Hodge Theory on Kahler Manifolds Presentation

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Contents

1 Introduction

In this I will introduce the relevant objects in complex analysis needed to define Kähler manifolds and how we can talk about forms on these objects. Now we define the Hodge star which we have seen before. I must mention here that this presentation is a very brief overview to the subject and is meant to present results for the topic. Everything that was used here was taken from the book 'Complex Geometry' by Daniel Huybrechts.

1.1 Hodge Star

To start let (V, g) be a real inner product space, then we use the inner product $g = g_0$ to define an inner product g_k on $\bigwedge^k V$. The way to do this is to first let $e_1, \ldots, e_n$ be an orthonormal basis and note that g is defined completely by its action on the $\{e_i\}_{1 \leq i \leq n}$. Taking inspiration from this, define g_k so that

$$g_k(e_I, e_J) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

where $I \subseteq \{1, \ldots, n\}$ such that $I = \{i_1 < \cdots < i_k\}$. So now g_k can be determined uniquely to be an inner product on $\bigwedge^k V$. We set $\mathrm{vol}_g = e_1 \wedge \cdots \wedge e_n$, then $\bigwedge^n V = \mathrm{span}\{\mathrm{vol}\}$. Note that this defines an orientation on V. We extend the inner product to $h : \bigwedge^* V \to \bigwedge^* V$ so that forms of different degrees are orthogonal and $h|_{\bigwedge^k V} = g_k$. Now we define the Hodge star as

Definition 1.1.1 (Hodge Star)**.** *Suppose (V, g) is a real inner product space then the Hodge star is the operator $\star : \bigwedge^* V \to \bigwedge^* V$ such that*

$$\alpha \wedge \star\beta = g_k(\alpha, \beta)\, vol_g$$

for every $\alpha, \beta \in \bigwedge^* V$.

I will state some basic facts about the Hodge star

Theorem 1.1.1. *Given (V, g) with the Hodge star $\star : \bigwedge^* V \to \bigwedge^* V$ then*

1. *Restricting $\star|_{\bigwedge^k V}$ we have that*
$$h(\alpha, \star\beta) = (-1)^{k(n-k)} h(\star\alpha, \beta)$$
2. *Restricting $\star|_{\bigwedge^k V}$ we have that*
$$(\star|_{\bigwedge^k V})^2 = (-1)^{k(n-k)}$$
hence
$$\star^{-1} = (-1)^{k(k-n)} \star$$

From the notation of this section we will let (V, g) be a real inner product space moreover we will denote by $\left(\bigwedge^* V^*, h\right)$ the extension onto the exterior algebra.

Now suppose we have a Riemannian manifold (M, g) then the Hodge star defined on forms is given by $\alpha \wedge \star\beta = g_k(\alpha, \beta)\,\mathrm{vol}_g$ where vol_g and g_k is the induced metric on the space of k-forms. This defines an inner product on the space of forms called the Hodge inner product given by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star\beta = \int_M g_k(\alpha, \beta)\,\mathrm{vol}_g$$

From here we define the operator $\mathrm{d}^* = \delta$ to be the adjoint of d with respect to the Hodge inner product. One can show that $\delta\alpha = (-1)^{m(k+1)+1} \star \mathrm{d} \star \alpha$ where α is a k form. Finally we define the Laplacian on a Riemannian manifold by

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}$$

We denote by

$$H^k(M, g) = \{\alpha \in \Gamma^k(M) | \Delta\alpha = 0\}$$

the space of harmonic k forms.

2 Introductory Complex Analysis

We first give some conventions, the book I am following uses the basis of the topology given by polydisks, that is

$$B(w, \epsilon) = \{z \in \mathbb{C}^n \mid |z_i - w_i| < \epsilon_i\}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{C}^n$. We write coordinates on $\mathbb{C}^n$ with $z = (z_1, \cdots, z_n)$ where $z_i = x_i + iy_i$ for $x_i, y_i \in \mathbb{R}$. Now we start with the definition of holomorphicity. We need this to be able to define complex manifolds. After that I will mention some important theorems that hold in this Multivariable case. Now

Definition 2.0.1. *Let $U \subseteq \mathbb{C}^n$ be open and let $f : U \to \mathbb{C}$ be a be function. Then f is holomorphic if for variable $z_i = x_i + iy_i$ we have that*

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}$$

and

$$\frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i}$$

where $f = u + iv$.

Holomorphic functions are harmonic and that follows from the Cauchy-Riemann Equations. We also define two very important operators called the Wirtinger Operators

Definition 2.0.2 (Wirtinger Operators)**.** *For each z_i define*

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y}\right)$$

From here we can see that a function f is holomorphic if and only if for every z_i we have

$$\frac{\partial f}{\partial \overline{z}_i} = 0$$

In the study of many complex variables a very strong theorem holds which does not hold in several real variables. That is

Theorem 2.0.1 (Hartog's Theorem)**.** *If $f : \mathbb{C}^n \to \mathbb{C}$ is a function such that for each $j \in \{1, \ldots, n\}$ the function $g_j(z) = f(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n)$, for fixed $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$, is analytic then f is continuous.*

This doesn't hold true over $\mathbb{R}$, an example of this is the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$$

We also have the following

Theorem 2.0.2 (Useful Osgood's Theorem)**.** *Suppose $f : U \to \mathbb{C}$ is a continuous function on an open set $U \subseteq \mathbb{C}^n$ such that for each $j \in \{1, \ldots, n\}$ the function $g_j(z) = f(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n)$, for fixed $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$, is analytic then f is analytic.*

Putting Hartog's and Osgood's together we get that f is analytic if and only if for each $j \in \{1, \ldots, n\}$ the function $g_j(z) = f(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n)$, for fixed $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$, is analytic. Hartog's is hard to prove. Osgood's follows from the Cauchy Integral Formula

Theorem 2.0.3 (Cauchy Integral Formula)**.** *Let $f : \overline{B(w, \epsilon)} \to \mathbb{C}$ be continuous and for each $j \in \{1, \ldots, n\}$ the function $g_j(z) = f(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n)$, for every fixed $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$, is analytic then for any $z \in B(w, \epsilon)$ the formula*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\alpha_i - w_i| < \epsilon} \frac{f(\alpha_1, \ldots, \alpha_n)}{(\alpha_1 - z_1) \cdots (\alpha_n - z_n)} \, \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n$$

We also note that the fact that a series expansion exists, that is

Theorem 2.0.4. *Let $f : U \to \mathbb{C}$ be holomorphic then for each $z \in U$ there is an $\epsilon \in \mathbb{C}^n$ such that $B(w, \epsilon) \subseteq U$ such that for each $\alpha \in B(w, \epsilon)$ we have*

$$f(\alpha) = \sum_{i_1,\ldots,i_n=0}^{\infty} a_{i_1,\ldots,i_n} \prod_{j=1}^{n} (z_j - w_j)^{i_j}$$

where

$$a_{i_1,\ldots,i_n} = \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1+\cdots+i_n} f}{z_1^{i_i} \cdots z_n^{i_n}}$$

This follows from the Cauchy integral formula just like in the single variable case. Now one that holds only for $n \geq 2$

Theorem 2.0.5 (Riemann Extension)**.** *Let $f : U \to \mathbb{C}$ be holomorphic on some $U \subseteq \mathbb{C}^n$. Let $Z(f) = \{z \in U \mid f(z) = 0\}$. Let $g : U \setminus Z(f) \to \mathbb{C}$, if g is locally bounded near $Z(f)$ and holomorphic then there is a function $\tilde{g} : U \to \mathbb{C}$ such that $\tilde{g} \mid_{U \setminus Z(f)} = g$ and $\tilde{g}$ is holomorphic.*

With this we can finally move on to $\mathbb{C}^n$-valued function of several variables. We start with

Definition 2.0.3. *Let $f : U \to \mathbb{C}^m$ where $U \subseteq \mathbb{C}^n$ is open. Then f is holomorphic if and only if each component function f_i is holomorphic.*

We can also define the jacobian matrix of a holomorphic mapping.

Definition 2.0.4. *let $f : U \to \mathbb{C}^m$ be holomorphic, then for each $z \in U$ we define the matrix*

$$Df(z) = \left(\frac{\partial f_i}{\partial z_j}(z) \right)_{i,j}$$

Here we take some time to determine how the matrices and tangent maps interact in this setting. Take a function $f : U \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ that is differentiable, then $\mathrm{d}f_z : T_z\mathbb{R}^{2n} \to T_z\mathbb{R}^{2m}$ then with respect to the standard basis fo the tangent spaces($x_1, \ldots, x_n, y_1, \ldots, y_n$ being the coordinates on $\mathbb{R}^{2n}$), the jacobian matrix of this real differentiable function looks liek

$$Df_{\mathbb{R}}(z) = \begin{bmatrix} \left(\frac{\partial u_i}{\partial x_j}(z) \right)_{i,j} & \left(\frac{\partial u_i}{\partial y_j}(z) \right)_{i,j} \\ \left(\frac{\partial v_i}{\partial x_j}(z) \right)_{i,j} & \left(\frac{\partial v_i}{\partial y_j}(z) \right)_{i,j} \end{bmatrix}$$

now we may complexify this map to a map $\mathrm{d}f_{\mathbb{C}}(z) : T_z\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} \to T_z\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{C}$ then in the coordinates $z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n$ the matrix is given by

$$Df_{\mathbb{C}}(z) = \begin{bmatrix} \left(\frac{\partial f_i}{\partial z_j}(z) \right)_{i,j} & \left(\frac{\partial f_i}{\partial \overline{z}_j}(z) \right)_{i,j} \\ \left(\frac{\partial \overline{f}_i}{\partial z_j}(z) \right)_{i,j} & \left(\frac{\partial \overline{f}_i}{\partial \overline{z}_j}(z) \right)_{i,j} \end{bmatrix}$$

and if f is holomorphic, by the Cauchy-Riemann equations, this reduces to

$$Df_{\mathbb{C}}(z) = \begin{bmatrix} Df(z) & 0 \\ 0 & \overline{Df(z)} \end{bmatrix}$$

for later use, one has that $\det Df_{\mathbb{R}}(z) = \det Df(z)$ and that $\det Df_{\mathbb{C}}(z) = |\det Df(z)|^2 \geq 0$. Both Inverse and Implicit function theorems hold.

Theorem 2.0.6. *Let $f : U \to V$ is a bijective holomorphic function between open sets $U, V \subseteq \mathbb{C}^n$. Then for all $z \in U$ we have that $\det Df(z) \neq 0$, moreover f is a biholomorphism.*

Finally we note that Riemann mapping theorem does not hold in general. This is very sad :(But the counter example is that the polydisk and the unit ball are the biholomorphic.

3 Complex Structures

Suppose that V is a real vector space. Then we can complexify V to a vector space $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. Now we make $V_{\mathbb{C}}$ a complex vector space where the addition is defined normally and the scalar multiplication is defined as

$$\lambda(v \otimes \alpha) = v \otimes (\lambda \alpha)$$

Given this we also define complex conjugation as

$$\overline{v \otimes \lambda} = v \otimes \overline{\lambda}$$

extended linearly. Given a linear map $T : V \to W$ we can define the $\mathbb{C}$-linear extension of T onto the complexification hence we define $T_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ as

$$T_{\mathbb{C}}(v \otimes \lambda) = T(v) \otimes \lambda$$

We will omit the tensor product symbol in the complexfication. Now we define an almost complex structure on V.

Definition 3.0.1 (Almost Complex Structure)**.** *Suppose V is a real vector space, then an isomorphism $I : V \to V$ so that $I^2 = -id_V$ is called a almost complex structure on I.*

Now let us take the complexification of V, $V_{\mathbb{C}}$. Then the map $v \otimes 1 \to v \otimes i$ is an almost complex structure. We can work backwards as well, so suppose that we have an almost complex structure I on V then we can make V into a complex vector space by defining

$$(a + bi) \cdot v = av + bI(v)$$

If we are given an almost complex structure on V we will call the above construction the complexification, $V_{\mathbb{C}}$. Since $\mathbb{C}^n$ is canonically oriented, we get that the complex vector space given by the complex structure I is also canonically oriented. The minimal polynomial of I is $x^2 + 1$ hence the only roots of this are $\pm i$. So we can ask whether V can be decomposed into direct sums of eigenspaces. As it turns out, that's exactly what happens

Theorem 3.0.1. *Suppose V is a real vector space and I is an almost complex structure on V then we define*

$$V^{1,0} = \{v \in V_{\mathbb{C}} \mid I(v) = iv\}$$

and

$$V^{0,1} = \{v \in V_{\mathbb{C}} \mid I(v) = -iv\}$$

then we have that

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

moreover, conjugation gives an isomorphism $V^{1,0} \cong V^{0,1}$

Given an almost complex structure on V we define a natural almost complex structure on V^* by

$$I(f)(v) = f(I(v))$$

under this complex structure we get that

$$(V^{1,0})^* \cong (V^*)^{1,0}$$

and

$$(V^{0,1})^* \cong (V^*)^{0,1}$$

Now we want to talk about forms on manifolds so we need to talk about the exterior algebra. We denote that by

$$\bigwedge^* V = \bigoplus_{k=0}^{n} \bigwedge^k V$$

we also have that $\bigwedge^* V_{\mathbb{C}} = \bigwedge^* V \otimes_{\mathbb{R}} \mathbb{C}$ which can be seen directly from

$$\begin{aligned} \bigwedge^* V_{\mathbb{C}} &= \bigoplus_{k=0}^{n} \bigwedge^k V_{\mathbb{C}} \\ &= \bigoplus_{k=0}^{n} \bigwedge^k (V \otimes_{\mathbb{R}} \mathbb{C}) \end{aligned}$$

given the decomposition of the $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ we can define

Definition 3.0.2. *Denote by* $\bigwedge^{p,q} V$ *the set*

$$\bigwedge^{p,q} V = \bigwedge^{p} V^{1,0} \otimes_{\mathbb{C}} \bigwedge^{q} V^{0,1}$$

where we take the exterior products as if $V^{1,0}$ *and* $V^{0,1}$ *are complex vector spaces. An element* $\alpha \in \bigwedge^{p,q} V$ *is of bidegree* (p, q).

the important aspect of this is that

Theorem 3.0.2. *For a real vector space* V *endowed with an almost complex structure* I *one has:*

1. $\bigwedge^{p,q} V$ *is in a canonical way a subspace of* $\bigwedge^{p+q} V_{\mathbb{C}}$.
2. $\bigwedge^{k} V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$.
3. *Complex conjugation on* $\bigwedge^{*} V_{\mathbb{C}}^{*}$ *defines a* $\mathbb{C}$*-antilinear isomorphism*
$$\bigwedge^{p,q} V^{*} \cong \bigwedge^{q,p} V^{*}$$
that is
$$\overline{\bigwedge^{p,q} V^{*}} = \bigwedge^{q,p} V^{*}$$
Conjugation is defined to be multiplicative and sesquilinear. That is $\overline{\alpha_1 \wedge \alpha_2} = \overline{\alpha_1} \wedge \overline{\alpha_2}$.
4. *The exterior product extends to the space of bidegree forms and map*
$$\bigwedge^{p,q} V \times \bigwedge^{r,s} V \to \bigwedge^{p+r,s+q} V$$

We have a canonical volume form on V so we have

Definition 3.0.3. *For any* $m \leq \dim_{\mathbb{C}} V^{1,0}$ *we have*

$$(-2i)^m (z_1 \wedge \overline{z}_1) \wedge \cdots \wedge (z_m \wedge \overline{z}_m) = (x_1 \wedge y_1) \wedge \cdots \wedge (x_m \wedge y_m)$$

This defines a volume form on the dual space as well. But we replace $-2i$ with $\frac{i}{2}$ and replace the basis $z_i, \overline{z}_i$ for the corresponding dual. We also define more notation

Definition 3.0.4. *With respect to the decomposition given above we define the natural projection maps as*

$$\Pi^k : \bigwedge^{*} V_{\mathbb{C}} \to \bigwedge^{k} V_{\mathbb{C}}$$

and

$$\Pi^{p,q} \bigwedge^{*} V_{\mathbb{C}} \to \bigwedge^{p,q} V$$

moreover we extend the almost complex structure I *to* $\boldsymbol{I} : \bigwedge^{*} V_{\mathbb{C}} \to \bigwedge^{*} V_{\mathbb{C}}$ *defined by*

$$\boldsymbol{I} = \sum_{p,q} i^{p-q} \Pi^{p,q}$$

Note that $\boldsymbol{I}$ is not an almost complex structure, just an extension of one. We extend Π^k and $\Pi^{p,q}$ and $\boldsymbol{I}$ to $\bigwedge^{*} V_{\mathbb{C}}^{*}$ by defining

$$\boldsymbol{I}(\alpha)(v_1, \ldots, v_k) = \alpha(\boldsymbol{I}(v_1), \ldots, \boldsymbol{I}(v_k))$$

for $\alpha \in \bigwedge^{*} V_{\mathbb{C}}^{*}$ and $v_i \in V_{\mathbb{C}}$. But the most important class of vector spaces we would like to talk about are the ones that have an inner product on them. Just like in the case of $\mathbb{R}^2$ we would like that the inner product of two vectors is unchanged under a multiplication by i, this can be made more rigorous by requiring that the complex structure be *compatible* with our inner product.

Definition 3.0.5 (Compatible Almost Complex Structure)**.** *Suppose* I *is an almost complex structure on* V *then it is said to be compatible with the inner product* g *if and only if* $g(I(v), I(w)) = g(v, w)$.

We also have a canonical form we can define on (V, g, I) given as

Definition 3.0.6 (Fundamental Form). *Given an almost complex inner product space (V, g, I) we define the form*

$$\omega(v, w) = g(I(v), w) = -g(v, I(w))$$

ω is called the fundamental form associated to (V, g, I).

It turns out that $\omega \in \bigwedge^2 V_{\mathbb{C}}^* \cap \bigwedge^{1,1} V_{\mathbb{C}}^*$ if I is compatible with g. Since we deal with the complexification of the vector space a lot we note that the

Theorem 3.0.3. *Given almost complex real inner product space (V, g, I) the form*

$$h(v, w) = g(v, w) - i\omega(v, w)$$

is a positive definite hermitian form on (V, I).

From here we may also take the inner product g and define a $\mathbb{C}$ extension of g defined by

$$g_{\mathbb{C}}(v \otimes \lambda, w \otimes \alpha) = (\lambda\overline{\alpha})g(v, w)$$

One can check that this is positive definite and hermitian on $V_{\mathbb{C}}$. Using this hermitian form on $V_{\mathbb{C}}$ we have the following decomposition

Theorem 3.0.4. *Given a compatible almost complex real inner product space (V, g, I) we have that the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is an orthogonal decomposition with respect to $g_{\mathbb{C}}$.*

Now, along with the Hodge star operator there is another operator that is very important in the study. This is known as the Lefschetz operator, it is defined as

Definition 3.0.7 (Lefschetz Operator). *Let (V, g, I) be a compatible almost complex real inner product space. Let ω be its fundamental form then the Lefschetz operator $L : \bigwedge^* V_{\mathbb{C}}^* \to \bigwedge^* V_{\mathbb{C}}^*$ defined by*

$$L\alpha = \omega \wedge \alpha$$

We note just for later use that if $\omega \in \bigwedge^k V_{\mathbb{C}}^*$ then $L\omega \in \bigwedge^{k+2} V^*$. Now we can define the dual of the Lefschetz operator.

Definition 3.0.8 (Dual Lefschetz Operator). *We will define the dual Lefschetz operator by $\Lambda : \bigwedge^* V_{\mathbb{C}}^* \to \bigwedge^* V_{\mathbb{C}}^*$ that is the adjoint to L defined*

$$h(\Lambda\alpha, \beta) = h(\alpha, L\beta)$$

Now we give the action of Λ in terms of L and $\star$. Before this we need to extend the definition of the Hodge star from the real vector space to its complexification by the almost complex structure. Now let (V, g, I) be an almost complex real inner product. Let $(V_{\mathbb{C}}, g_{\mathbb{C}})$ be the complexification then the same way we extended the inner product g to $\bigwedge^* V^*$ we do the same for $\bigwedge^* V_{\mathbb{C}}^*$. After this we define the Hodge star as

$$\alpha \wedge \star\overline{\beta} = h_{\mathbb{C}}(\alpha, \beta)\,\mathrm{vol}$$

Using this we can now show the following action of Λ.

Theorem 3.0.5. *Now for $\omega \in \bigwedge^k V^*$ we get $\Lambda\omega \in \bigwedge^{k-2} V^*$ and*

$$\Lambda\omega = \star^{-1} L \star \omega$$

There is one last operator that we need to define. So

Definition 3.0.9 (Hodge-Riemannian Pairing). *Let (V, g, I) be a real almost complex inner product space and let ω be the associated fundamental form. Then we define the Hodge-Riemannian Pairing is the bilinear form $Q : \bigwedge^k V^* \times \bigwedge^k V^* \to \mathbb{R}$ defined by*

$$Q(\alpha, \beta) = (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k}$$

where $\bigwedge^{2n} V^* \cong \mathbb{R}$ by the volume form.

4 Differential Forms

Before we talk about complex manifolds and forms over those manifolds it is important to study the motivating example. Those are precisely the open subsets of $\mathbb{R}^n$. So let $U \subseteq \mathbb{C}^n$ be open, then we may regard U as a real $2n$ dimensional manifold with local coordinates given by $z = x_i + iy_i$. This corresponds to the following smooth frame for TU that is

$$\frac{\partial}{\partial x_i} \quad \frac{\partial}{\partial y_i}$$

Now for each $p \in U$ we have that T_pU is a real vector space and admits an almost complex structure. There is a canonical choice for this almost complex structure given by

$$\left.\frac{\partial}{\partial x_i}\right|_p = \left.\frac{\partial}{\partial y_i}\right|_p$$
$$\left.\frac{\partial}{\partial y_i}\right|_p = -\left.\frac{\partial}{\partial x_i}\right|_p$$

We now extend this to each point of U and hence regard $I : TU \to TU$ as a vector bundle isomorphism. Similar to the vector space case we extend this fibrewise almost complex structure to a fibrewise almost complex structure on the cotangent bundle T^*U. Now we allude back to the vector space case again where we had a decomposition of $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. We know that this holds fibrewise: $T_pU_{\mathbb{C}} = T_pU^{1,0} \oplus T_pU^{0,1}$ and using this one might theorize that

Theorem 4.0.1. *Let $U \subseteq \mathbb{C}^n$ be an open subset that is seen as a $2n$ dimensional real manifold then the complexified tangent bundle $T_{\mathbb{C}}U = TU \otimes \mathbb{C}$ decomposes as a direct sum of complex vector bundles as*

$$T_{\mathbb{C}}U = T^{1,0}U \oplus T^{0,1}U$$

such that the complex linear extension of I satisfies

$$I \mid_{T^{1,0}U} = i \cdot id$$

and

$$I \mid_{T^{0,1}U} = -i \cdot id$$

Moreover the bundle $T^{1,0}U$ has a frame given by the vector fields $\dfrac{\partial}{\partial z_i} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_i} - i\dfrac{\partial}{\partial y_i}\right)$ and the bundle $T^{0,1}U$ has a frame given the vector fields $\dfrac{\partial}{\partial \bar{z}_i} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_i} + i\dfrac{\partial}{\partial y_i}\right)$. We also note that $T^*_{\mathbb{C}}U = T^*U \otimes \mathbb{C}$ has a similar decomposition given by

$$T^*_{\mathbb{C}}U = (T^*U)^{1,0} \oplus (T^*U)^{0,1}$$

Moreover the bundle $(T^*U)^{1,0}$ has a coframe given by the covectors $\mathrm{d}z_i = \mathrm{d}x_i + i\mathrm{d}y_i$ and the bundle $(T^*U)^{0,1}$ has a coframe given by the covectors $\mathrm{d}\bar{z}_i = \mathrm{d}x_i - i\mathrm{d}y_i$.

One of the most advantages things about this decomposition is that

Theorem 4.0.2. *Given that $f : U \to V$ is a holomorphic map between open sets in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively then we have that the $\mathbb{C}$-linear extension of the differential $\mathrm{d}f_p : T_pU \to T_{f(p)}V$ has the property that*

$$\mathrm{d}f_p(T^{1,0}_pU) \subseteq \mathrm{d}f_p(T^{1,0}_{f(p)}V)$$

and

$$\mathrm{d}f_p(T^{0,1}_pU) \subseteq \mathrm{d}f_p(T^{0,1}_{f(p)}V)$$

Now we define the space of forms this is again motivated by the definition of the vector space. Note that

Definition 4.0.1. *Let $U \subseteq \mathbb{C}^n$ be an open subset. Over U one defined the complex vector bundles*

$$\bigwedge^{p,q} U = \bigwedge^{p} (T^*U)^{1,0} \otimes \bigwedge^{p} (T^*U)^{0,1}$$

Then we denote by $\Gamma^k_{\mathbb{C}}(U)$ the space of sections of $\bigwedge^k_{\mathbb{C}} U = \bigwedge^k T^*_{\mathbb{C}}U$ and denote by $\Gamma^{p,q}(U)$ the space of sections of $\bigwedge^{p,q} U$

We again have similar decompositions of these spaces as:

$$\bigwedge\nolimits_{\mathbb{C}}^{k} U = \bigoplus_{p+q=k} \bigwedge\nolimits^{p,q} U$$

and

$$\Gamma_{\mathbb{C}}^{k}(U) = \bigoplus_{p+q=k} \Gamma^{p,q}(U)$$

we denote by $\Pi^{p,q}$ the projections of the spaces $\bigwedge_{\mathbb{C}}^{k} U$ and $\Gamma_{\mathbb{C}}^{k}(U)$ onto the respective subspaces in the decomposition above. Now we define the most important operators in all of differential geometry in my opinion. Those are the differential operators on the space of differential forms and in this case we have extra maps as well. We will define them below

Definition 4.0.2. *Let* $\mathrm{d} : \Gamma_{\mathbb{C}}^{k}(U) \to \Gamma_{\mathbb{C}}^{k+1}(U)$ *be the* $\mathbb{C}$ *linear extension of the usual exterior differential. Moreover we define*

$$\partial : \Gamma^{p,q}(U) \to \Gamma^{p+1,q}(U)$$

given by $\partial = \Pi^{p+1,q} \circ \mathrm{d}$ *and*

$$\overline{\partial} : \Gamma^{p,q}(U) \to \Gamma^{p,q+1}(U)$$

given by $\overline{\partial} = \Pi^{p,q+1} \circ \mathrm{d}$ *and*

The action of the above oeprators can be seen in the following example. Take $f : U \to \mathbb{C}$ be any function then in the usual local coordinates one has that

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i + \sum_i \frac{\partial f}{\partial y_i}\,\mathrm{d}y_i = \sum_i \frac{\partial f}{\partial z_i}\,\mathrm{d}z_i + \sum_i \frac{\partial f}{\partial \overline{z}_i}\,\mathrm{d}\overline{z}_i$$

and as the exposition in the complex analysis section happened we get that f is holomorphic if and only if $\overline{\partial} f = 0$. We can also get the local expression for the ∂ and $\overline{\partial}$ operators. We only need to do this for a scaled basis form. Note that

$$\partial(f\,\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z}_{i_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{i_q}) = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}\,\mathrm{d}z_k \wedge \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z}_{i_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{i_q}$$

and

$$\overline{\partial}(f\,\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z}_{i_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{i_q}) = \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z}_k}\,\mathrm{d}\overline{z}_k \wedge \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z}_{i_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{i_q}$$

from this we see something very important. That $\mathrm{d} = \partial + \overline{\partial}$. Moreover the ∂ and $\overline{\partial}$ are derivations. Hence

Theorem 4.0.3. *For the differential operator* ∂ *and* $\overline{\partial}$ *we have*

1. $\mathrm{d} = \partial + \overline{\partial}$
2. $\partial^2 = 0 = \overline{\partial}^2$ *and* $\partial\overline{\partial} = -\overline{\partial}\partial$
3. *We have that*

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \partial\beta$$

and

$$\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^{p+q}\alpha \wedge \overline{\partial}\beta$$

for $\alpha \in \Gamma^{p,q}(U)$ *and* $\beta \in \Gamma^{r,s}(U)$.

Now we ask the question about whether we can ask when a ∂-closed function is ∂-exact and when a $\overline{\partial}$-closed function is $\overline{\partial}$-exact. Turns out a version of the Poincaré Lemma holds in certain spaces.

Theorem 4.0.4 ($\overline{\partial}$-Poincaré lemma in several variables)**.** *Let* U *be an open neighborhood of the closed of a bounded polydisk* $B(0, \epsilon) \subseteq \overline{B(0, \epsilon)} \subseteq U \subseteq \mathbb{C}^n$. *If* $\alpha \in \Gamma^{p,q}(U)$ *is so that* $\overline{\partial}\alpha = 0$ *and* $q > 0$, *then there is a form* $\beta \in \Gamma^{p,q-1}(B(0, \epsilon))$ *with* $\alpha = \overline{\partial}\beta$ *on* $B(0, \epsilon)$.

this also holds for open disks over $\mathbb{C}^n$. Finally we endow U with a riemannian metric g.

Definition 4.0.3. *Let* (U, g) *be Riemannian, then* g *is said to be compatible with the almost complex structure* $I : TU \to TU$ *if for all* $p \in U$ *we have*

$$g_p(v, w) = g_p(I_p(v), I_p(w))$$

for all $v, w \in T_pU$.

We also define the fundamental form

Definition 4.0.4. *Let (U, g) be Riemannian then $\omega \in \Gamma^{1,1}(U) \cap \Gamma^2(U)$ be defined by $\omega_p(v, w) = g_p(I_p(v), w)$ for all $p \in U$ and all $v, w \in T_pU$.*

Just like in the case of vector spaces the form $h_p = g_p - i\omega_p$ defines a smoothly varying hermitian form on U. We now move to the theory of complex manifolds.

5 Introduction to Complex Manifolds

The definition of a complex manifold is not that different than that of real manifolds. So let X be a real manifold with an atlas (U_i, φ_i).

Definition 5.0.1. *The atlas (U_i, φ_i) is a holomorphic atlas if the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is holomorphic, where the sets are identified with the complex counterparts with the standard map into complex space.*

From this we see that

Definition 5.0.2. *A complex manifold X of dimension n is a real differential manifold of dimension $2n$ such that the differentiable atlas is actually a maximal holomorphic atlas.*

Moreover

Definition 5.0.3. *A function $f : X \to \mathbb{C}$ is said to be holomorphic if $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{C}$ is holomorphic for every chart (U_i, φ_i).*

Due to the maximum principle for holomorphic functions we get that if X is compact and $f : X \to \mathbb{C}$ is holomorphic then f is constant on each connected component. Now we define holomorphic functions between manifolds.

Definition 5.0.4. *Let X and Y be two complex manifolds. Then a continuous map $f : X \to Y$ is holomorphic if for any holomorphic charts (U, φ) of X and (V, ψ) of Y the map*

$$\psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \to \psi(V)$$

is holomorphic.

We can also define complex submanifolds.

Definition 5.0.5. *Let X be a complex manifold of complex dimension n and let $Y \subseteq X$ be a differentiable submanifold of real dimension $2k$. Then Y is a complex submanifold if there exists a holomorphic atlas $\{(U_i, \varphi_i)\}$ of X such that $\varphi_i : U_i \cap Y \to \varphi_i(U_i) \cap \mathbb{C}^k$ is a biholomorphism.*

5.1 Holomorphic Vector Bundles

To talk about differential forms on a manifold we require a vector bundle structure on the manifold. In our case we require holomorphic forms hence we might want the vector bundle to also be compatible with this idea. Hence we require Holomorphic vector bundles

Definition 5.1.1. *Let X be a complex manifold. A holomorphic vector bundle of rank r over X is a complex manifold E along with a holomorphic surjection $\pi : E \to X$ such that for each $x \in X$ we have that $\pi^{-1}(x) \subseteq E$ is an r dimensional complex vector space satisfying the following: There is an open cover $\{U_i\}_{i \in I}$ of X along with biholomorphisms $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ such that the map $\psi_i|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \mathbb{C}^r$ is a linear isomorphism. We call the maps*

$$\psi_{ij}(x) = (\psi_i \circ \psi_j^{-1})|_{\pi^{-1}(x)} : \mathbb{C}^r \to \mathbb{C}^r$$

transition maps. Moreover they are linear for all $x \in U_i \cap U_j$.

We note that we don't always need E to be able to define a vector bundle in differential geometry. It is enough to provide an open cover and a set of transition functions for that open cover. We call that data a cocyle. It turns out that this is sufficient in the complex holomorphic case too. The following are holomorphic vector bundles(these are examples of ones we will use): Let X be a complex n manifold(real dimension $2n$)

1. The tangent bundle $\mathcal{T}_X$ and the cotangent bundle $\mathcal{T}^*M$.[1]
2. The exterior powers of $\mathcal{T}_M^*$, $\bigwedge^k \mathcal{T}_M^*$ which is the space of holomorphic k forms.

The holomorphic tangent bundle of X can be constructed by the data $\{(U_i, \varphi_i)\}$ which is a holomorphic atlas of X and the transition maps are the complex jacobian. The rest just follow.

[1]The textbook makes a big deal about how the bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ is apriori just a differentiable bundle. Let we will see that that $\mathcal{T}_X$ is isomorphic to a sub-bundle of $T_{\mathbb{C}}X$

5.2 Almost Complex Manifolds

Similar to the case of vector space we define an almost complex structure but in this case it will be on the manifolds.

Definition 5.2.1. *An almost complex manifold is a differentiable manifold M together with a vector bundle isomorphism $I : TM \to TM$ with $I^2 = -id_{TM}$.*

One would imagine that complex manifolds are canonically almost complex manifolds. This does happen to be the case. Now pretty much all the decompositions that held in the case of vector spaces with the almost complex structures hold here as well.

Theorem 5.2.1. *Let M be an almost complex manifold, let I be the $\mathbb{C}$-linear extension of the almost complex structure. Then*

1. *We have that*
$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$
where $T^{1,0}M$ is the subbundle of TM where $I|_{T^{1,0}M}$ at each point is just multiplication by i map on the $T_x^{1,0}M$ and $T^{0,1}M$ is the subbundle of TM where $I|_{T^{0,1}M}$ at each point is just multiplication by $-i$ map on the $T_x^{0,1}M$.

2. *Moreover, if M is a complex manifold then $\mathcal{T}_M \cong T^{1,0}M$.*

From this decomposition we make the following defintions

Definition 5.2.2. *From the decomposition above, we call the bundle $T^{1,0}M$ the holomorphic tangent bundle and we call the bundle $T^{0,1}M$ the antiholomoprhic tangent bundle.*

Now that we have all this we move on to the next topic.

6 Forms on (Almost) Complex Manifolds

We give the following definitions

Definition 6.0.1. *Let M be an almost complex manifold then*

$$\bigwedge\nolimits_{\mathbb{C}}^{k} M = \bigwedge\nolimits^{k}(T_{\mathbb{C}}M)$$

moreover we denote the sheaf of sections of this space by $\Gamma_{M,\mathbb{C}}^k$ and the space of global sections of this $\Gamma_{M,\mathbb{C}}^k$ by $\Gamma^k(M)$. Now we also have

$$\bigwedge\nolimits^{p,q} M = \bigwedge\nolimits^{p}(T^{1,0}M)^* \otimes_{\mathbb{C}} \bigwedge\nolimits^{q}(T^{0,1}M)^*$$

moreover we denote the sheaf of sections of this space by $\Gamma_M^{p,q}$. We denote the space of global sections of $\Gamma_M^{p,q}$ by $\Gamma^{p,q}(M)$. We call $\alpha \in \Gamma^{p,q}(M)$ a form of type (p,q) and degree $p+q$. Denote by

$$\Gamma^*(M) = \bigoplus_{k=0}^{\infty} \Gamma^k(M)$$

Again we have direct sum decomposition as

Theorem 6.0.1. *We have the following*

$$\bigwedge\nolimits_{\mathbb{C}}^{k} M = \bigoplus_{p+q=k} \bigwedge\nolimits_{\mathbb{C}}^{p,q} M$$

and

$$\Gamma_{M,\mathbb{C}}^k = \bigoplus_{p+q=k} \Gamma_M^{p,q}$$

Now given this decomposition we have projection maps from $\Gamma_{\mathbb{C}}^*(M) \to \Gamma_{\mathbb{C}}^k(M)$ denoted by Π^k and we also have projection maps $\Gamma_{\mathbb{C}}^*(M) \to \Gamma^{p,q}(M)$ by $\Pi^{p,q}$. Then if we let $\mathrm{d} : \Gamma^k(M) \to \Gamma^{k+1}(M)$ be the $\mathbb{C}$-linear extension of the exterior differential. Then we can define the ∂ and $\overline{\partial}$ operator like we did for the open sets of $\mathbb{C}^n$.

Definition 6.0.2. *Define by* $\partial : \Gamma^{p,q}(M) \to \Gamma^{p+1,q}(M)$ *defined by*

$$\partial = \Pi^{p+1,q} \circ \mathrm{d}$$

and define by $\overline{\partial} : \Gamma^{p,q}(M) \to \Gamma^{p,q+1}(M)$ *defined by*

$$\overline{\partial} = \Pi^{p,q+1} \circ \mathrm{d}$$

We now define a special type of almost complex structure on M.

Definition 6.0.3. *Let* (M, I) *be an almost complex manifold. Then* I *is said to be integrable if either*

$$\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$$

for all $\alpha \in \Gamma^*_{\mathbb{C}}(M)$ *or equivalently*

$$\Pi^{0,2} \circ \mathrm{d} = 0$$

on $\Gamma^{0,2}_{\mathbb{C}}(M)$

Now all the things that we mentioned that held for arbitrary open subset of $\mathbb{C}^n$ also hold here. One final thing we do define is a cohomology on an almost complex manifold that is different from the deRham cohomology.

Definition 6.0.4. *Suppose* (M, I) *be an integrable almost complex manifold then we define the* (p, q)*-Dolbeault cohomology is the vector space*

$$H^{p,q}(M) = \frac{\ker(\overline{\partial} : \Gamma^{p,q}(M) \to \Gamma^{p,q+1}(M))}{\ker(\overline{\partial} : \Gamma^{p,q-1}(M) \to \Gamma^{p,q}(M))}$$

We finally have the background to talk about the main items

7 Kähler Manifolds

Let us begin with the first definition

Definition 7.0.1. *Let* (M, g, I) *an almost complex Riemannian manifold then* g *is a hermitian structure on* M *if* $\forall p \in M$ *we have* g_p *is compatible with* I_p*. We also define the* $(1, 1)$ *form* $\omega_p(v, w) = g_p(I_p(v), w)$*.* ω *is called the Fundamental Form. If* M *is a complex manifold then* (M, g) *is called a Hermitian Manifold.*

Once we have this, we are finally ready to state the definition of a Kähler Manifold.

Definition 7.0.2. *Let* (M, g, I) *be a Riemannian manifold with* g *hermitian. Then* (M, g, I) *is a Kähler manifold if and only if the associated fundamental form* ω *is closed. Here* g *is a Kähler metric.*

Now we define the Lefschetz operators the same way we did before

Definition 7.0.3. *The Lefschetz operator is defined as*

$$L : \bigwedge^k M \to \bigwedge^{k+2} M$$

given by $L\alpha = \alpha \wedge \omega$*. Moreover we define the dual to be*

$$\Lambda : \bigwedge^k M \to \bigwedge^{k-2} M$$

by $\Lambda\alpha = \star^{-1} L \star \alpha$*.*

We already defined the Hodge star operator above, that definition works for any real differentiable manifold. We extend all these operators to the complexified versions of the vector bundles and call them all by the same name. Also similar to how we defined the adjoint of the differential we may do the same with the ∂ and $\overline{\partial}$ operators.

Definition 7.0.4. *If* (M, g) *is a hermitian manifold then define*

$$\partial^*\alpha = - \star \overline{\partial} \star \alpha$$

$$\overline{\partial^*}\alpha = - \star \partial \star \alpha$$

and we define the corresponding Laplacians

Definition 7.0.5. *If (M,g) is a hermitian manifold then define*

$$\Delta_\partial = \partial^*\partial + \partial\partial^*$$
$$\Delta_{\overline{\partial}} = \overline{\partial^*}\,\overline{\partial} + \overline{\partial}\,\overline{\partial^*}$$

With this we now introduce three equalities that are very important to the study, so important that they are called the Kähler identities.

Theorem 7.0.1. *Let X be a complex manifold with a Kähler metric. Then the following are true*

1. $[\partial, L] = 0 = [\overline{\partial}, L]$ *and* $[\partial^*, \Lambda] = 0 = [\overline{\partial^*}, \Lambda]$.
2. $[\partial^*, L] = -i\overline{\partial}$, $[\overline{\partial^*}, L] = -i\partial$, $[\Lambda, \overline{\partial}] = -i\partial^*$ *and* $[\Lambda, \partial] = -i\overline{\partial^*}$.
3. $\Delta_\partial = \Delta_{\overline{\partial}} = \dfrac{1}{2}\Delta$. *Moreover*

$$\begin{aligned} 0 &= [\Delta, \partial] \\ &= [\Delta, \overline{\partial}] \\ &= [\Delta, \partial^*] \\ &= [\Delta, \overline{\partial^*}] \\ &= [\Delta, L] \\ &= [\Delta, \Lambda] \end{aligned}$$

We finally have all the tools to talk about Hodge Theory on Kähler Manifolds.

8 Hodge Theory on Kähler Manifolds and The Decomposition Theorems

Just like in the case of real riemannian manifolds, we can define the Hodge inner product on $\Gamma^*_{\mathbb{C}}(M)$ where (M,g) is a compact hermitian manifold with complex extension g_C. So define

$$(\alpha, \beta) = \int_M \alpha \wedge \star\overline{\beta} = \int_M g_{\mathbb{C}}(\alpha, \beta)\,\mathrm{vol}_g$$

Using this inner product we can justify why we use the symbols ∂^* and $\overline{\partial^*}$, this is because they are the adjoints of the corresponding maps ∂ and ∂^*. This can be seen for the case of ∂ as follows: Let $\alpha \in \Gamma^{p-1,q}(M)$ and $\beta \in \Gamma^{p,q}(M)$ then

$$\begin{aligned} (\partial\alpha, \beta) &= \int_M \partial\alpha \wedge \star\overline{\beta} \\ &= \int_M \partial(\alpha \wedge \star\overline{\beta}) - (-1)^{p+q+1} \int_M \alpha \wedge \partial \star \overline{\beta} \end{aligned}$$

This is from integration by parts. Now we note that on the space $\Gamma^{n-1,n}(M)$ the map ∂ and d coincide. So we have

$$\int_M \partial(\alpha \wedge \star\overline{\beta}) = \int_M \mathrm{d}\big(\alpha \wedge \star\overline{\beta}\big) = 0$$

since M has no boundary. Now the last integral note that we use the fact that $\star^2 = (-1)^k$ on $\Gamma^k(M)$ then we note that

$$\begin{aligned} \int_M \alpha \wedge \partial \star \overline{\beta} &= \epsilon \int_M g_{\mathbb{C}}(\alpha, \star\overline{\partial} \star \beta)\,\mathrm{vol}_g \\ &= \epsilon \int_M g_{\mathbb{C}}(\alpha, -\partial^*\beta)\,\mathrm{vol}_g \\ &= -\epsilon(\alpha, \partial^*\beta) \end{aligned}$$

here we have that $\epsilon = (-1)^{2n-(p+q)+1}$ hence we get that $(\partial\alpha, \beta) = (\alpha, \partial^*\beta)$. The whole point of Hodge theory is to be able to talk about harmonic forms and get a sense of how the space of spaces is related to the closed and harmonic forms. In the case of complex manifolds we have a larger number of types of forms and this enriches the theory a great amount. So we define space of harmonic forms for each of the operators ∂ and ∂^*.

Definition 8.0.1. *We denote by*

$$\mathcal{H}^k_\partial(M,g) = \{\alpha \in \Gamma^k_{\mathbb{C}}(M) | \Delta_\partial \alpha = 0\}$$

and

$$\mathcal{H}^k_{\overline{\partial}}(M,g) = \{\alpha \in \Gamma^k_{\mathbb{C}}(M) | \Delta_{\overline{\partial}} \alpha = 0\}$$

and

$$\mathcal{H}^{p,q}_\partial(M,g) = \{\alpha \in \Gamma^{p,q}(M) | \Delta_\partial \alpha = 0\}$$

and

$$\mathcal{H}^{p,q}_{\overline{\partial}}(M,g) = \{\alpha \in \Gamma^{p,q}(M) | \Delta_{\overline{\partial}} \alpha = 0\}$$

Now for any real riemannian manifold we note that we have the Hodge decomposition as

$$\Gamma^k(M) = \mathcal{H}^k(M) \oplus \mathrm{d}\Gamma^{k-1}(M) \oplus \delta\Gamma^{k+1}(M)$$

since hermitian manifolds can also be considered as real riemannian manifolds, we have the same decomposition hold for them. However we have additional classes of forms, namely of type (p,q). So

Theorem 8.0.1. *Let (M,g) be a (compact) hermitian manifold then*

$$\Gamma^{p,q}(M) = \mathcal{H}^{p,q}_\partial(M) \oplus \partial\Gamma^{p-1,q}(M) \oplus \partial^*\Gamma^{p+1,q}(M)$$

and

$$\Gamma^{p,q}(M) = \mathcal{H}^{p,q}_{\overline{\partial}}(M) \oplus \overline{\partial}\Gamma^{p-1,q}(M) \oplus \overline{\partial^*}\Gamma^{p+1,q}(M)$$

which is already over nice. But there is more, take a form α such that $\Delta_\partial \alpha = 0$ then

$$\begin{aligned}
0 &= (\Delta_\partial \alpha, \alpha) \\
&= \int_M \Delta_\partial \alpha \wedge \star\alpha \\
&= \int_M (\partial^*\partial\alpha + \partial\partial^*\alpha) \wedge \star\alpha \\
&= \int_M \partial^*\partial\alpha \wedge \star\alpha + \int_M \partial\partial^*\alpha \wedge \star\alpha \\
&= (\partial^*\partial\alpha, \alpha) + (\partial\partial^*\alpha, \alpha) \\
&= (\partial\alpha, \partial\alpha) + (\partial^*\alpha, \partial^*\alpha) \\
&= \|\partial\alpha\|^2 + \|\partial^*\alpha\|^2
\end{aligned}$$

this means that both $\partial\alpha = 0$ and $\partial^*\alpha = 0$. We have a similar thing for ∂^*. Moreover this is an if and only if condition. So all ∂-harmonic forms are both ∂ and ∂^* closed. Now we mention a very strong lemma that holds only for Kähler manifolds

Theorem 8.0.2. *Let X be a compact Kähler Manifold and let α be of type (p,q) such that $\mathrm{d}\alpha = 0$ then the following are equivalent*

1. *There is a form β such that $\mathrm{d}\beta = \alpha$*
2. *There is a form β such that $\partial\beta = \alpha$*
3. *There is a form β such that $\overline{\partial}\beta = \alpha$*
4. *There is a form β such that $\partial\overline{\partial}\beta = \alpha$*