Question 1

Let $det(F_n) = (a+ib) \in \mathbb{C}$

It is known $F_n\overline{F_n} = nI$

Thus:

$$\begin{aligned} det(nI) = n &= det(F_n\overline{F_n}) \\ &= det(F_n)\ det(\overline{F_n}) \\ &= det(F_n)\ \overline{det(F_n)} \\ &= (a+ib)(a-ib) \\ &= a^2+b^2-2abi \end{aligned}$$

$$\begin{aligned} &Thus: \\ &\implies a^2+b^2 = n\ and\ -2abi = 0 \\ &\implies ab = 0 \\ &\implies a = 0\ or\ b = 0 \\ &\implies a = \pm\sqrt{n} \text{ if b=0 or } b = \pm\sqrt{n} \text{ if a=0} \\ &\implies det(F_n) = \pm\sqrt{n}\ or\ \pm\sqrt{n}i \end{aligned}$$

$$As\ |det(F_n)| = \sqrt{a^2+b^2}:$$
$$|det(F_n)| = \sqrt{(\pm\sqrt{n})^2} = \sqrt{n}$$

Question 2

Part a

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

Part b

Consider:

$$\begin{aligned} &\frac{du}{dt} = f(t,u) = \lambda u \\ &\implies u_{k+1} = u_k + h\lambda(u_k + \frac{h}{2}f(t_k,u_k)) \\ &\qquad = u_k + h\lambda(u_k + \frac{h}{2}\lambda u_k) \\ &\qquad = u_k + h\lambda u_k + \frac{(h\lambda)^2}{2}u_k \\ &\qquad = \rho(h\lambda)u_k \\ &where\ \rho(z) = (1+z+\frac{z^2}{2}) \text{ is the amplification factor} \end{aligned}$$

Part c

For $u_k \longrightarrow 0$: $|\rho(h\lambda)| < 1$
That is:

$$\begin{aligned} &|1+h\lambda+\frac{(h\lambda)^2}{2}| < 1 \\ \implies &1+h\lambda+\frac{(h\lambda)^2}{2} < 1 \\ \implies &h\lambda+\frac{(h\lambda)^2}{2} < 0 \\ \implies &h\lambda(1+\frac{1}{2}h\lambda) < 0 \end{aligned}$$

Now:

$$
\begin{aligned}
f(t,u) = -5u \implies \ & \lambda = -5 \\
& \implies 0 > -5h(1-\frac{5}{2}h) \\
& \implies 0 > h(\frac{5}{2}h-1) \\
& \implies h \in (0, \frac{2}{5})
\end{aligned}
$$

Thus the smallest value of h such that u_k does not converge to 0 is $h = \frac{2}{5} \implies N = 4$

Question 3

Part a

$$
\begin{aligned}
& x'(t) = -sin(t) && y'(t) = cos(t) \\
& \quad \implies x(t) = cos(t) + c_x && \quad \implies y(t) = sin(t) + c_y \\
& && \\
& x(0) = 1 && y(0) = 0 \\
& \quad \implies cos(0) + c_x = 1 && \quad \implies sin(0) + c_y = 0 \\
& \quad \implies c_x = 0 && \quad \implies c_y = 0 \\
& \quad \implies x(t) = cos(t) && \quad \implies y(t) = sin(t)
\end{aligned}
$$

$$
\begin{aligned}
& Thus: \\
& \qquad P(t) = \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix}
\end{aligned}
$$

Part b

$$
\begin{aligned}
& E(t) := x(t)^2 + y(t)^2 \\
& \quad \implies E(t) = sin^2(t) + cos^t(t) \\
& \quad \implies E(t) = 1 \ \forall t
\end{aligned}
$$

Part c

We have the following:

1. $u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix}$
2. $u_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$
3. $u_{k+1} = u_k + hf(t_k, u_k) = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + h \begin{bmatrix} -sin(t_k) \\ cos(t_k) \end{bmatrix} = \begin{bmatrix} x_k - hsin(t_k) \\ y_k + hcos(t_k) \end{bmatrix}$
4. $E_k = x_k^2 + y_k^2$
5. $E_{k+1} = (x_k - hsin(t_k))^2 + (y_k + hcos(t_k))^2$

Now

$$
\begin{aligned}
\|u_{k+1}\|_2^2 &= \sqrt{(x_k - hsin(t_k))^2 + (y_k + hcos(t_k))^2}^2 \\
&= E_{k+1} \\
&= |(x_k - hsin(t_k))^2 + (y_k + hcos(t_k))^2| \\
&= |x_k^2 - 2hsin(t_k)x_k + h^2 sin(t_k)^2 + y_k^2 + 2hcos(t_k)y_k + h^2 cos(t_k)^2| \\
&= |x_k^2 + y_k^2 + h^2(sin^2(t_k) + cos^2(t_k)) + 2h(cos(t_k)y_k - sin(t_k)x_k)| \\
&= |E_k + h^2 + 2h(cos(t_k)y_k - sin(t_k)x_k)| \\
&\leq E_k + h^2 + 2h|cos(t_k)y_k - sin(t_k)x_k| \\
&\leq E_k + h^2 + 2h(|cos(t_k)y_k| + |sin(t_k)x_k|) \\
&\leq E_k + h^2 + 2h(|y_k| + |x_k|) \\
&= E_k + h^2 + 2h\|u_k\|_1 \\
&\leq E_k + h^2 + 2h\sqrt{2}\|u_k\|_2 \\
&= E_k + h^2 + 2h\sqrt{2}\sqrt{E_k} \\
\implies E_k + 1 &\leq E_k + 2\sqrt{2}h\sqrt{E_k} + h^2
\end{aligned}
$$

Part d

$$
\begin{aligned}
E_{k+1} &\leq E_k + 2\sqrt{2}h\sqrt{E_k} + h^2 \\
&\leq E_k + 2\sqrt{2}hE_k + h^2 \ (as \ E_k > 0) \\
&= (1 + 2\sqrt{2}h)E_k + h^2
\end{aligned}
$$

Then, by the lemma on slide 19 of ODEs lecture:

$$\begin{aligned} E_n &= (1+2\sqrt{2}h)^n E_0 + h^2(\frac{(1+2\sqrt{2}h)^n - 1}{1+2\sqrt{2}h - 1}) \\ &= (1+2\sqrt{2}h)^n E_0 + h^2(\frac{(1+2\sqrt{2}h)^n - 1}{2\sqrt{2}h}) \\ h = \frac{T}{n} &\implies (1+2\sqrt{2}h) = (1+2\sqrt{2}\frac{T}{n}) \\ 1+x \le e^x &\implies (1+2\sqrt{2}\frac{T}{n}) \le e^{2\sqrt{2}\frac{T}{n}} \\ \implies E_n &= (e^{2\sqrt{2}\frac{T}{n}})^n E_0 + h\frac{(e^{2\sqrt{2}\frac{T}{n}})^n - 1}{2\sqrt{2}} \\ &= e^{2\sqrt{2}T}E_0 + \frac{T}{n}\frac{e^{2\sqrt{2}T} - 1}{2\sqrt{2}} \end{aligned}$$

Question 4

Part a

We have:

$$\begin{aligned} f(x) &= (x-1)^2 \\ f'(x) &= 2(x-1) \\ f''(x) &= 2 \\ f^{(k)}(x) &= 0 \qquad\qquad \forall k > 2 \end{aligned}$$

For Newtons method

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ \implies x_{n+1} &= x_n - \frac{(x_n-1)^2}{2(x_n-1)} \\ &= x_n - \frac{1}{2}(x_n - 1) \end{aligned}$$

Substituting $x_n = r - \epsilon_n$ where $r = 1$ is the true root of $f(x)$:

$$\begin{aligned} r - \epsilon_{n+1} &= r - \epsilon_n - \frac{1}{2}(r - \epsilon_n - 1) \\ \implies \epsilon_{n+1} &= \epsilon_n + \frac{1}{2}(1 - \epsilon_n - 1) \\ &= \epsilon_n - \frac{1}{2}\epsilon_n \\ &= \frac{1}{2}\epsilon_n \\ \\ \implies |\epsilon_{n+1}| &= C|\epsilon_n| \quad for\ C = \frac{1}{2} \end{aligned}$$

Thus Newton's method for $f(x) = (x+1)^2$ converges linearly.

Part b

In the lectures, the following argument was made:

$$\text{For some } \xi_k \text{ between } x_n \text{ and } r: \qquad e_{k+1} = e_k + \frac{f(x_k)}{f'(x_k)} = -\frac{f''(\xi_k)}{2f'(x_k)}e_k^2$$

$$\begin{aligned} x_k \text{ close to r} &\implies \xi_k \text{ close to r} \\ &\implies \frac{f''(\xi_k)}{2f'(x_k)} \approx \frac{f''(r)}{2f'(r)} \\ &\implies e_{k+1} \approx -\frac{f''(r)}{2f'(r)}e_k^2 \end{aligned}$$

However,

$$\begin{aligned} f'(x) = 2(x-1)\ &and\ r = 1 \\ \implies &f'(r) = f'(1) = 0 \\ \implies &\frac{f''(\xi_k)}{2f'(x_k)} \not\approx \frac{f''(r)}{2f'(r)} \end{aligned}$$

As dividing by $f'(r)$ leads to a division by 0

Part c

$$\begin{aligned} x_{n+1} &= x_n - 2\frac{f(x_n)}{f'(x_n)} \\ \implies x_{n+1} &= x_n - 2\frac{(x_n-1)^2}{2(x_n-1)} \\ &= x_n - (x_n - 1) \\ &= 1 \end{aligned}$$

Substituting $x_n = r - \epsilon_n$ where $r = 1$ is the true root of $f(x)$:

$$\begin{aligned} r - \epsilon_{n+1} &= r - \epsilon_n - (r - \epsilon_n - 1) \\ \implies \epsilon_{n+1} &= \epsilon_n + (1 - \epsilon_n - 1) \\ &= \epsilon_n - \epsilon_n \\ &= 0 \end{aligned}$$

Thus unlike regular Newtons method, this method will converge in a single iteration for any value of x.

Question 5

Part a

$$\begin{aligned} &f(x) = \frac{1}{2}x_1^2 + \frac{c}{2}x_2^2 \\ &\nabla f(x) = \begin{bmatrix} x_1 \\ cx_2 \end{bmatrix} \\ &\nabla f(x^*) = 0 \\ &\quad \implies x_1 = 0 \; and \; x_2 = 0 \\ &\quad \implies x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\quad \implies x^* = (0,0)^T \text{ is a unique local minimum} \end{aligned}$$

Part b

$$\begin{aligned} \nabla^2 f(x) &= \nabla \begin{bmatrix} x_1 \\ cx_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \kappa(\nabla^2 f(x)) &= \left\| \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}^{-1} \right\|_2 \\ &= \left\| \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \right\|_2 \left\| \frac{1}{c} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \right\|_2 \\ &= \frac{1}{c} \left\| \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \right\|_2 \end{aligned}$$

$$\begin{aligned} \left\| \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \right\|_2 &= \sqrt{\lambda_{max} \left(\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}^T \right)} \\ &= \sqrt{\lambda_{max} \begin{bmatrix} 1 & 0 \\ 0 & c^2 \end{bmatrix}} \\ &= \sqrt{\max\{\lambda \mid (\lambda - 1)(\lambda - c^2) = 0\}} \\ &= c \text{ as c} > 1 \end{aligned}$$

$$\begin{aligned} \left\| \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \right\|_2 &= \sqrt{\lambda_{max} \left(\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}^T \right)} \\ &= \sqrt{\lambda_{max} \begin{bmatrix} c^2 & 0 \\ 0 & 1 \end{bmatrix}} \\ &= \sqrt{\max\{\lambda \mid (\lambda - 1)(\lambda - c^2) = 0\}} \\ &= c \text{ as c} > 1 \end{aligned}$$

$$\begin{aligned} \implies \kappa(\nabla^2 f(x)) &= \frac{1}{c}(c)(c) \\ &= c \end{aligned}$$

Part c

$f(x)$ $L_{\nabla f}$-Lipschitz continuous if $\|\nabla f(y) - \nabla f(y)\|_2 \leq L_{\nabla f}\|y - x\|_2$
Further if $f(x)$ $L_{\nabla f}$-Lipschitz continuous then gradient descent converges globally if 0 < α < $\frac{2}{L_{\nabla f}}$
Thus, for $\alpha = \frac{1}{c}$, gradient descent converges if $f(x)$ is c-Lipschitz continuous as $\alpha < \frac{2}{c}$.

$$\begin{aligned}\|\nabla f(y) - \nabla f(y)\|_2 &= \sqrt{(x_1 - y_1)^2 + c^2(x_2 - y_2)^2} \\ &= c\sqrt{\frac{1}{c^2}(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &< c\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\ &= c\|y - x\|_2\end{aligned}$$

Thus $f(x)$ is c-Lipshitz continuous and thus converges globally to x^* with step size $\frac{1}{c}$

Now, for gradient descent we have

$$\begin{aligned} & x_{n+1} \leq x_n - \alpha\nabla f(x_n) \\ \implies & x_{n+1} \leq x_n - \frac{1}{c}\nabla f(x_n) \\ & \quad = \begin{bmatrix} x_{n_1} \\ x_{n_2} \end{bmatrix} - \frac{1}{c}\begin{bmatrix} x_{n_1} \\ cx_{n_2} \end{bmatrix} \\ & \quad = \begin{bmatrix} x_{n_1}(1 - \frac{1}{c}) \\ 0 \end{bmatrix}\end{aligned}$$

Thus, x_{n_2} converges to x_2^* after a single iteration. However, $x_{n+1_1} \leq x_{n_1}(1 - \frac{1}{c})$ which converges linearly as c > 1. Thus the overall convergence is linear.

Part d

For the convergence above we have $x_{n+1_1} \leq x_{n_1}(1 - \frac{1}{c})$. For larger values of c, the value of $(1 - \frac{1}{c})$ increases, approaching 1 and thus the value of x_{n+1_1} will decrease more slowly.

Furthermore, gradient descent performs poorly when $\nabla^2 f(x)$ is ill conditioned.
As $\kappa(\nabla^2 f(x)) = c$ the hessian becomes more ill conditioned as c increases. Leading to slower convergence for larger values of c.

Question 6

Part a

$$\|x\|_2^2 \leq \Delta^2 \iff g(x) = \Delta^2 - \|x\|_2^2 \geq 0$$

For Slater's CQ:

1. $g(x)\ concave$
2. $\exists x \in \mathbb{R}^n\ such\ that\ g(x) > 0$

For 1:

$$\begin{aligned}
-g(\boldsymbol{x})\ convex &\implies g(\boldsymbol{x})\ concave \\
-g(\boldsymbol{x}) &= -(\Delta^2 - \|\boldsymbol{x}\|_2^2) \\
&= \|\boldsymbol{x}\|_2^2 - \Delta^2 \\
&= \sum_{i=1}^{n} x_i^2 - \Delta^2 \\
\\
\nabla(-g(\boldsymbol{x})) &= \begin{bmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{bmatrix} \\
\nabla^2(-g(\boldsymbol{x})) &= \begin{bmatrix} 2 & & & \\ & 2 & & \Huge 0 \\ & & 2 & \\ & \Huge 0 & & \ddots \\ & & & & 2 \end{bmatrix} \\
&\implies -g(x)\ convex\ as\ \nabla^2(-g(\boldsymbol{x}))\ positive\ semidefinate \\
&\implies g(x)\ concave
\end{aligned}$$

For 2 consider $\boldsymbol{x} = \boldsymbol{0}$:

$$\begin{aligned}
g(\boldsymbol{0}) &= \Delta^2 - \|\boldsymbol{0}\|_2^2 \\
&= \Delta^2 - 0 \\
&= \Delta^2 \\
&> 0\ as\ \Delta > 0
\end{aligned}$$

Thus Slater's CQ holds.

Part b

As $f(x)$ and $g(x)$ are continuously differentiable and Slater's CQ holds, local minimizer x^* is a KKT point.

Thus $\exists \lambda^*$ such that:

1. $\nabla f(\boldsymbol{x}^*) = \lambda^* \nabla g(\boldsymbol{x}^*)$
2. $\lambda^* \geq 0$
3. $\lambda^* g(\boldsymbol{x}^*) = 0$
4. $g(x) \geq 0 \implies g(\boldsymbol{x}^*) \geq 0 \implies \|\boldsymbol{x}^*\|_2^2 \leq \Delta^2 \implies \|\boldsymbol{x}^*\|_2 \leq \Delta$

From 3 we have:

$$\lambda^* = 0 \quad or \quad g(\boldsymbol{x}^*) = 0$$

$$\begin{aligned}
&g(\boldsymbol{x}^*) = 0 \\
&\implies \Delta^2 - \|\boldsymbol{x}^*\|_2^2 = 0 \\
&\implies (\Delta - \|\boldsymbol{x}^*\|_2)(\Delta + \|\boldsymbol{x}^*\|_2) = 0 \\
&\implies \Delta - \|\boldsymbol{x}^*\|_2 = 0 \quad as\ \Delta, \|\boldsymbol{x}^*\|_2 \geq 0
\end{aligned}$$

$$\begin{aligned}
Thus:\ &\lambda^* = 0 \quad or \quad \Delta - \|\boldsymbol{x}^*\|_2 = 0 \\
&\implies \lambda^*(\Delta - \|\boldsymbol{x}^*\|_2) = 0
\end{aligned}$$

Now:

$$\begin{aligned}
\nabla g(\boldsymbol{x}) &= -2 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = -2\boldsymbol{x} \\
&\implies \nabla g(\boldsymbol{x}^*) = -2\boldsymbol{x}^*
\end{aligned}$$

$$\begin{aligned}
And:\ f(x) &= \frac{1}{2}\boldsymbol{x}^T H \boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{g} \\
&\implies \nabla f(x) = \frac{1}{2}\nabla(\boldsymbol{x}^T H \boldsymbol{x}) + \nabla(\boldsymbol{x}^T \boldsymbol{g})
\end{aligned}$$

Considering $\boldsymbol{x}^T \boldsymbol{y}$:

$$
\begin{aligned}
\nabla(\boldsymbol{x}^T\boldsymbol{y}) &= \nabla\left(\begin{bmatrix}x_1 & x_2 & \dots & x_n\end{bmatrix}\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}\right)\\
&= \nabla\begin{bmatrix}x_1y_1\\x_2y_2\\\vdots\\x_ny_n\end{bmatrix}\\
&= \begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}\\
&= \boldsymbol{y}\\
\implies \nabla(\boldsymbol{x}^T\boldsymbol{g}) &= \boldsymbol{g}
\end{aligned}
$$

Now let $\boldsymbol{y} = A\boldsymbol{x}$ for some symmetric matrix A:

$$
\begin{aligned}
\nabla(\boldsymbol{x}^T\boldsymbol{y}) &= (\nabla\boldsymbol{x}^T)\boldsymbol{y} + \boldsymbol{x}^T(\nabla\boldsymbol{y})\\
&= \boldsymbol{y} + \boldsymbol{x}^T\nabla(A\boldsymbol{x})\\
&= \boldsymbol{y} + \boldsymbol{x}^T\nabla(A)\boldsymbol{x} + A\nabla\boldsymbol{x}\\
&= \boldsymbol{y} + \boldsymbol{x}^TA\\
&= A\boldsymbol{x} + \boldsymbol{x}^TA\\
&= A\boldsymbol{x} + A^T\boldsymbol{x}\\
&= (A + A^T)\boldsymbol{x}\\
&= 2A\boldsymbol{x}\ as\ A\ symmetric\\
\implies \nabla(\boldsymbol{x}^TH\boldsymbol{x}) &= \frac{1}{2}2H\boldsymbol{x}
\end{aligned}
$$

Thus:

$$
\begin{aligned}
&\nabla f(x) = \frac{1}{2}2H\boldsymbol{x} + \boldsymbol{g}\\
&\quad\implies f(\boldsymbol{x}^*) = H\boldsymbol{x}^* + g
\end{aligned}
$$

$$
\begin{aligned}
&\nabla f(x) = \lambda^*\nabla g(x)\\
&\quad\implies H\boldsymbol{x}^* + \boldsymbol{g} = \lambda^*\boldsymbol{x}^*\\
&\quad\implies H\boldsymbol{x}^* + \lambda^*\boldsymbol{x}^* = -\boldsymbol{g}\\
&\quad\implies (H + \lambda^*I)\boldsymbol{x}^* = -g
\end{aligned}
$$