2019年合肥工业大学第十一届高等数学竞赛试题及解答

一、（15分）设 $\lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}-\left(A+Bx+Cx^2\right)}{x^3}=D$，求常数 A,B,C,D.

解：泰勒展开 $(1+x)^{\frac{1}{x}}=\exp\left(\dfrac{\ln(1+x)}{x}\right)\overset{\text{Taylor}}{=}\exp\left(1-\dfrac{1}{2}x+\dfrac{1}{3}x^2-\dfrac{1}{4}x^3+o\left(x^3\right)\right)$

$$=\exp\left(1+\left(-\frac{1}{2}x+\frac{1}{3}x^2-\frac{1}{4}x^3\right)+\frac{1}{2!}\left(-\frac{1}{2}x+\frac{1}{3}x^2-\frac{1}{4}x^3\right)^2+\frac{1}{3!}\left(-\frac{1}{2}x+\frac{1}{3}x^2-\frac{1}{4}x^3\right)^3+o\left(x^3\right)\right)$$

$$=\exp\left(1-\frac{1}{2}x+\left(\frac{1}{3}+\frac{1}{8}\right)x^2+\left(-\frac{1}{4}-\frac{1}{6}-\frac{1}{48}\right)x^3+o\left(x^3\right)\right)$$

$=\exp\left(1-\dfrac{1}{2}x+\dfrac{11}{24}x^2-\dfrac{7}{16}x^3+o\left(x^3\right)\right)$，因此得到 $A=e,B=-\dfrac{1}{2}e,C=\dfrac{11}{24}e,D=-\dfrac{7}{16}e$.

二、（15分）设 $f(x)=\arctan\dfrac{1-x}{1+x}$，求 $f^{(2019)}(0)$.

解：由于 $f'(x)=-\dfrac{1}{1+x^2}=\sum\limits_{n=0}^{\infty}(-1)^{n+1}x^{2n}$，即 $f(x)=f(0)+\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{2n+1}x^{2n+1}$，因此有

$$f^{(n)}(0)=\begin{cases}\dfrac{\pi}{4}, & n=0\\ 0, & n=2k\\ (-1)^{\frac{n+1}{2}}(n-1)!, & n=2k-1\end{cases}\qquad\left(k\in\mathbb{Z}^+\right)$$，故 $f^{(2019)}(0)=2018!$.

三、（15分）设 $f(x)$ 在 $[a,b]$ 上连续可导，$f(a)=f(b)=0$，证明：

$$\int_a^b|f(x)|\,\mathrm{d}x\le\frac{(b-a)^2}{4}\max_{a\le x\le b}\left|f'(x)\right|.$$

证明：令 $M=\max\limits_{a\le x\le b}\left|f'(x)\right|$，由题设易知 $\int_a^b|f(x)|\,\mathrm{d}x=\int_a^{\frac{a+b}{2}}|f(x)|\,\mathrm{d}x+\int_{\frac{a+b}{2}}^b|f(x)|\,\mathrm{d}x$.

法1：利用 Taylor 公式二阶展开 $\int_a^{\frac{a+b}{2}}|f(x)\mathrm{d}x|=\int_a^{\frac{a+b}{2}}\left|f(a)+f'(\xi)(x-a)\right|\mathrm{d}x$

$\le M\int_a^{\frac{a+b}{2}}\left|x-a\right|\mathrm{d}x=M\int_a^{\frac{a+b}{2}}(x-a)\mathrm{d}x=M\left(\dfrac{(a+b)^2}{8}-\dfrac{ab}{2}\right)$，同理可得

$\int_{\frac{a+b}{2}}^b|f(x)|\,\mathrm{d}x\le M\left(\dfrac{(a+b)^2}{8}-\dfrac{ab}{2}\right)$，两式相加得 $\int_a^b|f(x)|\,\mathrm{d}x\le\dfrac{1}{4}M(b-a)^2$.

法 2：由拉格朗日中值定理 $f(x)=f'\left(\xi_1\right)(x-a), a<\xi_1<x$，

$f(x)=f'\left(\xi_2\right)(x-b), x<\xi_2<b$，因此

$$\int_a^b |f(x)|\,\mathrm{d}x=\int_a^{\frac{a+b}{2}}\left|f'\left(\xi_1\right)\right|(x-a)\mathrm{d}x+\int_{\frac{a+b}{2}}^b\left|f'\left(\xi_2\right)\right|(b-x)\mathrm{d}x$$

$$\leq M\left(\int_a^{\frac{a+b}{2}}(x-a)\mathrm{d}x+\int_{\frac{a+b}{2}}^b(b-x)\mathrm{d}x\right)=\frac{1}{4}M(b-a)^2.$$

四、(10 分) 计算 $\displaystyle\int\frac{x\ln\left(x+\sqrt{1+x^2}\right)}{\left(1-x^2\right)^2}\mathrm{d}x$.

解：凑微分 $\displaystyle\int\frac{x\ln\left(x+\sqrt{1+x^2}\right)}{\left(1-x^2\right)^2}\mathrm{d}x=-\frac{1}{2}\int\ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}\left(\frac{1}{1-x^2}\right)$

$\displaystyle=-\frac{\ln\left(x+\sqrt{1+x^2}\right)}{2\left(1-x^2\right)}+\frac{1}{2}\int\frac{1}{\left(1-x^2\right)\sqrt{1+x^2}}\mathrm{d}x$，又有

$$\int\frac{1}{\left(1-x^2\right)\sqrt{1+x^2}}\mathrm{d}x\overset{x=\frac{1}{t}}{=}-\int\frac{t}{\left(t^2-1\right)\sqrt{t^2+1}}\mathrm{d}t\overset{u^2=t^2+1}{=}-\int\frac{t}{u^2-(\sqrt{2})^2}\mathrm{d}u$$

$$=-\frac{1}{2\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|+C=-\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{t^2+1}-\sqrt{2}}{\sqrt{t^2+1}+\sqrt{2}}\right|+C=-\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{x^2+1}-\sqrt{2}x}{\sqrt{x^2+1}+\sqrt{2}x}\right|+C,$$

因此 $\displaystyle\int\frac{x\ln\left(x+\sqrt{1+x^2}\right)}{\left(1-x^2\right)^2}\mathrm{d}x=-\frac{\ln\left(x+\sqrt{1+x^2}\right)}{2\left(1-x^2\right)}-\frac{1}{4\sqrt{2}}\ln\left|\frac{\sqrt{x^2+1}-\sqrt{2}x}{\sqrt{x^2+1}+\sqrt{2}x}\right|+C$.

五、(15 分) 计算积分 $\int_0^1 x^m(\ln x)^n\mathrm{d}x$.（$m,n$ 为自然数）

解：易知 $\displaystyle\int_0^1 x^m(\ln x)^n\mathrm{d}x\overset{x=e^{-t}}{=}\int_0^{\infty}e^{-mt}(-t)^n e^{-t}\mathrm{d}t=(-1)^n\int_0^{\infty}t^n e^{-(m+1)t}\mathrm{d}t$

$\displaystyle\overset{u=(m+1)t}{=}\frac{(-1)^n}{(m+1)^{n+1}}\int_0^{\infty}u^n e^{-u}\mathrm{d}u=\frac{(-1)^n n!}{(m+1)^{n+1}}$.

六、(15 分) 设函数 $f(x,y)$ 在区域 $D:x^2+y^2\leq 1$ 上有二阶连续偏导数，且 $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=$

$e^{-\left(x^2+y^2\right)}$，计算 $\iint_D\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y$.

解：利用极坐标得 $\iint_D\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(\rho\cos\theta\cdot f_x'+\rho\sin\theta\cdot f_y'\right)\rho\mathrm{d}\rho$

$=\int_0^1\rho\mathrm{d}\rho\int_0^{2\pi}\left(\rho\cos\theta\cdot f_x'+\rho\sin\theta\cdot f_y'\right)\mathrm{d}\theta$，记 L_ρ 是半径为 ρ 的圆周，D_ρ 为圆周 L.包围的区域.易知 $\rho\cos\theta\mathrm{d}\theta=\mathrm{d}y$，$\rho\sin\theta\mathrm{d}\theta=-\mathrm{d}x$.于是上式的内层积分可以看作沿闭曲线 L_ρ（逆时针方向）的曲线积分 $\oint_{L_p}-f_y'\mathrm{d}x+f_x'\mathrm{d}y$，则有

$$\int_0^1\rho\mathrm{d}\rho\int_0^{2\pi}\left(\rho\cos\theta\cdot f_x'+\rho\sin\theta\cdot f_y'\right)\mathrm{d}\theta=\int_0^1\rho\left(\oint_{L_p}-f_y'\mathrm{d}x+f_x'\mathrm{d}y\right)\mathrm{d}\rho$$

$$=\int_0^1\rho\left[\iint_{D_p}\left(f_{xx}''+f_{yy}''\right)\mathrm{d}x\mathrm{d}y\right]\mathrm{d}\rho=\int_0^1\rho\left(\int_0^{2\pi}\mathrm{d}\theta\int_0^\rho \mathrm{e}^{-s^2}s\mathrm{d}s\right)\mathrm{d}\rho=\frac{\pi}{2\mathrm{e}}.$$

七、（15 分）计算三重积分 $I=\iiint_\Omega\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\left(1+x^2+y^2+z^2\right)^2}$，其中 Ω 为 $0\le x\le 1,0\le y\le 1,0\le z\le 1$.

解：采用“先二后一”，利用对称性得 $I=2\int_0^1\mathrm{d}z\iint_D\frac{\mathrm{d}x\mathrm{d}y}{\left(1+x^2+y^2+z^2\right)^2}$，其中 $D:0\le x\le 1,0\le y\le x$.利用极坐标计算二重积分得 $I=2\int_0^1\mathrm{d}z\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\sec\theta}\frac{r\mathrm{d}r}{\left(1+r^2+z^2\right)^2}$

$=\int_0^1\mathrm{d}z\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+z^2}-\frac{1}{1+\sec^2\theta+z^2}\right)\mathrm{d}\theta$，交换积分次序得

$I=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^1\left(\frac{1}{1+z^2}-\frac{1}{1+\sec^2\theta+z^2}\right)\mathrm{d}z=\frac{\pi^2}{16}-\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^1\frac{1}{1+\sec^2\theta+z^2}\mathrm{d}z$，作变量代换 $z=\tan t$ 并利用对称性得 $\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^1\frac{1}{1+\sec^2\theta+z^2}\mathrm{d}z=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2 t}{\sec^2\theta+\sec^2 t}\mathrm{d}t$

$=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{\sec^2\theta+\sec^2 t}\mathrm{d}t=\frac{1}{2}\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta+\sec^2 t}{\sec^2\theta+\sec^2 t}\mathrm{d}t=\frac{1}{2}\times\frac{\pi^2}{16}=\frac{\pi^2}{32}$，因此

$$I=\frac{\pi^2}{16}-\frac{1}{2}\times\frac{\pi^2}{16}=\frac{\pi^2}{32}.$$