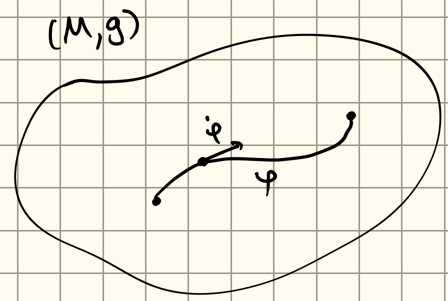
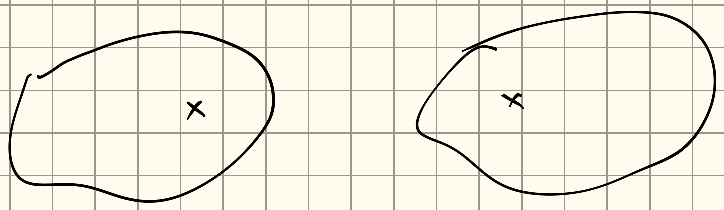
$$L(\gamma) = \int_{t_i}^{t_f} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

$$= \int_{t_i}^{t_f} \sqrt{\delta_{ij}\dot{x}^i\dot{x}^j}\, dt$$



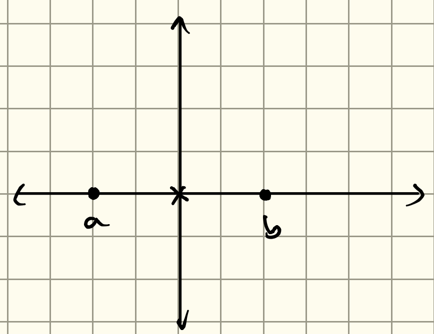


$$\|\dot{\varphi}(p)\|^2 = \langle \dot{\varphi}(p), \dot{\varphi}(p) \rangle_p = g_{ij}\dot{\varphi}^i\dot{\varphi}^j$$

$$\Rightarrow \|\dot{\varphi}(p)\| = \sqrt{g_{ij}\dot{\varphi}^i\dot{\varphi}^j}$$

$$\Rightarrow L(\varphi) := \int_{t_i}^{t_f} \sqrt{g_{ij}\dot{\varphi}^i\dot{\varphi}^j}\, dt$$

There might not be a curve that minimizes distance like in the above.



Geodesics are curves that attain minimum length.

length function takes smooth curves to a real number.

$$S[\varphi] = \int_{t_i}^{t_f} L(\varphi, \dot{\varphi})\, dt$$

$$L(\varphi + \delta\varphi, \dot{\varphi} + \delta\dot{\varphi})$$

$$= L(\varphi, \dot{\varphi}) + (DL)(\varphi, \dot{\varphi})(\delta\varphi, \delta\dot{\varphi}) + \text{other}$$

$$\varphi \to \varphi + \delta\varphi$$
$$\dot{\varphi} \to \dot{\varphi} + \delta\dot{\varphi}$$

$$\Leftrightarrow L(\varphi + \delta\varphi, \dot{\varphi} + \delta\dot{\varphi}) - L(\varphi, \dot{\varphi}) = \frac{\partial L}{\partial \varphi}\delta\varphi + \frac{\partial L}{\partial \dot{\varphi}}\delta\dot{\varphi} + \text{other}$$

$$S[\varphi + \delta\varphi] - S[\varphi] = \int_{t_i}^{t_f} L(\varphi + \delta\varphi, \dot{\varphi} + \delta\dot{\varphi}) - L(\varphi, \dot{\varphi})\, dt$$

$$\approx \int_{t_i}^{t_f} \frac{\partial L}{\partial \varphi}\delta\varphi + \frac{\partial L}{\partial \dot{\varphi}}\delta\dot{\varphi}\, dt$$

$$= \int_{t_i}^{t_f} \frac{\partial L}{\partial \varphi}\delta\varphi + \frac{\partial L}{\partial \dot{\varphi}}\frac{d}{dt}\delta\varphi\, dt$$

$$= \int_{t_i}^{t_f} \frac{\partial L}{\partial \varphi} \delta\varphi \, dt + \frac{\partial L}{\partial \dot\varphi} \delta\varphi \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} \frac{d}{dt} \frac{\partial L}{\partial \dot\varphi} \delta\varphi \, dt$$

$$= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial \varphi} - \frac{d}{dt} \frac{\partial L}{\partial \dot\varphi} \right) \delta\varphi \, dt + \frac{\partial L}{\partial \dot\varphi} \delta\varphi \Big|_{t_i}^{t_f}$$

$$= 0$$

In the case that the end points stay fixed $\delta\varphi(t_i) = 0 = \delta\varphi(t_f)$ hence no boundary terms.

Hence $\int_{t_i}^{t_f} \left(\frac{\partial L}{\partial \varphi} - \frac{d}{dt} \frac{\partial L}{\partial \dot\varphi} \right) \delta\varphi \, dt = 0$ for all variations keeping endpoints fixed.

Exercise: Show that if $\varphi \in C_c^\infty((t_i, t_f))$ and f is integrable and $\int_{t_i}^{t_f} f\varphi = 0$ $\forall$ such φ then $f = 0$.

$$\Rightarrow \begin{cases} \frac{\partial L}{\partial \varphi} - \frac{d}{dt} \frac{\partial L}{\partial \dot\varphi} = 0 \\ \varphi(t_i) = a \\ \varphi(t_f) = b \end{cases}$$

This is called the Euler-Lagrange equation.

Going back to the distance function. Note $L(x, y, \dot x, \dot y) = \sqrt{\dot x^2 + \dot y^2}$

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial \dot x} = \dot x (\dot x^2 + \dot y^2)^{-1/2}$$

$$\frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial \dot y} = \dot y (\dot x^2 + \dot y^2)^{-1/2}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot x} = \frac{\partial L}{\partial x} \Rightarrow \ddot x (\dot x^2 + \dot y^2)^{-1/2} + \dot x \frac{d}{dt}\left((\dot x^2 + \dot y^2)^{-1/2}\right) = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot y} = \frac{\partial L}{\partial y} \Rightarrow \ddot y (\dot x^2 + \dot y^2)^{-1/2} + \dot y \frac{d}{dt}\left((\dot x^2 + \dot y^2)^{-1/2}\right) = 0$$

$x(t) = tx_1 + (1-t)x_2 \Rightarrow \dot{x} = x_1 - x_2 \Rightarrow \ddot{x} = 0$
$y(t) = ty_1 + (1-t)y_2 \Rightarrow \dot{y} = y_1 - y_2 \Rightarrow \ddot{y} = 0$

Hence this satisfies the E-L equations.

Exercise: How do we show this is a minimum?

From a mathematical standpoint $L: TM \to \mathbb{R}$. we call M the configuration space and L the lagrangian.

In classical physics we usually take $L = K.E - P.E$. which in standard $\mathbb{R}^n$ is: $K.E = \frac{1}{2}mv^2$ and $P.E: TM \to \mathbb{R}$

If we had a particle of mass m in free space then its configuration space is $\mathbb{R}^n$ with a Riemannian metric $g_{ij} = m\delta_{ij}$

$K.E = \frac{1}{2} g_{ij} v^i v^j$

If we have a free particle then $L = K.E = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j = \frac{1}{2} m \delta_{ij} \dot{x}^i \dot{x}^j$

$$\frac{\partial L}{\partial x^k} = 0$$

$$\frac{\partial L}{\partial \dot{x}^k} = \frac{1}{2} m \delta_{ij} \frac{\partial}{\partial \dot{x}^k}(\dot{x}^i \dot{x}^j) = \frac{1}{2} m \delta_{ij} (2\delta^i{}_k \dot{x}^j) = m\delta_{kj}\dot{x}^j$$

hence $m\delta_{kj}\ddot{x}^j = 0 \Rightarrow \ddot{x}^k = 0$ hence the particle moves at constant velocity. we get that $m\dot{x}^k$ is conserved along the motion.

Exercise: let $M = S^1(r)$ and $g = r^2 d\theta^2$ what does the coordinates here mean? what does θ mean? What is the kinetic energy? Find Equations of motion.

Hint: $x = r\cos\theta$
$y = r\sin\theta$

On a general Riemannian manifold we have a nice Lagrangian which in local coordinates is given by $\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$

On each tangent space $g_p : T_pM \times T_pM \to \mathbb{R}$ is non-degenerate

this means the induced map $\hat{g}_p : T_pM \to T_p^*M$ is an isomorphism.

$$v \mapsto g_p(v, \cdot) = \iota_v g_p$$

$g = g_{ij} dx^i \otimes dx^j$ hence if $v = v^k \partial_k$ then $\hat{g}(v, \cdot) = g_{ij} dx^i (v^k \partial_k) dx^j$

$$= g_{ij} v^i dx^j$$

We get an inverse $\hat{g}_p^{-1} : T_p^*M \to T_pM$ s.t. $\hat{g}_p^{-1} = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$

and $g^{ij} g_{jk} = \delta^i_k$

$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$ what are the E-L equations?

$$\frac{\partial L}{\partial x^k} = \frac{1}{2} (\partial_k g_{ij}) \dot{x}^i \dot{x}^j$$

$$\frac{\partial L}{\partial \dot{x}^k} = \frac{1}{2} g_{ij} \partial_k (\dot{x}^i \dot{x}^j) = \frac{1}{2} g_{ij} \delta^i{}_k \dot{x}^j = g_{kj} \dot{x}^j = \dot{x}_k$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^k} = \ddot{x}^j g_{kj} + \dot{x}^j (\partial_i g_{kj}) \dot{x}^i$$

Hence $\ddot{x}^j g_{kj} + (\partial_i g_{kj}) \dot{x}^i \dot{x}^j = \frac{1}{2} (\partial_k g_{ij}) \dot{x}^i \dot{x}^j$

$$\Rightarrow \ddot{x}^j g_{kj} + \frac{1}{2} (\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}) \dot{x}^i \dot{x}^j = 0$$

$$\Rightarrow g^{k\alpha} \ddot{x}^j g_{kj} + \underbrace{\frac{1}{2} g^{k\alpha} (\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})}_{\Gamma^\alpha_{ij}} \dot{x}^i \dot{x}^j = 0$$

$$\Rightarrow \delta^\alpha{}_j \ddot{x}^j + \Gamma^\alpha_{ij} \dot{x}^i \dot{x}^j = 0$$

$$\Rightarrow \ddot{x}^\alpha + \Gamma^\alpha_{ij} \dot{x}^i \dot{x}^j = 0$$

The n^3 functions Γ^α_{ij} are Christoffel symbols.