Recap:

Def: A group is a set G with a binary operation $\cdot : G \times G \to G$ s.t. $\forall x, y, z \in G$

(i) $\exists! e \in G$ s.t. $x \cdot e = e \cdot x = x$

(ii) $\exists! x^{-1} \in G$ s.t. $x \cdot x^{-1} = x^{-1} \cdot x = e$

(iii) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Def: A group homomorphism is a map $\varphi : G \to H$ if $\forall g, h \in G$
$\varphi(gh) = \varphi(g)\varphi(h)$

[If $x \cdot y = y \cdot x$ $\forall x, y$ then G is called Abelian (group)]

Def: A group isomorphism is a homomorphism that is bijective and φ^{-1} is also a homomorphism.

$GL(n, \mathbb{R}) \ni A$ is a $n \times n$ matrix. Assume $e_1, \ldots, e_n \in \mathbb{R}^n$ is the standard basis. I now have a natural "action" of A on $\mathbb{R}^n$.

$A \cdot v \leftarrow$ usual matrix multiplication

A has done the role of a linear map on $\mathbb{R}^n$.

Def: let G be group and let A be any set. We say G acts on A from the left if there is a map
$G \times A \to A$
$(g, a) \mapsto g \cdot a$

(i) $e \cdot a = a$

(ii) $g \cdot (h \cdot a) = (gh) \cdot a$

We say G acts on A from the right if there is a map
$A \times G \to A$
$(a, g) \mapsto a \cdot g$

(i) $a \cdot e = a$ $\qquad e \cdot a = a$

(ii) $(a \cdot h) \cdot g = a \cdot (hg)$ $\quad g \cdot (h \cdot a) = (hg) \cdot a$

Remark: Say G acts on A from the left. $(g, a) \mapsto g \cdot a$
we can make this into a right action $(a, g) \mapsto g^{-1} \cdot a$

↳ Check this !!

Example: Let G be any group, we then have a map $G \times G \to G$
$(g, h) \mapsto ghg^{-1}$

$e \cdot g = ege^{-1} = g$

$h \cdot (g \cdot z) = h \cdot (gzg^{-1}) = hgzg^{-1}h^{-1} = (hg)z(hg)^{-1} = (hg) \cdot z$

⊛ $(hg)^{-1} = g^{-1}h^{-1}$

↳ Need to check inverses are unique: z, w are inverses of g
$gz = gw = e = wg = zg$

$z = ez = (wg)z = w(gz) = we = w$

↳ Need to check $(x^{-1})^{-1} = x$ ← Exercise

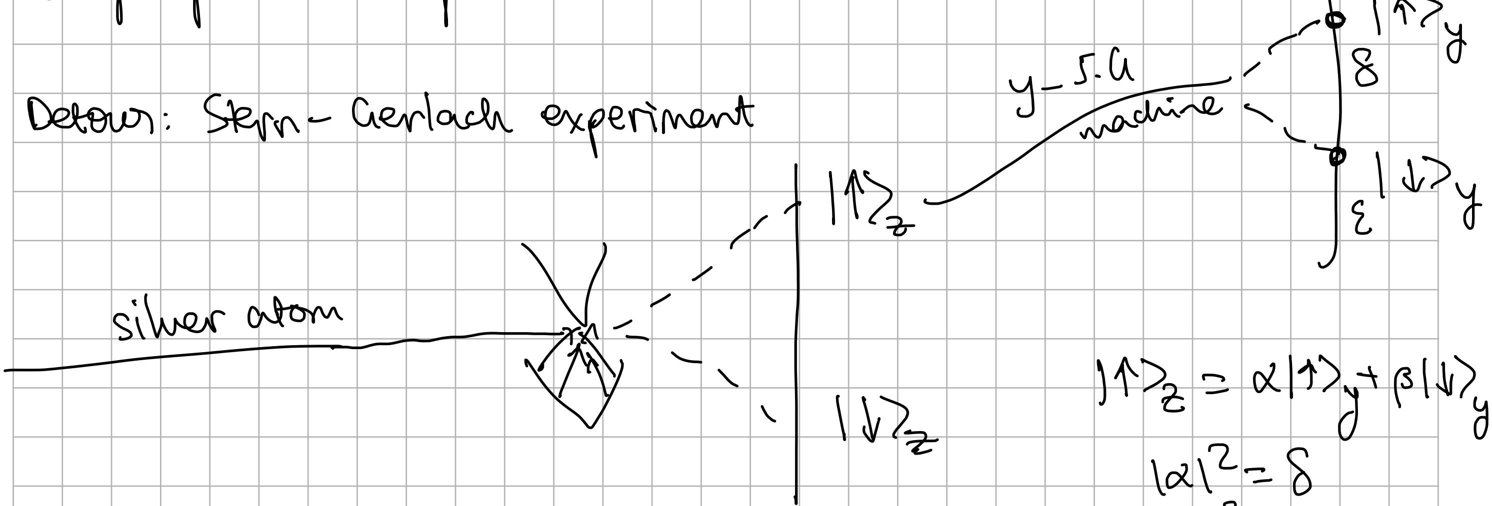
↳ $(g^{-1}h^{-1})(hg) = g^{-1}(h^{-1}h)g = g^{-1}eg = e$

↳ Conjugation is a left action.

$\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}$

Def: A representation of a group G on a vector space V is a group homomorphism $\pi: G \to GL(V)$

Detour: Stern-Gerlach experiment



$|\uparrow\rangle_z = \alpha|\uparrow\rangle_y + \beta|\downarrow\rangle_y$

$|\alpha|^2 = \delta$

$|\beta|^2 = \varepsilon$

$U(1)$ - Gauge freedom

$\Psi = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ where $\alpha, \beta \in \mathbb{C}$ s.t. $|\alpha|^2 + |\beta|^2 = 1$

$\sigma_z|\uparrow\rangle = |\uparrow\rangle$

probability of $|\uparrow\rangle$, $|\langle\uparrow|\Psi\rangle|^2 = |\alpha|^2$

$\sigma_z|\downarrow\rangle = -|\downarrow\rangle$

$|\downarrow\rangle$, $|\langle\downarrow|\Psi\rangle|^2 = |\beta|^2$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ← one of the pauli matrices

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma_x, \sigma_y, \sigma_z$ are linearly ind. over $\mathbb{R}$ $\to \mathbb{R}^3 \subseteq \mathbb{R}^4$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x\sigma_x + y\sigma_y + z\sigma_z$ is an isomorphism

$SU(2) := \{A \in GL(2, \mathbb{C}) \mid \det A = 1 \text{ and } A^\dagger = A^{-1}\}$

Check that every $A \in SU(2)$ has the form $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$

Check that $\sigma_i^2 = -1$ for $i = x, y, z$ and $\sigma_i\sigma_j + \sigma_j\sigma_i = i\sigma_k$
where (i,j,k) are cyclic permutation of $(1,2,3)$

$SU(2) \subseteq \mathbb{R}^4$ with all it objects having norm 1. $SU(2)$ $\underline{\underline{\text{is}}}$ a 3-sphere.

$SU(2)$ acts on $\mathbb{R}^3$ by conjugation.

$\downarrow$

$a\sigma_0 + b_1\sigma_x + b_2\sigma_y + b_3\sigma_z$ with $a^2 + b_1^2 + b_2^2 + b_3^2 = 1$

ATA^{-1} where $T = T_1\sigma_x + T_2\sigma_x + T_3\sigma_z$

$A \in SU(2)$