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Question 1

P	Q	R	(P V Q)	~(Q ∧ R)	W
0	0	0	0	1	1
0	0	1	0	1	1
0	1	0	1	1	1
0	1	1	1	0	0
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	0	0

Hence for

$$(P, Q, R) \in \{(0,1,1),(1,1,1)\}$$

Then

$$(P \wedge Q) \not\Rightarrow (Q \wedge R)$$

And W is false

Question 2

The Formula

$$\sim P \vee \sim (Q \vee R)$$

Is logically equivalent to W, below is a semantic proof by truth table

#	P	Q	R	~P	~(Q V R)	~P V ~(Q V R)	W
1	0	0	0	1	1	1	1
2	0	0	1	1	0	1	1
3	0	1	0	1	0	1	1
4	0	1	1	1	0	1	1
5	1	0	0	0	1	1	1
6	1	0	1	0	0	0	0
7	1	1	0	0	0	0	0
8	1	1	1	0	0	0	0

Proof W needs more than 1 binary connective
Assume W could be constructed with either 0 or 1 binary connectives.

If there are 0 connectives, then W must be either an atomic variable or the negation of an atomic variable. However, for each of P, Q and R it is clear that W does not rely on a single one alone as there are cases where either a change in the variable doesn't affect W or a change in W doesn't affect the variable.

- Considering lines 5 and 6: P is constant at 1 but the value of W changes
- Considering lines 2 and 3: W is constant at 1 but the value of Q changes
- Considering lines 1 and 2: W is constant at 1 but the value of R changes

Thus W cannot be made represented by a formula with no connectives.

If there is one connective, then at most two of P, Q and R could be used.
If a single variable is used on either side of the connective, WLOG assume that variable is P.
Then the possible options for a formula with a single connective are:

- $P \wedge P = P$
- $P \vee P = P$
- $P \implies P = \top$
- $P \iff P = \top$
- $P \wedge \sim P = \bot$
- $P \vee \sim P = \top$
- $P \implies \sim P = \sim P$
- $\sim P \implies P = P$
- $P \iff \sim P = \bot$
  As these are all either true, false or a single variable W cannot be represented using a single connective and a single variable.

Now, assume two variables are used, there are three combinations possible:

- Case (P, Q): In lines 5 and 6, (P, Q) = (1, 0) is constant but the value of W changes
- Case (P, R): In lines 5 and 7, (P, R) = (1, 0) is constant but the value of W changes
- Case (Q, R): in lines 3 and 7, (Q, R) = (1, 0) is constant but the value of W changes
  Thus in each case there is a contradiction, meaning W cannot be represented using a single connective between two of P, Q and R.

Thus, W cannot be represented with 0 or 1 connectives and therefore must contain at least two binary connectives.

Question 3

A) $(P \vee Q) \implies R \vdash (P \implies R) \wedge (Q \implies R)$

#	Assumptions	Formula	Rule
1	1	$(P \vee Q) \implies R$	A
2	2	P	A
3	2	$P \vee Q$	2 $\vee I$
4	1, 2	R	1, 2 MP
5	1	$P \implies R$	2, 4 CP
6	6	Q	A
7	6	P V Q	6 $\vee I$
8	1, 6	R	1, 6 MP
9	1	$Q \implies R$	6, 8 CP
10	1	$(P \implies R) \wedge (Q \implies R)$	5, 9 $\wedge I$

B) $P \implies Q, \ \sim(\sim R \wedge S) \vdash (Q \implies S) \implies (P \implies R)$

#	Assumptions	Formula	Rule
1	1	$P \implies Q$	A
2	2	$\sim(\sim R \wedge S)$	A
3	3	$Q \implies S$	A
4	4	p	A
5	1, 4	Q	1,4 MP
6	1, 3, 4	S	3, 5 MP
7	7	~R	A
8	1, 3, 4, 7	~R ∧ S	6, 7 $\wedge I$
9	1, 2, 3, 4, 7	$\sim(\sim R \wedge S) \wedge (\sim R \wedge S)$	2, 8 $\wedge I$
10	1, 2, 3, 4	R	7, 9 RAA
11	1, 2, 3	P $\implies$ R	4, 10 CP
12	1, 2	$(Q \implies S) \implies (P \implies R)$	3, 11 CP

C) $P \iff Q \vdash (P \vee Q) \iff (P \wedge Q)$

#	Assumptions	Formula	Rule
0	0	$P \iff Q$	A
1	0	$(P \implies Q) \wedge (Q \implies P)$	1 def
2	2	$P \wedge Q$	A
3	2	P	2 $\wedge E$
4	2	$P \vee Q$	3 $\vee E$
5		$(P \wedge Q) \implies (P \vee Q)$	2, 4 CP
6	6	$P \vee Q$	A
7	7	Q	A
8	8	P	A
9	0	$P \implies Q$	1 $\wedge E$
10	0, 6	Q	8, 9, 7, 7, 6 $\vee E$
11	0	$Q \implies P$	1 $\wedge E$
12	0, 6	P	8, 8, 7, 11, 6 $\vee E$
13	0, 6	$P \wedge Q$	10, 12 $\wedge I$
14	0	$(P \vee Q) \implies (P \wedge Q)$	6, 13 CP
15	0	$(P \implies Q) \wedge (Q \implies P) \vdash ((P \vee Q) \implies (P \wedge Q)) \wedge ((P \wedge Q) \implies (P \vee Q))$	5, 14 $\wedge I$
16	0	$P \iff Q \vdash (P \vee Q) \iff (P \wedge Q)$	15 def

Question 4

A) $(P \wedge (Q \vee \sim P)) \implies Q$

P	Q	$Q \vee \sim P$	$(P \wedge (Q \vee \sim P))$	$(P \wedge (Q \vee \sim P)) \implies Q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

#	Assumptions	Formula	Rule
1	1	$(P \wedge (Q \vee \sim P))$	A
2	1	P	1 $\wedge E$
3	1	$Q \vee \sim P$	1 $\wedge E$
4	4	Q	A
5	5	~P	A
6	6	~Q	A
7	5, 6	$\sim P \wedge \sim Q$	5, 6 $\wedge I$
8	5, 6	~P	7 $\wedge E$
9	1, 5, 6	$P \wedge \sim P$	2, 8 $\wedge I$
10	1, 5	Q	6, 10 RAA
11	1	Q	4, 4, 5, 10, 3
12		$(P \wedge (Q \vee \sim P)) \implies Q$	1, 11 CP

Thus the formula is always true and hence a theorem

B) $(Q \wedge (Q \vee \sim P)) \implies P$

P	Q	$Q \vee \sim P$	$Q \wedge (Q \vee \sim P)$	$(Q \wedge (Q \vee \sim P)) \implies P$
0	0	1	0	1
0	1	1	1	0

P	Q	$Q \vee \sim P$	$Q \wedge (Q \vee \sim P)$	$(Q \wedge (Q \vee \sim P)) \implies P$
1	0	0	0	1
1	1	1	1	1

Thus for P = 0 and Q = 1 the formula is false

Question 5

$W_1 = \ \sim P_1$ and $W_{k+1} = \sim (\sim W_k \vee P_{k+1})$

Proving by mathematical induction:

$$W_n = 1 \iff (\forall i \in [1, n])\ P_i = 0$$

For n = 1:
$W_1 = \sim P_1$
$\implies (W_1 = 1 \iff P_1 = 0)$
Thus the statement is true for n = 1

For n = k assume the statement is true

Then, for n = k + 1:
$W_{k+1} = \sim (\sim W_k \vee P_{k+1})$
$= W_k \wedge (\sim P_{k+1})$ (by De Morgan's rule)

Now, from the assumption:
$W_k = 1 \iff (\forall i \in [1, n])\ P_i = 0$

Thus, if $W_{k+1} = 1$, then $W_k = 1$ and $\sim P_{k+1} = 0$
$\implies P_{k+1} = 0$
$\implies \forall i \in [0, k+1] P_i = 0$

But if $W_{k+1} = 0$ then one or both of $W_k = 0$ or $\sim P_{k+1} = 0$ must be true
$W_k = 0 \implies \exists i \in [0, k] : P_i = 1$ (by the previous assumption)
Additionally, $\sim P_{k+1} = 0 \implies P_{k+1} = 1$
Thus, $(W_k + 1 = 0) \implies \exists i \in [0, k+1]\ :\ P_i = 1$
Thus, by contrapositive, $(\forall i \in [0, k+1]) P_i = 0 \implies (W_k + 1 = 1)$

Hence:
$(\forall i \in [0, k+1]) P_i = 0 \iff (W_k + 1 = 1)$

Thus if true for n = k, the statement is true for n = k + 1
As it is true for n = 1, it is true for all n

Question 6

Proposition:
All WFF are equivalent to one only using implication and negation

The proof will use the following theorems:
De Morgan's laws:

$$\sim (P \wedge Q) \equiv \sim P \vee \sim Q \ and\ \sim (P \vee Q) \equiv \sim P \wedge \sim Q$$

Implication to disjunction:

$$P \implies Q \equiv \sim P \vee Q$$

Definition of bi-implication:

$$P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$$

The proof of the proposition by natural deduction is as follows

Base case:
Any atomic variable has no connectives and thus satisfies the proposition

Now, assume P and Q are WFF containing only negations and implications or can be expressed as logically equivalent formula that only use those connectives.

By definition, if P and Q are well formed formula each of the following cases are well formed formula:

1. ~P
2. ~Q
3. $P \wedge Q$
4. $P \vee Q$
5. $P \implies Q$
6. $P \iff Q$

Thus, it suffices to prove each of the following have equivalent formulas using only negation and implication.

Cases 1, 2 and 5 already satisfy the proposition.

Case 3:
$P \wedge Q$
= $\sim\sim(P \wedge Q)$
= $\sim(\sim P \vee \sim Q)$ (by De Morgan's law)
= $\sim(P \implies \sim Q)$
Thus $P \wedge Q$ can be expressed with only negation and implication

Case 4:
$P \vee Q$
= $\sim\sim P \vee Q$
= $\sim P \implies Q$

Case 6:
$P \iff Q$
= $(P \implies Q) \wedge (Q \implies P)$
= $\sim\sim(P \implies Q) \wedge (Q \implies P)$
= $\sim(\sim(P \implies Q) \vee \sim(Q \implies P))$
= $\sim((P \implies Q) \implies \sim(Q \implies P))$

Truth tables:

P	Q	$P \wedge Q$	$\sim(P \implies \sim Q)$	$P \vee Q$	$\sim P \implies Q$	$P \iff Q$	$\sim((P \implies Q) \implies \sim(Q \implies P))$
0	0	0	0	0	0	1	1
0	1	0	0	1	1	0	0
1	0	0	0	1	1	0	0
1	1	1	1	1	1	1	1

Thus, if P and Q are well formed formula that either satisfy the proposition or can be expressed using logically equivalent formula that use only negation and implication then so too can any other WFF that can be made from P and Q as any connective used will have an equivalent WFF that satisfies the proposition.

All WFF are defined recursively from atomic variables and the connectives above that can each be converted to an equivalent formula that satisfies the proposition. As All atomic variables satisfy the proposition so too does any WFF formula.

Hence the proposition is true by natural deduction

Question 7

A)

1. 112 = 2(39) + 34 $\implies$ 112 - 2(39) = 34
2. 39 = 34 + 5 $\implies$ 39 - 34 = 5
3. 34 = 6(5) + 4 $\implies$ 34 - 6(5) = 4
4. 5 = 4 + 1 $\implies$ 5 - 4 = 1

RHS = 1
= 5 - 4 (by 5)
= 5 - (34 - 6(5)) (by 4)
= 7(5) - 34
= 7(39 - 34) - 34 (by 2)
= 7(39) - 8(34)

= 7(39) - 8(112 - 2(39)) (by 1)
=7(39) - 8(112) - 16(39)
=-8(112) + 23(39)
=LHS

Thus α = -8 and β = 23

B)
Considering $\mathbb{Z}_{112}$:
1 = -8(112) + 23(39) = 23(39)
$\implies$ 23 is the multiplicative inverse of 39 in $\mathbb{Z}_{112}$

Considering $\mathbb{Z}_{39}$:
1 = -8(112) + 23(39)
= 31(112)
= 31(2(39) + 34)
= 31(34)
$\implies$ 31 is the multiplicative inverse of 34 in $\mathbb{Z}_{39}$