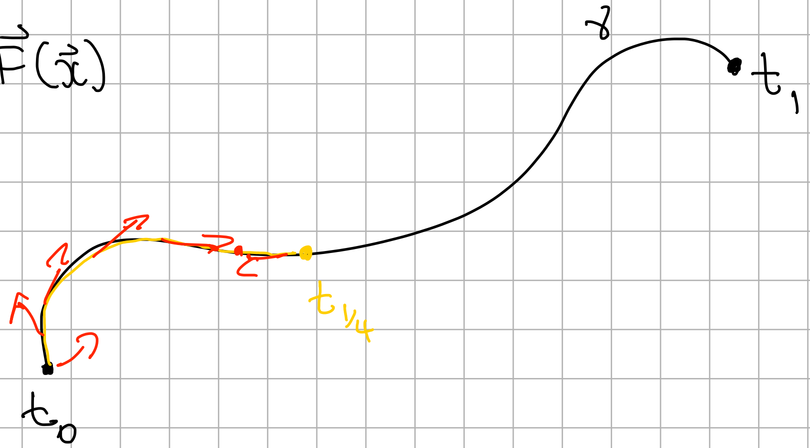
Integration in multiple dimensions:

1) It generalizes integrals

2) Work done by Force

let $\vec{F}(\vec{x})$



move just a little bit $\gamma(t) \to \gamma(t+\delta t) \approx \gamma(t) + \gamma'(t)\delta t$

$$\vec{F}(\gamma(t)) \cdot \delta\gamma(t) \approx \vec{F}(\gamma(t)) \cdot \gamma'(t)\,\delta t$$

$$W = \int_{t_0}^{t_1} \vec{F}(\gamma(t)) \cdot \gamma'(t)\,dt = \int_C \vec{F} \cdot d\vec{l}$$ $\leftarrow$ line integral of a vector field along C

line integral of $f: \mathbb{R}^n \to \mathbb{R}$ along a curve C with respect to length

$$W = \int_{t_0}^{t_1} f(\gamma(t))\,|\gamma'(t)|\,dt$$

Example: $f(\vec{x}) = 1$ along a circle, $\gamma(t) = (r\cos(t), r\sin(t))$

$$C = \int_0^{2\pi} 1 \cdot |\gamma'(t)|\,dt$$

$\gamma'(t) = (-r\sin(t), r\cos(t))$
$|\gamma'(t)| = r$

$$= \int_0^{2\pi} r\,dt$$

$$= 2\pi r$$

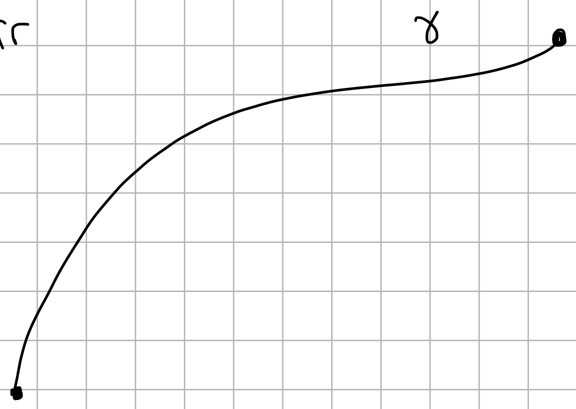
$\gamma(t) = (r\cos(2\pi t), r\sin(2\pi t)) \qquad t \in [0,1]$

$\gamma'(t) = 2\pi r(-\sin t, \cos t)$

$$C = \int_0^1 1 \cdot |\gamma'(t)|\, dt$$

$$= \int_0^1 2\pi r\, dt$$

$$= 2\pi r$$



$\gamma(t) = \gamma^i(t) e_i \quad$ where $\{e_i\}$ is standard basis of $\mathbb{R}^n$

$$\delta s^2 = (\delta\gamma^1)^2 + \dots + (\delta\gamma^n)^2$$

$$\delta s = \sqrt{(\delta\gamma^1)^2 + \dots + (\delta\gamma^n)^2}$$

$$ds = \sqrt{((\gamma^1)')^2 + \dots + ((\gamma^n)')^2}\, dt = |\gamma'(t)|\, dt$$

$$L = \int_C ds = \int_{t_i}^{t_f} |\gamma'(t)|\, dt$$

$f: \mathbb{R}^n \to \mathbb{R}$ what is $\int_C f\, dx^i := \int_{t_i}^{t_f} f(\gamma(t))(\gamma^i)'(t)\, dt$

Exercise: check this is independent of choice of parametrization

Note is this integral cares about orientation but the arclength integral didn't.

$\vec{F} = f^i e_i$ and $\gamma = \gamma^i e_i \Rightarrow \gamma' = (\gamma^i)' e_i$

$\vec{F} \cdot \gamma' = f_i (\gamma^i)'$

over curves we are integrating objects that look like

$f_1 dx^1 + f_2 dx^2 + \cdots + f_n dx^n \leftarrow$ differential forms

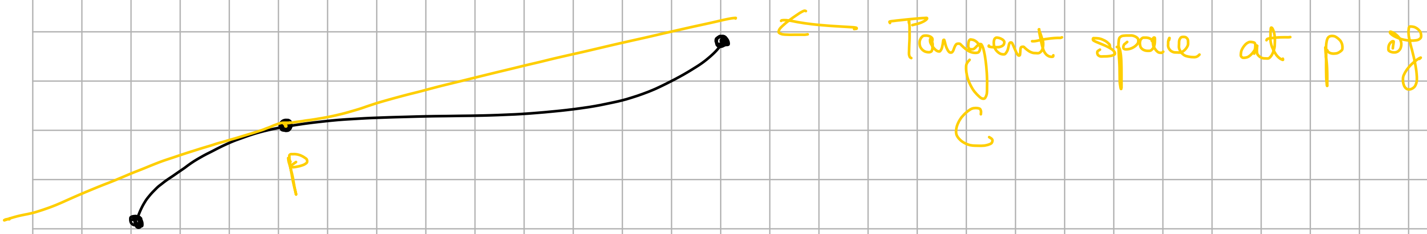
$$\int_C f_i \, dx^i = \int_{t_{init}}^{t_{final}} f_i(\gamma(t)) (\gamma^i)'(t) \, dt$$

integral over the curve

integral over $\mathbb{R}$

$$\gamma^*(f_i \, dx^i) = f_i(\gamma(t)) \, d\gamma^i = f_i(\gamma(t)) (\gamma^i)'(t) \, dt$$

taking the pull back



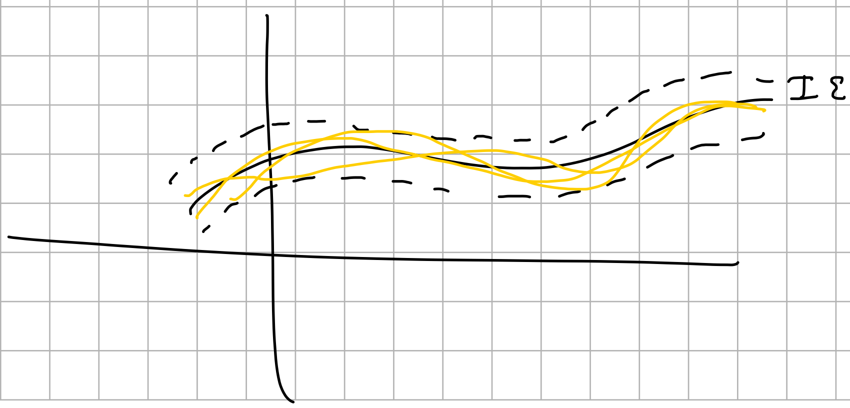
Collection of all tangent spaces will be called the tangent bundle

Integration Theory

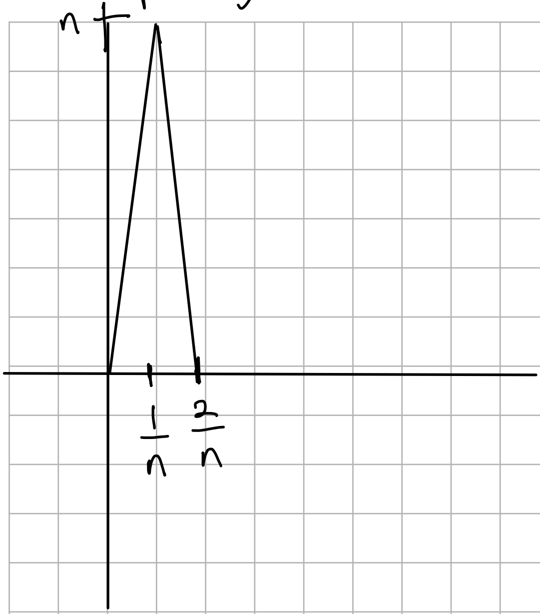
$f_1, \ldots, f_n, \ldots$

$$f(x) = \lim_{n \to \infty} f_n(x)$$

$$\sup |f_n - f| < \varepsilon$$



Example 1)



$$f(x) = \begin{cases} n^2 x & ; x \in [0, \frac{1}{n}] \\ -n^2 x + 2n & ; x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & ; \text{else} \end{cases}$$

$$\int_0^1 f_n(x)dx = \frac{1}{2}\left(\frac{2}{n}\right)n = 1$$

pointwise limit of $f_n \to 0$

$$\lim_{n\to\infty} \int_0^1 f_n \, dx \neq \int_0^1 \lim_{n\to\infty} f_n \, dx$$

Example 2)

$$f_{jK}(x) = \left(\cos(j!\pi x)\right)^{2K} \quad \text{[Henri lebesgue)}$$

$$\lim_{j\to\infty} \lim_{k\to\infty} f_{jk} = \chi_{\mathbb{Q}} = \begin{cases} 1 & ; x \in \mathbb{Q} \\ 0 & ; \text{else} \end{cases}$$

$\chi_{\mathbb{Q}}$ is not riemann integrable. To see this $\int_0^1 \chi_{\mathbb{Q}} \, dx$

$$L(\chi_{\mathbb{Q}}, P) = \sum_i \left(\inf_{P_i} \chi_{\mathbb{Q}}\right) \ell(P_i) = 0$$

$$U(\chi_{\mathbb{Q}}, P) = \sum_i \left(\sup_{P_i} \chi_{\mathbb{Q}}\right) \ell(P_i) = 1$$

What does it mean to measure a set in $\mathbb{R}^n$?

$\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$

1) $\mu([0,1]^n) = 1$

2) $\mu(A + t) = \mu(A)$ $\quad A \subseteq \mathbb{R}^n$ and $A + t = \{a + t \mid a \in A\}$

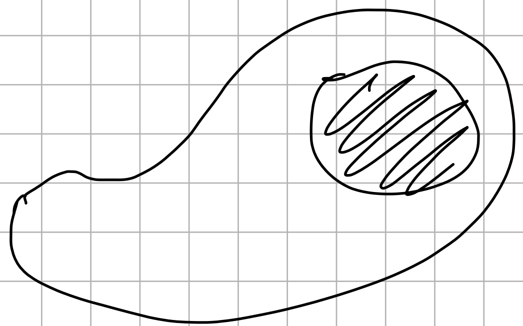
3) $\mu\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

Unfortunately no such function exists.*

Banach Tarski Paradox

This forces us to restrict the collection of sets we can measure.

I should be able to measure $\emptyset, X$



X

If I know the volume of $E \subseteq X \Rightarrow$ I know the volume of $X \setminus E$.

If I know the volumes of $E_i \subseteq X \Rightarrow$ I know the volume of $\bigcup_{i=1}^{\infty} E_i$

let X be a set, $\Sigma \subseteq P(X)$ is called a σ-algebra on X if

(i) $\emptyset, X \in \Sigma$

(ii) $E \subseteq X$ with $E \in \Sigma \Rightarrow X \setminus E \in \Sigma$

(iii) $E_1, \ldots, E_n, \ldots \subseteq X$ with $E_i \in \Sigma \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \Sigma$

De Morgan's laws to show that $\bigcap_{i=1}^{\infty} E_i \in \Sigma$

* $E \in \Sigma$ is called measurable and we call (X, Σ) a measure space

Exercise: If Σ_α are σ-algebras on X then $\bigcap_\alpha \Sigma_\alpha$ is also a σ-algebra

Take any collection $\mathcal{A}$ of subsets of X then define the σ-algebra generated by $\mathcal{A}$ to be $\Sigma_{\mathcal{A}} = \bigcap_{\substack{\beta \subseteq P(X) \\ \beta \text{ is a } \sigma\text{-algebra}}} \beta$. Note $\Sigma_{\mathcal{A}}$ is the

$\mathcal{A} \subseteq \mathcal{B}$

Smallest σ-algebra containing $\mathcal{A}$.

So let $(X, \mathcal{T})$ be a top. space so the σ-algebra $\Sigma_{\mathcal{T}}$ is called the Borel σ-algebra

Can we describe the Borel σ-algebra of $(\mathbb{R}^n, \mathcal{T}_{std})$?

I don't know.

In $\mathbb{R}$ we will show $\{x\}$ is a Borel set. Consider

$$\bigcap_{n=1}^{\infty} \left[x - \frac{1}{n}, x + \frac{1}{n}\right] = \{x\}$$

A measure on (X, Σ) is a function $\mu : \Sigma \to [0, \infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) if $E_1, \ldots, E_n, \ldots$ are in Σ and disjoint then $\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$

(X, Σ, μ) is called a measure space

$\Rightarrow$ (X, Σ) admits at least 1 measure: let $x \in X$ now define $\mu(E) = \begin{cases} 1 & ; x \in E \\ 0 & ; \text{else} \end{cases}$ this is called the Dirac measure

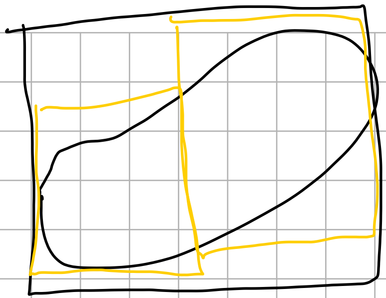
$\Rightarrow$ (X, Σ, μ) and $\mu(X) = 1$ this is called a probability space and $E \in \Sigma$ are called events.

How do we define a measure on $\mathbb{R}^n$?

$$\mu^*((a_1, b_1) \times \cdots \times (a_n, b_n)) = (b_1 - a_1) \cdots (b_n - a_n)$$

$$\mu^*(A) = \inf \left\{ \sum_{k=0}^{\infty} \mu(I_k) \;\middle|\; I_k \text{ are open boxes and } A \subseteq \bigcup_{i=1}^{\infty} I_k \right\}$$

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i) \longleftarrow$$ if μ satisfies this it is called a premeasure



If we restrict to the Borel σ-algebra of $\mathbb{R}^n$ then μ^* is a measure.

Lebesgue measure

We may find Borel sets of measure 0 whose subsets are not measurable. So we complete the Borel σ-algebra of $\mathbb{R}^n$: Add in all these subsets and find the σ-algebra generated by this new collection (Borel + subsets of Borel measure 0)

This new σ-algebra is called the Lebesgue σ-algebra extend μ^* to this.

Every countable set has measure 0:

countable set

let $a_1, \dots, a_n, \dots$ be enumeration of $A \subseteq \mathbb{R}^n$

consider $I_j = \left[a_j - \frac{1}{2^{j-1}}, a_j + \frac{1}{2^{j-1}}\right]^n$

$$\mu(I_j) = \left[\left(a_j + \frac{1}{2^{j-1}}\right) - \left(a_j - \frac{1}{2^{j-1}}\right)\right]^n = \left(\frac{2}{2^{j-1}}\right)^n = \left(\frac{1}{2^j}\right)^n$$

$\bigcup_{j=1}^{\infty} I_{jn} \supseteq A$ note

$$\mu\left(\bigcup_{j=1}^{\infty} I_{jn}\right) \leq \sum_{j=1}^{\infty} \mu(I_{jn})$$

$$= \sum_{j=1}^{\infty} \left(\frac{1}{2^j}\right)^n$$

$$= \sum_{j=1}^{\infty} \left(\frac{1}{2^n}\right)^j$$

$$= \frac{\frac{1}{2^n}}{1 - \frac{1}{2^n}}$$

$$= \frac{1}{2^n}\left(\frac{2^n}{2^n - 1}\right) = \frac{1}{2^n - 1} \xrightarrow{n \to \infty} 0$$

$\mu(A) = 0.$