

PHY484 - Final Project

Spinors in Curved Spacetimes

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1 Introduction

Spinors are a very useful piece of mathematical formalism, used widely in physics. But the math that is used to describe them is considerably more complicated than what is needed to describe the rules of tensor calculus. In this exposition I will be introducing the various mathematical objects that are needed to talk about spinors. After that, we will talk about an example from flat spacetime that will set up the main topic of this paper: an application of spinors in arbitrary curved spacetimes. This will serve as an important stepping stone in introducing several equations that are fundamental in the study of relativistic quantum mechanics such as the Dirac and Weyl equations. Moreover, I aim to make it so that this is more of an introduction to the mathematics of the subject and how we may apply it to certain fields. I will show how we can extend the Levi-Civita connection that is defined on classical fields to the spin connection defined on spinor fields that we will later see are needed to describe spin- $\frac{1}{2}$ particles in the way the Dirac equation does in flat spacetime.

1.1 Conventions

In this paper I will use the mostly minus metric convention, and I will work in natural units of $c = \hbar = 1$. Moreover I will reserve the symbol $\eta_{\mu\nu}$ to be the Minkowski metric and in local cartesian coordinates it will be $[\eta_{\mu\nu}] = \text{diag}(1, -1, -1, -1)$. I will also use Latin alphabet to denote indices $1, \dots, n$ and I will use Greek alphabet to denote indices $0, 1, \dots, n$, unless otherwise stated.

I will denote the flat spacetime γ matrices by γ^μ . $\mathcal{M} = \gamma^\mu M_\mu$ for any relevant object. We will use the standard representation of the Pauli matrices unless otherwise stated, that is

$$\begin{aligned}\sigma_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \sigma_x &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \sigma_y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ \sigma_z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\end{aligned}$$

We will also, unless otherwise stated, use the standard representation of the γ matrices in flat spacetime, that being

$$\begin{aligned}\gamma^0 &= \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix} \\ \gamma^1 &= \begin{bmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{bmatrix} \\ \gamma^2 &= \begin{bmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{bmatrix} \\ \gamma^3 &= \begin{bmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{bmatrix}\end{aligned}$$

We also have the convention that $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Whenever I mention Lagrangian, I really mean a Lagrangian density.

2 Mathematical Background

The general purpose of this section will be to introduce new notions like Clifford algebras and spin connections. Along the way there is more math jargon. So to begin, let us start with the first topic.

2.1 Clifford Algebras

To talk about spinors in a satisfactory manner, it is important to see what is happening in the background. To do this, we will introduce the idea of a Clifford algebra. But what is an algebra? It is simply a vector space V over a field \mathbb{F} with a bilinear mapping $\cdot : V \times V \rightarrow V$, called multiplication, denoted by $\cdot(v, w) = v \cdot w$. With this, suppose we have a real (or complex) vector space V . This will be the tangent or the cotangent space later on. Suppose further we have a symmetric bilinear form¹ $Q : V \times V \rightarrow \mathbb{F}$. Now take the tensor algebra

$$T(V) = \bigoplus_{n \in \mathbb{N}} \left(\bigotimes_{k=1}^n V \right) = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

To define the Clifford algebra, take the two sided ideal generated by the set

$$I = \{v \otimes w + w \otimes v - 2Q(v, w)\} \subseteq T(V)$$

After this we define $\text{Cl}(V, Q) = T(V)/I$. This definition comes from [1, pg. 30] though it is slightly modified to make it so that there are whole numbers to work with rather than fractions. This definition is very terse and hard to work with. One of the issues that the author of the citation brings up is the nontrivial nature of this algebra. I think that this question is best answered by the mathematician and we will just take that the Clifford algebra of a finite dimensional real vector space is a vector space $\text{Cl}(V, Q)$ of dimension $2^{\dim V}$. Taking inspiration from the ideal quotient definition, we can define the structure of a \mathbb{Z}_2 -graded algebra on the space (V, Q) by

$$vw + wv = 2Q(v, w)$$

Now we take a basis $e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}$ such that

$$\begin{cases} Q(e_i, e_j) = 0 & \text{for } i \neq j \\ Q(e_i, e_i) = 1 & \text{for } 1 \leq i \leq p \\ Q(e_i, e_i) = -1 & \text{for } p+1 \leq i \leq p+q \end{cases}$$

then the Clifford Algebra

$$\text{Cl}(V, Q) = \text{span}\{e_1, \dots, e_{p+1}, e_1e_2, \dots, e_{p+q-1}e_{p+q}, e_1e_2e_3, \dots, e_{p+q-2}e_{p+q-1}e_{p+q}, \dots, e_1e_2 \cdots e_{p+q}\}$$

¹For all purposes here we restrict to those bilinear forms that are nondegenerate along with the properties mentioned. But all the things that are mentioned here should work just fine, albeit with some extra care.

Now this looks like a lot, so let us do a very famous example. This example will show that \mathbb{C} is a Clifford algebra over \mathbb{R} . To do this, let $V = \text{span}\{v\}$ with the bilinear form defined by $Q(v, v) = -1$. By the definition above we have that $v^2 + v^2 = -2 \implies v^2 = -1$. This now means that $\text{Cl}(V, Q) = \text{span}\{1, v\} \cong \mathbb{C}$ where this isomorphism is actually an algebra isomorphism. Another very important example of a Clifford algebra that is used throughout the field of Relativistic Quantum Mechanics is the Dirac Algebra. To define the algebra we will take 4 matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\} \subseteq M_{4 \times 4}(\mathbb{C})$ such that

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$$

A particular choice of the γ -matrices is said to be a representation, though this is a very informal naming. The corresponding Clifford algebra is the Dirac algebra given by

$$\text{Cl}(\mathbb{C}^2, \eta) = \text{span}\{\text{id}_{4 \times 4}, \gamma^\mu, \gamma^\mu \gamma^\nu, \gamma^\mu \gamma^\nu \gamma^\sigma, \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho\}$$

The motivation of this algebra is given below when we talk about the Dirac equation in flat spacetime (3.2) so we can extend it to curved spacetimes. As it turns out, the classification theory for real Clifford algebras is horrible. The study of the classification can be found in [3, pg. 18-19]. Regardless, in section 3.2 we will see that we require our Clifford algebra to be defined over the complex numbers anyways. So let us take a Clifford algebra $\text{Cl}(V, Q)$. We defined the complexification of the algebra by $\mathbb{C}l(V, Q) = \text{Cl}(V, Q) \otimes_{\mathbb{R}} \mathbb{C}$ and we define the scalar multiplication by the rule

$$(a + bi)(v \otimes \lambda) = v \otimes (\lambda(a + bi))$$

Just like if we were over the complex numbers, we shall denote $v \otimes \lambda$ by λv . So we get an algebra over the complex numbers. The Dirac algebra is given as the complexification of $\text{Cl}(\mathbb{R}^4, \eta)$.

2.1.1 Subalgebras

Suppose we have some algebra A over the field \mathbb{F} . We define a subalgebra to be a linear subspace $V \subseteq A$ such that the multiplication between elements in V is another element in V . Given the structure of a Clifford algebra above, there are some important subalgebras that I will now present. We first need some naming. An element $v = v_1 \cdots v_n \in \text{Cl}(V, Q)$ with the $v_i \in V$ is said to have degree n . This leads to a natural decomposition of the space $\text{Cl}(V, Q)$ into two pieces

$$\text{Cl}(V, Q) = \text{Cl}_+(V, Q) \oplus \text{Cl}_-(V, Q)$$

where

$$\text{Cl}_+(V, Q) = \text{span}\{v \in \text{Cl}(V, Q) \mid v \text{ has even degree}\}$$

Similarly,

$$\text{Cl}_-(V, Q) = \text{span}\{v \in \text{Cl}(V, Q) \mid v \text{ has odd degree}\}$$

We will also take, without proof, that $\text{Cl}_+(V, Q)$ is a subalgebra of $\text{Cl}(V, Q)$.

2.2 Important Groups

In the discussion of spinors it is important to keep certain groups we have used a lot up to this point in physics. I will also treat this as a place to define a standard definition in terms of notations and conventions. The first of the many groups I will mention is the group of Lorentz Transformations. To define this, let V be a vector space and let Q be a bilinear form so that there is an orthonormal basis in which the matrix of Q agrees with the matrix of $\eta^{\mu\nu}$ in the standard cartesian coordinates. Let $\text{Cl}(V, Q)$ be the corresponding Clifford algebra. Then a map $L : V \rightarrow V$ so that $Q(Lv, Lw) = Q(v, w)$ is called an isometry of (V, Q) and in this specific case is called a Lorentz Transformation of the space. One can check that products of Lorentz transformations are again Lorentz transformations, they are invertible, and identity is a Lorentz transformation. Hence they form a group

$$L(V, Q) = \{\Lambda : V \rightarrow V \mid Q(\Lambda v, \Lambda w) = Q(v, w) \forall v, w \in V\}$$

This definition comes from [2, pg. 66]. A non trivial fact that is mentioned in the text is that the mapping $v \rightarrow \Lambda v \Lambda^{-1}$ for some fixed $\Lambda \in \text{Cl}(V, Q)$ is an isometry only if $\Lambda v \Lambda^{-1} \in V$. They also show that an isometry can be decomposed into a product of vectors so $\Lambda = u_1 \cdots u_n$ and we can hence define $\tilde{\Lambda} = u_n \cdots u_1$.

From the definition of the Lorentz group we can see that $\det \Lambda = \pm 1$ for all $\Lambda \in L(V, Q)$ hence we see that this group is not connected topologically. We will denote the connected component that is also a subgroup by $L_0(V, Q)$.

This leads us to one of the many definitions of the pin group³ that we will be using. We now define another group, called the pin group

$$\text{Pin}(V, Q) = \{\Lambda \in \text{Cl}(V, Q) \mid \forall v \in V \text{ we have that } \Lambda v \Lambda^{-1} \in V \text{ and } \Lambda \tilde{\Lambda} = 1\}$$

By another nontrivial fact we get that the map $\text{Pin}(V, Q) \ni \Lambda \mapsto [a^\alpha_\beta]$ where $\Lambda e_\beta \Lambda^{-1} = a^\alpha_\beta e_\alpha$ is a 2-1 group homomorphism into the Lorentz group. This idea about the spin group being a two to one representation of the Lorentz group has profound physical consequences. In the specific case when $(V, Q) = (\mathbb{R}^n, \eta)$ we will denote this group with $\text{Pin}(n)$. In some sense we can decompose certain classes of vectors into a product of two vectors so that an overall rotation of the vector reduces to a half rotation of the decomposed vectors. An example of where we see this is in the polarization of light. This is seen in the construction of the Jones vectors which are an integral representation of the polarization of light.

Now we move on to another, seemingly unrelated, group. This is also a subgroup of the Clifford Algebra. We look at the following group

$$\text{Pin}(V, Q) = \{v \in \text{Cl}(V, Q) \mid v = v_1 \cdots v_n \text{ such that } v_i \in V \text{ and } Q(v_i, v_i) = \pm 1\}$$

²This is a bit of a misnomer since here I mean that η acts on \mathbb{R}^{n+1} and not just \mathbb{R}^4 .

³This is gotten from [2, pg. 67] but as I was reading through it, I realized that at some points they call the spin group the pin group. This is one of those places, they say there is a 2-1 homomorphism into the Lorentz group but the spin group is only the even elements so the determinant must be positive so it only goes into the special orthogonal group and not into the entire Lorentz group. This is backed by [1, pg. 43-44].

This is group of units of $\text{Cl}(V, Q)$. So $\text{Pin}(V, Q) = \text{Cl}(V, Q)^\times$. This definition is from [3, pg. 21]. Now we introduce the spin group:

$$\text{Spin}(V, Q) = \text{Pin}(V, Q) \cap \text{Cl}_+(V, Q)$$

The elements of the spin group are just the elements of even order within the group of units of the Clifford algebra. We can decompose these sets further by defining

$$S^\pm(V, Q) = \{v \in V \mid Q(v, v) = \pm 1\}$$

From here if we take

$$\text{Spin}^\pm(V, Q) = \{v_1 \cdots v_{2p} w_1 \cdots w_{2q} \mid v_i \in S^+(V, Q) \text{ and } w_i \in S^-(V, Q)\}$$

it turns out that this is also a subgroup of the Clifford algebra. It is called the orthochronous spin group as defined in [9, pg. 350].

Now the map $\rho : \text{Spin}(n) \rightarrow \text{SO}(n)$ defined by

$$\rho(a)x = axa^{-1}$$

where $x \in \mathbb{R}^n$ is a 2-1 Lie group homomorphism as shown in [6, pg. 77]. Therefore $\text{Spin}(n)$ is called the double cover of $\text{SO}(n)$.

I will be using the the Clifford algebra definitions for the remaining parts.

2.3 Representations

We will not really need much representation theory but there are some very important concepts that we will need. We will first start off with the definition of a representation. To do this we need to take some algebra/group A and some vector space V then a representation of A over V is an algebra/group homomorphism $\rho : A \rightarrow \text{GL}(V)$.

Now this seems like a useless object, but as it turns out, having a representation of an algebra/group allows us to study a certain phenomenon a lot more in depth. But before we can talk about the useful presentations on the pinor and spinor groups we need to define another concept. That of an irreducible representation. So let $U \subseteq V$ be a linear subspace then U is called A -invariant if $\rho(g)(u) \in U$ for all $u \in U$ and all $A \in G$. Then we say that the representation ρ is irreducible if and only if the only A -invariant subspaces of V are $\{0\}$ and V . But what is the point of all this? The funny answer is that we can use these to define even more things.

As it turns out, in physics it is more important that we deal with the complexification of the Clifford Algebra. This connection will be made more clearly from section 3.2 and onward. We saw the classification of complex Clifford algebras when we introduced them. So we will call \mathbb{C}^N the space of spinors⁴. Then the classification given in [3, pg. 18-19] defines a representation of the complex Clifford algebra $\mathbb{C}l(p, q)$ ⁵ $\rho : \mathbb{C}l(p, q) \rightarrow \mathbb{C}^N$.

⁴It is more useful for this to be a representation of spinors as we will see here.

⁵Where this the complexification of the real Clifford of \mathbb{R}^d with an symmetric bilinear form of signature (p, q) .

This is from the classification theorems given in [3, pg. 18-19]. This representation is called the spinor representation of the Clifford algebra.

We also require another sort of representation. Let G be a group and let F be any set. Suppose G acts on F . Then we may think of the group action on F as a map $\rho : G \rightarrow \text{End}(F)$ here $\text{End}(F)$ is the group of invertible maps from F to itself.

2.4 Fiber and Principle Bundles

In this I will introduce the various bundles that are associated with manifolds that are used a lot in the theory. The main bundles we will need are principle bundles. We have already seen many types of bundles when doing General Relativity. The main ones are the tangent and cotangent bundle. We have also taken various tensor products of these bundles to get higher order tensor bundles. My goal here is to simply generalize this definition so I can introduce principle bundles which are central to the theory of spinors even in flat spacetimes. So let us start with the general definition of a fiber bundle. A fiber bundle is a collection of data that allows us to study the structure of the underlying space better. Just like how the tangent and cotangent bundles allow us to study the motion of particles in the space, different fiber bundles allow us to study similar things.

In the definitions that are to follow, I will talk about manifolds only. The theory can be generalized to other topological spaces but in physics we mainly care about objects that are manifolds. Moreover all manifolds are assumed to be smooth and all maps between manifolds are also smooth. To define a fiber bundle we will need some ingredients. So let M be a manifold (this will be the space over which the bundle is defined) called the base space, let G be a Lie group called the structural group, let F be a manifold, called the typical fiber, and let there be a smooth action of G on F (with ρ its corresponding map as defined in 2.3). Finally, let E be a manifold, called the total space, with a surjection $\pi : E \rightarrow M$. Then a fiber bundle with structure group G and typical fiber F is a map $\pi : E \rightarrow M$ such that for every $p \in M$ there is a neighborhood of p such that there is a diffeomorphism $\varphi : \pi^{-1}(U) \rightarrow U \times F$ so that $\pi = \pi_U \circ \varphi$. Here $\pi_U : U \times F \rightarrow U$ is the projection onto the first component. From the fiber bundle structure we also require that the transition maps are compatible in the sense that the transition map $\varphi_{UV} = \varphi_U \circ \varphi_V^{-1}|_{\{p\} \times F} = \rho(g_{UV}(p))$ where $g_{UV} : U \cap V \rightarrow G$. The maps g_{UV} are called transition maps. This definition is from [3, pg. 29-30] and so is the definition of principle bundles. If the typical fiber and the structural group are the same and moreover G acts on itself via left multiplication then the corresponding fiber bundle is called a principle G bundle. To every principle G bundle $P^G(M)$ we have an associated vector bundle (see below) $P^G(M) \times_\rho V$ where $\rho : G \rightarrow \text{GL}(V)$ is a representation. The definition of this is extremely involved and is defined in [9, pg. 237-240].

2.4.1 Spin bundles

In physics we mostly see vector bundles and frame bundles. From the definition above if F is a vector space and ρ maps into $\text{GL}(F)$, then $\pi : E \rightarrow M$ is a vector bundle. Frame bundles are a little more delicate of an object. To define it we first need to know how to construct fiber bundles from local data, but the construction is involved, so it is advised to check [3], [6] and [9]. But to every vector bundle $\pi : E \rightarrow M$ there is an associated

principle $\text{GL}(n)$ fiber bundle whose typical fiber is the space of frames over $\pi^{-1}(p)$. This is called the principle frame bundle. If (M, g) is an orientable pseudo-riemannian manifold then the frame bundle associated with the tangent bundle can be reduced to a principle $\text{SO}(p, q)$ frame bundle⁶. We note here that the associated fibers are the space of oriented frames on the tangent space at a point. To agree with the notation in [6] we will denote the principle $\text{SO}(p, q)$ bundle associated with the tangent bundle $P^{\text{SO}}(M)$ ⁷. If this principle frame bundle can be lifted to the double cover $\text{Spin}(p, q)$ of $\text{SO}(p, q)$ then we say that the manifold is spinnable. Moreover the principle bundle $P^{\text{Spin}}(M)$ is called the spinor bundle. In physics, as motivated in 3.4, we complexify the associated bundle of $P^{\text{Spin}}(M)$ to a complex bundle $P^{\text{Spin}}(M) \times_{\rho} \mathbb{C}^N$ the formal definition of this is given [9, pg. 237-240]. Due to the classification and decompositions of the complex Clifford algebras, the structure in even dimensional spaces (like the one we live in) is much more interesting. It allows us to define the chirality. Sections of $P^{\text{Spin}}(M) \times_{\rho} \mathbb{C}^N$ are called spinor fields. We can also take the appropriated tensor products and exterior products to get the tensors and forms. I will not be talking much about them though. The transformation laws for spinor fields, mathematically, are simply compositions with transitions functions. Just like how vector fields transform under Lorentz transformations, and those are the $\text{SO}(1, 3)$ transition maps in special relativity.

2.5 Connection on the Spin bundle

For the connection on the spinor bundle we note that the affine connection induces a connection on the spinor bundle. But we will talk about this in terms of coordinates since it will be the more useful concept. It can be shown that, in local coordinates, we have local frame $\{e_A\}$ and corresponding one forms $\{e^A\}$ such that

$$e_A{}^\alpha e^A{}_\beta = \delta^\alpha{}_\beta$$

and

$$e^A{}_\alpha e_B{}^\beta = \delta^A{}_B$$

The important thing about the tetrads is that

$$g_{\alpha\beta} = e_A{}^\alpha e_B{}^\beta \eta_{AB}$$

and

$$\eta_{\alpha\beta} = e_A{}^\alpha e_B{}^\beta g_{AB}$$

The definition for the tetrad basis is given in [11, pg. 1828]. Moreover in this coordinate system the torsion free spinor connection is given by

$$\omega_\mu{}^{AB} = e_\nu{}^A \partial_\mu e^\nu{}^B + e_\nu{}^A \Gamma^\nu{}_{\sigma\mu} e^\sigma{}^B$$

We will see a more in depth motivation of this definition in section 3.4. We will define more operators that act on spinors, but for now this is sufficient.

⁶The more general case of general frame bundles for oriented vector bundles can be found here [10, pg. 1].

⁷A more formal definition of associated bundles can be found at [6, pg. 94].

3 Dirac and Weyl Equations

Before we talk about the Dirac Equation, it is important to mention that the motivation behind this equation is the Klein-Gordon equation which was one of the first real attempts at unifying relativity and quantum mechanics.

3.1 Short detour to Klein-Gordon

The Klein-Gordon equation is mainly used to talk about spinless particles, or, in other words, scalar fields. This is because the derivation comes from the mass-energy constraint and replacing the relevant quantities with their quantum equivalent as is done here [4, pg. 1-2]. This gives us the following

$$(\partial^\mu \partial_\mu + m^2)\psi = 0$$

It is useful to note that this is Lorentz invariant since each operator and field here is. I would also like to note that this is the equation over Minkowski spacetime and we can make this over an arbitrary spacetime manifold by replacing the partials with the connection $\partial^\mu \rightarrow \nabla^\mu$ hence we get that for some scalar field ψ on a manifold M satisfies the Klein-Gordon equation if

$$(\nabla^\mu \nabla_\mu + m^2)\psi = 0$$

There are many issues with this equation, the blaring one is that the wave function ψ here cannot be used to represent the probability density of the particle as pointed out in [5]. However this mysterious Dirac equation seems to have the desired properties.

3.2 Dirac Equation in Flat Spacetime

If we start with the Klein Gordon equation in flat spacetime we can note that the main issue came from the fact that the equation is second order in time. So Dirac stared at this and thought about ‘factoring’ the second order differential operator $\partial^\mu \partial_\mu$. So the idea is that there is an operator $(\gamma^\mu \partial_\mu)^2 = \partial^\mu \partial_\mu$ so we have

$$\gamma^\mu \gamma_\nu \partial_\mu \partial^\nu = \partial^\mu \partial_\mu$$

From here we get that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. From here we can get the standard representation given in 1.1. So we rewrite the Klein Gordon equation as

$$(\not{D}\not{D} + m^2)\psi = 0$$

Now using $x^2 + y^2 = (ix + y)(ix - y)$ we get that

$$(i\not{D} + m)(i\not{D} - m)\psi = 0$$

Now we want to choose solutions ψ so that

$$(i\not{D} - m)\psi = 0$$

This is because this corresponds to the positive energy solutions. This can be seen more directly by taking the mass shell constraint $p^\mu p_\mu = m^2$ and replacing the classical momentum operators by their quantum counterparts. Since we are taking the ‘square

root' of this expression we end with $i\partial\psi = m\psi$. An interesting thing to note is that the 'dimensionality' of the wave function has now 'jumped'. We are no longer happy with scalar solutions but rather ψ has to be a 'vector' solution here because of the nature of the Dirac Algebra. For what is to follow, we use the exposition done in [7, pg. 81-96]. An important thing to mention is that the subspace of second order elements in the Dirac algebra defines a representation of the Lorentz group under the standard representation of the Dirac algebra. So for notation we have that⁸

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$$

Moreover, it turns out that this representation is in fact reducible. This has significant physical significance as we shall see. The sub-representations act on only two out of the four components of the wavefunction. This means that

$$\psi = \psi_L \oplus \psi_R$$

where R and L are the two parts that transform the same under rotations but under Lorentz transformations we get a phase shift. More explicitly, under this representation of the Lorentz group we see that under rotations $\psi_R \rightarrow \exp(-i\vec{\varphi} \cdot \vec{\sigma}/2)\psi_R$ and $\psi_L \rightarrow \exp(-i\vec{\varphi} \cdot \vec{\sigma}/2)\psi_L$ while under Lorentz boosts we have that $\psi_R \rightarrow \exp(i\vec{\chi} \cdot \vec{\sigma}/2)\psi_R$ and $\psi_L \rightarrow \exp(-i\vec{\chi} \cdot \vec{\sigma}/2)\psi_L$ ⁹.

Moreover under a rotation of 2π the components $\psi^\mu \rightarrow -\psi^\mu$. This is only half the rotation one might expect for a vector. Hence the fields used to describe these particles are not even vector fields, they are 'spin- $\frac{1}{2}$ '. This ties back to the discuss at the end of 2.2 where we found a double cover for $\text{SO}(n)$ and this was called the spin group. It is only fair that we call ψ_R and ψ_L (Weyl) spinors and we call $\psi = \psi_L \oplus \psi_R$ a (Dirac bi)spinor. I will introduce the equations of motion that govern the Weyl spinors in flat spacetime in section 3.3.

The Dirac equation can be derived from variational methods as well. The corresponding Lagrangian will be the key to defining the Dirac equation in curved spacetimes. This is where the mathematical foundation about spin manifolds and the spin connection will come in handy. Say we have a spinor field ψ then we denote by ψ^\dagger the conjugate spinor field¹⁰. Then we define the Dirac conjugate to be $\bar{\psi} = \psi^\dagger \gamma^0$. Then from [7, pg. 88] we see that $\bar{\psi}\psi$ is a Lorentz scalar and $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector. From this the only Lagrangian we could possibly construct for linear equations of motion is

$$\mathcal{L} = \bar{\psi}(i\partial - m)\psi$$

Given all this it is not reasonable to simply replace the partials with the Levi-Cevita connection as we have been doing. This is where the theory of Spin Geometry comes into play, but we will discuss the spin connection in coordinates in section 3.4.

⁸For the below, I will also define $\sigma^{\mu\nu} = iS^{\mu\nu}$.

⁹Using the notation for the representation shown in [7, pg. 85-87]

¹⁰In the standard representation this corresponds to the regular complex conjugate of a vector.

3.3 Weyl Equation in Flat Spacetime

The Dirac Lagrangian can be derived another way. Instead of dealing with the bispinor field ψ we deal with the Weyl spinors ψ_R and ψ_L . This is the approach taken in [8, pg. 131-135] and it differs slightly from the approach found in [7, pg. 91-93]. The idea with the Weyl Lagrangian is derived in [7, pg. 91] as well but the approach taken jumps through some mathematical magic. Before jumping into that we are required to talk about an additional symmetry that we see strong interaction uphold, that is of parity.

3.3.1 Parity Symmetry

But what exactly is a parity symmetry? Parity is closely related to time reflection. It is a transformation that takes the form $x^i \rightarrow -x^i$. We now justify the use of ψ_R and ψ_L when talking about the decomposition of a Dirac bispinor into component Weyl spinors. We define the following Lorentz invariant projection operators

$$P_R = \frac{1}{2}(1 + \gamma^5)$$

$$P_L = \frac{1}{2}(1 - \gamma^5)$$

Moreover we may note that $P_L\psi = \psi_L$ and $P_R\psi = \psi_R$. Moreover it is important to note that Weyl spinors cannot have parity symmetry and this is because they ‘spin’ in a specified direction. Hence under a parity operation we should get that left and right handed particles get swapped. But the most important thing is that to have parity symmetry we must have both left and right handed spinors. This is seen in the Dirac equation, since the Dirac field can be decomposed into two Weyl spinors.

From the fact that Dirac particles obey parity symmetry, the Lagrangian must have a coupled term to respect the parity symmetry hence we get that

$$\mathcal{L} = i\overline{\psi}_R \not{\partial} \psi_R + i\overline{\psi}_L \not{\partial} \psi_L - m(\overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R)$$

We now see that the theory of a massless and massive spin- $\frac{1}{2}$ particle, even in flat spacetime, differ by quite a bit. For a massless particle we have that one of the fields is sufficient to describe the physics. This gives rise to the massless Weyl equations

$$\not{\partial} \psi_R = 0$$

and

$$\not{\partial} \psi_L = 0$$

But for massive particles, they couple to give

$$\not{\partial} \psi_R = m\psi_L$$

and

$$\not{\partial} \psi_L = m\psi_R$$

An interesting thing is that for these particles the Klein Gordan equation is also satisfied. In the case when we are dealing only the right (or left) handed particles it is more instructive to project onto the corresponding subspaces and write the corresponding equations as

$$\sigma^\mu \partial_\mu \psi_R = 0$$

and

$$\overline{\sigma^\mu} \partial_\mu \psi_L = 0$$

So it is clear that the theory of initially ‘simple’ particles like electrons even in flat space-time are just horrible to deal with. To make things worse it is not the case that we can simply do $\partial_\mu \rightarrow \nabla_\mu$ in the equations since we are not dealing with vectors that transform the way we would like.

In section 3.2 we talked about the double cover of $\text{SO}(n)$ given by the spin group defined by the Clifford algebra of \mathbb{R}^n with the standard Euclidean bilinear form. So to describe spin particles we must use a connection on a $\text{Spin}(n)$ -principle bundle.

3.4 Dirac Equation in Curved Spacetime

Taking inspiration from the flat spacetime case, we should start off by defining the Clifford algebra of ‘gamma’ matrices that relate to the metric tensor. Take a set of tetrad $\{e^A\}$ and let γ^μ be the flat spacetime Dirac basis. Following the notation used in [11]¹¹ we define the following

$$\bar{\gamma}^A = e^A{}_\mu \gamma^\mu$$

this gives that $\{\bar{\gamma}^A, \bar{\gamma}^B\} = 2g^{AB}$. Hence this defines a Clifford algebra and so following all the theory above we define a spinor bundle over the spacetime manifold. Hence we have the notion of spinor fields ψ . Just as is the case with tensors, the partials of spinor fields don’t, in general, transform as such. Hence we need to define a new operator D_μ that plays the role of that the covariant derivative played in the classical theory of fields in curved spacetimes. Once we develop such an operator, we will be able to take the Dirac Lagrangian and replace the partials with the the covariant derivative of spinor fields. Hence we have that

$$\mathcal{L} = \bar{\psi}(iD - m)\psi$$

Again we define ¹² $\bar{\psi} = \psi^\dagger \gamma^0$. So we begin the hunt for the connection. Since a spinor field is a section of the spinor bundle defined on the manifold, the transformation law is given by $\psi \rightarrow \psi' = L\psi$ where L is some spacetime dependent spinor transformation. Furthermore the transformation of the partial of ψ can be seen by chain rule to be

$$\partial_\mu \psi' = L \partial_\mu \psi + \psi \partial_\mu L$$

so we want D_μ so that

$$(D_\mu \psi)' = L D_\mu \psi$$

From this we see that we can keep the partial term since it already is transformed by L so we may say that

$$D_\mu \psi = (\partial_\mu + \Gamma_\mu) \psi$$

Now to determine Γ_μ , these are called the Fock-Ivanenko coefficients, we do as indicated in [12, pg. 1-6] and let $\Gamma'_\mu = L \Gamma_\mu L^{-1} - (\partial_\mu L)L^{-1}$. This gives us right transformation law for $D_\mu \psi$. So all that is left is to figure out what Γ_μ is. Taking inspiration from the the

¹¹The notation is unfortunately contradicting.

¹²Here $\bar{\gamma}^\mu D_\mu = iD$.

tensor theory in curved spacetime we note that this construction has something to do with the spin connection ω_μ^{AB} and as it turns out, there is a connection¹³ and it is that

$$\Gamma_\mu = \frac{1}{2}(\omega_{AB})_\mu S^{AB}$$

Hence we get that $D_\mu\psi = \partial_\mu\psi + \frac{1}{2}(\omega_{AB})_\mu S^{AB}\psi$. As I mentioned in section 2.5 the affine connection induces the spin connection. From the formulas I have presented, this connection¹⁴ doesn't really seem apparent. So let us make it more clear. We note that

$$\omega_\mu^{AB} = e_\nu^A \partial_\mu e^\nu B + e_\nu^A \Gamma^\nu{}_{\sigma\mu} e^\sigma B = e_\nu^A \nabla_\mu e^\nu B$$

hence we get that $(\omega_{AB})_\mu = (e_A)_\nu \nabla_\mu e_B^\nu$. So in this formulation it is a lot more clear as to what the spin connection is measuring physically. The spinor derivative D_μ essentially measures how much the object is rotating about certain axes. Using the description from [13, pg. 4] we see that the coordinate independent description of Γ_μ is given by

$$\Gamma_\mu = \frac{1}{4} \bar{\gamma}^\nu (\partial_\mu \bar{\gamma}_\nu - \Gamma^\alpha{}_{\mu\nu} \bar{\gamma}_\alpha)$$

Now let us look more carefully at the Dirac equation in curved spacetime

$$(i\cancel{D} - m)\psi = 0$$

Now we try to define the adjoint Dirac equation. To do this we define the action $D_\mu \bar{\psi}$ as

$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \Gamma_\mu \bar{\psi}$$

To get rid of any confusion it is common to denote the action of D_μ on ψ as $\overrightarrow{D}_\mu \psi$ and the action of D_μ on $\bar{\psi}$ as $\overleftarrow{\bar{\psi}} D_\mu$. We will denote

$$\overleftarrow{\bar{\psi}} \overrightarrow{D}_\mu \psi = \overleftarrow{\bar{\psi}} D_\mu \psi + \overleftarrow{\bar{\psi}} \overrightarrow{D}_\mu \psi$$

Then the equations of motion for spinor fields are

$$(i\cancel{D} - m)\psi = 0$$

and for the adjoint it is

$$\overleftarrow{\bar{\psi}} (i\cancel{D} - m) = 0$$

We see below that $j^\mu = \bar{\psi} \gamma^\mu \psi$ is spinor covariantly conserved. But this is also true in the free Dirac case.

If we look at $\omega_{AB\mu}$ it is important to note that the Latin indices here are not physical. They merely tell us how much the ‘spin’ is projected onto the plane spanned by the corresponding basis vectors. It can therefore be shown that the spin connection is simply a one form that connects different spins together. The role of this object is similar to the one of the covariant derivative in many ways. Moreover I suspect that the massless Dirac equation can be seen as a parallel transport equation but for the spin connection.

¹³I can't keep going like this, the puns are too funny.

¹⁴omg again.

3.4.1 Significance of Dimension

The spinor bundle $P^{\text{Spin}}(M) \times_{\rho} \mathbb{C}^n$ over even dimensional spacetime manifolds can be decomposed into two bundles, due to the classification of the even dimensional complex Clifford algebras. This small quirk of the bundle means that any spinor field can then be decomposed into two more fundamental fields called the Weyl spinors.

$$\psi = \psi_R \oplus \psi_L$$

As we saw in the case of flat spacetime, this decomposition actually allows us to talk about more sophisticated symmetries such as chiral and parity symmetry. So the theory is more complicated in even dimensional spacetimes since we require that our fields obey the various symmetries that we impose on them.

4 What's Next

There is a lot more that we can do with the theory of spinors in curved spacetime than talk about vacuum solutions. The most interesting of these is coupling spinor fields to gravity in different spacetimes and seeing how they interact. We can also talk about charged spin- $\frac{1}{2}$ particles interacting with an external A_μ field. The coupling for this is even more straightforward than it is for gravity so this is what I will do now.

4.1 Very Brief Intro to Charged Spin- $\frac{1}{2}$ Particles in A_μ Fields

The coupling is straight forward and all we have to do is make charged spin- $\frac{1}{2}$ particles interact in the simplest way possible that keeps the properties of the Lagrangian we would otherwise like. Adding a coupling term to the Lagrangian we had for the spinor field with no interactions gives us

$$\mathcal{L}_{EM} = \bar{\psi}(i\overleftrightarrow{D} - q\mathcal{A} - m)\psi$$

and the corresponding equation of motion for ψ is

$$(i\overleftrightarrow{D} - q\mathcal{A} - m)\psi = 0$$

From this information, we can use Noether's theorem to find the conversed spinor current of the system. As it turns out, this is $j^\mu = \bar{\psi}\gamma^\mu\psi$, this is taken from [14, pg. 2] and one can derive this using the fact that D_μ is a derivation of spinors hence follows the product rule. Now taking the Dirac equation and the adjoint Dirac equation

$$\bar{\psi}(i\overleftrightarrow{D} + q\mathcal{A} + m) = 0$$

Now multiplying the Dirac equation by $\bar{\psi}$ on the left and the adjoint Dirac equation by ψ on the right and adding together we get that

$$\bar{\psi}\overleftrightarrow{D}\psi = 0$$

hence

$$D_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

This is the conserved current density. Moreover in flat spacetime $\rho = \bar{\psi} \gamma^0 \psi$ is actually a probability density. I conjecture that given our metric convention ρ in curved spacetimes is also a probability density. It is not feasible to solve these in the arbitrary case, but there are reasonable approximations we can make to simplify the equations of motion. A very common approximation that is made is the WKB approximation which is covered in [14, pg. 2-4]. In this paper, they state that if we are working with $(i\vec{D} - qA - m)\Psi = 0$ then the approximations we can make are

$$\begin{aligned}\Psi(x) &= \psi(x, \nabla_\mu S, \delta x) \exp\left(\frac{iS(x)}{\delta x}\right) \\ \psi(x, \nabla_\mu S, \delta x) &= \psi_0(x, \nabla_\mu S) + \delta x \psi_1(x, \nabla_\mu S) + \dots\end{aligned}$$

where S is some function, ψ is an amplitude. This approximation is taken directly from [14, pg. 2]. This approximation has the effect of turning a problem from a spinor field equation of motion to the scalar complex field ψ as the Euler-Lagrange equations in the lowest order of δx gives us

$$\begin{aligned}\vec{D}\psi_0 &= 0 \\ \bar{\psi}_0 \vec{D} &= 0 \\ \nabla_\mu j_0^\mu &= 0\end{aligned}$$

where $\vec{D} = \not{k} + m$ and $k_\mu = \nabla_\mu S + qA_\mu$.

4.2 Other possibilities

With the concepts and equations presented here, it is now possible to consider Dirac fields around black holes and in other more useful scenarios like in the FRW metric. An interesting thing that I wanted to cover was the coupling of spinor fields to gravity as done in [13, pg. 13] however the Lagrangian felt highly unmotivated and I couldn't find many resources to motivate the subject. To summarize the results however, we note that we have a coupling Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_{\text{Einstein-Hilbert}} + \mathcal{L}_D$$

and this results in a more nontrivial theory of gravity.

5 Conclusion

To conclude this we note that the abstract formulations needed to describe fairly simple particles is immense. Along the way we have to describe anti-commutation relations in order to break down a wave equation into a simpler form. We noted that spin- $\frac{1}{2}$ particles cannot be adequately talked about using only scalar fields and hence we needed to introduce the theory of spinors to make sense of them. Moreover a lot of the theory used in classical field theory does not apply one to one and some work needed to be done mathematically to describe these particles. Moreover we defined the connection which is central after the introduction of spinor fields. The classifications of even dimensional complex Clifford algebras have an immense impact on certain symmetries that our field theories can have and this seemingly random fact leads to the theory of Weyl particles and the possible existence of Majorana fermions. So the theory of spinors is very rich and I haven't even scratched the surface.

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