

小学一年级数学基础知识趣味竞答

小朋友快来一起玩，比一比谁做的快又准！

题号	1	2	3	4	5	6	7	8	9	总分
得分										

一、探索数字与代表符号的神奇规律

1.

Let $T \subset \mathbb{N}_{>0}$ be a finite set of positive integers. For each integer $n > 0$, define a_n to be the number of all finite sequences (t_1, \dots, t_m) with $m \leq n$, $t_i \in T$ for all $i = 1, \dots, m$ and $t_1 + \dots + t_m = n$. Prove that the infinite series

$$1 + \sum_{n \geq 1} a_n z^n \in \mathbb{C}[[z]]$$

is a *rational* function in z , and find this rational function.

2.

Let $V \cong \mathbb{C}^2$ be the standard representation of $SL_2(\mathbb{C})$.

(a) Show that the n -th symmetric power $V_n = \text{Sym}^n V$ is irreducible.

(b) Which V_n appear in the decomposition of the tensor product $V_2 \otimes V_3$ into irreducible representations?

3. 选做

Let G_1, \dots, G_n denote finite groups. For each $m \in \{1, \dots, n\}$, let $\rho_m: G_m \rightarrow \text{GL}(V_m)$ denote a finite dimensional, complex representation of G_m . Use χ_m to denote the character of ρ_m . Set $G = G_1 \times \dots \times G_n$ and $V = V_1 \otimes \dots \otimes V_n$.

- a) Define $\rho: G \rightarrow \text{GL}(V)$ by the rule $\rho(g_1, \dots, g_n) = \rho_1(g_1) \otimes \dots \otimes \rho_n(g_n)$. Write the character of ρ in terms of the characters $\{\chi_m\}_{1 \leq m \leq n}$.
- b) Prove that (V, ρ) is an irreducible representation of G if and only if, for all m , each (V_m, ρ_m) is an irreducible representation of G_m .

二、算一算，分析问题我最棒！

4.

Let $K(x, y) \in C([0, 1] \times [0, 1])$. For all $f \in C[0, 1]$, the space of continuous functions on $[0, 1]$, define a function

$$Tf(x) = \int_0^1 K(x, y)f(y)dy$$

Prove that $Tf \in C([0, 1])$. Moreover $\Omega = \{Tf \mid \|f\|_{sup} \leq 1\}$ is precompact in $C([0, 1])$, i.e. every sequence in Ω has a converging subsequence, here $\|f\|_{sup} = \sup\{|f(x)| \mid x \in [0, 1]\}$.

5.

Consider the equation

$$\ddot{x} + (1 + f(t))x = 0.$$

We assume that $\int^{\infty} |f(t)|dt < \infty$. Study the Lyapunov stability of the solution $(x, \dot{x}) = (0, 0)$.

6.

Suppose $\Omega \subset \mathbf{R}^3$ to be a simply connected domain and $\Omega_1 \subset \Omega$ with boundary Γ . Let u be a harmonic function in Ω and $M_0 = (x_0, y_0, z_0) \in \Omega_1$. Calculate the integral:

$$II = - \int \int_{\Gamma} \left(u \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial u}{\partial n} \right) dS,$$

where $\frac{1}{r} = \frac{1}{\sqrt{(x-x_0)^2 + (y-x_0)^2 + (z-x_0)^2}}$ and $\frac{\partial}{\partial n}$ denotes the out normal derivative with respect to boundary Γ of the domain Ω_1 .
(Hint: use the formula $\frac{\partial v}{\partial n} dS = \frac{\partial v}{\partial x} dy \wedge dz + \frac{\partial v}{\partial y} dz \wedge dx + \frac{\partial v}{\partial z} dx \wedge dy$.)

三、几何图形真美丽

7.

Let M be a smooth 4-dimensional manifold. A symplectic form is a closed 2-form ω on M such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.

- (a). Construct a symplectic form on \mathbb{R}^4 .
- (b). Show that there are no symplectic forms on S^4 .

8.

Let M be a compact smooth manifold of dimension d . Prove that there exists some $n \in \mathbb{Z}^+$ such that M can be regularly embedded in the Euclidean space \mathbb{R}^n .

四、想一想，事情多大可能发生呢？

9.

Let X_1, \dots, X_n be independently and identically distributed random variables with $X_i \sim N(\theta, 1)$. Suppose that it is known that $|\theta| \leq \tau$, where τ is given. Show

$$\min_{a_1, \dots, a_{n+1}} \sup_{|\theta| \leq \tau} E \left(\sum_{i=1}^n a_i X_i + a_{n+1} - \theta \right)^2 = \frac{\tau^2 n^{-1}}{\tau^2 + n^{-1}}.$$

Hint: Carefully use the sufficiency principle.