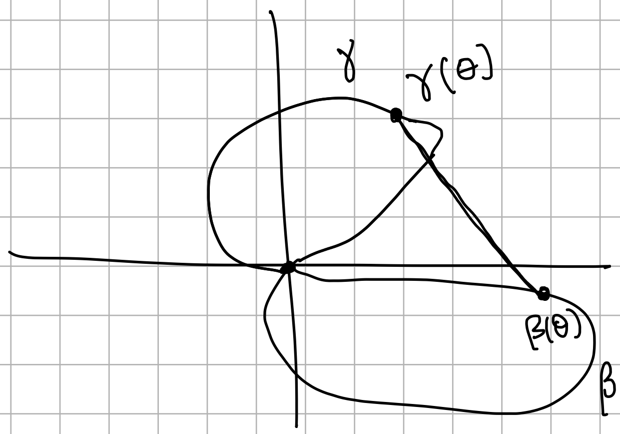
let X be a top. space. let $p \in X$. then we define

$\underbrace{\pi_1(X, x)}_{\text{First Fundamental Group}} = \{[\gamma] \mid \gamma : S^1 \to X$ s.t. $\gamma(0) = x$, $\gamma \sim \beta$ if the path homotopic$\}$



γ
concatonation
(check that this is well defined)

Ex: $\pi_1(\mathbb{R}^2, (0,0)) = \{0\}$



$H: [0,1] \times S^1 \longrightarrow \mathbb{R}^2$

$(1-t)\gamma(\theta) + t\beta(\theta) \qquad t \in [0,1]$

$H(0,1) = \gamma(1)$
$H(1,1) = \beta(1)$

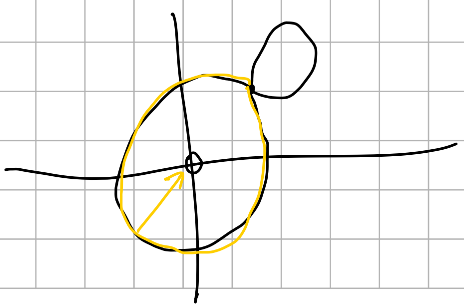
$H(t,\theta) = (1-t)\gamma(\theta) + t\beta(\theta)$

Def: We say X is simply connected if X is connected and $\pi_1(X,x)$ is trivial

If X is simply connected then $\pi_1(X,x) = \pi(X,y)$ for $x,y \in X$.

More generally if X is path connected then $\pi_1(X)$ is well defined.

Ex: $\mathbb{R}^2 - \{(0,0)\}$ is <u>not</u> simply connected, $\pi_1(\mathbb{R}^2 \setminus \{0,0\}) \cong \mathbb{Z} \cong \pi_1(S^1)$



$\mathbb{R}^2 - \{(0,0)\}$ and S^1 are equivalent up to homotopy [Homotopy Equivalence]

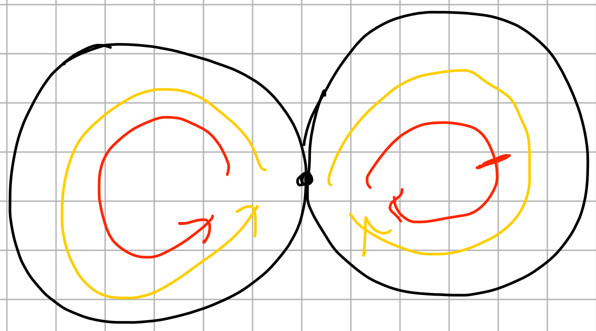
Quick proof that $\mathbb{R}^2 \not\cong \mathbb{R}^n$ unless $n=2$.
$\mathbb{R} \not\cong \mathbb{R}^n$ unless $n=1$

Fact: Equivalence of domain that says $\mathbb{R}^n \cong \mathbb{R}^m \iff n=m$.

Fact: $\times$ and $\otimes$ that unital binary operations on set M.

$(a\times b)\otimes(c\times d) = (a\otimes c)\times(b\otimes d)$ — $\exists e$ s.t. $a\times e = a$, $\exists w$ s.t. $a\otimes w = a$

then $\times$ and $\otimes$ are the same. $\times$ and $\otimes$ are associative and abelian.

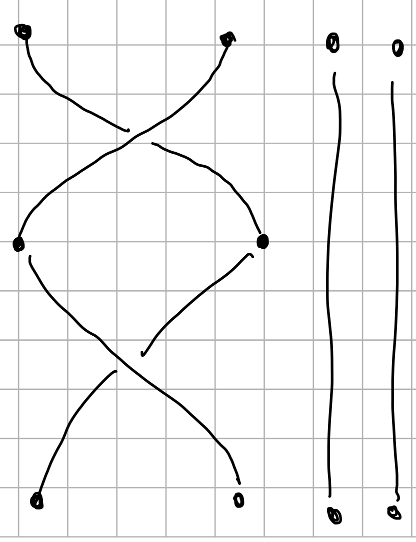


$\leftarrow S^1 \vee S^1$

this has non abelian $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$

Theorem: Seifert Vankampen theorem

$\mathbb{R}^3$



action of S_2

$\psi(r_1, r_2) \longrightarrow e^{i\theta}\psi(r_1, r_2)$

$\downarrow$

$e^{2i\theta}\psi(r_1, r_2)$

$\psi(r_1, r_2) = e^{2i\theta}\psi(r_1, r_2)$

$\Rightarrow e^{2i\theta} = 1$

$\Rightarrow 2\theta = 2n\pi$ where $n \in \mathbb{Z}$

$\Rightarrow \theta = n\pi$

n is even $\psi(r_1,r_2) \to e^{i\theta}\psi(r_1,r_2) = e^{in\pi}\psi(r_1,r_2)$

$$= \underbrace{e^{i(2m\pi)}}_{1}\psi(r_1,r_2)$$

$$= \psi(r_1,r_2)$$

Bosons

n is odd $\Rightarrow$ $n=2m+1$ $\psi(r_1,r_2) \to \cancelto{1}{e^{i(2m\pi)}} e^{i\pi}\psi(r_1,r_2) = -\psi(r_1,r_2)$

Fermions

let X be some top. space define the configuration space of n ordered points in X

$$\hat{C}_n(X) = \mathrm{Conf}_n(X) = \{(x_1,\dots,x_n) \in \overbrace{X \times \cdots \times X}^{n \text{ times}} \mid x_i \neq x_j \text{ for } i \neq j\}$$

$S_n := \{\sigma : \{1,\dots,n\} \to \{1,\dots,n\} \mid \sigma \text{ bijective}\}$ ← permutation group

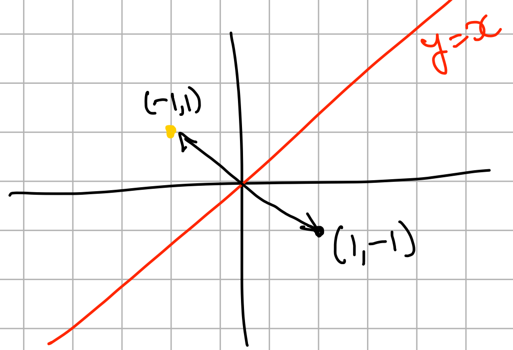
S_n acts on $\mathrm{Conf}_n(X)$: $\sigma \cdot (x_1,\dots,x_n) = (x_{\sigma(1)},\dots,x_{\sigma(n)})$

Unordered configuration space of n points :=

$$C_n(X) = U\mathrm{Conf}_n(X) = \hat{C}_n(X)/S_n$$

Ex: $\hat{C}_2(\mathbb{R})$ (x,y) s.t. $x \neq y$

distinguishable here →



$C_2(\mathbb{R})$

← indistinguishable here

Exchanging particles is a path in the ordered configuration space

But this path is not closed and hence we bring it down to the unordered configuration space.

Hence we invoke the fundamental group: $\pi_1(\mathrm{UConf}_n(X), p) = B_n(X)$

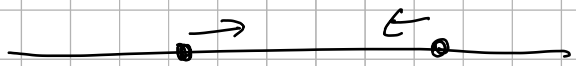
Main Examples: In $\mathbb{R}^3$ it turns out that

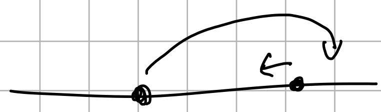
$\pi_1(\mathrm{UConf}_n(\mathbb{R}^3), p) = S_n$

In $\mathbb{R}^2$ things are different. $\pi_1(\mathrm{UConf}_n(\mathbb{R}^2), p) = \underbrace{B_n}_{\text{Braid Group}}$

Exercise: Show that $|S_n| = n!$. Hence S_n is finite

Using what we talk about below show that B_n is not finite.





In $\mathbb{R}^3$ we look at actions of S_n on the particles

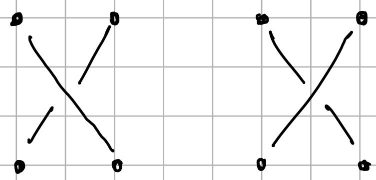
In $\mathbb{R}^2$ we look at actions of B_n on the particles

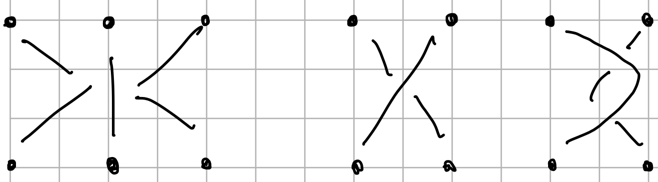
Exercise: define σ_i to be the permutation that swaps $i \leftrightarrow i+1$

Show that S_n has following properties: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i \in \{1, \dots, n-2\}$

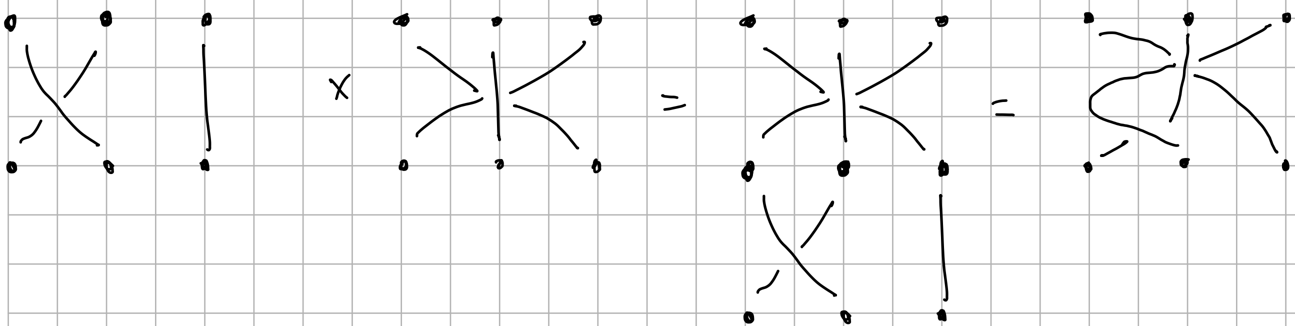
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1$$

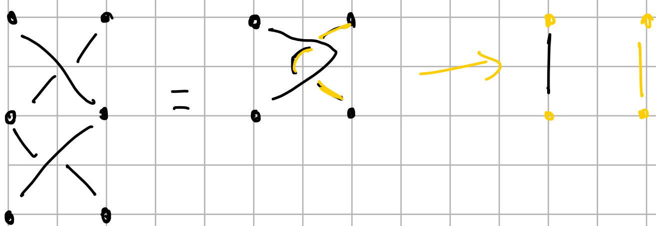
$$\sigma_i^2 = 1$$



 ← Example of braids

Define operation on B_n

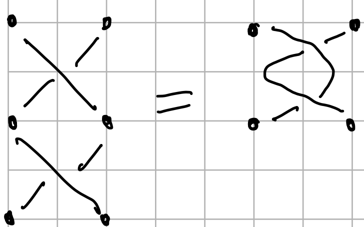




define σ_i to be the permutation that swaps $i \leftrightarrow i+1$

Show that B_n has following properties: $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ $\quad i \in \{1, \dots, n-2\}$

$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1$



$B_n = \pi_1(UConf_n(\mathbb{R}^2), p)$

What are the possible representations of S_n and B_n that are unitary and that act on $\mathbb{C}$.

S_n: $\rho \leftarrow$ representation

$\rho(\underbrace{\sigma_i \sigma_i}_{I}) \overset{\text{by group homomorphism}}{=} \rho(\sigma_i)\rho(\sigma_i) = e^{2i\theta}$

$\rho(I) = e^{2i\theta} \Rightarrow e^{2i\theta} = 1 \Rightarrow$ Fermion/Boson representatione

B_n: Allowed any θ

1) $\theta = 0 \leftarrow$ Boson

2) $\theta = \pi \leftarrow$ Fermion

3) $\theta =$ else $\leftarrow$ Anyons