

## Manifolds (Smooth)

- ↪ Fiber Bundles (Things like tangent bundle, cotangent bundle, tensor bundles etc)
- ↪ Lie Groups
- ↪ Principle  $\mathcal{A}$ -bundle

## (Pseudo) Riemannian manifolds

- ↪ Connections
- ↪ Curvature
- ↪ Torsion

## Symplectic Manifold

- ↪ Hamiltonian Mechanics
- ↪ Fibration \*

## Lagrangian Mechanics

### \* Spinors and Spin Geometry

### \* Poisson Geometry

(real)

A top. space  $(M, \tau)$  is a topological manifold if

- (i) Hausdorff ( $T_2$ )
- (ii) 2<sup>nd</sup> countable
- (iii) For every  $p \in M$  there is an open neighborhood  $U \subset M$  and a homeomorphism  $\varphi: U \rightarrow \tilde{U} \subseteq \mathbb{R}^n$   
The pair  $(U, \varphi)$  is called a chart.

Ex:  $-\mathbb{R}^n$  is a top. manifold

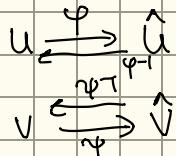
$$-S^n = \{(\alpha_0, \dots, \alpha_n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^n \alpha_i^2 = 1\}$$

$-\mathbb{R}^n$  ↪ matrices in  $\mathbb{R}^n$

- ↪  $U(n, \mathbb{R})$  is a manifold
- ↪  $SL(n, \mathbb{R})$  is a manifold
- ↪  $U(n)$  is a manifold
- ↪  $SU(n)$  is a manifold
- ↪  $O(p, q)$  is a manifold
- ↪  $SO(p, q)$  is a manifold

An atlas for a manifold  $\mathcal{M}$  is the collection of all charts on that manifold.

Given charts  $(U, \varphi)$  and  $(V, \psi)$  the maps  $\varphi \circ \psi^{-1}$  and  $\psi \circ \varphi^{-1}$  are called transition maps



We could have charts that maps us into  $\mathbb{C}^n, X$  where  $X$  is a Banach space.

If  $\varphi: U \rightarrow \hat{U} \subseteq \mathbb{C}^n$  then  $M$  is complex manifold

If  $\varphi: U \rightarrow \hat{U} \subseteq X$  then  $M$  is a Banach manifold

### Smooth Manifolds:

Transition maps are maps from open subsets of  $\mathbb{R}^n$  into open subsets of  $\mathbb{R}^m$ . We can now ask if these are smooth.

If  $(U, \varphi)$  and  $(V, \psi)$  are charts with  $\varphi \circ \psi^{-1}$  and  $\psi \circ \varphi^{-1}$  both smooth. Then the charts are said to be  $C^\infty$ -compatible.

We define a smooth structure on a top. manifold  $M$  to be a maximal  $C^\infty$ -compatible atlas on  $M$ .

$$x \mapsto x^3$$

Ex:  $\mathbb{R}$ . Note that  $\text{id}: \mathbb{R} \rightarrow \mathbb{R}$  and  $f: \mathbb{R} \rightarrow \mathbb{R}$  are homeomorphisms. But they are not  $C^\infty$  compatible.  $f \circ \text{id}^{-1} = f$  but  $f(x) = x^3$  is not smooth at 0.

Note however  $\text{id}: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \setminus \{0\}$  then  $\text{id}$  and  $g$  are  $C^\infty$ -compatible.

A smooth manifold is a top. manifold  $M$  with a smooth structure.

Ex:  $\mathbb{R}^n$  with the chart  $(\mathbb{R}^n, \text{id})$

$\mathbb{R}$  with the chart  $(\mathbb{R}, f)$

$S^n$  is a smooth manifold (determined by stereographic projection)

All the matrix groups above are smooth manifolds with the standard charts.

$\mathbb{C}^n$  is a smooth manifold determined by  $\varphi(z_1, \dots, z_n) = (x_1, y_1, \dots, x_n, y_n)$   
 $z_j = x_j + iy_j$

Let  $M$  be a smooth manifold then  $M$  is said to be  $m$  dimensional if all charts have domain in  $\mathbb{R}^m$ .

NOTE:  $\mathbb{R}^n \cong \mathbb{R}^m \Leftrightarrow n=m$  (Homeology)



Maps in the category of smooth manifolds should "smooth".  
dimensions

Suppose  $f: M^m \xrightarrow{\text{smooth}} N^n$ , this is smooth at  $p \in M$  if there is a chart  $(U, \varphi)$  around  $p$  and a chart  $(V, \psi)$  around  $f(p)$  s.t. the map  $\psi \circ f \circ \varphi^{-1}$  is smooth.

$f$  is smooth if  $f$  is smooth at all points  $p \in M$ .

charts around  $p$       charts around  $f(p)$   
/ \                    / \

Note: If  $f$  is smooth then choice of chart does not matter:  $(U, \varphi)$ ,  $(U', \varphi')$  and  $(V, \psi)$ ,  $(V', \psi')$  we have  $\psi \circ f \circ \varphi^{-1}$  is smooth,  $\psi' \circ \psi^{-1}$  and  $\varphi \circ \varphi'^{-1}$  are smooth then note that the map  $(\psi' \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ \varphi'^{-1})$  is smooth.

$$\psi^r \circ f \circ \varphi^{-1}$$

Projective space: let  $V$  be a vector space. Then we define an equivalence relation  $\sim$  on  $V$  as follows: we say  $v \sim u$  if there is a  $\lambda \in \mathbb{F}^*$  s.t.  $v = \lambda u$ . Then  $PV = \{[v] \mid v \in V\}$

We show a vector space  $V$  is a manifold. So a chart here will be a choice of basis. This is a chart since a choice of basis gives a linear map  $T: V \rightarrow \mathbb{R}^n$  by map  $v \mapsto (a_1, \dots, a_n)$  s.t.  $a_1v_1 + \dots + a_nv_n = v$  ( $v_1, \dots, v_n$  is the choice of basis)