

Section 2:

Mathematics

2016 Exam 1 Qu. 1

1. [11 points] The statements “A” and “B” described in this problem do not have to have anything to do with economics.
 - (a) Give an example of a statement “A” and another statement “B” such that the following is true: “A” is a necessary but not sufficient condition for “B.”
 - (b) Give an example of a statement “A” and another statement “B” such that the following is true: “A” is a sufficient but not a necessary condition for “B.”
 - (c) Give an example of a true statement and its converse where the converse is false, or explain why that is impossible to do.
 - (d) Give an example of a true statement and its contrapositive where the contrapositive is false, or explain why that is impossible to do.
 - (e) Consider the following statements: “A” is “ $1 + 1 = 2$ ” and “B” is “ $1 + 2 = 17$.” Is “A or B” true or false? Why?
 - (f) Briefly describe what the following term means: “proof by contradiction.”
 - (g) Briefly describe what the following term means: “proof by induction.”

Answers to Econ. 7005 Midterm Exam

Fall 2016

①

a) "The lawn is getting wet" is a necessary but not sufficient condition
 $A \uparrow$

for "It is raining."
 $B \uparrow$

A is not sufficient for B because A could be true while B is false

(as when lawn sprinklers artificially water the grass).

A is necessary for B because the only way for B to be true is that A is also true.

b) "It is raining" is a sufficient but not necessary condition for
 $A \uparrow$
"The lawn is getting wet."
 $B \uparrow$

A is sufficient for B because if A is true then B has to be true.

A is not a necessary condition for B because it's possible for A to be false while B is true (as when lawn sprinklers artificially water the grass).

c) True: If it is raining, ^{then} \downarrow the lawn is getting wet.

Converse: If the lawn is getting wet, then it is raining.

The converse is false. There are other ways for the lawn to be getting wet besides rain.

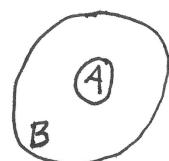
d) Statement: If it is raining, then the lawn is getting wet.

Contrapositive: If the lawn is not getting wet, then it is not raining.

This contrapositive is true, and in fact if a statement is true then its contrapositive will also be true.

" $A \Rightarrow B$ " has a contrapositive of " $\text{not } B \Rightarrow \text{not } A$ ".

" $A \Rightarrow B$ " has this Venn diagram:



From this diagram, it is clear that if one is

outside of B ("not B"), one is not going to be in A ("not A").

e) $A : 1+1=2$ True

$B : 1+2=17$ False

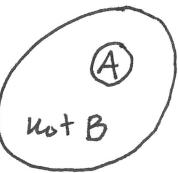
" $A \text{ or } B$ " is True. The truth table for "or" is

A	B	$A \text{ or } B$
T	T	T
T	F	T
F	T	T
F	F	F

(This is the definition of "or", sometimes called the "inclusive or.")

f) To prove: $A \Rightarrow B$.

Method of proof: Suppose that $A \Rightarrow \text{not } B$. Prove that statement is false (by showing that it implies something false, that is, it implies a contradiction). Then this Venn diagram



must be wrong, so A cannot lie in "not B ", so

A has to lie in B , which means $A \Rightarrow B$.

g) Suppose one is trying to prove that a certain statement is true for every natural number $N = \{1, 2, 3, 4, \dots\}$.

optional example

$$\text{"For every } i \in \mathbb{N}, \sum_{j=1}^i j = \frac{i(i+1)}{2}$$

The left-hand side is the sum of the first i integers,

$$1 + 2 + 3 + 4 + \dots + i.$$

"Proof by induction" means to prove it by:

1) Showing that it is true for $i = 1$; and

2) "assuming it is true for i

and proving that it is true for $i+1$ ".

Optional: if the set of natural numbers of interest starts at a number > 1 , it can be re-numbered, or the first step in the induction proof can be carried out on the smallest natural number of interest.

$$\sum_{j=1}^i j = \frac{i(i+1)}{2}$$

\uparrow $\uparrow \frac{1 \cdot 2}{2} = 1 \quad \text{OK}$

Assume $\sum_{j=1}^i j = \frac{1}{2} i(i+1)$.

Prove $\sum_{j=1}^{i+1} j = \frac{1}{2} (i+1)(i+2)$.

$$\begin{aligned} \text{LHS: } \sum_{j=1}^i j + (i+1) &= \frac{1}{2} i(i+1) + (i+1) \\ &= (i+1)\left(\frac{1}{2} i + 1\right) \end{aligned}$$

$$\text{RHS: } (i+1) \cdot \frac{i+1+1}{2} = (i+1)\left(\frac{1}{2} i + 1\right).$$

Final Exam
1999

Question 1

(6)

Answer all of the following five questions.

1. [15 points] In parts (b) through (e) below, assume that $f : \mathbf{R}^3 \rightarrow \mathbf{R}^1$.
- (a) Suppose a 3×3 matrix A has entries a_{ij} . Under what conditions is A negative definite? (Another way of asking this question is: under what conditions is the function $g(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ negative $\forall \mathbf{x} \neq \mathbf{0}$?)
 - (b) What conditions are sufficient to guarantee that the function $f(\mathbf{x})$ (which was described at the beginning of this problem) is strictly concave?
 - (c) Suppose \mathbf{x}^* satisfies the first-order conditions for the problem of maximizing $f(\mathbf{x})$. What conditions are sufficient to guarantee that \mathbf{x}^* is a local maximum of $f(\mathbf{x})$?
 - (d) Suppose \mathbf{x}^* satisfies the first-order conditions for the problem of maximizing $f(\mathbf{x})$ subject to the linear constraint

$$x_1 + x_2 + x_3 = 0.$$

What conditions are sufficient to guarantee that \mathbf{x}^* is a local maximum of $f(\mathbf{x})$ subject to $x_1 + x_2 + x_3 = 0$? Your answer should not include the Lagrangian function \mathcal{L} .

- (e) A student wrote, "to determine if $f(\mathbf{x})$ is strictly concave, one first has to find the second-order conditions." What is wrong with this statement?

Answers to Final Exam, Econ. 6710, Spring 1999

① a) $a_{11} < 0$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0$$

b) $f''_{11} < 0$

$$\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} > 0$$

$$\begin{vmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{vmatrix} < 0$$

$\forall x$ in the domain
of f

Final Exam
1999

Answer 1

c) D_{2m+1} of $\nabla^2 \mathcal{L}$ should have the same sign as $(-1)^{m+1}$, then 0's should alternate
in sign.

$$m=0, \mathcal{L}=f, (-1)^{m+1} = -1$$

$$D_1 = f''_{11} < 0$$

$$D_2 = \begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} > 0 \quad \text{and} \quad D_3 = \begin{vmatrix} f''_{11} & f''_{12} & f''_{13} \\ f''_{21} & f''_{22} & f''_{23} \\ f''_{31} & f''_{32} & f''_{33} \end{vmatrix} < 0$$

d) $m=1, \mathcal{L}=f(\underline{x}) + \lambda(x_1 + x_2 + x_3)$

$$2m+1=3, (-1)^{m+1} = 1. \text{ So because}$$

$$\begin{bmatrix} \mathcal{L}_{\lambda\lambda}'' & \mathcal{L}_{\lambda 1}'' & \mathcal{L}_{\lambda 2}'' & \mathcal{L}_{\lambda 3}'' \\ \mathcal{L}_{1\lambda}'' & \mathcal{L}_{11}'' & \mathcal{L}_{12}'' & \mathcal{L}_{13}'' \\ \mathcal{L}_{2\lambda}'' & \mathcal{L}_{21}'' & \mathcal{L}_{22}'' & \mathcal{L}_{23}'' \\ \mathcal{L}_{3\lambda}'' & \mathcal{L}_{31}'' & \mathcal{L}_{32}'' & \mathcal{L}_{33}'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & f''_{11} & f''_{12} & f''_{13} \\ 1 & f''_{21} & f''_{22} & f''_{23} \\ 1 & f''_{31} & f''_{32} & f''_{33} \end{bmatrix},$$

The conditions are

$$D_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & f_{11}'' & f_{12}'' \\ 1 & f_{21}'' & f_{22}'' \end{vmatrix} > 0$$

and

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & f_{11}'' & f_{12}'' & f_{13}'' \\ 1 & f_{21}'' & f_{22}'' & f_{23}'' \\ 1 & f_{31}'' & f_{32}'' & f_{33}'' \end{vmatrix} < 0.$$

Final Exam
1999

Answer 1 cont...

- e) The conditions for concavity are given in (b). This has nothing to do with second-order conditions, which are given in (c) and (d). This is true even though the conditions in (b) and (c) are identical. This latter fact implies that if the conditions in (b)
 are true for all x in the domain of f , then local extreme points are local maxima. (Actually, the implication is much stronger, but you were not asked that. If (b) is true throughout the relevant domain, then x^* is the unique global maximum.)

Except that (b) must hold for x in f 's domain, whereas (c), a local result, need only hold at x^*

To repeat: the issue of concavity is in essence unrelated to optimization (and second-order conditions only relate to optimization.)

3. [11 points] Suppose we wish to maximize a function $f(\mathbf{x})$ over \mathbf{x} (interpret \mathbf{x} as a column vector). Suppose we have found a point \mathbf{x}^* which satisfies the first-order conditions for maximizing $f(\mathbf{x})$ over \mathbf{x} . Finally, suppose that the following “Taylor Series approximation” of $f(\mathbf{x})$ for any value of \mathbf{x} is so good that we can ignore any error it leads to:

$$f(\mathbf{x}) \approx f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*) (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \nabla^2 f(\mathbf{x}^*) (\mathbf{x} - \mathbf{x}^*)$$

where the “ T ” superscript denotes “transpose.”

- (a) What is the numerical value of $\nabla f(\mathbf{x}^*)$? (The second sentence of this problem should make this easy to answer.)
- (b) If \mathbf{x}^* really is a maximum point of f , what can be said about the value of $f(\mathbf{x}) - f(\mathbf{x}^*)$?
- (c) What condition on the matrix $\nabla^2 f(\mathbf{x}^*)$ would ensure that the criterion of part (b) is satisfied for any arbitrary value of \mathbf{x} ?
- (d) How would your answer to part (c) change if the Taylor Series approximation was not very good, so that one could not drop the “higher order terms” (“H.O.T.”) in

$$\begin{aligned} f(\mathbf{x}) \approx \\ f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*) (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \nabla^2 f(\mathbf{x}^*) (\mathbf{x} - \mathbf{x}^*) + \text{H.O.T.} \end{aligned}$$

?

(3)

a) the F.O.C. for maximizing $f(\underline{x})$ over \underline{x} set $\partial f / \partial x_i = 0$ for all i .

Hence $\nabla f = (\partial f / \partial x_1, \partial f / \partial x_2, \dots, \partial f / \partial x_n) = 0$ at \underline{x}^* .

b) $f(\underline{x}^*) \geq f(\underline{x})$ so

$$0 \geq f(\underline{x}) - f(\underline{x}^*)$$

c) From the Taylor Series approximation,

$$\underbrace{f(\underline{x}) - f(\underline{x}^*)}_{\text{Should be } \leq 0} \approx \underbrace{\nabla f(\underline{x}^*) (\underline{x} - \underline{x}^*)}_{\substack{= 0 \text{ from} \\ \text{part a}}} + \frac{1}{2} (\underline{x} - \underline{x}^*)^T \nabla^2 f(\underline{x}^*) (\underline{x} - \underline{x}^*)$$

$$\Rightarrow \frac{1}{2} (\underline{x} - \underline{x}^*)^T \nabla^2 f(\underline{x}^*) (\underline{x} - \underline{x}^*) \leq 0 ; \text{ letting } \underline{y} = \underline{x} - \underline{x}^*$$

$$\Leftrightarrow \underbrace{\underline{y}^T \nabla^2 f(\underline{x}^*) \underline{y}}_{\text{for all } \underline{y}} \leq 0$$

$\Leftrightarrow \nabla^2 f(\underline{x}^*)$ is a negative semidefinite matrix.

(This gives rise to the second order conditions for a maximum.)

d) In this case we would have

$$\frac{1}{2} (\underline{x} - \underline{x}^*)^T \nabla^2 f(\underline{x}^*) (\underline{x} - \underline{x}^*) + \text{H.O.T.} \leq 0 \Leftrightarrow$$

$$\underbrace{\underline{y}^T \nabla^2 f(\underline{x}^*) \underline{y}}_{\text{H.O.T.}} \leq -\text{H.O.T.}$$

from which nothing can be concluded about $\nabla^2 f(\underline{x}^*)$.

Optional: the H.O.T. does not appear in the neighborhood of \underline{x}^* , so the condition in (c) is valid locally but not globally.

Exam 1
2004

Question 1

Answer all of the following three questions.

1. [11 points] Suppose $\mathbf{x} \in \mathbf{R}^n$.

- (a) State the second-order sufficient conditions for the problem

$$\max_{\mathbf{x}} f(\mathbf{x}).$$

- (b) State sufficient conditions for $f(\mathbf{x})$ to be strictly concave.

- (c) Do the conditions you found in (a) imply the conditions you found in (b)? Do the conditions you found in (b) imply the conditions you found in (a)?

- (d) State the second-order sufficient conditions for the problem

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{c} \cdot \mathbf{x} - m = 0$$

where \mathbf{c} and m are constants. By using information from the first-order conditions, express your answer without using \mathbf{c} .

- (e) State sufficient conditions for $f(\mathbf{x})$ to be quasiconcave.

- (f) Do the conditions you found in (d) imply the conditions you found in (e)? Do the conditions you found in (e) imply the conditions you found in (d)?

- (g) How would your answer to (f) change if, in (d), one were to replace $\mathbf{c} \cdot \mathbf{x} - m = 0$ with $g(\mathbf{x}) - m = 0$ for some general function g ?

Answers to Exam 1, Spring 2004, Econ 7005

① a) D_{2m+1}, \dots, D_{n+m} of $\nabla^2 L$ should alternate in sign beginning with $(-1)^{m+1}$
so with $m=0$,

D_1, \dots, D_n of $\nabla^2 f$ should alternate in sign beginning with $(-1)^1 = -1$

$$D_1 \text{ of } \nabla^2 f < 0$$

$$D_2 \text{ --- } > 0$$

:

b) D_1, \dots, D_n of $\nabla^2 f$ should alternate in sign beginning with < 0

c) (a) \Leftrightarrow (b) the conditions are the same

d) $L = f(x) + \lambda (\underline{c} \cdot \underline{x} - m)$

F.O.C. $0 = \partial L / \partial \lambda = \underline{c} \cdot \underline{x} - m$

$$0 = \partial L / \partial \underline{x} = \nabla f(\underline{x}) + \lambda \underline{c} \Rightarrow \lambda \underline{c} = -\nabla f(\underline{x}) \text{ and}$$

$$\underline{c} = \frac{-1}{\lambda} \nabla f(\underline{x}).$$

Exam 1 $\nabla^2 L = \begin{bmatrix} \mathcal{L}_{\lambda\lambda}'' & \mathcal{L}_{\lambda\underline{x}}'' \\ \mathcal{L}_{\underline{x}\lambda}'' & \mathcal{L}_{\underline{x}\underline{x}}'' \end{bmatrix} = \begin{bmatrix} 0 & \underline{c}^T \\ \underline{c} & \nabla^2 f \end{bmatrix}$ if one thinks of \underline{c}
as being a column vector

2004

Answer 1

$$= \begin{bmatrix} 0 & -\frac{1}{\lambda} \nabla f^T \\ \frac{-1}{\lambda} \nabla f & \nabla^2 f \end{bmatrix}$$

S.O.C. given in the first line of (a)'s answer, so here they are:

$$m=1, \quad D_3 \text{ of } \nabla^2 L \text{ has sign of } (-1)^{1+1} > 0$$

$$D_4 \text{ --- " --- } < 0$$

etc.
:

e) D_3 of $\begin{bmatrix} 0 & \nabla f^\top \\ \nabla f & \nabla^2 f \end{bmatrix} > 0$

$D_4 = - < 0$, etc. This is phrased in the notes with a different notation ($\delta_2 > 0, \delta_3 < 0$, etc.).

f) any D_i of $\nabla^2 \mathcal{L} = \underbrace{\left(\frac{-1}{\lambda}\right)\left(\frac{-1}{\lambda}\right)}_{+} D_i$ of $\begin{bmatrix} 0 & \nabla f^\top \\ \nabla f & \nabla^2 f \end{bmatrix}$

so the conditions are the same.

g) $\mathcal{L} = f(x) + \lambda(g(x) - m)$

F.O.C. $0 = \partial \mathcal{L} / \partial \lambda = g(x) - m$

$$0 = \partial \mathcal{L} / \partial x = \nabla f + \lambda \nabla g \Rightarrow \nabla g = -\frac{1}{\lambda} \nabla f$$

$$\nabla^2 \mathcal{L} = \begin{bmatrix} 0 & \nabla g^\top \\ \nabla g & \nabla^2 f + \lambda \nabla^2 g \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\lambda} \nabla f^\top \\ -\frac{1}{\lambda} \nabla f & \nabla^2 f + \lambda \nabla^2 g \end{bmatrix}$$

Exam 1

2004

Answer 1 cont...

which has no equality or general relationship with the conditions in (e). So in this case there is no link between S.O.C. and quasi-concavity.

1. [12 points] Suppose you are trying to determine if a function $\{f(x) : \mathbb{R}^n \rightarrow \mathbb{R}\}$ is convex. Suppose you have calculated all the leading principal minors of $\nabla^2 f$ and have found that they are *not* all positive. (Maybe they were all zero, or maybe some were negative.) How should you proceed? Carefully explain your answer.

① *

A necessary condition for convexity is that all of the principal minors are nonnegative: Δ_i of $\nabla^2 f \geq 0 \quad \forall i$.

Each Δ_i (each leading principal minor) is one of the Δ_i 's.

If any of the Δ_i 's were < 0 , then some $\Delta_i < 0$, violating the necessary condition for convexity; so the function would not be convex.

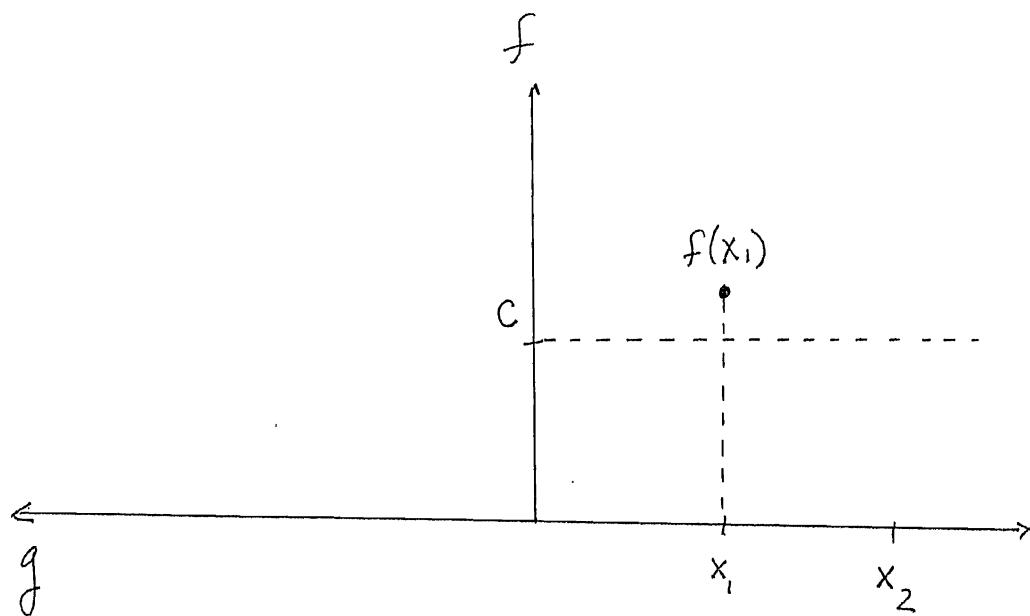
and none were < 0

If some of the Δ_i 's were $= 0$, proceed as follows. Calculate all the Δ_1 's, then all the Δ_2 's, then ..., then the Δ_n ($= D_n$). Stop immediately if any of these principal minors is negative, because that means the necessary condition for convexity is violated and f is not convex. If all the Δ_i 's are by contrast ≥ 0 , then convexity is assured.

* Positivity of all of the leading principal minors is a sufficient condition for convexity. (It's actually sufficient for strict convexity, but that implies convexity in turn.) So the sufficient condition for convexity has been violated.

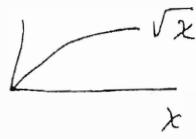
3. [11 points]

- (a) Show that the following is false: "a monotonically increasing function of a concave function is concave." A simple counterexample will suffice.
- (b) Show that a monotonically increasing function of a quasiconcave function is quasiconcave. [Hint 1: Figure 1 may or may not be useful to you. Hint 2: this is probably the hardest part of this question, so you might want to skip over it and answer the remaining parts first.]
- (c) Show that a monotonically increasing function of a concave function is quasiconcave. If it is helpful, you may use the result of part (b) even if you haven't proven it.
- (d) Show that $x_1^{1/2}x_2^{1/4}$ is concave on \mathbb{R}_+^2 .
- (e) Use parts (c) and (d) to argue that $\ln(x_1^{1/2}x_2^{1/4})^4 = 2\ln x_1 + \ln x_2$ is quasiconcave on \mathbb{R}_+^2 . (If you cannot do this, and you verify quasiconcavity some other way, you will get some partial credit but not full credit.)

Figure 1

(3)

a) $f(x) = \sqrt{x}$ is concave on \mathbb{R}_+^1

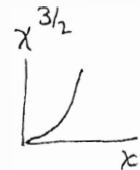


$g(f) = f^3$ is monotonically

increasing on \mathbb{R}_+^1 :



$$\text{But } g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^3 = x^{3/2}$$



which is not concave: $\frac{d}{dx} x^{3/2} = \frac{3}{2} x^{1/2}$ and

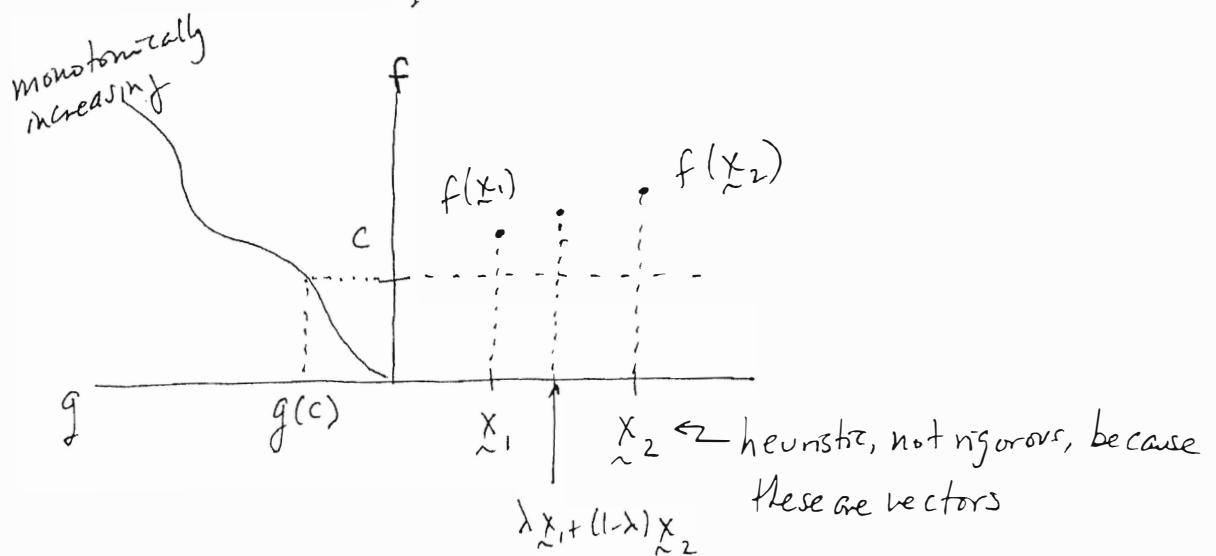
$$\frac{d^2}{dx^2} x^{3/2} = \frac{3}{4} x^{-1/2} > 0 \text{ on } \mathbb{R}_+^1$$

so $x^{3/2}$ is convex.

b) f quasiconcave \Leftrightarrow

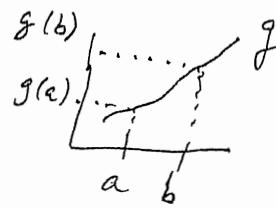
$$\left\{ \begin{array}{l} f(\tilde{x}_1) \geq c \\ f(\tilde{x}_2) \geq c \end{array} \right\} \Rightarrow f(\lambda \tilde{x}_1 + (1-\lambda) \tilde{x}_2) \geq c$$

(i.e., f 's upper level sets are convex sets)



g is monotonically increasing. This means that

if $a < b$ then $g(a) < g(b)$. Therefore:



$$f(\underline{x}_1) \geq c \Rightarrow g(f(\underline{x}_1)) \geq g(c) \quad (1)$$

$$f(\underline{x}_2) \geq c \Rightarrow g(f(\underline{x}_2)) \geq g(c). \quad (2)$$

$$f(\lambda \underline{x}_1 + (1-\lambda) \underline{x}_2) \geq c \Rightarrow g(f(\lambda \underline{x}_1 + (1-\lambda) \underline{x}_2)) \geq g(c) \quad (3).$$

Hence $g \circ f$ is quasiconcave.

c) Part b showed that if f is quasiconcave and if g is monotonically increasing then $g(f)$ is quasiconcave.

All concave functions are quasiconcave, so they can play the role of " f ", hence $g(f)$ is quasiconcave.

d)

$$j(\underline{x}) = x_1^{1/2} x_2^{1/4}$$

$$j'_1 = \frac{1}{2} x_1^{-1/2} x_2^{1/4}$$

$$j'_2 = \frac{1}{4} x_1^{1/2} x_2^{-3/4}$$

$$j''_{11} = -\frac{1}{4} x_1^{-3/2} x_2^{1/4}$$

$$j''_{12} = \frac{1}{8} x_1^{-1/2} x_2^{-3/4}$$

$$j''_{22} = -\frac{3}{16} x_1^{1/2} x_2^{-7/4}$$

5, 5

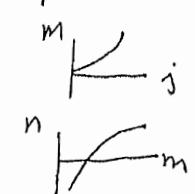
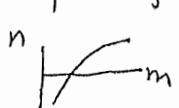
$$\nabla^2 j = \begin{bmatrix} -\frac{1}{4} x_1^{-3/2} x_2^{1/4} & \frac{1}{8} x_1^{-1/2} x_2^{-3/4} \\ \frac{1}{8} x_1^{-1/2} x_2^{-3/4} & \frac{-3}{16} x_1^{1/2} x_2^{-7/4} \end{bmatrix}$$

$$\mathbb{R}_+^2 = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$$

On \mathbb{R}_+^2 , D_1 of $\nabla^2 j$ is $-\frac{1}{4} x_1^{-3/2} x_2^{1/4} < 0$ and

$$\begin{aligned} D_2 \text{ of } \nabla^2 j \text{ is } & -\frac{1}{4} x_1^{-3/2} x_2^{1/4} \frac{(-3)}{16} x_1^{1/2} x_2^{-7/4} - \frac{1}{64} x_1^{-1} x_2^{-3/2} \\ & = \frac{3}{64} x_1^{-1} x_2^{-3/2} - \frac{1}{64} x_1^{-1} x_2^{-3/2} \\ & = \frac{1}{32} x_1^{-1} x_2^{-3/2} > 0 \end{aligned}$$

which suffices to show concavity.

- e) Let $j = x_1^{1/2} x_2^{1/4}$, which is concave from Part d and is positive on \mathbb{R}_+^2
 $m(j) = j^4$, which is increasing and positive on \mathbb{R}_+^1 
 $n(m) = \ln m$, — " — on \mathbb{R}_+^1 

So $\ln(x_1^{1/2} x_2^{1/4})^4$ is an increasing function of an increasing function
of a concave function, so it's also

"an increasing function of a concave function,"
which by Part c means it is quasi-concave.

Optional: The intuition is that preferences are: (a) invariant to monotonically increasing transformations; and (2) represented by quasi-concave utility functions, typically. So monotonically increasing transformations should preserve quasi-concavity. They don't preserve concavity, so concavity cannot be an important property of utility functions.

Summer
2006
Qualifiers

1 / 2

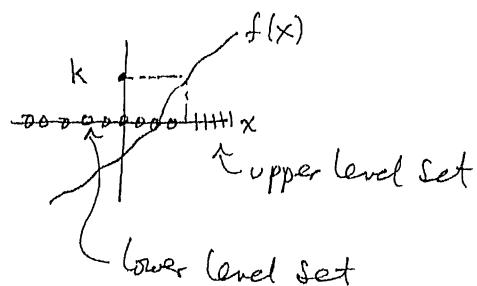
2. (a) Give an example of a function of one variable (not two or more variables) which is both quasiconcave and quasiconvex.
- (b) Give an example of a function of one variable (not two or more variables) which is not quasiconcave but is quasiconvex.
- (c) Give an example of a function of one variable (not two or more variables) which is quasiconcave but not quasiconvex.
- (d) Give an example of a function of one variable (not two or more variables) which is quasiconcave but not concave.
- (e) Give an example of a function of one variable (not two or more variables) which is quasiconvex but not convex.

You do not have to choose functions which are defined over the entire real line; a domain which is only part of the entire real line is acceptable.

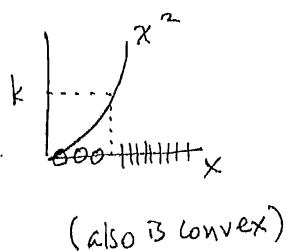
2/2

Qn 2.quasiconcave \Rightarrow convex upper level setsquasiconvex \Rightarrow convex lower level sets

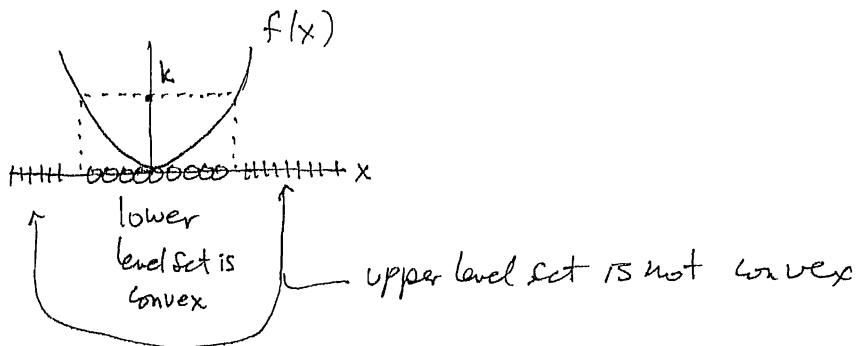
a)



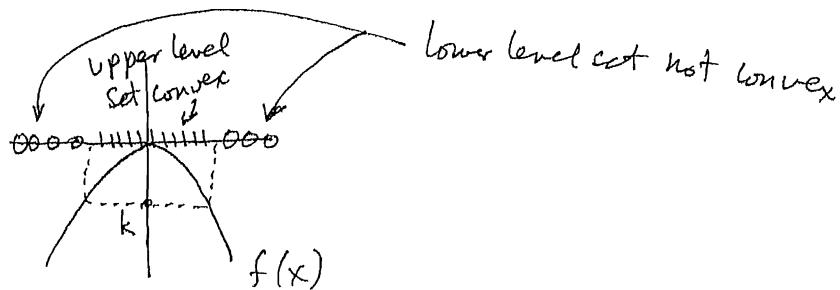
or



b)

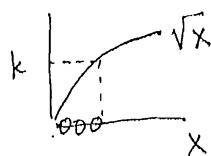


c)



d) Either of part (a)'s graphs (neither has $f'' \leq 0$ so neither is everywhere concave)

e) Part (a)'s first graph, or



2017 Exam 1 Qu. 2

2. [11 points] A function f is “homogeneous of degree k ” if “ $f(t\mathbf{x}) = t^k f(\mathbf{x})$.”

A function f is “homothetic” if “ $f(\mathbf{x}) = f(\mathbf{y})$ ” implies “ $f(t\mathbf{x}) = f(t\mathbf{y})$.”

Using the above definitions, prove that “if $f(\mathbf{x})$ is homogeneous and if $H(f): \mathbf{R}^1 \rightarrow \mathbf{R}^1$ is strictly increasing or strictly decreasing, then $H(f(\mathbf{x}))$ is homothetic.”

Answer to Qu. 2, Midterm Exam, Fall 2017 (Econ. 7005)

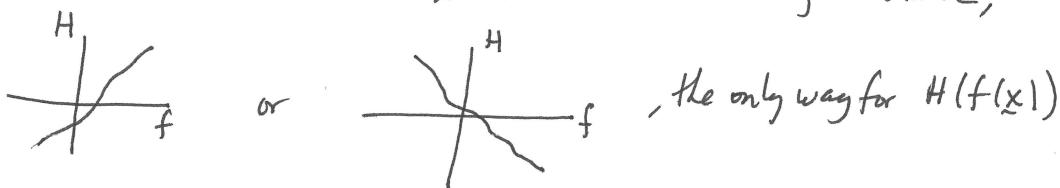
$f(x)$ homogeneous : $f(tx) = t^k f(x)$

$H(f) : \mathbb{R} \rightarrow \mathbb{R}$ strictly monotonic

To show : $H(f(x))$ homothetic, that is,

$$H(f(x)) = H(f(y)) \Rightarrow H(f(tx)) = H(f(ty)).$$

Proof. Assume $H(f(x)) = H(f(y))$. Since H is strictly monotonic,



to equal $H(f(y))$ is for $f(x)$ to equal $f(y)$. Now

$$f(tx) = t^k f(x) \text{ and}$$

$$f(ty) = t^k f(y) \text{ since } f \text{ is assumed to be homogeneous.}$$

But we already saw that $f(x) = f(y)$. So $f(tx) = f(ty)$. Then

$$H(f(tx)) = H(f(ty)). \blacksquare$$

2015 Exam 1 Qu. 1

1. [11 points] A function f is “homothetic” if “ $f(\mathbf{x}) = f(\mathbf{y})$ ” implies “ $f(t\mathbf{x}) = f(t\mathbf{y})$.” Use this definition of homotheticity in parts (a) and (b) of this question.

In part (c) of this question, you may use this result without proving it: “if $f(\mathbf{x})$ is homogeneous and if $H(f): \mathbf{R}^1 \rightarrow \mathbf{R}^1$ is strictly increasing or strictly decreasing, then $H(f(\mathbf{x}))$ is homothetic.” However, if you want to use this result in parts (a) or (b), you should prove it first.

- (a) Show that all homogenous functions are homothetic.
- (b) Use $f(x_1, x_2) = x_1 + x_2 + 1$ as a counterexample in showing that not all homothetic functions are homogeneous. You do need to include a proof that this f is homothetic.
- (c) Carefully explain what is going on in the derivation below, and what result it proves:

$$\frac{g'_i(t\mathbf{a})}{g'_j(t\mathbf{a})} = \frac{H'(f(t\mathbf{a})) f'_i(t\mathbf{a})}{H'(f(t\mathbf{a})) f'_j(t\mathbf{a})} = \frac{H'(f(\mathbf{a})) t^{k-1} f'_i(\mathbf{a})}{H'(f(\mathbf{a})) t^{k-1} f'_j(\mathbf{a})} = \frac{H'(f(\mathbf{a})) f'_i(\mathbf{a})}{H'(f(\mathbf{a})) f'_j(\mathbf{a})} = \frac{g'_i(\mathbf{a})}{g'_j(\mathbf{a})}.$$

In order to prove the second equality, it is helpful to observe that

$$\frac{H'(f(t\mathbf{a}))}{H'(f(t\mathbf{a}))} = \frac{H'(f(\mathbf{a}))}{H'(f(\mathbf{a}))}$$

(why is this true?); you not need compare $H'(f(t\mathbf{a}))$ with $H'(f(\mathbf{a}))$.

Answers to Question 1, Midterm Exam, Econ. 7005, Fall 2015

a) homogeneous: $f(tx) = t^k f(x)$ (1)

homothetic: $f(x) = f(y) \Rightarrow \underbrace{f(tx)}_{(2)} = \underbrace{f(ty)}_{(3)}$.

Suppose (1)

and (2); we need

to prove (3).

$$f(tx) = t^k f(x) = t^k f(y) \stackrel{\text{from (2)}}{=} f(ty) \stackrel{\text{from (1)}}{=} f(tx)$$

↑ ↑

proves (3).

b) $f(x_1, x_2) = x_1 + x_2 + 1$

This is homothetic. To prove this, suppose $f(x) = f(y)$, then prove that $f(tx) = f(ty)$.

The assumption that $f(x) = f(y)$ is equivalent to

$$x_1 + x_2 + 1 = y_1 + y_2 + 1. \quad (5)$$

(over →)

Is $f(t\tilde{x}) = f(\tilde{t}\tilde{y})$?

$$tx_1 + tx_2 + 1 \stackrel{?}{=} t\gamma_1 + t\gamma_2 + 1 ; \text{ subtract 1 from both sides:}$$

$$tx_1 + tx_2 \stackrel{?}{=} t\gamma_1 + t\gamma_2 ; \text{ divide both sides by } t, \text{ which is not zero:}$$

$$x_1 + x_2 \stackrel{?}{=} \gamma_1 + \gamma_2 ; \text{ add 1 to both sides:}$$

$$x_1 + x_2 + 1 \stackrel{?}{=} \gamma_1 + \gamma_2 + 1$$

This is (5), so we've assumed this to be true. \square

So f is homothetic. If it were homogeneous then $f(t\tilde{x}) = t^k f(\tilde{x})$, that is,

$$tx_1 + tx_2 + 1 \stackrel{?}{=} t^k(x_1 + x_2 + 1)$$

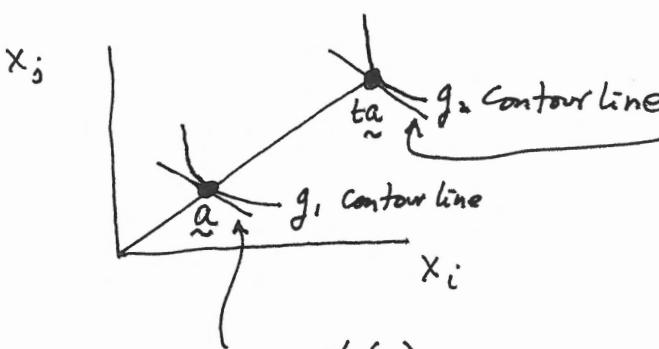
This is not true in general. For example,

$$tx_1 + tx_2 + 1 \stackrel{?}{=} t^k x_1 + t^k x_2 + t^k$$

$$1 - t^k \stackrel{?}{=} (t^k - t)x_1 + (t^k - t)x_2 ,$$

which is false even taking $k=1$ because that would imply $1-t=0$, whereas t cannot be restricted to a single value (such as $t=1$). Taking $k=0$ does not work either: $1-1 \stackrel{?}{=} (1-t)x_1 + (1-t)x_2$ is not true in general.

c)



$$\text{Slope of this line} = \frac{g'_i(ta)}{g'_j(ta)}, \text{ the left-}$$

hand side of the
long equation in (c)

$$\text{Slope of this line here} = \frac{g'_i(a)}{g'_j(a)}, \text{ the right-hand side of}$$

the long equation in (c)

So the assertion is that \rightarrow

for function g , along rays from the origin the slopes of contour lines are constant.

This is true for g homothetic. To show this, use the result from the second paragraph of this question: $g(\underline{a}) = H(f(\underline{a}))$.

$\begin{matrix} \uparrow & \uparrow \\ \text{homothetic} & \text{strictly} \\ & \text{monotonic} \end{matrix}$
homogeneous

Then by the Chain Rule, $g' = H'f'$, or, more explicitly,

$$\frac{\partial g}{\partial x_i} = \frac{\partial H}{\partial f} \frac{\partial f}{\partial x_i}, \text{ or even more explicitly,}$$

$$\frac{\partial g(x)}{\partial x_i} = \frac{\partial H(f(x))}{\partial f} \frac{\partial f(x)}{\partial x_i} \quad \text{or}$$

$$g'_i(t\underline{a}) = H'(f(t\underline{a})) \cdot f'_i(t\underline{a})$$

$$g'_j(t\underline{a}) = H'(f(t\underline{a})) \cdot f'_j(t\underline{a})$$

$$g'_i(\underline{a}) = H'(f(\underline{a})) \cdot f'_i(\underline{a})$$

$$g'_j(\underline{a}) = H'(f(\underline{a})) \cdot f'_j(\underline{a}).$$

This proves the first equality in (c); it also proves the fourth. To prove the second, recall that since f is homogeneous of some degree k , f' is homogeneous of degree $k-1$, so $f'_i(t\underline{a}) = t^{k-1}f'_i(\underline{a})$ and $f'_j(t\underline{a}) = t^{k-1}f'_j(\underline{a})$.

More explicitly, the left-hand side of the second equality is

$$\frac{H'(f(t\underline{a})), f'_i(t\underline{a})}{H'(f(t\underline{a})), f'_j(t\underline{a})} = \frac{f'_i(t\underline{a})}{f'_j(t\underline{a})} = \frac{t^{k-1}f'_i(t\underline{a})}{t^{k-1}f'_j(t\underline{a})} = \frac{H'(f(\underline{a})) \cdot t^{k-1}f'_i(\underline{a})}{H'(f(\underline{a})) \cdot t^{k-1}f'_j(\underline{a})} = 1$$

This completes the proof of the second equality.

The third equality follows trivially from $\frac{t^{k-1}}{t^{k-1}} = 1$. ■

A brief summary:

first equality: Chain Rule, $g = H(f)$

second " : homogeneity of f

third " : trivial

fourth " : Chain Rub, $g = H(f)$.