Yichun Xie

A Generalized Model for Cellular Urban Dynamics

Cities have long been recognized as complex and dynamic entities. However, the mathematical description of cities has been traditionally based on aggregate observations and on their static properties. Although some new approaches derived from the science of complex systems, particularly Cellular Automata (CA), have been explored in recent years, many urban researchers think that current forms of these models are cursory, fragmentary, and unrealistic in terms of theoretical and practical solidity. This paper generalizes an urban model to deal with questions of dynamic urban evolutionary modeling (DUEM). The DUEM situates its conceptual and technical foundations on Couclelis's CA spaces, Dendrinos' urban selection, and contemporary techniques of GIS. The DUEM provides a generic modeling technique that is capable of capturing the processes of evolutionary reproduction at the finest level and handling interactions between automaton self-reproduction and environmental and socioeconomic influences through the feedback among neighborhood, field, and region. Here we first review developments of dynamic systems theories and CA in particular. We then discuss emerging issues of CA applications in urban fields, and lay out conceptual foundations for this research. Next, we propose a generic urban CA model—DUEM for describing urban complexities and dynamics. We finally show how this generic model can be used to simulate growth dynamics for multisector land uses in a suburban area of Buffalo, New York.

INTRODUCTION

Cities have long been recognized as having basic, nonlinear, dynamic properties (Crosby 1983). However, capturing the dynamics of complex urban systems is one of the most delicate problems in urban modeling. Only very recently, conceptual and mathematical foundations for a substantive scientific inquiry into urban dynamics have been rapidly introduced with advances in our understanding of open systems and the nature of human decision processes. The application of systems theory to dynamic urban systems was largely motivated by Ilya Prigogine's dissipative structure theory (Nicolis and Prigogine 1977; Prigogine, Allen, and Herman 1977; Allen 1980; Prigogine 1980; Straussfogel

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1991). The continuous flow of matter and energy in open chemical systems results in a continuous process of spontaneous self-organization. This self-organization is characterized by a series of structural configurations called *dissipative structures*. Peter Allen has brought this notion into urban systems and has suggested that the self-organization of a system represents a process whereby two successive phases can be distinguished: stability and instability, or in other words, organization and reorganization (Allen et al. 1978, 1979; Allen 1982). Allen further employed the terminology of a bifurcation tree to describe an evolutionary system. Wilson (1981) proposed that catastrophe theory, derived from the mathematical work of Thom (1975), and bifurcation theory are similar mathematical schemas built on differential equations but applicable in different systems. The catastrophe concept is suitable for analyzing sudden and discrete changes in gradient systems.

Dendrinos and Mullally (1985) suggested that although the underlying processes in urban dynamics are very complicated, the macroscopic state of the urban system may be easily described and understood if the approach to mathematical ecology developed by Robert May (1971, 1973) is used to examine the system. Dendrinos and Mullally proposed a set of nonlinear dynamic models based on the Volterra-Lotka formalism as appropriate to interurban and intraurban dynamics. They gradually came to the conclusion that urban and natural systems display a great analogy in terms of evolution. In the urban context, this evolution is symbolized as selection. More recently, Dendrinos (1992) synthesized a very broad array of ideas about the dynamics of cities from the perspectives of ecological determinism, dualism, and chaos. Although these efforts have contributed greatly to improve our understanding of complex urban dynamics, there remain many unsolved issues: (1) the aggregate approach violates the principle of the uniqueness of individuals and smooths out variation in urban complexity; (2) isolation of temporal and spatial changes fails to capture the nature of urban dynamics; and (3) emphasis on "structure" neglects "process" of urban evolution (Batty 1993). All of these problems imply the need for much finer analysis at the level of more basic, and "genetic" system dynamics. The paradigm of cellular automata (CA) is acquiring momentum in urban modeling for these reasons (Xie 1994).

The principle of cellular automata was first applied to urban land use applications by Chapin and Weiss at the University of North Carolina in the early 1960s although they did not explicitly adopt the term "cellular automata." They employed grid cells to represent the components of settlement and assigned degrees of attractiveness to neighboring cells according to adjacency relationships (Chapin and Weiss 1968). This index of attractiveness was then used to determine which neighboring cell should be first converted into urban settlement. Tobler (1970) is perhaps the first geographer who formally recognizes the advantages of cellular models. Considering the land uses at the location i, j from the geographical perspective, Tobler (1979) demonstrated that one of the important features of the CA-like geographical model is the delineation of a district or a neighborhood. The neighborhood defines the geographical domain of influence and its spatiotemporal changes can be nicely illustrated by Conway's *Game of Life*, in which the temporal change of a place depends on its current state and the states of its neighboring places (Gardner 1970). Tobler concluded that the geographical model based on the cell-space representation is an innovative approach for analyzing dynamic geographical phenomena. Albin (1975) developed a very appealing approach by laying down a framework about automata representation of socioeconomic systems, which he labeled "smallest unit representation (SUR)." He applied automata to represent individuals characterized by demography, labor force, and class-state descriptions, and treated ensembles of automata as counterparts of social associations and institutions. Albin also developed several CA prototypes to simulate a primitive agricultural economy, urban racial tipping, and technical change, and demonstrated that "the SUR approach provides a practical working technique for modeling detailed socioeconomic relationships" (p. 2, 1975).

Couclelis (1985) initiated a thorough discussion of the concept of *cellular spaces* in the context of geographic modeling. She contended that the standard cell-space models are too constrained to be useful in realistic geographic applications because of their underlying conventions, such as the infinite plane, neighborhood stationarity, spatial homogeneity, spatial and temporal invariance of transition rules, and closure to external events. Couclelis, hoping to overcome these restrictions, proposed a generalized form of cell-space based on the discrete structure theory of modeling and simulation developed by Zeigler (1976). Couclelis envisions that these revisions could overcome major weak points of the early (simple) models of cellular automata and make them into an impressive framework for modeling micro-macro dynamics (Couclelis 1985, 1987, 1988, 1989).

Recently, more realistic CA applications have been added to the literature. White (1991) and his colleagues (1993) constructed a transition potential look-up table to represent the transition parameters, and successfully simulated an aggregate land use pattern in the city of Cincinnati. Hogeweg (1988), Phipps (1989), Campos (1991), Cecchini and Viola (1992), Nottola, Leroy, and Davalo (1992), and Bura et al. (1993) have made simulations of urban systems from a variety of approaches. Encouraged by Couclelis' work, Xie (1994) proposed that, to foster a rapid dissemination of vigorous CA research, urban systems scientists and modelers have to address three critical issues: (1) conceptual legitimacy of applying CA in urban fields; (2) technical merit of CA models in urban applications; and (3) broad applicability of generic urban CA models. Some preliminary work in this direction has been done by Xie (1994), and Batty and Xie (1994a, 1996). This paper is a continuing effort, attempting to demonstrate the theoretical integrity and technical merit of the CA approach through proposing a generic model for dynamic urban evolutionary modeling.

EMERGING ISSUES OF CA APPLICATIONS IN URBAN STUDIES

"Cellular automata are systems of cells interacting in a simple way but displaying complex overall behavior" (Wolfram 1986). Each cell displays any one of n possible discrete states (Figure 1). These states may transform through discrete time following some well-defined transition rules. These transition rules are deterministic in most cases but can incorporate stochastic elements. All cells are updated simultaneously by taking into account the current state of a cell together with the states of its neighbors.

CA can be generically characterized by a few salient features: (1) CA imbed the dynamics at the finest (bottom) level—the cells; (2) cells communicate with their environments through the overlay of neighborhoods; (3) the transition rules depend on topological arrangements instead of direct associations with conventional observed variables and constraints; (4) CA can generate complex global patterns through simple local reproduction processes; and (5) CA can provide a modeling formalism—an alternative to dynamics based on differential equations (Toffoli 1984; Couclelis 1985). These features make CA an ideal framework for dynamic modeling in the urban context (Couclelis 1985; Xie 1994).

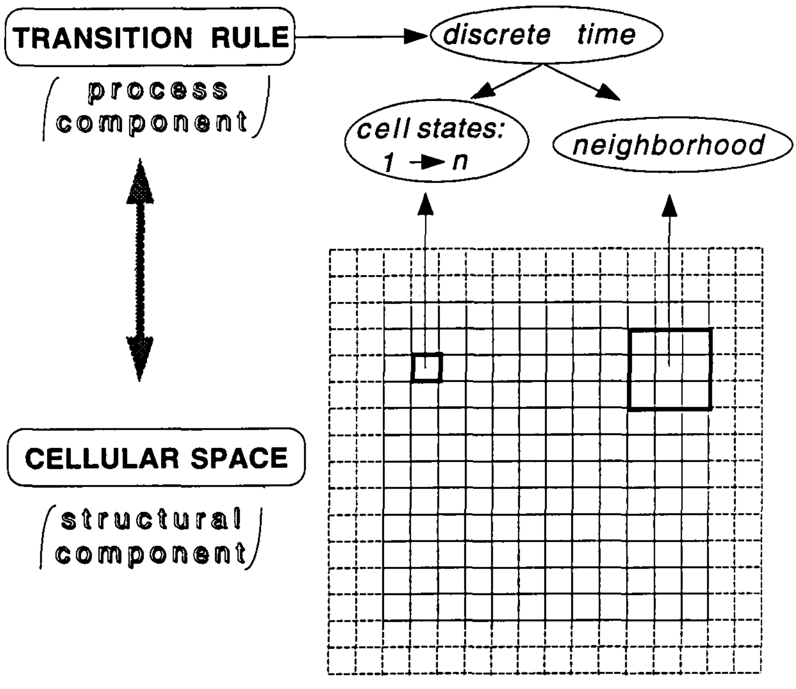

FIG. 1. Basic Components of Cellular Automata

Cities display two basic growth dynamics, life cycling and transition, or mutation in terms of ecological growth. For instance, buildings undergo a life cycle of construction and demolition. New developments reflect current regional economic trends, locational allowance or policy preference. They are "young" and "energetic" automata which have "reproduction capability" (attractiveness in terms of urban modeling) and stimulate immediate urban growth. Some of them may even attract development of other types or categories of new constructions (through economic agglomeration). However, in a specific location, a growth zenith period is usually short because of land availability and density constraints. Some urban automata become inactive with regard to future growth, but remain in the market for reasons of being part of existing urban development. In the view of simulation, the existence of these "survivals" affects regional economics and thus global constraint functions. Moreover, as the clock ticks, sooner or later they deteriorate and become vacant from the perspective of natural process. Once these automata are removed from the market, the locations (cells) occupied previously by them become available for a new cycle of growth in terms of "new construction," "renovation," or "gentrification." There are two important issues worth mentioning here. First, urban land use experiences transitional state changes, upgrading from vacant land to housing or to industrial or commercial use or degrading from a specific use to vacancy. This type of conversion can happen to any land category and at any time period. Second, the natural process of urban evolution is frequently interrupted and distorted by social and economic forces.

CA microscopic dynamics well represent many kinds of urban processes because CA exploit the dynamics at the finest level—automata located in cells. Automata can be directly employed to represent individuals of various socioeconomic status, while groups of automata can be used to signify urban institutions and associations (Albin 1975; Couclelis 1985). It is also very intuitive to use automata and their ensembles to represent physical components of cities such as land parcels and construction blocks (Xie 1994). Therefore, automata can be meaningfully manipulated to exemplify practical urban textures.

Another pragmatic assertion to support CA applications in urban studies lies in succinct representation of complex dimensions by CA models, an application that has not been widely recognized so far. Urban evolution involves multidimensional intricate interactions among a large number of human-made factors and environmental components as well. A concise, robust, and generic representation of this complex system is a critical task. The symbolic formulation of dynamic models based on the concept of progressive adaptation provides an ideal solution. The genetic algorithm (Holland 1975), the evolution strategy (Schwefel 1981), the cellular worlds (Couclelis 1985), and models derived from artificial life (Hoffmeister and Bäck 1991) are all constructed in this fashion. The symbolic model employs a small number of parameters which represent "the intrinsic complexity of individual components of the system, the complexity of various chains of interaction, and the effective range of influence of a single component" (Albin 1975, p. 7). This chained interaction fulfills feedback between involved components of the system, and updates automata states and external constraints iteratively. An adaptive learning capacity is thus rendered through these rewarding interactions. More importantly, these computable parameters can be extended to cover characteristics and activities taking place within subsystems [see Figure 2 and equations (1)–(16)]. For all of these reasons, CA models can be used effectively to explore urban dynamics.

However, there are a few typical problems, such as the lack of predictive power, the nature of a homogeneous neighborhood, the closure to external events, and the lack of spatiotemporal variance of transition rules, which need to be addressed cautiously in the urban context. The impression that the CA paradigm could be used only as a kind of qualitative metaphor is misleading but it can be ascribed to the following reasons. Early CA advocates in geography and urban studies placed an emphasis on the introduction of CA concepts rather than on technical details. Moreover, current CA applications in geosciences are still confined to descriptive representation of dynamic geoprocesses. In most cases these applications are either simplistic or hypothetical. In fact, CA was originally proposed in systems sciences as alternatives to models with predictive capacity. Newly developed CA-based evolutionary-type algorithms, particularly the Evolution Strategy (Schwefel 1981, 1988, 1989) and the Genetic Algorithm (Holland 1975), treat automata as a population of individuals. The individuals or automata in these models are able to adapt to a given environment (static or dynamically changing) by randomized processes of reproduction, recombination, and mutation. A record-keeping capability is built into these algorithms to track the number of individuals in the evolution process. This type of predictive power is incorporated in the design of DUEM [equations (15)–(16)], in which each individual is stored in a dynamically allocated population vector, optimizing the "dynamic allocation of memory" in terms of computation (Xie 1994). However, the complex nature of urban dynamics cannot be solved solely by quantitative approaches. The problem "complexity" is inherently associated with the problem of "uncertainty," which is deeply rooted in the process of the individual decision-making process. The social value of individualism or alternativism creates a dilemma in social studies because there is no clear-cut numerical solution. Therefore, the possibility of a synthesis between both quantitative and qualitative approaches is addressed in

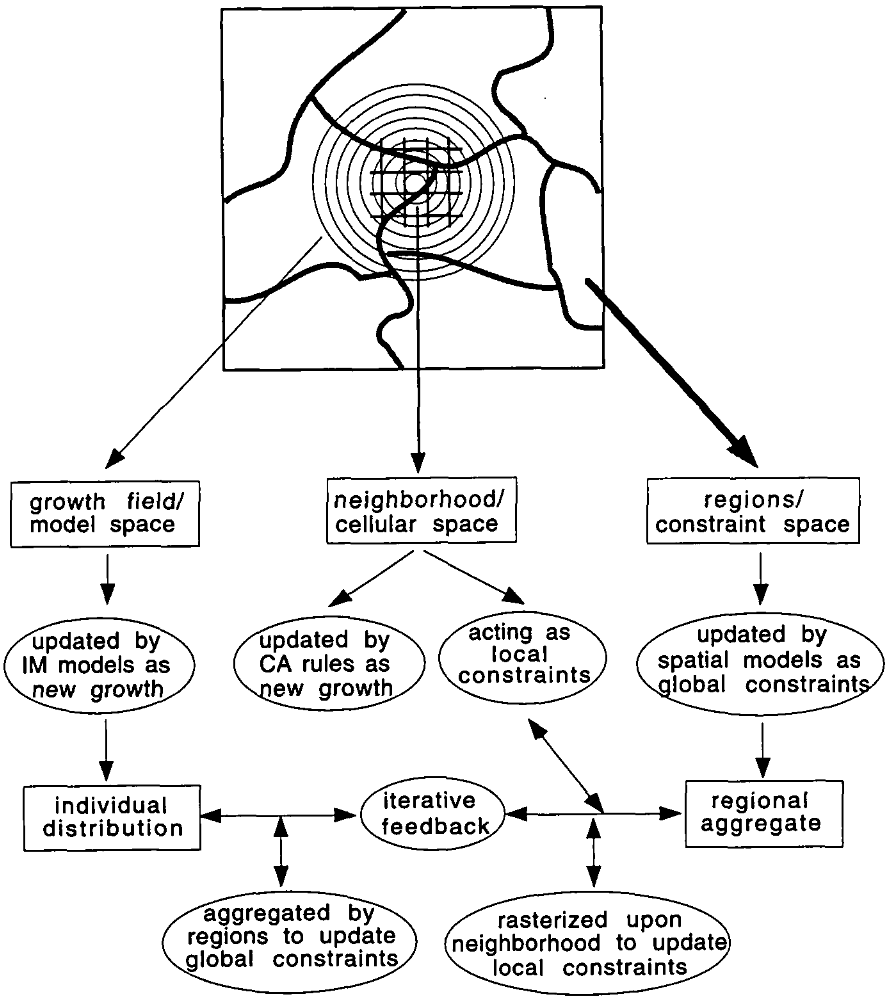

FIG. 2. Logical Operations between Three Types of Space in DUEM

the design of DUEM. The quantitative or numerical framework of DUEM is derived from the "evolution strategy" proposed by Schwefel (1981), and the qualitative function is realized through integration of "three types of space" and "interactive visualization" which will be further described in later sections.

The CA neighborhood is a physical frame which describes CA's discrete nature of space. Inspired by the experiments of the changeable CA neighborhoods (Toffoli and Margolus 1987), the zonal CA spaces (Couclelis 1985), and the neighborhood coherence principle (NCP) (Phipps 1989), we propose a hierarchical system of CA spaces consisting of neighborhood, field, and region, and simulating interactions between three types of space: *cell space*, *model space*, and *geographical space*. A good model should be able to identify discrepancies among these three types of space and perceive possible trade-offs and feedback between these conflicting and overlapping alternatives (Densham and Goodchild 1990). Cell space is an idealized space which shows an optimal state and a homogeneous nature. It is consistent with the neighborhood concept in terms of homogeneity (Phipps 1989). Model space is more pertinent to conceptual space on which theoretical models are built. Many urban and regional models (particularly those based on Newton's gravity law) show this affinity. The radiant zone (Dodd 1950) is one good example of this kind of space. We use the term *field* to name the model space because it resembles the gradient field in physics. Geographic space encompasses our surrounding environments. These environments impose impacts upon human societies in the form of physical, socioeconomic, and political boundaries, namely, *regions*. With the technical support of contemporary GIS, we can easily implement the linkages between the three types of space upon which the iterative feedbacks between three types of models can be gauged to enhance our capability of exploring urban complexity (Figure 2).

Search for CA transition rules involves modeling, parameterizing and simulation, an open-ended task of scientific enquiries (Langton 1989; Xie 1994). This analogy between CA rules and modeling strongly supports CA applications in modeling dynamic urban systems. This notion provides both theoretical and operational justifications for the design of DUEM. Moreover, the feedback between the local and the outside world is easy to establish through linking model parameters because transition rules can be treated as parameterized models (Langton 1992). The interaction between CA neighborhoods and geographic constraint regions can be activated through the calibration of model parameters in DUEM simulations. Thus, the design of CA's hierarchical spaces supports both internal dynamics and the feedback with external information.

The dynamics of urban growth signify recurrent temporal changes of spatial patterns of growth occurrence. In order to reveal the causes of changes in spatial patterns at a certain temporal scale, the DUEM model employs two tactics: (1) a hierarchical space and different scales of models, as discussed previously; and (2) the explicit specification of a *growth generator* and a *growth locator* in the design [see equation (1)]. The growth generator consists of individual-based models or transition rules to update internal automaton states, representing the temporal dimension of DUEM simulations. The spatial aspect of dynamic growth is monitored by the growth locator through modeling of three parameterized functions, the step length operator, the growth direction operator and the growth shape operator. The interface or synthesis of the temporal and the spatial dimensions is achieved by a set of constraint functions, which determine the local and the global as well as the ecological and the socioeconomic reactions to the new development [see equation (5)]. This dialectical unity of various dimensional factors operating upon a hierarchical system of spaces follows the concept of conflict resolution in terms of artificial intelligence programming (Xie 1994).

Modeling spatial phenomena and integrating external information can be greatly enhanced through the integration of GIS technology. GIS can make four contributions to dynamic urban modeling: (1) the provision of abundant data sources; (2) the power of processing spatial data; (3) the support of heuristic modeling approaches; and (4) the enhancement of both system and user interfaces. These GIS capacities are built into the DUEM design to develop a heuristic approach, facilitating communications between internal automata dynamics and external constraints (Figure 3). Urban automata reside on a grid system and update their states by following "natural reproduction processes." External influences in the form of physical environments, social-economic conditions, or policy-oriented interferences are areal-based aggregates. Although they are incompatible in terms of spatial structure, urban automata and external constraints can be turned into compatible systems through GIS rasteriza-

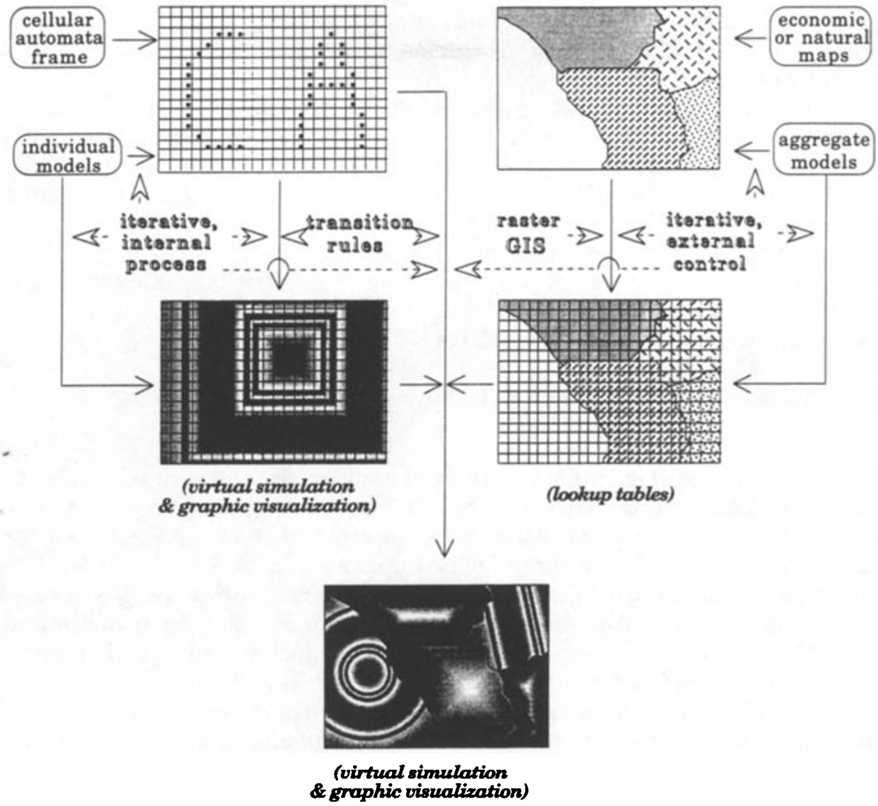

FIG. 3. The Heuristic Approach of Integrating External Information with CA Internal Dynamics

tion and vectorization processes. The iterations of rasterization and vectorization build communication channels between automata and their environments, and between the three level hierarchical DUEM spaces (Figures 2, 3). Finally, GIS modularity and visualization help connect GIS systems, user-developed specific modeling modules, and other supporting software packages into a coherent system with graphic user interface, which enables users to handle data selection and analysis, model specification, calibration and prediction, and system adjustment, interactively and flexibly.

THE GENERAL FORM OF DYNAMIC URBAN EVOLUTIONARY MODEL (DUEM)

This section outlines the formal structure of the dynamic urban evolutionary model described above. The model proposed here is a two-dimensional cellular automata framework, incorporating the *Evolution Strategies* of individuals and the cellular automata concepts, and using raster GIS methodology for generic application to urban dynamics (Figure 3). The DUEM formalism is stated in symbolic and set-theoretic notations (Linz 1990) as follows:

$$DUEM = \{\Psi, p_{ji}, \eta, \omega, \vartheta, \delta, P_{ki}\} \tag{1}$$

where Ψ is a finite state set of land categories,

p_{ji} is the initial set of land use units ($j \in \Psi$),
η is a composite function describing urban growth dynamics called the growth generator,
ω is a joint function locating new developments called the growth locator,
ϑ is a finite set of the synthetic effects of urban development controlling functions called the constraint input alphabet,
δ: $p_{ji} \cap \eta \cap \omega \cap \vartheta \rightarrow P_{ki}(k \in \Psi)$ is a composite function called the transition function,
P_{ki} is the final set of land use units ($k \in \Psi$),
j represents finite numbers of land categories at the beginning of a simulation,
k represents finite numbers of land categories at the end of a simulation ($j \in k$), and
i is an index ranging from 0 to the maximum size of a predominant land use.

Compared with the standard CA formula, the major distinction in the DUEM is a further delineation of the transition function by incorporating important components of urban dynamics discussed in the previous section. The transition function (δ) is an intersection of three DUEM operators (η, ω, ϑ) which simulate various aspects of urban dynamics by taking the initial land use units p_{ji} as the seeds and generating new land use units (urban automata) P_{ki}. For prediction purposes in the DUEM model, P_{ki} is a specific urban automaton and $\sum_i P_{ki}$ represents the quantity of land use units of a land category k, and thus the total size of $P_{ki}(\sum_k \sum_i P_{ki})$ is a direct measurement of total urban developments at a specific time. This notation differs from the conventional notation because i has a varying size for each land category rather than being a constant. If we arrange land use units $P[k][i]$ (or $p[j][i]$) as a "rectangular" collection of elements in rows and columns, the varying row lengths give the "rectangle" a ragged look, thus, a "ragged array" (Kelley and Pohl 1990). The adoption of ragged arrays of land use units in the DUEM model facilitates dynamic memory allocation for data storage in computer programming.

We first discuss the growth generator (η), which implements the concept of urban evolution and determines urban land use development. η takes the initial urban land units as inputs and differentiates them into four subgroups,

$$\eta(p_{ji}) = R(p_{ji}) + U(p_{ji}) + D(p_{ji}) + M(p_{ji}) \qquad (2)$$

where

$R(p_{ji})$ denotes the portion of the initial land units that attracts new growth, and thus is capable of reproducing new growth of the same categories;

$U(p_{ji})$ stands for the portion which has passed the zenith of its natural development and is unable to stimulate new growth because of lack of viability but still remains in the market stocks;

$D(p_{ji})$ represents the deteriorated portion of the current stocks, or simply the "deaths," which will be removed at the next simulation; and

$M(p_{ji})$ is used to signify the portion of initial land units that can generate new growth of different land categories or introduce new categories into the existing land stocks, a process analogous to "natural mutation." In fact, $M(p_{ji})$ involves sophisticated decision rules concerning transition of various land categories.

The growth generator (η) operates on the basis of the internal characteristics of the seeds (such as construction year, construction value, land value, physical condition, etc.) and the regional socioeconomic environments. One simple case takes into account a single piece of information—"the construction year" (Yr),

$$\eta(p_{ji}) = R[p_{ji}(Yr)] + U[p_{ji}(Yr)] + D[p_{ji}(Yr)] + M[\gamma \cdot^{*} p_{ji}(Yr)], \tag{3}$$

where γ is a probability set that determines the opportunities of upgrading or changing of land categories within a subgroup. We assume in this case a place experienced an initial twenty-five-year (0–25) period of growth, then a fifteen-year (26–40) sustainable growth period, and later a ten-year (41–50) period of decay. Suppose, moreover, that the rapid growth and diversification of land uses happened during the most prosperous period of the twelfth to the sixteenth (12–16) years. Then equation (3) can be written as

$$\eta(p_{ji}) = R[p_{ji}(25)] + U[p_{ji}(40)] + D[p_{ji}(50)] + M[\gamma \cdot^{*} p_{ji}(12-16)]. \tag{4}$$

The growth generator performs a task analogous to "natural selection" in the urban evolution process. Stocks in $D(p_{ji})$ will be removed from the land stocks. Elements in $U(p_{ji})$ are no longer considered active for the next round of simulation though they remain as an important part affecting the growth environment. Members in $R(p_{ji})$ and $M(p_{ji})$ are vigorous automata which carry dynamics in future urban growth processes. Moreover, $M(p_{ji})$ determines the upgrading of "states," that is, land categories in the DUEM simulation through involving itself with a set of knowledge-based rules that govern transition of land categories.

The growth locator (ω) determines where new growth is to be located. ω extends the concept of CA space from a universal neighborhood to a three-level hierarchy of space, neighborhood, field, and region. The neighborhood is defined as Ω_1, the field as Ω_2, and the region as Ω_3. They usually imply a systematic change in scale, but do not strictly nest within each other. In most cases, ω consists of three subfunctions acting upon the field or the region,

$$\omega = \begin{cases} \sigma_{R(p)|M(p)} \cap \theta_{R(p)|M(p)} \in \Omega_2 & \text{(point growth)} \\ \sigma_{R(p)|M(p)} \cap \theta_{R(p)|M(p)} \cap \varsigma \in \Omega_2|\Omega_3 & \text{(linear and areal growth)} \end{cases} \tag{5}$$

where σ is the step-length operator, θ is the growth-direction operator, and ς is the growth shape operator. σ and θ jointly determine a potential site or birthplace for a new development of "point" nature such as houses and factories depending on scale (or resolution). ς determines the shape of linear and areal artifacts such as transport networks, large shopping malls, or city blocks. ς is integrated here for simulations concerning object-oriented automata (Xie 1994).

Stochasticity and human intervention can be integrated into the growth processes through the manipulation of the step-length and the growth-direction parameters. Differentiated growth can be generated if the step length is described by a probability distribution, a function of relative position along either the x axis or the y axis, $\sigma_\ell(x)$. (It does not matter here which axis is chosen because the direction is controlled by θ.) A step-length function can be defined in many ways. Four basic types (linear, exponential, negative power, and gamma) are presented here:

(i) the linear gradient probability function

$$\sigma_\ell(x_\ell) = \frac{\binom{x_{max}}{x_\ell}}{x_{max}!}\, x\colon (1, \Omega_2) \tag{6}$$

where x_{max} is the maximum growth step length within a space, usually being equal to the radius of the growth field, and x_ℓ is a specific step length. This equation defines a linear probability function stating that the closer a cell is located to the bearing parent's location, the greater its chance is for capturing the "child."

(ii) the negative exponential probability function

$$6_\ell(x_\ell) = \frac{\exp(-cx_\ell)}{\sum \exp(-cx_\ell)} \quad x\colon (1, \Omega_2), \tag{7}$$

(iii) the negative power probability function

$$6_\ell(x_\ell) = \frac{x_\ell^{-\tau}}{\sum x_\ell^{-\tau}} \quad x\colon (1, \Omega_2), \tag{8}$$

(iv) the gamma probability function

$$6_\ell(x_\ell) = \frac{x_\ell^{-\tau}\exp(-cx_\ell)}{\sum x_\ell^{-\tau}\exp(-cx_\ell)} \quad x\colon (1, \Omega_2). \tag{9}$$

In the latter three functions, c and τ are parameters of the associated probability distribution functions which can take any value from $-\infty$ to $+\infty$. These probability functions are derived from the urban population density models, which reflect the inverse distance law for population density distributions (Batty and Xie 1994b).

In the cellular (grid) space, these probability sets are transformed into sets of integers for describing randomness of selecting a step length,

$$INT[x_1, \ldots, x_1, x_\ell, \ldots, x_\ell, \ldots, x_{max}] \in (x_1, x_{max}) \tag{10}$$

where INT is a set of integers,

x_1 is one cell distance from a growth origin in the x-axis direction,
x_{max} is the maximum cell distance from the origin in the x-axis direction,
x_ℓ is the lth cell distance from the origin in the x-axis direction, $(x_1, x_{max}) = (1, \Omega_2)$,
$COUNT(x_1, \ldots, x_1, x_\ell, \ldots, x_\ell, \ldots, x_{max}) = N$,
N is a large integer usually predetermined as 1,000 or 10,000,
$COUNT(x_\ell, \ldots, x_\ell) = 6_\ell(x_\ell) \times N$, indicating that there are $6_\ell(x_\ell) \times N$ number of x_ℓ in the set INT.

Then the step length can be determined by a random generator,

$$INDEX = random(N), \tag{11}$$

$$6 = INT_{INDEX}. \tag{12}$$

The stochasticity of growth directions determines an anisotropic growth. The scheme of eight nearest-neighbor directions is adopted here (Figure 4). The general rule for directional stochasticity is

$$Pb(\theta) = \sum_{l=1}^{n=8} Pb(\theta_\ell) = 1.0. \tag{13}$$

A "planned" growth or "selective birth," however, does not secure its sur-

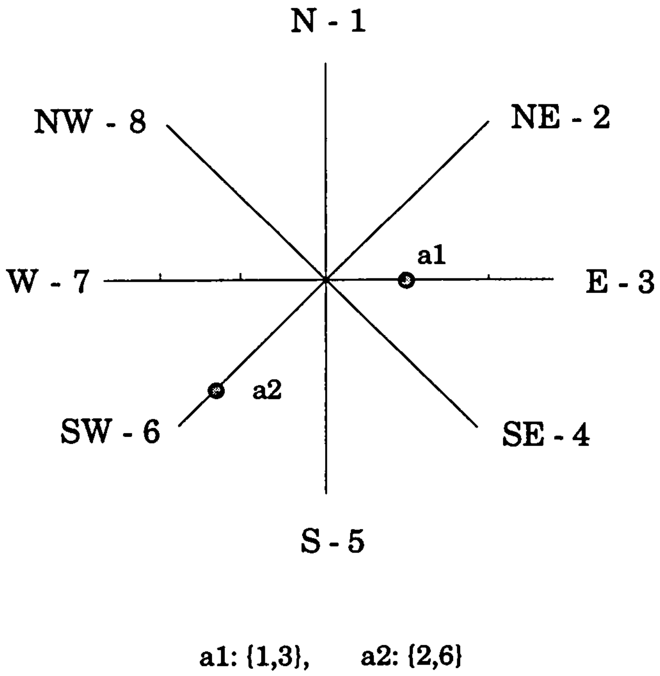

FIG. 4. Eight Neighbor Directions

vival. Its life relies on where the birth takes place, that is, the availability of the space, and the congeniality of local and global environmental and socio-economic conditions. That is the base for the rationale that a set of internal and external constraints should be brought into consideration. In other words, the constraint input alphabet (ϑ) represents the "unplanned operation of ecological and social processes of urban growth" through the interaction between three levels of "spatial units"—neighborhood, field, and region. A potential growth site is chosen within the field. Local constraints are checked over the immediate neighborhood of the growth site. Global factors are investigated over the regions. Operationally, local and global constraints can be constructed as probabilistic functions or look-up tables, and their resulting interactions are formulated as a finite set of alphabets. An application discussed in Xie's dissertation involves a binary set of alphabets $\vartheta[0,1]$ to represent the constraint functions (Xie 1994),

$$\vartheta = \begin{cases} 1, & \text{if } Pb[L_m(\;)] \cap Pb[G_n(\;)] > \sigma \\ 0, & \text{if } Pb[L_m(\;)] \cap Pb[G_n(\;)] \leq \sigma \end{cases} \tag{14}$$

where

$L_m(\;)$: mth local constraint function,
$G_n(\;)$: nth global constraint function, and
σ is the system-wide constraint checking criterion.

The probabilistic rules are set up to ensure that the actual constraint criterion will approach σ. The constraint rules used in this paper are listed in Table 1.

TABLE 1
The Probability Set of the Transitional Function for Residential Growth

```
if (Built-Year < 2.0) Vitality = 1.0;
else if (Built-Year > = 2.0 && Built-Year < 4.0) Vitality = 0.9;
else if (Built-Year > = 4.0 && Built-Year < 6.0) Vitality = 0.8;
else if (Built-Year > = 6.0 && Built-Year < 8.0) Vitality = 0.7;
else if (Built-Year > = 8.0 && Built-Year < 10.0) Vitalit = 0.6;
else if (Built-Year > = 10.0 && Built-Year < 12.0) Vitality = 0.5;
else if (Built-Year > = 12.0 && Built-Year < 14.0) Vitality = 0.4;
else Vitality = 0.3;
if (Density < 0.2) Den_Prob = 1.0;
else if (Density > = 0.2 && Density < 0.3) Den_Prob = 0.9;
else if (Density > = 0.3 && Density < 0.4) Den_Prob = 0.8;
else if (Density > = 0.4 && Density < 0.5) Den_Prob = 0.7;
else if (Density > = 0.5 && Density < 0.6) Den_Prob = 0.6;
else if (Density > = 0.6 && Density < 0.7) Den_Prob = 0.5;
else if (Density > = 0.7 && Density < 0.8) Den_Prob = 0.4;
else Den_Prob = 0.3;
Joint_Prob = Vitality*Den_Prob;
Flag1 = (int)(Joint_Prob*1000.0);
Flag2 = random_int(1000);
if(Flag2 > Flag1) Survival = 0;
else Survival = 1.
```

The sequential operations performed by δ simulate the "processes" of urban land use development. δ takes the initial land units p_{ji} as inputs, synthesizes resulting decisions from the generator η, the locator ω, and the constraint alphabet ϑ, and then determines the reproduction of new growth P_{ki}. P_{ki} is the overall urban land use developments over a finite time and can be calculated as

$$P_{ki}(t+1) = \sum_{ki} \eta[p_{ji}(t)], \quad \text{or} \tag{15}$$

$$P_{ki}(t+1) = \sum_{ji} R[p_{ji}(t)] + \sum_{ki} M[p_{ji}(t)] + \sum_{ji} U[p_{ji}(t)] - \sum_{ji} D[p_{ji}(t)], \tag{16}$$

where

$$\sum_{ki} \eta[p_{ji}(t)] \quad \forall \omega \cap \vartheta \subseteq \delta,$$

$$\sum_{ji} R[p_{ji}(t)] \quad \forall \omega \cap \vartheta \subseteq \delta,$$

$$\sum_{ki} M[p_{ji}(t)] \quad \forall \omega \cap \vartheta \subseteq \delta,$$

$$\sum_{ji} U[p_{ji}(t)] \quad \forall \omega \cap \vartheta \subseteq \delta,$$

$$\sum_{ji} D[p_{ji}(t)] \quad \forall \omega \cap \vartheta \subseteq \delta,$$

$$j \in k \in \Psi.$$

TABLE 2
The Six Periods of Simulation for Multisector Land Use in Amherst, New York

BLT-YR	PERIOD	RESID.	COMME.	INDUS.
1777–1880	initial (seeds)	238	21	0
1881–1935	simulation-1	3020	133	0
1936–1945	simulation-2	2338	41	2
1946–1960	simulation-3	9861	345	0
1961–1980	simulation-4	11576	2232	8
>1980	simulation-5	4112	419	10

MULTISECTOR LAND USE SIMULATIONS IN AMHERST, NEW YORK

The preliminary tests of the DUEM model are conducted using historical data for building construction in Amherst, New York, a suburb of Buffalo. The major database was compiled from The Property Assessment Tape, obtained from the State Board of Equalization and Assessment of New York which contains information about land properties registered in the State of New York before the end of March 1989. In total, there were 35,193 registered properties in Amherst (Table 2). There are two important variables, the construction year of the property (*BLT-YR*) and the land use type (*LD-TP*), which are used as grouping variables in the simulations and statistical analysis. *BLT-YR* is grouped into six periods according to the significant historical events in Buffalo's social and economic development. Major land uses are depicted in Figure 5, showing clear patterns. Amherst is predominately a residential suburb. The large proportion of middle- and upper-middle-class residents in Amherst's population has attracted rapid commercial growth. Commercial land uses, such as shopping malls, plazas, and commercial strips have been built and clustered at the important transport intersections or along the major roads. Industries are concentrated in two corners of the town, the northwest and the southeast. The second data source comes from the postcensus version of the TIGER-LINE files, from which we created digital political boundaries and the traffic network. These linear features have significant impacts on current land use patterns which will become apparent as we proceed in the discussion.

Here we attempt to demonstrate how the generalized DUEM model can be applied easily to describe urban dynamics in real cases rather than attempting to generate realistic predictions. Therefore, many functions or tuples in equation (1), $DUEM = (\Psi, p_{ji}, \eta, \omega, \vartheta, \delta, P_{ki})$, are either simplified or omitted. The *final state set* only includes two land categories, residential (r), and commercial (c),

$$\Psi = (r, c). \tag{17}$$

The *initial set of urban developments* can be described as

$$\sum_j p_{ji} = (238, 21). \tag{18}$$

The *growth generator* is defined as a probabilistic attraction function depending on the year of construction,

$$\eta = attr(p_{ji}(Yr)). \tag{19}$$

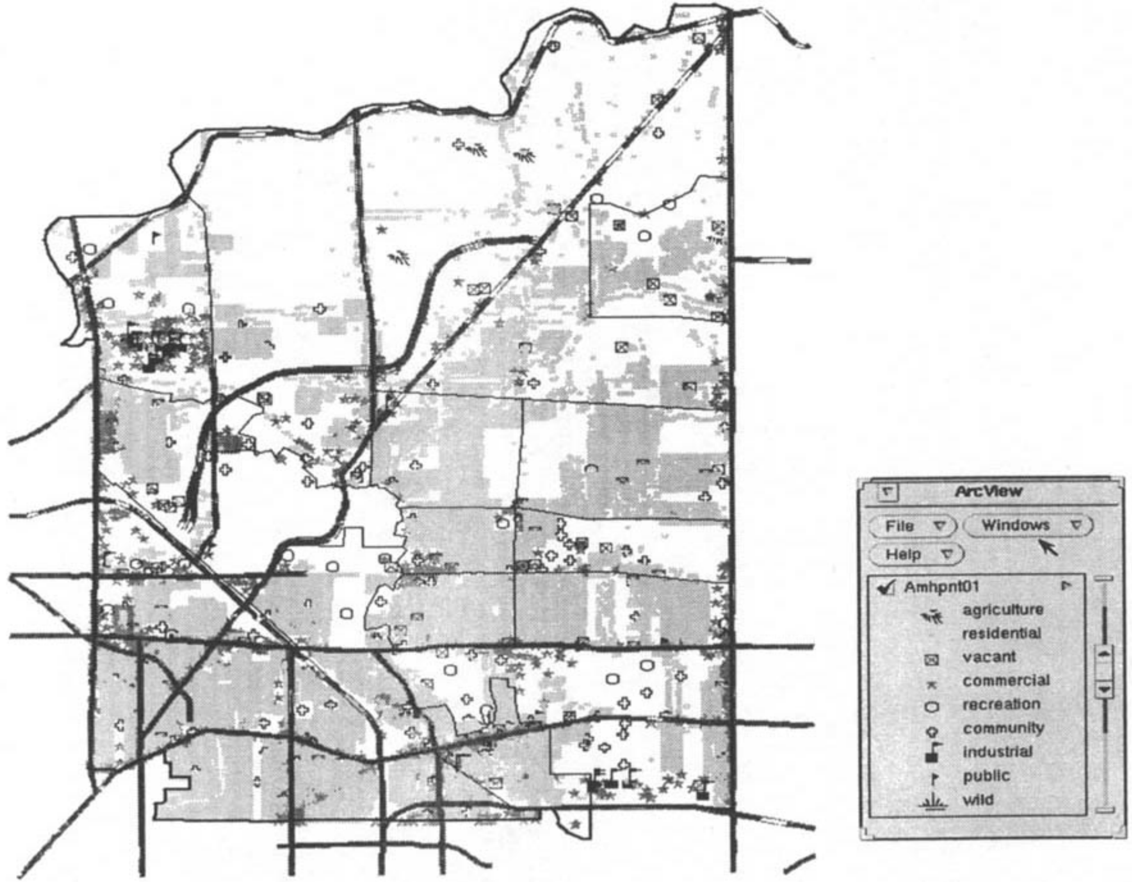

FIG. 5. The Land Use Pattern in Amherst, New York

The *growth locator* is determined by two random generators,

$$\omega = [\delta = random(\Omega_2) \cap \theta = random(8)], \tag{20}$$

where Ω_2 represents the growth field that has a fixed length (of 50 cells), though a variable length can reflect the impact of technical change particularly in transport technology (Xie 1994).

The *constraint input alphabet* includes two subsets, dealing with residential and commercial land uses respectively,

$$\vartheta = \begin{cases} \vartheta_r \text{ (the residential land use)} \\ \vartheta_c \text{ (the commercial land use).} \end{cases} \tag{21}$$

For the residential development, the *constraint input alphabet* ϑ_r is an intercept set of the neighborhood density function and the global constraint function,

$$\vartheta_r = dens\left(\sum p_{ji}\right) \in \Omega_1 \cap glob(\ ) \in \Omega_3. \tag{22}$$

It is proper to use neighborhood density as an impeding factor to future growth. The global constraints involve environmental factors (such as land suitability, slope, groundwater, flooding, soil infiltration, etc.), and socioeconomic factors (zoning, city master plan, urban infrastructure, community development, economic sustainability, etc.) as well. Ideally these influences should be examined jointly so that a comprehensive index set can be synthesized. For demonstration purposes, we have constructed natural development regions based on the historical observations as a substitute for the external constraint regions by employing a cartographic smoothing method (Xie 1994). This method plots constructions on a fifty-meter grid map and then applies an algorithm to merge clustered constructions and to wipe out isolated ones. In this case, a grid is merged into a construction cluster if the count of constructions within its neighborhood (a 10×10 grid square) is equal to or more than 5, and a site is removed if the count is less than 5. The newly merged clusters are much wider than the original ones, thus creating substantially large "composite natural areas" as proper surrogates for external constraint regions (Figure 6). With contemporary GIS, these digital maps can be converted easily into a series of binary $(0, 1)$ data sets or look-up tables. Figure 6 also involves the *periodical updating* of external constraint regions. The digital table of the natural areas in each period acts as a global regulator controlling births (new growth). In other words, new growth has a great opportunity of surviving if it is positioned at a point falling within a cluster of natural areas. Otherwise it usually dies. The periodical updating of global constraints is a critical technique for realistic simulation of dynamic urban complexity.

In the case of commercial development, the *constraint input alphabet* ϑ_c is a joint function of the neighborhood density function and the global constraint function as well as the distance function from the nearest transport lines. Commercial land use is mainly located along the existing transport lines. This association is intuitive and straightforward but not exhaustive. There are many other factors, such as landscape, neighborhood amenity, and parking space, that can play important roles in the decision making for commercial development. Here the attention is focused, however, on the accessibility of transportation for the simplicity of the demonstration. ϑ_c can be formulated as

$$\vartheta_c = dens\left(\sum p_{ji}\right) \in \Omega_1 \cap glob(\ ) \in \Omega_3 \cap dist(near\_trans). \tag{23}$$

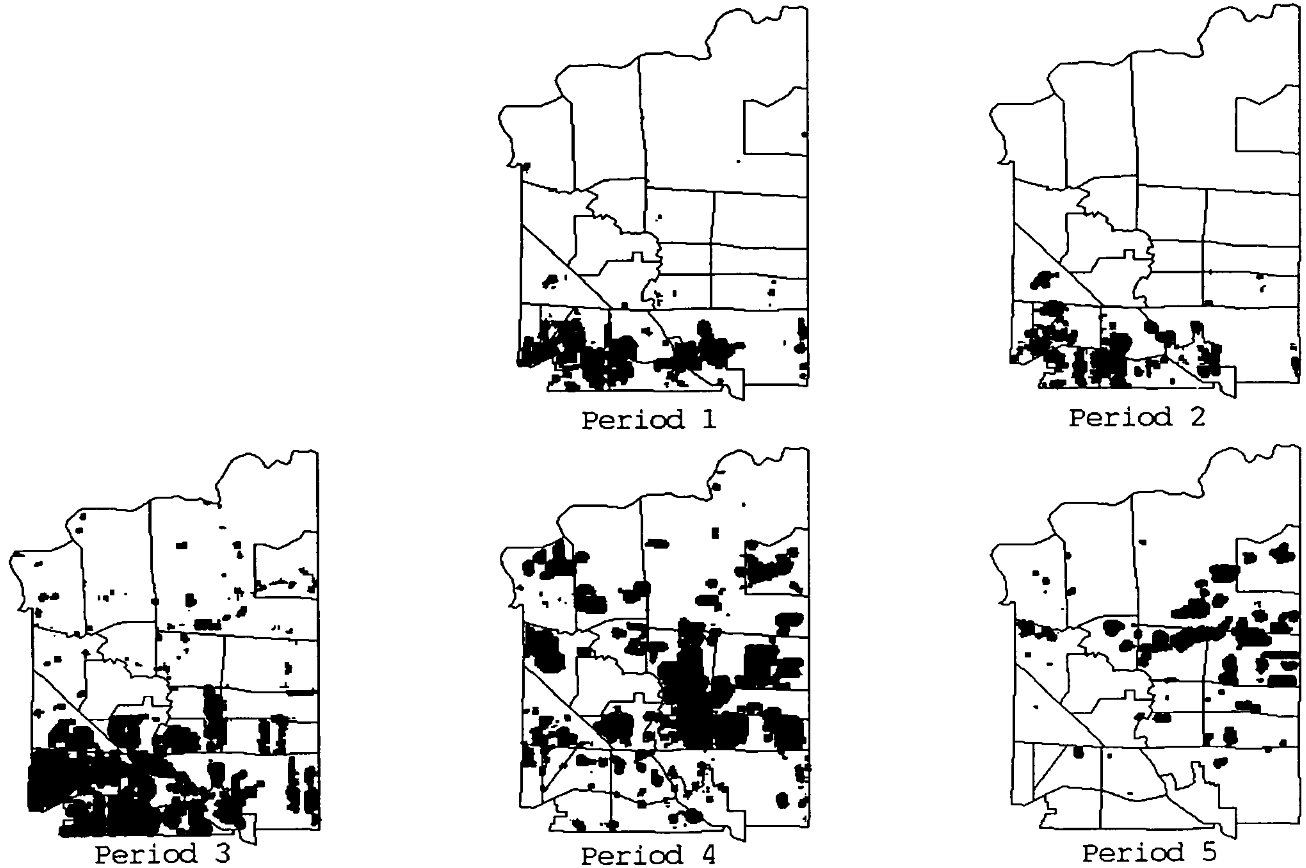

FIG. 6. The External Constraint Regions: "Natural Areas." The natural areas are derived from the Map of Perodical Dispersion of Constructions by adopting the GIS Technique of "surface smoothing."

The *transition function* also consists of two subsets, one for the residential land use and the other for the commercial land use,

$$\delta_r = \begin{cases} \delta_r \text{ (the residential land use)} \\ \delta_c \text{ (the commercial land use).} \end{cases} \tag{24}$$

In the case of residential growth, the transition function can be expressed as

$$\delta_r\colon p_{ri} \cap \eta \cap \omega \cap \vartheta_r \rightarrow P_{ri}, \tag{25}$$

where P_{ri} is the *final set of residential land use units* $(r \in \Psi)$. Since the global constraint in the constraint input alphabet ϑ_r is handled by a binary look-up table with GIS techniques, δ_r defines the composite operations of the attraction function and the housing density function in the neighborhood, which can be described as a stochastic set of rules (Table 1).

For commercial development, the transition function is defined as

$$\delta_c\colon p_{ci} \cap \eta \cap \omega \cap \vartheta_c \rightarrow P_{ci}. \tag{26}$$

The probabilistic nature of commercial land development was not considered in this illustration. Thus, the transition function for commercial land use can be simply formulated as a group of *if-then* instructions:

1. New commercial growth is allowed to be generated and survive *if* the construction year of a commercial site is less than τ; and
2. *If* the potential commercial growth happens within a threshold distance d (in cell units) from a transport line; and
3. *If* the land use intensity within a developing commercial area (the commercial neighborhood) does not exceed ρ; and
4. *If* the potential commercial growth is located within periodically updated *external constraint regions* $(P_{ci} \in \Omega_3)$.

In the first simulation, we gave the following assignments to these parameters:

$\tau \leq 40$,
$d \leq 5$,
$\rho \leq 0.4$, and
$P_{ci} \not\subseteq \Omega_3$ (growth was not limited within the constraint regions).

Under these conditions, commercial development forms small clusters in a string of beads along transport lines (Figure 7).

The parameters of the second simulation are listed as follows:

$\tau \leq 40$,
$d \leq 10$ (the width increased compared with Simulation 1),
$\rho \leq 0.4$, and
$P_{ci} \in \Omega_3$ (growth was only allowed within the constraint regions).

The growth pattern generated by following these rules is obviously different from the first simulation (Figure 8). Some larger clusters of commercial growth are formed along the transport lines. The main reason is that commercial growth is restricted within the natural areas but allowed within relatively enlarged neighborhoods (the permissible width of growth increased from five cell units to ten units).

We also conducted a crude statistical analysis about the performance of the multisector simulations. The r-square values of the simulation results regressed

FIG. 7. The Multisector Land Use: Simulation I

FIG. 8. The multisector Land Use: Simulation II

TABLE 3
The R-square Values from the Multisector Simulations

Test	Residential Growth r^2	Commercial Growth r^2
1	0.72289	0.0360
2	0.75097	0.10167

against the observed values over the Census tracts are reported in Table 3. The r-square values for the residential simulations are above 0.70, indicating that the simulation outcomes are significantly stable. The performance of commercial sector simulations, however, is quite poor although Simulation 2 has a slightly improved r-square value. One explanation for the poor performance is that commercial developments often occur along major transport lines or concentrate in a limited number of locations. Thus the regression analysis based on the Census tract aggregates may not be able to capture spatial characteristics of commercial developments. Other causes for the poor performance of the commercial sector simulations and the potential approaches for improvement will be presented in the next section.

DISCUSSION

The Amherst multiple-sector land use simulations are based on three types of spaces (the neighborhood, the growth field, and the geographical region) and operationalized through four procedures (the selection of growth direction, the decision of growth step length, the simulation of growth, or reproduction of growth seeds, and the check of locational conditions for new growth). Each of the spaces represents an important component of the urban mosaic while the procedures signify critical phenomena of urban growth. The Amherst simulations demonstrated that urban growth dynamics were properly interpreted, and represented by the components and processes encoded in the DUEM model. Any growth events pertaining to land use, either normal or abnormal, can be controlled or modeled in the DUEM simulations. Therefore, the design of DUEM framework is transparent, comprehensive, and flexible.

The performance levels of the simulations depend on domain-specific knowledge in the form of transition rules. In the multiple-sector simulations, the performance levels for commercial development in general were poor because the transition rules adopted were very simplistic. The transition rules in Simulation 1 simply allocated commercial development to the sides of transport lines without considering residential locations. The rules in Simulation 2 strictly confined commercial development within residential areas along the major transport lines. These were two extremes. A better solution should incorporate more detailed rules or relevant information into the simulations. For instance, a deterministic mapping approach based on constructing natural regions of multiple categories may improve simulation performance. Instead of the current binary character, the natural regions can be distinguished by various categories of land use. In the sense of practical applications, the function of natural regions should be updated to determine *what kind of new growth is allowed* in addition to *where new growth is permitted.* The adoption of multiple category natural regions will definitely improve modeling capacity and enhance control over the simulation processes. This is just one possible suggestion to refine the transition rules. A better solution is to gather all relevant social, economic, and physical information and use them directly as composite external constraints. The

DUEM model provides many flexible ways at multiple levels to approach the same endpoint which will be explored in a systematic manner elsewhere. In short, the search for better transition rules relies on a researcher's knowledge of particular problems and available data, rather than being a design issue in the DUEM model.

The DUEM model is proposed here as a generic paradigm for building computer-based quantitative models concerning dynamic geographic and decision-making processes. Needless to say, further empirical experiments with the DUEM approach should be undertaken. Moreover, several potential contributions by the DUEM approach to methodologies of geographic research are most appealing. The DUEM is the first operational design capable of integrating cellular space, model space, and geographic space, and various types (or levels) of models running within these spaces. The DUEM manages the interactions between spatial and temporal dimensions through the conflict resolution of the growth generator and the growth locator. The DUEM also incorporates contemporary information technologies particularly GIS and scientific visualization into modeling practices. Finally, the DUEM promotes a new perception of complexity in science applied to modeling complex systems, and human and social systems in particular.

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