Why care about orientation?

$$\int_a^b f = -\int_b^a f$$

we have oriented $[a,b]$ so that $a \to b$ is positive and $b \to a$ is negative.

In $\mathbb{R}^3$ we have basis vector $\hat{x}, \hat{y}, \hat{z}$. $\hat{x} \times \hat{y} = \hat{z}$
$\hat{y} \times \hat{x} = -\hat{z}$

$f: \mathbb{R}^3 \to \mathbb{R}^3$ in matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \xrightarrow{\det} -1 \det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1$

$\hat{x} \mapsto \hat{y}$
$\hat{y} \mapsto \hat{x}$
$\hat{z} \mapsto -\hat{z}$

let V be a (real) vector space and let $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ be two bases. Then these are oriented with the same orientation if the linear map $f: v_i \mapsto w_i$ has positive determinant.

They have opposite orientations if $\det f < 0$.

We can define an equivalence on the set of ordered bases i.e. on $GL(V)$ by $(v_1, \ldots, v_n) \sim (w_1, \ldots, w_n)$ if $\det f > 0$ where f is defined as above.

We therefore partition the set of ordered bases into 2 pieces. A choice of equivalence class $[v_1, \ldots, v_n]$ is then called an orientation.

Given $[v_1, \ldots, v_n]$ we can define $\omega \in \Lambda^n V^*$ by $\omega = v_1^* \wedge \cdots \wedge v_n^*$ (This is ill defined) But if we choose a particular basis $(v_1, \ldots, v_n) \in [v_1, \ldots, v_n]$ then ω is well defined.

Note if $\omega \in \Lambda^n V^*$ and $\omega \neq 0$ then we can find a basis $v_1, \ldots, v_n$ s.t. $\omega(v_1, \ldots, v_n) = 1$. Then we can choose this basis as our orientation.

An orientation on V is a choice of $\omega \in \Lambda^n V^* \setminus \{0\}$.

Now an orientation of a manifold is a choice of $\omega \in \Omega^n(M)$ with $\omega_p \neq 0$ $\forall p \in M$.

Such a form is called a volume form.

Does such ω exist on all manifolds? The answer is No!!!

There is no way to choose an "upward" pointing vector field on the möbius band in a continuous way.

If such ω exists we say M is orientable.

Every manifold is locally orientable.

Proof: let (U, φ) be a chart on M. let $\omega = dx^1 \wedge \dots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ then we restrict $\omega|_{\varphi(U)}$. Note that since φ is a diffeomorphism $d\varphi : TU \to T\varphi(U)$ is invertible. Hence $(\varphi^{-1})^* \omega$ is a volume form on U.

This means that if $F: M \to N$ is a local diffeomorphism and $\underbrace{\omega \in \Omega^n(N)}_{\text{volume form}}$ then $F^*\omega \in \Omega^n(M)$ is a volume form.

F is orientation preserving if $\det dF_p > 0$.
F is orientation reversing if $\det dF_p < 0$

Define $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$

Then M is a manifold with boundary if $\forall p \in M \; \exists U \subseteq M$ open and homeomorphism $\varphi : U \to \hat{U} \subseteq \mathbb{R}^n, \mathbb{H}^n$. And M is 2nd countable and Hausdorff.

If $\varphi(U) \cap \mathbb{H}^n = \emptyset$ then we say p is an interior point

If $\varphi(U) \cap \mathbb{H}^n \neq \emptyset$ then we say p is a boundary point.

we define smooth compatibility to be: $\varphi \circ \psi^{-1}$ is extendable to a smooth function on an open set in $\mathbb{R}^n$.

We can then analogously define a smooth atlas.

We call $\partial M = \{p \in M \mid p \text{ boundary points}\}$ the boundary of M.

Exercise: show ∂M is a $\dim M - 1$ smooth manifold without boundary.

We also have $\operatorname{Int} M = \{p \in M \mid p \text{ interior point}\}$, the interior of M.

Exercise: Show $\operatorname{Int} M$ is a smooth manifold. What is its dimension?

$\text{Int } M \cap \partial M = \emptyset$.

let M and N be manifolds with boundary. let $F: M \to N$ then F is smooth if $\forall p \in M$ there are charts (U, φ) in M containing p and (V, ψ) in N containing $F(p)$ s.t. the map $\psi \circ f \circ \varphi^{-1}$ is smooth.

we then have the category of smooth manifolds with boundary.

we have a map ∂: smooth manifolds with boundary $\to$ smooth manifolds in such a way that $\partial(\partial M) = \emptyset$

$d: \Omega^n(M) \to \Omega^{n+1}(M)$ with $d(d\omega) = 0$

Given this is every manifold without boundary the boundary of some manifold.
Yes!! $M \times [0,1)$

let $p \in \mathbb{H}^n$ and $i: \mathbb{H}^n \to \mathbb{R}^n$ be inclusion then $di_p: T_p\mathbb{H}^n \to T_p\mathbb{R}^n$ is an isomorphism i.e. $T_p\mathbb{H}^n \cong \mathbb{R}^n$.

we can then define T_pM for $p \in \partial M$ to be $\mathbb{R}^n$. By taking chart and using that to define T_pM.

One thing to note is that $\underbrace{T_p\partial M}_{m-1 \text{ dimensional}} \neq \underbrace{T_pM}_{m \text{ dimensional}}$

$X_p \in T_pM$ is said to be tangent to ∂M if $X_p \in T_p\partial M$

We say X_p is outward pointing if $\exists \gamma: (-\varepsilon, 0] \to M$ with $\gamma(0) = p$ and $\gamma'(0) = X_p$
we say X_p is inward pointing if $\exists \gamma: [0, \varepsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = X_p$

Fact: For any manifold with boundary $\exists N \in \mathfrak{X}(M)$ s.t. N_p is outward pointing

Exercise: Show that $i^*(\iota_N \omega) \in \Omega^{n-1}(\partial M)$ and if ω is a volume form then so is $i^*(\iota_N \omega)$.

Hence we have an orientation on ∂M given by M.

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let M and N be orientable manifolds with volume forms ω_M and ω_N then $\pi_M^*\omega_M \wedge \pi_N^*\omega_N$ is a volume form on $M \times N$.

Proof: Exercise

We can now define integration on manifolds.

Let $\omega \in \Omega^n(\mathbb{R}^n)$ be compactly supported. Let U be some open set.
Note $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$.

Define $\int_U \omega = \int_U f\, dx^1 dx^2 \cdots dx^n$

If $F: U \to V$ is a diffeomorphism then $\int_V F^*\omega = \begin{cases} \int_U \omega & \text{if } F \text{ is orientation preserving} \\ -\int_U \omega & \text{if } F \text{ is orientation reversing} \end{cases}$

Let M be a manifold and let (U, φ) be a chart. Let $\omega \in \Omega^n(M)$ then

$$\int_U \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

Now let $(U_\alpha, \varphi_\alpha)$ be an atlas and let $\{\psi_\alpha\}$ be a partition of unity subordinate to U_α then $\int_M \omega = \sum_\alpha \int_{U_\alpha} \psi_\alpha \omega$

Exercise: Read the proof of well definedness.

Example: Let $M = T^2 = S^1 \times S^1$ and $\omega = xyz\, dw \wedge dy$ where $T^2 = \{(w,x,y,z) \mid w^2 + x^2 = 1 = y^2 + z^2\}$ with orientation given by S^1.

We have positively oriented parametrization of S^1: $\varphi: (0, 2\pi) \to S^1$, $\theta \mapsto (\cos\theta, \sin\theta)$

We have positively oriented parametrization of T^2: $\varphi: (0, 2\pi)^2 \to T^2$, $(\theta, \varphi) \mapsto (\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)$

$\int_{T^2} \omega = \int_{(0,2\pi)^2} \varphi^*\omega$ what is $\varphi^*\omega = \sin\theta \cos\varphi \sin\varphi \; d(\cos\theta) \wedge d(\cos\varphi)$
$= \sin^2\theta \sin^2\varphi \cos\varphi \; d\theta \wedge d\varphi$

$$= \int_0^{2\pi}\int_0^{2\pi} \sin^2\theta \sin^2\varphi \cos\varphi \; d\varphi\, d\theta$$

$$= 0$$

If $\omega \in \Omega^n(M)$ and ω is a volume form then $\int_M \omega \neq 0$.

Stokes' Theorem: Let M be a oriented smooth manifold with boundary and let $\omega \in \Omega^{n-1}(M)$ be compactly supported then: $\int_M d\omega = \int_{\partial M} \omega$

If $\partial M = \emptyset$ then $\int_M d\omega = 0$.

Corollary: let $\omega \in \Omega^n(M)$ be a volume form. Let M be compact and without boundary. Then $\omega \neq d\eta$ for any $\eta \in \Omega^{n-1}(M)$

Proof: $\int_M \omega \neq 0$ But $\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0$. Contradiction.