BULGARIAN NATIONAL OLYMPIAD IN MATHEMATICS

Third round

1995

Problem 1. Let p and q be positive numbers such that the parabola $y = x^2 - 2px + q$ has no common point with the x-axis. Prove that there exist points A and B on the parabola such that the segment AB is parallel to the x-axis and $\angle AOB = 90°$ (O is the coordinate origin) if and only if $p^2 < q \leq \frac{1}{4}$. Find the values of p and q for which the points A and B are defined in an unique way.

Solution. Since the parabola has no common point with the x-axis, then the roots of the equation $x^2 - 2px + q = 0$ are not real and hence $p^2 < q$. Let the points $A(x_1, y_0)$ and $B(x_2, y_0)$ (Figure 1) be with the required properties. Then x_1 and x_2 are the roots of the equation $x^2 - 2px + q - y_0 = 0$ and $y_0 > q - p^2$, because the vertex of the parabola has coordinates $(p, q - p^2)$. On the other hand $OA^2 = x_1^2 + y_0^2$, $OB^2 = x_2^2 + y_0^2$, $AB^2 = (x_1 - x_2)^2$ and it follows from the Pythagorean theorem that $y_0^2 + x_1x_2 = 0$. But $x_1x_2 = q - y_0$ and thus $y_0^2 - y_0 + q = 0$. Consequently the existence of the points A and B is equivalent to the assertion that the equation $f(y) = y^2 - y + q = 0$ has a solution $y_0 > q - p^2$. (A and B are defined in an unique way if this is the only solution.) A necessary condition is that the discriminant of the equation is not negative, i.e. $q \leq \frac{1}{4}$. The last condition is sufficient because $f(q - p^2) = (q - p^2) + p^2 > 0$ and $\frac{1}{2} > \frac{1}{4} \geq q - p^2$. The corresponding solution y_0 is unique iff $q = \frac{1}{4}$.

Figure 1.

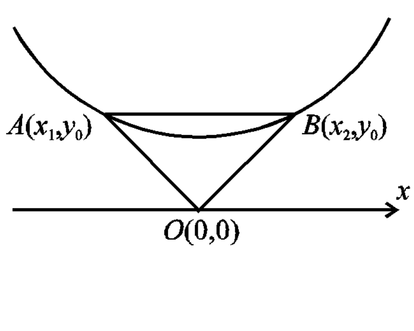
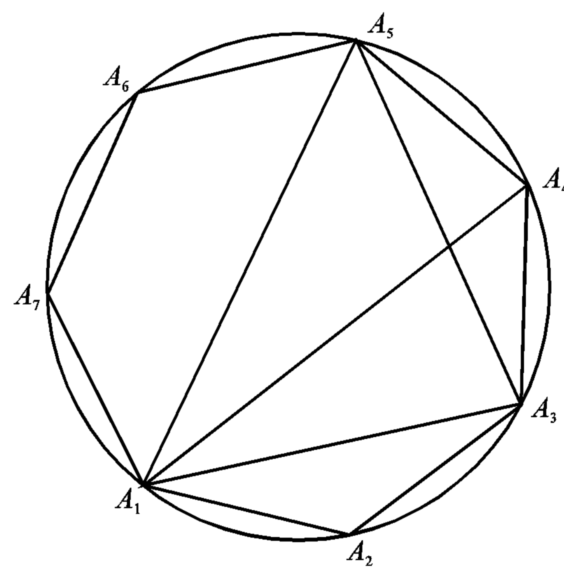


Figure 2.



Problem 2. Let $A_1A_2A_3A_4A_5A_6A_7$, $B_1B_2B_3B_4B_5B_6B_7$, $C_1C_2C_3C_4C_5C_6C_7$ be regular heptagons with areas S_A, S_B and S_C, respectively. Let $A_1A_2 = B_1B_3 = C_1C_4$. Prove that

$$\frac{1}{2} < \frac{S_B + S_C}{S_A} < 2 - \sqrt{2}.$$

Solution. Let $A_1A_2 = a$, $A_1A_3 = b$, $A_1A_4 = c$ (Figure 2). By the Ptolomeus theorem for the quadrangle $A_1A_3A_4A_5$ it follows that $ab+ac = bc$, i.e. $\frac{a}{b}+\frac{a}{c} = 1$. Since $\triangle A_1A_2A_3 \cong \triangle B_1B_2B_3$, then $\frac{B_1B_2}{B_1B_3} = \frac{a}{b}$ and hence $B_1B_2 = \frac{a^2}{b}$. Analogously $C_1C_2 = \frac{a^2}{c}$. Therefore $\frac{S_B + S_C}{S_A} = \frac{a^2}{b^2}+\frac{a^2}{c^2}$. Then $\frac{a^2}{b} + \frac{a^2}{c} > \frac{1}{2}(\frac{a}{b} + \frac{a}{c})^2 = \frac{1}{2}$ (equality is not possible because $\frac{a}{b} \neq \frac{a}{c}$). On the other hand

$$\frac{a^2}{b^2} + \frac{a^2}{c^2} = \left(\frac{a}{b} + \frac{a}{c}\right)^2 - \frac{2a^2}{bc} = 1 - \frac{2a^2}{bc}. \tag{1}$$

By the sine theorem we get $\frac{a^2}{bc} = \frac{\sin^2\frac{\pi}{7}}{\sin^2\frac{\pi}{7}\sin\frac{4\pi}{7}} = \frac{1}{4\cos\frac{2\pi}{7}(1+\cos\frac{2\pi}{7})}$. Since $\cos\frac{2\pi}{7} < \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, then $\frac{a^2}{bc} > \frac{1}{4\frac{\sqrt{2}}{2}(1+\frac{\sqrt{2}}{2})} = \sqrt{2}-1$. From here and from (1) we get the right hand side inequality of the problem.

Problem 3. Let $n > 1$ be an integer. Find the number of the permutations $(a_1, a_2, \ldots, a_n)$ of the numbers $1, 2, \ldots, n$ with the following property: there exists only one index $i \in \{1, 2, \ldots, n-1\}$ such that $a_i > a_{i+1}$.

Solution. Denote by p_n the number of the permutations with the given properties. Obviously, $p_1 = 0$ and $p_2 = 1$. Let $n \geq 2$. The number of the permutations with $a_n = n$ is equal to p_{n-1}. Consider all the permutations $(a_1, a_2, \ldots, a_n)$ with $a_i = n$, where $1 \leq i \leq n-1$ is fixed. Their number is $\binom{n-1}{i-1}$. Consequently

$$p_n = p_{n-1} + \sum_{i=1}^{n-1}\binom{n-1}{i-1} = p_{n-1} + 2^{n-1} - 1.$$

From here

$$\begin{aligned} p_n &= (2^{n-1}-1) + (2^{n-2}-1) + \cdots + (2-1) \\ &= 2^n - n - 1. \end{aligned}$$

Problem 4. Let $n \geq 2$ and $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, n$. Prove the inequality

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1) \leq \left[\frac{n}{2}\right].$$

When is there an equality?

Solution. Denote by $S(x_1, x_2, \ldots, x_n)$ the left hand side of the inequality. This function is linear with respect to each of the variables x_i. Particularly,

$$S(x_1, x_2, \ldots, x_n) \leq \max\left(S(0, x_2, \ldots, x_n), S(1, x_2, \ldots, x_n)\right).$$

From here it follows by induction that it is enough to prove the inequality when all x_i are equal to 0 or 1. On the other hand for arbitrary x_i we have

$$\begin{aligned} 2S(x_1, x_2, \ldots, x_n) &= n - (1-x_1)(1-x_2) - (1-x_2)(1-x_3) - \cdots - \\ &-(1-x_n)(1-x_1) - x_1x_2 - x_2x_3 - \cdots - x_nx_1 \end{aligned} \qquad (*)$$

i.e. $S(x_1, x_2, \ldots, x_n) \leq \dfrac{n}{2}$, when $x_i \in [0,1]$. In the case when x_i are equal to 0 or 1, the left hand side of the last inequality is an integer. Consequently $S(x_1, x_2, \ldots, x_n) \leq \left[\dfrac{n}{2}\right]$. It follows from $(*)$ that

- when n is even, the equality is satisfied iff $(x_1, x_2, \ldots, x_n) = (0,1,0,1,\ldots,0,1)$;
- when n is odd, the equality is satisfied iff $(x_1, x_2, \ldots, x_n) = (x,0,1,0,1,\ldots,0,1)$, where $x \in [0,1]$ is arbitrary.

Problem 5. The points A_1, B_1, C_1 lie on the sides BC, CA, AB of the triangle ABC respectively and the lines AA_1, BB_1, CC_1 have a common point M. Prove that if the point M is center of gravity of $\triangle A_1B_1C_1$, then M is the center of gravity of $\triangle ABC$.

Solution. Let M be the center of gravity of $\triangle A_1B_1C_1$. Let A_2 be a point on $MA^{\rightarrow}$ such that $B_1A_1C_1A_2$ is a parallelogram. The points B_2 and C_2 are constructed analogously. Since $A_1C_1 \| A_1B_1 \| C_1B_2$, then the points A_2, C_1, B_2 are colinear and C_1 is the midpoint of A_2B_2. The same is true for the points A_2, B_1, C_2 and C_2, A_1, B_2. We shall prove that $A_2 = A$, $B_2 = B$ and $C_2 = C$, which will solve the problem.

Assume that $A_2 \neq A$ and let A be between A_2 and M. Then C_2 is between C and M, B is between B_2 and M and consequently A_2 is between A and M, which is a contradiction.

Problem 6. Find all pairs of positive integers (x,y) for which $\dfrac{x^2+y^2}{x-y}$ is an integer which is a divisor of 1995.

Solution. It is enough to find all pairs (x,y) for which $x > y$ and $x^2 + y^2 = k(x-y)$, where k divides $1995 = 3.5.7.19$. We shall use the following well-known fact: if p is a prime number of the type $4q+3$ and if it divides x^2+y^2, then p divides x and y. (For $p = 3, 7, 19$ this can be proved directly.) If k is divisible by 3 then x and y are divisible by 3 too. Simplifying by 9 we get an equality $x_1^2 + x_1^2 = k_1(x_1 - y_1)$, where k_1 divides 5.7.19. Considering 7 and 19 in an analogous way we get an equality $a^2 + b^2 = 5(a-b)$ (it is not possible to get an equality $a^2 + b^2 = (a-b)$), where $a > b$. From here $(2a-5)^2 + (2b-5)^2 = 50$, i.e. $a = 3$, $b = 1$ or $a = 2$, $b = 1$.

The above considerations imply that the pairs we are looking for are of the type $(3c, c)$, $(2c, c)$, $(c, 3c)$, $(c, 2c)$, where $c = 1, 3, 7, 19, 3.7, 3.19, 7.19, 3.7.19$.

BULGARIAN NATIONAL OLYMPIAD IN MATHEMATICS

Fourth round

1995

Problem 1. Find the number of all integers $n > 1$, for which the number $a^{25} - a$ is divisible by n for every integer a.

Solution. Let n be with the required property. Then p^2 (p prime) does not divide n since p^2 does not divide $p^{25} - p$. Hence n is a product of pairwise different prime numbers. On the other hand $2^{25} - 2 = 2.3^2.5.7.13.17.241$. But n is not divisible by 17 and 241 because $3^{25} \equiv -3 \pmod{17}$ and $3^{25} \equiv 32 \pmod{241}$. The Fermat theorem implies that $a^{25} \equiv a \pmod{p}$ when $p = 2, 3, 5, 7, 13$. Thus n should be equal to the divisor of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$, different from 1. Therefore the number we are looking for is $2^5 - 1 = 31$.

Problem 2. A triangle ABC with semiperimeter p is given. Points E and F lie on the line AB and $CE = CF = p$. Prove that the excircle k_1 of $\triangle ABC$ to the side AB touches the circumcircle k of $\triangle EFC$.

Solution. Let P and Q be the tangent points of k_1 with the lines CA and CB, respectively. Since $CP = CQ = p$, then the points E, P, Q and F lie on the circle with center C and radius p. We denote by i the inversion defined by this circle. Since $i(P) = P$, $i(Q) = Q$, then $i(k_1) = k_1$. On the other hand $i(E) = E$ and $i(F) = F$. Hence $i(k)$ is the line AB. But k_1 touches AB and thus k touches k_1.

Problem 3. Two players **A** and **B** take stones one after the other from a heap with $n \geq 2$ stones. **A** begins the game and takes at least 1 stone but no more then $n - 1$ stones. Each player on his turn must take at least 1 stone but no more than the other player has taken before him. The player who takes the last stone is the winner. Find who of the players has a winning strategy.

Solution. Consider the pair (m, l), where m is the number of the stones in the heap and l is the maximal number of stones that could be taken by the player on turn. We must find for which n the position $(n, n-1)$ is winning (i.e. **A** wins) and for which n it is losing (**B** wins). We shall apply the following assertion several times: If (m, l) is a losing position and $l_1 < l$, then (m, l_1) is losing too.

Now we shall prove that $(n, n-1)$ is a losing position iff n is a power of 2.

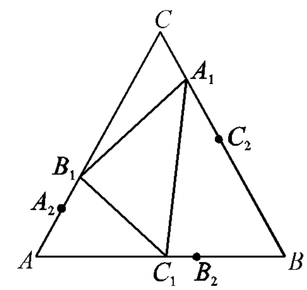
Sufficiency: Let $n = 2^k$, $k \geq 1$. If $k = 1$ then **B** wins on his first move. Assume that $(2^k, 2^k - 1)$ is a losing position and let consider the position $(2^{k+1}, 2^{k+1} - 1)$. If **A** takes at least 2^k stones on his first move, then **B** wins at ones. Let **A** take l stones, where $1 \leq l < 2^k$. By the inductive assumption **B** could play in such a way that he could win the game $(2^k, l)$ since $l \leq 2^k - 1$; the last move will be the move of **B**. After this move we get the position $(2^k, m)$ with $m \leq l$, which is losing for **A**, according to the inductive assumption.

Necessity: It is enough to prove that if n is not a power of 2, then $(n, n-1)$ is a winning position. Let $n = 2^k + r$, where $1 \leq r \leq 2^k - 1$. On his first move **A** takes r stones and **B** is faced to the position $(2^k, r)$, which is losing for **B**.

Problem 4. The points C_1, A_1 and B_1 lie on the sides AB, BC and CA of the equilateral triangle ABC respectively in such a way that the inradii of the triangles C_1AB_1, B_1CA_1, A_1BC_1 and $A_1B_1C_1$ are equal. Prove that A_1, B_1 and C_1 are the midpoints of the corresponding sides.

Solution. We shall prove that $BA_1 = CB_1 = AC_1$ (Figure 1). Assume the contrary and let $BA_1 \geq CB_1 > AC_1$. Let ρ be the rotation at 120^0 which center coincides with the in center of the incircle of $\triangle ABC$. This rotation transforms the incircles of the triangles C_1BA_1, A_1CB_1 and B_1AC_1 to the incircles of the triangles A_1CB_1, B_1AC_1 and C_1BA_1, respectively. Let $A_2 = \rho(A_1)$, $B_2 = \rho(B_1)$ and $C_2 = \rho(C_1)$. It follows that $BB_2 < BC_1$ and $BC_2 < BA_1$. But the incircles of the triangles BC_1A_1 and BC_2B_2 have equal radii (because $\rho(\triangle AC_1B_1) = \triangle BC_2B_2$), which is a contradiction.

Figure 1.



Let r be the radius of the incircles of the triangles C_1AB_1, B_1CA_1, A_1BC_1 and $A_1B_1C_1$. From the triangle B_1AC_1 we have $r = \dfrac{1 - B_1C_1}{2}.\dfrac{\sqrt{3}}{3}$, and from $\triangle A_1B_1C_1$ which is equilateral we have $r = B_1C_1.\dfrac{\sqrt{3}}{6}$. From here $B_1C_1 = \dfrac{1}{2}$ and consequently A_1, B_1, C_1 are midpoints of the corresponding sides.

Problem 5. Let $A = \{1, 2, \ldots, m+n\}$, where m and n are positive integers and let the function $f : A \to A$ be defined by the equations:

$$f(i) = i + 1 \quad \text{for} \quad i = 1, 2, \ldots, m-1, m+1, \ldots, m+n-1$$

$$f(m) = 1 \quad \text{and} \quad f(m+n) = m+1.$$

a) Prove that if m and n are odd then there exists a function $g : A \to A$ such that $g(g(a)) = f(a)$ for all $a \in A$.

b) Prove that if m is even then $m = n$ iff there exists a function $g : A \to A$ such that $g(g(a)) = f(a)$ for all $a \in A$.

Solution. a) Let $m = 2p+1$, $n = 2q+1$ and $g(i) = p+i+1$ for $i = 1, 2, \ldots, p$; $g(i) = q+i+1$ for $i = m+1, m+2, \ldots, m+q$; $g(2p+1) = p+1$; $g(p+1) = 1$; $g(m+2q+1) = m+q+1$; $g(m+q+1) = m+1$. It is easy to check that $g(g(a)) = f(a)$ for all $a \in A$.

b) Let $m = n$ and $g(i) = m+i$ for $i = 1, 2, \ldots, m$; $g(m+i) = i+1$ for $i = 1, 2, \ldots, m-1$; $g(2m) = 1$.

For the converse let $M = \{1, 2, \ldots, m\}$. It follows by the definition of f that the elements of M remain in M after applying the powers of f with respect to superposition. Moreover, these powers scoop out the whole M. The same is true for the set $A \setminus M$. The function f is bijective in A and if there exists g verifying the condition, then g is bijective too. We shall prove that $g(M) \bigcap M = \emptyset$. It follows from the contrary that there exists $i \in M$ such that $g(i) \in M$. Consider the sequence $i, g(i), g^2(i), \ldots$ and the subsequence $i, f(i), f^2(i), \ldots$. It is easy to see that $g(M) = M$. We deduce that there exists a permutation $a_1, a_2, \ldots, a_m$ of elements of M, such that $g(a_i) = a_{i+1}$ for $i = 1, 2, \ldots, m-1$; $g(a_m) = a_1$ and $f(a_{2i-1}) = a_{2i+1}$ for $i = 1, 2, \ldots, s-1$; $f(a_{2s-1}) = a_1$, where $m = 2s$. The last contradicts to the properties of f which were mentioned already. Thus $g(M) \bigcap M = \emptyset$. Analogously $g(A \setminus M) = A \setminus M$, if $g(i) \in A \setminus M$ for $i \in A \setminus M$. At last let us observe that when starting from an element of M and applying g we go to $A \setminus M$, but when applying g for a second time we go back to M. The same is true for the set $A \setminus M$.

From here and from the bijectivity of g it follows that M and $A \setminus M$ have one and the same number of elements, i.e. $n = m$.

Problem 6. Let x and y be different real numbers such that $\dfrac{x^n - y^n}{x - y}$ is an integer for some four consecutive positive integers n. Prove that $\dfrac{x^n - y^n}{x - y}$ is integer for all positive integers n.

Solution. Let $t_n = \dfrac{x^n - y^n}{x - y}$. Then $t_{n+2} + b.t_{n+1} + c.t_n = 0$ for $b = -(x + y)$, $c = xy$, where $t_0 = 0$, $t_1 = 1$. We shall show that $b, c \in \mathbb{Z}$. Let $t_n \in \mathbb{Z}$ for $n = m, m+1, m+2, m+3$. Since $c^n = (xy)^n = t_{n+1}^2 - t_n.t_{n+2} \in \mathbb{Z}$ when $n = m, m+1$, then $c^m, c^{m+1} \in \mathbb{Z}$. Therefore c is rational and from $c^{m+1} \in \mathbb{Z}$ it follows that $c \in \mathbb{Z}$. On the other hand

$$b = \frac{t_m t_{m+3} - t_{m+1} t_{m+2}}{c^m},$$

i.e. b is rational. From the recurrence equation it follows by induction that t_n could be represented in the following way $t_n = f_{n-1}(b)$, where $f_{n-1}(X)$ is a monic polynomial with integer coefficients and $\deg f_{n-1} = n - 1$. Since b is a root of the equation $f_m(X) = t_{m+1}$, then $b \in \mathbb{Z}$. Now from the recurrence equation it follows that $t_n \in \mathbb{Z}$ for all n.

BULGARIAN NATIONAL OLYMPIAD IN MATHEMATICS

Third round

1996

Problem 1. Prove that for all positive integers $n \geq 3$ there exist an odd positive integers x_n and y_n, such that

$$7x_n^2 + y_n^2 = 2^n.$$

Solution. If $n = 3$ we have $x_3 = y_3 = 1$.

Suppose that for an integer $n \geq 3$ there are odd positive integers x_n, y_n, such that $7x_n^2 + y_n^2 = 2^n$. We shall prove that for each pair

$$\left(X = \frac{x_n + y_n}{2}, Y = \frac{|7x_n - y_n|}{2}\right) \quad \text{and} \quad \left(X = \frac{|x_n - y_n|}{2}, Y = \frac{7x_n + y_n}{2}\right)$$

we have $7X^2 + Y^2 = 2^{n+1}$. Indeed,

$$7\left(\frac{x_n \pm y_n}{2}\right)^2 + \left(\frac{7x_n \mp y_n}{2}\right)^2 = 2\left(7x_n^2 + y_n^2\right) = 2 \cdot 2^n = 2^{n+1}.$$

Since x_n and y_n are odd, i.e. $x_n = 2k + 1$ and $y_n = 2l + 1$ (k, l are integers), then $\frac{x_n + y_n}{2} = k + l + 1$ and $\frac{|x_n - y_n|}{2} = |k - l|$, which shows that one of the numbers $\frac{x_n + y_n}{2}$ and $\frac{|x_n - y_n|}{2}$ is odd. Thus, for $n + 1$ there are odd natural numbers x_{n+1} and y_{n+1} with the required property.

Problem 2. The circles k_1 and k_2 with centers O_1 and O_2 respectively are externally tangent at the point C, while the circle k with center O is externally tangent to k_1 and k_2. Let ℓ be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k, which is perpendicular to ℓ, the points A and O_1 lie in one and the same semiplane with respect to the line ℓ. Prove that the lines AO_2, BO_1 and ℓ have a common point.

Figure 1.

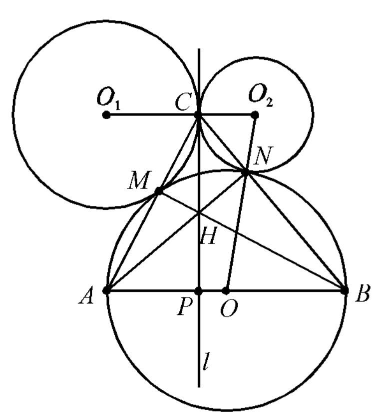
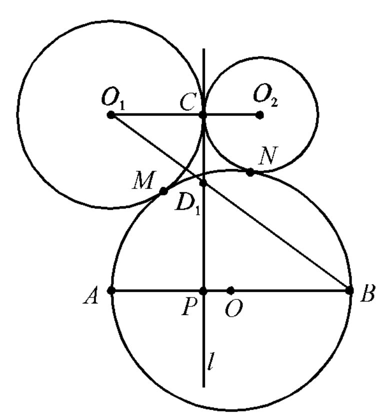


Figure 2.



Solution. Denote by r, r_1 and r_2 the radii of k, k_1 and k_2, by M and N the tangent points of k with k_1 and k_2, respectively and by P the common point of ℓ and AB (Figure 1).

It follows from $O_1O_2 \perp \ell$ and $AB \perp \ell$ that $\triangle BON \sim \triangle CO_2N$. Then $\angle CNO_2 = \angle ONB$, and consequently the points C, N and B are colinear. Also, $\dfrac{BN}{CN} = \dfrac{BO}{CO_2} = \dfrac{r}{r_2}$. Analogously, A, M and C are colinear and $\dfrac{AM}{MC} = \dfrac{r}{r_1}$.

The lines AN, BM and ℓ have a common point H, which is the altitude center of $\triangle ABC$. By Ceva's theorem we have: $\dfrac{AP}{PB} \cdot \dfrac{BN}{NC} \cdot \dfrac{CM}{MA} = \dfrac{AP}{PB} \cdot \dfrac{r}{r_2} \cdot \dfrac{r_1}{r} = 1$, from where:

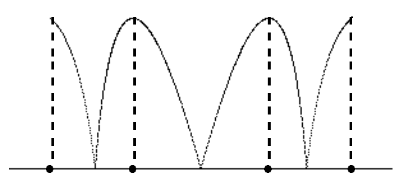
$$\frac{r_1}{PB} = \frac{r_2}{AP}. \tag{1}$$

Let now D_1 and D_2 be the common points of the line ℓ with the lines BO_1 and AO_2, respectively. Obviously, $\triangle O_1CD_1 \sim \triangle BPD_1$ (Figure 2), from where $\dfrac{CD_1}{D_1P} = \dfrac{r_1}{PB}$. Analogously, $\dfrac{CD_2}{D_2P} = \dfrac{r_2}{AP}$ and according to (1) we have $\dfrac{CD_1}{D_1P} = \dfrac{CD_2}{D_2P}$, which shows that $D_1 \equiv D_2$. Thus, the lines AO_2, BO_1 and ℓ have a common point.

Problem 3. a) Find the maximal value of the function $y = |4x^3 - 3x|$ in the interval $[-1, 1]$.

b)Let a, b and c be real numbers and M be the maximal value of the function $y = |4x^3 + ax^2 + bx + c|$ in the interval $[-1, 1]$. Prove that $M \geq 1$. For which a, b, c is the equality reached?

Figure 3.



Solution. a) Using that $\left(4x^3 - 3x\right)' = 12x^2 - 3 = 12\left(x - \dfrac{1}{2}\right)\left(x + \dfrac{1}{2}\right)$, we find that the function

$$y = \left|4x^3 - 3x\right|$$

has a local maximums when $x = \pm\dfrac{1}{2}$. Then its maximal value in the interval $[-1, 1]$ is the biggest among the numbers $y(-1)$, $y(1)$, $y\left(-\dfrac{1}{2}\right)$ and $y\left(\dfrac{1}{2}\right)$ (Figure 3).

But $y(-1) = y\left(-\dfrac{1}{2}\right) = y\left(\dfrac{1}{2}\right) = y(1) = 1$, thus the maximal value is equal to 1.

b) Let $f(x) = 4x^3 + ax^2 + bx + c$. Assume that there exist numbers a, b, c, for which the maximal value M of the function $y = |f(x)|$ in $[-1, 1]$ is less than 1, i.e. $M < 1$. Then $-1 < f(x) < 1$ for all $x \in [-1, 1]$.

Consider the function $g(x) = f(x) - (4x^3 - 3x) = ax^2 + (b + 3)x + c$. We have $g(-1) > 0$, $g\left(-\dfrac{1}{2}\right) < 0$, $g\left(\dfrac{1}{2}\right) > 0$ and $g(1) < 0$. Consequently $g(x)$ changes its sign at least 3 times, which means that the quadratic equation $ax^2 + (b + 3)x + c = 0$ has at least 3 different roots. This is possible only if $a = b + 3 = c = 0$, i.e. if $f(x) = 4x^3 - 3x$. According to a) the maximal value of $y = \left|4x^3 - 3x\right|$ is ≤ 1.

The equality $M = 1$ is reached only when $a = 0$, $b = -3$ and $c = 0$.

Problem 4. The real numbers $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ form an arithmetic progression. There exists a permutation $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ of the same numbers, which is a geometric progression.

Find the numbers $a_1, a_2, \dots, a_n$, if they are pairwise different and the biggest among them is equal to 1996.

Solution. Let $a_1 < a_2 < \cdots < a_n = 1996$ and q be the quotient of the geometric progression $a_{i_1}, a_{i_2}, \dots, a_{i_n}$. We have $q \neq 0$ and $q \neq 1$. The numbers $a_{i_n}, a_{i_{n-1}}, \dots, a_{i_1}$ form also a geometric progression which quotient is $\dfrac{1}{q}$. Thus, we can assume that $|q| > 1$, i.e. $q > 1$ or $q < -1$. Then $|a_{i_1}| < |a_{i_2}| < \cdots < |a_{i_n}|$, from where $a_i \neq 0$ for all i.

More exactly, either all numbers are positive ($q > 1$) and then $a_{i_1} < a_{i_2} < \cdots < a_{i_n}$, which together with $a_1 < a_2 < \cdots < a_n$ shows that $a_{i_k} = a_k$, i.e. the numbers $a_1, a_2, \dots, a_n$ form an arithmetic as well as a geometric progression, or the numbers $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ change their signs alternatively ($q < -1$) and then the positive ones form an increasing geometric progression with quotient q^2, and the order is the same as in the arithmetic progression. (The numbers $a_1, a_2, \dots, a_n$ could not be all negative, because $a_n = 1996 > 0$.)

Assume now that 3 among the numbers $a_1, a_2, \dots, a_n$ are positive. Then $0 < a_{n-2} < a_{n-1} < a_n$ and they form a geometric as well as an arithmetic progression. Therefore $2a_{n-1} = a_{n-2} + a_n$ and $a_{n-1}^2 = a_{n-2}a_n$. From here $a_{n-2} = a_{n-1} = a_n$, which is a contradiction.

Thus at most two among the numbers are positive. Analogously, at most two among the numbers are negative. Consequently, $n \leq 4$.

Let $n = 4$. Then $a_1 < a_2 < 0 < a_3 < a_4$ and $2a_2 = a_1 + a_3$, $2a_3 = a_2 + a_4$. But $q < -1$ and the geometric progression is either a_3, a_2, a_4, a_1 or a_2, a_3, a_1, a_4. Let it be a_3, a_2, a_4, a_1. Then $a_2 = a_3q$, $a_4 = a_3q^2$ and $a_1 = a_3q^3$. Thus, $2a_3q = a_3q^3 + a_3$ and $2a_3 = a_3q + a_3q^2$. From here $q = 1$, which contradicts to $q < -1$.

So $n = 3$. There are two possibilities:

I. $a_1 < a_2 < 0 < a_3 = 1996$. Then the geometric progression is a_2, $a_3 = a_2q$, $a_1 = a_2q^2$. It follows from $2a_2 = a_1 + a_3$ that $2a_2 = a_2q^2 + a_2q$, i.e. $q^2 + q - 2 = 0$. Thus, $q = -2$, $-2a_2 = 1996$, $a_2 = -998$ and the numbers are $(-3992; -998; 1996)$.

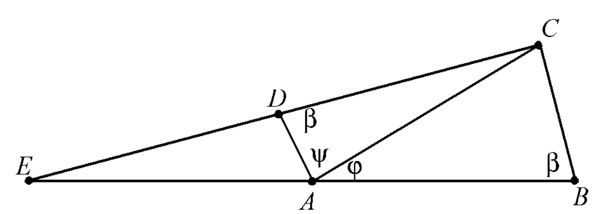
II. $a_1 < 0 < a_2 < a_3 = 1996$. Now the geometric progression is a_2, $a_1 = a_2q$, $a_3 = a_2q^2$. From $2a_2 = a_1 + a_3$ we obtain $2a_2 = a_2q + a_2q^2$, i.e. again $q = -2$. Therefore, $a_3 = 4a_2 = 1996$ and $a_2 = 499$. The numbers are $(-998; 499; 1996)$.

Problem 5. A convex quadrilateral $ABCD$, for which $\angle ABC + \angle BCD < 180°$, is given. The common point of the lines AB and CD is E. Prove that $\angle ABC = \angle ADC$ if and only if

$$AC^2 = CD \cdot CE - AB \cdot AE.$$

Solution. Let $\angle ABC = \beta$, $\angle ADC = \delta$, $\angle BAC = \varphi$ and $\angle CAD = \psi$ (Figure 4). The point A is between E and B and the point D is between E and C. Also, $\angle AEC = \delta + \varphi + \psi - 180°$.

Figure 4.



Applying the sine theorem to the triangles ACD, ACE, ABC and again to $\triangle ACE$, we obtain:

$$\frac{CD}{AC} = \frac{\sin\psi}{\sin\delta},$$
$$\frac{CE}{AC} = -\frac{\sin\varphi}{\sin(\delta+\varphi+\psi)},$$
$$\frac{AB}{AC} = \frac{\sin(\beta+\varphi)}{\sin\beta},$$
$$\frac{AE}{AC} = -\frac{\sin(\delta+\psi)}{\sin(\delta+\varphi+\psi)}.$$

From here the equation $AC^2 = CD \cdot CE - AB \cdot AE$ is equivalent to the equations:

$$\frac{CD}{AC} \cdot \frac{CE}{AC} - \frac{AB}{AC} \cdot \frac{AE}{AC} - 1 = 0;$$

$$\sin(\beta+\varphi)\sin(\delta+\psi)\sin\delta - \sin\psi\sin\varphi\sin\beta \\ - \sin\delta\sin\beta\sin(\varphi+\psi+\delta) = 0;$$

$$(\cos(\beta+\varphi-\delta-\psi) - \cos(\beta+\varphi+\delta+\psi))\sin\delta \\ - (\cos(\varphi-\psi) - \cos(\varphi+\psi))\sin\beta \\ - (\cos(\beta-\delta) - \cos(\beta+\delta))\sin(\varphi+\psi+\delta) = 0$$

$$\sin(\beta+\varphi+\psi) - \sin(2\delta+\varphi+\psi-\beta) \\ + \sin(2\delta+\psi-\beta-\varphi) - \sin(\beta+\psi-\varphi) = 0;$$

$$\sin(\beta-\delta)\cos(\delta+\varphi+\psi) + \sin(\delta-\beta)\cos(\delta+\psi-\varphi) = 0;$$

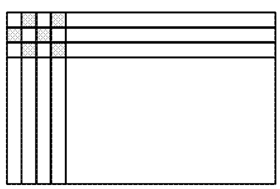
$$\sin(\beta-\delta)\sin(\delta+\psi)\sin\varphi = 0$$

But $\sin\varphi \neq 0$ and $\sin(\delta+\psi) \neq 0$. Consequently, $\sin(\beta-\delta) = 0$ and $\beta = \delta$.

Problem 6. A rectangle $m \times n$ ($m > 1$, $n > 1$) is divided into mn squares 1×1 with lines, parallel to its sides. In how many ways could two of the squares be canceled and the remaining part be covered with dominoes 2×1?

Solution. Denote by $F(m,n)$ the number we are looking for. Since every domino covers exactly two squares, then $F(m,n) = 0$ if m and n are odd.

Figure 5.



Let at least one of the numbers m and n be even. We color the squares in two colors — white and black in such a way that every two neighbor squares (with common side) are of different colors (Figure 5). The number S_0 of the white squares is equal to the number S_1 of the black ones and $S_0 = S_1 = \dfrac{mn}{2}$.

Each domino covers one white and one black square. If two white or two black squares are canceled, then it is impossible to cover by dominoes the remaining part of the rectangle. Now we shall show that if one white and one black squares are canceled then the remaining part can be covered by dominoes.

Since mn is even, then $mn = 2t$, where $t \geq 2$. We make induction with respect to t. The case $t = 2$ is obvious. Let $t_0 > 2$ and the proposition is true for all $2 \leq t \leq t_0$. Let $mn = 2(t_0+1)$. Denote by T_1 the rectangle consisted of the first two rows of the considered rectangle, and by T_2 — the rectangle consisted of the remaining rows. If the two canceled squares are in T_1 or in T_2, then we can cover each of the rectangles by dominoes and consequently we can cover the given rectangle.

Let one of the canceled squares be in T_1, and the other — in T_2. We place a domino in such a way that it covers one square from T_1 and one square from T_2. It is also possible that the canceled square from T_1 and the covered square from T_1 are of different colors. Thus, the canceled square from T_2 and the covered square from T_2 are of different colors. According to the inductive assumption the remaining part of T_1 and the remaining part of T_2 can be covered. Thus, the rectangle can be covered too.

Finally one white and one black squares can be chosen in $S_0 \cdot S_1 = \left(\dfrac{mn}{2}\right)^2$ ways. Therefore, in this case $F(m,n) = \dfrac{m^2n^2}{4}$.

BULGARIAN NATIONAL OLYMPIAD IN MATHEMATICS

Fourth round

1996

Problem 1. Find all primary numbers p and q, for which $\dfrac{(5^p-2^p)(5^q-2^q)}{pq}$ is an integer.

Solution. Let p be prime number and $p|\,(5^p-2^p)$. It follows by the Fermat theorem that $5^p-2^p\equiv 3 \pmod p$. Consequently $p=3$.

Let now p and q be such prime numbers that $\dfrac{(5^p-2^p)(5^q-2^q)}{pq}$ is an integer. If $p|\,(5^p-2^p)$, then $p=3$. Since $5^3-2^3=3\cdot 3\cdot 13$, then either $q|\,(5^q-2^q)$, i.e. $q=3$, or $q=13$. Therefore the pairs (3,3), (3,13), (13,3) satisfy the problem condition. It remains the case when $p\neq 3, q\neq 3$. Now $p|\,(5^q-2^q)$ and $q|\,(5^p-2^p)$. We can assume that $p>q$. It is clear that $(p,q-1)=1$ and consequently, there are positive integers a and b, for which $ap-b(q-1)=1$ (Bezou theorem). Since $(q,5)=(q,2)=1$, it follows by the Fermat theorem that $5^{q-1}\equiv 2^{q-1} \pmod q$. From $5^p\equiv 2^p(\pmod q)$ we deduce that $5^{ap}\equiv 2^{ap} \pmod q$ and therefore, $5^{b(q-1)+1}\equiv 2^{b(q-1)+1}$ $\pmod q$. But $5^{b(q-1)+1}\equiv 5 \pmod q$ and $2^{b(q-1)+1}\equiv 2 \pmod q$. Thus, $q=3$, which is a contradiction. Finally, $(p,q)=(3,3),(3,13),(13,3)$.

Problem 2. Find the side length of the smallest equilateral triangle in which three disks with radii 2, 3 and 4 without common inner points can be placed.

Solution. Let in a equilateral $\triangle ABC$ two disks with radii 3 and 4 without common inner points be placed. It is clear that a line ℓ exists, which separates them, i.e. the disks are in different semiplanes with respect to ℓ (Figure 1).

Figure 1.

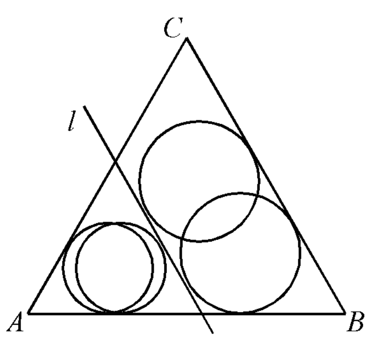
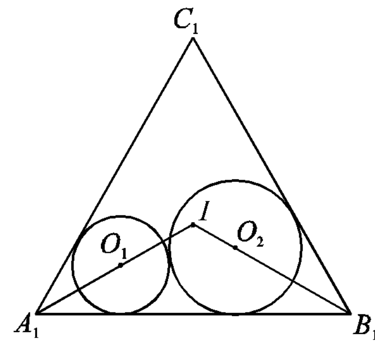
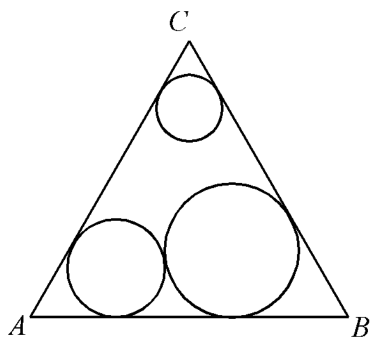


Figure 2.



This line divides the triangle into a triangle and a quadrilateral or into two triangles. In both cases the disks can be replaced in the figure in a way that each of them is tangent to two of the sides of $\triangle ABC$. It is clear that the new disks have no common inner point. Let the disks be inscribed in $\angle A$ and $\angle B$ of $\triangle ABC$, respectively. We translate the side BC parallelly to itself towards the point A, till the disk which is inscribed in $\angle B$ touches the disk which is inscribed in $\angle A$ (Figure 2). Thus, we get an equilateral $\triangle A_1B_1C_1$ with a smaller sides, in which two disks with radii 3 and 4 and without common inner points are placed.

Figure 3.



Let $A_1B_1 = x$, I be the incenter of $\triangle A_1B_1C_1$, while O_1 and O_2 be the centers of the two disks. Then $A_1I = B_1I = \dfrac{x}{\sqrt{3}}$, $A_1O_1 = 6, B_1O_2 = 8$. Since the disk with radius 4 is inside the $\triangle A_1B_1C_1$, then $O_2 \in IB_1$. Thus, $B_1O_2 \le B_1I$, i.e. $x \ge 8\sqrt{3}$. On the other hand $O_1I = \dfrac{x}{\sqrt{3}} - 6, O_2I = \dfrac{x}{\sqrt{3}} - 8, O_1O_2 = 7$ and by the cosine theorem for $\triangle O_1O_2I$ we find that $\left(\dfrac{x}{\sqrt{3}} - 6\right)^2 + \left(\dfrac{x}{\sqrt{3}} - 8\right)^2 + \left(\dfrac{x}{\sqrt{3}} - 6\right)\left(\dfrac{x}{\sqrt{3}} - 8\right) = 49.$ But $x \ge 8\sqrt{3}$, and from here $x = 11\sqrt{3}$. Consequently, $AB \ge 11\sqrt{3}$. On the other hand in the equilateral $\triangle ABC$ with side length $11\sqrt{3}$ three disks with radii 2, 3 and 4 (without common inner points) can be placed inscribing circles with these radii in the angles of the triangle (Figure 3).

Note that the disks wit radii 3 and 4 are tangent to each other. It follows from the above considerations that the solution of the problem is $11\sqrt{3}$.

Problem 3. The quadratic functions $f(x)$ and $g(x)$ are with real coefficients and have the following property: if the number $g(x)$ is integer for a positive x, then the number $f(x)$ is integer too. Prove that there are such integers m and n, that $f(x) = mg(x) + n$ for all real x.

Solution. Let $g(x) = px^2 + qx + r$. We can assume that $p > 0$. Since $g(x) = p(x + \dfrac{q}{2p})^2 + r - \dfrac{q^2}{4p}$ after the variable change of x by $x + \dfrac{q}{2p}$ we reduce the problem for the following quadratic functions $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + s, p > 0$. Let k be such an integer that $k > s$ and $\sqrt{\dfrac{k-s}{p}} > \dfrac{q}{2p}$. Since $g\left(\sqrt{\dfrac{k-s}{p}}\right) = k$ is integer, then $f\left(\sqrt{\dfrac{k-s}{p}}\right) = \dfrac{a(k-s)}{p} + b\sqrt{\dfrac{k-s}{p}} + c$ is an integer too. Consequently, the number

$$f\left(\sqrt{\frac{k+1-s}{p}}\right) - f\left(\sqrt{\frac{k-s}{p}}\right) = \frac{b}{\sqrt{p}}.\frac{1}{\sqrt{k+1-s}+\sqrt{k-s}} + \frac{a}{p} \qquad (1)$$

is an integer for all k which are sufficiently big. It follows from here that $\dfrac{a}{p}$ is an integer. Indeed, suppose that $\dfrac{a}{p}$ is not an integer. If $b > 0$, we chose k in a way that

$$\frac{b}{\sqrt{p}}.\frac{1}{\sqrt{k+1-s}+\sqrt{k-1}} < \left[\frac{a}{p}\right] + 1 - \frac{a}{p},$$

and if $b < 0$, we chose k in a way that

$$\frac{b}{\sqrt{p}}.\frac{1}{\sqrt{k+1-s}+\sqrt{k-1}} > \left[\frac{a}{p}\right] - \frac{a}{p}.$$

In both cases there is a contradiction with the fact that (1) is an integer. Now, it follows that

$$\frac{b}{\sqrt{p}}.\frac{1}{\sqrt{k+1-s}+\sqrt{k-s}}$$

for all k, which are sufficiently big. This is possible only when $b = 0$.

Let $\frac{a}{p} = m$. Then, $f\left(\sqrt{\frac{k-s}{p}}\right) = m(k-s) + c$ is an integer (when k is sufficiently big), i.e. $c - ms$ is an integer. Let $n = c - ms$. Now it is clear that $f(x) = mg(x) + n$ for all x.

Problem 4. The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by

$$a_1 = 1, \quad a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}, \quad n \geq 1.$$

Prove that $\lfloor a_n^2 \rfloor = n$ when $n \geq 4$ (it is denoted by $\lfloor x \rfloor$ the integer part of the number x).

Solution. Let $f(x) = \frac{x}{n} + \frac{n}{x}$. Since $f(a) - f(b) = \frac{(a-b)(ab-n^2)}{abn}$, it follows that the function $f(x)$ is decreasing in the interval $(0, n)$.

Firstly, by induction we shall prove that $\sqrt{n} \leq a_n \leq \frac{n}{\sqrt{n-1}}$ when $n \geq 3$. We have $a_1 = 1, a_2 = 2$ and $a_3 = 2$, i.e. $\sqrt{3} \leq a_3 \leq \frac{3}{\sqrt{2}}$. Let $\sqrt{n} \leq a_n \leq \frac{n}{\sqrt{n-1}}$ for an integer $n \geq 3$. Then, $a_{n+1} = f(a_n) \leq f(\sqrt{n}) = \frac{n+1}{\sqrt{n}}$ and $a_{n+1} = f(a_n) \geq f\left(\frac{n}{\sqrt{n-1}}\right) = \frac{n}{\sqrt{n-1}} > \sqrt{n+1}$ and thus the induction finishes.

Since $a_n \geq \sqrt{n}$, it remains to prove, that $a_n < \sqrt{n+1}$. We have $a_{n+1} = f(a_n) \geq f\left(\frac{n}{\sqrt{n-1}}\right) = \frac{n}{\sqrt{n-1}}$ when $n \geq 3$. Consequently, $a_n \geq \frac{n-1}{\sqrt{n-2}}$ when $n \geq 4$. Then,

$$a_{n+1} = f(a_n) < f\left(\frac{n-1}{\sqrt{n-2}}\right) = \frac{(n-1)^2 + n^2(n-2)}{(n-1)n\sqrt{n-2}} < \sqrt{n+2}$$

when $n \geq 4$. (The last inequality is equivalent to $2n^2(n-3) + 4n - 1 > 0$.) Therefore, $\sqrt{n} \leq a_n < \sqrt{n+1}$ when $n \geq 4$, i.e. $\lfloor a_n^2 \rfloor = n$.

Problem 5. The quadrilateral $ABCD$ is inscribed in a circle. The lines AB and CD meet each other in the point E, while the diagonals AC and BD — in the point F. The circumcircles of the triangles AFD and BFC have a second common point, which is denoted by H. Prove that $\angle EHF = 90°$.

Solution. Let O be the circumcenter of $ABCD$. We shall prove that O is the second common point of the circumcircles of $\triangle AHB$ and $\triangle CHD$. (Since AB and CD are not parallel, then $O \neq H$.) After that we shall prove that the points E, H and O are colinear and $\angle OHF = 90°$.

We shall consider the possible positions of H.

Let G be the common point of AD and BC (these lines are not parallel because the circumcircles of $\triangle AFD$ and $\triangle BFC$ are not tangent). It is clear that H is in the interior of $\angle AGB$.

1) H is in $\triangle CGD$. Then, $\angle CHD = \angle CHF + \angle DHF = 180° - \angle CBF + 180° - \angle DAF = 360° - \overset{\frown}{CD} > 180°$ (from $\overset{\frown}{CD} < \overset{\frown}{AB}$ and $\overset{\frown}{AB} + \overset{\frown}{CD} < 360°$), which is impossible.

2) H is in $\triangle CFD$. Then, $\angle CHD = 360° - \angle CHF - \angle DHF = \angle CBF + \angle DAF = \overset{\frown}{CD} = \angle COD$ and $\angle AHB = \angle AHF + \angle BHF = \angle ADF + \angle BCF = \overset{\frown}{AB} = \angle AOB$.

3) H is in $\triangle ABF$. Analogously, $\angle CHD = \angle COD$ and $\angle AHB = \angle AOB$.

4) H is on AB. Again $\angle CHD = \angle COD$ and $180° = \angle AHB = \angle AOB$. Consequently, O is the midpoint of AB.

5) H is not in $\triangle ABG$. Then $\angle CHD = \angle COD$ and $\angle AHB = 360° - \angle AOB$.

Note that in the cases 2), 3) and 5) the points O and H are in one and the same semiplane with respect to the line AB and with respect to the line CD. Consequently, the points A, B, H and O are concyclic. We have the same for the points C, D, H and O.

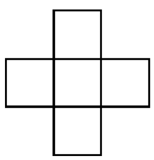
Now we shall prove that the line EH passes through the point O. Let this line meet the circumcircles of $\triangle AHB$ and $\triangle CHD$ at the points O_1 and O_2, respectively. Since these points lie on the ray $EH^{\rightarrow}$, it follows from the equalities $EH.EO_1 = EA.EB = EC.ED = EH.EO_2$ that $O_1 \equiv O_2 \equiv O$.

The points O and H are in one and the same semiplane with respect to CD and we can assume that the quadrilateral $COHD$ is convex. Let H be in $\triangle CFD$. Then, $\angle OHF = \angle FHC - \angle OHC = 180° - \angle FBC - \angle ODC = 180° - \frac{1}{2}.\angle COD - (90° - \frac{1}{2}.\angle COD) = 90°$. The other possibility for H is to be inside $\angle AGB$ and outside the non-convex quadrilateral $AFBG$.

Hence $\angle OHF = \angle OHC + \angle FHC = \angle ODC + \angle FBC = 90°$.

Problem 6. A square table of size 7×7 with the four corner squares deleted is given.

Figure 4.



a) What is the smallest number of squares which need to be colored black so that a 5-square entirely uncolored Greek cross (Figure 4) cannot be found on the table?

b) Prove that it is possible to writte integers in each square in a way that the sum of the integers in each Greek cross is negative while the sum of all integers in the square table is positive.

Solution. Denote the square in row i and column j by (i, j). Note that a cross is uniquely determined by its central cell. The cross with central cell (i, j) is denoted by C_{ij}, $2 \le i, j \le 6$. The number of all crosses is 25.

a) The squares $(1, i)$, $(i, 1)$, $(7, i)$, $(i, 7)$, $i = 2, \ldots, 6$, are included in exactly one cross; the squares (2,2), (2,6), (6,2), (6,6) are included in exactly 3 crosses; the squares $(2, i)$, $(i, 2)$, $(6, i)$, $(i, 6)$, $i = 3, 4, 5$ — in exactly 4 crosses and finally, the squares $(i, j), 3 \le i, j \le 5$, are included in exactly 5 crosses. Since C_{22} contains a colored square, then at least one of (1,2), (2,1), (2,2), (2,3), (3,2) should be colored. Similarly, at least one of (2,7), (1,6), (2,6), (2,5), (3,6), at least one of (7,2), (6,1), (6,2), (5,2), (6,3) and at least one of (6,7), (7,6), (6,6), (6,5), (5,6) are colored. Any of them is contained in at most four crosses (the first two in each quintuple in 1 cross, the third one in 3 crosses and the remaining two — in 4 crosses). Denote by x the number of the colored squares among $(i, j), 3 \le i, j \le 5$. The number of crosses containing a colored square is not greater than $4.4 + 5x$, whence

$$16 + 5x \ge 25. \tag{1}$$

Thus, $x \ge 2$ and the number of the colored cells is at least 6.

Suppose that the number of the colored squares is 6. Then $x = 2$. Moreover, acording (1) there exists at most one Greek cross containing more than one colored square. If two squares with a common side or a common vertex are colored, then it is easy to check that there are two crosses with at least two colored squares each. Therefore, the squares (4,4), (3,3), (3,5), (5,3), (5,5) should be uncolored. With no loss of generality, let the two colored squares be (4,3) and (4,5). But now the crosses C_{34} and C_{54} do not contain colored squares.

To prove that the minimal number of colored squares is 7, color for example the squares (2,5), (3,2), (3,3), (4,6), (5,4), (6,2), (6,5).

b) Let us write -5 in each of the colored squares from a) and 1 in the remaining squares. Since every Greek cross contains a colored square, the sum of the numbers in its squares does not exceed $1+1+1+1-5 = -1 < 0$. The sum of all numbers in the table is $7.(-5)+(45-7).1 = 3 > 0$ and we are done.

XLV National Mathematics Olympiad: 3rd round, April 1997

Problem 1. Find all natural numbers a, b and c such that the roots of the equation

$$\begin{aligned} x^2 - 2ax + b &= 0 \\ x^2 - 2bx + c &= 0 \\ x^2 - 2cx + a &= 0 \end{aligned}$$

are natural numbers.

Solution: Let $\{x_1, x_2\}$, $\{x_3, x_4\}$ and $\{x_5, x_6\}$ be the roots of the first, second and third equation respectively, and let all of them be natural numbers.

Assume that $x_i \geq 2$ for all $i = 1, 2, \ldots, 6$. Then $2a = x_1 + x_2 \leq x_1 x_2 = b$, $2b = x_3 + x_4 \leq x_3 x_4 = c$ and $2c = x_5 + x_6 \leq x_5 x_6 = a$. Thus $2(a + b + c) \leq a + b + c$, which is impossible, since a, b, c are natural numbers.

Therefore at least one of the numbers x_i equals 1. Without loss of generality suppose $x_1 = 1$, so $1 - 2a + b = 0$.

If $x_i \geq 2$ for $i = 3, 4, 5, 6$, then

$$2(b + c) = (x_3 + x_4) + (x_5 + x_6) \leq x_3x_4 + x_5x_6 = c + a,$$

whence $2(2a - 1 + c) \leq c + a \Rightarrow c \leq 2 - 3a$, which is impossible when a, b, c are natural numbers.

So at least one of x_3, x_4, x_5, x_6 equals 1. Let $x_3 = 1$. Now $1 - 2b + c = 0$. Assuming that $x_5 \geq 2$ and $x_6 \geq 2$, we get $2c = x_5 + x_6 \leq a$, so $2(2b - 1) \leq \dfrac{b+1}{2}$. Thus $7b \leq 5$, a contradiction.

Therefore at least one of the numbers x_5, x_6 is 1 and it follows that $1 - 2c + a = 0$. Further

$$0 = (1 - 2a + b) + (1 - 2b + c) + (1 - 2c + a) = 3 - (a + b + c)$$

and since a, b, c are natural numbers, it follows that $a = b = c = 1$.

Direct verification shows that $a = b = c = 1$ satisfy the conditions of the problem.

Problem 2. Given a convex quadrilateral $ABCD$ which can be inscribed in a circle. Let F be the intersecting point of diagonals AC and BD and E be the intersecting point of the lines AD and BC. If M and N are the midpoints of AB and CD, prove that

$$\frac{MN}{EF} = \frac{1}{2} \cdot \left| \frac{AB}{CD} - \frac{CD}{AB} \right|.$$

Solution: Let $\angle AEB = \gamma$, $EC = c$, $ED = d$, $\overrightarrow{i} = \frac{1}{c} \cdot \overrightarrow{EC}$ and $\overrightarrow{j} = \frac{1}{d} \cdot \overrightarrow{ED}$. Since $ABCD$ is an inscribed quadrilateral, $\frac{AB}{CD} = \frac{AE}{CE} = \frac{BE}{DE} = k$. Therefore $\overrightarrow{EA} = kc\overrightarrow{j}$ and $\overrightarrow{EB} = kd\overrightarrow{i}$. Since $F \in AC$ and $F \in BD$ there exist x and y such that

$$\overrightarrow{EF} = x\overrightarrow{EA} + (1-x)\overrightarrow{EC} = xkc\overrightarrow{j} + (1-x)c\overrightarrow{i}$$

and

$$\overrightarrow{EF} = y\overrightarrow{EB} + (1-y)\overrightarrow{ED} = ykd\overrightarrow{i} + (1-y)d\overrightarrow{j}.$$

Comparing the coefficients of $\overrightarrow{i}$ and $\overrightarrow{j}$ in these equalities gives $xkc = (1-y)d$ and $ykd = (1-x)c$. This implies $x = \frac{kd-c}{(k^2-1)c}$. Therefore

$$\overrightarrow{EF} = \frac{k}{k^2-1}\left((kd-c)\overrightarrow{j} + (kc-d)\overrightarrow{i}\right)$$

and thus

$$EF^2 = \left(\frac{k}{k^2-1}\right)^2 \left((kd-c)^2 + (kc-d)^2 + 2(kd-c)(kc-d)\cos\gamma\right).$$

On the other hand

$$\begin{aligned}\overrightarrow{MN} = \frac{1}{2}\cdot\left(\overrightarrow{AD} + \overrightarrow{BC}\right) &= \frac{1}{2}\cdot\left(\overrightarrow{ED} - \overrightarrow{EA} + \overrightarrow{EC} - \overrightarrow{EB}\right)\\ &= \frac{1}{2}\cdot\left((d-kc)\overrightarrow{j} + (c-kd)\overrightarrow{i}\right)\end{aligned}$$

and it follows that

$$MN^2 = \frac{1}{4}\cdot\left((d-kc)^2 + (c-kd)^2 + 2(d-kc)(c-kd)\cos\gamma\right).$$

Therefore $\dfrac{MN^2}{EF^2} = \dfrac{1}{4}\left(\dfrac{k^2-1}{k}\right)^2 = \dfrac{1}{4}\left(k - \dfrac{1}{k}\right)^2$ and so

$$\frac{MN}{EF} = \frac{1}{2} \cdot \left|\frac{AB}{CD} - \frac{CD}{AB}\right|$$

Problem 3. Prove that the equation

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0$$

has no solution in rational numbers.

Solution: It is easy to see that the equation is equivalent to

$$(2x + 3)^2 + (2y + 3)^2 + (2z + 3)^2 = 7.$$

It has a solution in rational numbers if and only if there exist integer numbers a, b, c and a natural number m such that

$$(*) \qquad a^2 + b^2 + c^2 = 7m^2.$$

Suppose such numbers exist and give m its smallest possible value. There are two cases:

(I) $m = 2n$ is an even number. Now $a^2 + b^2 + c^2$ is divisible by 4. This implies that all numbers a, b, c are even ones, so $a = 2a_1$, $b = 2b_1$, $c = c_1$. It follows now that $a_1^2 + b_1^2 + c_1^2 = 7n^2$, which contradicts the way m have been chosen.

(II) $m = 2n + 1$ is an odd number. Now $m^2 \equiv 1 \pmod 8$ and therefore $a^2 + b^2 + c^2 \equiv 7 \pmod 8$, which is impossible when a, b, c are integer.

Problem 4. Find all continuous functions $f(x)$ defined in the set of real numbers and such that

$$f(x) = f\left(x^2 + \frac{1}{4}\right)$$

for all real x.

Solution: Let $f(x)$ be a function satisfying the conditions of the problem. Obviously $f(x)$ is an even function.

Let $x_0 \geq 0$. There are two cases:

(I) $0 \leq x_0 \leq \frac{1}{2}$. Consider the sequence

$$x_0, x_1, \ldots, x_n, \ldots \tag{1}$$

defined by the equalities $x_{n+1} = x_n^2 + \frac{1}{4}$.

It is easy to see by induction that $0 \leq x_n \leq \frac{1}{2}$ for all n. Moreover

$$x_{n+1} - x_n = x_n^2 - x_n + \frac{1}{4} = \left(x_n - \frac{1}{2}\right)^2 \geq 0,$$

which implies that (1) is a monotone increasing function. Since it is bounded, it follows that it is a convergent function. Let $\lim_{n\to\infty} x_n = \alpha$. Now $\alpha^2 - \alpha + \frac{1}{4} = 0$, so $\alpha = \frac{1}{2}$.

On the other hand, since $f(x)$ is a continuous function,

$$\lim_{n\to\infty} f(x_n) = f\left(\frac{1}{2}\right).$$

But

$$f(x_{n+1}) = f\left(x_n^2 + \frac{1}{4}\right) = f(x_n)$$

for all n. Thus $f(x_0) = f(x_1) = \cdots$ which means that $f(x_{0)} = f\left(\frac{1}{2}\right)$ for all $x_0 \in \left[0, \frac{1}{2}\right]$.

(II) $x_0 > \frac{1}{2}$. Consider the following sequence:

$$x_0, x_1, \ldots, x_n, \ldots \tag{2}$$

defined by $x_{n+1} = \sqrt{x_n - \frac{1}{4}}$.

As in the previous case, we show that (2) is a convergent function and $\lim_{n\to\infty} x_n = \frac{1}{2}$. Further $\lim_{n\to\infty} f\left(x_n\right) = f\left(\frac{1}{2}\right)$ and since

$$f(x_{n+1}) = f\left(x_{n+1}^2 + \frac{1}{4}\right) = f(x_n)$$

for all n we get that $f(x_{0)} = f\left(\frac{1}{2}\right)$.

Therefore $f(x)$ is a constant function in the interval $[0, +\infty)$, and since it is even, it is a constant for all x.

Conversely, any constant function satisfies the conditions of the problem.

Problem 5. Two squares K_1 and K_2 with centres M and N and sides of length 1 are placed in the plane in a way that $MN = 4$, two of the sides of K_1 are parallel to MN and one of the diagonals of

K_2 lies on the line MN. Find the locus of midpoints of segments XY where X is an interior point for K_1 and Y is an interior point for K_2.

Solution: The locus is the interior of a regular hexagon centred at the midpoint of the segment MN and with a side length of $\frac{1}{2}$.

To prove this, proceed as follows.

Fix the point Y in the interior of K_2. When X varies in the interior of K_1 the locus of the midpoints of XY is a square K_1' which is homothetic to K_1 by homothecy with centre Y and coefficient $\frac{1}{2}$. Obviously the side of this square is $\frac{1}{2}$ and its centre is the midpoint of MY. When Y varies in the interior of K_2 then the locus of the midpoints of MY is a square K_2' which is homothetic to K_2 by homothecy with centre M and coefficient $\frac{1}{2}$. The side of this square equals $\frac{1}{2}$ and its centre Q is the midpoint of MN. Finally, when the centres of K_1' vary in the interior of K_2', the squares vary in the interior of a regular hexagon centred at Q and with a side length of $\frac{1}{2}$.

Problem 6. Find the number of non-empty sets of $S_n = \{1, 2, \ldots, n\}$ such that there are no two consecutive numbers in one and the same set.

Solution: Denote the required number by f_n. It is easy to see that $f_1 = 1,\ f_2 = 2,\ f_3 = 4$.

Divide the subsets of S_n having no two consecutive numbers into two groups—those that do not contain the element n and those that do. Obviously the number of subsets in the first group is f_{n-1}.

Let T be a set of the second group. Therefore either $T = \{n\}$ or $T = \{a_1, \ldots, a_{k-1}, n\}$, where $k > 1$. It is clear that $a_{k-1} \neq n - 1$, so $\{a_1, \ldots a_{k-1}\} \subset S_{n-2}$, whence the number of sets in the second group is $f_{n-2} + 1$. Therefore

$$f_n = f_{n-1} + f_{n-2} + 1.$$

After substituting $u_n = f_n + 1$ we get

$$u_1 = 2, u_2 = 3, \qquad u_n = u_{n-1} + u_{n-2}.$$

Therefore the sequence $\{u_n\}_{n=1}^{\infty}$ coincides with the Fibonacci sequence from its third number onwards. Thus we obtain

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right) - 1.$$

XLV National Mathematics Olympiad: 4th round, May 1997

Problem 1. Consider the polynomial

$$P_n(x) = \binom{n}{2} + \binom{n}{5}x + \binom{n}{8}x^2 + \cdots + \binom{n}{3k+2}x^k,$$

where $n \geq 2$ is a natural number and $k = \left[\dfrac{n-2}{3}\right]$.

(a) Prove that $P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$;

(b) Find all integer numbers a such that $P_n(a^3)$ is divisible by $3^{[\frac{n-1}{2}]}$ for all $n \geq 2$.

Solution: (a) Compare the coefficients in front of x^m, $0 \leq m \leq \left[\dfrac{n+1}{3}\right]$. It suffices to show that

$$\binom{n+3}{3m+2} = 3\binom{n+2}{3m+2} - 3\binom{n+1}{3m+2} + \binom{n}{3m+2} + \binom{n}{3m-1}.$$

Using the identity $\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1}$ we get that

$$\begin{aligned}
&\left(\binom{n+3}{3m+2} - \binom{n+2}{3m+2}\right) - 2\left(\binom{n+2}{3m+2} - \binom{n+1}{3m+2}\right) + \\
&+ \left(\binom{n+1}{3m+2} - \binom{n}{3m+2}\right) - \binom{n}{3m-1} = \\
= \; & \binom{n+2}{3m+1} - 2\binom{n+1}{3m+1} + \binom{n}{3m+1} - \binom{n}{3m-1} = \\
= \; & \left(\binom{n+2}{3m+1} - \binom{n+1}{3m+1}\right) + \\
&- \left(\binom{n+1}{3m+1} - \binom{n}{3m+1}\right) - \binom{n}{3m-1} = \\
= \; & \binom{n+1}{3m} - \binom{n}{3m} - \binom{n}{3m-1} = \\
= \; & 0.
\end{aligned}$$

(b) Suppose a satisfies the condition of (b). Then $P_5(a^3) = 10 + a^3$ is divisible by 9 and so $a \equiv -1 \pmod 3$. On the contrary, let $a \equiv -1 \pmod 3$. Now $a^3 + 1 \equiv 0 \pmod 9$. Since $P_2(a^3) = 1, P_3(a^3) = 3$ and $P_4(a^3) = 6$ it follows by induction from (a) that $P_n(a^3)$ is divisible by $3^{\left[\frac{n-1}{2}\right]}$ for any n. Therefore the required values of a are all integer numbers congruent to 2 modulo 3.

Problem 2. Let M be the centroid of $\triangle ABC$. Prove the inequality

$$\sin\angle CAM + \sin\angle CBM \le \frac{2}{\sqrt{3}}$$

(a) if the circumscribed circle of $\triangle AMC$ is tangent to the line AB;

(b) for any $\triangle ABC$.

Solution: We use the standard notation for the elements of $\triangle ABC$. Let G be the midpoint of AB.

(a) It follows from the conditions of the problem that

$$\left(\frac{c}{2}\right)^2 = GA^2 = GM \cdot GC = \frac{1}{3}m_c^2 = \frac{1}{12}(2a^2 + 2b^2 - c^2),$$

and therefore $a^2 + b^2 = 2c^2$. Using the median formula we get $m_a = \frac{\sqrt{3}}{2}b$ and $m_b = \frac{\sqrt{3}}{2}a$. Further

$$A = \sin \angle CAM + \sin \angle CBM = S\left(\frac{1}{bm_a} + \frac{1}{am_b}\right) = \frac{(a^2+b^2)\sin\gamma}{\sqrt{3}ab}.$$

From the Cosine Law $a^2 + b^2 - 2ab\cos\gamma = c^2 = \frac{a^2+b^2}{2}$, so $a^2 + b^2 = 4ab\cos\gamma$. Therefore $A = \frac{2}{\sqrt{3}}\sin 2\gamma \leq \frac{2}{\sqrt{3}}$.

(b) There are two circles through C and M tangent to the line AB. Denote the contact points by A_1 and B_1 and let $A_1 \in GA^{\rightarrow}$, $B_1 \in GB^{\rightarrow}$. Since G is the midpoint of A_1B_1 and $CM : MG = 2 : 1$, M must be the centroid of $\triangle A_1B_1C$. Furthermore it is clear that $\angle CAM \leq \angle CA_1M$ and $\angle CBM \leq \angle CB_1M$. Suppose $\angle CA_1M \leq 90^\circ$ and $\angle CB_1M \leq 90^\circ$. Now $\sin\angle CAM + \sin\angle CBM \leq \sin\angle CA_1M + \sin\angle CB_1M \leq \frac{2}{\sqrt{3}}$.

It remains to consider the case $\angle CA_1M > 90^\circ$, $\angle CB_1M \le 90^\circ$ (the above angles could not both be obtuse). It follows from $\triangle CA_1M$ that $CM^2 > CA_1^2 + A_1M^2$, so

$$\frac{1}{9}(2b_1^2 + 2a_1^2 - c_1^2) > b_1^2 + \frac{1}{9}(2b_1^2 + 2c_1^2 - a_1^2)$$

(a_1, b_1, c_1 are sides of $\triangle A_1B_1C$). We know from (a) that $a_1^2+b_1^2 = 2c_1^2$ and the above inequality becomes $a_1^2 > 7b_1^2$. Again from (a) we obtain

$$\sin\angle CB_1M = \frac{b_1\sin\gamma_1}{a_1\sqrt{3}} = \frac{b_1}{a_1\sqrt{3}}\sqrt{1 - \left(\frac{a_1^2+b_1^2}{4a_1b_1}\right)^2}.$$

Substituting $\dfrac{b_1^2}{a_1^2} = x$ we get that

$$\sin\angle CB_1M = \frac{1}{4\sqrt{3}}\sqrt{14x - x^2 - 1} < \frac{1}{4\sqrt{3}}\sqrt{2 - \frac{1}{49} - 1} = \frac{1}{7}$$

since $x < \dfrac{1}{7}$. Therefore

$$\sin\angle CAM + \sin\angle CBM < 1 + \sin\angle CB_1M < 1 + \frac{1}{7} < \frac{2}{\sqrt{3}}.$$

Note: The inequality holds only for $\triangle ABC$ with angles $\alpha = 22.5^\circ$, $\beta = 112.5^\circ$, $\gamma = 45^\circ$ or $\alpha = 112.5^\circ$, $\beta = 22.5^\circ$, $\gamma = 45^\circ$.

Problem 3. Let n and m be natural numbers such that $m + i = a_ib_i^2$ for $i = 1, 2, \ldots, n$, where a_i and b_i are natural numbers and a_i is not divisible by a square of a prime number. Find all n for which there exists an m such that $a_1 + a_2 + \cdots + a_n = 12$.

Solution: It is clear that $n \leq 12$. Since $a_i = 1$ if and only if $m+i$ is a perfect square, at most three of the numbers a_i equal 1 (prove it!). It follows now from $a_1 + a_2 + \cdots + a_n = 12$ that $n \leq 7$.

We show now that the numbers a_i are pairwise distinct. Assume the contrary and let $m + i = ab_i^2$ and $m + j = ab_j^2$ for some $1 \leq i < j \leq n$. Therefore $6 \geq n - 1 \geq (m + j) - (m + i) = a(b_j^2 - b_i^2)$. It is easy to see that the former is true only if $(b_i, b_j, a) = (1, 2, 2)$ or $(2, 3, 1)$ and in either case $a_1 + a_2 + \cdots + a_n > 12$.

All possible values of a_i are 1, 2, 3, 5, 6, 7, 10 and 11. There are three possibilities for n: $n = 2$ and $\{a_1, a_2\} = \{1, 11\}, \{2, 10\}, \{5, 7\}$; $n = 3$ and $\{a_1, a_2, a_3\} = \{1, 5, 6\}, \{2, 3, 7\}$; $n = 4$ and $\{a_1, a_2, a_3, a_4\} = \{1, 2, 3, 6\}$. Suppose $n = 4$ and $\{a_1, a_2, a_3, a_4\} = \{1, 2, 3, 6\}$. Now $(6b_1b_2b_3b_4)^2 = (m+1)(m+2)(m+3)(m+4) = (m^2+5m+5)^2 - 1$, which is impossible. Therefore $n = 2$ or $n = 3$.

If $n = 3$ and $(a_1, a_2, a_3) = (1, 5, 6)$, then $m = 3$ has the required property, and if $n = 2$ and $(a_1, a_2) = (11, 1)$, then $m = 98$ has the required property. (It is not difficult to see that the remaining cases are not feasible.)

Problem 4. Let a, b and c be positive numbers such that $abc = 1$. Prove the inequality

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

Solution: Let $x = a + b + c$ and $y = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = ab + bc + ca$ $(abc = 1)$. It follows from Cauchy's Inequality that $x \geq 3$ and $y \geq 3$. Since both sides of the given inequality are symmetric functions of

a, b and c, we transform the expression as a function of x, y and $abc = 1$. After simple calculations we get

$$\frac{3 + 4x + y + x^2}{2x + y + x^2 + xy} \le \frac{12 + 4x + y}{9 + 4x + 2y},$$

which is equivalent to

$$3x^2y + xy^2 + 6xy - 5x^2 - y^2 - 24x - 3y - 27 \ge 0.$$

Write the last inequality in the form

$$\begin{aligned}(\frac{5}{3}x^2y - 5x^2) + (\frac{xy^2}{3} - y^2) + (\frac{xy^2}{3} - 3y) + (\frac{4}{3}x^2y - 12x) + \\ + (\frac{xy^2}{3} - 3x) + (3xy - 9x) + (3xy - 27) \quad \ge \quad 0.\end{aligned}$$

When $x \ge 3$, $y \ge 3$, all terms in the left hand side are non-negative and the inequality is true. Equality holds when $x = 3$, $y = 3$, which implies $a = b = c = 1$.

Problem 5. Given a $\triangle ABC$ with bisectors BM and CN ($M \in AC$, $N \in AB$). The ray $MN^{\rightarrow}$ intersects the circumcircle of $\triangle ABC$ at point D. Prove that

$$\frac{1}{BD} = \frac{1}{AD} + \frac{1}{CD}.$$

Solution: Let A_1, B_1 and C_1 be the orthogonal projections of D on the lines BC, CA and AB, respectively. It follows from $\triangle DAB_1$ and the Sine Law that $DB_1 = DA \cdot \sin \angle DAB_1 = DA \cdot \sin \angle DAC = \dfrac{DA \cdot DC}{2R}$ (R is the circumradius of $\triangle ABC$). Analogously $DA_1 =$

$\dfrac{DB \cdot DC}{2R}$ and $DC_1 = \dfrac{DA \cdot DB}{2R}$. Our equality is now equivalent to $AD \cdot CD = BD \cdot CD + AD \cdot BD$ and so it suffices to prove that

$$DB_1 = DA_1 + DC_1 \tag{1}$$

Denote by m the distance from M to AB and BC and by n the distance from N to AC and BC. Let $\dfrac{DM}{MN} = x$ $(x > 1)$. Further, $\dfrac{DB_1}{n} = x$, $\dfrac{DC_1}{m} = x - 1$ and $\dfrac{DA_1 - m}{n - m} = x$. Therefore $DB_1 = nx$, $DC_1 = m(x-1)$ and $DA_1 = nx - m(x-1) = DB_1 - DC_1$ and (1) holds.

Problem 6. Let X be a set of $n+1$ elements, $n \geq 2$. Ordered n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ formed from distinct elements of X are called 'disjoint' if distinct indices i and j exist such that $a_i = b_j$. Find the maximal number of n-tuples any two of which are 'disjoint'.

Solution: For $n \geq 2$ denote by $A(n+1)$ the maximum number of ordered n-tuples such that any two of them are 'disjoint'. Also let $S(X)$ be a set of such n-tuples for which $|S(X)| = A(n+1)$. It is clear that for any $\alpha \in X$ the following holds:

$$|\{(a_1, a_2, \ldots, a_n) \in S(X) | a_1 = \alpha\}| \leq A(n).$$

Thus $A(n+1) \leq (n+1)A(n)$. Therefore

$$A(n+1) \leq (n+1)n \ldots A(3).$$

Direct verification shows that $A(3) = 3$, so $A(n+1) \leq \dfrac{(n+1)!}{2}$.

We prove now that $A(n+1) = \dfrac{(n+1)!}{2}$ by constructing a set of $\dfrac{(n+1)!}{2}$ ordered n-tuples, any two of which are 'disjoint'. We may assume that $X = \{1, 2, \ldots, n+1\}$. Consider a set E of all even permutations of $1, 2, \ldots, n+1$. (A permutation $(a_1, a_2, \ldots, a_{n+1})$ is called even if the number of pairs (i, j) such that $i < j$ and $a_i > a_j$ is an even number.) The set

$$\{(a_1, a_2, \ldots, a_n) | (a_1, a_2, \ldots, a_n, a_{n+1}) \in E\}$$

has $\dfrac{(n+1)!}{2}$ ordered n-tuples, any two of which are 'disjoint'.

XLVII National Mathematics Olympiad: 3rd round, 25–26 April 1998

Problem 1. Find the least positive integer number n ($n \geq 3$) with the following property: for any colouring of n different points $A_1,\ A_2, \ldots,\ A_n$ on a line and such that $A_1A_2 = A_2A_3 = \ldots = A_{n-1}A_n$ in two colours, there are three points $A_i, A_j, A_{2j-i} (1 \leq i < 2j - i \leq n)$ which have the same colour.

Solution: Assume the two colours are white and black. Consider 8 points coloured as follows: A_1, A_2, A_5, A_6 (white), A_3, A_4, A_7, A_8 (black). Obviously no three points A_i, A_j, A_{2j-i} $(1 \leq i < 2j - i \leq n)$ have the same colour and therefore $n \geq 9$.

If we can show that $n = 9$ has the required property, we will be done. Suppose there are 9 points coloured black or white and no three points A_i, A_j, A_{2j-i} $(1 \leq i < 2j - i \leq n)$ have the same colour.

First assume that for $i = 3$, $i = 4$ or $i = 5$ points A_i and A_{i+2} have the same colour (say white). Then the points $A_{i-2}, A_{i+1}, A_{i+4}$

should be black (note that $i-2 \geq 1$ and $i+4 \leq 9$), which is a contradiction.

Suppose now that for $i = 3, 4, 5$ the points A_i and A_{i+2} have different colours. Without loss of generality assume A_5 is a white point. Then A_3 and A_7 are black. Because of the symmetry we may suppose that A_4 is white and A_6 is black. Consiquently A_8 is white, A_2 is black $(2+8 = 2 \cdot 5)$ and A_9 is white $(7+9 = 2 \cdot 8)$. Therefore A_1 should be both white $(1+3 = 2 \cdot 2)$ and black $(1+9 = 2 \cdot 5)$, which is again a contradiction.

Consequently the assumption is not true and so $n = 9$.

Problem 2. Let $ABCD$ be a quadrilateral such that $AD = CD$ and $\angle DAB = \angle ABC < 90°$. The line passing through D and the midpoint of the segment BC intersects the line AB at the point E. Prove that $\angle BEC = \angle DAC$.

Solution: Let M be the midpoint of BC and let AD and BC meet at point N and AN and EC meet at point P. It follows from Menelaus' Theorem aplied to $\triangle DMN$ and $\triangle DEN$ that $DP \cdot NC \cdot ME = PN \cdot CM \cdot ED$ and $DA \cdot NB \cdot ME = AN \cdot BM \cdot ED$. Combining the above equalities with $AN = BN$, $BE = CE$ and $AD = CD$, we get $DP \cdot NC = DC \cdot PN$. Therefore CP is bisector of $\angle DCN$.

Consequently $\angle ACP = \angle ACD + \angle DCP = \frac{1}{2}(\angle NDC + \angle DCN) = \angle NAB$ so $\angle DCP = \angle CAB$ and we obtain $\angle BEC = \angle ABC - \angle BCE = \angle BAD - \angle DCP = \angle DAC$, *Q. E. D.*

Note: The assertion is also true if the condition $\angle ABC < 90°$ is left out.

Problem 3. Let $\mathbb{R}^+$ be the set of all positive real numbers. Prove that there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(f(x))^2 \geq f(x+y)(f(x)+y)$$

for arbitrary positive real numbers x and y.

Solution: Suppose there exists a function satisfying the conditions of the problem. Write the initial equality in the form

$$f(x) - f(x+y) \geq \frac{f(x)y}{f(x)+y}.$$

First we prove that $f(x) - f(x+1) \geq \frac{1}{2}$ for $x > 0$. Obviously f is a (strictly) monotone non-increasing function. Fix $x > 0$ and choose a natural number n, such that $n \cdot F(x+1) \geq 1$. When $k = 0, 1, \ldots, n-1$, we obtain that

$$f(x+\frac{k}{n}) - f(x+\frac{k+1}{n}) \geq \frac{f(x+\frac{k}{n})\frac{1}{n}}{f(x+\frac{k}{n})+\frac{1}{n}} \geq \frac{1}{2n}.$$

Adding the above inequalities gives $f(x) - f(x+1) \geq \frac{1}{2}$.

Let the natural number m be such that $m \geq 2f(x)$. Therefore

$$f(x) - f(x+m) = \sum_{i=0}^{m-1}(f(x+i) - f(x+i+1)) \geq \frac{m}{2} \geq f(x),$$

and so $f(x+m) \leq 0$. But this contradicts the fact that f is strictly positive.

Problem 4. Let $f(x) = x^3 - 3x + 1$. Find the number of different real solutions of the equation $f(f(x)) = 0$.

Solution: Since $f'(x) = 3(x-1)(x+1)$ it follows that f is strictly monotone non-decreasing in the intervals $(-\infty, -1]$ and $[1, \infty)$ and strictly monotone non-increasing in the interval $[-1, 1]$. Moreover $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty, f(-1) = 3, f(1) = -1, f(3) = 19 > 0$ and so the equation $f(x) = 0$ has three distinct roots x_1, x_2, x_3 such that $x_1 < -1 < x_2 < 1 < x_3 < 3$. Therefore $f(x) = x_1$ has only one real root (which is less than -1) and $f(x) = x_2$ and $f(x) = x_3$ have three distinct real roots each (one in each of the intervals $(-\infty, -1), (-1, 1)$ and $(1, \infty)$). Since the roots of $f(f(x)) = 0$ are exactly the roots of these three equations, we conclude that it has seven distinct real roots.

Problem 5. The convex pentagon $ABCDE$ is inscribed in a circle with radius R. The inradii of the triangles ABC, ABD, AEC and AED are denoted by r_{ABC}, r_{ABD}, r_{AEC} and r_{AED}. Prove that

a.) $\cos\angle CAB + \cos\angle ABC + \cos\angle BCA = 1 + \dfrac{r_{ABC}}{R}$;

b.) If $r_{ABC} = r_{AED}$ and $r_{ABD} = r_{AEC}$, then $\triangle ABC \cong \triangle AED$.

Solution: a.) Using the standard notation for the elements of $\triangle ABC$, we obtain that

$$\frac{r}{R} = \frac{S}{pR} = \frac{4S^2}{pabc} = \frac{4(p-a)(p-b)(p-c)}{abc} =$$

$$= \frac{1}{2abc}(a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2)-2abc) =$$
$$= \cos\angle CAB + \cos\angle ABC + \cos\angle BCA - 1.$$

b.) Applying continuously the equality from a.) and using the fact that $ABCDE$ is inscribed in a circle convex pentagon, it is easy to see that

$$r_{ABC} + r_{AEC} + r_{EDC} = r_{AED} + r_{ABD} + r_{BCD}.$$

The condition of the problem and the above equality imply $r_{BCD} = r_{EDC}$. Since $\triangle BCD$ and $\triangle EDC$ have a side in common and equal circumradiuses we get that $\triangle BCD \cong \triangle EDC$ (prove it using a.)). In particular $BC = ED$ and using $r_{ABC} = r_{AED}$ again, we get $\triangle ABC \cong \triangle AED$.

Problem 6. Show that the equation

$$x^2y^2 = z^2(z^2-x^2-y^2)$$

has no solution in positive integer numbers.

Solution: Suppose x, y, z is a solution for which $\frac{xy}{z}$ (which is a natural number—why?) has its minimum value.

Write x, y, z in the form $x = dx_1, y = dy_1, z = dz_1$, where $d = (x, y, z)$. Our equation is equivalent to $x_1^2y_1^2 = z_1^2(z_1^2-x_1^2-y_1^2)$. Let $u = (x_1, z_1), v = (y_1, z_1), x_1 = ut, y_1 = vw$. Since z_1 divides x_1y_1, we get $z_1 = uv$. A substitution in the last equality gives

$(u^2+w^2)(v^2+t^2) = 2u^2v^2$. Further, since $(x_1, y_1, z_1) = 1$, it follows that $(u,w) = 1, (v,t) = 1$. Therefore

$$u^2+w^2 = v^2, v^2+t^2 = 2u^2$$

or

$$u^2+w^2 = 2v^2, v^2+t^2 = u^2.$$

Without loss of generality we may assume that the first pair of equalities hold. It is easily seen that v and u are odd integer numbers. It follows now from $u^2+w^2 = v^2$ that $u = m^2-n^2, w = 2mn, v = m^2+n^2$, where m and n are coprime natural numbers (of distinct parity). Substitution in $v^2+t^2 = 2u^2$ shows that $t^2+(2mn)^2 = (m^2-n^2)^2$ and so $t = p^2-q^2, mn = pq, m^2-n^2 = p^2+q^2$ for some natural numbers p and q. Therefore

$$p^2q^2 = m^2(m^2-p^2-q^2),$$

which shows that p, q, m is a solution of the original equation. It remains to be seen that $\dfrac{pq}{m} = n < d(p^2-q^2)2mn = \dfrac{xy}{z}$, which contradicts the way we have chosen x, y, z.

XLVII National Mathematics Olympiad: 4th round, 16–17 May 1998

Problem 1. Let n be a natural number. Find the least natural number k for which there exist k sequences of 0's and 1's of length $2n+2$ with the following property: any sequence of 0's and 1's of length $2n+2$ coincides in at least $n+2$ positions with some of these k sequences.

Solution: We shall prove that $k = 4$. Assume that $k \leq 3$ and let the respective sequences be $a_1^i, a_2^i, \ldots, a_{2n+2}^i$ for $i = 1, \ldots, k$. Since $k \leq 3$ there is a sequence $b_1, b_2, \ldots, b_{2n+2}$ such that $(b_{2l+1}, b_{2l+2}) \neq (a_{2l+1}^i, a_{2l+2}^i)$ for $l = 0, 1 \ldots, n$ and $i = 1, \ldots, k$. This is a contradiction. For $k = 4$ it is easily seen that the sequences $000\ldots0$, $011\ldots1$, $100\ldots0, 111\ldots1$ have the required property.

Problem 2. The polynomials $P_n(x, y)$, $n = 1, 2, \ldots$ are defined by

$$P_1(x, y) = 1, P_{n+1}(x, y) = (x+y-1)(y+1)P_n(x, y+2)+(y-y^2)P_n(x, y).$$

Prove that $P_n(x,y) = P_n(y,x)$ for all x, y and n.

Solution: We know that $P_1(x,y) = 1$ and $P_2(x,y) = xy + x + y - 1$. Assume that $P_{n-1}(x,y)$ and $P_n(x,y)$, $(n \geq 2)$ are symmetric polynomials. Then

$$\begin{aligned} P_{n+1}(x,y) &= (x+y-1)(y+1)P_n(x,y+2) + (y-y^2)P_n(x,y) \\ &= (x+y-1)(y+1)P_n(y+2,x) + (y-y^2)P_n(y,x) \\ &= (x+y-1)(y+1)\left(\begin{array}{l} (x+y+1)(x+1)P_{n-1}(y+2,x+2) \\ +(x-x^2)P_{n-1}(y+2,x) \end{array} \right) \\ &\quad + (y-y^2)\left(\begin{array}{l} (y+x-1)(x+1)P_{n-1}(y,x+2) \\ +(x-x^2)P_{n-1}(y,x) \end{array} \right) \\ &= (x+y-1)(y+1)(x+y+1)(x+1)P_{n-1}(y+2,x+2) \\ &\quad + (y-y^2)(x-x^2)P_{n-1}(y,x) \\ &\quad + (x+y-1)(y+1)(x-x^2)P_{n-1}(y+2,x) \\ &\quad + (y-y^2)(x+y-1)(x+1)P_{n-1}(x+2,y). \end{aligned}$$

and by induction it follows that all the polynomials are symmetric.

Problem 3. On the sides of a non-obtuse triangle ABC a square, a regular n-gon and a regular m-gon $(n, m > 5)$ are constructed externally, so that their centres are vertices of a regular triangle. Prove that $m = n = 6$ and find the angles of ABC.

Solution: Let the square, the n-gon and the m-gon be constructed on the sides AB, BC and CA, respectively. Denote their centres by O_1, O_2 and O_3; denote by A_1, B_1 and C_1 the centres of the equilateral triangles constructed externally on BC, CA and AB.

The lines O_1C_1, O_2A_1 and O_3B_1 intersect at the circumcentre O of $\triangle ABC$. Since $\triangle A_1A_2A_3$ is equilateral, it follows straightforwardly that $\triangle O_1O_2O_3$ is equilateral if and only if $C_1A_1||O_1O_2$, $A_1B_1||O_2O_3$ and $B_1C_1||O_1O_3$. This is equivalent to

$$\frac{OC_1}{C_1O_1} = \frac{OA_1}{A_1O_2} = \frac{OB_1}{B_1O_3} = k.$$

On the other hand,

$$\begin{aligned} \frac{OC_1}{C_1O_1} &= \frac{\cot C + \tan 30^\circ}{\cot 45^\circ - \tan 30^\circ}, \\ \frac{OA_1}{A_1O_2} &= \frac{\cot A + \tan 30^\circ}{\cot \frac{180^\circ}{n} - \tan 30^\circ}, \\ \frac{OB_1}{B_1O_3} &= \frac{\cot B + \tan 30^\circ}{\cot \frac{180^\circ}{m} - \tan 30^\circ}. \end{aligned}$$

Set $\cot \dfrac{180^\circ}{n} = x$ and $\cot \dfrac{180^\circ}{m} = y$. The above identities imply that

$$\cot A = kx - \frac{k+1}{\sqrt{3}}, \cot B = ky - \frac{k+1}{\sqrt{3}}, \cot C = k - \frac{k+1}{\sqrt{3}}.$$

From the identity $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$ we get

$$\begin{aligned} k &= \frac{2(x+y+1-\sqrt{3})}{\sqrt{3}xy + (\sqrt{3}-2)(x+y) + \sqrt{3} - 2}, \\ \cot C &= \frac{x+y-xy+3-2\sqrt{3}}{\sqrt{3}xy + (\sqrt{3}-2)(x+y) + \sqrt{3} - 2}. \end{aligned}$$

Since $m, n \geq 6$ it follows that $x, y \geq \sqrt{3}$,i. e. $xy \geq \sqrt{3}(x+y) - 3$. The inequality $\cot C \geq 0$ implies $x + y - xy + 3 - 2\sqrt{3} \geq 0$. and therefore $x + y + 3 - 2\sqrt{3} \geq xy \geq \sqrt{3}(x+y) - 3$,i. e. $x + y \leq 2\sqrt{3}$. This shows that $x = y = \sqrt{3}$, i. e., $m = n = 6$. Hence $\cot C = 0, \cot A = \cot B = 1$, so $\angle C = 90^\circ, \angle A = \angle B = 45^\circ$.

Problem 4. Let $a_1, a_2, \ldots, a_n$ be real numbers, not all of them zero. Prove that the equation

$$\sqrt{1+a_1x} + \sqrt{1+a_2x} + \ldots + \sqrt{1+a_nx} = n$$

has at most one nonzero real root.

Solution: The given equation is equivalent to $x\sum_{i=1}^{n}\frac{a_i}{\sqrt{1+a_ix}} = n$. Since the function $\frac{a_i}{\sqrt{1+a_ix}}$ $(a_i \neq 0)$ is strictly decreasing, it follows that this equation has at most one nonzero root.

Problem 5. Let m and n be natural numbers such that $A = ((m+3)^n+1)/3m$ is integer. Prove that A is an odd integer number.

Solution: Assume that A is an even integer number, i.e., $(m+3)^n + 1 = 6km$. Then m is an even integer number. Moreover $m|3^n+1$, which shows that $m = 3t+2$ and n is odd. Let $m = 2^\alpha m_1$, where $\alpha \geq 1$ and m_1 is odd. Then $2^\alpha|3^n+1$ and therefore $\alpha \leq 2$. Since $m_1|3^n+1$, it follows that $m_1|a^2+3$, where $a = 3^{\frac{n+1}{2}}$. It is well-known that in this case $m_1 = 6t_1+1$. Since $m = 2^\alpha(6t_1+1)$ has the form $3t+2$ and $1 \leq \alpha \leq 2$ we see that $\alpha = 1$. Then $m = 12t_1+2$ and from $(m+3)^n+1 = 6km$ it follows that $4|5^n+1$, which is impossible.

Problem 6. The sides and the diagonals of a regular n-gon X are coloured in k colours so that:

(i) for each colour a and any two vertices A and B of X, the segment AB is coloured in colour a or there is a vertex C such that AC and BC are coloured in colour a;

(ii) the sides of any triangle with vertices among the vertices of X are coloured in at most two colours.

Prove that $k \leq 2$.

Solution: Assume that the colouring involves at least three different colours a, b, c. We shall construct an infinite subset of vertices of X, which will imply a contradiction.

Let $Z \in X$ and A_1 is a vertex such that the colour of A_1Z is a. From (i) it follows that there is a vertex B_1, such that the colour of B_1Z and B_1A_1 is b. Analogously there is a vertex C_1 such that the colour of C_1Z and C_1B_1 is c. Considering the triangles C_1A_1Z and $C_1A_1B_1$ we see (using condition (ii)) that the colour of C_1A_1 is c. Let $A_2 \in X$ be such that the colour of A_2C_1 and A_2Z is a. It is easily seen that $A_2 \neq A_1$ and the colour A_2A_1 and A_2B_1 is a. Now we shall proceed by induction. Let the vertices $A_2, B_2, C_2, \ldots, A_{k-1}, B_{k-1}, C_{k-1}$ be such that the colour of A_iA_j, A_iB_j, A_iC_j is a, the colour of B_iA_j, B_iB_j, B_iC_j is b and the colour C_iA_j, C_iB_j, C_iC_j is c, $(2 \leq j < i < k)$.

Take a vertex A_k such that the colour of A_kC_{k-1} and A_kZ is a. Then considering the triangles ZA_kB_j and $C_{k-1}A_kB_j$ $(j < k)$, ZA_kC_j and $B_{j+1}A_kC_j$ $(j+1 < k)$, $A_kA_jB_j$ and $A_kA_jB_j$ we see that the colour of A_kB_j, A_kC_j and A_kC_j is a. The vertices B_k and C_k are constructed in a similar way.

XLVIII National Mathematics Olympiad: 3rd round, 17–18 April 1999

Problem 1. Find all triples (x, y, z) of natural numbers such that y is a prime number, y and 3 do not divide z, and $x^3 - y^3 = z^2$.

Nikolay Nikolov

Solution: Since $(x-y)((x-y)^2+3xy)) = x^3 - y^3 = z^2$, it follows from the conditions of the problem that $x-y$ and $(x-y)^2+3xy$ are relatively prime. Therefore $x - y = u^2$ and $x^2 + xy + y^2 = v^2$. Thus $3y^2 = (2v - 2x - y)(2v + 2x + y)$ and since y is a prime number, there are three cases to consider:

1. $2v - 2x - y = y, 2v + 2x + y = 3y$. Now $x = 0$, which is impossible.

2. $2v - 2x - y = 1, 2v + 2x + y = 3y^2$. Now $3y^2 - 1 = 2(2x + y) = 2(2u^2 + 3y)$ and it follows that 3 divides $u^2 + 1$, which is impossible.

3. $2v - 2x - y = 3, 2v + 2x + y = y^2$. Now $y^2 - 3 = 2(2x + y) = 2(2u^2 + 3y)$ and it follows that $(y-3)^2 - (2u)^2 = 12$. Therefore $y = 7, u = 1, x = 8, z = 13$. Direct verification shows that $(8, 7, 13)$ is a solution of the problem.

Problem 2. A convex quadrilateral of area S is inscribed in a circle whose centre is a point interior to the quadrilateral. Prove that the area of the quadrilateral whose vertices are the projections of the point of intersection of the diagonals on the sides does not exceed $\frac{S}{2}$. *Christo Lesov*

Solution: Let $ABCD$ be a quadrilateral inscribed in a circle with centre O and radius R. Denote by E the point of intersection of AC and BD. Denote further by M, N, P, Q and F the projections of E on AB, BC, CD, DA and MN, respectively. We know that $MN = BE \sin \angle ABC = \frac{BE \cdot AC}{2R}$. Also,

$$EF = EM \sin \angle EMN = \frac{AE \cdot BE \sin \angle AEB}{AB} \sin \angle CBE.$$

Since $BE \sin \angle CBE = CE \sin \angle BCE = CE \frac{AB}{2R}$ and $AE \cdot CE = R^2 - OE^2$, it follows that $EF = \frac{R^2 - OE^2}{2R} \sin \angle AEB$. Therefore

$$S_{\triangle MEN} = \frac{MN \cdot EF}{2} = \frac{AC \cdot BE \sin \angle AEB(R^2 - OE^2)}{8R^2}.$$

Similar equalities hold for $S_{\triangle NEP}, S_{\triangle PEQ}$ and $S_{\triangle QEM}$. By combining the above we obtain

$$S_{MNPQ} = \frac{AC \cdot BD \sin \angle AEB(R^2 - OE^2)}{4R^2} \leq \frac{S_{ABCD}}{2},$$

Q. E. D.

Problem 3. In a competition 8 judges marked the contestants by *yes* or *no*. It is known that for any two contestants, two judges gave both a *yes*; two judges gave the first one a *yes* and the second one a *no*; two judges gave the first one a *no* and the second one a *yes*, and finally, two judges gave both a *no*. What is the greatest possible number of contestants? *Emil Kolev*

Solution: Denote the number of contestants by n. Consider a table with 8 rows and n columns such that the cell in the ith row and jth column contains 0 (1) if the ith judge gave the jth contestant a *no* (a *yes*). The conditions of the problem now imply that the table formed by any two columns contains among its rows each of the pairs 00, 01, 10 and 11 twice. We shall prove that 8 columns having this property do not exist. Assume the opposite. It is easily seen that if in any column all 0s are replaced by 1s and *vice versa*, the above property is retained. Therefore without loss of generality suppose that the first row consists of 0s. Denote the number of 0s in the ith row by a_i. It is clear that the total number of 0s is $8 \cdot 4 = 32$. Further, the number of occurrences of 00 is $\binom{8}{2} \cdot 2 = 56$. On the other hand the same number is $\sum_{i=1}^{8} \binom{a_i}{2}$. Since $a_1 = 8$, it follows that $\sum_{i=2}^{8} a_i = 24$. It is easy to prove now that $\sum_{i=2}^{8} \binom{a_i}{2} \geq 30$. Therefore $56 = \sum_{i=1}^{8} \binom{a_i}{2} \geq 58$, which is false.

The diagram on the right shows that it is possible to have exactly 7 contestants:

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{matrix}$$

Problem 4. Find all pairs (x, y) of integer numbers such that $x^3 = y^3 + 2y^2 + 1$. *Nikolay Nikolov and Emil Kolev*

Solution: It is obvious that $x > y$. On the other hand $x < y + 1 \iff (y+1)^3 > y^3 + 2y^2 + 1 \iff y(y+3) > 0$. Therefore if $y > 0$ or $y < -3$ the problem has no solution. Direct verification yields all pairs (x, y) which satisfy the equality $x^3 = y^3 + 2y^2 + 1$, namely $(-2, -3)$, $(1, -2)$ and $(1, 0)$.

Problem 5. Let B_1 and C_1 be points on the sides AC and AB of $\triangle ABC$. The straight lines BB_1 and CC_1 intersect at point D. Prove that the quadrilateral AB_1DC_1 is circumscribed if and only if the incircles of $\triangle ABD$ and $\triangle ACD$ are tangent.

Rumen Kozarev and Nikolay Nikolov

Solution: Note that the incircles of $\triangle ABD$ and $\triangle ACD$ are tangent if and only if $AB+AD-BD = AC+AD-CD$, so $AB+CD = AC+BD$.

Suppose AB_1DC_1 is circumscribed and the incircle touches AB_1, B_1D, DC_1, C_1A in the points M, N, P, Q, respectively. Therefore $AB+CD = AQ+BQ+CP-DP = AM+BN+CM-DN = AC+BD$.

Conversely, let the incircles of $\triangle ABD$ and $\triangle ACD$ be tangent. Denote the point of intersection of the tangent through C (different from CA) with the incircle of $\triangle ABB_1$ by D'. It follows from the above that $BD'-CD' = AB-AC = BD-CD$, so $DD' = |CD-CD'|$. Therefore $D' \equiv D$, which completes the proof.

Problem 6. Each interior point of an equilateral triangle of side 1 lies in one of six circles of the same radius r. Prove that $r \geq \dfrac{\sqrt{3}}{10}$.

Nikolay Nikolov and Emil Kolev

Solution: Divide each side of the triangle into five equal parts and draw lines parallel to the sides through these points. Thus the triangle is divided into 25 equilateral triangles of side $\dfrac{1}{5}$. The total number of vertices is $21 > 6 \cdot 3$. Therefore there exist 4 points which are interior to one and the same circle. It is easy to see now that $r \geq \dfrac{\sqrt{3}}{10}$, which solves the problem.

XLVIII National Mathematics Olympiad: 4th round, 18–19 May 1999

Problem 1. The faces of an orthogonal parallelepiped whose dimensions are natural numbers are painted green. The parallelepiped is partitioned into unit cubes by planes parallel to its faces. Find the dimensions of the parallelepiped if the number of cubes having no green face is one third of the total number of cubes.

Sava Grozdev

Solution: Let $x \leq y \leq z$ be the dimensions of the parallelepiped. It follows from the conditions of the problem that $x \geq 3$ and $(x-2)(y-2)(z-2) = \frac{xyz}{3}$. Since $\frac{x-2}{x} \leq \frac{y-2}{y} \leq \frac{z-2}{z}$, when $x \geq 7$, we obtain that $\frac{(x-2)(y-2)(z-2)}{xyz} \geq (\frac{5}{7})^3 > \frac{1}{3}$. Therefore $x \leq 6$ and thus $x = 3$, $x = 4$, $x = 5$ or $x = 6$.

1. If $x = 3$, then $(y-2)(z-2) = yz$, which is impossible.

2. If $x = 4$, then $2(y-2)(z-2) = \dfrac{4yz}{3}$, so $(y-6)(z-6) = 24$. In this case the only solutions are $(4,7,30)$, $(4,8,18)$, $(4,9,14)$ and $(4,10,12)$.

3. If $x = 5$, then $3(y-2)(z-2) = \dfrac{5yz}{3}$, so $(2y-9)(2z-9) = 45$. Therefore the solutions are $(5,5,27)$, $(5,6,12)$ and $(5,7,9)$.

4. If $x = 6$, then $4(y-2)(z-2) = 2yz$, so $(y-4)(z-4) = 8$. Thus there is an unique solution $(6,6,8)$.

Answer: The problem has 8 solutions—$(4,7,30)$, $(4,8,18)$, $(4,9,14)$, $(4,10,12)$, $(5,5,27)$, $(5,6,12)$, $(5,7,9)$ and $(6,6,8)$.

Problem 2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of integer numbers such that

$$(n-1)a_{n+1} = (n+1)a_n - 2(n-1)$$

for any $n \geq 1$. If 2000 divides a_{1999}, find the smallest $n \geq 2$ such that 2000 divides a_n. *Oleg Mushkarov, Nikolai Nikolov*

Solution: It is obvious that $a_1 = 0$ and $a_{n+1} = \dfrac{n+1}{n-1}a_n - 2$ when $n \geq 2$. Therefore the sequence is uniquely determined by its second term. Furthermore the sequence $a_n = (n-1)(cn+2)$ (where $c = \dfrac{a_2}{2} - 1$ is an arbitrary real number) satisfies the equality from the conditions of the problem. We conclude now that all sequences which satisfy this equality are of the kind given above. Since all terms are integer numbers and 2000 divides a_{1999}, it is easy to see that c is integer and $c = 1000m + 2$. Therefore 2000 divides a_n if and only if

1000 divides $(n-1)(n+1)$. Thus $n = 2k+1$ and $k(k+1)$ is divisible by $250 = 5^3 \cdot 2$. Since k and $k+1$ are relatively prime, we get that the smallest $n \geq 2$ equals $2 \cdot 124 + 1 = 249$.

Problem 3. The vertices of a triangle have integer coordinates and one of its sides is of length $\sqrt{n}$, where n is a square-free natural number. Prove that the ratio of the circumradius and the inradius is an irrational number. *Oleg Mushkarov, Nikolai Nikolov*

Solution: Suppose that $\dfrac{R}{r} = q$, where q is a rational number. Without loss of generality assume that one of the ends of the side of length $\sqrt{n}$ is at the origin of the coordinate system. Let the remaining two vertices have coordinates (x, y) and (z, t), where x, y, z and t are integers. The sides of our triangle have lengths $a = \sqrt{A}, b = \sqrt{B}$ and $c = \sqrt{C}$, where $n = A = x^2 + y^2, B = z^2 + t^2$ and $C = (x-z)^2 + (y-t)^2$. It follows from the conditions of the problem that

$$q = \frac{R}{r} = \frac{abc}{4S} \cdot \frac{p}{S} = \frac{abc(a+b+c)}{8S^2},$$

where S is the area of the triangle. Since S is rational (prove it!), it follows that $\sqrt{ABC}(\sqrt{A} + \sqrt{B} + \sqrt{C}) = 8S^2 q$ is a rational number. Therefore $A\sqrt{BC} + B\sqrt{AC} = 8S^2q - C\sqrt{AB}$ and after squaring we obtain that $\sqrt{AB}$ is a rational number. Thus AB is a perfect square. By analogy both BC and CA are perfect squares. Let $AB = E^2$, $BC = F^2$ and $CA = G^2$, where E, F and G are integer. Write A, B and C in the following form: $A = a_1 a_2^2, B = b_1 b_2^2$ and $C = c_1 c_2^2$, where a_1, b_1 and c_1 are square-free integers. So $a_1 b_1 (a_2 b_2)^2 = m^2$ and therefore $a_1 b_1$ is a perfect square, whence $a_1 = b_1$. By analogy

$a_1 = c_1$. Thus $A = ma_1^2, B = mb_1^2, C = mc_1^2$, where m is square-free. It follows from $ma_1^2 = n$ that $m = n, a_1 = 1$ and we obtain

$$\left| \begin{array}{ccccc} x^2 & + & y^2 & = & n \\ z^2 & + & t^2 & = & nb_1^2 \\ (x-z)^2 & + & (y-t)^2 & = & nc_1^2 \end{array} \right.$$

Since both b_1 and c_1 are integer, it follows from the Triangle Inequality that $1+b_1 > c_1$ and $1+c_1 > b_1$, whence $b_1 = c_1$. It is easy to determine now that $x^2+y^2 = 2(xz+yt)$ and consequently $2(xz+yt) = n$. Let $2(xt - yz) = k$. Then $n^2 + k^2 = 4(x^2+y^2)(z^2+t^2) = 4n^2b_1^2$, so $k^2 = n^2(4b_1^2 - 1)$. Therefore $4b_1^2 - 1$ is a perfect square, which is impossible.

Problem 4. Find the number of all natural numbers $n, 4 \leq n \leq 1023$, such that their binary representations do not contain three consecutive equal digits. *Emil Kolev*

Solution: Denote by $a_n, n \geq 3$, the number of sequences of zeroes and ones of length n which begin with 1 and do not contain three consecutive equal digits. Also, for any $a, b \in \{0, 1\}$ denote by x_{ab}^n the number of sequences of zeroes and ones of length n which begin with 1 and do not contain three consecutive equal digits, such that the last two terms are respectively a and b. It is easy to see that for $n \geq 5$

$$x_{00}^n = x_{10}^{n-1}; x_{01}^n = x_{00}^{n-1} + x_{10}^{n-1}; x_{10}^n = x_{11}^{n-1} + x_{01}^{n-1}; x_{11}^n = x_{01}^{n-1}.$$

Adding up the above equalities, we obtain $a_n = a_{n-1} + x_{10}^{n-1} + x_{01}^{n-1} = a_{n-1} + a_{n-2}$. Since $a_3 = 3$ and $a_4 = 5$, it follows that $a_5 = 8$, $a_6 = 13$, $a_7 = 21$, $a_8 = 34$, $a_9 = 55$ and $a_{10} = 89$.

Since the required number is equal to $a_3+a_4+\ldots+a_{10}$, it follows from the above that the answer is 228.

Problem 5. The vertices A, B and C of an acute triangle ABC lie on the sides B_1C_1, C_1A_1 and A_1B_1 of $\triangle A_1B_1C_1$ and $\angle ABC = \angle A_1B_1C_1$, $\angle BCA = \angle B_1C_1A_1$, $\angle CAB = \angle C_1A_1B_1$. Prove that the orthocentres of $\triangle ABC$ and $\triangle A_1B_1C_1$ are equaly remote from the circumcentre of $\triangle ABC$. *Nikolai Nikolov*

Solution: Denote by H the orthocentre of $\triangle ABC$. Since $\angle CHB = 180^\circ - \angle CAB = 180^\circ - \angle C_1A_1B_1$, we have that A_1 lies on the circumcircle k_1 of $\triangle BHC$. Similarly, B_1 and C_1 lie on circumcircles k_2 and k_3 of $\triangle CHA$ and $\triangle AHB$. Therefore $\angle B_1HC_1 = \angle B_1HA + \angle C_1HA = \angle B_1CA + \angle C_1BA = 2\angle B_1A_1C_1$ and likewise $\angle C_1HA_1 = 2\angle C_1B_1A_1$ and $\angle A_1HB_1 = 2\angle A_1C_1B_1$, so H is the circumcentre of $\triangle A_1B_1C_1$.

Let us draw straight lines passing through the vertices of $\triangle ABC$ and parallel to the corresponding sides and denote their points of intersection by A_0, B_0 and C_0. Since $\angle A_0B_0C_0 = \angle A_1B_1C_1, \angle B_0C_0A_0 = \angle B_1C_1A_1$ and $\angle C_0A_0B_0 = \angle C_1B_1A_1$, it follows from the above that the segments A_0H, B_0H and C_0H are of equal length and are diameters of k_1, k_2 and k_3. It is clear now that there exists a composition of a rotation and a homothecy, both centred at H, such that the image of $\triangle A_1B_1C_1$ is $\triangle A_0B_0C_0$. Therefore the image of orthocentre H_1 of $\triangle A_1B_1C_1$ is the orthocentre H_0 of $\triangle A_0B_0C_0$. Thus $\angle HH_1H_0 = \angle HA_1A_0 = 90^\circ$, and to solve the problem we have to show that the circumcentre O of $\triangle ABC$ is the midpoint of HH_0.

Indeed, the image of $\triangle ABC$ by a homothecy centred at the centroid of $\triangle ABC$ and with coefficient -2 is $\triangle A_0B_0C_0$. Therefore $\overrightarrow{MH_0} = -2\,\overrightarrow{MH}$ and since $\overrightarrow{MH} = -2\,\overrightarrow{MO}$, we obtain $\overrightarrow{OH_0} = -\,\overrightarrow{OH}$.

Problem 6. Prove that the equation

$$x^3 + y^3 + z^3 + t^3 = 1999$$

has infinitely many integer solutions. *Grigor Grigorov*

Solution: Since $10^3 + 10^3 + (-1)^3 + 0^3 = 1999$, we are looking for solutions of the form $x = 10 - k, y = 10 + k, z = -1 - l, t = l$, where k and l are integer. After simple calculations we obtain that our equation is equivalent to $l(l+1) = 20k^2$, whence $(2l+1)^2 - 80k^2 = 1$. The latter is Pell's equality. Since $l = 4, k = 1$ is a solution, all solutions are of the form (l_n, k_n), where $2l_n + 1 + k_n\sqrt{80} = (9 + \sqrt{80})^n, n = 1, 2, \ldots$. Therefore the original equation has infinitely many integer solutions.

Union of Bulgarian Mathematicians

Sava Grozdev Emil Kolev

BULGARIAN

MATHEMATICAL COMPETITIONS

2000

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Problem 8.1 Given the inequality $(n^2-1)x < -3n^3-4n^2+n+2$, where n is an integer.

a) Factorize the expression $-3n^3-4n^2+n+2$.

b) Find all n, for which the inequality holds true for any positive number x.

Solution: a) $-3n^3-4n^2+n+2=(n+1)^2(2-3n)$.

b) Since $0.x<0$ is not true for any x it follows that $n\neq -1$. If $n>-1$ then the inequality is equivalent to $(n-1)x<(n+1)(2-3n)$. If $n=1$ then $0.x<-2$ which is not true for any x. Let $n\neq 1$. If $n-1>0$, then $\dfrac{(n+1)(2-3n)}{n-1}<0$ and the inequality is not true for any positive x. If $n-1<0$, then the inequality becomes $x>\dfrac{(n+1)(2-3n)}{n-1}$ and $n=0$ is the only solution. If $n<-1$ then $\dfrac{(n+1)(2-3n)}{n-1}>0$ and there exists $x>0$, which is not a solution. Therefore the inequality has no solution when $n<-1$.

Problem 8.2 In an isosceles $\triangle ABC (AC=BC)$ the points A_1, B_1 and C_1 are midpoints of BC, AC and AB respectively. Points A_2 and B_2 are symmetric points of A_1 and B_1 with respect to AB. Let M be the intersecting point of CA_2 and A_1C_1, and let N be the intersecting point of CB_2 and B_1C_1. The intersecting point of AN and BM is denoted by P. Prove that $AP=BP$.

Solution: Since $CC_1 \parallel A_1A_2$ and $CC_1 = A_1A_2$, we have that $CC_1A_2A_1$ is a parallelogram. Thus, $A_1M = C_1M$. But $A_1B_1C_1B$ is also a parallelogram and therefore the intersecting point of BM and AC is B_1. Hence P lies on the median BB_1. Analogously P lies on the median AA_1. In the isosceles $\triangle ABC$ the medians AA_1 and BB_1 are of the same length. Therefore $AP = \frac{2}{3}AA_1 = \frac{2}{3}BB_1 = BP$.

Problem 8.3 Find all pairs of prime numbers p and q, such that $p^2 + 3pq + q^2$ is:

a) a perfect square;

b) a power of 5.

Solution: a) Let $p^2 + 3pq + q^2 = r^2$, where p and q are prime numbers. If $p \neq 3, q \neq 3$, then $p^2 + 3pq + q^2 \equiv 2(\text{mod } 3)$ and $r^2 \equiv 2(\text{mod } 3)$, a contradiction. Without loss of generality $p = 3$ and we get that $q^2 + 9q + 9 = r^2$ and $4q^2 + 36q + 36 = (2r)^2$. Therefore $(2q-2r+9)(2q+2r+9) = 45$. We may assume that $r > 0$ and so $2q+2r+9 = 15$ or $2q+2r+9 = 45$. In the first case $q+r = 3$, which is impossible and in the second case solving the system

$$\left| \begin{array}{rcl} q+r & = & 18 \\ 2q-2r+9 & = & 1, \end{array} \right.$$

we find $q = 7$. Because of the symmetry the only solutions are $p = 3, q = 7$ and $p = 7, q = 3$.

b) Let $p^2 + 3pq + q^2 = 5^n$, where n is a natural number. Since $p \geq 2, q \geq 2$, we have $p^2 + 3pq + q^2 \geq 20$ and so $n \geq 2$. It follows now that $25/(p^2 + 3pq + q^2)$ and $5/(p^2 + 3pq + q^2) = (p-q)^2 + 5pq$. Thus, $5/(p-q)^2$ and $25/(p-q)^2$. Therefore $25/5pq$, showing that $p = 5$ or $q = 5$. But if $p = 5$ then $q = 5$ (and vice versa). We obtain $p^2 + 3pq + q^2 = 125 = 5^3$. The only solution of the problem is $p = q = 5$.

Problem 9a.1 Given the equation $\dfrac{1}{|x-2|} = \dfrac{1}{|x-52a|}$, where a is a parameter.

a) Solve the equation.

b) If a is the square of a prime number prove that the equation has a solution which is a compose integer.

Solution: After squaring and simple calculations we obtain $(26a-1)x = (26a-1)(26a+1)$.

a) If $a = \dfrac{1}{26}$ then every $x \neq 2$ is a solution. If $a \neq \dfrac{1}{26}$ then the only solution is $x = 26a + 1$.

b) Let $a = p^2$ where p is a prime number. If $p = 3$ then $x = 235$ is not a prime. If $p \neq 3$ then $p = 3k \pm 1$ and $x = 26(3k \pm 1)^2 + 1 = 3A + 27$, which is divisible by 3.

Problem 9a.2 The quadrilateral $ABCD$ is inscribed in a circle with diameter BD. Let M be the symmetric point of A with respect to BD and let N be the intersecting point of the straight lines AM and BD. The line passing through N, which is parallel to AC, intersects CD and BC in P and Q respectively. Prove that the points P, C, Q and M are vertices of a rectangle.

Solution: It follows from the condition of the problem that M lies on the circumcircle of $ABCD$. Since $\sphericalangle MAC = \sphericalangle MBC = \frac{\overset{\frown}{MC}}{2}$ and $\sphericalangle MNQ = \sphericalangle MAC$ i.e. $\sphericalangle MNQ = \sphericalangle MBC$, we get that the points M, N, B and Q lie on a circle. Since $\sphericalangle MNB = 90^\circ$ we conclude that $\sphericalangle BQM = 90^\circ$. Also, since $\triangle BDC$ is a right angle triangle we have that $MQ \parallel PC$. From the other hand $\sphericalangle MDC = \sphericalangle MAC = \frac{\overset{\frown}{MC}}{2}$, and therefore $\sphericalangle MDC = \sphericalangle MNQ$, so the points N, P, M and D lie on a circle. Thus, $\sphericalangle MPD = \sphericalangle MND = 90^\circ$ and $MP \parallel CQ$. Therefore P, C, Q and M are vertices of a rectangle.

Problem 9a.3 See Problem 8.3.

Problem 9b.1 Given the system:

$$\left|\begin{array}{rcl} \dfrac{1}{x+y}+x & = & a-1 \\[2ex] \dfrac{x}{x+y} & = & a-2, \end{array}\right.$$

where a is a real parameter.

a) Solve the system if $a=0$.

b) Find all values of a, such that the system has an unique solution.

c) If $a \in (2,3)$ and (x,y) is a solution of the system find all values of a such that the expression $\dfrac{x}{y}+\dfrac{y}{x}$ takes its minimal value.

Solution: It follows easily that x and $\dfrac{1}{x+y}$ are roots of the equation $t^2-(a-1)t+a-2=0$. There are two cases to be considered:

$$(1) \quad \left|\begin{array}{rcl} x & = & 1 \\[2ex] \dfrac{1}{x+y} & = & a-2 \end{array}\right. \Longleftrightarrow \left|\begin{array}{rcl} x & = & 1 \\[2ex] y & = & \dfrac{3-a}{a-2}, \end{array}\right. \quad \text{when} \quad a \neq 2$$

and

$$(2) \quad \left|\begin{array}{rcl} x & = & a-2 \\[2ex] \dfrac{1}{x+y} & = & 1 \end{array}\right. \Longleftrightarrow \left|\begin{array}{rcl} x & = & a-2 \\[2ex] y & = & 3-a. \end{array}\right.$$

a) When $a=0$ we have $(x,y)=(1;-\dfrac{3}{2})$ or $(x,y)=(-2;3)$.

b) The sytem has an unique solution $(0,1)$ when $a=2$ and $(1,0)$ when $a=3$.

c) If (x,y) is a solution and $a\in(2;3)$ then $\frac{x}{y}$ and $\frac{y}{x}$ are positive. Further $\frac{x}{y}+\frac{y}{x}\geq 2$, and equality occures when $\frac{x}{y}=\frac{y}{x}$. It follows from (1) and (2) that $\frac{x}{y}=\frac{a-2}{3-a}$. Using the equality $\frac{a-2}{3-a}=\frac{3-a}{a-2}$ we find the only value of $a=\frac{5}{2}$.

Problem 9b.2 Given an acute $\triangle ABC$. The bisector of $\sphericalangle ACB$ intersects AB at point L. The feet of the perpendiculars from L to AC and BC are denoted by M and N respectively. Let P be the intersecting point of AN and BM. Prove that $CP\perp AB$.

Solution: Let l be the line through C which is parallel to AB. Let F and E be respectively the intersecting points of AN and BM with l. The intersecting point of CP and AB is denoted by D. We obtain $\frac{AD}{CF}=\frac{PD}{PC}=\frac{BD}{CE}$, so $\frac{AD}{BD}=\frac{CF}{CE}$. From the other hand $\frac{AM}{CM}=\frac{AB}{CE}$ and $\frac{BN}{CN}=\frac{AB}{CF}$. But $CM=CN$ and we get $\frac{AM}{BN}=\frac{CF}{CE}$. Therefore $\frac{AD}{BD}=\frac{AM}{BN}$, which implies

$$\frac{AM}{AD}=\frac{BN}{BD} \tag{1}$$

Further if $CH\perp AB(H\in AB)$ then $\triangle ALM\sim\triangle AHC$ and so $\frac{AL}{AC}=\frac{AM}{AH}$. In the same manner $\frac{BL}{BC}=\frac{BN}{BH}$. But CL is a bisector and therefore $\frac{AL}{AC}=\frac{BL}{BC}$, so $\frac{AM}{AH}=\frac{BN}{BH}$. The last equation combined with (1) gives $D\equiv H$ which implies $CP\perp AB$.

Problem 9b.3 Prove that the digit of the hundreds of $2^{1999}+2^{2000}+$

2^{2001} is even.

Solution: Write the number $2^{1999}+2^{2000}+2^{2001}$ in the form $2^{1999}(1+2+4) = 7.2^9.2^{1990} = 7.2^9.2^{10}.2^{1980} = 7.2^9.2^{10}.(2^{20})^{99}$. Since $2^9 = 512, 2^{10} = 1024$ and $2^{20} = (2^{10})^2$ we have that the last two digits of 2^{20} coincide with the last two digits of 24^2, so the last two digits of 2^{20} are 76. Moreover the last two digits of 76.76 are also 76. Therefore the last two digits of the given number are the last two digits of the product 7.12.24.76, which are 1 and 6. Since $2^{1999} + 2^{2000} + 2^{2001}$ is divisible by 8 and it ends by 16, the digit of the hundreds is even.

Problem 10.1 Find all values of the real parameter a such that the nonnegative solutions of the equation $(2a-1)\sin x+(2-a)\sin 2x = \sin 3x$ form an infinite arithmetic progression.

Solution: Since $\sin 2x = 2\sin x\cos x$ and $\sin 3x = \sin x(4\cos^2 x-1)$, we may write the equation in the form
$\sin x\,(2\cos^2 x-(2-a)\cos x-a) = 0$. Thus $\sin x = 0, \cos x = 1$ or $\cos x = -\dfrac{a}{2}$. The nonnegative solutions of the equations $\sin x = 0$ and $\cos x = 1$ are $x = k\pi$ and $x = 2k\pi, k = 0, 1, 2, \ldots$ respectively. Let $|a| > 2$. In this case the equation $\cos x = -\dfrac{a}{2}$ has no solution and therefore the nonnegative solutions of the initial equation are $0, \pi, 2\pi, \ldots$, which form an arithmetic progression. Let now $|a| \le 2$ and let x_0 be the only solution of the equation $\cos x = -\dfrac{a}{2}$ in the interval $[0, \pi]$. In this case the nonnegative solutions of the last equation are $x = x_0 + 2k\pi$ and $x = 2\pi - x_0 + 2k\pi$. It is clear now that the nonnegative solutions form an arithmetic progression only when $x_0 = 0, x_0 = \dfrac{\pi}{2}$ and $x_0 = \pi$, so giving $a = -2, a = 0$ and $a = 2$. The values of a are $a = 0$ and $|a| > 2$.

Problem 10.2 Let O, I and H be respectively the circumcenter, incenter and orthocenter for an acute nonequilateral $\triangle ABC$. Prove that if the circumcircle of $\triangle OIH$ passes through one of the vertices of $\triangle ABC$ then it passes through another vertex of $\triangle ABC$.

Solution: Assume that O, I, H and C lie on a circle. It is well known that CI is the bisector of $\sphericalangle HCO$. Thus $\sphericalangle IHO = \sphericalangle ICO = \sphericalangle ICH = \sphericalangle HOI$ and it follows from $\triangle IHO$ that $IH = IO = t$.

We shall prove that if $\sphericalangle BAC \neq 60°$ then O, I, H and A lie on a circle. Denote by M and N the projection points of I respectively on AO and AH. Let O_1 and O_2 be such that $IO_1 = IO_2 = t$ and O_1 lies between A and M, and M lies between O_1 and O_2. Analogously let H_1 and H_2 be such that $IH_1 = IH_2 = t$ and H_1 lies between A and N, and N lies between H_1 and H_2. If $O \equiv O_1; H \equiv H_1$ or $O \equiv O_2; H \equiv H_2$, then $\triangle AIO \sim \triangle AIH$ and therefore $AO = AH$. But $AH = 2AO \cos \sphericalangle BAC$ and so $\sphericalangle BAC = 60°$. If $O \equiv O_1; H \equiv H_2$ or $O \equiv O_2; H \equiv H_1$, then it follows from $\sphericalangle IO_1O_2 = \sphericalangle IO_2O_1 = \sphericalangle IH_1H_2 = \sphericalangle IH_2H_1$ that $AOIH$ is inscribed.

Suppose now that A and B do not lie on the circumcircle of $\triangle OIH$. In this case $\sphericalangle BAC = \sphericalangle ABC = 60°$ and therefore $\triangle ABC$ is equilateral which is a contradiction.

Problem 10.3 In each of the cells of a 3×3 table is written a real number. The element in the $i-$th row and $j-$th column equals to the modulus of the difference of the sum of the elements from the $i-$th row and the sum of the elements from the $j-$th column. Prove that every element of the table equals either to the sum or to the difference of two other elements of the table.

Solution: Let p_1, p_2, p_3 and q_1, q_2, q_3 be the sum of the elements in the first, second and third row and in the first, second and third column respectively. It is clear that $p_1 + p_2 + p_3 = q_1 + q_2 + q_3$. Therefore the element in the first row and the first column equals to $|p_1 - q_1|$. From the other hand $|p_1 - q_1| = |p_2 + p_3 - q_2 - q_3|$ which implies $|p_1 - q_1| = \epsilon_1 |p_2 - q_2| + \epsilon_2 |p_3 - q_3|$ where $\epsilon_1, \epsilon_2 \in +1, -1$. Since

$|p_1 - q_1| \geq 0$ it is clear that $\epsilon_1 = \epsilon_2 = -1$ is impossible. Therefore $|p_1 - q_1|$ is either the sum or the difference of $|p_2 - q_2|$ and $|p_3 - q_3|$. By analogy every element of the table is the sum or the difference of two other elements.

Problem 11.1. Prove that for every positive number a the sequence $\{x_n\}_{n=1}^{\infty}$, such that $x_1 = 1, x_2 = a, x_{n+2} = \sqrt[3]{x_{n+1}^2 x_n}, n \geq 1$, is convergent and find its limit.

Solution: It follows by induction that the terms of the sequence $\{x_n\}_{n=1}^{\infty}$ can be expressed as $x_n = a^{\alpha_n}$, where $\{\alpha_n\}_{n=1}^{\infty}$, is a siquence defined by $\alpha_1 = 0, \alpha_2 = 1, \alpha_{n+2} = \dfrac{2\alpha_{n+1} + \alpha_n}{3}, n \geq 1$. Thus, $\alpha_{n+2} - \alpha_{n+1} = -\dfrac{1}{3}(\alpha_{n+1} - \alpha_n)$ and therefore $\alpha_{n+2} - \alpha_{n+1} = \left(-\dfrac{1}{3}\right)^n (\alpha_2 - \alpha_1) = \left(-\dfrac{1}{3}\right)^n$. Adding the equalities $\alpha_{k+2} - \alpha_{k+1} = \left(-\dfrac{1}{3}\right)^k$ for $k = 0, 1, \ldots, n$ after simple calculations we obtain $\alpha_{n+2} - \alpha_0 = \left(-\dfrac{1}{3}\right)^0 + \left(-\dfrac{1}{3}\right)^1 + \ldots + \left(-\dfrac{1}{3}\right)^n = \dfrac{1 - \left(-\frac{1}{3}\right)^{n+1}}{1 + \left(\frac{1}{3}\right)} = \dfrac{3}{4}(1 - \left(-\dfrac{1}{3}\right)^{n+1})$. Since $\alpha_0 = 0$ and $\lim\limits_{n\to\infty} \left(-\dfrac{1}{3}\right)^n = 0$ it follows that $\lim\limits_{n\to\infty} \alpha_n = \dfrac{3}{4}$. This shows that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} x_n = a^{\frac{3}{4}}$.

Problem 11.2 Given a convex quadrilateral $ABCD$ where M is the intersecting point of its diagonals. It is known that $DB = 3DM, AM = MC$.

a) Express BC and CD by the sides of $\triangle ABD$.

b) Prove that if $2 \sphericalangle ADB - \sphericalangle ABD = 180^\circ$, then $\sphericalangle DBC = 2 \sphericalangle BDC$.

Solution: a) Denote $AB = c, BC = p, CD = q, DA = b$ and

$DB = a$. It follows from the condition of the problem that $DM = \frac{1}{3}a$, $MB = \frac{2}{3}a$. The formula for the median in a triangle applied for $\triangle ABC$ and $\triangle ACD$ gives

$$(1) \qquad \frac{16}{9}a^2 = 2p^2 + 2c^2 - 4AM^2 \frac{4}{9}a^2 = 2q^2 + 2b^2 - 4AM^2$$

Let $\sphericalangle AMB = \alpha$. From the Law of Cosines for $\triangle AMB$ and $\triangle AMD$ we obtain

$$c^2 = AM^2 + \frac{4a^2}{9} - \frac{4}{3}AM.a\cos\alpha b^2 = AM^2 + \frac{a^2}{9} + \frac{2}{3}AM.a\cos\alpha$$

Thus $2b^2 + c^2 = 3AM^2 + \frac{2}{3}a^2$ and a substitution in (1) gives

$$(2) \qquad p^2 = \frac{4a^2 + 12b^2 - 3c^2}{9} q^2 = \frac{3b^2 + 6c^2 - 2a^2}{9}$$

b) It follows from the condition of the problem that $\sphericalangle ADB > 90^\circ$ and so $c > a$. Let D_1 be a point on AB, such that $D_1B = DB$. Further $\sphericalangle AD_1B = \sphericalangle ABD + \frac{180^\circ - \sphericalangle ABD}{2} = \frac{180^\circ + \sphericalangle ABD}{2} = \sphericalangle ADB$. Therefore $\triangle AD_1D \sim \triangle ADB$ and thus $b^2 = c(c-a)$. A substitution in (2) implies $p = \frac{3c-2a}{3}$ and $q^2 = \frac{9c^2 - 3ac - 2a^2}{9}$. Hence $q^2 = p(p-a)$. Let B_1 be a point on BC (B lies between C and B_1), such that $BB_1 = BD = a$. Since $\frac{q}{p} = \frac{p+a}{q}$ it follows that $\triangle DBC \sim \triangle B_1DC$. Therefore $\sphericalangle DBC = \sphericalangle BB_1D + \sphericalangle BDB_1 = 2\sphericalangle BB_1D = 2\sphericalangle BDC$.

Problem 11.3 See problem 10.3.

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Problem 8.1. Let $f(x)$ be a linear function such that $f(0) = -5$ and $f(f(0)) = -15$. Find all values of m, for which the set of the solutions of the inequality $f(x).f(m-x) > 0$ is an interval of length 2.

Solution: Let $f(x) = ax+b$. It follows from $f(0) = -5$ that $b = -5$, and from $f(f(0)) = -15$ that $a = 2$. Therefore the function is $f(x) = 2x-5$. Consider the inequality $(2x-5)(2(m-x)-5) > 0 \iff (2x-5)(2m-5-2x) > 0$. The solution of the last inequality is an interval with end points $\frac{5}{2}$ and $\frac{2m-5}{2}$. Therefore $\left|\frac{5}{2} - \frac{2m-5}{2}\right| = 2$, so $|5-m| = 2$. Finally, we obtain that $m = 3$ and $m = 7$.

Problem 8.2 Given an isosceles right angle triangle ABC with $\sphericalangle ACB = 90^\circ$. Point P lies on BC, M is the midpoint of AB and let L and N be points from the segment AP such that $CN \perp AP$ and $AL = CN$.

a) Find $\sphericalangle LMN$;

b) If the area of $\triangle ABC$ is 4 times greater than the area of $\triangle LMN$ find $\sphericalangle CAP$.

Solution: a) Let $\sphericalangle CAP = \alpha$. We find $\sphericalangle ACN = 90^\circ - \alpha$ and $\sphericalangle MCN = \sphericalangle ACN - 45^\circ = 45^\circ - \alpha = \sphericalangle LAM$. Since $AM = CM$ and $AL = CN$ it follows that $\triangle AML \cong \triangle CMN$. Therefore $\sphericalangle AML = \sphericalangle CMN$ and so $\sphericalangle LMN = 90^\circ - \sphericalangle AML + \sphericalangle CMN = 90^\circ$.

b) It follows from $\triangle AML \cong \triangle CMN$ that $LM = MN$ and since

$\sphericalangle LMN = 90^\circ$ we get $S_{LMN} = \dfrac{MN^2}{2}$. Also $S_{ABC} = \dfrac{AC^2}{2}$. Now $\dfrac{4MN^2}{2} = \dfrac{AC^2}{2}$ applies that $MN = \dfrac{AC}{2}$. Denote by Q the midpoint of AC. Thus $QM = QN = \dfrac{AC}{2} = MN$. Therefore $\triangle QMN$ is equilateral and $\sphericalangle QNM = 60^\circ$, which implies $\sphericalangle QNA = 60^\circ - 45^\circ = 15^\circ$. But $AQ = QN$ and so $\sphericalangle CAP = \sphericalangle QNA = 15^\circ$.

Problem 8.3 There are 2000 white balls in a box. There are also sufficiently many white, green and red balls. The following operations are allowed:

1) Replacement of two white balls with a green ball;

2) Replacement of two red balls with a green ball;

3) Replacement of two green balls with a white bal and a red ball;

4) Replacement of a white ball and a green ball with a red ball;

5) Replacement of a green ball and a red ball with a white ball;

a) After finitely many of the above opperations there are three balls left in the box. Prove that at least one of them is a green ball.

b) Is it possible after finitely many opperations to have only one ball left in the box?

Solution: Consider a box with x white, y green and z red balls. Direct verification shows that after applying any of the allowed operations the sum $x + 2y + 3z$ does not change modulo 4. Since the initial values are $x = 2000, y = z = 0$, we obtain that this sum is congruent to 0 modulo 4.

a) There are 3 balls in the box and therefore $x + y + z = 3$. Moreover $x+2y+3z \equiv 0(\mathrm{mod} 4)$. If a green ball is not in the box then $y = 0$ and so $x+z = 3$ and $x+3z \equiv x+3(3-x) \equiv 1-2x \equiv 0(\mathrm{mod} 4)$, which is impossible.

b) Suppose that there is only one ball left in the box. Therefore $x + y + z = 1$ and $x + 2y + 3z \equiv 0 (\mathrm{mod} 4)$, which is impossible.

Problem 9a.1 Find all values of m such that the equation

$$\left(\frac{1}{x+m} + \frac{m}{x-m} - \frac{2m}{m^2 - x^2}\right)(|x-m| - m) = 0$$

has exactly one nonnegative root.

Solution: The equation $\frac{1}{x+m} + \frac{m}{x-m} - \frac{2m}{m^2 - x^2} = 0$ is equivalent to (when $x \neq \pm m$) to $(m+1)x = -m(m+1)$. When $m = -1$ it has infinitely many roots – all numbers $x \neq \pm 1$. If $m \neq -1$ we obtain $x = -m$. The equation $|x - m| - m = 0$ has two roots: $x = 0$ and $x = 2m$ when $m > 0$; an unique root: $x = 0$ when $m = 0$ and has no roots when $m < 0$.

Let $m < 0$ and $m \neq -1$. In this case the equation has an unique root $x = -m$ and it is nonnegative. Let $m = 0$. Then the equation becomes $\frac{1}{x}|x| = 0$, which obviously has no roots. Let $m > 0$. The equation has three roots $x = -m, x = 0, x = 2m$ and two of them are nonnegative.

Thus, the desired values of m are $m < 0$ and $m \neq -1$.

Problem 9a.2 Given an acute-angled triangle ABC and let α, β and γ be its angles respectively to A, B and C. For an arbitrary interior point M denote by A_1, B_1 and C_1 respectively the feet of the perpendiculars from M to BC, CA and AB. Find the locus of M for which the triangle $A_1B_1C_1$ is a right angle triangle.

Solution: Let M be a point such that $\sphericalangle A_1C_1B_1 = 90^\circ$. Since the quadrilateral AC_1MB_1 is inscribed we have $\sphericalangle B_1C_1M = \sphericalangle B_1AM = x$. Analogously $\sphericalangle A_1C_1M = \sphericalangle A_1BM = y$. Denote $z = \sphericalangle AMB$. It follows now from the quadrilateral $AMBC$ that $x + \gamma + y + (360^\circ -$

$z) = 360^\circ$. Since $x + y = \sphericalangle A_1C_1B_1 = 90^\circ$ we get $z = 90^\circ + \gamma$. Therefore $\sphericalangle AMB = 90^\circ + \gamma$. Thus, the locus of M such that $\sphericalangle A_1C_1B_1 = 90^\circ$ is an arc G_1 in the interior of $\triangle ABC$ for which the segment AB is seen under angle $90^\circ + \gamma$. Analogously one can prove that the locus of M such that $\sphericalangle C_1B_1A_1 = 90^\circ$ is an arc G_2 in the interior of $\triangle ABC$, for which AC is seen under angle $90^\circ + \beta$. Further, the locus of M such that $\sphericalangle B_1A_1C_1 = 90^\circ$ is an arc G_3 in the interior of $\triangle ABC$ for which BC is seen under angle $90^\circ + \alpha$. The desired locus is the union of the three arcs $G_1 \cup G_2 \cup G_3$.

Problem 9a.3 See Problem 8.3

Problem 9b.1 The real numbers x and y are such that $x^2+xy+y^2 = 1$. If $F = x^3y + xy^3$,

a) prove that $F \geq -2$;

b) find the greatest possible value of F.

Solution: a) It follows from the condition of the problem that $(x+y)^2 = 1 + xy$ and therefore $xy \geq -1$. From the other hand $x^2 + y^2 = 1 - xy$. Thus, $xy \leq 1$, so $z = xy \in [-1; 1]$. Hence $F = xy(x^2 + y^2) = z(1 - z)$ and the inequality $F \geq -2$ is equivalent to the inequality $z^2 - z - 2 \leq 0$. The latter one is true for any $z \in [-1; 2]$.

b) Assume that the greatest value of F exists and denote it by A. This implies that the system

$$(1) \qquad \left| \begin{array}{rcl} xy(x^2 + y^2) & = & A \\ xy + (x^2 + y^2) & = & 1 \end{array} \right.$$

has a solution. It is clear that $t_1 = xy$ and $t_2 = x^2 + y^2$ are roots of the quadratic equation $t^2 - t + A = 0$. But $t_2 - t_1 = \frac{1}{2}(x - y)^2 + \frac{1}{2}(x + y)^2 \geq 0$ and the equality is impossible since if so then $x = y = 0$ which is a contradiction to $x^2 + xy + y^2 = 1$. Therefore $t_2 > t_1$ and so $D = 1 - 4A > 0$, i.e. $A < \frac{1}{4}$. The system

(1) is equivalent to $\left|\begin{array}{rcl} xy & = & t_1 \\ x^2+y^2 & = & t_2, \end{array}\right.$ and the last is equivalent to $\left|\begin{array}{rcl} (x-y)^2 & = & t_2-2t_1 \\ (x+y)^2 & = & t_2+2t_1. \end{array}\right.$ The latter system has a solution iff the inequalities $t_2-2t_1 \geq 0$ and $t_2+2t_1 \geq 0$ hold true. It suffices to prove that when $A < \dfrac{1}{4}$ the roots $t_1 < t_2$ of the equation $g(t) = t^2 - t + A$ satisfy the inequalities $t_2 - 2t_1 \geq 0$ and $t_2 + 2t_1 \geq 0$. If $A = 0$ then $t_1 = 0$ and $t_2 = 1$ and the inequalities hold true. It is clear that we can consider only the case $A \in (0; \dfrac{1}{4})$. Now $t_1 > 0, t_2 > 0$ and the inequality $t_2 + 2t_1 \geq 0$ holds true. From the other hahd $t_1 + t_2 = 1$ and the inequality $t_2 - 2t_1 \geq 0$ is equivalent to $t_2 \geq \dfrac{2}{3}$. The same inequality is equivalent to $t_1 \leq \dfrac{1}{3}$. For the roots t_1 and t_2 we obtain $t_1 \leq \dfrac{1}{3}$ $t_2 \geq \dfrac{2}{3}$, which is equivalent to $\left|\begin{array}{rcl} g(\frac{1}{3}) & \leq & 0 \\ g(\frac{2}{3}) & \leq & 0. \end{array}\right.$ Therefore $A \leq \dfrac{2}{9}$. Conversely, if $A = \dfrac{2}{9}$ then $x = y = \pm\dfrac{\sqrt{3}}{3}$ satisfy the condition of the problem.

Problem 9b.2 A line l is drown through the orthocenter of an acute-angled triangle ABC. Prove that the lines symmetric to l with respect to the sides of the triangle intersect in a point.

Solution: Let the intersecting points of l with the sides AC and BC be Q and P respectively and let the intersecting point of l with the extention of AB be R. Without loss of generality A lies between R and B. Denote the symmetric points of H with respect to AC, BC and AB by B_1, A_1 and C_1 respectively. It is well known that A_1, B_1 and C_1 lie on the circumcircle k of $\triangle ABC$. It is clear also that the symmetric lines of l are B_1Q, A_1P and C_1R. Let $B_1Q \cap A_1P = S$ and $B_1Q \cap C_1R = T$. We obtain $\sphericalangle CB_1Q + \sphericalangle CA_1P = \sphericalangle CHQ + \sphericalangle CHP = 180^\circ$ and therefore SA_1CB_1 is inscribed. Thus, $S = k \cap B_1Q$. From the other hand $\sphericalangle RC_1A = \sphericalangle RHA = \sphericalangle AB_1T$ and therefore B_1ATC_1 is inscribed. Hence, $T = k \cap B_1Q$. This implies that $T \equiv S$ which shows that the three lines intersect in a point.

Problem 9b.3 See Problem 8.3.

Problem 10.1 Solve the equation $\sqrt{x} + \sqrt[3]{x+7} = \sqrt[4]{x+80}$.

Solution: It is clear that $x \geq 0$. Note that $x = 1$ is a root of the equation. We shall prove that there are no other roots. Rasing the equation to the forth power we get $x^2 + 4(\sqrt{x})^3\sqrt[3]{x+7} + 6(\sqrt{x})^2(\sqrt[3]{x+7})^2 + 4(\sqrt{x})(\sqrt[3]{x+7})^3 + (\sqrt[3]{x+7})^4$
$= x + 80$.

Let $f(x) = 4(\sqrt{x})^3\sqrt[3]{x+7} + 6(\sqrt{x})^2(\sqrt[3]{x+7})^2$
$+4(\sqrt{x})(\sqrt[3]{x+7})^3 + (\sqrt[3]{x+7})^4$. Obviously $f(x)$ is an increasing function. If $x > 1$ then it follows from the inequalities $x^2 > x, f(x) > f(1) = 80$ that $x^2 + f(x) > x + 80$, a contradiction. If $x < 1$ then it follows from the inequalities $x^2 < x, f(x) < f(1) = 80$ that $x^2 + f(x) < x + 80$, a contradiction. This completes the prove.

Problem 10.2 The incircle of an isosceles $\triangle ABC$ touches the legs AC and BC at points M and N respectively. A tangent t is drawn to the smaller of the arcs $\overset{\frown}{MN}$ and let t intersects NC and MC at points P and Q respectively. Let T be the intersecting point of the lines AP and BQ.

a) Prove that T lies on the segment MN;

b) Prove that the sum of the areas of triangles ATQ and BTP is the smallest possible when t is parallel to AB.

Solution: a) Let the incircle touches AB and PQ at points R and S respectively. Let MN and SR intersect QB in points T_1 and T_2 respectively. Since $\sphericalangle T_1MQ = \sphericalangle T_1NC = 180° - \sphericalangle T_1NB$ and $\sphericalangle MT_1D = \sphericalangle BT_1N$ it follows from the Law of Sines for $\triangle MT_1D$ and

$\triangle BT_1N$ that $\dfrac{DT_1}{MD} = \dfrac{BT_1}{BN}$, so $\dfrac{QT_1}{BT_1} = \dfrac{MQ}{BN}$. By analogy $\dfrac{QT_2}{BT_2} = \dfrac{SQ}{BR}$. It follows now from $MQ = SQ$ and $BN = BR$ that $\dfrac{QT_1}{BT_1} = \dfrac{QT_2}{BT_2}$, so $T_1 = T_2$. In the same manner one can prove that AP passes through the intersecting point of MN and SR. This implies that AP, BQ, MN and SR intersect in T.

b) We have $S_{ATQ} + S_{BPT} = S_{ABQ} + S_{ABP} - 2S_{ABT}$. Since $\triangle ABC$ is isosceles we get $MN \parallel AB$ and therefore S_{ABT} is constant. Thus, $S_{ATQ} + S_{BPT}$ is minimal exactly when $S_{ABQ} + S_{ABP}$ is minimal. The latter sum equals to $\dfrac{AQ.AB\sin\alpha + BP.AB\sin\alpha}{2} =$ $\frac{(AQ+BP)AB\sin\alpha}{2}$, where α is the angle to the base of the triangle. Therefore it suffices to find the minimum of $AQ + BP$. It is easily seen that $AQ + BP = AM + BN + PQ$ and therefore we have to find when PQ is minimal. Let r be the inradius and O be the center of the incircle of $\triangle ABC$. Using that $PQ = r(\text{cotg}\phi + \text{cotg}\psi)$ where $\phi = \sphericalangle OQP, \psi = \sphericalangle OPQ$ we obtain from $ABPQ$ that $2\phi + 2\psi + 2\alpha = 360^\circ$. Therefore $\phi + \psi = 180^\circ - \alpha$. Thus $PQ = \dfrac{2r\sin\alpha}{\cos(\phi - \psi) + \cos\alpha}$. It is clear now that PQ is minimal exactly when $\cos(\phi - \psi) = 1$, so $\phi = \psi \Longrightarrow PQ \parallel AB$.

Problem 10.3 There are $n \geq 4$ points in the plane such that the distance between any two of them is an integer. Prove that at least $\dfrac{1}{6}$ from the distances between them are divisible by 3.

Solution: We show first that the assertion from the problem is true for $n = 4$ i.e. for 4 points with integer distances between them at least one distance is divisible by 3. Denote the points by A, B, C and D (it is easy to be seen that WLOG $\sphericalangle BAD = \sphericalangle BAC + \sphericalangle CAD$). By the Law of Cosines for $\triangle ABC, \triangle ACD, \triangle ABD$ we obtain

$$BC^2 = AB^2 + AC^2 - 2.AB.AC\cos\alpha$$

$$CD^2 = AD^2 + AC^2 - 2.AD.AC\cos\beta$$
$$BD^2 = AB^2 + AD^2 - 2.AB.AD\cos\gamma$$
where $\alpha = \measuredangle BAC, \beta = \measuredangle CAD, \gamma = \measuredangle BAD = \alpha + \beta$.

Suppose that all distances are integers not divisible by 3. Therefore $AB^2 \equiv AC^2 \equiv AD^2 \equiv BC^2 \equiv CD^2 \equiv BD^2 \equiv 1 \pmod 3$ and so $2.AB.AC\cos\alpha \equiv 2.AD.AC\cos\beta \equiv 2.AB.AD\cos\gamma \equiv 1 \pmod 3$. Thus, $2.AB.AC\cos\alpha.2.AD.AC\cos\beta \equiv 4.AC^2.AB.AD.\cos\alpha\cos\beta$ $\equiv AC^2.AB.AD.\cos\alpha\cos\beta \equiv 1 \pmod 3$.

Note that $\cos\alpha, \cos\beta$ and $\cos\gamma$ are rational numbers. Moreover, if $\cos\alpha = \frac{p}{q}, \cos\beta = \frac{r}{s}$, where p, q and r, s are relatively prime then p, q, r and s are not divisible by 3. Hence $p^2 \equiv q^2 \equiv r^2 \equiv s^2 \equiv 1 \pmod 3$ and $\cos\gamma = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos\alpha\cos\beta - \frac{\sqrt{q^2-p^2}}{q}\frac{\sqrt{s^2-r^2}}{s}$. Therefore $2.AC^2.AB.AD.\sin\alpha\sin\beta$ is divisible by 3 and after multiplying $2.AB.AD\cos\gamma \equiv 1 \pmod 3$ by AC^2 we obtain $2.AC^2.AB.AD.\cos\alpha\cos\beta \equiv 1 \pmod 3$, a contradiction to $AC^2.AB.AD.\cos\alpha\cos\beta \equiv 1 \pmod 3$. Therefore at least one of the distances is divisible by 3.

Let $n \geq 4$. Since there exist $\binom{n}{4}$ sets with four elements each there exist at least $\binom{n}{4}$ distances (counted more than once) divisible by 3. Each such distance is counted exactly $\binom{n-2}{2}$ times and we get that the desired number is at least $\frac{\binom{n}{4}}{\binom{n-2}{2}} = \frac{1}{6}\binom{n}{2}$.

Problem 11.1 Let $f(x) = \dfrac{x^2+4x+3}{x^2+7x+14}$.

a) Find the greatest value of $f(x)$;

b) Find the greatest value of the function $\left(\dfrac{x^2-5x+10}{x^2+5x+20}\right)^{f(x)}$.

Solution: a) We shall prove that the greatest value of $f(x)$ equals 2. Since $x^2+7x+14>0, \forall x$, we obtain $f(x) \le 2 \Longrightarrow (x+5)^2 \ge 0$, and an equality occurs only when $x=-5$.

b) Let $g(x) = \dfrac{x^2-5x+10}{x^2+5x+20}$. Since $x^2+5x+20>0, \forall x$, we obtain $g(x) \le 3 \Longrightarrow (x+5)^2 \ge 0$, and equality occurs only when $x=-5$. Since $x^2-5x+10>0, \forall x$, we have that the function $h(x)=g(x)^{f(x)}$ is correctly defined. Further, when $f(x)>0$, i.e. $x \in [-3,-1]$, then $h(x) \le 3^2 = 9$. From the other hand $g(x) \ge 1 \Longrightarrow x \le -1$ and so $h(x) \le 1$ when $f(x) \le 0$. Therefore the greatest value of $h(x)$ equals 9 and $h(x)=9$ when $x=-5$.

Problem 11.2 A point A_1 is chosen on the side BC of a triangle ABC such that the inradii of $\triangle ABA_1$ and $\triangle ACA_1$ are equal. Denote the diameters of the incircles of $\triangle ABA_1$ and $\triangle ACA_1$ by d_a. In the same manner define d_b d_c. If $BC=a, CA=b, AB=c, p=\dfrac{a+b+c}{2}$ and h_a, h_b, h_c are the altitudes of the triangle ABC and d is the diameter of the incircle of $\triangle ABC$ prove that:

a) $d_a + \dfrac{\sqrt{p(p-a)}}{a} d = h_a$;

b) $d_a + d_b + d_c + p \ge h_a + h_b + h_c$.

Solution: a) Let O_1 and O_2 be the incenters of $\triangle ABA_1$ and $\triangle ACA_1$ and p_1 and p_2 be semiperimeters of the same triangles. We have $S_{ABC} = S_{ABA_1} + S_{ACA_1} = r_a.p_1 + r_a.p_2 = r_c(p_1+p_2) = r_c.(p+CC_1)$, where $r_a = \dfrac{d_a}{2}$. Therefore $r_a(p+AA_1) = S_{ABC}$. If P and Q are the touching points of the incircles of $\triangle ABA_1$ and $\triangle ACA_1$ with

BC then the quadrilateral O_1PQO_2 is rectangle and so $O_1O_2 = PQ = PA_1 + QA_1 = p_1 - AB + p_2 - AC = p_1 + p_2 - AC - BC = p + AA_1 - AC - BC$. It follows from the similarity of $\triangle IO_1O_2$ and $\triangle IBC$, where I is the incenter of $\triangle ABC$ that $\frac{O_1O_2}{BC} = \frac{r - r_a}{r} \Longrightarrow \frac{AA_1 + p - AC - BC}{BC} = \frac{r - r_c}{r}$. We obtain the system

$$\left|\begin{array}{ccc} r_a(p + AA_1) & = & S \\ \dfrac{AA_1 + AB - p}{c} & = & \dfrac{r - r_a}{r} \end{array}\right.$$

Thus, $AA_1 = \sqrt{p(p-a)}$ and $r_a = \frac{rp}{p + \sqrt{p(p-a)}} = \frac{r\sqrt{p}(\sqrt{p} - \sqrt{p-a})}{a} = \frac{h_a}{2} - \frac{r\sqrt{p(p-a)}}{a}$, which implies a).

b) By a) the inequality is equivalent to $\frac{\sqrt{p-a}}{a} + \frac{\sqrt{p-b}}{b} + \frac{\sqrt{p-c}}{c} \leq \frac{\sqrt{p}}{d}$. Further $\frac{\sqrt{p-a}}{a} + \frac{\sqrt{p-b}}{b} + \frac{\sqrt{p-c}}{c} = \frac{\sqrt{p-a}}{(p-b)+(p-c)} + \frac{\sqrt{p-b}}{(p-a)+(p-c)} + \frac{\sqrt{p-c}}{(p-a)+(p-b)} \leq \frac{\sqrt{p-a}}{2\sqrt{p-b}\sqrt{p-c}} + \frac{\sqrt{p-b}}{2\sqrt{p-a}\sqrt{p-c}} + \frac{\sqrt{p-c}}{2\sqrt{p-a}\sqrt{p-b}} = \frac{p}{\sqrt{(p-a)(p-b)(p-c)}} = \frac{\sqrt{p}}{d}$.

Problem 11.3 See problem 10.3.

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Problem 1. Find all value of the real parameter a such that the equation
$$9^t - 4a3^t + 4 - a^2 = 0$$
has an unique root in the interval $(0, 1)$.

Solution: After the substitutions $x = 3^t$ and $f(x) = x^2 - 4ax + 4 - a^2$ the problem is equivalent to: find all values of a such that the equation $f(x) = 0$ has an unique root in the interval $(1, 3)$. It is easily seen that if $f(1) = 0$ or $f(3) = 0$, then $a = 1, a = -5$ or $a = -13$ are not solutions of the problem. Therefore the equation $f(x) = 0$ has an unique root in the interval $(1, 3)$ when $f(1).f(3) < 0$ or $D_f = 0, 2a \in (1, 3)$. It follows now that $f(1).f(3) < 0 \iff (a-1)^2(a+5)(a+13) < 0 \iff a \in (-13, -5)$ or $5a^2-4 = 0, 2a \in (1, 3)$, and so $a = \dfrac{2\sqrt{5}}{5}$.

Problem 2. In $\triangle ABC, CH(H \in AB)$ is altitude and CM and $CN(M, N \in AB)$ are bisectors respectively of $\sphericalangle ACH$ and $\sphericalangle BCH$. The circumcenter of $\triangle CMN$ coincides with the incenter of $\triangle ABC$. Prove that $S_{\triangle ABC} = \dfrac{AN.BM}{2}$.

Solution: Let I be the incenter of ABC. Denote by P, Q and R the common points of the incircle of ABC respectively with AB, BC and CA. It follows from $IP = IQ = IR$ and $IC = IM = IN$ that $\triangle IMP, \triangle INP, \triangle ICQ$ and $\triangle ICR$ are congruent. Let $\sphericalangle MIP = \sphericalangle NIP = \sphericalangle QIC = \sphericalangle CIR = \delta$. We have that $\sphericalangle MCN = \frac{1}{2} \sphericalangle MIN = \delta$. Since CM and CN are bisectors we get $\sphericalangle ACB = 2 \sphericalangle MCN = 2\delta$. It follows now from the quadrilateral $IQCR$ that $\sphericalangle QIR+$

$\measuredangle QCR = 180^\circ \iff 4\delta = 180^\circ \iff \delta = 45^\circ$. Therefore $\triangle ABC$ is a right angle triangle which implies $\measuredangle BCM = \measuredangle BMC = \measuredangle BAC + \dfrac{\measuredangle ABC}{2}$ and $\measuredangle ANC = \measuredangle ACN = \measuredangle ABC + \dfrac{\measuredangle BAC}{2}$. Therefore $BC = BM$ and $AC = AN$ and so $S_{ABC} = \dfrac{AC.BC}{2} = \dfrac{AN.BM}{2}$.

Problem 3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_1 = 43, a_2 = 142, a_{n+1} = 3a_n + a_{n-1}$ for $n \geq 2$. Prove that:

a) a_n and a_{n+1} are relatively prime for all n;

b) for every natural number m there exist infinitely many natural numbers n, such that $a_n - 1$ and $a_{n+1} - 1$ both are divisible by m.

Solution: a) Suppose that there exist natural numbers n and $m > 1$, such that m divides both a_n and a_{n+1}. It follows from $a_{n-1} = a_{n+1} - 3a_n$ that m divides a_{n-1}. By induction m divides both a_1 and a_2, which is impossible since a_1 and a_2 are relatively prime.

b) Consider the sequence $\{a_n\}$ defined by $a_{n-1} = a_{n+1} - 3a_n$ for negative indices. Compute $a_0 = a_2 - 3a_1 = 13; a_{-1} = a_1 - 3a_0 = 4; a_{-2} = a_0 - 3a_1 = 1; a_{-3} = a_{-1} - 3a_{-2} = 1$ and so on. We find a sequences $\{a_n\}_{-\infty}^{\infty}$ of integers such that $a_{n+1} = 3a_n + a_{n-1}$ for every n. Since the pairs $(p(\text{mod } m), q(\text{mod } m))$ are finitely many we get that there exist integers r and $s > r$ such that $a_r \equiv a_s(\text{mod } m)$ and $a_{r+1} \equiv a_{s+1}(\text{mod } m)$. It is clear now that $a_{r+i} \equiv a_{s+i}(\text{mod } m)$ for every i. Therefore the sequence $\{a_n\}_{-\infty}^{\infty}$ is periodic. Since $a_{-3} \equiv a_{-2} \equiv 1(\text{mod } m)$, there exist infinitely many natural numbers n such that both $a_n - 1$ and $a_{n+1} - 1$ are divisible by m.

Problem 4. Given a convex quadrilateral $ABCD$, such that $\measuredangle BCD = \measuredangle CDA$. The bisector of $\measuredangle ABC$ intersects the segment CD in point E. Prove that $\measuredangle AEB = 90^\circ$ if and only if $AB = AD + BC$.

Solution: Let $\sphericalangle AEB = 90^\circ$. Since $\sphericalangle CEB < 90^\circ$ there exists a point F on the side AB such that $\sphericalangle BEF = \sphericalangle BEC$. Thus $\triangle BCE \cong \triangle BFE$, which implies $BF = BC$ and $\sphericalangle BFE = \sphericalangle BCE$. From the other hand

$$(*) \qquad \frac{AE}{\sin \sphericalangle AFE} = \frac{AF}{\sin \sphericalangle AEF}, \frac{AE}{\sin \sphericalangle ADE} = \frac{AD}{\sin \sphericalangle AED}.$$

Since $\sphericalangle AED = \sphericalangle AEF$ and $\sphericalangle AFE + \sphericalangle ADE = 180^\circ$, we obtain $AF = AD$ and so $AB = AD + BC$.

Conversely, let $AB = AD + BC$. There exists a point F on the segment AB such that $AF = AD$ and $BF = BC$. Therefore $\triangle BFE \cong \triangle BCE$, so $\sphericalangle BFE = \sphericalangle BCE$ and $\sphericalangle BEF = \sphericalangle BEC$. It follows from $(*)$ and $AF = AD$ that $\sin \sphericalangle AED = \sin \sphericalangle AEF$. Since $\sphericalangle AED + \sphericalangle AEF < 180^\circ$, we have $\sphericalangle AED = \sphericalangle AEF$ and therefore $\sphericalangle AEB = 90^\circ$.

Problem 5. Prove that for any two real numbers a and b there exists a real number $c \in (0,1)$, such that

$$\left| ac + b + \frac{1}{c+1} \right| > \frac{1}{24}.$$

Solution: Consider $f(x) = ax + \dfrac{1}{x+1}$ and let m and M be respectively the minimum and the maximum value of the function f in the interval $[0,1]$. Since $f(0) = 1, f(1) = a + \frac{1}{2}$ and $f'(x) = a - \dfrac{1}{(x+1)^2}$ there are four cases for the difference $M - m$.

1. $a \leq \dfrac{1}{4}$. Thus $f'(x) \leq 0$ for $x \in [0,1]$ and so $M - m = f(0) - f(1) = \dfrac{1}{2} - a \geq \dfrac{1}{4}$.

2. $a \geq 1$. Thus $f'(x) \geq 0$ for $x \in [0,1]$ and so $M - m = f(1) - f(0) = a - \frac{1}{2} \geq \frac{1}{2}$.

3. $\frac{1}{4} \leq a \leq 1$. Thus $d = \frac{1}{\sqrt{a}} - 1 \in [0,1]$, $f'(x) \leq 0$ for $x \in [0,d]$ and $f'(x) \geq 0$ for $x \in [d,1]$.

3.1 $\frac{1}{4} \leq a \leq \frac{1}{2}$. Since $f(0) \geq f(1)$, we have $M - m = f(0) - f(d) = (1 - \sqrt{a})^2 \geq (1 - \frac{\sqrt{2}}{2})^2$.

3.2 $\frac{1}{2} \leq a \leq 1$. Since $f(1) \geq f(0)$, we have $M - m = f(1) - f(d) = \frac{1}{2}(2\sqrt{a} - 1)^2 \geq \frac{1}{2}(\sqrt{2} - 1)^2$.

In all four cases $M - m > \frac{1}{12}$, which implies $M + b > \frac{1}{24}$ or $m + b < \frac{1}{24}$. The assertion of the problem follows now by continuity.

Problem 6. Find all sets S of four points in the plane such that: for any two circles k_1 and k_2, having diameters with endpoints - points from S there exists a point $A \in S \cap k_1 \cap k_2$.

Solution: Let $S = \{A, B, C, D\}$. Consider the circles k_1 and k_2 with diameters respectively AB and CD. It follows that at least one of the angles ACB, ADB, CAD and CBD is right angle. Without loss of generality $\sphericalangle ACB = 90°$. Let k_3 and k_4 be circles with diameters respectively AC and BD. Since $\sphericalangle ABC < 90°$, we obtain that the common point of k_3 and k_4 is one of the points A, C or D.

1. Let $A \in k_3 \cap k_4$. Now $\sphericalangle BAD = 90°$ and therefore $D \in l_1$, $l_1 \perp AB, A \in l_1$. It is easily seen that $\sphericalangle BDC < 90°$ and since $\sphericalangle BAC < 90°, \sphericalangle ABD < 90°$, it follows that the common point of

the circles with diameters AD and BC is the point C and $\sphericalangle ACD = 90^\circ$. Therefore $D = l_1 \cap BC$.

2. Let $C \in k_3 \cap k_4$. Now $\sphericalangle BCD = 90^\circ$ and therefore D lies on the line AC. Since $\sphericalangle BDC < 90^\circ, \sphericalangle BAC < 90^\circ$ and $\sphericalangle ACD < 90^\circ$, we obtain that the common point of the circles with diameters AD and BC is B and $\sphericalangle ABD = 90^\circ$. Therefore $D = l_2 \cap AC$, where $l_2 \perp AB, B \in l_2$.

3. Let $D \in k_3 \cap k_4$. Now $\sphericalangle ADC = 90^\circ$ and therefore $D \in k_3$. Since $\sphericalangle BAC < 90^\circ$ and $\sphericalangle ACD < 90^\circ$, we obtain that the common point of the circles with diameters AD and BC is B or D. In the first case $\sphericalangle ABD = 90^\circ$ and thus $D \in l_2$; from the other hand $l_2 \cap k_3 = \emptyset$, which is a contradiction. In the second case $\sphericalangle BDC = 90^\circ$, and we get that D is the orthogonal projection of C on AB.

In conclusion, all sets S, satisfying the condition of the problem are those consisting of the vertices of a right triangle and the foot of the altitude to the hypothenusis.

$XLIX$ National Mathematical Olympiad
Fourth Round, 16-17 May 2000

Problem 1. In an orthogonal coordinate system xOy a set consisiting of 2000 points $M_i(x_i, y_i)$, is called "good" if $0 \leq x_i \leq 83, 0 \leq y_i \leq 1 \quad i = 1, 2, \ldots, 2000$ and $x_i \neq x_j$ for $i \neq j$. Find all natural numbers n with the following properties: : a) For any "good" set some n of its points lie in a square of side length 1.

b) There exists a "good" set such that no $n + 1$ of its points lie in a square of side length 1.

(A point on a side of a square lies in the square).

Solution: We shall prove that $n = 25$ is the only solution of the problem. We show first that for a "good" set some 25 points lie in a square of side length 1. All points lie in the rectangle $1 \leq x \leq 83, 0 \leq y \leq 1$. Divide this rectangle to 83 squares of side length 1. If some 25 points lie in one of these rectangles then we are done. Conversely, in every square there are at least 26 or at most 24 points. We prove now that there exists a square with at least 26 points and there exists a square with at most 24 points. Indeed, if a square with 26 points does not exist then the points are at most $83.24 = 1992 < 2000$. If there is no square with less than 25 points then the points are $83.26 - 82 > 2000$. Further, move the square with more than 25 points towards the square with less that 25 points. Since the number of points in this square changes at most by one we get the assertion of the problem.

To prove b) let $x_1 = 0, x_i = x_{i-1} + \dfrac{83}{1999}$ and $y_i = 0$ for $i = 0, 2, 4, \ldots, 2000$ while $y_i = 1$ for $i = 1, 3, \ldots, 1999$. Let $XYZT$ be an unit square. WLOG we assume that it intersects the lines

$y = 0$ and $y = 1$ in the points P, Q and R, S respectively. Then

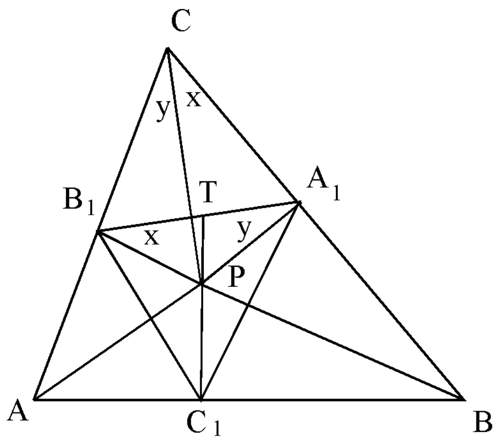
$$PQ+QR+RS+SP < PY+YQ+QZ+ZR+RT+TS+SX+XP = 4.$$

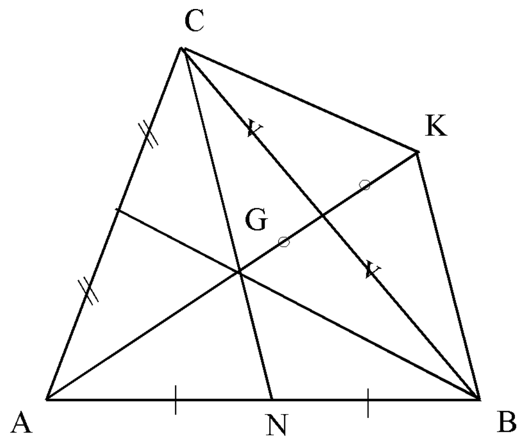
But $SP, QR \geq 1$ and we get $PQ + RS < 2$. This implies that in the square $XYZT$ there are no more than 25 points.

We found that $n = 25$ is a solution of the problem. It is unique since $24 < \dfrac{2000}{83} < 25$.

Problem 2. Given an acute $\triangle ABC$. Prove that there exist unique points A_1, B_1 and C_1 on BC, AC and AB respectively with the following property: Each of the points is the midpoint of the segment with ends the orthogonal projections of the other two points on the corresponding side. Prove that $\triangle A_1B_1C_1$ is similar to the triangle formed by the medians of $\triangle ABC$.

Solution: It is easy to be seen that the perpendicular through C_1 to AB intersects A_1B_1 at its midpoint. The same is applied for A_1 and B_1. Therefore the tree perpendiculars intersect at the medcenter of $\triangle A_1B_1C_1-$ the point P.





Let T be the midpoint of A_1B_1, $\sphericalangle PB_1A_1 = x = \sphericalangle PCA_1$; $\sphericalangle PA_1B_1 = y = \sphericalangle PCB_1$. The Law of Sines for $\triangle B_1TP$ and $\triangle A_1TP$ gives:

$$\frac{\sin\alpha}{\sin x} = \frac{B_1T}{TP} = \frac{A_1T}{AP} = \frac{\sin\beta}{\sin y},$$

where $\dfrac{\sin x}{\sin y} = \dfrac{\sin\alpha}{\sin\beta}$. Further, if CN is the median in $\triangle ABC$, similar arguments imply that $\dfrac{\sin\sphericalangle ACG}{\sin\sphericalangle BCG} = \dfrac{\sin\alpha}{\sin\beta}$. Since $x + y = \sphericalangle ACG + \sphericalangle BCG = \gamma$ and γ is acute it follows that $x = \sphericalangle ACG, y = \sphericalangle BCG$. It is clear now that CP is symmetric to the median in $\triangle ABC$ through C with respect to the bisector of angle γ. The same is true for AP and BP. Therefore the point P is unique (and therefore A_1, B_1 and C_1 are unique). Further, $\sphericalangle B_1C_1A_1 = \sphericalangle B_1C_1P + \sphericalangle A_1C_1P = \sphericalangle B_1AP + \sphericalangle A_1BP = \sphericalangle BAG + \sphericalangle ABG = \sphericalangle BGK$, where K is the symmetric point of G with respect to the midpoint of BC. By analogy $\sphericalangle C_1A_1B_1 = \sphericalangle GBK$ and $\sphericalangle A_1B_1C_1 = \sphericalangle GKB$. Therefore $\triangle A_1B_1C_1$ is similar to the triangle formed by the medians of $\triangle A_1B_1C_1$.

Problem 3. Let $p \geq 3$ be a prime number and $a_1, a_2, \ldots, a_{p-2}$ be a sequence of natural numbers such that p does not divide both a_k and $a_k^k - 1$ for all $k = 1, 2, \ldots, p-2$. Prove that the product of some elements of the sequence is congruent to 2 modulo p.

Solution: Consider the sequence $1, a_1, a_2, \ldots, a_{p-2}$. We shall prove by induction that for any $i = 2, 3, \ldots, p-1$ there exist integers $b_1, b_2, \ldots, b_i$ each of which is a product of some elements from the above sequence and $b_m \not\equiv b_n (\bmod\ p)$ for $m \neq n$. Indeed, for $i = 2$ we can choose $b_1 = 1, b_2 = a_1 (a_1 \not\equiv 1 (\bmod\ p))$. Suppose we have chosen $b_1, b_2, \ldots, b_i$ such that $b_m \not\equiv b_n (\bmod\ p)$ for $m \neq n$. Consider $b_1 a_i, b_2 a_i, \ldots, b_i a_i$. It is easily seen that any two of them are not congruent modulo p. Further, if $b_j a_i$ for any j is congruent to b_l for some l we get that $b_1 a_i, b_2 a_i, \ldots, b_i a_i$ modulo p is a permutation of $b_1, b_2, \ldots, b_i$ modulo p. Thus,

$$(b_1 a_i).(b_2 a_i). \ldots .(b_i a_i) \equiv b_1.b_2. \ldots .b_i (\bmod\ p)$$

and therefore $a_i^i \equiv 1 (\bmod\ p)-$ a contradiction.

It follows now that for any $s = 2, 3, \ldots, p-1$ there exist few elements of the sequence from the condition of the problem such that their product is congruent to s modulo p.

Problem 4. Find all polynomials $P(x)$ with real coeficients such that

$$P(x).P(x+1) = P(x^2)$$

for any real x.

Solution: We show first that for any natural number n there exists at most one polynomial $P(x)$ of degree n such that $P(x).P(x+1) =$

$P(x^2)$. Indeed, if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

comparing the coefficients in front of the same degrees of x in $P(x).P(x+1)$ and $P(x^2)$, we get $a_n = 1$ and $a_{n-1} = -\frac{n}{2}$. It is easily seen that each of the subsequent coefficients is a solution of an equation of first degree of the form $2x + b = 0$, where b is a function of already chosen coefficients. Therefore, if a polynomial of degree n with the desired property exists it is unique one.

We prove now that a polynomial $P(x)$ of odd degree such that $P(x).P(x+1) = P(x^2)$ does not exist. Let $P(x) = x^m(x-1)^n.Q(x)$ (m or n could be zero), where $Q(x)$ is a polynomial such that $Q(0) \neq 0$ and $Q(1) \neq 0$. A substitution gives

$$(x+1)^m.x^n.Q(x).Q(x+1) = x^m.(x+1)^n.Q(x^2).$$

If $m \neq n$ the substitution $x = 0$ leeds to a contradiction. Therefore $m = n$ and so $Q(x).Q(x+1) = Q(x^2)$, where $Q(x)$ is of odd degree. Therefore there exists a real number $x_0 \neq 0, 1$, such that $Q(x_0) = 0$. It is clear that $x_0 = -1$ since otherwise the substitution $x = -1$ shows that $Q(1) = 0$ which is impossible. A substitution $x = x_0$ gives that $Q(x_0^2) = 0$, thus $Q(x_0^4) = 0$ and so on. Since $x_0 \neq \pm 1, 0$ it follows that there are infinitely many distinct terms in the sequence $x_0, x_0^2, \ldots, x_0^{2^k}$, which is impossible.

Direct verification shows that for any even natural number $n = 2k$, the polynomial $P(x) = x^k(x-1)^k$ is of degree n and is a solution of the problem.

Therefore all polynomials such that $P(x).P(x+1) = P(x^2)$ are $P(x) = x^k(x-1)^k$.

Problem 5. Let D be the midpoint of the base AB of an isosceles acute $\triangle ABC$. A point E is chosen on AB and O is the circumcenter of $\triangle ACE$. Prove that the line through D perpendicular to DO, the line through E, perpendicular to BC and the line through B, parallel to AC intersect in a point.

Solution: Let $\sphericalangle ABC = \sphericalangle BAC = \alpha$ and let G be the circumcenter of $\triangle ABC$. Consider points F' and F'' on the line through B parallel to AC such that $OD \perp DF'$ and $BC \perp EF''$. Denote by H' and H'' respectively the projection points of F' and F'' on the line AB. Since O is inner point for $\sphericalangle ADC$ and $\sphericalangle ACB < 90°$, we have that F' and F'' lie in the interior of $\sphericalangle BAC$. It suffices to prove that $F' \equiv F''$, i.e. $F'H' = F''H''$. Let O' and G' be respectively the projection points of O on AB and of G on OO'. Since $\triangle DH'F' \sim \triangle OO'D$, it follows that $\dfrac{DH'}{F'H'} = \dfrac{OO'}{DO'}$. Also, $\sphericalangle GOG' = \alpha$ and so $\dfrac{B'H'}{F'H'} = \dfrac{OG'}{GG'}$. It follows from $GG' = DO', O'G' = DG$ and $\sphericalangle DBG = 2\alpha - 90°$ that $F'H' = \dfrac{BD.O'D'}{GD} = -\mathrm{tg}2\alpha.O'D$. Denote $BC \cap EF'' = I$. Since $\sphericalangle CBF'' = 180° - 2\alpha$ and $BE = 2O'D$, it is clear that $F''H'' = BF'' \sin\alpha = \dfrac{BI \sin\alpha}{\cos(180° - 2\alpha)} = -BE\dfrac{\sin\alpha\cos\alpha}{\cos 2\alpha} = -O'D\mathrm{tg}2\alpha = F'H'$. This completes the prove.

Problem 6. Let $\mathcal{A}$ be the set of all binary sequences of length n and let $\mathbf{0} \in \mathcal{A}$ be the sequence with zero elements. The sequence $c = c_1, c_2, \dots, c_n$ is called sum of $a = a_1, a_2, \dots, a_n$ and $b = b_1, b_2, \dots, b_n$ if $c_i = 0$ when $a_i = b_i$ and $c_i = 1$ when $a_i \neq b_i$. Let $f : \mathcal{A} \to \mathcal{A}$ be a function such that $f(\mathbf{0}) = \mathbf{0}$ and if the sequences a and b differ in exactly k terms then the sequences $f(a)$ and $f(b)$ differ also exactly in k terms. Prove that if a, b and c are sequences from $\mathcal{A}$ such that $a + b + c = \mathbf{0}$, then $f(a) + f(b) + f(c) = \mathbf{0}$.

Solution: Consider the sequence $e_1 = 1, 0, \dots, 0, 0$; $e_2 = 0, 1, \dots, 0$; $\dots, e_{n-1} = 0, 0, \dots, 1, 0; e_n = 0, 0, \dots, 0, 1$. It follows from the condition of the problem that (since $f(\mathbf{0}) = \mathbf{0}$) for all $p, 1 \leq p \leq n$ there exists $q, 1 \leq q \leq n$, such that $f(e_p) = e_q$.

It is clear also that $f(e_p) \neq f(e_q)$ for $1 \leq p, q \leq n; p \neq q$. Therefore

$$(1) \qquad \{f(e_1), f(e_2), \ldots, f(e_n)\} = \{e_1, e_2, \ldots, e_n\}.$$

Consider an arbitrary sequence $a = a_1, a_2, \ldots, a_n$ with t ones. If $f(e_p) = e_q$, and $a_p = 1$, then the $q-$th term of the sequence $f(a)$ is also 1 (otherwise e_p and a differ at $t-1$ terms whereas $f(e_p) = e_q$ and $f(a)$ differ at $t+1$ terms). By analogy if $a_p = 0$, then the $q-$th term of $f(a)$ is also 0.

Finaly, consider the sequences $a = a_1, a_2, \ldots, a_n; b = b_1, b_2, \ldots, b_n$ and $c = c_1, c_2, \ldots, c_n$ such that $a + b + c = \mathbf{0}$. This means that for every $i, 1 \leq i \leq n$ the sum $a_i + b_i + c_i$ is even number. Fix $i, 1 \leq i \leq n$ and let $f(e_i) = e_j$. It follows now that the $j-$th terms of the sequences $f(a), f(b), f(c)$ coincide with a_i, b_i, c_i respectively and using (1) we obtain that $f(a) + f(b) + f(c) = \mathbf{0}$.

Union of Bulgarian Mathematicians

Sava Grozdev

Emil Kolev

BULGARIAN

MATHEMATICAL COMPETITIONS

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Problem 9.1. a) Draw all points in the plane with coordinates $(x; y)$ such that

$$(|3x - y| - 3)(|3x + y| - 3) = 0;$$

b) Find all x and y for which

$$\left|\begin{array}{rcl} (|3x - y| - 3)(|3x + y| - 3) & = & 0 \\ y - \{4x\} & = & 0 \\ -1 \leq x \leq 1 & & \end{array}\right.$$

(For a real number x we denote the unique number in the interval $[0, 1)$ for which $x - \{x\}$ is an integer by $\{x\}$).

Solution: a) It is easy to see that these are the points on the four lines $l_1 : 3x - y = 3, l_2 : 3x - y = -3, l_3 : 3x + y = 3$ and $l_4 : 3x + y = -3$.

b) There are two types of solutions:

1. Solutions of $\left|\begin{array}{ccc} y & = & -3x + 3 \\ y & = & \{4x\} \\ -1 & \leq x \leq & 1 \end{array}\right.$

2. Solutions of $\left|\begin{array}{ccc} y & = & 3x + 3 \\ y & = & \{4x\} \\ -1 & \leq x \leq & 1 \end{array}\right.$

Denote the integer part of $4x$ by $[4x]$, i.e. $4x = [4x] + \{4x\}$. In the first case we have $-3x + 3 = \{4x\} = 4x - [4x]$, i.e $7x = 3 + [4x]$.

Since the right hand side is an integer all posiible values of x are $0, \pm\frac{1}{7}, \pm\frac{2}{7}, \ldots, \pm 1$. Direct verification shows that only $x = \frac{5}{7}, y = \frac{6}{7}$; $x = \frac{6}{7}, y = \frac{3}{7}$; $x = 1, y = 0$ are solutions.

In the second case $3x + 3 = \{4x\} = 4x - [4x]$, i.e. $x = 3 + [4x]$. It follows that x is an integer, i.e. $x = -1, 0$ or 1. Direct verification shows that solution is only $x = -1, y = 0$.

Thus, there are four solutions:

$$x = \frac{5}{7}, y = \frac{6}{7}; x = \frac{6}{7}, y = \frac{3}{7}; x = 1, y = 0; x = -1, y = 0.$$

Problem 9.2. Points A_1, B_1 and C_1 are chosen on the sides BC, CA and AB of a triangle ABC. Point G is the centroid of $\triangle ABC$, and G_a, G_b and G_c are centroids of $\triangle AB_1C_1, \triangle BA_1C_1$ and $\triangle CA_1B_1$ respectively. The centroids of $\triangle A_1B_1C_1$ and $\triangle G_aG_bG_c$ are denoted by G_1 and G_2 respectively. Prove that:

a) the points G, G_1 and G_2 lie on a straight line;

b) lines AG_a, BG_b and CG_c intersect in a point if and only if AA_1, BB_1 and CC_1 intersect in a point.

Solution: a) Let O be an arbitrary point and

$$\vec{OA_1} = \alpha\vec{OB} + (1-\alpha)\vec{OC}; \vec{OB_1} = \beta\vec{OA} + (1-\beta)\vec{OC};$$
$$\vec{OC_1} = \gamma\vec{OA} + (1-\gamma)\vec{OB},$$

where $\alpha, \beta, \gamma \in (0,1)$. The existence of α, β and γ follows from the fact that A_1, B_1 and C_1 lie on the sides BC, CA and AB of $\triangle ABC$. Then we have

$$\vec{OG} = \frac{1}{3}\left(\vec{OA} + \vec{OB} + \vec{OC}\right) = \frac{1}{9}\left(3\vec{OA} + 3\vec{OB} + 3\vec{OC}\right);$$
$$\vec{OG_1} = \tfrac{1}{3}\left(\vec{OA_1} + \vec{OB_1} + \vec{OC_1}\right) =$$

$\frac{1}{3}\left[(\beta+\gamma)\vec{OA}+(\alpha-\gamma+1)\vec{OB}+(2-\alpha-\beta)\vec{OC}\right].$

By analogy $\vec{OG_a}=\frac{1}{3}\left[(1+\beta+\gamma)\vec{OA}+(1-\gamma)\vec{OB}+(1-\beta)\vec{OC}\right];$

$\vec{OG_b}=\frac{1}{3}\left[\gamma\vec{OA}+(2+\alpha-\gamma)\vec{OB}+(1-\alpha)\vec{OC}\right];$

$\vec{OG_c}=\frac{1}{3}\left[\beta\vec{OA}+\alpha\vec{OB}+(3-\alpha-\beta)\vec{OC}\right].$

Thus, $\vec{OG_2}=\frac{1}{3}\left(\vec{OA_1}+\vec{OB_1}+\vec{OC_1}\right)=$

$\frac{1}{9}\left[(1+2\beta+2\gamma)\vec{OA}+(3+2\alpha-2\gamma)\vec{OB}+(5-2\alpha-2\beta)\vec{OC}\right].$

Since $\vec{GG_1}=\vec{OG_1}-\vec{OG}$ we have

$$\vec{GG_1}=\frac{1}{3}\left[(\beta+\gamma-1)\vec{OA}+(\alpha-\gamma)\vec{OB}+(1-\alpha-\beta)\vec{OC}\right].$$

Using the same arguments we obtain $\vec{GG_2}=\vec{OG_2}-\vec{OG}=$ $\frac{1}{9}\left[(2\beta+2\gamma-2)\vec{OA}+(2\alpha-2\gamma)\vec{OB}+(2-2\alpha-2\beta)\vec{OC}\right]$. It follows from the last two equalities that $\vec{GG_1}=\frac{3}{2}\vec{GG_2}$ and we are done.

b) Since the lines AA_1, BB_1 and CC_1 intersect in a point Ceva's theorem gives $\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}=1.$

Denote the intersecting points of AG_a, BG_b and CG_c with the sides BC, CA and AB by A_2, B_2 and C_2 respectively. A necessary and sufficient condition for lines AG_a, BG_b and CG_c to intersect in a point is $\frac{CA_2}{A_2B}\cdot\frac{BC_2}{C_2A}\cdot\frac{AB_2}{B_2C}=1.$

Denote the midpoint of B_1C_1 by A_3. Let h_1 and h_2 be the altitudes of $\triangle AA_2C$ and AA_3B_1 from A_2 and A_3. We have

$$\frac{S_{AA_2C}}{S_{AA_3B_1}} = \frac{h_1.AC}{h_2.AB_1} = \frac{AA_2.AC}{AA_3.AB_1}.$$

By analogy

$$\frac{S_{AA_2B}}{S_{AA_3C_1}} = \frac{AA_2.AB}{AA_3.AC_1}.$$

Dividing the above equalities and using that $S_{AA_3C_1} = S_{AA_3B_1}$ and $\frac{S_{AA_2C}}{S_{AA_2B}} = \frac{CA_2}{BA_2}$ we obtain

$$\frac{CA_2}{A_2B} = \frac{AC}{AB} \cdot \frac{AC_1}{AB_1}.$$

By analogy

$$\frac{BC_2}{C_2A} = \frac{CB}{CA} \cdot \frac{CB_1}{CA_1}, \frac{AB_2}{B_2C} = \frac{BA}{BC} \cdot \frac{BA_1}{BC_1}.$$

Multiplying the above equalities gives

$$\frac{CA_2}{A_2B}\frac{BC_2}{C_2A}\frac{AB_2}{B_2C} = \frac{AC}{AB}\frac{AC_1}{AB_1} \cdot \frac{BC}{AC}\frac{CB_1}{CA_1} \cdot \frac{BA}{BC}\frac{BA_1}{BC_1} = \frac{AC_1}{C_1B}\frac{BA_1}{A_1C}\frac{CB_1}{B_1A}.$$

Therefore the lines AG_a, BG_b and CG_c intersect in a point iff the lines AA_1, BB_1 and CC_1 interect in a point.

Problem 9.3. Let A_n be the number of sequences from 0's and 1's of length n, such that no four consequtive elements equal 0101. Find the parity of A_{2001}.

Solution: Denote the number of sequences of length n, the last three terms of which are ijk, where $i, j, k \in \{0, 1\}$ and no four consequtive elements equal to 0101 by a^n_{ijk}.

Obviously $a_{ijk}^{n+1} = a_{0ij}^n + a_{1ij}^n$ when $ijk \neq 101$ and $a_{101}^{n+1} = a_{110}^n$. Adding the above equalities for all values of ijk we obtain $A_{n+1} = 2A_n - a_{010}^n$.

Therefore $A_{n+1} \equiv a_{010}^n \equiv a_{001}^{n-1} + a_{101}^{n-1} \equiv a_{000}^{n-2} + a_{100}^{n-2} + a_{110}^{n-2} \equiv a_{000}^{n-3} + a_{100}^{n-3} \equiv a_{000}^{n-4} + a_{100}^{n-4} + a_{010}^{n-4} + a_{110}^{n-4} \equiv a_{000}^{n-5} + a_{001}^{n-5} + a_{010}^{n-5} + a_{100}^{n-5} + a_{111}^{n-5} + a_{110}^{n-5} + a_{101}^{n-5} + a_{011}^{n-5} + \equiv A_{n-5} \pmod{2}$.

Thus, $A_{k+6m} \equiv A_k \pmod{2}$ for all k and m.

Therefore $A_{2001} \equiv A_3 = 8 \equiv 0 \pmod{2}$.

Problem 10a.1. Find all pairs $(a;b)$ of integers such that the system

$$\left| \begin{array}{rcl} x^2 + 2ax - 3a - 1 & = & 0 \\ y^2 - 2by + x & = & 0 \end{array} \right.$$

has exactly three real solutions.

Solution: Let $(a;b)$ satisfy the condition of the problem. Since there are exactly three real solutions the first equation has two distinct roots $x_1 < x_2$, which implies $D_1 = 4(a^2 + 3a + 1) > 0$. The discriminant of the second equation in respect to y equals $D_2 = 4(b^2 - x)$ and therefore there are exactly three real solutions iff $x_1 < x_2 = b^2$.

These conditions are satisfied iff a and b are integers such that $a^2 + 3a + 1 > 0$ and $b^2 = -a + \sqrt{a^2 + 3a + 1}$. It follows from the latter equality that $a^2 + 3a + 1 = c^2$, where c is a positive integer. Therefore the discriminant of $a^2 + 3a + 1 - c^2 = 0$ is a perfect square, i.e. $9 - 4(1 - c^2) = d^2$, where d is nonnegative integer. The last equality can be written in the form $(d - 2c)(d + 2c) = 5$ which implies that $d - 2c = 1, d + 2c = 5$, i.e. $d = 3, c = 1$. Hence, $a^2 + 3a + 1 = 1$ with roots $a = 0$ and $a = -3$. Respectively $b = \pm 1$ or $b = \pm 2$. Direct verification shows that all pairs $(a, b) = (0, 1), (0, -1), (-3, 2)$ and $(-3, -2)$ satisfy the condition of the problem.

Problem 10a.2. The tangential point of a circle k through the vertex C of a $\triangle ABC$ and the line AB is the vertex B. The circle k intersects for a second time the side AC snd the median of $\triangle ABC$ through C at points D and E respectively. Prove that if the intersecting point of the tangents to k through C and E lies on the line BD then $\sphericalangle ABC = 90^\circ$.

Solution: Let F be the intersecting point of the tangents t_C and t_E to k at C and $E, G = BD \cap CE$ and $ACBH$ is parallelogram. We have that $\sphericalangle ABC = 90^\circ \iff t_C \parallel AB \iff \dfrac{FD}{FB} = \dfrac{CD}{CA}$. From the other hand $\dfrac{CD}{CA} = \dfrac{CD}{BH} = \dfrac{GD}{GB}$, i.e. it suffices to show that $\dfrac{FD}{FB} = \dfrac{GD}{GB}(*)$. Since $\triangle FBE \sim \triangle FED$ and $\triangle FBC \sim \triangle FCD$, we obtain $\dfrac{FB}{FE} = \dfrac{BE}{ED}$ and $\dfrac{FB}{FC} = \dfrac{BC}{CD}$. Further, $FC^2 = FE^2 = FB.FD$ and so $\dfrac{FD}{FB} = \dfrac{CD.ED}{CB.EB}$. Now the equality from $(*)$ follows from $\dfrac{CD.ED}{CB.EB} = \dfrac{S_{CED}}{S_{CEB}} = \dfrac{GD}{GB}$, which completes the proof.

Problem 10a.3. Ivan and Peter alternatively write down 0 or 1 until each of them has written 2001 digits. Peter is a winner if the number, which binary representation has been obtained, cannot be expressed as a sum of two perfect squares. Prove that Peter has a winning strategy.

Solution: First we prove that if the binary representation of a positive integer ends with two ones and even number of zeroes then this integer cannot be represented as sum of two squares. Indeed, such a number is of the form $4^k(4s + 3)$ and if we suppose that $x^2 + y^2 = 4^k(4s+3)$, then using the fact that $x^2 + y^2$ equals 0 modulo 4 iff x and y are even we get that there exist integers p, q for which $p^2 + q^2 = 4s + 3$, which is a contradiction.

The winning strategy of Peter could be:

If one of Ivan's digits is 1 then Peter simply repeats all digits written by Ivan. The final number is of the form $4^k(4s+3)$ and cannot be written as x^2+y^2. If all Ivan's digits are zeroes then the first three digits of Peter are $1,1,1$ after which he writes only zeroes. The final number is $(0101010\ldots 0)_2 = 21.4^{1998}$, and cannot be represented as p^2+q^2 since 21 cannot be written in that form.

Problem 10b.1. Find all values of the real parameter a such that the equation

$$\log_x (x^2+x+a)^2 = 4$$

has unique solution.

Solution: The equation is equivalent to $(x^2+x+a)^2 = x^4, x > 0, x \neq 1$. Further, $(x^2+x+a)^2 = x^4 \iff (x+a)(2x^2+x+a) = 0$ with roots $x_1 = -a$ and $x_{2/3} = \dfrac{-1 \pm \sqrt{1-8a}}{4}$, provided $1-8a \geq 0$. Since $\dfrac{-1-\sqrt{1-8a}}{4} < 0$, we obtain that only $x_1 = -a, x_2 = \dfrac{-1+\sqrt{1-8a}}{4}$ can be roots of the equation from the problem. If $a \geq 0$ then $x_1 \leq 0$ and $x_2 \leq 0$, which implies that the equation has no roots. If $a < 0$ then $x_1 > 0$ and $x_2 > 0$. Note that $x_1 = x_2$ implies $a = 0$ which is a contradiction. Therefore if the equation has unique solution then it is necessary to have $a < 0$ and one of the two roots equals 1. Thus, $x_1 = 1 \Rightarrow a = -1$ and $x_2 = 1 \Rightarrow a = -3$.

Problem 10b.2. On each side of a right isosceles triangle with legs of length 1 is chosen a point such that the triangle formed from these three points is a right triangle. What is the least value of the hypotenuse of this triangle?

Solution: Consider a right isosceles triangle ABC with right angle at C. Let $A_1 \in BC, B_1 \in CA$ and $C_1 \in AB$ be such that $\triangle A_1B_1C_1$ is right triangle.

1. Suppose $\sphericalangle B_1A_1C_1 = 90°$. Assume that the circle k with diameter B_1C_1 intersects BC at point $X \neq A_1$. It is easy to be seen that X is an interior point for the line segment BC. Draw a tangent l to k which is parallel to BC and the tangential point Y belongs to the smaller of the arcs $\overset{\frown}{A_1X}$. Denote the intersecting points of l with AB and AC by C_2 and B_2 respectively. Consider a homothety of center A such that the image of $B_2(C_2)$ is $C(B)$. It is clear that the image of $\triangle B_1C_1Y$ is inscribed in $\triangle ABC$ and its hypotenuse is less than B_1C_1. Therefore wlog we may assume that BC is tangent to k. Thus, if $\sphericalangle C_1B_1A_1 = \alpha$, then $\sphericalangle C_1A_1B = \alpha$. From the Sine Low for $\triangle BC_1A_1$ we obtain $BA_1 = \dfrac{B_1C_1 \sin\alpha \sin(\alpha + 45°)}{\sin 45°} = B_1C_1 \sin\alpha(\sin\alpha + \cos\alpha)$. Hence

(1) $1 = BA_1 + A_1C = B_1C_1 \cos\alpha \sin\alpha + B_1C_1 \sin\alpha(\sin\alpha + \cos\alpha)$

Therefore $1 = B_1C_1(\sin^2\alpha + \sin 2\alpha) \iff B_1C_1(1 + 2\sin 2\alpha - \cos 2\alpha) = 2$. From the other hand $2\sin 2\alpha - \cos 2\alpha =$
$\sqrt{5}\left(\dfrac{2}{\sqrt{5}}\sin 2\alpha - \dfrac{1}{\sqrt{5}}\cos 2\alpha\right) = \sqrt{5}\sin(2\alpha - \delta) \leq \sqrt{5}$, where δ is such that $\sin\delta = \dfrac{1}{\sqrt{5}}$ and $\cos\delta = \dfrac{2}{\sqrt{5}}$. Note that the equality holds when $2\alpha - \delta = 90°$ and since $\delta < 90°$ we have that $\alpha < 90°$.

Further $B_1C_1(1 + \sqrt{5}) \geq B_1C_1(1 + 2\sin 2\alpha - \cos 2\alpha) = 2$ and so $B_1C_1 \geq \dfrac{\sqrt{5}-1}{2}$. It is easily seen that if $B_1C_1 = \dfrac{\sqrt{5}-1}{2}$ and $2\alpha - \delta = 90°$ then (1) holds true and therefore there exists triangle with $B_1C_1 = \dfrac{\sqrt{5}-1}{2}$.

2. Suppose $\sphericalangle A_1C_1B_1 = 90°$. Denote the projection of A_1 and

B_1 on AB by A_2 and B_2 respectively. Obviously $AB_2 = B_1B_2$ and $BA_2 = A_1A_2$. Let P be the midpoint of A_1B_1 and Q – the midpoint of A_2B_2. Thus, $A_1B_1 = 2PC_1 \geq 2PQ = A_1A_2 + B_1B_2$ and $A_1B_1 \geq A_2B_2 = AB - AB_2 - BA_2 = AB - B_1B_2 - A_1A_2$. Adding the above equalities gives $2A_1B_1 \geq AB = \sqrt{2}$ and so $A_1B_1 \geq \dfrac{\sqrt{2}}{2}$.

Since $\dfrac{\sqrt{2}}{2} > \dfrac{\sqrt{5}-1}{2}$ we obtain that the least possible value equals $\dfrac{\sqrt{5}-1}{2}$.

Problem 10b.3. An element x is chosen from the set $A = \{1, 2, \ldots, 2^n\}, n \geq 3$. Questions of the type: Does x belong to $B \subset A$ where the sum of the elements of B equals $2^{n-2}(2^n + 1)$ are allowed? Prove that one can find x with exactly n questions stated in advance.

Solution:

Lemma There exist n sets each with 2^{n-1} elements, the sum of the elements of each set equals $2^{n-2}(2^n + 1)$ with the following property: the elements from the set $\{1, 2, \ldots, 2^n\}$ get as answers distinct $n-$tuples from "yes" and "no".

Proof: Induction by $n \geq 3$. For $n = 3$ the sum of the elements of B is 18 and we use the sets $B_1 = \{1, 2, 7, 8\}$, $B_2 = \{1, 3, 6, 8\}$ and $B_3 = \{1, 4, 6, 7\}$. The table shows that the elements from A get as answers distinct triples.

	1	2	3	4	5	6	7	8
$B_1 = \{1, 2, 7, 8\}$	+	+	-	-	-	-	+	+
$B_2 = \{1, 3, 6, 8\}$	+	-	+	-	-	+	-	+
$B_3 = \{1, 4, 6, 7\}$	+	-	-	+	-	+	+	-

+ means "yes", $-$ means "no".

The number of elements in each set is $2^2 = 2^{3-1}$.

Suppose the assertion is true for some m. Therefore there exist sets $B_1, B_2, \ldots, B_m$ each with 2^{m-1} elements, the sum of the elements of each set equals $2^{m-2}(2^m+1)$ and every element from $\{1, 2, \ldots, 2^m\}$ gets distinct $m-$tuple.

Consider the set $\{1, 2, \ldots, 2^{m+1}\}$. For any $i, 1 \leq i \leq m$ if $B_i = \{a_{1i}, a_{2i}, \ldots, a_{2^{m-1}i}\}$, let

$$D_i = \{a_{1i}, a_{2i}, \ldots, a_{2^{m-1}i}, a_{1i} + 2^m, a_{2i} + 2^m, \ldots, a_{2^{m-1}i} + 2^m\}.$$

It is clear that each set $D_i, 1 \leq i \leq m$ has exactly 2^m elements and the sum of the elements of D_i is equal to: $2(a_{1i} + a_{2i} + \ldots + a_{2^{m-1}i}) + 2^{m-1}.2^m = 2.2^{m-2}(2^m+1) + 2^{2m-1} = 2^{2m-1} + 2^{m-1} + 2^{2m-1} = 2^{m-1}(2^{m+1}+1)$. It is easily seen that only the elements t and $t+2^m$ for $1 \leq t \leq 2^m$ get equal $m-$tuples. Let P and Q be nonintersecting sets such that $|P| = |Q| = 2^{m-1}$ and $P \cup Q = \{1, 2, \ldots, 2^m\}$. Consider a set $\bar{Q}$ obtained from Q by adding 2^m to each of its elements. The set $D_{m+1} = P \cup \bar{Q}$ has 2^m elements and the sum of its elements is equal to $\sum_{s \in P} s + \sum_{s \in Q} s + 2^{m-1}.2^m = 2^{m-1}(2^m+1) + 2^{2m-1} = 2^{m-1}(2^{m+1}+1)$. It is clear that exactly one element of each pair $(t, t+2^m)$ for $1 \leq t \leq 2^m$ belongs to D_{m+1}. Hence, we have found the desired sets $D_1, D_2, \ldots, D_{m+1}$.

It is obvious now that the sets, given by the Lemma solve the problem.

Problem 11.1. A sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by

$$a_1 = k; a_2 = 5k - 2 \text{ and } a_{n+2} = 3a_{n+1} - 2a_n, n \geq 1,$$

where k is a real number.

a) Find all values of k, such that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

b) Prove that if $k = 1$ then

$$a_{n+2} = \left[\frac{7a_{n+1}^2 - 8a_n a_{n+1}}{1 + a_n + a_{n+1}}\right], n \geq 1,$$

where $[x]$ denotes the integer part of x.

Solution: a) Write the given reccurent relation in the form $a_{n+2} - a_{n+1} = 2(a_{n+1} - a_n)$ and consider the sequence $c_n = a_{n+1} - a_n$. It is obvious that if the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent then the sequence $\{c_n\}_{n=1}^{\infty}$ is also convergent. From the other hand it follows from $c_1 = a_2 - a_1 = 4k - 2$ and $c_{n+1} = 2c_n$ that the sequence $\{c_n\}_{n=1}^{\infty}$ is an arithmetic progression and $c_n = (4k-2).2^{n-1}$ for any positive integer n. Thus, if $4k-2 \neq 0$ then the sequence $\{c_n\}_{n=1}^{\infty}$ is unbounded and therefore is not convergent one. Therefore $4k-2 = 0$, i.e. $k = \frac{1}{2}$. For this value of k all terms of the sequence $\{a_n\}_{n=1}^{\infty}$ are equal to $\frac{1}{2}$ and therefore the sequence is convergent.

b) It easily follows by induction that $a_n = 2^n - 1$. From the other hand the equality from the condition of the problem is satisfied for $n = 1, 2$. Suppose it is true for $n = m$ and $n = m + 1$. Then

$$\left[\frac{7a_{m+1}^2 - 8a_m a_{m+1}}{1 + a_m + a_{m+1}}\right] = \left[\frac{7(2^{m+1}-1)^2 - 8(2^m - 1)(2^{m+1} - 1)}{1 + 2^m - 1 + 2^{m+1} - 1}\right] =$$

$$\left[\frac{3.2^{2m+2} - 2^{m+2}}{3.2^m - 1} + \frac{-1}{3.2^m - 1}\right] = 2^{m+2} - 1 = a_{m+1},$$

which completes the proof.

Problem 11.2. On each side of a triangle with angles $30°, 60°$ and $90°$ and hypotenuse 1 is chosen a point such that the triangle formed from these three points is a right triangle. What is the least value of the hypotenuse of this triangle?

Solution: Consider triangle ABC with angles α, β and γ. Let $A_1 \in BC, B_1 \in CA$ and $C_1 \in AB$ be such that $\triangle A_1B_1C_1$ is right with $\sphericalangle A_1C_1B_1 = 90^\circ$. Suppose that the circle k of diameter A_1B_1 intersects AB at point $X \neq C_1$. It is easy to see that if $\alpha \leq 90^\circ$ and $\beta \leq 90^\circ$ then X is an interior point for line segment AB. Draw a tangent l to k which is parallel to AB and the tangential point Y belongs to the smaller of the arcs $\overset{\frown}{C_1X}$. Denote the intersecting points of l and lines AC and BC by B_2 and A_2 respectively. Consider homothety of center C for which the image of $B_2(A_2)$ is the point $A(B)$. It is obvious that the image of $\triangle B_1A_1Y$ is inscribed in $\triangle ABC$ and its hypotenuse is less than B_1A_1. Therefore wlog we may suppose that AB is tangent to k. In this case if $\sphericalangle B_1A_1C_1 = \delta$ then $\sphericalangle B_1C_1A = \delta$.

The Sine Low for $\triangle AC_1B_1$ and $\triangle BC_1A_1$ gives

$$AC_1 = \frac{B_1A_1 \sin\delta \sin(\alpha+\delta)}{\sin\alpha}; BC_1 = \frac{B_1A_1 \cos\delta \sin(90^\circ - \delta + \beta)}{\sin\beta}.$$

Since $AB = AC_1 + C_1B$ we obtain

$$(1) \qquad AB = \frac{B_1A_1 \sin\delta \sin(\alpha+\delta)}{\sin\alpha} + \frac{B_1A_1 \cos\delta \sin(90^\circ - \delta + \beta)}{\sin\beta}$$

This equality is equivalent to

$$2AB = A_1B_1(\mathrm{cotg}\alpha + \mathrm{cotg}\beta + 2\sin 2\delta - (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)\cos 2\delta).$$

From the other hand

$$2\sin 2\delta - (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)\cos 2\delta =$$

$$\sqrt{4 + (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)^2}\sin(2\delta - \phi) \leq \sqrt{4 + (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)^2},$$

where

$$\cos\phi = \frac{2}{\sqrt{4 + (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)^2}}, \sin\phi = \frac{\mathrm{cotg}\alpha - \mathrm{cotg}\beta}{\sqrt{4 + (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)^2}}.$$

It is easily seen that such an angle δ always exists.

Finally, the least value is

$$B_1A_1 = \frac{2AB}{\operatorname{cotg}\alpha + \operatorname{cotg}\beta + \sqrt{4 + (\operatorname{cotg}\alpha - \operatorname{cotg}\beta)^2}}.$$

It is clear that if $B_1A_1 = \dfrac{2AB}{\operatorname{cotg}\alpha + \operatorname{cotg}\beta + \sqrt{4 + (\operatorname{cotg}\alpha - \operatorname{cotg}\beta)^2}}$ and $2\delta - \phi = 90^\circ$ then (1) holds true and therefore there exists a triangle with $B_1A_1 = \dfrac{2AB}{\operatorname{cotg}\alpha + \operatorname{cotg}\beta + \sqrt{4 + (\operatorname{cotg}\alpha - \operatorname{cotg}\beta)^2}}$.

It remains to compute the above expression for the three possible ways to inscribe a right triangle in triangle of angles $30^\circ, 60^\circ$ and 90° and hypotenuse 1. We obtain $\dfrac{\sqrt{39}-\sqrt{3}}{12}, \dfrac{\sqrt{21}-3}{4}, \dfrac{\sqrt{3}}{4}$. The least value is $\dfrac{\sqrt{39}-\sqrt{3}}{12}$ which is the answer of the problem.

Problem 11.3. The plane is divided into unit squares by lines parallel to coordinate axes of an orthogonal coordinate system. Find the number of paths of length n from the point with coordinates $(0;0)$ to the point with coordinates $(a;b)$ moving along the sides of the unit squares.

Solution: Divide the path into unit paths, i.e. paths between two neighbouring points of integer coordinates. Denote the number of moves up by x_1, down by y_1, right by x_2 and left by y_2. The condition of the problem gives $x_1 + x_2 + y_1 + y_2 = n, x_1 - y_1 = b, x_2 - y_2 = a$. Thus, $y_1 = x_1 - b; x_2 = \dfrac{n+a+b}{2} - x_1; y_2 = \dfrac{n-a+b}{2} - x_1$. Since $|a| + |b| = |x_2 - y_2| + |x_1 - y_1| \le x_1 + x_2 + y_1 + y_2 = n$ we obtain that it is necessary to have $a + b \le n$ and $a + b \equiv n \pmod{2}$. Let us first fix the moves up and rigth. This can be done by $\binom{n}{\frac{n+a+b}{2}}$ ways. After that in these already fixed $\dfrac{n+a+b}{2}$ positions fix the moves up and finally, in the remaining $\dfrac{n-a-b}{2}$ positions fix the moves left. We obtain

$$\binom{n}{\frac{n+a+b}{2}} \sum_{i=b}^{\frac{n+a-b}{2}} \binom{\frac{n+a+b}{2}}{i} \binom{\frac{n-a-b}{2}}{\frac{n-a+b}{2}-i}.$$

The sum $\displaystyle\sum_{i=b}^{\frac{n+a-b}{2}} \binom{\frac{n+a+b}{2}}{i} \binom{\frac{n-a-b}{2}}{\frac{n-a+b}{2}-i}$ equals to the coefficient in front of $x^{\frac{n-a+b}{2}}$ in the expansion of $(1+x)^{\frac{n+a+b}{2}}(1+x)^{\frac{n-a-b}{2}} = (1+x)^n$. Therefore the sum equals $\binom{n}{\frac{n-a+b}{2}}$. The number of paths is:

$$\binom{n}{\frac{n+a+b}{2}} \binom{n}{\frac{n-a+b}{2}}.$$

Answer: If $|a|+|b|>n$ or $a+b \not\equiv n (\mathrm{mod}\ 2)$, the number of paths is 0, otherwise the number of paths is $\binom{n}{\frac{n+a+b}{2}} \binom{n}{\frac{n-a+b}{2}}$.

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Problem 8.1. Let a be a real parameter such that $0 \le a \le 1$. Prove that the solutions of the inequality

$$|x| + |ax + \frac{1}{2}| \le 1$$

form an interval of length greater or equal to 1.

Solution: If $a = 0$ then all numbers from the interval $\left[\frac{1}{2}, -\frac{1}{2}\right]$ of length 1 are solutions of the problem. If $a = 1$ then the inequality becomes $|x| + |x + \frac{1}{2}| \le 1$. The solutions of this inequality are all numbers from the interval $\left[-\frac{3}{4}, \frac{1}{4}\right]$, which is of length 1.

Further, assume that $0 < a < 1$ and let $x \ge 0$. In this case $ax + \frac{1}{2} > 0$ and therefore the solutions are $x \le \frac{1}{2(1+a)}$. Let $x < 0$. If $ax + \frac{1}{2} \ge 0$, i.e. $x \ge -\frac{1}{2}$, the solutions satisfy $x \ge -\frac{1}{2(1-a)}$. If $ax + \frac{1}{2} < 0$, i.e. $x < -\frac{1}{2a}$, the solutions satisfy $x \ge -\frac{1}{3(1+a)}$.

Let $a \le \frac{1}{2}$. Then $-\frac{1}{2(1-a)} \ge -\frac{1}{2a}$ and so all numbers $x \ge -\frac{1}{2(1-a)}$ are solutions of the inequality. Since $-\frac{1}{3(1+a)} > -\frac{1}{2a}$ it has no other solutions. Therefore all solutions form an interval $\left[-\frac{1}{2(1-a)}, \frac{1}{2(1+a)}\right]$, of length $\frac{1}{2(1+a)} + \frac{1}{2(1-a)} = \frac{1}{1-a^2} > 1$.

Let $a > \frac{1}{2}$. Then $-\frac{3}{2(1+a)} < -\frac{1}{2a}$ and the solutions are $x \ge$

$-\dfrac{3}{2(1+a)}\left(x \geq -\dfrac{1}{2a} \text{ or } x \in \left[-\dfrac{3}{2(1+a)}, -\dfrac{1}{2a}\right]\right)$. Therefore all solutions form an interval $\left[-\dfrac{3}{2(1+a)}, \dfrac{1}{2(1+a)}\right]$, of length $\dfrac{2}{1+a} \geq 1$.

Problem 8.2. Given a square $ABCD$ of side length 1. Point $M \in BC$ and point $N \in CD$ are such that the perimeter of $\triangle MCN$ is 2.

a) Find $\sphericalangle MAN$;

b) If P is the foot of the perpendicular from A to MN, find the locus of the point P.

Solution: a) Let K be a point on the extension of CB such that $BK = DN$. Then $MK = MB + BK = MB + DN = 1 - CM + 1 - CN = 2 - (CM + CN) = MN$ (using that $CM + CN + MN = 2$). Since $\triangle ABK \cong \triangle ADN$ we have that $AK = AN$. Therefore $\triangle AMN \cong \triangle AMK$ and it follows that $\sphericalangle MAN = \sphericalangle MAK$, i.e. $\sphericalangle MAN = \dfrac{1}{2} \sphericalangle KAN$. Furthermore $\sphericalangle KAB = \sphericalangle NAD$, which implies that $\sphericalangle KAN = \sphericalangle BAD = 90^\circ$. Therefore $\sphericalangle MAN = 45^\circ$.

b) It follows from $\triangle AMN \cong \triangle AMK$ that $\sphericalangle AMN = \sphericalangle AMK$. Thus, $\triangle APM \cong \triangle ABM$ and $AP = AB = 1$. Therefore P lies on a circle of center A and radius 1. Finally, the locus of P is an arc of a circle with center A and radius 1 excluding points B and D.

Problem 8.3. a) Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

holds true for any positive integer n.

b) Find the least integer $n, n > 1$ for which

$$\frac{1^2 + 2^2 + \cdots + n^2}{n}$$

is a perfect square.

Solution:

a) Direct verification shows that

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

for any integer n. Using this equality the proof is easily done by induction.

b) It follows from a) that $n(n+1)(2n+1) = 6m^2$. Since $2n+1$ is odd then $n+1$ is even, i.e. n is odd. Let $n = 2k-1$. Then $k(4k-1) = 3m^2$. Therefore $3/k$ or $3/4k-1$ i.e. $3/k$ or $3/k-1$. Let $k = 3l$. Then $l(12l-1) = m^2$. Since $(l, 12l-1) = 1$, we have $l = n^2$ and $12l-1 = v^2$. The latter equality is impossible (contradiction both modulo 4 and modulo 3). Let $k = 3l+1$. Then $(3l+1)(4l+1) = m^2$. Since $(3l+1, 4l+1) = 1$ we get that $3l+1 = u^2, 4l+1 = v^2, u > 1, v > 1$. Verifying consequently for $v = 3, 5, 7, 9, 11, 13, \ldots$ we obtain $l = 2, 6, 12, 20, 30, 42, 56, \ldots$. Thus, $3l+1 = 7, 19, 37, 61, 91, 127, 169$ (first perfect square). So, the least n is $2k-1$, where $k = 3l+1 = 3.56+1 = 169$ and therefore $n = 337$.

Problem 9.1. Let $f(x) = x^2 + 6ax - a$ where a is a real parameter.

a) Find all values of a for which the equation $f(x) = 0$ has at least one real root.

b) If x_1 and x_2 are the real roots of $f(x) = 0$ (not necessarily distinct) find the least value of the expression

$$A = \frac{9a - 4a^2}{(1+x_1)(1+x_2)} - \frac{70a^3 + 1}{(1-6a-x_1)(1-6a-x_2)}.$$

Solution: a) The discriminant of $f(x)$ is $D = 4(9a^2 + a)$. Therefore the equation $f(x) = 0$ has at least one real root iff $9a^2 + a \geq 0$, so giving $a \in \left(-\infty, -\frac{1}{9}\right] \cup [0, +\infty)$.

b) Since $f(x) = (x - x_1)(x - x_2)$ we obtain $f(-1) = (-1 - x_1)(-1 - x_2) = (1 + x_1)(1 + x_2) = (-1)^2 + 6a(-1) - a = 1 - 7a$ and $f(1-6a) = (1-6a-x_1)(1-6a-x_2) = (1-6a)^2 + 6a(1-6a) - a = 1 - 7a$. Therefore the denominators of the two fractions of A are equal to $1 - 7a$ and

$$A = \frac{-70a^3 - 4a^2 + 9a - 1}{1 - 7a} = \frac{(1-7a)(10a^2 + 2a - 1)}{1 - 7a} = 10a^2 + 2a - 1.$$

The quadratic function $g(a) = 10a^2 + 2a - 1$ attains its minimal value for $a = -\frac{1}{10} \notin \left(-\infty, -\frac{1}{9}\right] \cup [0, +\infty)$. Therefore the minimal value of A equals to the smallest of the numbers $g(0)$ and $g\left(-\frac{1}{9}\right)$, i.e. this value is $g\left(-\frac{1}{9}\right) = -\frac{89}{81}$.

Problem 9.2. Given a convex quadrilateral $ABCD$ such that $OA = \frac{OB.OD}{OC + OD}$, where O is the intersecting point of its diagonals. The circumcircle of $\triangle ABC$ intersects the line BD at point Q. Prove that CQ is the bisector of $\sphericalangle DCA$.

Solution: Let $CQ_1, Q_1 \in BD$ be the bisector of $\sphericalangle DCO$. Therefore

$$\frac{DQ_1}{Q_1O} = \frac{DC}{CO}$$

This equality, combined with the condition of the problem gives

$$OA(OC + OD) = OB.OD \iff OA\frac{Q_1O + DQ_1}{Q_1O}CD = OB.OD$$

$$\iff OA.CO.\frac{DO}{Q_1O} = OB.OD \iff OA.CO = Q_1O.OB.$$

Therefore the quadrilateral $ABCQ_1$ is cyclic. Thus, $Q_1 \equiv Q$.

Problem 9.3. Prove that there exist eight consecutive positive integers such that non of them can be written in the form $|7x^2 + 9xy - 5y^2|$, where x and y are integers.

Solution: Denote $f(x, y) = 7x^2 + 9xy - 5y^2$. Since $f(0, 0) = 0$, $f(0, 1) = 5$, $f(1, 0) = 7$, $f(1, 1) = 11$ and $f(0, 2) = 20$, first possible sequence of eight positive integers is $12, 13, \ldots, 19$. We shall prove that non of these integers can be written in the form $|7x^2+9xy-5y^2|$, where x and y are integers.

Let $f(x, y) = \pm k$, where x and y are integers. It suffices to prove that $f(x, y) = \pm k$ has no solutions for $k \in \{12, 13, \ldots, 19\}$.

Suppose k is even. Then x and y are also even. If $x = 2x_1$ and $y = 2y_1$ we get the equality $4f(x_1, y_1) = \pm k$ which implies that k is divisible by 4. Thus, $k \neq 14$ and $k \neq 18$. Let $k = 16$ and consider the equation $4f(x_1, y_1) = \pm 16$ which is equivalent to $f(x_1, y_1) = \pm 4$. As above we conclude that x_1 and y_1 are both even and let $x_1 = 2x_2$ and $y_1 = 2y_2$. Therefore $f(x, y) = \pm 1$. By analogy if $k = 12$ we get the equation $f(x, y) = \pm 3$.

Multiply the equation $f(x, y) = \pm k$ by 28 and write it in the form

$$(14x + 9y)^2 - 221y^2 = \pm 28k.$$

Since $221 = 13.17$ it is appropriate to consider modules 13 and 17.

Denote $t = 14x + 9y$ and consider all possibilities for k, i.e. $k \in \{1, 3, 13, 15, 17, 19\}$.

1) If $k = 13$ then $t^2 \equiv \pm 28.13 \pmod{17}$ and so $t^2 \equiv \pm 7 \pmod{17}$. Raising this congruence to 8-th power gives $t^{16} \equiv (\pm 7)^8 \equiv -1$

(mod 17) which is a contradiction to Fermat's theorem. Therefore $k \neq 13$.

2) If $k = 15$ then $t^2 \equiv \pm 28.15 \equiv \mp 5 \pmod{17}$. Raising in 8-th power gives $t^{16} \equiv (\mp 5)^8 \equiv -1 \pmod{17}$, a contradiction.

The cases $k = 17, 19, 1, 3$ are treated similarly.

Problem 10a.1. Let a and b be positive numbers such that both of the equations $(a+b-x)^2 = a-b$ and $(ab+1-x)^2 = ab-1$ have two distinct real roots. Prove that if the two bigger roots are equal then the two smaller roots are also equal.

Solution: Let a and b satisfy the condition of the problem. Both equations have two distinct roots iff $a > b$ and $ab > 1$. Since $a > 0$ it follows that $a^2 > ab > 1$, i.e. $a > 1$. Therefore $a > b > \dfrac{1}{a}$ and $a > 1$. It follows from the condition of the problem that $a + b + \sqrt{a-b} = ab + 1 + \sqrt{ab-1}$, i.e. $\sqrt{a-b} = (a-1)(b-1) + \sqrt{ab-1}$. If $a > b \geq 1$ then $\sqrt{a-b} \geq \sqrt{ab-1} \iff (a+1)(b-1) \leq 0 \iff b \leq 1$. Therefore $b = 1$. Conversely if $\dfrac{1}{a} < b \leq 1$ then $\sqrt{a-b} \leq \sqrt{ab-1} \iff (a+1)(b-1) \geq 0 \iff b \geq 1$, i.e. $b = 1$. Thus, the two bigger roots are equal iff $b = 1$ and $a > 1$. In this case the two smaller roots are also equal.

Problem 10a.2. Let A_1 and B_1 be points respectively on the sides BC and AC of $\triangle ABC$, $D = AA_1 \cap BB_1$ and $E = A_1B_1 \cap CD$. Prove that if $\sphericalangle A_1EC = 90^\circ$ and the points A, B, A_1, E lie on a circle, then $AA_1 = BA_1$.

Solution: Let $F = AE \cap BC$. We prove that EA_1 is the bisector of $\sphericalangle BEF$, which solves the problem. Indeed, then we have $\sphericalangle BAA_1 = \sphericalangle BEA_1 = \sphericalangle FEA_1 = \sphericalangle ABA_1$, i.e. $AA_1 = BA_1$.

Using Ceva's and Menelaus's theorems for $\triangle AA_1C$ we obtain $\frac{AD}{A_1D} \cdot \frac{A_1F}{CF} \cdot \frac{CB_1}{AB_1} = 1$, $\frac{AD}{A_1D} \cdot \frac{A_1B}{CB} \cdot \frac{CB_1}{AB_1} = 1$ and so

$$(1) \qquad \frac{A_1F}{A_1B} = \frac{CF}{CB}$$

Let B' be a point on the ray $A_1B^{\rightarrow}$ such that $\sphericalangle B'EA_1 = \sphericalangle A_1EF$. Then EA_1 is the bisector of $\sphericalangle B'EF$, and since $\sphericalangle A_1EC = 90^\circ$ it follows that EC is the external bisector of the same angle. Therefore $\frac{A_1F}{A_1B'} = \frac{CF}{CB'}$ and it follows from (1) that $\frac{A_1B}{CB} = \frac{A_1B'}{CB'}$, i.e. $B = B'$ which completes the proof.

Problem 10a.3. Find all positive integers x and y such that

$$\frac{x^3 + y^3 - x^2y^2}{(x+y)^2}$$

is a nonnegative integer.

Solution: Let x and y be positive integers such that

$$z = \frac{x^3 + y^3 - x^2y^2}{(x+y)^2}$$

is a nonnegative integer. Substitute $a = x + y$ and $b = xy$ and write the expression in the form $b^2 + 3ab - a^2(a - z) = 0$. The discriminant of this quadratic equation $a^2(4a + 9 - 4z)$ is a perfect square, so $(4a + 9 - 4z) = (2t + 1)^2$. Thus, $a = t^2 + t + z - 2$ and from the equation for b we obtain that $b = a(t - 1)$. Since $t \geq 2$ we have $(x - y)^2 = a^2 - 4a(t - 1) < (a - 2(t - 1))^2$. From the other hand $a \geq t^2$ and therefore $a^2 - 4a(t - 1) \geq (a - 2(t - 1) - 2)^2$. Since $a^2 - 4a(t-1) \neq (a - 2(t-1) - 1)^2$ (the two numbers are of different parity) it follows that $(x - y)^2 = a^2 - 4a(t - 1) = (a - 2(t - 1) - 2)^2$.

Thus, $a = t^2$ i.e. $t + z = 2$ and so $t = 2, z = 0$, which implies $a = b = 4$. Therefore $x = y = 2$ and these are the only positive integers satisfying the condition of the problem.

Problem 10b.1. Solve the equation:

$$3^{\log_3(\cos x+\sin x)+\frac{1}{2}} - 2^{\log_2(\cos x-\sin x)} = \sqrt{2}.$$

Solution: Note that the admissible values are those x for which $\cos x + \sin x > 0, \cos x - \sin x > 0$. After simple calculations the equation becomes

$$(\sqrt{3} - 1)\cos x + (\sqrt{3} + 1)\sin x = \sqrt{2}.$$

This equation is equivalent to

$$\frac{\sqrt{6} - \sqrt{2}}{4}\cos x + \frac{\sqrt{6} + \sqrt{2}}{4}\sin x = \frac{1}{2}.$$

Since $\cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}$ and $\sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$ we have

$$\sin(x + 15^\circ) = \frac{1}{2}.$$

From the roots of the later equation only $x = 15^\circ \pm k360^\circ$ are admissible. Therefore the roots are $x = 15^\circ \pm k360^\circ$.

Problem 10b.2. Given a triangle ABC. Let M be such an interior point of $\sphericalangle BAC$ that $\sphericalangle MAB = \sphericalangle MCA$ and $\sphericalangle MAC = \sphericalangle MBA$. Analogously, let N be such an interior point of $\sphericalangle ABC$ that $\sphericalangle NBA = \sphericalangle NCB$ and $\sphericalangle NBC = \sphericalangle NAB$, and let P be such an interior point of $\sphericalangle ACB$ that $\sphericalangle PCA = \sphericalangle PBC$ and $\sphericalangle PCB = \sphericalangle PAC$. Prove that lines AM, BN and CP intersect in a point on circumcircle of $\triangle MNP$.

Solution: Suppose $\measuredangle C < 90^\circ$. The case $\measuredangle C > 90^\circ$ is treated similarly. Denote the intersecting point of line CP and line segment AB by C_1. Since PC_1 is the bisector of $\measuredangle APB(\measuredangle APC_1 = \measuredangle BPC_1 = \gamma = \measuredangle ACB)$ we get $\dfrac{AC_1}{BC_1} = \dfrac{AP}{BP}$. Triangles APC and BPC are similar which gives $\dfrac{AP}{CP} = \dfrac{AC}{BC}$ and $\dfrac{BP}{CP} = \dfrac{BC}{AC} \Longrightarrow \dfrac{AP}{BP} = \dfrac{AC^2}{BC^2} = \dfrac{b^2}{a^2}$. By analogy, if A_1 and B_1 are the intersecting points of AM and BN with BC and AC respectively then $\dfrac{BA_1}{A_1C} = \dfrac{c^2}{b^2}$ and $\dfrac{CB_1}{B_1A} = \dfrac{a^2}{c^2}$. Thus, $\dfrac{AC_1.BA_1.CA_1}{C_1B.A_1C.B_1A} = 1$, i.e. according to Ceva's theorem lines AM, BN and CP intersect in a point. Dnote this point by K. Since $\measuredangle APB = \measuredangle AOB = 2\gamma$, where O is the circumcenter of $\triangle ABC$ we get that A, P, O and B lie on a circle and PC_1 intersects the arc $\overset{\frown}{AB}$ at its midpoint C_1. Therefore OC_2 is a diameter of this circle and $\measuredangle OPC_2 = \measuredangle OPC = 90^\circ$. It easily follows now that M, N and P lie on a circle of diameter OK.

Problem 10b.3. Consider a set P of six four-letter words over an alphabet of two letters a and b. Denote by Q_P the set of all words over the same alphabet which do not contain as subwords the words from P. Prove that:

a) if Q_P is finite then it does not contain words of length ≥ 11;

b) there exists a set P such that Q_P is finite and it contains word of length 10.

Solution: a) Suppose that Q_P contains a word of length 11. We will show that Q_P contains words of any length. Let ω be a word of length 11. There are 9 subwords of ω of length 3. Since there are 8 distinct words of length 3 it follows that there exists a word α that appears as subword of ω twise. Let $\alpha = \alpha_1\alpha_2\alpha_3$ where $\alpha_i \in \{a, b\}$.

Consider the subword of ω obtained after the second appearance of α, i.e. consider the word

$$\ldots \alpha_1\alpha_2\alpha_3\gamma_1 \ldots \alpha_1\alpha_2\alpha_3$$

Write γ_1 after α_3. Obviously the new word does not contain subwords from P.

$$\ldots \alpha_2\alpha_3\gamma_1\gamma_2 \ldots \alpha_2\alpha_3\gamma_1$$

By analogy write γ_2 and so on. Thus, we find words of any length without subwords from P. This contradiction shows that there are no words of length ≥ 11 in Q_P.

b) A direct verification shows that the set

$$P = \{0000, 1000, 1001, 1010, 1101, 1111, \}$$

is such that Q_P is finite and $0001011100 \in Q_P$.

Problem 11.1. Prove that there exist unique numbers α and β such that $\cos\alpha = \alpha^2$, $\beta\mathrm{tg}\beta = 1$ and $0 < \alpha < \beta < 1$.

Solution: The function $f(x) = \cos x - x^2$ is continuous in the interval $(0;1)$, $f(0) = 1 > 0$ and $f(1) = \cos 1 - 1 < 0$. Therefore there exists α such that $\cos\alpha = \alpha^2$. In the interval $(0;1)$ the function $\cos x$ is decreasing one, whereas the function x^2 is increasing. Therefore there exists unique $\alpha \in (0;1)$ such that $\cos\alpha = \alpha^2$.

We show now that in the interval $[0;1]$ there exists a unique β such that $g(\beta) = 0$ where $g(x) = x\mathrm{tgx} - 1$. The function $g(x)$ is increasing one because $x\mathrm{tgx}$ is increasing as product of two increasing functions. Moreover $\mathrm{tg}\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1 < 0, \mathrm{g}(1) = \mathrm{tg}1 - 1 = \mathrm{tg}1 - \mathrm{tg}\frac{\pi}{4} > 0$ and the uniquennes of β follows. Since $\sin x < x$ for positive

x it follows that $g(\alpha) = \alpha \mathrm{tg}\alpha - 1 = \dfrac{\sin\alpha}{\alpha} - 1 < 0$. Further, $g(x)$ is increasing in the interval $[0;1]$ and so $\alpha < \beta$.

Problem 11.2. Let AA_1 and BB_1 be the altituds of obtuse non-isosceles $\triangle ABC$, and O and O_1 are circumcenters of $\triangle ABC$ and $\triangle A_1B_1C$ respectively. A line through C intersects the line segments AB and A_1B_1 at points D and D_1 respectively and E is point on the line OO_1 such that $\sphericalangle ECD = 90°$. Prove that $\dfrac{EO_1}{EO} = \dfrac{CD_1}{CD}$.

Solution: Let F and F_1 be the feet of the perpendiculars from O and O_1 to CD. Since $CO_1 \perp AB$ we have

(1)
$$\frac{EO_1}{EO} = \frac{CF_1}{CF} = \frac{CO_1 \cos \sphericalangle O_1CF_1}{CO \cos \sphericalangle OCF} = \frac{CO_1 \sin \sphericalangle BDC}{CO \sin(\sphericalangle DCB + \sphericalangle BAC)}.$$

From the other hand the Low of Sine's for $\triangle A_1B_1C, \triangle ABC$, $\triangle A_1D_1C$ and $\triangle BDC$ gives

(2)
$$CO_1 = \frac{A_1C}{2\sin \sphericalangle A_1B_1C} = \frac{A_1C}{2\sin \sphericalangle ABC}$$

(3)
$$CO = \frac{BC}{2\sin \sphericalangle BAC}$$

(4)
$$CD_1 = CA_1 \frac{\sin \sphericalangle B_1A_1C}{\sin(\sphericalangle B_1A_1C + \sphericalangle D_1CA_1)} = \frac{CA_1 \sin \sphericalangle BAC}{\sin(\sphericalangle BAC + \sphericalangle DCB)}$$

(5)
$$CD = \frac{BC \sin \sphericalangle ABC}{\sin \sphericalangle BDC}$$

It follows now from (1),(2),(3),(4) and (5) that

$$\frac{EO_1}{EO} = \frac{A_1C}{BC}\frac{\sin\sphericalangle BAC\sin\sphericalangle BDC}{\sin\sphericalangle ABC\sin(\sphericalangle BAC+\sphericalangle DCB)} = \frac{CD_1}{CD},$$

which completes the proof.

Problem 11.3. There are 2001 towns in a country every one of which is connected with at least 1600 towns by direct bus line. Find the largest n for which there exist n towns any two of which are connected by direct bus line.

Solution: Let S_1 snd S_1 be two towns connected by direct bus line. If k is the number of towns connected to both S_1 and S_2 by bus line then $(1599-k)+(1599-k)+k \leq 1999$ which implies that $k \geq 1198$. Therefore there exists town S_3 connected to both S_1 and S_2. Further, let k be the number of towns connected to all S_1, S_2 and S_3. Therefore $(1197-k)+(1598-k)+k \leq 1998$, so giving $k \geq 797$. Therefore there exists town S_4 connected to all S_1, S_2 and S_3. By analogy let k be the number of towns connected to all S_1, S_2, S_3 and S_4. We have $(796-k)+(1597-k)+k \leq 1997$, which implies $k \geq 396$. Therefore there exists S_5 connected to all S_1, S_2, S_3 and S_4. For the number n from the condition of the problem we obtain $n \geq 5$. We show that $n = 5$. For, number the towns by $S_1, S_2, \ldots, S_{2001}$ and connect S_k and S_m with direct line for all k and m for which $k \not\equiv m(mod 5)$. Since $\left[\frac{2001}{5}\right] = 400$, we have that each town is connested to 1600 or 1601 other towns, i.e. the condition of the problem is satisfied.

For arbitrary 6 towns the numbers of at least two are equal modulo 5 and therefore they are not connected to each other. Therefore $n < 6$ and so $n = 5$.

L National Mathematics Olympiad
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Problem 1. For which values of the real parameter a the equation

$\lg(4x^2-(8a-1)x+5a^2)+x^2+(1-2a)x+2a^2 = \lg(x^2-2(a+1)x-a^2)$

has exactly one root?

Solution: Write the equation in the form:

$$\lg(4x^2-(8a-1)x+5a^2)+\frac{4x^2-(8a-1)x+5a^2}{3} =$$
$$\lg(x^2-2(a+1)x-a^2)+\frac{x^2-2(a+1)x-a^2}{3}.$$

Since $\lg t + \dfrac{t}{3}$ is an increasing function this equality is equivalent to

$$\left| \begin{array}{l} 4x^2-(8a-1)x+5a^2 = x^2-2(a+1)x-a^2 \\ x^2-2(a+1)x-a^2 > 0 \end{array} \right.$$

The equation in the above system is equivalent to $f(x) = 0$ where $f(x) = x^2-(2a-1)x+2a^2$. Substitution $x^2 = 2ax-x-2a^2$ in the inequality gives $x < -a^2$. Therefore we have to find those a for which the equation $f(x) = x^2-(2a-1)x+2a^2 = 0$ has exactly one root such that $x < -a^2$. There are three cases to be considered:

1. $f(x) = 0$ has two roots and $-a^2$ is between the roots. In this case $f(-a^2) < 0 \Rightarrow a^2(a+1)^2 < 0$, which is impossible.

2. $f(x) = 0$ has two roots one of which equals $-a^2$ and the second one is less than $-a^2$. in this case $f(-a^2) = 0$ and so $a = -1$

or $a = 0$. When $a = -1$ we obtain $x_1 = -2, x_2 = -1$, which implies that $a = -1$ is a solution. If $a = 0$ we get $x_1 = -1, x_2 = 0$, which implies that $a = 0$ is also a solution.

3. $f(x) = 0$ has one root which is less than $-a^2$. Since $D = 0$ we get that $-4a^2 - 4a + 1 = 0$, and so $a_{1/2} = \dfrac{-1 \pm \sqrt{2}}{2}$. It is easy to check that for both values of a the corresponding root is less than $-a^2$.

Therefore the solutions are $a = -1, a = 0, a = \dfrac{-1 \pm \sqrt{2}}{2}$.

Problem 2. Diagonals AC and BD of a cyclic quadrilateral $ABCD$ intersect in a point E. Prove that if $\measuredangle BAD = 60^\circ$ and $AE = 3CE$ then the sum of two of the sides of the quadrilateral equals the sum of the other two.

Solution: Set $\measuredangle ABD = x$ and $\measuredangle CBD = y$. From the Low of Sine's we obtain

$$\frac{AE}{AB} = \frac{\sin x}{\sin(120^\circ + y - x)}, \frac{CE}{BC} = \frac{\sin y}{\sin(60^\circ + x - y)}$$

and $\dfrac{AB}{BC} = \dfrac{\sin(120^\circ - x)}{\sin(60^\circ - y)}$. Therefore $3 = \dfrac{AE}{CE} = \dfrac{\sin x . \sin(120^\circ - x)}{\sin y . \sin(60^\circ - y)}$.

Hence $3(\cos(2y - 60^\circ) - \cos 60^\circ) = \cos(2x - 120^\circ) - \cos 120^\circ$, i.e. $1 - \cos(2x - 120^\circ) = 3(1 - \cos(2y - 60^\circ))$ and so $\sin^2(x - 60^\circ) = 3\sin^2(y - 30^\circ)$. Therefore $\sin(x - 60^\circ)\cos 60^\circ = \pm \cos 30^\circ \sin(y - 30^\circ)$, i.e. $\sin x - \sin(120^\circ - x) = \pm(\sin y - \sin(60^\circ - y))$. Again fom the Low of Sine's we get $AD - AB = \pm(CD - BC)$, i.e.

$$AD + BC = AB + CD \text{ or } AD + CD = AB + BC.$$

Problem 3. Find the least positive integer n such that there exists a group of n people such that:

1. There is no group of four every two of which are friends;

2. For any choice of $k \geq 1$ people among which there are no friends there exists a group of three among the remaining $n-k$ every two of which are friends.

Solution: Consider a group of 7 people $A_1, A_2, \ldots, A_7$, such that $A_i, i = 1, 2 \ldots, 7$ is not friend only with A_{i+1} and A_{i-1} (we set that $A_8 = A_1$ and $A_0 = A_7$). It is easily seen that there are no four every two of which are friends. Also, for any choice of $k \geq 1$ people (in this case k is 1 or 2) every two of which are not friends there exists a group of three among remaining $7-k$ people every two of which are friends. Therefore $k \leq 7$.

We prove that for any group of 6 which satisfy the condition 1) it is possible to choose a group of $k \geq 1$ every two of which are not friends such that among remaining $6-k$ a group of three every two of which are friends does not exist.

Denote the people by $A_1, A_2, \ldots A_6$. If some of them is friend with the other 5 (wlog suppose this is A_1) it is clear that there are no three among $A_2, A_3, \ldots A_5$ every two of which are friends. Therefore the choice of A_1 solves the problem.

Suppose one of then is friend with exactly four others (wlog assume A_1 is friend with A_2, A_3, A_4 and A_5) Then the choice of A_1 and A_6 solves the problem.

Therefore each person has $0, 1, 2$ or 3 friends. It is obviuos that there exists a group of three any two of which are friends (otherwise the problem is trivial). Assume that the group A_1, A_2 and A_3 has this property. Wlog A_1 and A_4 are not friends. Therefore among A_2, A_3, A_5 and A_6 there is a group of three any two of which are friends. If this is A_5 and A_6 together with one of A_2 or A_3 then there exists a person who is friend with at least four others, a contradiction. Therefore wlog suppose that this group is A_2, A_3 and A_5. Since A_1 and A_5 are not friends and since among the others there is a group of three friends we obtain that either A_2 or A_3 has at least four friends which is a contradiction.

Therefore $k = 7$.

Problem 4. Given a right triangle ABC with hypotenuse AB. A point D distinct from A and C is chosen on the ray $AC^{\rightarrow}$ such that the line through incenter of $\triangle ABC$ parallel to the bisector of $\sphericalangle ADB$ is tangent to the incircle of $\triangle BCD$. Prove that $AD = BD$.

Solution: First we show that C lies on the line segment AD. For, suppose the contrary, i.e. $\sphericalangle ADB > 90°$. Denote the tangential point of incircle $k(I, r)$ of $\triangle BCD$ and the side BD by P and the tangential point of k and the line through the center J parallel to bisector l of $\sphericalangle ADB$ by T. Since $l \perp DI$ we get $T \in DI$. Thus, $\sphericalangle IJT = \dfrac{\sphericalangle CBD}{2} = \sphericalangle IBP$ and since $IT = r = IP$ we have that $\triangle IJT$ and $\triangle IBP$ are congruent. In particular $IJ = IB$ which implies that $\sphericalangle BJC < 90°$, a contradiction. Let A' be such point on the ray $DA^{\rightarrow}$ that $DA' = DB$ and E is the midpoint of $A'B$. Denote the tangential point of incircle of $\triangle A'BC$ by F and its incenter by J'. Then we have

$$\begin{aligned}
&EF = |A'E - A'F| = \frac{1}{2}|A'B - (A'B + A'C - BC)| = \\
&= \frac{1}{2}|BC - A'C| = \frac{1}{2}|BC - (A'D - CD)| = \\
&\tfrac{1}{2}|BC + CD - BD| = r.
\end{aligned}$$

Therefore the line $J'F$ parallel to l is tangent to k. This implies that $J' \in JF$. From the other hand $J' \in JC$ and since JC is not parallel to $JF(\sphericalangle ACJ = 45^\circ > \frac{1}{2}\sphericalangle ADB = \sphericalangle ADI)$, we get $J = J'$. Thus, $\sphericalangle ABJ = \sphericalangle CBJ = \sphericalangle A'BJ$, i.e. $A = A'$ which completes the proof.

Problem 5. Find all triples of positive integers (a, b, c) such that $a^3 + b^3 + c^3$ is divisible by a^2b, b^2c and c^2a.

Solution: If $d = gcd(a, b, c)$ it is easy to see that $\left(\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right)$ is also a solution of the problem. Thus, it suffices to find all triples (a, b, c) such that $gcd(a, b, c) = 1$. Let $gcd(a, b) = s$ and suppose $s > 1$. If p is a prime divisor of s then p divides a; p divides b and p divides $a^3 + b^3 + c^3$. Hence p divides c^3 and so p divides c. This is a contradiction to $gcd(a, b, c) = 1$. Therefore $gcd(a, b) = 1$ and by analogy $gcd(a, c) = gcd(b, c) = 1$. It follows from $a^2/(a^3 + b^3 + c^3), b^2/(a^3 + b^3 + c^3), c^2/(a^3 + b^3 + c^3)$ and $gcd(a^2, b^2) = gcd(a^2, c^2) = gcd(b^2, c^2) = 1$ that $a^3 + b^3 + c^3$ is divisible by $a^2b^2c^2$. In particular $a^3 + b^3 + c^3 \geq a^2b^2c^2$. Wlog we may assume that $a \leq b \leq c$. Then $3c^3 \geq a^3 + b^3 + c^3 \geq a^2b^2c^2 \Rightarrow c \geq \frac{a^2b^2}{3}$. Suppose $a > 1$. Then $\frac{a^2b^2}{3} > b^2 \geq b^2 + a(a-b) = a^2 - ab + b^2$ and the inequality (1), $c > a^2 - ab + b^2$ holds. From the other hand since $b \geq a \geq 2$ we have that $b^2 \geq 2b \geq a + b$ and since $c > b^2$ we obtain (2), $c > a + b$. Now (1) and (2) give $\frac{(a^2 - ab + b^2)(a + b)}{c^2} < 1 \iff \frac{a^3 + b^3}{c^2} < 1$, which is impossible since $\frac{a^3+b^3}{c^2}$ is an integer. Thus, $a = 1$. In this case we have that $1 + b^3 + c^3$ is divisible by b^2c^2. Consider the following cases:

1) if $b = c$ it is easy to see that $b = c = 1$. Indeed, $(1, 1, 1)$ is a solution of the problem.

2) if $b = 1$ we obtain the same solution.

3) if $b = 2$ we obtain $c = 3$. Triple $(1, 2, 3)$ is a solution of the

problem.

Supose now that $c > b \geq 3$. Since $1 + b^3 + c^3 \geq b^2c^2$, it follows that $2c^3 > 1+b^3+c^3$ and so $2c > b^2$ or $c > \frac{b^2}{2}$. It follows from $2c > b^2$ that $2c > b^2 - b + 1$, i.e. (3), $\frac{b^2-b+1}{c} < 2$. From the other hand when $b \geq 5$ the inequalities $\frac{c}{2} > \frac{b^2}{4} > b+1 \Rightarrow (4)\frac{b+1}{c} < \frac{1}{2}$ hold. Multiplying (3) and (4) gives $\frac{b^3+1}{c^2} < 1$, which is impossible since this number is an integer. Direct verificstion shows that when $b = 3$ or $b = 4$ we get no new solutions. Therefore all solutions of the problem are triples (k, k, k) and $(k, 2k, 3k)$ (and its permutations) for arbitrary positive integer k.

Problem 6. Given a pack of 52 cards. The following opperations are allowed:

1. Swap the first two cards;
2. Put the first card on the last place.

Prove that using these opperations one can order the cards in arbitrary manner.

Solution: First we show that it is possible to change any two neighbouring cards. Indeed, using 2) we can move the two cards on the first two positions. After that apply 1) and again 2) to put all remaining cards on their initial positions.

Consider two arbitrary cards a_i and a_{i+j}. We can change these two cards by changing a_i and a_{i+1} then a_i and a_{i+2} and so on, up to a_i and a_{i+j}. After that we change a_{i+j} and a_{i+j-1} and so on, up to a_{i+j} and a_{i+1}.

Since we can change any two cards we can order the cards in arbitrary manner.

L National Mathematics Olympiad
4th round, 19-20 May 2001

Problem 1. Consider the sequence $\{a_n\}$ such that $a_0 = 4$, $a_1 = 22$ and $a_n - 6a_{n-1} + a_{n-2} = 0$ for $n \geq 2$. Prove that there exist sequences $\{x_n\}$ and $\{y_n\}$ of positive integers such that $a_n = \dfrac{y_n^2 + 7}{x_n - y_n}$ for any $n \geq 0$.

Solution: Let $x_n = \dfrac{a_n + a_{n-1}}{2}$, $x_0 = 3$ and $y_n = \dfrac{a_n - a_{n-1}}{2}$, $y_0 = 1$. Then $x_n = 3x_{n-1} + 4y_{n-1}$ and $y_n = 2x_{n-1} + 3y_{n-1}$. Since $a_n = x_n + y_n$, it suffices to prove that $x_n + y_n = \dfrac{y_n^2 + 7}{x_n - y_n}$, i.e. $x_n^2 = 2y_n^2 + 7$. We prove this by induction. The assertion is obvious for $n = 0$. Suppose that $x_{n-1}^2 = 2y_{n-1}^2 + 7$. Writing this equality in the form $(3x_{n-1} + 4y_{n-1})^2 = 2(3y_{n-1} + 2x_{n-1})^2 + 7$ gives $x_n^2 = 2y_n^2 + 7$, which completes the proof.

Problem 2. Given nonisosceles triangle ABC. Denote the tangential points of the inscribed circle k of center O with the sides AB, BC and CA by C_1, A_1 and B_1 respectively. Let $AA_1 \cap k = A_2$, $BB_1 \cap k = B_2$ and let A_1A_3, B_1B_3 be bisectors in triangle $A_1B_1C_1$ $(A_3 \in B_1C_1, B_3 \in A_1C_1)$. Prove that:

a) A_2A_3 is bisector of $\sphericalangle B_1A_2C_1$;

b) if P and Q are the intersecting points of circumcircles of triangle $A_1A_2A_3$ and triangle $B_1B_2B_3$ then the point O lies on the line PQ.

Solution: a) From the Low of Sine's we obtain

$$\frac{AB_1}{AA_1} = \frac{\sin \sphericalangle B_1A_1A_2}{\sin \gamma_1}, \quad \frac{AC_1}{AA_1} = \frac{\sin \sphericalangle C_1A_1A_2}{\sin \beta_1},$$

where $\beta_1 = \sphericalangle A_1B_1C_1$, $\gamma_1 = A_1C_1B_1$. Since $AB_1 = AC_1$ we get

$$\frac{A_2B_1}{A_2C_1} = \frac{\sin \sphericalangle B_1A_1A_2}{\sin \sphericalangle C_1A_1A_2} = \frac{\sin \gamma_1}{\sin \beta_1} = \frac{A_1B_1}{A_1C_1} = \frac{A_3B_1}{A_3C_1},$$

which implies that A_2A_3 is the bisector of $\sphericalangle B_1A_2C_1$.

b) Let $M = AA_1 \cap BB_1$. It follows from $MA_1 \,.\, MA_2 = MB_1 \,.\, MB_2$ that M lies on the line PQ. Therefore it suffices to prove that $OM \perp O_1O_2$, where O_1 and O_2 are circumcenters of $\triangle A_1A_2A_3$ and $\triangle B_1B_2B_3$. It follows from a) that the diametrically opposite point of A_3 in k_1 – the circumcircle of $\triangle A_1A_2A_3$, lies on the line B_1C_1. Therefore $O_1 \in B_1C_1$. Moreover $\sphericalangle B_1B_3A_1 = \sphericalangle CA_1A_3 = \gamma + \dfrac{\alpha_1}{2}$. It easily follows now that O_1 coinsides with the intersecting point of B_1C_1 and BC. Let $OO_1 \cap A_1A_2 = N$ and $OO_2 \cap B_1B_2 = K$. It follows from $\triangle OA_1O_1$ that $ON \,.\, OO_1 = OA_1^2 = r^2$ and by analogy $OK \,.\, OO_2 = r^2$ where r is the radius of k. Since O, N, M and K lie on the circle k_3 of diameter OM we have that the line O_1O_2 is the image of k_3 by inversion of center O and degree r^2, i.e. $OO_1 \perp OM$.

Problem 3. For a permutation $a_1, a_2, \ldots a_n$ of the numbers $1, 2 \ldots n$ it is allowed to change the places of any two consecutive blocks, i.e. from

$$a_1, \ldots, a_i, \underbrace{a_{i+1}, a_{i+2} \ldots, a_{i+p}}_{A}, \underbrace{a_{i+p+1}, a_{i+p+2} \ldots, a_{i+q}}_{B}, a_{i+q+1}, \ldots, a_n$$

by replacing A and B one can obtain

$$a_1, \ldots, a_i, \underbrace{a_{i+p+1}, a_{i+p+2} \ldots, a_{i+q}}_{B}, \underbrace{a_{i+1}, a_{i+2} \ldots, a_{i+p}}_{A}, a_{i+q+1}, \ldots, a_n.$$

Find the least number of such changes after which from $n, n-1, \ldots 1$ one can obtain $1, 2, \ldots n$.

Solution: Call the change of two blocks a move. We shall prove that the least number of moves such that from $n, n-1, \ldots 1$ one can obtain $1, 2, \ldots n$ is $\left\lceil \frac{n+1}{2} \right\rceil$.

Consider the number of pairs a_i, a_{i+1} such that $a_i < a_{i+1}$. This number is 0 in the initial permutation $n, n-1, \ldots 2, 1$ and is $n-1$ in the final permutation $1, 2, \ldots, n-1, n$.

First we show that a move changes the number of pairs a_i, a_{i+1} such that $a_i < a_{i+1}$ at most by two. For, consider a move of the blocks $a \ldots, b$ and $c, \ldots d$ of the permutation

$$\ldots p, a \ldots, b, c, \ldots d, q \ldots$$

As a result we obtain

$$\ldots p, c \ldots, d, a, \ldots b, q \ldots$$

It is clear that at most three pairs can change the ordering. Suppose the elements in all three pairs change the ordering, i.e. from $p > a$, $b > c$ and $d > q$ we get $p < c$, $d < a$ and $b < q$. Adding the first three inequalities gives $p + b + d > a + c + q$ and adding the last three implies $p + b + d < a + c + q$, a contradiction. Therefore a move changes the number of pairs a_i, a_{i+1} for which $a_i < a_{i+1}$ at most by two. It is easily seen that the first and the last moves change this number by one. Therefore if x is desired number we have $2 + 2(x-2) \geq n-1$ which implies $x \geq \left\lceil \frac{n+1}{2} \right\rceil$. It remaines to find a sequence of $\left\lceil \frac{n+1}{2} \right\rceil$ moves such that from $n, n-1, \ldots 2, 1$ we get $1, 2, \ldots, n-1, n$. Let n be even number, i.e. $n = 2k$. Number the positions from right to left by $1, 2 \ldots n$. First change the places of the blocks from positions $1, 2 \ldots, k-1$ and $k, k+1$. Next, change the places of the blocks from positions $2, 3 \ldots, k$ and $k+1, k+2$. Third, change the blocks $3, 4 \ldots, k+1$ and $k+2, k+3$ and so on. On the k-th step change the blocks from positions $k, k+1 \ldots, 2k-2$ and $2k-1, 2k$. As a result we obtain $k+1, k+2, \ldots, 2k, 1, 2, \ldots, k$. The last change is $1, 2, \ldots, k$ and $k+1, k+2 \ldots, 2k$. When n is odd, i.e. $n = 2k+1$ the first change is of the blocks from positions $1, 2, \ldots, k$ and $k+1, k+2$ after that $2, 3, \ldots, k+1$ and $k+2, k+3$ and so on. The last $k+1$ change is $1, 2 \ldots k$ and $k+1, \ldots, 2k$ and we obtain $1, 2, \ldots, 2k+1$.

Problem 4. Let $n \geq 2$ be fixed integer. At any point with integer coordinates $(i,\, j)$ we write $i + j$ modulo n. Find all pairs $(a; b)$ of positive integers such that the rectangle with vertices $(0,\, 0)$, $(a,\, 0)$, $(a,\, b)$, $(0,\, b)$ has the following properties:

1) the remainders $0,\, 1,\, \ldots, n-1$ written in its interior points appear equal number of times;

2) the remainders $0,\, 1,\, \ldots, n-1$ written on its boundary appear equal number of times.

Solution: Let p_i and s_i for $i = 0,\, 1,\, \ldots,\, n-1$ be the number of residues i on the sides and in the interior of the rectangle respectively. If $a' = a + kn$ for an integer k then the corresponding numbers for $(a',\, b)$ are $p_i' = p_i + 2k$ and $s_i' = s_i + k(b-1)$.

Therefore if $(a,\, b)$ is a solution then $(a + kn,\, b)$ and by analogy $(a,\, b + ln)$ are also solutions. Thus, wlog we may suppose that $1 \leq a,\, b \leq n$.

If $n = 2$ then all possible values of $(a,\, b)$ are

$$(a,\, b) = (1,\, 1),\ (1,\, 2),\ (2,\, 1),\ (2,\, 2).$$

Pair $(2,\, 2)$ is not a solution because there is unique interior point $(s_0 = 1,\ s_1 = 0)$ for the rectangle. The remaining three cases give solutions.

Therefore for $n = 2$ solutions are all pairs $(a; b)$ where either a or b is an odd number.

Let $n > 2$ and suppose $(a,\, b)$ is a solution of the problem for which $1 \leq a,\, b < n$. In the vertices $(0,\, 0)$ and $(a,\, b)$ are written 0 and $a + b$ modulo n respectively, whereas all residues $1,\, 2,\, \ldots,\, a + b - 1$ appear on the boundary even number of times. (once on the boundary $(1,\, 0),\, \ldots,\, (a,\, 0),\, (a,\, 1),\, \ldots,\, (a,\, b-1)$ and once on the boundary $(0,\, 1),\, \ldots,\, (0,\, b),\, (1,\, b),\, \ldots,\, (a-1,\, b)$). If n does not divide $a + b$ then 0 and the residue of $a + b$ appear odd number of times whereas at least one of the remaining residues $(n > 2)$ appears even number of times. Therefore n divides $a + b$ and so $a + b = n$ or $a + b = 2n$. When $a + b = 2n$ we have that $a = b = n$. This pair is not a solution since the number of interior points is $(n-1)^2$ and is not divisible by n. It remains to consider the case $a + b = n$. If $a > 1$ and $b > 1$ then the rectangle has interior points. For any such point $(i,\, j)$ we have $0 < i < a$, $0 < j < b$, $0 < i + j < a + b = n$ and therefore the residue modulo n is not 0. Therefore $a = 1$, $b = n - 1$ (or $a = n - 1$, $b = 1$) which is a solution. Therefore for $n > 2$ all solutions are $a = 1 + kn$, $b = n - 1 + ln$ and $a = n - 1 + kn$, $b = 1 + ln$, where $k,\, l = 0,\, 1,\, 2,\, \ldots$.

Problem 5. Find all real numbers t for which there exist real numbers x, y, z such that

$$3x^2 + 3xz + z^2 = 1,\ 3y^2 + 3yz + z^2 = 4,\ x^2 - xy + y^2 = t.$$

Solution. We shall prove that the answer is $t \in \left(\dfrac{3 - \sqrt{5}}{2}, 1 \right)$.

Let x, y, t, α satisfy the conditions of the problem. Consider four points A, B, C, O in the plane such that $AO = x, BO = y, CO = \dfrac{z}{\sqrt{3}}$ and $\sphericalangle AOB = 60^\circ, \sphericalangle BOC = \sphericalangle COA = 150^\circ$. By the cosine theorem it follows that $t > 0, AB = \sqrt{t}, BC = \dfrac{2}{\sqrt{3}}, CA = \dfrac{1}{\sqrt{3}}$. Since $\sphericalangle ACB < \sphericalangle AOB = 60^\circ$ and $\sphericalangle BAC < \sphericalangle BOC = 150^\circ$, we have

$$(1) \qquad \frac{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2 - (\sqrt{t})^2}{2\frac{1}{\sqrt{3}}\frac{2}{\sqrt{3}}} = \cos \sphericalangle ACB > \frac{1}{2},$$

$$(2) \qquad \frac{(\sqrt{t})^2 + \left(\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{2}{\sqrt{3}}\right)^2}{2\sqrt{\alpha}\frac{2}{\sqrt{3}}} = \cos \sphericalangle BAC > -\frac{\sqrt{3}}{2}.$$

These inequalities are equivalent to $t < 1$ and $t + \sqrt{t} - 1 > 0$, i.e. $t \in \left(\dfrac{3-\sqrt{5}}{2}, 1\right)$.

Conversely, let $t \in \left(\dfrac{3-\sqrt{5}}{2}, 1\right)$. Then we can construct a triangle ABC such that $AB = \sqrt{t}, BC = \dfrac{2}{\sqrt{3}}, CA = \dfrac{1}{\sqrt{3}}$. By (1) and (2) it follows that $\sphericalangle ACB < 60^\circ$ and $\sphericalangle ABC < \sphericalangle BAC < 150^\circ$. Let k_A be the circle through B and C such that the arc BC, lying in the half-plane (with respect to BC) containing the point A, is equal to 60°. Denote by k_B the analogous circle trough A and C. Then it is easy to see that the second common point O of k_A and k_B lies in the interior of $\triangle ABC$. (Indeed, assume the contrary. If O lies in $\sphericalangle ACB$, then $\sphericalangle AOB = \sphericalangle AOC + \sphericalangle BOC = 150^\circ + 150^\circ = 300^\circ$ which is impossible. If O lies in the opposite angle of $\sphericalangle ACB$, then $\sphericalangle AOB = \sphericalangle AOC + \sphericalangle BOC = 30^\circ + 30^\circ = 60^\circ$ which contardicts to the inequalities $\sphericalangle AOB < \sphericalangle ACB < 60^\circ$. The other cases for the position of O can be rejected in the same manner.) Hence $\sphericalangle AOB = 360^\circ - \sphericalangle AOC - \sphericalangle BOC = 360^\circ - 150^\circ - 150^\circ = 60^\circ$. Set $x = AO, y = BO, z = CO\sqrt{3}$. Applying the cosine theorem for triangles AOB, BOC and COA it follows that x, y, z, t satisfy the conditions of the problem.

Problem 6. Given the equation

$$(p+2)x^2-(p+1)y^2+px+(p+2)y=1,$$

where p is fixed prime number of the form $4k+3$. Prove that:

a) If (x_0, y_0) is a solution of the equation where x_0 and y_0 are positive integers, then p divides x_0;

b) The given equation has infinitely many solutions (x_0, y_0) where x_0 and y_0 are positive integers.

Solution: a) Set $y-1=z$ and write the equation in the form

$$x^2=(z-x)((p+1)(z+x)+p). \tag{3}$$

If $z-x$ and $(p+1)(z+x)+p$ are relatively prime then they are perfect squares which is impossible since the second one is of the form $4k+3$. Let q be a common divisor of these numbers. It follows from (3) that q/x and so q/z. Since $q/(p+1)(z+x)+p$ we have q/p giving $q=p$ which completes the proof.

b) It suffices to prove that (3) has infinitely many solutions in positive integers. Let $x=px_1$ and $z=pz_1$. Then $x_1^2=(z_1-x_1)((p+1)(z_1+x_1)+1)$ and therefore there exist positive integers a and b such that $z_1-x_1=a^2$, $x_1=ab$ and $(p+1)(z_1+x_1)+1=b^2$. It follows now that

$$(p+2)b^2-(p+1)(a+b)^2=1. \tag{4}$$

Let $\left(\sqrt{p+2}+\sqrt{p+1}\right)^{2k+1}=m_k\sqrt{p+2}+n_k\sqrt{p+1}$ for any $k=0,1,\ldots,$ where m_k and n_k are positive integers. It is obvious that

$$(\sqrt{p+2}-\sqrt{p+1})^{2k+1}=m_k\sqrt{p+2}-n_k\sqrt{p+1}$$

and after multiplying we obtain that $(p+2)m_k^2-(p+1)n_k^2=1$, i.e. $b=m_k$ and $a+b=n_k$ are solutions of (4). Hence, $x=pm_k(n_k-m_k)$ and

$z = pn_k(n_k - m_k)$ are solutions of (3). The assertion of b) follows from the fact that both sequences $m_1, m_2, \ldots$ and $n_1, n_2, \ldots$ are strictly increasing.

SPRING MATHEMATICAL COMPETITION

1995

Grade 8.

Problem 1. Find all values of a, for which the system

$$\left| \begin{array}{rcl} x+4|y| & = & |x| \\ |y|+|x-a| & = & 1 \end{array} \right.$$

has exactly two solutions.

Solution. Let (x,y) be a solution with $x \geq 0$. Then $y=0$ and $|x-a|=1$, i.e. $x=a \pm 1$. It is obvious that when $a \geq 1$ the system has two solutions with $x \geq 0$, namely $(a-1,0)$, $(a+1,0)$. When $-1 \leq a < 1$ the system has only one solution $(a+1,0)$ with $x \geq 0$. Let $x<0$. Then $|y| = -\dfrac{x}{2}$ and consequently $|x-a| = 1+\dfrac{x}{2}$. Since $|x-a| \geq 0$, then $1+\dfrac{x}{2} \geq 0$ or $x \geq -2$. We have $x-a = 1+\dfrac{x}{2}$ and $x=2(a+1)$, $x-a=-1-\dfrac{x}{2}$. From here $x=\dfrac{2}{3}(a-1)$. From $-2 \leq 2(a+1) < 0$ we get$-2 \leq a < -1$ and from $-2 \leq \dfrac{2}{3}(a-1) < 0$ we get $-2 \leq a < 1$. Obviously if (x,y) is a solution of the system with $x<0$, then $y \neq 0$ and therefore $(x,-y)$ is a solution too. Thus, when $a<-2$ the system has no solution. When $a=-2$ we get $x=2(a+1)=-2$ and $x=\dfrac{2}{3}(a-1)=-2$, i.e. the system has exactly two solutions $(-2,\pm 1)$. If $1>a>-2$, then $2(a+1) \neq \dfrac{2}{3}(a-1)$ and the two values of x give solutions, thus we get four solutions with $x<0$.

Therefore the system has two solutions only when $a \geq 1$, namely $(a-1,0)$ and $(a+1,0)$ and when $a=-2$, namely $(-2,\pm 1)$.

Problem 2. Let M be the midpoint of the side BC of the parallelogram $ABCD$, N be the common point of AM and BD, while P be the common point of AD and CN. Prove that

a) $AP=AD$;

b) $CP=BD$ iff $AB=AC$.

Solution. a) Consider $\triangle ABC$. The lines AM and BD are medians and thus N is the center of gravity. If Q is the intersection point of CP and AB, then Q is the midpoint of AB. It's easy to see that $\triangle APQ \cong \triangle BCQ$ (Figure. 1), from where $AP=BC=AD$.

b) Let $AB=AC$. Then $\triangle ABC$ is isosceles and AM is a median in it. It is easy to see that $\triangle NBC$ is isosceles and $BN=CN$. Analogously NA is a median and an altitude in $\triangle PND$, thus $PN=DN$, i.e. $PC=PN+CN=DN+BN=BD$ (Figure 2). Let $CP=BD$. Trough B we draw a line BF, parallel to CP. Obviously $PFBC$ is a parallelogram and therefore $\angle CPD = \angle BFP$. Since $CP=BF$, then $\triangle DBF$ is isosceles. Consequently

$\angle NDA = \angle BFP = \angle CPD$ and $DN = NP$. A is the midpoint of PD and NA is perpendicular to AD and BC. We get that AM is perpendicular to BC and we deduce from here that $\triangle ABC$ is isosceles, i.e. $AB = AC$.

Figure 1.

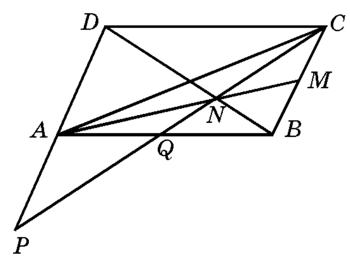
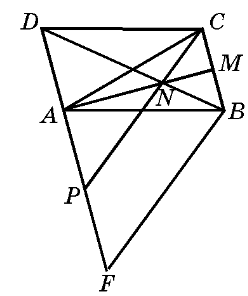


Figure 2.



Problem 3. A convex polygon with n sides, $n \geq 4$, is given. No four vertices of it lie on one and the same circle.

a) Prove that there exists a circle through 3 vertices of the polygon which contains the remaining vertices in its interior.

b) Prove that there exists a circle through 3 consecutive vertices of the polygon which contains the remaining vertices in its interior.

Solution. a) Let AB be a side of the polygon. All the vertices lie in one of the half-planes with respect to AB. The segment AB is seen under different angles from the vertices which are different from A and B. Let C be the vertex from which AB is seen under the smallest angle. The circle we are looking for is defined by A, B and C.

b) We shall use the following two lemmas.

Lemma 1. Let the segment AB be seen from the point X under the angle α and from the point Y under the angle β, where $0 < \alpha < \beta < 90°$. Then the radius of the circumcircle of $\triangle ABX$ is greater than the radius of the circumcircle of $\triangle ABY$.

Lemma 2. For each convex polygon there exist a side and a vertex from which this side is seen under an acute angle.

Proof. Let A_i, A_{i+1} and A_{i+2} be three consecutive vertices of the polygon (Figure 3). At least one of the angles $\alpha A_{i+1}A_iA_{i+2}$ and $\alpha A_iA_{i+2}A_{i+1}$ is acute. This is enough for the proof.

Figure 3.

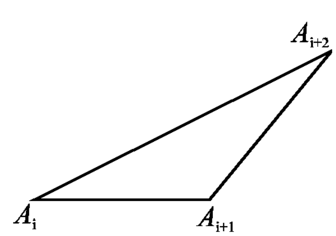
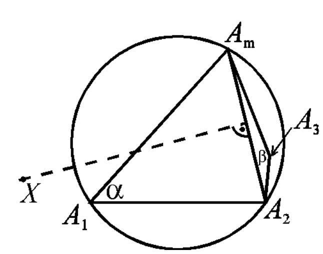


Figure 4.



Let now M be the set of all pairs, that are consisted of a side and a vertex from which this side is seen under an acute angle. Let (AB, C) be an element of M. Consider the circumcircle of $\triangle ABC$ and let N be the set of all such circles. We denote by k the circle in N with the biggest radius. Let k be the circumcircle of $\triangle A_1A_2A_m$, where A_1A_2 is a side of the polygon (Figure

4). We shall show that k is the circle we are looking for. Assume that there exists a vertex A_p which is outside k. Then $\angle A_1A_pA_2 < \angle A_1A_mA_2 < 90°$ and the circumcircle of $\triangle A_1A_2A_p$ is from N and its radius is greater than the radius of k (Lemma 1). We get a contradiction.

Consider one of the acute angles in $\triangle A_1A_2A_m$. Let $\alpha = \angle A_2A_1A_m$ be acute. We shall prove that $A_m = A_3$, which means that k passes through 3 consecutive vertices of the polygon. Assume that $A_m \neq A_3$. Then A_3 is situated in the way which is shown in the Figure 4. If $\beta = \angle A_2A_3A_m$, then $\alpha + \beta > 180°$, i.e. $180° - \beta < \alpha < 90°$. Thus $\angle A_2A_mA_3 < 90°$, which means that (A_2A_3, A_m) is an element of M. Let c be the circumcircle of $\triangle A_2A_3A_m$ and X be the intersection point of c and the segment bisector of A_2A_m. Since $\angle A_2XA_m = 180° - \beta < \alpha = \angle A_2A_1A_m$, then the radius of c is greater than the radius of k (Lemma 1). This is a contradiction.

Grade 9.

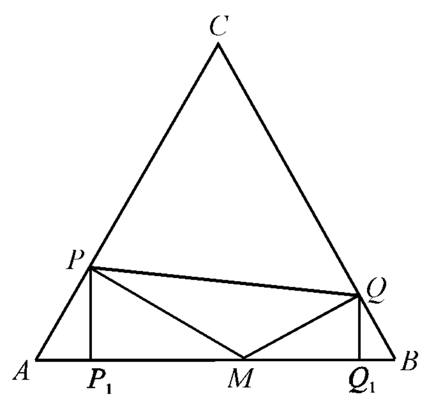
Problem 1. Let M be an arbitrary point on the side $AB = 1$ of the equilateral triangle ABC. The points P and Q are orthogonal projections of M on AC and BC, while P_1 and Q_1 are orthogonal projections of P and Q on AB.

a) Prove that $P_1Q_1 = \dfrac{3}{4}$.

b) Find the position of M for which the segment PQ is with the smallest length.

Solution. a) We have $S_{ABC} = S_{ACM} + S_{BCM} = \dfrac{1}{2}(AC.MP + BC.MQ) = \dfrac{1}{2}(MP + MQ)$ (Figure 5). On the other hand $S_{ABC} = \dfrac{\sqrt{3}}{4}$. Thus $MP + MQ = \dfrac{\sqrt{3}}{2}$. Now from the rectangular triangles P_1MP and MQ_1Q we evaluate $P_1M = \dfrac{\sqrt{3}}{2}MP$ and $MQ_1 = \dfrac{\sqrt{3}}{2}MQ$. From here $P_1Q_1 = P_1M + MQ_1 = \dfrac{\sqrt{3}}{2}(MP + MQ) = \dfrac{3}{4}$.

Figure 5.



b) The orthogonal projection of the segment PQ on AB is the segment P_1Q_1 and thus $PQ \geq P_1Q_1$. Therefore PQ is with minimal length when it is parallel to P_1Q_1. The last is true exactly when $AP = BQ$. We get that $\triangle AMP \cong \triangle BMQ$ and hence $AM = BM$. So, PQ is minimal when M is the midpoint of AB.

Problem 2. The quadratic function $f(x) = -x^2 + 4px - p + 1$ is given. Let S be the area of the triangle with vertices at the intersection points of the parabola $y = f(x)$ with the x-axis and the vertex of the same parabola. Find all rational p, for which S is an integer.

Solution. The discriminant of $f(x)$ is $D = 4(4p^2-p+1)$ and $D > 0$ for all real p. Consequently $f(x)$ has two real roots x_1 and x_2, i.e. $f(x)$ intersects the x-axis in two different points — A and B. The vertex C of the parabola has coordinates $2p$ and $h = f(2p) = 4p^2 - p + 1 > 0$. We have

$$AB = |x_1 - x_2| = \sqrt{(x_1 - x_2)^2} = \sqrt{(x_1 + x_2)^2 - 4x_1x_2} = 2\sqrt{4p^2 - p + 1}.$$

Now we find $S = S_{ABC} = \dfrac{AB.h}{2} = (4p^2 - p + 1)^{\frac{3}{2}}$. Denote $q = 4p^2 - p + 1$. Since q is rational and $q^3 = S^2$ is an integer, then q is an integer too. Then $\dfrac{S}{q}$ is rational and $\left(\dfrac{S}{q}\right)^2 = q$ is integer, thus $\dfrac{S}{q}$ is integer too. Therefore $q = n^2$, where n is a positive integer, i.e. $4p^2 - p + 1 - n^2 = 0$. The quadratic equation (with respect to p) has a rational root exactly when its discriminant $16n^2 - 15$ is a square of a rational number. Consequently $16n^2 - 15 = m^2$, and we can consider m to be a positive integer. From the equality $(4n - m)(4n + m) = 15$ we get $4n - m = 1$, $4n + m = 15$ or $4n - m = 3$, $4n + m = 5$. From here $n = 2$, $m = 7$ or $n = 1$, $m = 1$. The rational numbers we are looking for are $0, 1, \dfrac{1}{4}, -\dfrac{3}{4}$.

Problem 3. Let n be a positive integer and X be a set with n elements. Prove that

a) The number of all subsets of X (X and $\emptyset$ included) is equal to 2^n.

b) There exist 2^{n-1} subsets of X each pair of which is with common element.

c) There do not exist $2^{n-1} + 1$ subsets of X, each pair of which is with common element.

Solution. a) We use induction with respect to n. The base of the induction is obvious. Assume that the assertion is true for a set with $n - 1$ elements and let X be with n elements. We can assume that $X = \{1, 2, \dots, n\}$. Let $Y = \{1, 2, \dots, n - 1\}$. All subsets of X are divided into two groups: I group — those which do not contain n and II group — those, which contain n. Both groups have one and the same number of elements because each set of the II group is obtained from exactly one set of the I group by annexing n. Thus the number of the elements of X is twice greater than the number of the subsets of the I group. But the subsets of the I group are exactly the subsets of Y and according to the inductive assumption their number is 2^{n-1}. Thus the number of the subsets of X is 2^n.

b) According to a) the number of the subsets of X from the II group is 2^{n-1} and each pair of them has a common element — the number n.

c) If $A \subseteq X$, let $\overline{A} = X \setminus A$. All subsets of X are divided into pairs $\{A, \overline{A}\}$ and the number of these pairs is 2^{n-1}. Now if we have $2^{n-1} + 1$ arbitrary subsets of X, according to the pigeonhole principle, it is not possible that they are in different pairs of the type $\{A, \overline{A}\}$. Consequently among the given $2^n + 1$ subsets there exist two pairs of the type $\{A, \overline{A}\}$, which obviously have no common element.

Grade 10.

Problem 1. Find all values of the real parameters p and q, for which the roots of the equations $x^2 - px - 1 = 0$ and $x^2 - qx - 1 = 0$ form (in a suitable order) an arithmetic progression with four members.

Solution. Denote by x_1, x_2 the roots of the equation $x^2 - px - 1 = 0$ and by y_1, y_2 the roots of $x^2 - qx - 1 = 0$. It is clear that for all p and q the numbers x_1, x_2, y_1, y_2 are real and $x_1x_2 = y_1y_2 = -1$. Assume that $x_1 < 0 < x_2$ and $y_1 < 0 < y_2$. If four numbers a, b, c and d form an arithmetic progression then the numbers d, c, b, a form an arithmetic progression too. So, we can assume that x_1, x_2, y_1, y_2 in a suitable order form an increasing arithmetic progression. If $x_1 < y_1$ (the case $y_1 < x_1$ is analogous) then there are two possibilities:

I. The arithmetic progression is x_1, y_1, y_2, x_2. Then $x_1 + x_2 = y_1 + y_2$, from where $p = q$, i.e. $x_1 = y_1$, $x_2 = y_2$ which is impossible.

II. The arithmetic progression is x_1, y_1, x_2, y_2. Then $x_2 - x_1 = y_2 - y_1$, from where $\sqrt{p^2 + 4} = \sqrt{q^2 + 4}$, i.e. $p^2 = q^2$ and since $p \neq q$, then $p = -q \neq 0$.

In the same way we have $y_1 = \dfrac{x_1 + x_2}{2} = \dfrac{p}{2}$ and therefore $\dfrac{p^2}{4} - \dfrac{pq}{2} - 1 = 0$. Since $p = -q$, then $p^2 = \dfrac{4}{3}$ and $p = \pm\dfrac{2}{\sqrt{3}}$. Hence $(p, q) = \left(\dfrac{2}{\sqrt{3}}, -\dfrac{2}{\sqrt{3}}\right)$, $\left(-\dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}}\right)$.

Problem 2. Triangle ABC with $AB = 22$, $BC = 19$, $CA = 13$ is given.

a) If M is the center of gravity of $\triangle ABC$, prove that $AM^2 + CM^2 = BM^2$.

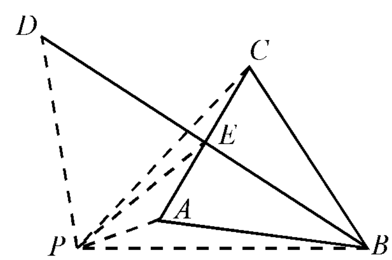
b) Find the locus of points P from the plane of $\triangle ABC$, for which $AP^2 + CP^2 = BP^2$.

c) Find the minimal and maximal values of BP, if $AP^2 + CP^2 = BP^2$.

Solution. a) We have

$$\begin{aligned} AM = \frac{2}{3}m_a &= \frac{2}{3}\cdot\frac{1}{2}\sqrt{2b^2+2c^2-a^2} = \sqrt{105}, \\ CM = \frac{2}{3}m_c &= \frac{2}{3}\cdot\frac{1}{2}\sqrt{2a^2+2b^2-c^2} = 8, \\ BM = \frac{2}{3}m_b &= \frac{2}{3}\cdot\frac{1}{2}\sqrt{2a^2+2c^2-b^2} = 13. \end{aligned}$$

Figure 6.



Hence $AM^2 + CM^2 = 105 + 64 = 169 = BM^2$.

b) Let E be the midpoint of AC and D be symmetric to B with respect to E (Figure 6). We shall prove that the locus we are looking for is a circle k with center D and radius 26. For an arbitrary point P we have $4PE^2 = 2PA^2 + 2PC^2 - AC^2$ and $4PE^2 = 2PB^2 + 2PD^2 - BD^2$, from where $2(PA^2 + PC^2 - PB^2) = 2PD^2 - (BD^2 - AC^2)$. But $BD = 2BE = 39$, i.e. $PA^2 + PC^2 - PB^2 = PD^2 - (26)^2$. It follows from here that the equality $PA^2 + PC^2 = PB^2$ is equivalent to $PD = 26$.

c) Let the circle k intersects the line BD at the points M and N. It follows from b) that BP is minimal when P coincides with M and then $BP = BM = 13$. BP is maximal when P coincides with N, which gives $BN = 65$.

Problem 3. Find the smallest positive integer n, for which there exist n different positive integers $a_1, a_2, \ldots, a_n$ satisfying the conditions:

a) the smallest common multiple of $a_1, a_2, \ldots, a_n$ is 1995;

b) for each $i, j \in \{1, 2, \ldots, n\}$ the numbers a_i and a_j have a common divisor $\neq 1$;

c) the product $a_1 a_2 \ldots a_n$ is a perfect square and is divisible by 243.

Find all n-ples $(a_1, a_2, \ldots, a_n)$, satisfying a), b) and c).

Solution. Since $1995 = 3.5.7.19$ and $a_i/1995$, then for all i

$$a_i = 3^{\alpha_i} 5^{\beta_i} 7^{\gamma_i} 19^{\delta_i} \tag{$*$}$$

where the numbers α_i, β_i, γ_i, δ_i are equal to 0 or 1. There is no a_i which is divisible by 9. We have $a_1 a_2 \ldots a_n = k^2$, where k is a positive integer and since $243 = 3^5$ divides k^2, then 3^6 divides k^2 and since among the numbers $(*)$ there is no one which is divisible by 9, then $n \geq 6$. The numbers $(*)$, which are divisible by 3 are

$$3;\quad 3.5;\quad 3.7;\quad 3.19;\quad 3.5.7;\quad 3.5.19;\quad 3.7.19;\quad 3.5.7.19 \tag{$**$}$$

They are 8 in number. Let $n = 6$. Then the numbers $a_1, a_2, \ldots, a_n$ are among $(**)$. It is easy to see that the product of any 6 numbers from $(**)$ is not a perfect square. Thus $n \geq 7$. Let $n = 7$. It is not possible that all the numbers $a_1, a_2, \ldots, a_n$ are divisible by 3, because in such a case $3^7/k^2$ and $3^8 \not| \, k^2$. Therefore 6 numbers from $a_1, a_2, \ldots, a_n$ are divisible by 3, i.e. they are from $(**)$ and at least one of them (for example a_1) is not divisible by 3. It follows from a) that among $a_1, a_2, \ldots, a_7$ there is at least one which is divisible by 5, at least one which is divisible by 7 and at least one which is divisible by 19. Since each pair of these numbers has a common divisor, and $3 \not| \, a_1$, then a_1 must be divisible by 5.7.19, i.e. $a_1 = 5.7.19$. At last note that the product of all numbers (**) is equal to $3^8.5^4.7^4.19^4$. The only possibility is $a_1a_2\dots a_7 = 3^6.5^4.7^4.19^4$. From here $a_2a_3\dots a_7 = 3^6.5^3.7^3.19^3$ and the only possibility is

$$\{a_2, a_3, \dots, a_7\} = \{3\cdot 5,\ 3\cdot 7,\ 3\cdot 19,\ 3\cdot 5\cdot 7,\ 3\cdot 5\cdot 19,\ 3\cdot 7\cdot 19\}.$$

Therefore $n = 7$ and

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\} = \{3\cdot 5,\ 3\cdot 7,\ 3\cdot 19,\ 3\cdot 5\cdot 7,\ 3\cdot 5\cdot 19,\ 3\cdot 7\cdot 19\ 5\cdot 7\cdot 19\}.$$

Grade 11.

Problem 1. Let $a_n = \dfrac{n+1}{2^{n+1}}\left(\dfrac{2^1}{1} + \dfrac{2^2}{2} + \cdots + \dfrac{2^n}{n}\right)$, $n = 1, 2, 3, \dots$. Prove that
a) $a_{n+1} \le a_n$ for all $n \ge 3$;
b) the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent and find its limit.
Solution. a) We have

$$a_{n+1} = \frac{n+2}{2^{n+2}}\left(\frac{2^1}{1} + \frac{2^2}{2} + \cdots + \frac{2^{n+1}}{n+1}\right) = \frac{n+2}{2(n+1)}(a_n + 1).$$

From here

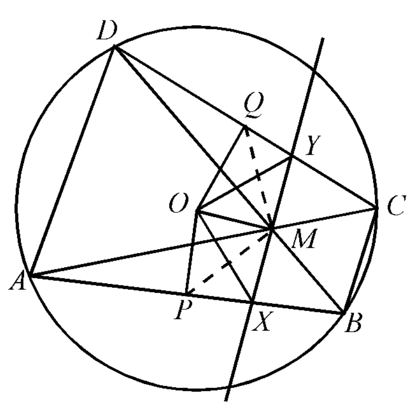
$$a_{n+2} - a_{n+1} = \frac{(n+2)^2(a_{n+1} - a_n) - (a_n + 1)}{2(n+1)(n+2)}, \quad n = 1, 2, 3 \dots.$$

Since $a_n > 0$ for all n, then if $a_{n+1} - a_n \le 0$, we have $a_{n+2} - a_{n+1} \le 0$. But $a_3 = \dfrac{5}{3}$ and $a_4 = \dfrac{5}{3}$, i.e. $a_4 - a_3 = 0$. Hence $a_5 - a_4 \le 0$, $a_6 - a_5 \le 0$, ..., $a_{n+1} - a_n \le 0$.

b) The sequence $a_1, a_2, a_3, \dots$ is decreasing when $n \ge 3$ and it is bounded ($a_n > 0$ for all n). Therefore this sequence is convergent. Let $\lim\limits_{n\to\infty} a_n = a$. From the equality $a_{n+1} = \dfrac{n+2}{2(n+1)}(a_n+1)$ after passing to infinity we get $a = \dfrac{1}{2}(a+1)$, i.e. $a = 1$.

Problem 2. The quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC and BD intersect each other in the point M, $M \ne O$. The line through M which is perpendicular to OM intersects the sides AB and CD of the quadrilateral $ABCD$ in the points X and Y, respectively. Prove that $AB = CD$ iff $BX = CY$.

Figure 7.



Solution. If $AB = CD$, then $ABCD$ is isosceles trapezoid. Hence $OM \perp AD$ and $OM \perp BC$, from where $XY \| BC$ and $BX = CY$.

Let $BX = CY$. Denote by P and Q the midpoints of AB and CD, respectively. The quadrilaterals $OPXM$ and $OQYM$ are inscribed, thus $\angle OPM = \angle OXM$ and $\angle OQM = \angle OYM$. We shall prove that $\angle OPM = \angle OQM$. The triangles ABM and DCM are congruent and

MP and MQ are medians in them. Therefore $\triangle MPB \sim \triangle MQC$ and $\angle MPB = \angle MQC$. Then $\angle OPM = 90° - \angle MPB = 90° - \angle MQC = \angle OQM$. (Or $\angle OPM = \angle MPB - 90° = \angle MQC - 90° = \angle OQM$.)

Hence $\angle OXM = \angle OYM$ and $OX = OY$. Therefore $\triangle OXB \cong \triangle OYC$. From here $\angle OBA = \angle OCD$ and the isosceles triangles ABO and DCO are equal, i.e. $AB = CD$

Remark. If $X = P$ and $Y = Q$, then $PB = QC = \dfrac{1}{2}AB = \dfrac{1}{2}CD$.

Problem 3. Let n be a positive integer and let

$$f(x) = x^n + (k+1)x^{n-1} + (2k+1)x^{n-2} + \cdots + ((n-1)k+1)x + nk + 1.$$

a) Prove that $f(1-k) = n+1$.

b) Prove that if $n \geq 3$ and k is an integer ($k \neq 0$), then the equation $f(x) = 0$ has no integer solution.

Solution. a) Since

$$\begin{aligned} & x^{n-1} + 2x^{n-2} + \cdots + (n-1)x + n \\ = \ & (x^{n-1} + \cdots + x + 1) + (x^{n-2} + \cdots + x + 1) + \cdots + (x^2 + x + 1) + (x+1) + 1 \\ = \ & \frac{x^n - 1}{x-1} + \frac{x^{n-1}-1}{x-1} + \cdots + \frac{x^3-1}{x-1} + \frac{x^2-1}{x+1} + \frac{x-1}{x-1}, \end{aligned}$$

then

$$\begin{aligned} x^{n-1} + 2x^{n-2} + \cdots + (n-1)x + n &= \frac{1}{x-1}\left(\frac{x^{n+1}-1}{x-1} - (n+1)\right) \\ &= \frac{x^{n+1} - (n+1)x + n}{(x-1)^2} \end{aligned}$$

when $x \neq 1$. Thus

$$\begin{aligned} f(x) &= x^n + x^{n-1} + \cdots + x + 1 + k(x^{n-1} + 2x^{n-2} + \cdots + n) \\ &= \frac{x^{n+1}-1}{x-1} + k\frac{x^{n+1} - (n+1)x + n}{(x-1)^2}, \end{aligned}$$

i.e.

$$f(x) = \frac{x^{n+2} + (k-1)x^{n+1} - [k(n+1)+1]x + kn + 1}{(x-1)^2}$$

when $x \neq 1$.

From here

$$f(1-k) = \frac{-[k(n+1)+1](1-k) + kn + 1}{k^2} = n+1,$$

when $1 - k \neq 1$. If $1 - k = 1$, i.e. $k = 0$, then $f(1) = n + 1$.

b) Let $n \geq 3$ and k be an integer ($k \neq 0$). Assume that $f(a) = 0$. Obviously $a \neq 0$. We have

$$a^n + a^{n-1} + \cdots + a + 1 = -k(a^{n-1} + 2a^{n-2} + \cdots + (n-1)a + n).$$

If $a = -1$, the left hand side of the equation is equal to 0 or ± 1, while the right hand side is neither 0 nor ± 1, because $|n - (n-1) + (n-2) - \cdots| > 1$ when $n \geq 3$ ($k \neq 0$). Thus $a \neq -1$. The equation $f(a) = 0$ can be written in the following way:

$$(-a - k + 1)(a^{n-1} + 2a^{n-2} + \cdots + (n-1)a + n) = n + 1.$$

Hence $r_n(a) = a^{n-1} + 2a^{n-2} + \cdots + (n-1)a + n$ divides $n+1$. If $a > 1$, then $r_n(a) > n+1$ when $n \geq 3$ and this is impossible. From $a \neq 0$ and $a \neq -1$ it follows that $a \leq -2$. We shall prove that the inequality $|r_n(t)| \geq n+2$ is satisfied for all integers $t \leq -2$ and for all $n \geq 3$ except $t = -2$, $n = 3$ and $t = -2$, $n = 4$. Since $r_n(-2) = \dfrac{(-2)^{n+1} + 3n + 2}{9}$, then $r_3(-2) = 3$, $r_4(-2) = -2$ and $r_5(-2) = 9$, i.e. $|r_5(-2)| \geq 7$. From $r_3(t) = t^2 + 2t + 3$ it follows that $|r_3(t)| \geq 5$ when $t \leq -3$.

Now we shall use induction. If $|r_n(t)| \geq n+2$ for $t \leq -2$ and $n \geq 3$, then $r_{n+1}(t) = t.r_n(t) + n + 1$ and $|r_{n+1}(t)| \geq |t|.|r_n(t)| - (n+1) \geq 2(n+2) - n - 1 = n+3$, i.e. $|r_{n+1}(t)| \geq n+3$. Hence $|r_n(t)| \geq n+2$ when $n \geq 3$ and $t \leq -2$ except the cases $n = 3, t = -2$ and $n = 4, t = -2$. Thus $r_n(a)$ does not divide $n+1$ when $n \geq 3$ and $a \leq -2$, because $r_3(-2) = 3$, $r_4(-2) = -2$. For all others $n \geq 3$ and $a \leq -2$ we have $|r_n(a)| \geq n+2$.

Grade 12.

Problem 1. The function $f(x) = \sqrt{1-x}$ $(x \leq 1)$ is given. Let $F(x) = f(f(x))$.

a) Solve the equations $f(x) = x$ and $F(x) = x$.

b) Solve the inequality $F(x) > x$.

c) If $a_0 \in (0,1)$, prove that the sequence $\{a_n\}_{n=0}^{\infty}$, determined by $a_n = f(a_{n-1})$ for $n = 1, 2, \ldots$, is convergent and find its limit.

Solution. The functions $f(x), F(x)$ and $F(x) - x$ are defined for all $x \in [0,1]$.

a) The equation $f(x) = x$, i.e. $\sqrt{1-x} = x$ has only one root $\alpha = \dfrac{-1+\sqrt{5}}{2}$. It is clear that $\alpha \in (0,1)$ and the roots of the equation $F(x) = x$ are $x_1 = 0$, $x_2 = 1$, $x_3 = \alpha$.

b) In $[0, \alpha]$ the function $F(x) - x$ has a constant sign. The contrary would imply that there is $\beta \in (0, \alpha)$ such that $F(\beta) = \beta$ and this contradicts the result from a). Analogously in $[\alpha, 1]$ the function $F(x) - x$ has a constant sign. On the other hand $\dfrac{1}{4} \in (0, \alpha)$, $\dfrac{3}{4} \in (\alpha, 1)$ and $F\left(\dfrac{1}{4}\right) - \dfrac{1}{4} > 0$, $F\left(\dfrac{3}{4}\right) - \dfrac{3}{4} < 0$. From here $F(x) > x$ iff $x \in \left(0, \dfrac{-1+\sqrt{5}}{2}\right)$.

c) Let $a_0 = \alpha$. It follows from a) that $a_n = \alpha$ for $n = 0, 1, 2, \ldots$ and hence the sequence is convergent and its limit is α. Let now $a_0 < \alpha$. Since $f'(x) = -\dfrac{1}{2\sqrt{1-x}} < 0$ for all $x \in [0,1)$, then $f(x)$ is decreasing. From here $frac(a_0) > frac(\alpha) = \alpha$, i.e. $a_1 > \alpha$. By induction $a_{2n} \in (0, \alpha)$ and $a_{2n+1} \in (\alpha, 1)$ for all $n = 0, 1, 2, \ldots$. On the other hand it follows from the result of b) that for all $x \in (0, \alpha)$ we have $F(x) > x$, while for $x \in (\alpha, 1)$ we have $F(x) < x$, respectively. Also $F'(x) = f'(f(x)).f'(x) > 0$, i.e. $F(x)$ is increasing and hence $F(x) \in (x, \alpha)$ if $x \in (0, \alpha)$ and $F(x) \in (\alpha, x)$ if $x \in (\alpha, 1)$. By induction we get

$$a_0 < a_2 < \cdots < a_{2n} < \cdots < \alpha,$$
$$a_1 > a_3 > \cdots > a_{2n+1} > \cdots > \alpha.$$

Both sequences are convergent and let their limits be α_1 and α_2, respectively. We have $F(a_{2n}) = a_{2n+2}$ and $F(a_{2n+1}) = a_{2n+3}$ for $n = 0, 1, 2, \ldots$. The function $F(x)$ is continuous and thus $F(\alpha_1) = \alpha_1$ and $F(\alpha_2) = \alpha_2$. We get $\alpha_1 = \alpha_2 = \alpha$ because the only solution of $F(x) = x$ in $(0,1)$ is α. Therefore the sequence $\{a_n\}_{n=0}^{\infty}$ is convergent and its limit is α. The case $\alpha < a_0$ is analogous.

Problem 2. The sides AC and BC of the triangle ABC are diameters of two circles, each of which touches internally a circle k, which is concentric to the incircle of $\triangle ABC$.

a) Prove that $AC = BC$.

b) If $\cos \angle BAC = \dfrac{3}{4}$, find the ratio of the radii of k and the incircle of $\triangle ABC$.

Solution. a) Let I be the center of the incircle of $\triangle ABC$, N be the common point of this circle with AC and M be the midpoint of AC. Let r_1 be the radius of k and r be the radius of the incircle of $\triangle ABC$. From the condition it follows that $IM = \dfrac{b}{2} - r_1$ and from the rectangular $\triangle INM$ we get $MN^2 = \left(\dfrac{b}{2} - r_1\right)^2 - r^2$. On the other hand $AN = \dfrac{b+c-a}{2}$, i.e. $MN = |AM - AN| = \dfrac{|c-a|}{2}$. Therefore

$$(c-a)^2 = (b - 2r_1)^2 - 4r^2. \tag{1}$$

Analogously

$$(c-b)^2 = (a - 2r_1)^2 - 4r^2. \tag{2}$$

Assume that $a \neq b$. From (1) and (2) we get $(b-a)(2c-a-b) = (b-a)(a+b-4r_1)$, i.e. $2c - a - b = a + b - 4r_1$. From here $b - 2r_1 = c - a$ and from (1) it follows that $r = 0$, which is impossible. Thus $a = b$.

b) Let $\dfrac{r_1}{r} = t$ and $\angle BAC = \alpha$. Then $c = 2b\cos\alpha = \dfrac{3}{2}b$ and

$$r = \frac{c}{2}\tan\frac{\alpha}{2} = b\cos\alpha\tan\frac{\alpha}{2} = b\cos\alpha\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{3b}{4\sqrt{7}}.$$

Since $b > 2r_1$, from (1) it follows that $t = \dfrac{b - \sqrt{(c-b)^2 + 4r^2}}{2r}$ and substituting c and r we get $t = \dfrac{2}{3}(\sqrt{7} - 2)$.

Problem 3. n points ($n > 4$), no three of which are colinear are given in the plane. More than n triangles are constructed with vertices among these points. Prove that at least two triangles have exactly one common vertex.

Solution. Assume the contrary and let k be the smallest number for which the assertion is not true. This means that there are constructed at least $(k+1)$ triangles using k points. It follows from the pigeonhole principle that there exists a point A which is a vertex of at least 4 triangles. Let ABC be the first triangle. At least one of the points B and C is a vertex of the second triangle, which we denote by ABD. If ACX is the third triangle then $X = D$. Thus the forth triangle must contain B or C, which is impossible. Therefore if A and B are vertices of two triangles then they are vertices of all the four triangles. Let A be a vertex of t triangles, $t \geq 4$. These triangles are of the kind ABA_1, ABA_2, $\ldots$, ABA_t, where all the points $A_1, A_2, \ldots, A_t$ are pairwise different. Obviously it is not possible to exist a triangle of the type BXY, where X and Y are points which are different from $A_1, A_2, \ldots, A_t$. Triangles BA_iA_j and $A_iA_jA_m$ do not exist too. Hence the points A, B, A_1, A_2, $\ldots$, A_t are vertices only of the triangles ABA_1, ABA_2, $\ldots$, ABA_t. In such a way we use $t+2$ points and get t triangles. It is not possible that $t + 2 = k$, because all triangles are $t < k$. The number of remaining points is $k_0 = k - t - 2$ and by them there are constructed at least $k + 1 - t > k_0$ triangles, such that no two of them have exactly one common vertex. The triangles are more than the points k_0, and thus $k_0 > 4$. We have found a number $k_0 < k$ for which the assertion is not true. This contradicts the choice of k.

SPRING MATHEMATICAL COMPETITION

1996

Grade 8

Problem 1. Prove that for all real $a \in (1,2)$ the area of the figure encountered by the graphs of the functions $y = 1 - |x-1|$ and $y = |2x-a|$ is less than $\frac{1}{3}$.

Solution. Firstly, we shall find the common points of the given functions. For this purpose we solve the equation

$$|2x-a| = 1 - |1-x|. \tag{1}$$

Since $1 < a < 2$, then $\frac{a}{2} < 1$, and we shall consider the cases: $x \leq \frac{a}{2}$, $\frac{a}{2} < x < 1$ and $x \geq 1$. We have:

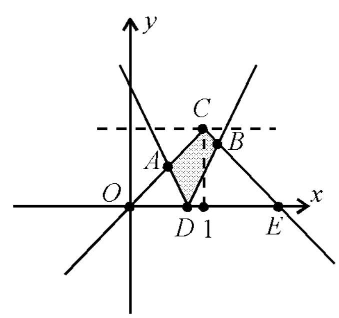
1. When $x \leq \frac{a}{2}$ the equation (1) takes the form $a - 2x = x$. Then $x = \frac{a}{3}$, which satisfies (1), because $\frac{a}{3} < \frac{a}{2}$.

2. When $\frac{a}{2} < x < 1$ the equation (1) takes the form $2x - a = x$, i.e. $x = a$, which does not satisfy (1), because $a > 1$.

3. When $x \geq 1$ the equation (1) takes the form $2x - a = 2 - x$. Then $x = \frac{a+2}{3}$, which satisfies (1), because $\frac{a+2}{3} > 1$ when $a > 1$.

Thus, the graphs of the two functions have two common points (Figure 1), the first of which (denoted by A) has coordinates $x_A = \frac{a}{3}$, $y_A = \frac{a}{3}$, while the second one (denoted by B) has coordinates $x_B = \frac{a+2}{3}$, $y_B = 2 - \frac{a+2}{3} = \frac{4-a}{3}$. Denote by C, D and E the points, with coordinates $x_C = 1$, $y_C = 1$; $x_D = \frac{a}{2}$, $y_D = 0$ and $x_E = 2$, $y_E = 0$, respectively.

Figure 1.

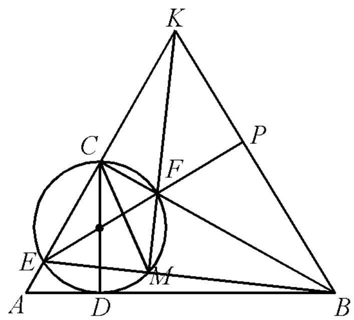


Then, the figure encountered by the two graphs is the quadrilateral $ACBD$ and its area S is obtained by subtracting the areas of the triangles ODA and BDE from the area of the triangle OEC. Therefore,

$$\begin{aligned} S &= S_{OEC} - S_{ODA} - S_{BDE} \\ &= 1 - \frac{1}{2} \cdot OD \cdot y_A - \frac{1}{12} \cdot DE \cdot y_B \\ &= 1 - \frac{1}{2} \cdot \frac{a}{2} \cdot \frac{a}{3} - \frac{1}{2} \cdot \left(2 - \frac{a}{2}\right) \cdot \frac{4-a}{3} \end{aligned}$$

$$
\begin{aligned}
&= 1-\frac{1}{12}\cdot a^2-\frac{1}{12}\cdot(4-a)^2 \\
&= \frac{1}{6}\cdot\left(-a^2+4a-2\right) \\
&= \frac{1}{6}\cdot\left(2-(a-2)^2\right)<\frac{1}{3}.
\end{aligned}
$$

Figure 2.



Problem 2. The altitude CD of the rectangle triangle ABC ($\angle ACB = 90^\circ$) is a diameter of the circle k, which meets the sides AC and BC in E and F, respectively. The intersection point of the line BE and the circle k, which is different from E, is denoted by M. Let the intersection point of the lines AC and MF be K, and the intersection point of the lines EF and BK be P.

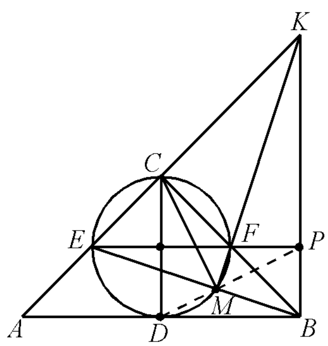
a) Prove that the points B, F, M and P are concyclic;

b) If the points D, M and P are colinear, find the angles A and B of the triangle ABC.

Solution. a) Since $\angle ECF = 90^\circ$, then EF is a diameter of the circle k and consequently $\angle EMF = 90^\circ = \angle BMK$. Then BC and KM are altitudes in $\triangle BEK$ (Figure 2), which means that the point F is the altitude center of this triangle. It follows from here that $EP \perp BK$, i.e. $\angle BPF = 90^\circ$.

Thus, $\angle BPF = \angle BMF = 90^\circ$ and consequently the points B, F, M and P lie on a circle with diameter BF.

Figure 3.



b) We have:

$$
\begin{aligned}
\angle BDM &= 90^\circ-\angle MDC = 90^\circ-\angle MEC \\
&= \angle CBE = \angle FBM = \angle FPM.
\end{aligned}
$$

If the points D, M and P are colinear (Figure 3), then the equality between the angles $\angle BDM$ and $\angle FPM$ implies that the lines AB and EF are parallel, i.e. EF and CD are diameters of the circle k, which are perpendicular to each other. This means that CD is the bisector of $\angle ACB$. Consequently, $AC = BC$ and $\angle BAC = \angle ABC = 45^\circ$.

Problem 3. In a state every town is connected with the nearest town by a straight way. The distances between the pairs of towns are pairwise different. Prove that

a) no two ways have common points;

b) every town is connected by ways with at most 5 other towns;

c) there is no closed piecewise line, consisted of ways.

Solution. a) Suppose that the ways AC and BD meet each other (Figure 4) and let C be the nearest town to A, while D be the nearest town to B. Then $AC < AD$ and $BD < BC$, from where $AC + BD < AD + BC$.

On the other hand, if O is the common point of AC and BD, then $AO + OD > AD$ and $BO + OC > BC$, from where $AC + BD > AD + BC$. This is a contradiction.

b) Let the town X be connected by ways with the towns A and B (Figure 5). Then AB is the longest side of $\triangle XAB$. Indeed, if we assume that for example AX is the longest side, then

A should not be the nearest town to X, and X should not be the nearest town to A as well. Consequently, the way AX should not exist.

Figure 4.

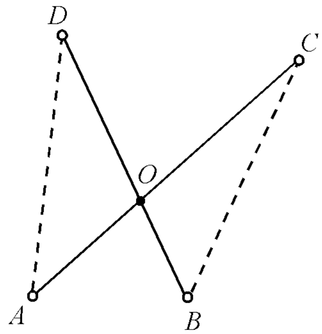


Figure 5.

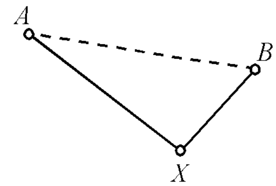
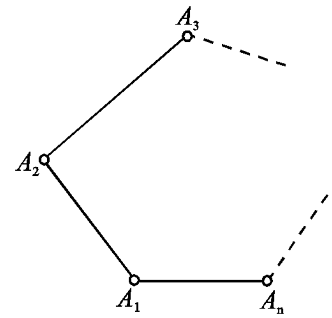


Figure 6.



Therefore $\angle AXB$ is the biggest angle in $\triangle XAB$, from where $\angle AXB > 60°$. Now, if we assume that the town X is connected with at least 6 towns, then the sum of the angles at X would be greater than $6 \cdot 60° = 360°$, which is impossible. Thus, every town is connected with at most 5 other towns.

c) Assume that there exists a closed piecewise line $A_1A_2 \ldots A_n$, consisted of ways (Figure 6). The distances A_1A_n and A_1A_2 are different. Let $A_1A_n < A_1A_2$. Then A_2 is not the nearest town to A_1 and consequently (because the way A_1A_2 exists) A_1 is the nearest town to A_2. It follows from here that $A_1A_2 < A_2A_3$.

Proceeding in this way we obtain the chain: $A_1A_n < A_1A_2 < A_2A_3 < \ldots < A_nA_1$, which leads to contradiction.

Grade 9

Problem 1. Find the values of the real parameter b, for which the difference between the maximal and the minimal values of the function $f(x) = x^2 - 2bx + 1$ in the interval $[0, 1]$ is equal to 4.

Figure 7.

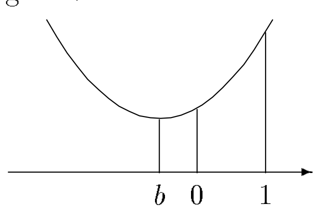


Figure 8.

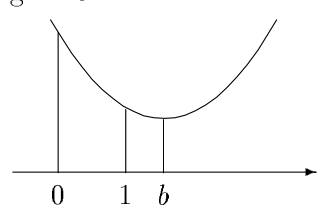
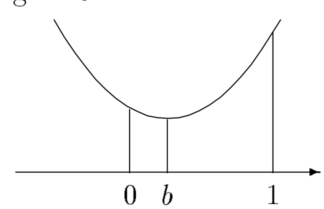


Figure 9.



Solution. It is clear that the minimal value of the quadratic function $f(x)$ is obtained when $x = b$. We shall consider the following three cases:

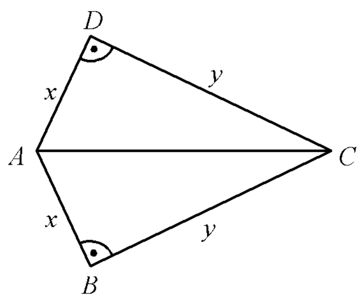
1. Let $b < 0$. In this case the function $f(x)$ is increasing in the interval $[0, 1]$ (Figure 7) and the maximal value is $f(1) = 2 - 2b$, while the smallest one is $f(0) = 1$. From the condition $f(1) - f(0) = 1 - 2b = 4$ we find $b = -\dfrac{3}{2}$, which is a solution of the problem.

2. Let $b > 1$. Now the function $f(x)$ is decreasing in the interval $[0, 1]$ (Figure 8) and the maximal value is $f(0) = 1$, while the minimal one is $f(1) = 2 - 2b$. From the condition $f(0) - f(1) = 2b - 1 = 4$ we find $b = \frac{5}{2}$, which is a solution of the problem.

3. Let $0 \leq b \leq 1$ (Figure 9). The minimal value of $f(x)$ in the interval $[0, 1]$ is $f(b) = 1 - b^2$, while the the maximal one is $f(0)$ or $f(1)$. From $f(0) - f(b) = b^2 = 4$ we find $b = \pm 2$, which are not solutions (because in this case $0 \leq b \leq 1$). From $f(1) - f(b) = (1 - b)^2 = 4$ we find $b = 3$ or $b = -1$, which are not solutions too.

Finally, the answers are: $b = -\frac{3}{2}$ and $b = \frac{5}{2}$.

Figure 10.



Problem 2. The quadrilateral $ABCD$ is inscribed in a circle with radius 1, a circle can be inscribed in it and $AB = AD$. Prove that:

a) the area of the quadrilateral $ABCD$ does not exceed 2;

b) the inradius of the quadrilateral $ABCD$ does not exceed $\frac{\sqrt{2}}{2}$.

Solution. We have (Figure 10) $AB + CD = AD + BC$ and from $AB = AD$ we get: $BC = CD$. Let $AB = AD = x$, $BC = CD = y$. The triangles ACB and ACD are equal (by **SSS**), from where $\angle B = \angle D$. But $ABCD$ is inscribed in a circle and consequently $\angle B + \angle D = 180°$. Therefore $\angle B = \angle D = 90°$. Then AC is diameter of the circumcircle of the quadrilateral and particularly $AC = 2$ and $x^2 + y^2 = AC^2 = 4$.

Let S, p and r be the area of the quadrilateral, its semiperimeter and inradius, respectively.

a) We have $S = S_{ABC} + S_{ACD} = 2S_{ACB} = xy$ and

$$xy = \sqrt{x^2y^2} \leq \frac{x^2 + y^2}{2} = \frac{AC^2}{2} = 2$$

Thus, $S \leq 2$ (and $S = 2$ only if $x = y$ and then the quadrilateral $ABCD$ is a square).

b) We shall make use of the formula $S = pr$, which is true for all polygons that can be inscribed in a circle. We have:

$$r^2 = \frac{S^2}{p^2} = \frac{S^2}{(x + y)^2} = fracS^2x^2 + y^2 + 2xy = \frac{S^2}{4 + 2S}.$$

Now

$$\begin{aligned} r \leq \frac{\sqrt{2}}{2} \quad &\Longleftrightarrow \quad r^2 \leq \frac{1}{2} \quad \Longleftrightarrow \quad \frac{S^2}{4 + 2S} \leq \frac{1}{2} \\ &\Longleftrightarrow \quad S^2 - S - 2 \leq 0 \quad \Longleftrightarrow \quad (S + 1)(S - 2) \leq 0 \\ &\Longleftrightarrow \quad S \leq 2 \end{aligned}$$

(because $S > 0$). The last inequality is true according to a) and consequently the inequality $r \leq \frac{\sqrt{2}}{2}$ is true too.

Problem 3. This problem is the same as problem 3, grade 8.

Grade 10

Problem 1. Find in the plane the locus of points with coordinates (x, y), for which there exists exactly one real number z, satisfying the equality:

$$xz^4 + yz^3 - 2(x + |y|)z^2 + yz + x = 0.$$

Solution. Let (x, y) be the coordinates of a point from the locus, we are looking for. This means that there exists exactly one real number z, for which

$$xz^4 + yz^3 - 2(x + |y|)z^2 + yz + x = 0.$$

If $x = 0$, the above equality takes the form

$$z\left(yz^2 - 2|y|z + y\right) = 0.$$

This equality is satisfied for at least two different values of z (for example $z = 0$ and $z = 1$, if $y \geq 0$ and $z = 0$ and $z = -1$, if $y < 0$), which shows that the condition of the problem is not verified. Consequently, the points from the y-axis do not belong to the locus.

Let $x \neq 0$. Thus, if z satisfies the given equality, then $z \neq 0$. It is easy to see that in this case the number $\frac{1}{z}$ satisfies the same equality and consequently $z = \frac{1}{z}$. From here $z^2 = 1$, i.e. $z = \pm 1$.

Case 1. Let $z = 1$. Then $x + y - 2(x + |y|) + y + x = 0$, from where $y - |y| = 0$. This shows that $y \geq 0$. If $y = 0$ and then the given equality takes the form $x\left(z^2 - 1\right)^2 = 0$, and consequently it is satisfied also by $z = -1$. Thus, we can assume that $y > 0$. Then,

$$xz^4 + yz^3 - 2(x + y)z^2 + yz + x = 0,$$

from where

$$(z - 1)^2\left(xz^2 + (y + 2x)z + x\right) = 0.$$

We have one of the following possibilities:

(i) The number $z = 1$ is the only to satisfy the equality $xz^2 + (y + 2x)z + x = 0$. This is true if $x + y + 2x + x = 0$ and $D = (y + 2x)^2 - 4x^2 = y(y + 4x) = 0$, from where $y = -4x$ and $y > 0$.

(ii) There is no real z, which satisfy the equality $xz^2 + (y + 2x)z + x = 0$. Then, $D = (y + 2x)^2 - 4x^2 = y(y + 4x) < 0$, from where $y < -4x$ and $y > 0$.

Therefore, every time when $y \leq -4x$ and $y > 0$, the point (x, y) belongs to the locus.

Case 2. Let $z = -1$. Analogously to the previous case we find that $y < 0$ and then the given equality takes the form

$$(z + 1)^2\left(xz^2 + (y - 2x)z + x\right) = 0.$$

We have the following possibilities:

(i) the number $z = -1$ is double root of the equation

$$xz^2 + (y - 2x)z + x = 0. \qquad (*)$$

(ii) the equation (*) has no real root.

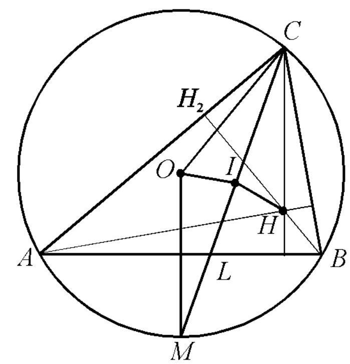
By computing we deduce that here $y \geq 4x$ and $y < 0$.

Finally, the locus (Figure 11) consists of the internal as well as of the boundary points of the angle which is defined by the graphs of the linear functions $y = 4x$ and $y = -4x$ when $x < 0$, without the points from the negative part of the x – axis.

Problem 2. In the triangle ABC, h_a and h_b are the altitudes from A and B respectively, ℓ_c is the internal bisector of $\angle ACB$, while O, I and H are the circumcenter, the incenter and the altitude center, respectively. Prove that if $\dfrac{\ell_c}{h_a} + \dfrac{\ell_c}{h_b} = 2$, then $OI = IH$.

Solution. Let $BC = a$, $AC = b$, $\angle ACB = \gamma$, $CL = \ell_c$ (Figure 12). Since $S_{ABC} = S_{ALC} + S_{BLC}$, then $\dfrac{1}{2} \cdot \ell_c a \sin\dfrac{\gamma}{2} + \dfrac{1}{2} \cdot \ell_c b \sin\dfrac{\gamma}{2} = \dfrac{1}{2} \cdot ab\sin\gamma$, from where $\ell_c = \dfrac{2ab\cos\dfrac{\gamma}{2}}{a+b}$. But $h_a = b\sin\gamma$ and $h_b = a\sin\gamma$. Then $\dfrac{\ell_c}{h_a} + \dfrac{\ell_c}{h_b} = \dfrac{2a\cos\dfrac{\gamma}{2}}{(a+b)\sin\gamma} + \dfrac{2b\cos\dfrac{\gamma}{2}}{(a+b)\sin\gamma} = \dfrac{1}{\sin\dfrac{\gamma}{2}}\left(\dfrac{a}{a+b} + \dfrac{b}{a+b}\right) = \dfrac{1}{\sin\dfrac{\gamma}{2}}$, which shows that $\sin\dfrac{\gamma}{2} = \dfrac{1}{2}$, i.e. $\dfrac{\gamma}{2} = 30°$, because $\dfrac{\gamma}{2} < 90°$. Consequently, $\gamma = 60°$.

Figure 12.



Let CL meet the circumcircle of $\triangle ABC$ in M. Then $OM \perp AB$. But $CH \perp AB$, and therefore $\angle OMC = \angle MCH$. On the other hand $OM = OC = R$ (R is the circumradius). Consequently, $\angle OMC = \angle OCM = \angle MCH$. Let H_2 be the foot of the altitude from B. From $\triangle HCH_2$ we have $CH = \dfrac{CH_2}{\sin\alpha}$ (because $\angle H_2HC = 90° - \angle ACH = \angle CAB = \alpha$). But $CH_2 = a\cos\gamma$, i.e. $CH = \dfrac{a}{\sin\alpha} \cdot \cos\gamma = 2R\cos\gamma$. Since $\gamma = 60°$, then $CH = R$.

We consider $\triangle COI$ and $\triangle CHI$. Since the point I lies on CM, then $\angle OCI = \angle HCI$. Also, $CO = R = CH$ and consequently $\triangle COI \cong \triangle CHI$. Thus, $OI = IH$.

Problem 3. Let A_1, A_2, …, A_n ($n \geq 4$) be n points in the plane, no 3 of which are colinear.

a) Prove that there is at most one point A_s, such that all triangles $A_sA_iA_j$ $(i, j = 1, 2, \dots, n)$ are acute.

b) Let among A_1, A_2, …, A_n be a point, which is a vertex of an acute triangles only. We consider the angles defined by the given points. Denote by N_k the number of the acute angles $\angle A_iA_kA_j$ $(i, j = 1, 2, \dots, n)$ for which the point A_k is their vertex. Find the minimal value of N_k.

Solution. a) Assume that there is more than one point with the given property and let A and B be such two points, while X and Y be any two of the remaining points. There are two possibilities for the points A, B, X and Y:

(i) A, B, X and Y define a convex quadrilateral. Because the sum of its internal angles is $360°$, then at least one of these angles is not acute. Consequently, at least one of the triangles with vertex A or B is not acute.

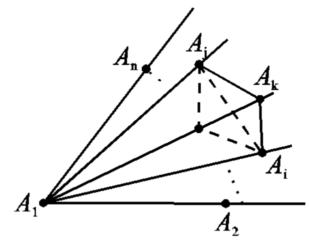
(ii) One of the points A, B, X and Y is inside the triangle defined by the other three. Then, the sum of the three angles with a vertex inside is $360°$. Consequently, in this case a triangle with vertex in A or B is not acute again.

The contradictions show that it is not possible to exist more than one point which is a vertex of acute triangles only.

b) Let A_1 be the point which is a vertex of acute triangles only. We consider the angles $\angle A_kA_1A_s$ (Figure 13). Let $A_2A_1A_n$ be the biggest one. Because all angles with vertex A_1 are acute, the points A_3, A_4, …, A_{n-1} are inside the acute angle $\angle A_2A_1A_n$. We can assume that the points are enumerate in such a way that $\angle A_2A_1A_k < \angle A_2A_1A_{k+1}$ for $k = 3, 4, \ldots, n$.

Consider $\angle A_iA_kA_j$, where $2 \le i < k < j \le n$.

Figure 13.



We assume that A_k is internal for $\triangle A_1A_iA_j$. Then

$$\angle A_1A_kA_i + \angle A_1A_kA_j > 180^\circ$$

and at least one of the angles $\angle A_1A_kA_i$ and $\angle A_1A_kA_j$ will not be acute, which is a contradiction.

Therefore, the points A_1, A_i, A_k, A_j define a convex quadrilateral. If we assume that $\angle A_iA_kA_j \le 90^\circ$, then there exists an angle of this quadrilateral with vertex A_1, A_i or A_j which is $\ge 90^\circ$. This is impossible.

Therefore, $\angle A_iA_kA_j > 90^\circ$, and it is clear that $\angle A_iA_kA_j < 90^\circ$ when $i, j < k$ or $i, j > k$.

Particularly, it follows from here that no of the angles $A_iA_kA_j$ is right. The number of all angles with vertex A_k is equal to $\dfrac{(n-1)(n-2)}{2}$. If T_k is the number of the obtuse angles with vertex A_k, then $N_k = \dfrac{(n-1)(n-2)}{2} - T_k$. Thus, N_k is minimal when T_k is maximal.

It is easy to see that $T_1 = T_2 = T_n = 0$. Let $3 \le k \le n-1$. It is clear that the number of the points A_i for which $2 \le i < k$ is $k-2$, and the number of the points A_j for which $k < j \le n$ is $n-k$. Then $T_k = (k-2)(n-k)$ for $k = 3, 4, \ldots, n-1$.

But $(k-2)(n-k) \le \dfrac{(n-2)^2}{4}$, and the equality is reached when $k-2 = n-k$, i.e. when $k = \dfrac{n+2}{2}$. We have two possibilities:

(i) The number n is even. Then $\dfrac{n+2}{2}$ is integer and consequently the maximal value of T_k is $T_{\frac{n+2}{2}} = \dfrac{(n-2)^2}{2}$. Therefore, the minimal value of N_k in this case is $N_{\frac{n+2}{2}} = \dfrac{(n-1)(n-2}{2} - \dfrac{(n-2)^2}{2} = \dfrac{n(n-2)}{4}$.

(ii) The number n is odd. Then, the nearest integers to $\dfrac{n+2}{2}$ are $\dfrac{n+1}{2}$ and $\dfrac{n+3}{2}$. It is easy to see that now the maximal value of T_k is $T_{\frac{n+1}{2}} = T_{\frac{n+3}{2}} = \left(\dfrac{n+1}{2} - 2\right)\left(n - \dfrac{n+1}{2}\right) = \dfrac{(n-1)(n-3)}{4}$. Consequently, the minimal value of N_k in this case is

$$N_{\frac{n+1}{2}} = N_{\frac{n+3}{2}} = \frac{(n-1)(n-2)}{2} - \frac{(n-1)(n-3)}{4} = \frac{(n-1)^2}{4}.$$

Grade 11

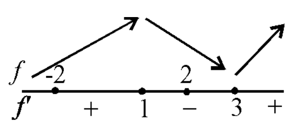
Problem 1. Find the values of the real parameter a, for which the inequality $x^6 - 6x^5 + 12x^4 + ax^3 + 12x^2 - 6x + 1 \ge 0$ is satisfied for all real x.

Solution. When $x=0$ the given inequality is satisfied for all a. Thus, it is enough to find such a, that $\left(x^3+\frac{1}{x^3}\right)-6\left(x^2+\frac{1}{x^2}\right)+12\left(x+\frac{1}{x}\right)+a\geq 0$ for all $x>0$ and $\left(x^3+\frac{1}{x^3}\right)-6\left(x^2+\frac{1}{x^2}\right)+12\left(x+\frac{1}{x}\right)+a\leq 0$ for all $x<0$.

Denote $t=x+\frac{1}{x}$. It is clear that $x>0 \iff t\geq 2$ and $x<0 \iff t\leq -2$. But $x^2+\frac{1}{x^2}=t^2-2$ and $x^3+\frac{1}{x^3}=t^3-3t$.

We consider the function $f(t)=\left(t^3-3t\right)-6\left(t^2-2\right)+12t+=t^3-6t^2+9t+12+a$. The problem is reduced to find such a, that $f(t)\geq 0$ for all $t\geq 2$ and $f(t)\leq 0$ for all $t\leq -2$ simultaneously.

Figure 14.



But $f'(t)=3t^2-12t+9=3(t-1)(t-3)$, from where it is easy to obtain that (Figure 14) $f(t)\geq 0$ for all $t\geq 2$ iff $f(3)\geq 0$ and $f(t)\leq 0$ for all $t\leq -2$ iff $f(-2)\leq 0$. Since $f(3)=a+12$ and $f(-2)=a-38$, we find for a: $a\geq -12$ and $a\leq 38$. Finally, $-12\leq a\leq 38$.

Problem 2. The point D lies on the arc $\overset{\frown}{BC}$ of the circumcircle of $\triangle ABC$ which does not contain the point A and $D\neq B$, $D\neq C$. On the rays $BD^{\rightarrow}$ and $CD^{\rightarrow}$ there are taken points E and F, such that $BE=AC$ and $CF=AB$. Let M be the midpoint of the segment EF.

a) Prove that $\angle BMC$ is right.

b) Find the locus of the points M, when D describes the arc $\overset{\frown}{BC}$.

Figure 15.

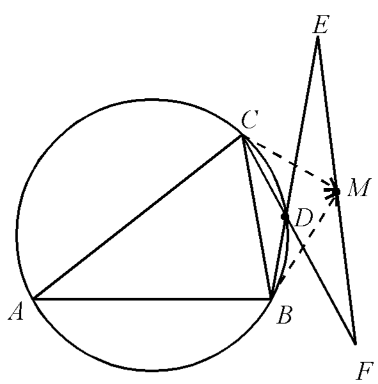
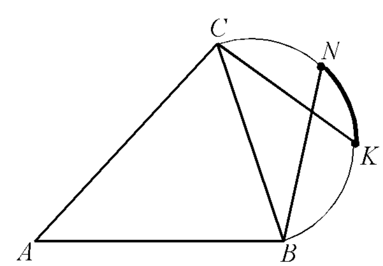


Figure 16.



Solution. Denote by α, β, γ the angles corresponding to the vertexes A, B, C of $\triangle ABC$, and by a, b, c the lengths of the sides BC, CA, AB, respectively. Let $\angle BCD=\varphi$. It follows from $D\in\overset{\frown}{BC}$ that $0<\varphi<\alpha$, i.e. $\varphi\in(0,\alpha)$.

a) We have (Figure 15) $\overrightarrow{BM}=\frac{1}{2}\cdot\left(\overrightarrow{BF}+\overrightarrow{BE}\right)=\frac{1}{2}\cdot\left(\overrightarrow{CF}+\overrightarrow{BE}+\overrightarrow{BC}\right)$ and $\overrightarrow{CM}=\frac{1}{2}\cdot\left(\overrightarrow{CF}+\overrightarrow{CE}\right)=\frac{1}{2}\cdot\left(\overrightarrow{CF}+\overrightarrow{BE}-\overrightarrow{BC}\right)$. Then we find for the scalar product $\overrightarrow{BM}\cdot\overrightarrow{CM}$:

$$\overrightarrow{BM}\cdot\overrightarrow{CM} = \frac{1}{4}\cdot\left(\left(\overrightarrow{CF}+\overrightarrow{BE}\right)^2-\overrightarrow{BC}^2\right)$$

$$
\begin{aligned}
&= \frac{1}{4}\cdot\left(\overrightarrow{CF}^2+\overrightarrow{BE}^2+2\cdot\overrightarrow{CF}\cdot\overrightarrow{BE}-a^2\right)\\
&= \frac{1}{4}\cdot\left(c^2+b^2+2bc\cos(\pi-\alpha)-a^2\right)\\
&= \frac{1}{4}\cdot\left(c^2+b^2-2bc\cos\alpha-a^2\right)=0.
\end{aligned}
$$

(We have used that $|BE|=|AC|=b$, $|CF|=|AB|=c$ and $\angle(\overrightarrow{BE},\overrightarrow{CF})=\angle BDC=\pi-\alpha$.)

Consequently, $\overrightarrow{BM}\perp\overrightarrow{CM}$, i.e. the angle $\angle BMC$ is right.

b) It is clear that the points M and A are in different semiplanes with respect to the line BC. According to a) $\angle BMC$ is a right angle and consequently if the point M is from the locus, then M lies on a semicircle k with diameter BC, k and A are in different semiplanes with respect to the line BC. Let $\angle BCM=\psi$. Then $CM=a\cos\psi$ (because $\angle BMC=\dfrac{\pi}{2}$) and

$$\overrightarrow{CM}\cdot\overrightarrow{CB}=a(a\cos\psi)\cdot\cos\psi=(a\cos\psi)^2.$$

On the other hand by the sine theorem we find:

$$
\begin{aligned}
\overrightarrow{CM}\cdot\overrightarrow{CB} &= \frac{1}{2}\cdot\left(\overrightarrow{CB}+\overrightarrow{BE}+\overrightarrow{CF}\right)\cdot\overrightarrow{CB}\\
&= \frac{1}{2}\cdot\left(a^2+ab\cos\left(\pi-(\alpha-\varphi)\right)+ac\cos\varphi\right)\\
&= \frac{1}{2}a^2\left(1-\frac{b}{a}\cos(\alpha-\varphi)+\frac{c}{a}\cos\varphi\right)\\
&= \frac{1}{2}a^2\left(1+\frac{\sin\gamma\cos\varphi-\sin\beta\cos(\alpha-\varphi)}{\sin\alpha}\right)\\
&= \frac{1}{2}a^2\left(1+\frac{\sin(\gamma-\varphi)-\sin(\beta-\alpha+\varphi)}{2\sin\alpha}\right)\\
&= \frac{1}{2}a^2\left(1+\cos(\beta+\varphi)\right)=\left(a\cos\frac{\beta+\varphi}{2}\right)^2.
\end{aligned}
$$

Consequently, $\cos^2\psi=\cos^2\dfrac{\beta+\varphi}{2}$. But $\dfrac{\beta+\varphi}{2}<\dfrac{\beta+\alpha}{2}<\dfrac{\pi}{2}$ and therefore $\psi=\dfrac{\beta+\varphi}{2}$, i.e. $\angle BCM=\dfrac{\beta+\varphi}{2}$ and $\angle BCM\in\left(\dfrac{\beta}{2},\dfrac{\beta+\alpha}{2}\right)$. In addition we have $\angle CBM=\dfrac{\pi}{2}-\angle BCM=\dfrac{\alpha+\gamma-\varphi}{2}$ and $\angle CBM\in\left(\dfrac{\gamma}{2},\dfrac{\alpha+\gamma}{2}\right)$.

Let K and N be points from the semicircle k, for which $\angle BCK=\dfrac{\beta}{2}$ and $\angle CBN=\dfrac{\gamma}{2}$. Then $angleBCN=\dfrac{\pi}{2}-\dfrac{\gamma}{2}=\dfrac{\alpha+\beta}{2}$ and $\angle CBK=\dfrac{\pi}{2}-\dfrac{\beta}{2}=\dfrac{\alpha+\gamma}{2}$.

It follows from the above considerations that when the point D describes the arc $\overset{\frown}{BC}$, then $CM^{\rightarrow}$ describes the interior of $\angle(CK^{\rightarrow},CN^{\rightarrow})$, while $BM^{\rightarrow}$ describes the interior of $\angle(BK^{\rightarrow},BN^{\rightarrow})$. Consequently, the locus, we are looking for is the arc $\overset{\frown}{NK}$ from the semicircle k (Figure 16).

Problem 3. is the same as problem 3, grade 10.

Spring mathematics tournament—1997

Problem 8'1. Given the equation $|x-a|+15=6|x+2|$, where a is a real parameter.

(a) Prove that for any value of a the equation has exactly two distinct roots x_1 and x_2.

(b) Prove that $|x_1-x_2|\geq 6$ and find all values of a for which $|x_1-x_2|=6$.

Solution: (a) When $x\leq -2$, the equation is equivalent to $|x-a|=-6x-27$, which has a solution only if $-6x-27\geq 0$, i. e., if $x\leq -\frac{9}{2}$. Considering the cases of both $x\geq a$ and $x\leq a$ shows that if $x\leq -2$, the given equation has a unique root x_1, and if $x>-2$, it has a unique root x_2:

$$x_1=\begin{cases}\dfrac{a-27}{7}, & a<-\dfrac{9}{2}\\ -\dfrac{a+27}{5}, & a\geq -\dfrac{9}{2}\end{cases}\qquad x_2=\begin{cases}\dfrac{3-a}{5}, & a\leq \dfrac{1}{2}\\ \dfrac{a+3}{7}, & a>\dfrac{1}{2}\end{cases}$$

(b) It follows from (a) that

$$|x_1 - x_2| = \begin{cases} \dfrac{156-12a}{35}, & a < -\dfrac{9}{2} \\ 6, & -\dfrac{9}{2} \le a \le \dfrac{1}{2} \\ \dfrac{12a+204}{35}, & a > \dfrac{1}{2} \end{cases}$$

It remains to be seen that $\dfrac{156-12a}{35} > 6$ when $a < -\dfrac{9}{2}$ and $\dfrac{12a+204}{35} > 6$ when $a > \dfrac{1}{2}$. Therefore $|x_1 - x_2| \ge 6$ and equality obtains only when $a \in [-\frac{9}{2}, \frac{1}{2}]$.

Problem 8'2. Let O be the intersecting point of the diagonals of the convex quadrilateral $ABCD$ and let $\angle DAC = \angle DBC$. The midpoints of AB and CD are respectively M and N and P and Q are points on AD and BC respectively such that $OP \perp AD$ and $OQ \perp BC$. Prove that $MN \perp PQ$.

Solution: Denote by E and F the midpoints of AO and BO. Then $\triangle PEM \cong \triangle MFQ$ since $MF = \frac{1}{2}AO = PE$, $QF = \frac{1}{2}OB = ME$ and $\angle MEP = \angle MEO + \angle OEP = \angle MEO + 2\angle DAC = \angle MFO + 2\angle DBC = \angle MFQ$ ($MFOE$ is a parallelogram). Therefore $MP = MQ$. The case when E and F are interior points for $\angle PMQ$ is treated similarly (prove that there are no other possibilities). By analogy we conclude that $NP = NQ$. Hence M and N lie on the axis of symmetry of PQ and so $MN \perp PQ$.

Problem 8'.3. Find all natural numbers n such that there exists an integer number x for which $499(1997^n + 1) = x^2+x$.

Solution: Let n be a solution of the problem. Then $(2x + 1)^2 = 1996 \cdot 1997^n + 1997$. If $n = 1$ we get $(2x + 1)^2 = 1997^2$ and so $2x + 1 = \pm 1997$. Therefore $x = 998$ and $x = -999$ satisfy the conditions of the problem. Let $n \geq 2$. Now $(2x + 1)^2$ is divisible by 1997, which is a prime number, and so $(2x + 1)^2$ is divisible by 1997^2. But this is impossible, since $1996 \cdot 1997^n + 1997$ is not divisible by 1997^2 when $n \geq 2$. The only solution is $n = 1$.

Problem 8.1. Find all values of the real parameter m such that the equation $(x^2 - 2mx - 4(m^2 + 1))(x^2 - 4x - 2m(m^2 + 1)) = 0$ has exactly three distinct roots.

Solution: Suppose m satisfies the conditions of the problem and the equations

$$\begin{aligned} x^2 - 2mx - 4(m^2 + 1) &= 0 \\ x^2 - 4x - 2m(m^2 + 1) &= 0 \end{aligned}$$

share the root x_0. After subtracting we get $(2m - 4)x_0 = (2m - 4)(m^2 + 1)$ and so $x_0 = m^2 + 1$ (note that if $m = 2$, the two equations coincide). Substituting x_0 in any of the equations gives the equation $(m^2 + 1)(m^2 - 2m - 3) = 0$ with roots $m = -1$ and $m = 3$. Direct verification shows that the condition is satisfied only for $m = 3$.

Let now (1) and (2) share no roots. Since $D_1 = 4 + 5m^2 > 0$ (1) always has two distinct roots and therefore (2) should have equal roots. Thus $D_2 = 4 + 2m(m^2+1) = 0$ and so $m = -1$. But this case has already been considered. Thus we determine that $m = 3$.

Problem 8.2. The area of the equilateral triangle ABC is 7. Points M and N are chosen respectively on AB and AC so that $AN = BM$. Denote by O the intersecting point of the straight lines BN and CM. The area of BOC is 2.

(a) Prove that $MB : AB = 1 : 3$ or $MB : AB = 2 : 3$.

(b) Find $\angle AOB$.

Solution: (a) Denote $\dfrac{MB}{AB} = x$. Therefore $S_{ABN} = 7x = S_{BMC}$ and so $S_{BOM} = 7x - 2$ and $S_{AMON} = S_{BOC} = 2$. Further $S_{CON} = 7 - 2 - 2 - (7x-2) = 5 - 7x$, $S_{ANO} = \dfrac{x}{1-x} \cdot S_{CNO} = \dfrac{x(5-7x)}{1-x}$, $S_{AMO} = \dfrac{1-x}{x} \cdot S_{BOM} = \dfrac{1-x}{x}(7x-2)$. It follows from $S_{AMON} = S_{ANO} + S_{AMO}$ that $2 = \dfrac{x(5-7x)}{1-x} + \dfrac{1-x}{x}(7x-2)$, and thus $9x^2 - 9x + 2 = 0$. The roots of the above equation are $x_1 = \dfrac{1}{3}$ and $x_2 = \dfrac{2}{3}$.

(b) Since $\triangle ABN \cong \triangle BMC$, we get $\angle BOM = \angle BCM + \angle CBO = \angle MBO + \angle CBO = 60^\circ$. Further $\angle MAN + \angle MON = 180^\circ$ and therefore the quadrilateral $AMON$ is inscribed in a circle. Let $\dfrac{MB}{AB} = \dfrac{1}{3}$, i. e., $AM = 2BM = 2AN$. Denote by Q the midpoint of AM. Triangle AQN is isosceles and has an angle equal to 60°, so it is equilateral. Therefore Q is the circumcentre of $AMON$ and $\angle AOM = \angle ANM = 90^\circ$. Thus $\angle AOB = 150^\circ$. Similarly, if

$\dfrac{MB}{AB} = \dfrac{2}{3}$, i. e., $2AM = MB = AN$, we get $\angle AMN = \angle AON = 90^\circ$, so $\angle AOB = 90^\circ$.

Problem 8.3. Given n points, $n \geq 5$, in the plane such that no three lie on a line. John and Peter play the following game: On his turn each of them draws a segment between any two points which are not connected. The winner is the one after whose move every point is an end of at least one segment. If John is to begin the game, find the values of n for which he can always win no matter how Peter plays.

Solution: Call a point isolated if it is not an end of a segment. John wins exactly when there are 1 or 2 isolated points before his last move. Peter is forced to reach the above only if before his move there are exactly 3 isolated points and any of the remaining $n-3$ points are connected by a segment. Indeed, if there are at least 4 isolated points he could connect one of them with a non-isolated point. If the isolated points are 3 but not all of the remainig $n-3$ points are connected he could draw a missing segment. Since the number of segments with ends in $n-3$ points is $\dfrac{(n-3)(n-4)}{2}$, we determine that John wins only when $\dfrac{(n-3)(n-4)}{2}$ is an odd integer number. This is true when n is of the form $4k+1$ or $4k+2$.

Problem 9.1. Let $f(x) = x^2 - 2ax - a^2 - \dfrac{3}{4}$ where a is real parameter. Find all values of a such that the inequality $|f(x)| \leq 1$ holds for any x in the interval $[0, 1]$.

Solution: Let M and m be the maximum and minimum values of $f(x)$ in the interval $[0,1]$. Then the condition of the problem is equivalent to $M \leq 1$ and $m \geq -1$. There are three cases to consider.

Case 1: $a \in [0,1]$. Then $m = f(a) = -2a^2 - \frac{3}{4}$ and $M = f(0) = -a^2 - \frac{3}{4}$ or $M = f(1) = -a^2 - 2a + \frac{1}{4}$. It follows from $m \geq -1$ and $M \leq 1$ that $a \in [-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}]$ and $a \in (-\infty, -\frac{3}{2}] \cup [-\frac{1}{2}, \infty)$. In this case the solution is $a \in [0, \frac{\sqrt{2}}{4}]$.

Case 2: $a < 0$. Now $m = f(0)$ and $M = f(1)$. From $m \geq -1$ and $M \leq 1$ we get that $a \in [-\frac{1}{2}, \frac{1}{2}]$. Therefore $a \in [-\frac{1}{2}, 0)$.

Case 3: $a > 1$. Now $m = f(1)$ and $M = f(0)$. It follows from $m \geq -1$ and $M \leq 1$ that $a \in [-\frac{5}{2}, \frac{1}{2}]$ which is a contradiction with $a > 1$.

Thus the solution is $a \in [-\frac{1}{2}, \frac{\sqrt{2}}{4}]$.

Problem 9.2. Let I and G be the incentre and the centre of $\triangle ABC$ with sides $AB = c$, $BC = a$, $CA = b$.

(a) Prove that (if $a > b$) the area of CIG equals $\frac{(a-b)r}{6}$ where r is the inradius of ABC.

(b) If $a = c+1$ and $b = c-1$, prove that the segment IG is parallel to AB and find its length.

Solution: (a) We shall use the usual notation for a triangle. Let CL and CM be respectively the bisector and the median from C. It follows from $CG = \frac{2}{3}CM$ that $S_{CIG} = \frac{2}{3}S_{CIM}$. Thus $S_{CIM} = S_{CLM} - S_{ILM} = \frac{LM \cdot h_c}{2} - \frac{LM \cdot r}{2} = \frac{LM}{2}(h_c - r)$. We find from $AL + BL = c$ and $\frac{AL}{BL} = \frac{b}{a}$ that $AL = \frac{bc}{a+b}$ and so $LM = AM - AL = \frac{c}{2} - \frac{bc}{a+b} = \frac{c(a-b)}{2(a+b)}$. Also, $h_c - r = \frac{2S}{c} - r = \frac{2pr}{c} - r = \frac{r(2p-c)}{c} = \frac{r(a+b)}{c}$. Therefore $S_{CIG} = \frac{2}{3}S_{CIM} = \frac{2}{3} \cdot \frac{c(a-b)}{4(a+b)} \cdot \frac{r(a+b)}{c} = \frac{(a-b)r}{6}$.

(b) The distances from I and G to AB are respectively r and $\frac{h_c}{3}$. Hence $r = \frac{S}{p} = \frac{ch_c}{2p} = \frac{ch_c}{3c} = \frac{h_c}{3}$ and so $IG \| AB$. Therefore the altitude from C of triangle CIG equals $\frac{2}{3}h_c = 2r$. Thus $S_{CIG} = IG \cdot r$. On the other hand $S_{CIG} = \frac{(a-b)r}{6} = \frac{r}{3}$ and so $IG = \frac{1}{3}$.

Problem 9.3. Let $n \geq n$ be an even number and A be a subset of $\{1, 2, \ldots, n\}$. Consider the sums of the form $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3$, where x_1, x_2, x_3 are integer numbers in A (not necessarily distinct), $\varepsilon_1, \varepsilon_2, \varepsilon_3$ (at least one of which is not 0) belong to $\{-1, 0, 1\}$ and none of the elements of A appears with coefficients 1 and -1 in any of the sums. Call A a 'free' set if n divides none of the above sums.

(a) Construct a 'free' set having $[\frac{n}{4}]$ elements ($[x]$ is the least integer number less than or equal to x).

(b) Prove that no set of $[\frac{n}{4}]+1$ elements is 'free'.

Solution: (a) The set $A=\{1,3,\ldots,2[\frac{n}{4}]-1\}$ is 'free' and has $[\frac{n}{4}]$ elements (prove it!).

(b) Let $n=4m$ and suppose that $A\subset\{1,2,\ldots,n\}$ is a 'free' set having $[\frac{n}{4}]+1=m+1$ elements. Without loss of generality assume that $A\subset\{1,2,\ldots,2m\}$ (since we can replace $x\in A$ by $n-x$). We shall show that there exist two elements of A whose sum is equal to another element of A. Indeed, let $a_1<a_2<\cdots<a_{m+1}$ be the elements of A and consider the set $B=\{a_1+a_i : i=1,2,\ldots,m+1\}$. There are $2m+2$ integer numbers $a_1,a_2,\ldots,a_{m+1},2a_1,a_1+a_2,\ldots,a_1+a_{m+1}$ from $A\cup B$ and they lie in the interval $[a_1,a_1+a_{m+1}]$, which contains exactly $a_{m+1}+1\le 2m+1$ integer numbers. This gives $a_i=a_1+a_j$ for some i,j. But then $a_1+a_j-a_i=0$, which is impossible. The case $n=4m+2$ is settled in a similar fashion. Notice that $2m+1$ cannot be an element of a 'free' set.

Problem 10.1. Find the least natural number a such that the equation $\cos^2\pi(a-x)-2\cos\pi(a-x)+\cos\frac{3\pi x}{2a}\cdot\cos(\frac{\pi x}{2a}+\frac{\pi}{3})+2=0$ has a root.

Solution: The roots of the our equations are the common roots of $\cos\pi(x-a)=1$ and $\cos\frac{3\pi x}{2a}\cdot\cos(\frac{\pi x}{2a}+\frac{\pi}{3})+1=0$. The roots of the first one are $x=a+2n,\ n=0,\pm1,\pm2,\ldots$ and the roots of the second one are $x=2a(k-\frac{1}{3}),\ k=0,\pm1,\ldots$. Therefore $a=\frac{6n}{6k-5}$

for some integer numbers n and k. It is easy to see now that the least natural number with the required property is $a = 6$.

Problem 10.2. Point F lies on the base AB of a trapezoid $ABCD$ and is such that $DF = CF$. Let E be the intersecting point of AC and BD and O_1 and O_2 are circumcentres of ADF and FBC respectively. Prove that the straight lines FE and O_1O_2 are orthogonal.

Solution: Let k_1 and k_2 be circles with centres O_1 and O_2 and let the intersecting points of the two circles be points P and Q. It is well known that $PQ \perp O_1O_2$. On the other hand, if L is an arbitrary point and two lines through L intersect k_1 and k_2 in points A, B and C, D respectively. Then $L \in PQ$ if and only if $LA \cdot LB = LC \cdot LD$. Let k_1 and k_2 be the circumscribed circles of $\triangle AFD$ and $\triangle FBC$ and let G be the intersecting point of FE with CD. Denote by C_1 and D_1 those points on DC for which $AD_1 || CF$ and $BC_1 || DF$, i. e., such that the quadrilaterals $AFCD_1$ and $BFDC_1$ are parallelograms. Using that $FD = FC$ we get $\angle CFB = \angle FCD = \angle FDC = 180^\circ - \angle BC_1C$. This means that F, B, C and C_1 lie on a circle and so the line DC intersects k_2 in C and C_1. By analogy $\angle AFD = \angle FDC = \angle FCD = \angle AD_1D$ and so line DC meets k_1 in points D and D_1. In accordance with the initial notes FE is perpendicular to OO_1 if and only if $GC \cdot GC_1 = GD \cdot GD_1$. It follows from $\triangle GCE \sim \triangle FAE$, $\triangle GDE \sim \triangle FBE$ and $\triangle DCE \sim \triangle BAE$ that $\dfrac{GC}{AF} = \dfrac{CE}{EA} = \dfrac{DC}{AB}$ and $\dfrac{GD}{BF} = \dfrac{DE}{EB} = \dfrac{DC}{AB}$. Thus $GC = \dfrac{DC \cdot AF}{AB}$ and $GD = \dfrac{BF \cdot DC}{AB}$. On the other hand $GC_1 = |DC_1 - DG| = |BF - DG| = BF|1 - \dfrac{DG}{BF}| = BF|1 - \dfrac{DC}{AB}| = \dfrac{BF}{AB} \cdot |AB - DC|$,

$GD_1 = |CD_1 - CG| = |AF - CG| = AF|1 - \frac{CG}{AF}| = AF|1 - \frac{DC}{AB}| = \frac{AF}{AB} \cdot |AB - DC|$. Therefore $GC \cdot GC_1 = \frac{DC \cdot AF \cdot BF}{AB^2} \cdot |AB - DC| = GD \cdot GD_1$.

Problem 10.3. Find all natural numbers n for which a convex n-gon can be partitioned into triangles through its diagonals in such a way that there is an even number of diagonals from each vertex. (If there is a vertex with no digonals through it, assume that there is an even number (zero) of diagonals from this vertex).

Solution: It is easy to see by induction that if an n-gon is partitioned into triangles through d non-intersecting diagonals then $n = d + 3$. Let n be a natural number and $A_1 A_2 \cdots A_n$ is a convex n-gon which can be partitioned into triangles through d diagonals in a way that there is an even number of diagonals through each vertex. Since $n = 3$ is a solution we may assume that $n \geq 4$.

It is clear that at least one side of each triangle is a diagonal of the n-gon. We say that a triangle is of type t_k, $(k = 1, 2, 3)$ if exactly k of its sides are diagonals. Denote by x_k the number of triangles of type t_k. It is easy to see that $2x_1 + x_2 = n = d + 3$ abd $x_1 + 2x_2 + 3x_3 = 2d$. It follows now that $x_1 = x_3 + 2$, so $x_1 > 0$. Therefore there exists a triangle two of whose sides are sides of the n-gon. Let that be $A_{j-1}A_jA_{j+1}$. Diagonal $A_{j-1}A_{j+1}$ is a side of another triangle—e. g., $A_{j-1}A_{j+1}A_s$. Assume that $A_{j-1}A_s$ or $A_{j+1}A_s$ is a side of the n-gon. If it is $A_{j-1}A_s$ then $s = j - 2$. It follows now that there are no diagonals from A_{j-1} distinct from $A_{j-1}A_{j+1}$ because such a diagonal intersects $A_{j+1}A_s$. This contradicts the premise that there is an even number of diagonals from each vertex. Therefore both $A_{j-1}A_s$ and $A_{j+1}A_s$ are diagonals so $A_{j-1}A_{j+1}A_s$ is of type t_3. Hence there is a triangle of type t_3 adjacent to each triangle of type t_1. If distinct triangles of type t_1 are adjacent to distinct triangles of type t_3 then $x_1 \leq x_3 = x_1 - 2 < x_1$, a contradiction. Therefore there are at least two triangles of type t_1 adjacent to one and the same triangle of type t_3. Without loss of generality assume these are the triangles $A_1A_nA_{n-1}$ and $A_{n-1}A_{n-2}A_{n-3}$. Consider the polygon $A_1A_2\cdots A_{n-3}$. Obviously the diagonals partition this polygon into triangles and there is an even number of diagonals through each vertex.

Conversely, if the polygon $A_1A_2\cdots A_{n-3}$ can be partitioned in the required way, then adding the vertices A_{n-2}, A_{n-1}, A_n and diagonals $A_{n-3}A_{n-1}$ and A_1A_{n-1} shows that the same is true for the polygon $A_1A_2\cdots A_n$.

Therefore a natural number $n \geq 6$ is a solution if and only if $n-3$ is a solution. It is easy to see that $n=3$ is a solution, whereas $n=4$ and $n=5$ are not. Thus all natural numbers satisfying the conditions of the problem are $n=3k$, $k=1,2,\ldots$.

Problem 11.1. For any real number b denote by $f(b)$ the maximal value of $|\sin x + \dfrac{2}{3+\sin x} + b|$. Find the minimal value of $f(b)$.

Solution: Substitute $t = \sin x$ and $g(t) = t + \dfrac{2}{3+t} + b$. Since $g(t)$ is an increasing function in the interval $[-1,1]$, it follows that $f(b) = \max(|g(-1)|, |g(1)|) = \max(|b|, |b+\frac{3}{2}|)$. Now from the graph of the function $f(b)$ we conclude that $\min f(b) = f(-\frac{3}{4}) = \frac{3}{4}$.

Problem 11.2. A convex quadrilateral $ABCD$ is such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O be respectively the orthocentre and the circumcentre of $\triangle ABC$. Prove that H, O and D lie on a line.

Solution: Let $\angle CAB = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$. Note that $\alpha < \beta$ and $\gamma < \beta$. There are three cases to consider: $\beta < 90°$, $\beta = 90°$ and $\beta > 90°$. Suppose first that $\beta < 90°$. Then O and H are interior points for $\triangle ABC$ and $\angle ACO = \angle CAO = \angle HCB = \angle HAB = 90° - \beta$. Therefore O is an interior point for $\triangle HAC$ and $\angle HAO = \beta - \gamma = \angle ACD$, $\angle HCO = \beta - \alpha = \angle CAD$, $\angle HAD = \angle HCD = 2\beta - 90°$. It follows from the Sine Theorem for $\triangle AHD$, $\triangle CHD$ and $\triangle ACD$ that $\dfrac{\sin\angle AHD}{\sin\angle HAD} = \dfrac{AD}{HD}$, $\dfrac{\sin\angle HCD}{\sin\angle CHD} = \dfrac{HD}{CD}$, $\dfrac{\sin\angle CAD}{\sin\angle ACD} = \dfrac{CD}{AD}$. By multiplying the above equalities we get

$$\sin\angle AHD\cdot\sin\angle HCO\cdot\sin\angle CAO = \sin\angle CHD\cdot\sin\angle HAO\cdot\sin\angle ACO.$$

It follows now from Ceva's Theorem that AO, CO and HD intersect in a point and so H, O and D lie on a line. In the case of $\beta = 90°$ we obtain that $H \equiv B$, O is the midpoint of AC and $AHCD$ is a rectangle. Therefore H, O and D lie on a line. Finally, let $\beta > 90°$. In this case B and O are interior points for $\triangle AHC$ and $\triangle ADC$ respectively. Similarly to the case $\beta < 90°$ we get that the points H, O and D lie on a line.

Problem 11.3. For any natural number $n \geq 3$ denote by $m(n)$ the maximum number of points which can be placed inside or on the outline of a regular n-gon with side 1 in a way that the distance between any two of them is greater than 1. Find all n for which $m(n) = n - 1$.

Solution: We prove first that $m(n) = n - 1$ for $n = 4, 5, 6$. Let n be one of the above numbers and let $B_1, B_2, \cdots, B_n$ be points satisfying the conditions of the problem for the regular n-gon $A_1A_2\cdots A_n$ of side 1 and centre O. It is obvious that $OB_i \leq 1$ and therefore no three points O, B_i, B_j, $1 \leq i \neq j \leq n$ lie on a line. Furthermore at least one of the angles OB_iB_j is less than $\dfrac{2\pi}{n} \leq 90°$ and it follows from the Cosine Theorem that $B_iB_j^2 \leq OB_i^2 + OB_j^2 - 2OB_iOB_j\cos\dfrac{2\pi}{n} = B_i'B_j'^2$ where B_i' and B_j' are points on the segments OA_1 and OA_2 such that $OB_i' = OB_i$, $B_j' = OB_j$. When $n = 4, 5, 6$ the greatest side in $\triangle OA_1A_2$ is $A_1A_2 = 1$ and therefore $B_iB_j \leq B_i'B_j' \leq 1$, which is a contradiction. Thus $m(n) \leq n - 1$ if $n = 4, 5, 6$. It is easily seen that for these n there exist $n - 1$ points on the outline of a regular n-gon $A_1A_2\cdots A_n$ with the required property. Therefore $m(n) = n - 1$ if $n = 4, 5, 6$. We shall prove now that if $n \geq 7$ then $m(n) \geq n$. Let $B_i \in A_iA_{i+1}$ be points such that $A_iB_i = 2^{i-1}\varepsilon$, $1 \leq i \leq n-1$ where $0 < \varepsilon < \dfrac{1}{2^{n-1}}$ is arbitrary chosen. From $n \geq 7$ we obtain $\cos\angle A_1A_2A_3 = -\cos\dfrac{2\pi}{n} < -\dfrac{1}{2}$. It follows now that ${B_1B_2}^2 > (1-\varepsilon)^2 + 4\varepsilon^2 + 2\varepsilon(1-\varepsilon)^2 = 1 + 3\varepsilon^2 > 1$. Similarly $B_iB_{i+1} > 1$ when $1 \leq i \leq n-2$. Further, it is clear that $B_1B_{n-1} > A_1A_n = 1$. Since $OA_1 = OA_2 = \cdots = OA_n > 1$, $A_iA_j > 1$ when $|i - j| \geq 2$ and $B_i \to A_i$ when $\varepsilon \to 0$, it follows that we can make ε so small that $OB_i > 1$ when $1 \leq i \leq n-1$ and $B_iB_j > 1$ when $|i-j| \geq 2$. Then the points $B_1, B_2, \cdots, B_{n-1}, O$ satisfy the conditions of the problem and so $m(n) \geq n$ when $n \geq 7$. Since it is obvious that $m(3) = 1$, we come to the conclusion that $m(n) = n - 1$ only when $n = 4, 5, 6$.

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Problem 8'.1. Find all values of the real parameter a such that the inequalities $|x+1|+|2-x|<a$ and $\dfrac{5a-8}{6x-5a+5}<-\dfrac{1}{2}$ are equivalent.

Solution: We begin with the first inequality. $|x+1|+|2-x|<a$. If $x<-1$, it is equivalent to $-x-1-x+2<a$ or $x>\dfrac{1-a}{2}$, i.e., $\dfrac{1-a}{2}<x<-1$, which has a solution if $\dfrac{1-a}{2}<-1$, i.e., if $a>3$. If $x\in[-1;2]$, then $|x+1|+|2-x|=x+1+2-x=3$ and the above equation has a solution only when $a>3$. Finally, if $x>2$, then $x+1+x-2<a$ or $x<\dfrac{a+1}{2}$, i.e., $2<x<\dfrac{a+1}{2}$, which has a solution when $a>3$. Therefore when $a\le 3$, the inequality has no solution, and when $a>3$, the solutions form the interval $\left(\dfrac{1-a}{2};\dfrac{1+a}{2}\right)$.

Let us rewrite the second inequality in the form $\frac{6x+5a-11}{6x-5a+5} < 0$. Its solutions form either the interval $\left(\frac{11-5a}{6}; \frac{5a-5}{6}\right)$ or the interval $\left(\frac{5a-5}{6}; \frac{11-5a}{6}\right)$, depending on which of the two numbers $\frac{5a-5}{6}$ and $\frac{11-5a}{6}$ is greater.

Therefore the two inequalities are equivalent if

$$\frac{1-a}{2} = \frac{11-5a}{6}, \qquad \frac{1+a}{2} = \frac{5a-5}{6}$$

or

$$\frac{1-a}{2} = \frac{5a-5}{6}, \qquad \frac{1+a}{2} = \frac{11-5a}{6}.$$

The first pair is satisfied by $a = 4$, the second one by $a = 1$. But if $a = 1$, the first inequality has no solution, whereas the second one does. Thus the only solution is $a = 4$.

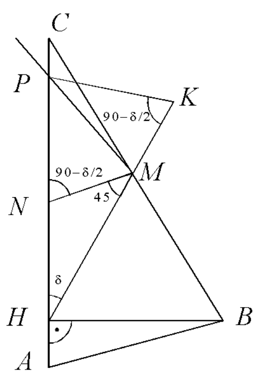
Problem 8'.2. Let M and N be the midpoints of the sides BC and AC and $BH, (H \in BC)$ the altitude in $\triangle ABC$. The straight line perpendicular to the bisector of $\angle HMN$ intersects the line AC in point P such that $HP = \frac{1}{2}(AB + BC)$ and $\angle HMN = 45\,\mathrm{deg}$.

a.) Prove that $\triangle ABC$ is isosceles.

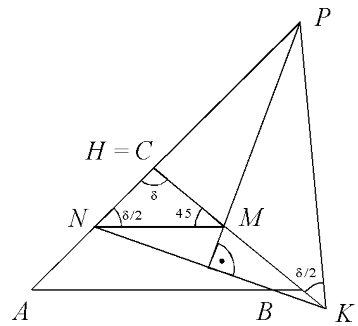
b.) Find the area of $\triangle ABC$ if $HM = 1$.

Solution: **a)** Since HM is a median to the hypothenuse BC of the right triangle $\triangle BHC$ and MN is a middle segment in $\triangle ABC$, it follows that $HM = \frac{1}{2}BC$ (if $H \equiv C$, again $HM = \frac{1}{2}BC$) and

$MN = \frac{1}{2}AB$. Therefore $HP = HM + MN$. Denote $\angle HMN = \delta$. There are two cases to consider for the points H, N and P: **1.** N lies between H and P; **2.** H lies between N and P.



1. Let us find a point K on the extension of HM such that $MK = MN$. Then $HP = HK$ and $\angle HKP = 90\deg - \frac{\delta}{2}$. The statement of the problem implies that MP is the bisector of $\angle NMK$ (external to $\triangle HMN$), i.e., $\angle NMP = \angle KMP$. Therefore $\triangle PNM \cong \triangle PKM$ and $\angle PNM = \angle PKM = 90\deg - \frac{\delta}{2}$. On the other hand, $\angle PNM = 45\deg + \delta$, i.e., $90\deg - \frac{\delta}{2} = 45\deg + \delta$, whence $\delta = 30\deg$ and $\angle HNM = 105\deg$. Also $\angle MHB = 60\deg$ and since $HM = MB$, it follows that $\angle HMB = 60\deg$. Thus $\angle BMN = 105\deg = \angle ANM$, i.e., $ABMN$ is a isosceles trapezoid $AN = BM$. Therefore $AC = BC$.



2. Let H lie between N and P and let K be a point such that $MK = MN$ and $HP = HK$. It follows from the isoscelesness of $\triangle KHP$ that $\angle HPK = \angle HKP = \dfrac{\delta}{2}$. Since MP is the bisector of $\angle NMK$ in the isosceles $\triangle NMK$, it follows that MP is the axis of symmetry of NK. Thus $NP = KP$ and $\triangle NMP \cong \triangle KMP$. Therefore $\angle PNM = \dfrac{\delta}{2}$ and since $\triangle NMP$, it follows that $\delta + \dfrac{\delta}{2} + 45\,\text{deg} = 180\,\text{deg}$. and hence $\delta = 90\,\text{deg}$. Therefore $MH \perp AC$ and since $BH \perp AC$, we get that H coincides with C and $\angle BCA = 90\,\text{deg}$, $\angle ABC = \angle BAC = 45\,\text{deg}$ and so $AC = BC$.

b) 1. When $\delta = 30\,\text{deg}$: $\angle ACB = 30\,\text{deg}$, $HM = HB = 1$, $AC = BC = 2$. Then $S_{ABC} = \dfrac{1}{2} AC \cdot BH = 1$.

2. When $\delta = 90\,\text{deg}$: $AC \perp BC$, $AC = BC = 2BH = 2$.

Then $S_{ABC} = \dfrac{1}{2} AC \cdot BC = 2$.

Problem 8'.3. (Problem for the UBM award)

Is it possible to find 100 straight lines in the plane such that there are exactly 1998 intersecting points?

Solution: Consider 99 lines such that 73 of them are parallel and the remaining 26 lines pass through a single point and intersect all 73 parallel lines. Then the total number of intersecting points is $73 \cdot 26 + 1 = 1899$. Choose the last line in such a way that it intersects all lines and does not pass through any of the points. Now there are $1899 + 99 = 1998$ intersecting points.

Problem 8.1. The graph of a linear function is parallel to the graph of $y = \frac{5}{4}x + \frac{95}{4}$, passing through $M(-1; -25)$, and intersects the coordinate axes Ox and Oy in A and B correspondingly.

(a) Find the coordinates of A and B.

(b) Consider the unity grid in the plane. Find the number of squares containing points of AB (in their interiors).

Solution: (a) The graph of a linear function is parallel to the graph of $y = \frac{5}{4}x + \frac{95}{4}$ when the linear function is of the form $y = \frac{5}{4}x + b$. From the condition that M belongs to its graph we determine $-25 = -\frac{5}{4} + b$, whence $b = -\frac{95}{4}$. The coordiantes of A and B are respectively the solutions of the systems

$$\left| \begin{array}{l} y = \frac{5}{4}x - \frac{95}{4} \\ y = 0 \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} y = \frac{5}{4}x - \frac{95}{4} \\ x = 0 \end{array} \right. ,$$

whence we get $A(19;0)$ and $B\left(0;-\dfrac{95}{4}\right)$.

(b) The coordinates of the points on AB satisfy the conditions

$$(1)\qquad \left|\begin{array}{l} y=\dfrac{5}{4}x-\dfrac{95}{4} \\ 0\le x\le 19 \end{array}\right. .$$

Find the number of points with integer coordinates lying on the segment AB, i. e., the number of integer solutions of (1). It follows from $y=\dfrac{5}{4}x-\dfrac{95}{4}$ that $y=x-23+\dfrac{x-3}{4}$ and if x and y are integer, then $\dfrac{x-3}{4}=t$ is integer. Conversely, if t is integer, then $x=4t+3$ and $y=5t-20$ are integer. Since $0\le x\le 19$, we find $0\le 4t+3\le 19$, i. e., $-\dfrac{3}{4}\le t\le 4$. This observation implies that the points with integer coordinates lying on AB satisfy the conditions $x=4t+3$, $y=5t-20$ for $t=0,\ 1,\ 2,\ 3,\ 4$. Therefore there are 5 such points.

Further: the segment AB exits a square and enters another one only when it intersects a line of the grid. The number of such intersections is $23+18=41$. But when AB passes through a knot (i. e., through a point with integer coordinates), the passage from a square to another one involves crossing two lines. This happens 4 times (not counting A). Therefore the required number is $41-4+1=38$.

Problem 8.2. Let l_1 and l_2 be the loci of the centroid G and the incentre I of the right triangle ABC whose hypotenuse AB is a given segment of length c.

(a) Find l_1 and l_2.

(b) Find the area of $\triangle ABC$ when the length of GI is minimal.

Solution: (a) The vertex C of $\triangle ABC$ could be placed in either semiplane with respect to AB. The centroid G lies on a circle k_1 of radius $\dfrac{c}{6}$ and centred at the midpoint O of segment AB. The incentre I lies on one of the two arcs which are the locus of the points X such that $\angle AXB = 135\deg$. One of the arcs is part of a circle k_2' with centre Q' and radius R and the other one is part of a circle k_2'' with centre Q'' and the same radius and Q' and Q'' lie in different semiplanes with respect to AB.

Conversely: Let G be a point on k_1 distinct from the intersecting points M and N of k_1 and AB. Then the vertex C of $\triangle ABC$ is uniquely determined by $CO = \dfrac{c}{2}$. Since $AO = BO = CO$ it follows that $\triangle ABC$ is a right triangle. Choose that of the points Q' and Q'' which does not lie in the same semiplane as C does with respect to AB. Without loss of generality this is Q'. Denote the intersecting point of the bisector of $\angle BAC$ and k_2' by I. We shall prove that I is the incentre $\triangle ABC$. Since $\angle ABI = 45\deg - \angle BAI$, we get $2 \cdot \angle ABI = 90\deg - 2 \cdot \angle BAI = \angle ABC$.

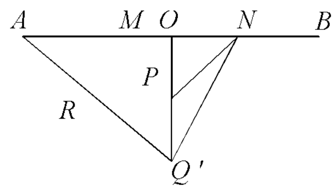
Therefore BI is the bisector of $\angle ABC$.

Let I be a point of k_2' such that $\angle AIB = 135\deg$. Then the vertex C is uniquely determined by: AI is the bisector of $\angle BAC$, BI is the bisector of $\angle ABC$ and C lies in one and the same semiplane with I with respect to AB. Since $\angle BAI + \angle ABI = 45\deg$, it follows that $\triangle ABC$ is a right triangle. Therefore $CO = \dfrac{c}{2}$ and hence the intersecting point G of CO and k_1 is the centroid of $\triangle ABC$.

These observations imply that l_1 is the circle k_1 without M and

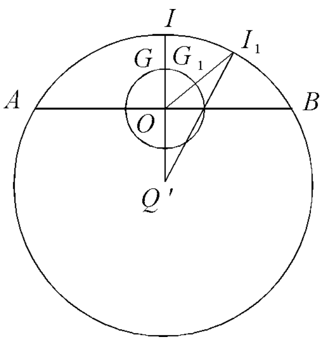
N, and l_2 consists of two arcs k_2' and k_2'' such that if $X \in k_2'$ or $X \in k_2''$, then $\angle AXB = 135\,\text{deg}$.

(b) We shall show that $R > \dfrac{2}{3}c$ and it will follow (how?) that k_1 lies in the interior of both k_2' and k_2''. Let P be the intersecting point of OQ' and k_1 which lies between O and Q'.

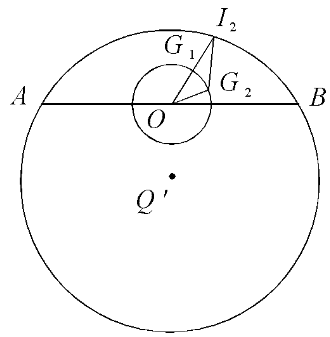


Since $PQ' = \dfrac{c}{3} = \dfrac{c}{6} + \dfrac{c}{6} = OP + ON$ and $\triangle PNO$, it follows that $OP + ON > PN$ and so $PQ' > PN$. Hence $\angle PNQ' > \angle PQ'N$. Therefore $\angle PNQ' + 45\,\text{deg} > \angle PQ'N + 45\,\text{deg}$, so $\angle ANQ' > \angle AQ'N$.

It follows for $\triangle AQ'N$ that $AQ' > AN$, i. e.,$R > \dfrac{2}{3}c$. Let $G \in l_1$ and $I \in l_2$ be such that Q', O, G and I lie in a line in this order. We shall show that GI is the least possible segment.



Case 1: $G_1 \in l_1$, $I_1 \in l_2$ and O lie on a line. Then $G_1I_1 > GI$. Indeed, it follows from $\triangle Q'I_1O$ that $OQ' + OI_1 > Q'I_1$, i. e., $OQ' + OG_1 + G_1I_1 > R = OQ' + OG + GI$ and so $G_1I_1 > GI$.



Case 2: $G_2 \in l_1$, $I_2 \in l_2$ and O are not on a line. Let OI_2 intersect k_1 in a point G_1. Then $G_2I_2 > G_1I_2$. Indeed, it follows from $\triangle OG_2I_2$ that $OG_2 + G_2I_2 > OI_2$. But $OI_2 = OG_1 + G_1I_2 = OG_2 + G_1I_2$ and therefore $G_2I_2 > G_1I_2$. But from case 1 we get that $G_1I_2 > GI$ and so $G_2I_2 > GI$.

Thus $\triangle ABC$ having the required property is such that Q', O, G and I lie on a straight line. Similarly, Q'' determines a triangle equal to the first one. Since $OQ' \perp AB$, $\triangle ABC$ is an isosceles triangle and so $S_{ABC} = \dfrac{c^2}{4}$.

Problem 8.3. (Problem for the UBM award)

Given n points on a circle such that no three chords with ends in the given points intersect in a point. Prove that there exists n such that there are $\dfrac{n^2 - 3n + 4}{2}$ chords with ends in the given points partitioning the interior of the circle into 1998 regions.

Solution: First we shall prove the following *Lemma:* The number of regions into which the interior of a circle is divided by drawing all $\binom{n}{2}$ chords with ends in n given points, provided no three chords intersect in a point, is $\binom{n}{4} + \binom{n}{2} + 1$.

Use induction by n. It is easy to see that the above formula holds for $n = 2$. Suppose it is true for some n. To obtain the result we count how many new regions are added when a new point appears on the circle. It is easily seen that if a chord intersects t other chords, then it 'adds' $t + 1$ new regions. Therefore the new regions are:

$$\sum_{k=0}^{n-1}(k(n-k-1)+1).$$

Given that $1+2+\cdots+n-1 = \dfrac{(n-1)n}{2}$ and $1^2+2^2+\cdots+(n-1)^2 = \dfrac{(n-1)n(2n-1)}{6}$, we easily obtain that the above sum equals

$$\binom{n+1}{4} + \binom{n+1}{2} + 1 - \left(\binom{n}{4} + \binom{n}{2} + 1\right),$$

which completes the proof.

We show now that if $n = 17$ it is possible to draw $\dfrac{n^2 - 3n + 4}{2} = 121$ chords such that there are 1998 regions. Let us draw all chords with ends in 16 of the given points (there are 120 such chords). It follows then that the interior of the circle is divided into $\binom{16}{4} + \binom{16}{2} + 1 = 1941$ regions. Draw a chord connecting the 17th point with one of the first 16 in a way that there are 8 and 7 points on the two sides of the drawn chord. This chord intersects $8 \cdot 7 = 56$ chords and therefore there are 57 new regions. Therefore the total number of regions is $1941 + 57 = 1998$.

Problem 9.1. Find all parameters a such that the inequality $|ax^2-3x-4| \leq 5-3x$ holds for any $x \in [-1;1]$.

Solution: Observe that the inequality is equivalent to the system

$$\left| \begin{array}{l} ax^2-9 \leq 0 \\ ax^2-6x+1 \geq 0. \end{array} \right.$$

It follows from the second inequality that $a > 0$, because if $a \leq 0$ then $ax^2-6x+1 \geq 0$ is not true for $x=1$. Further, the second inequality gives $x^2 \leq \dfrac{9}{a}$. Since this inequality is true for $x \in [-1;1]$, we get $1 \leq \dfrac{9}{a} \Rightarrow a \leq 9$. Let $D = 9-a$ be the discriminant of ax^2-6x+1. There are two cases to consider:

1) $D > 0$. The solution of the second inequality is $x \in (-\infty; x_1] \cup [x_2; +\infty)$, where $x_1 < x_2$ are the roots of $ax^2-6x+1 = 0$. Therefore $[-1;1] \subset (-\infty; x_1]$ or $[-1;1] \subset [x_2; +\infty)$.

1.1) $[-1;1] \subset (-\infty; x_1]$. It follows from the above that

$$\left| \begin{array}{l} a(-1)^2-6(-1)+1 \geq 0 \\ a \cdot 1^2 - 6 \cdot 1 + 1 \geq 0 \\ \dfrac{x_1+x_2}{2} > 1 \end{array} \right. \iff \left| \begin{array}{l} a \geq -7 \\ a \geq 5 \\ \dfrac{3}{a} > 1 \end{array} \right. \iff \left| \begin{array}{l} a \geq -7 \\ a \geq 5 \\ a < 3 \end{array} \right. ,$$

which is impossible.

1.2) $[-1;1] \subset [x_2; +\infty)$. Therefore

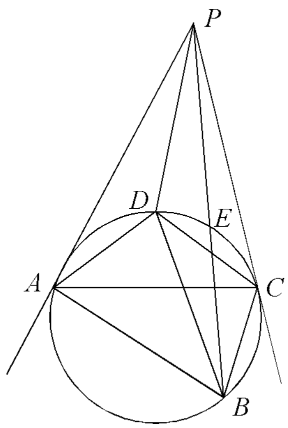
$$\left| \begin{array}{l} a(-1)^2-6(-1)+1 \geq 0 \\ a.1^2 - 6.1 + 1 \geq 0 \\ \dfrac{x_1+x_2}{2} < -1 \end{array} \right. \iff \left| \begin{array}{l} a \geq -7 \\ a \geq 5 \\ \dfrac{3}{a} < -1 \end{array} \right. ,$$

which is also impossible.

2) $D \leq 0 \Rightarrow$ the inequality $ax^2 - 6x + 1 \geq 0$ holds for any real value of x. It then follows that $9 - a \leq 0 \iff a \geq 9$. Therefore the solution is $a = 9$.

Problem 9.2. The quadrangle $ABCD$ is inscribed in a circle. The tangents to the circle passing through A and C intersect at the point P. If $PA^2 = PD \cdot PB$ and P does not lie on DB, prove that the intersecting point of AC and BD is the midpoint of AC.

Solution:



Let E be the second intersecting point of PB and the circle. Then $PA^2 = PE \cdot PB \Rightarrow PD = PE$. Hence $\angle APD = \angle EPC$. (If O is the centre of the circle, the above follows from the similarity of $\triangle ODP$ and $\triangle OEP$). Therefore $\triangle ADP \sim \triangle PCB$ because $\angle APD = \angle BPC$ and $\dfrac{AP}{BP} = \dfrac{DP}{CP}$, so $\dfrac{AD}{BC} = \dfrac{AP}{BP}$. (1)

Also, $\triangle APB \sim \triangle DCP$ because $\angle APB = \angle DPC = \angle APC - \angle APD$ and $\dfrac{AP}{DP} = \dfrac{BP}{CP}$ and so $\dfrac{AB}{DC} = \dfrac{BP}{CP}$. (2)

From (1) and (2) we get $\dfrac{AB \cdot AD}{BC \cdot DC} = \dfrac{AP \cdot BP}{BP \cdot CP} = 1$. On the other hand,

$$\frac{S_{ABD}}{S_{CBD}} = \frac{\frac{1}{2} AB \cdot AD \sin \angle DAB}{\frac{1}{2} BC \cdot DC \sin \angle DCB} = \frac{AB \cdot AD}{BC \cdot DC} = 1$$

$$(\angle DAB = 180 \deg - \angle DCB).$$

Therefore $S_{ABD} = S_{CBD}$, i.e., the diagonal BD halves AC.

Problem 9.3. (Problem for the UBM award)

See problem 8.3.

Problem 10.1. Find all values of the real parameter a such that the inequality $x^4 + ax^3 + (a+3)x^2 + ax + 1 > 0$ holds for all real values of x.

Solution: If $x = 0$, the inequality holds for any a. Suppose $x \neq 0$. Dividing both sides of the inequality by x^2 gives $x^2 + ax + a + 3 + \dfrac{a}{x} + \dfrac{1}{x^2} > 0$ or $\left(x + \dfrac{1}{x}\right)^2 + a\left(x + \dfrac{1}{x}\right) + a + 1 > 0$. Substitute $t = x + \dfrac{1}{x}$. If $x > 0$, it is true that $t \geq 2$, and if $x < 0$, it is true that $t \leq -2$. Therefore we want to find all a such that the inequality $t^2 + at + a + 1 > 0$ holds for any $t \in (-\infty; -2] \cup [2; +\infty)$. Denote $f(t) = t^2 + at + a + 1$. The discriminant of $f(t)$ is $D = a^2 - 4a - 4$.

Let $D < 0$, i.e., $a \in (2 - 2\sqrt{2}; 2 + 2\sqrt{2})$. Then the inequality $f(t) > 0$ holds for any real t, in particular for $t \in (-\infty; -2] \cup [2; +\infty)$.

Let $D \geq 0$, i.e., $a \in (-\infty; 2 - \sqrt{2}] \cup [2 + \sqrt{2}; +\infty)$. Then the inequality $f(t) > 0$ holds for any $t \in (-\infty; -2] \cup [2; +\infty)$ if and only if

$$\left|\begin{array}{l} -2 < -\dfrac{a}{2} < 2 \\ f(2) > 0 \\ f(-2) > 0 \end{array}\right.$$

The solutions of this system when $a \in (-\infty; 2-\sqrt{2}] \cup [2+\sqrt{2}; +\infty)$ are

$a \in \left(-\dfrac{5}{3}; 2-2\sqrt{2}\right]$. So $a \in \left(-\dfrac{5}{3}; 2+2\sqrt{2}\right)$.

Problem 10.2. A quadrangle with perpendicular diagonals AC and BD is inscribed in a circle with centre O and radius 1.

a.) Calculate the sum of the squares of the sides of the quadrangle.

b.) Find the area of $ABCD$ if a circle with centre I is inscribed in it and $OI = \dfrac{1}{\sqrt{3}}$.

Solution: a.) Denote $\angle BAC = \alpha$, $\angle CAD = \beta$. It follows from the Sine Theorem for $\triangle ABC$ and $\triangle ABD$ that $BC = 2\sin\alpha$, $AD = 2\sin\angle ABD = 2\sin(90\deg - \alpha) = 2\cos\alpha$. Analogously $CD = 2\sin\beta$, $AB = 2\cos\beta$. Therefore $AB^2 + BC^2 + CD^2 + DA^2 = 4\sin^2\alpha + 4\cos^2\alpha + 4\sin^2\beta + 4\cos^2\beta = 8$.

b.) The statement of the problem implies that $AB + CD = AD + BC$ or $2\cos\beta + 2\sin\beta = 2\cos\alpha + 2\sin\alpha$, i. e. $\sin(\beta + 45\deg) = \sin(\alpha + 45\deg)$. Therefore $\beta = \alpha$ or $\beta = 90\deg - \alpha$. We shall consider only the case $\beta = \alpha$ (the case $\beta = 90\deg - \alpha$ is settled in a similar fashion). Now $AD = AB$, $CD = BC$ and $\angle ABC = \angle ADC = 90\deg$. The points O and I lie on AC and when $\alpha \geq 45\deg$, I lies on the segment AO. In $\triangle ACD$ (DI is a bisector) determine $AI = \dfrac{2\cos\alpha}{\sin\alpha+\cos\alpha}$. Consequently $OI = AO - AI = \dfrac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha} = \dfrac{1}{\sqrt{3}}$. Raising both sides to the 2nd power gives $\dfrac{1-\sin 2\alpha}{1+\sin 2\alpha} = \dfrac{1}{3}$. Hence $\sin 2\alpha = \dfrac{1}{2}$. It then follows that the area of $ABCD$ is equal to $S = 2S_{ABC} = AB\cdot BC = 4\cos\alpha\sin\alpha = 2\sin 2\alpha = 1$.

Problem 10.3. (Problem for the Atanas Radev award)

Find all natural numbers n such that: If a and b are natural numbers and n divides a^2b+1, then n divides a^2+b.

Solution: Obviously the condition holds for $n=1$ and suppose $n\geq 2$. Let a be a natural number coprime to n. It follows from Bezout's Theorem that there exists a natural number b such that a^2b+1 is divisible by n. Further, n divides a^2+b and since $a^4-1 = a^2(a^2+b)-(a^2b+1)$, it follows that n divides a^4-1.

Let $n=2^\alpha k$, where k is an odd number and $\alpha\geq 0$. Suppose $k\geq 3$. Then n and $k-2$ are coprime and therefore $2^\alpha k$ divides $(k-2)^4-1$, so k divides 15. Hence n is of the form $n=2^\alpha\cdot 3^\beta\cdot 5^\gamma$, where $\alpha\geq 0$, $0\leq\beta$, $\gamma\leq 1$. Consequently n divides 11^4-1 and we conclude that $\alpha\leq 4$.

It is easy to see now that n divides $2^4\cdot 3\cdot 5 = 240$.

Conversely, let n be a divisor of 240. Then n satisfies the condition of the problem. Indeed, if 3 divides a^2b+1, then 3 divides both a^2-1 and a^2+1 and therefore 3 divides a^2+b. Similarly, if 5 divides a^2b+1, then 5 divides a^2+b. If we can show that the same property holds for 2, 4, 8 and 16, we will be done. Assume 2^k divides a^2b+1, where $1 \le k \le 3$. Then a is an odd integer number and therefore a^2-1 is divisible by 8. Consequently $b+1$ is divisible by 2^k and thus a^2+b is divisible by 2^k as well. Assume 16 divides a^2b+1. Then a is an odd integer number and it is easy to verify that a^2 is congruent to 1 or 9 modulo 16. Further, b should be congruent to 15 or 7 modulo 16 correspondingly and again a^2+b is divisible by 16.

The required numbers are all divisors of 240.

Problem 11.1.

a.) Let p be a positive real parameter. Find the least values of the functions $f(x) = x + \dfrac{p}{x}$ and $g(x) = x + \dfrac{p}{x^2}$ in the interval $(0; +\infty)$.

b.) Let $a_1,\ a_2,\ a_3$ be positive real numbers. Prove that $3(a_1 + \sqrt{a_1a_2} + \sqrt[3]{a_1a_2a_3}) \le 4(a_1+a_2+a_3)$.

Solution: a.) Since $f'(x) = 1 - \dfrac{p}{x^2} = \dfrac{(x-\sqrt{p})(x+\sqrt{p})}{x^2}$, we get that the function $f(x)$ decreases in the interval $(0; \sqrt{p})$ and increases in the interval $(\sqrt{p}; +\infty)$. Therefore the minimal value of $f(x)$ in $(0; +\infty)$ equals $f(\sqrt{p}) = 2\sqrt{p}$. Analogously it follows from $g'(x) = 1 - \dfrac{2p}{x^3}$ that the least value of $g(x)$ in $(0; +\infty)$ equals $g(\sqrt[3]{2p}) = \dfrac{3\sqrt[3]{2p}}{2}$.

b.) After substituting $x = a_1,\ y = \sqrt{a_1a_2},\ z = \sqrt[3]{a_1a_2a_3}$ our inequality becomes $0 \le \dfrac{1}{3}x + \dfrac{4}{3}\dfrac{y^2}{x} - y + \dfrac{4}{3}\dfrac{z^3}{y^2} - z$. It follows from a.)

that $\frac{4}{3}y \leq \frac{1}{3}x + \frac{4}{3}\frac{y^2}{x}$ and $\frac{1}{3}y + \frac{4}{3}\frac{z^3}{y^2} \geq z$. Therefore $\frac{1}{3}x + \frac{4}{3}\frac{y^2}{x} - y + \frac{4}{3}\frac{z^3}{y^2} - z \geq \frac{1}{3}y + \frac{4}{3}\frac{z^3}{y^2} - z \geq 0$, which completes the proof. Note that equality occurs when $x = 2y = 4z$, i. e., $a_1 = 4a_2 = 16a_3$. Note further that b.) could also be solved by the following inequalities:

$$a_1 + \sqrt{a_1a_2} + \sqrt[3]{a_1a_2a_3} = a_1 + \frac{1}{2}\sqrt{a_14a_2} + \frac{1}{4}\sqrt[3]{a_14a_216a_3} \leq$$

$$\leq a_1 + \frac{1}{4}(a_1 + 4a_2) + \frac{1}{12}(a_1 + 4a_2 + 16a_3) = \frac{4}{3}(a_1 + a_2 + a_3).$$

Problem 11.2. Let I and r are the incentre and inradius of $\triangle ABC$, and N is the midpoint of the median through C. Prove that if $r = CN - IN$, then $AC = BC$ or $\angle ACB = 90\deg$.

Solution: Use the standart notation for the elements of $\triangle ABC$ and apply the formula for a median in a triangle. Since IN and IM are medians in $\triangle CIM$ and $\triangle AIB$, respectively, we get $IN^2 = \frac{1}{4}(2CI^2 + 2MI^2 - CM^2) = = \frac{1}{4}(2CI^2 + AI^2 + BI^2 - \frac{1}{2}AB^2 - CM^2)$. Therefore

$$CN^2 - IN^2 = \frac{1}{4}(2CM^2 + \frac{1}{2}AB^2 - 2CI^2 - AI^2 - BI^2) =$$

$$\frac{1}{4}(a^2 + b^2 - 2(p-c)^2 - 2r^2 - (p-a)^2 - r^2 - (p-b)^2 - r^2) = \frac{(p-c)c}{2} - r^2.$$

It follows from the statement of the problem that $IN^2 = (CN - r)^2$, so $CN^2 - IN^2 + r^2 = 2CN \cdot r$. It follows from what we have proved above that $2CM \cdot r = (p-c)c$. Taking the square of both sides of this equality and using the formulæ $4CM^2 = 2a^2 + 2b^2 - c^2$ and $r^2 = \dfrac{(p-a)(p-b)(p-c)}{p}$, we get $(2a^2+2b^2-c^2)(p-a)(p-b) = p(p-c)c^2$. After some simple calculations the above equality becomes $(a^2 + b^2 - c^2)(a - b)^2 = 0$. Therefore $a = b$, i. e., $AC = BC$ or $a^2 + b^2 = c^2$, i. e., $\angle ACB = 90\,\text{deg}$.

Problem 11.3. (Problem for the Atanas Radev award)

See problem 10.3.

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Problem 8'1. Given an inequality $|x-1| < ax$, where a is a real parameter:

a) Solve the inequality.

b) Find all values of a such that the inequality has exactly two integer solutions.

Chavdar Lozanov, Kiril Bankov, Teodosi Vitanov

Solution: a) I. Let $x \geq 1$. Then the inequality is equivalent to $x - 1 < ax \iff (1-a)x < 1$.

1. $1 - a > 0, a < 1 \Longrightarrow x < \dfrac{1}{1-a}, \dfrac{1}{1-a} > 1 \iff a > 0$. Therefore $0 < a < 1, 1 \leq x < \dfrac{1}{1-a}$.

2. $1 - a = 0 \Longrightarrow a = 1 \Longrightarrow 0.x < 1$. Therefore $a = 1, x \geq 1$.

3. $1 - a < 0 \Longrightarrow 1 < a \Longrightarrow x \geq 1$. Therefore $1 < a, x \geq 1$.

II. Let $x < 1$. Then the inequality is equivalent to $1 - x < ax \iff 1 < (a+1)x$.

1. $a+1 > 0 \iff a > -1, \dfrac{1}{a+1} < x < 1, \dfrac{1}{a+1} < 1 \iff a > 0$. Therefore $a > 0, \dfrac{1}{a+1} < x < 1$, when $-1 < a \leq 0$ no solution exists.

2. $a + 1 = 0 \iff a = -1 \Longrightarrow 1 < 0 \cdot x$, so no solution exists.

3. $a + 1 < 0 \iff a < -1 \Longrightarrow x < \dfrac{1}{a+1} < 0$. Therefore $a < -1, x < \dfrac{1}{a+1}$.

So when $a < -1$, then $x < \frac{1}{a+1}$; for $-1 \leq a \leq 0$ no solution exists; when $0 < a < 1$, then $\frac{1}{a+1} < x < \frac{1}{1-a}$; when $1 \leq a$, then $1 \leq x$.

b) It follows from a) that the inequality could have two integer solutions only if $0 < a < 1$. Since in this case $0 < \dfrac{1}{a+1} < 1 < \dfrac{1}{1-a}$, we find that there are exactly two integer solutions if and only if

$$2 < \frac{1}{1-a} \leq 3.$$

Therefore the answer is $\dfrac{1}{2} < a \leq \dfrac{2}{3}$.

Problem 8'2. Let M be the midpoint of the side BC of $\triangle ABC$ and $\angle CAB = 45^\circ$; $\angle ABC = 30^\circ$.

a) Find $\angle AMC$.

b) Prove that $AM = \dfrac{AB \cdot BC}{2AC}$.

Chavdar Lozanov

Solution: a) Draw $CH \perp AB$. Now $\angle ACH = 45^\circ = \angle CAH$ and $\angle HCB = 60^\circ$. For $\triangle ACH$ it is true that $AH = HC$. Further it follows from $\triangle CHB$ that $CH = \dfrac{1}{2}CB = HM$. Therefore $AH = HM$, so $\angle MAH = \angle AMH = \dfrac{180^\circ - \angle AHM}{2}$. Note that $\angle CHM = 60^\circ$ and $\angle AHM = 90^\circ + 60^\circ = 150^\circ$. Therefore $\angle AMH = 15^\circ$. We obtain that $\angle AMC = \angle HMC - \angle AMH = 60^\circ - 15^\circ = 45^\circ$.

b) Let S be the area of $\triangle ABC$. We know that $S = \dfrac{AB \cdot CH}{2} = \dfrac{AB \cdot CB}{4}$. Since AM is a median, it follows that $S_{AMC} = \dfrac{S}{2}$. If $CP \perp AM$, then $\dfrac{S}{2} = \dfrac{AM \cdot CP}{2}$. But $\angle CAM = \angle CAB - \angle MAB = 45^\circ - 15^\circ = 30^\circ$, and thus $CP = \dfrac{AC}{2}$. Therefore $\dfrac{S}{2} = \dfrac{AM \cdot AC}{4}$. Now $\dfrac{AB \cdot CB}{4} = \dfrac{AM \cdot AC}{2}$, and we obtain $AM = \dfrac{AB \cdot BC}{2AC}$.

Problem 8'3. Consider all points in the plane whose coordinates (x, y) in an orthogonal coordinate system are integer numbers and $1 \leq x \leq 19, 1 \leq y \leq 4$. Each point is painted green, red or blue.

Prove that there exists a rectangle with sides parallel to the coordinate axes whose vertices are all of the same colour.

Kiril Bankov

Solution: Since the number of coloured points is $4 \cdot 19 = 76$ and there are three different colours, it follows that there are at least 26 points of the same colour (say blue). Denote by $p_1, p_2, \ldots, p_{19}$ the lines parallel to the ordinate axis and passing through the points $(1,0), (2,0), \ldots, (19,0)$, respectively. Let $n_1, n_2, \ldots, n_{19}$ be the number of blue points on the lines $p_1, p_2, \ldots, p_{19}$. It is clear that $0 \leq n_i \leq 4$ for $i = 1, 2, \ldots, 19$. Without loss of generality assume $n_1 \geq n_2 \geq \cdots \geq n_{19}$. Since $n_1 + n_2 + \cdots + n_{19} = 26$, we obtain that $n_1 \geq 2$.

1. Let $n_1 = 4$. Then the remaining 22 blue points lie on 18 lines and so $n_2 \geq 2$. Therefore there are two blue points on each of the lines p_1 and p_2 all having the same ordinates. These points form the required rectangle.

2. Let $n_1 = 3$. Then the remaining 23 blue points lie on 18 lines and therefore $n_2 \geq 2$.

   a) If $n_2 = 3$, then there is a blue rectangle with vertices on p_1 and p_2.

   b) Let $n_2 = 2$. Then $n_3 = n_4 = n_5 = n_6 = 2$. The number of ways of choosing two blue points on p_1 is 3 and the corresponding number for each of $p_2, p_3, \ldots, p_6$ is 1. The total number of blue pairs is $3 + 5 \cdot 1 = 8$, which is greater that 6—the number of ways of choosing two horizontal lines out of 4 lines. Therefore there exists a blue rectangle.

3. Let $n_1 = 2$. Then $n_2 = n_3 = \ldots = n_7 = 2$. Since $7 > 6$, we apply the same reasoning as in 2b).

Problem 8.1. Find all rational numbers a such that $|4a - 2| \leq 1$ and $A = \dfrac{4a-1}{27a^4}$ is integer. *Ivan Tonov*

Solution: It follows from $|4a-2| \leq 1$ that $\dfrac{1}{4} \leq a \leq \dfrac{3}{4}$. Also, it is clear that when $a \geq \dfrac{1}{4}$, then $A \geq 0$ and $A = 0$ only if $a = \dfrac{1}{4}$. Let k be a positive integer such that $A = k$. Then $27a^4 - 4al + l = 0$, where $l = \dfrac{1}{k}$. Multiply the above equality by 3 and write it in the following way:

$$81a^4 - 18a^2 + 1 + 18a^2 - 12al + 3l - 1 = 0 \iff$$

$$(9a^2-1)^2 + 2(3a-1)^2 - 12a(l-1) + 3(l-1) = 0,$$

so $(9a^2-1)^2 + 2(3a-1)^2 + 3(l-1)(1-4a) = 0$. Therefore $3(l-1)(1-4a) \leq 0$, which is possible (recall $a > \dfrac{1}{4}$) only if $l \geq 1$ or $k \leq 1$. But since k is a positive integer, it follows that $k = 1$ and $a = \dfrac{1}{3}$. Therefore the required values are $a = \dfrac{1}{4}$ and $a = \dfrac{1}{3}$.

Problem 8.2. Given a $\triangle ABC$. Let M be the midpoint of AB, $\angle CAB = 15°$ and $\angle ABC = 30°$.

a) Find $\angle ACM$.

b) Prove that $CM = \dfrac{AB \cdot BC}{2AC}$.

Chavdar Lozanov

Solution: a) Let $AH \perp BC$. Then $\angle HAB = 60^\circ$ and $AH = \dfrac{AB}{2} = HM$. It follows from $\angle HAC = \angle HAB - \angle CAB = 45^\circ$ that $AH = HC$. Thus $HM = HC$ and $\angle HCM = \dfrac{180^\circ - \angle MHC}{2}$. But $\angle MHC = \angle AHB - \angle AHM = 90^\circ - 60^\circ = 30^\circ$. Therefore $\angle HCM = 75^\circ$, so $\angle ACM = \angle HCM - \angle HCA = 75^\circ - 45^\circ = 30^\circ$.

b) Let S be the area of $\triangle ABC$. We know that $S = \dfrac{BC \cdot AH}{2} = \dfrac{BC \cdot AB}{4}$. Since CM is a median, we obtain $S_{ACM} = \dfrac{S}{2}$. If $MP \perp AC$, then $\dfrac{S}{2} = \dfrac{AC \cdot PM}{2}$. It follows from $\triangle PMC$ that $PM = \dfrac{1}{2}MC(\angle PCM = 30^\circ)$. Therefore $\dfrac{S}{2} = \dfrac{AC \cdot MC}{4}$, so $\dfrac{AC \cdot MC}{2} = \dfrac{BC \cdot AB}{4}$, which implies $MC = \dfrac{AB \cdot BC}{2AC}$.

Problem 8.3. Given n points on a circle denoted consecutively by $A_1, A_2, \ldots, A_n$ ($n \geq 3$). Initially 1 is written at A_1 and 0 at all remaining points. The following operation is allowed: choose a point A_i where a 1 is written and replace the numbers a, b and c written at the points A_{i-1}, A_i and A_{i+1} by $1-a$, $1-b$ and $1-c$, respectively. (Here A_0 means A_n and A_{n+1} means A_1.)

a) If $n = 1999$, is it possible to have a 0 in all points after performing the described operation a finite number of times?

b) Find all values of n such that it is not possible to have a 0 in all points after finite number of operations.

Kiril Bankov

Solution: a) After performing the transformation from the conditions of the problem consecutively for the points $A_1, A_2, A_3, \dots, A_{n-2}$, we have the following distribution of 0s and 1s:

$$\begin{array}{cccccccc} A_1 & A_2 & A_3 & \dots & A_{n-3} & A_{n-2} & A_{n-1} & A_n \\ 1 & 1 & 1 & \dots & 1 & 0 & 1 & 1 \end{array} \qquad (1)$$

If $n = 1999$, then arrange the obtained 1998 ones in 666 groups of three 1s and then perform the operation on each of the groups. We obtain a zero in every point.

b) If $n = 3k + 1$ we can repeat the steps from a) and we get only zeroes in the points.

If $n = 3k+2$, then starting from (1) and performing the operation with A_{n-1} one obtains:

$$\begin{array}{ccccccc} A_1 & A_2 & A_3 & \dots & A_{n-2} & A_{n-1} & A_n \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 \end{array}$$

There are $3k$ ones which can be arranged in k groups of 3 ones each and again to obtain anly zeroes.

We shall prove now that if $n = 3k$, it is not possible to have only zeroes after a finite number of operations. Assume the opposite, i. e., that after a finite number of operations we have only zeroes. Denote the number of operations performed with the point A_i by a_i. Since each operation changes the number of ones by an odd number (1 or 3), it follows that the sum $S = a_1 + a_2 + \dots + a_n$ of all operations is an odd number. On the other hand $S = (a_1 + a_2 + a_3) + (a_4 + a_5 + a_6) + \ldots + (a_{3k-2} + a_{3k-1} + a_{3k})$. Note that $a_1 + a_2 + a_3$ is equal to the number of changes (from 0 to 1 or *vice versa*) of the number written at A_2. Since at the beginning there is a 0 written at A_2 and a 0 again at the end, it follows that $a_1 + a_2 + a_3$ is an even number. The same applies for $a_4 + a_5 + a_6$ and so on. Therefore S is a sum of even numbers, a contradiction to the fact that S is odd. Answer to b): all numbers divisible by 3.

Problem 9.1. It is known that if the real parameter a equals any of the numbers $p < q < r$, then at least one of the remaining two is a root of the equation

$$x^2 - (2-a)x + a^2 - 2a - 7 = 0.$$

Prove that a) $p > -\frac{8}{3}$; b) $p < -1$. *Sava Grozdev*

Solution: a) It follows from the conditions of the problem that $D = (2-a)^2 - 4(a^2 - 2a - 7) \geq 0$, so $-\frac{8}{3} \leq a \leq 4$ and therefore $p \geq -\frac{8}{3}$. When $a = -\frac{8}{3}$, then the equation has an unique root $x = \frac{7}{3}$, and when $a = \frac{7}{3}$, the roots are $x = -\frac{8}{3}$ and $x = \frac{7}{3}$. Thus in the case of $p = -\frac{8}{3}$ three distinct numbers p, q and r satisfying the condition do not exist. Therefore $p > -\frac{8}{3}$.

b) Suppose that when $a = p$, then $x = q$ is a root. The case $x = r$ is treated in the same fashion. Since a and x are symmetric in the equality $x^2-(2-a)x+a^2-2a-7=0$, we get that when $a=q$, then $x=p$ is a root. When $a=r$, at least one of the numbers p and q is a root. Let that be p. Now when $a=p$, the equation has $x=r$ as a root (because of the symmetry). We obtain that when $a=p$, the roots are q and r. We conclude now that in all cases $p+q+r=2$. It is clear that the roots of $x^2-(2-p)x+p^2-2p-7=0$ are greater than p and therefore $p^2-(2-p)p+p^2-2p-7>0$, which gives $p<-1$ or $p>\frac{7}{3}$. But $3p<p+q+r=2$ and $p<\frac{2}{3}$. Therefore $p<-1$.

Problem 9.2. Through an interior point K of the non-equilateral $\triangle A_1A_2A_3$ lines $Q_2P_3 \parallel A_2A_3$, $Q_3P_1 \parallel A_3A_1$ and $Q_1P_2 \parallel A_1A_2$ are drawn. (Q_1, Q_2, Q_3 lie on A_3A_1, A_1A_2 and A_2A_3, respectively). Points $P_1, P_2, P_3, Q_1, Q_2, Q_3$ lie on a circle k. Prove that:

a) $\triangle P_1P_2P_3 \cong \triangle Q_1Q_2Q_3 \sim \triangle A_1A_2A_3$;

b) point K, the centre of k and the circumcentre $\triangle A_1A_2A_3$ lie on a line.

Rumen Kozarev

Solution: a) It follows from the conditions of the problem that the arcs $\overset{\frown}{P_1Q_1}, \overset{\frown}{P_2Q_2}$ and $\overset{\frown}{P_3Q_3}$ are equal. Therefore $\angle P_3P_1P_2 = \frac{\overset{\frown}{P_3Q_3P_2}}{2} = \frac{\overset{\frown}{P_3Q_3}}{2} + \frac{\overset{\frown}{Q_3P_2}}{2} = \frac{\overset{\frown}{P_1Q_1}}{2} + \frac{\overset{\frown}{Q_3P_2}}{2}$. Since the quadrilateral $A_1P_1KQ_1$ is a parallelogram we get $\angle A_3A_1A_2 = \angle Q_1KP_1 = \frac{\overset{\frown}{Q_1P_1}}{2} + \frac{\overset{\frown}{Q_3P_2}}{2}$, so $\angle P_3P_1P_2 = \angle A_3A_1A_2$. Similarly, one can show that $\angle Q_3Q_1Q_2 = \angle A_3A_1A_2$, and the same equalities for the remaining pairs of angles. Thus $\triangle P_1P_2P_3 \sim \triangle Q_1Q_2Q_3 \sim \triangle A_1A_2A_3$ and since the first two triangles have the same circumcircle, they are identical.

b) Since $A_1P_1KQ_1$ is a parallelogram, we obtain $\angle A_1P_1Q_1 = \angle KQ_1P_1 = \dfrac{P_2\hat{Q}_2}{2} + \dfrac{P_1\hat{Q}_2}{2} = \angle P_1P_2P_3 = \angle A_1A_2A_3$. Similarly $\angle A_1Q_1P_1 = \angle A_1A_2A_3$.

Let O be the circumcentre of $\triangle A_1A_2A_3$. It is easy to see that $OA_1 \perp P_1Q_1, OA_2 \perp P_2Q_2$ and $OA_3 \perp P_3Q_3$. Denote the midpoint of OK by S. If R_1, R_2 and R_3 are the midpoints of P_1Q_1, P_2Q_2 and P_3Q_3 then $SR_1 \parallel OA_1; SR_2 \parallel OA_2; SR_3 \parallel OA_3$ (SR_1 is a middle segment in $\triangle OKA_1 \Longrightarrow SR_1 \parallel OA_1$). Therefore S lies on the axes of symmetry of $P_1Q_1; P_2Q_2$ and P_3Q_3, so S is the centre of the circle through the points $P_1, P_2, P_3, Q_1, Q_2, Q_3$.

Problem 9.3. Find all polynomials $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, a_0 \neq 0$ with integer coefficients such that $f(a_i) = 0, i = 0, \ldots, n-1$.
Sava Grozdev

Solution: It is clear that $n > 1$. Since $f(x) = (x - a_0)(x - a_1)\ldots(x - a_{n-1})$, it follows from $f(0) = a_0 = (-1)^n a_0 \ldots a_{n-1}$ that $|a_i| = 1$ for $i = 1, 2, \ldots, n-1$.

<u>First case:</u> $|a_0| = 1$. Now $f(x) = (x-1)^p(x+1)^q, p + q = n > 1$. We know that $(x-1)^p(x+1)^q = (x^p - px^{p-1} + \ldots)(x^q + qx^{q-1} + \ldots)$ and comparing the coefficients in front of x^{n-1} and x^{n-2} we obtain

$$\left| \begin{array}{l} q - p = a_{n-1} = \pm 1 \\ \frac{q(q-1)}{2} + \frac{p(p-1)}{2} - pq = a_{n-2} = \pm 1 \end{array} \right. .$$

It is easy to find now that $p+q=3$ and so $p=1, q=2$ or $p=2, q=1$. In the first case we get that $f(x)=x^3+x^2-x-1$, which is a solution and in the second one $f(x)=x^3-x^2-x+1$, which is not.

Second case: $|a_0|\geq 2$. Now $0=f(a_0)=|a_0^n+a_{n-1}a_0^{n-1}+\cdots+a_1a_0+a_0|\geq |a_0|^n-|a_0|^{n-1}-\cdots|a_0|^2-|a_0|-|a_0|=\frac{|a_0|(|a_0|-2)(|a_0|^{n-1}-1)}{|a_0|-1}\geq 0$. Therefore $|a_0|=2$. Moreover $a_{n-1}a_0^{n-1},\ldots,a_1a_0, a_0$ have the same negativity—the opposite to those of a_0^n.

We conclude that $a_0=-2$, n is an even number and $a_i=(-1)^{i+1}$ for $i=1,2,\ldots,n-1$. If $n>2$, then $a_2=-1$ and $0=f(-1)=(-1)^n+(-1)^n(-1)^{n-1}+\cdots+(-1)^2(-1)-2=-n-2\neq 0$, which is impossible. Therefore $n=2$ and $f(x)=x^2+x-2$, which is a solution.

Answer:

$$f(x)=x^2+x-2 \qquad \text{and} \qquad f(x)=x^3+x^2-x-1.$$

Problem 10.1. Prove that the inequality

$$x(2\cdot 3^x-\frac{4x^2+x+2}{x^2+x+1})\geq 0$$

holds for any real number x. *Rumen Kozarev*

Solution: 1) Let $x\leq 0$. Then $2\cdot 3^x\leq 2$. We shall show that $\dfrac{4x^2+x+2}{x^2+x+1}\geq 2$. The last inequality is equivalent to $4x^2+x+2\geq 2x^2+2x+2 \iff x(2x-1)\geq 0 \iff x\in(-\infty;0]\cup\left[\frac{1}{2};+\infty\right)$ and therefore it holds for $x\leq 0$.

2) Let $x > 0$. We prove that $\frac{4x^2+x+2}{x^2+x+1}) < 2 \cdot 3^x$. Assume the opposite, i. e., $\frac{4x^2+x+2}{x^2+x+1} \geq 2 \cdot 3^x \Longrightarrow \frac{4x^2+x+2}{x^2+x+1} > 2 \cdot 3^0 = 2 \iff x(2x-1) > 0 \iff x \in (-\infty; 0) \cup (\frac{1}{2}; +\infty)$. Since $x > 0$, we obtain that $x \in \left(\frac{1}{2}; +\infty\right)$. Therefore $\frac{4x^2+x+2}{x^2+x+1} \geq 2 \cdot 3^x > 2 \cdot 3^{\frac{1}{2}} > 3 \iff 4x^2+x+2 > 3x^2+3x+2 \iff x^2-2x-1 > 0 \iff x \in (-\infty; 1-\sqrt{2}) \cup (1+\sqrt{2}; +\infty) \Longrightarrow x \in (1+\sqrt{2}; +\infty)$, because $x \geq \frac{1}{2}$. Thus $\frac{4x^2+x+2}{x^2+x+1} \geq 2 \cdot 3^x > 2 \cdot 3^{1+\sqrt{2}} > 2 \cdot 3^2 = 18$. Since obviously $\frac{4x^2+x+2}{x^2+x+1} < \frac{4x^2+4x+4}{x^2+x+1} = 4$ for any $x > 0$, we get a contradiction.

Problem 10.2. Let M be an interior point in the square $ABCD$. Denote the second points of intersection of the lines AM, BM, CM, DM with the circumcircle of $ABCD$ by A_1, B_1, C_1, D_1, respectively. Prove that

$$A_1B_1 \cdot C_1D_1 = A_1D_1 \cdot B_1C_1.$$

Emil Kolev

Solution: Since $\triangle ABM \sim \triangle A_1B_1M$; $\triangle BCM \sim \triangle B_1C_1M$; $\triangle CDM \sim \triangle C_1D_1M$; $\triangle DAM \sim \triangle D_1A_1M$, we obtain

$$\frac{AB}{A_1B_1} = \frac{BM}{A_1M}; \frac{BC}{B_1C_1} = \frac{BM}{C_1M}; \frac{CD}{C_1D_1} = \frac{DM}{C_1M}; \frac{DA}{D_1A_1} = \frac{DM}{A_1M}.$$

It is easy to see now that

$$\frac{AB}{A_1B_1} \cdot \frac{CD}{C_1D_1} = \frac{BM \cdot DM}{A_1M \cdot C_1M} = \frac{BC}{B_1C_1} \cdot \frac{DA}{D_1A_1}.$$

Using that $AB = BC = CD = DA$ it follows from the above that $A_1B_1 \cdot C_1D_1 = A_1D_1 \cdot B_1C_1$, *Q. E. D.*

Problem 10.3. Consider n points in the plane such that no three lie on a line. What is the least number of segments having their ends in the given points such that for any two points A and B there exists a point C connected to both A and B? *Emil Kolev*

Solution: Denote the points by $A_1, A_2, \ldots, A_n$. Draw segments connecting A_1 with all remaining points. Also, draw segments A_2A_3, A_4A_5, $\ldots$, $A_{n-1}A_n$ when n is odd and A_2A_3, A_4A_5, $\ldots$, $A_{n-2}A_{n-1}$, A_2A_n when n is even. It is easy to see that the condition of the problem is met and that there are $\left\lceil \frac{3n-3}{2} \right\rceil$ segments ($\lceil M \rceil$ denotes the least natural number which is greater or equal to M).

Suppose it is possible to draw less than $\left\lceil \frac{3n-3}{2} \right\rceil$ segments and to meet the condition of the problem. Obviously each point is connected by a segment with another one. If each point is connected to at least three others, we will have that the number of segments is at least $\frac{3n}{2}$, which is greater than $\left\lceil \frac{3n-3}{2} \right\rceil$. Therefore there exists a point (let that be A_1) connected with at most two others. If A_1 is connected to exactly one point (let that be A_2), a point connected to both A_1 and A_2 does not exist. Therefore A_1 is connected to exactly two points (let them be A_2 and A_3). It is easy to see that A_2 and A_3

are connected by a segment. Consider the pairs A_1 and A_i for any $i > 3$. It is clear that the point connected to both A_1 and A_i could be either A_2 or A_3. In both cases A_i is connected to A_2 or A_3. Since there are at least two segments from each point $A_i, i > 3$ then the number of segments from $A_i, i > 3$ is at least $2(n-3)$. Further, since at least $n-3$ from these points connect some point of $A_i, i > 3$ with A_2 or A_3 (and therefore they are counted once) the total number of drawn segments is at least $3 + n - 3 + \left\lceil \frac{n-3}{2} \right\rceil = \left\lceil \frac{3n-3}{2} \right\rceil$.

This is a contradiction with the number of drawn segments. Therefore the answer is $\left\lceil \frac{3n-3}{2} \right\rceil$.

Problem 11.1. Given a function $f(x)$ defined for any real x and $f(\mathrm{tg} x) = \sin 2x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Find the minimum and the maximum of the function $f(\sin^3 x) \cdot f(\cos^3 x)$.

Oleg Mushkarov, Nikolai Nikolov

Solution: Let $t = \mathrm{tg} x$. Then $\sin 2x = \frac{2t}{1+t^2}$ and it follows from the conditions of the problem that $f(t) = \frac{2t}{1+t^2}$ for any t. Therefore

$$f(\sin^3 x) \cdot f(\cos^3 x) = \frac{4 \sin^3 x \cdot \cos^3 x}{(1 + \sin^6 x)(1 + \cos^6 x)} =$$

$$\frac{4(\sin x \cdot \cos x)^3}{2 - 3(\sin x \cdot \cos x)^2 + (\sin x \cdot \cos x)^6}$$

Let $u = \sin x \cdot \cos x = \frac{1}{2}\sin 2x$. Then $u \in [-\frac{1}{2}, \frac{1}{2}]$ and we have to find the minimum and the maximum of the function $g(u) = \dfrac{4u^3}{2 - 3u^3 + u^6}$ in the interval $[-\frac{1}{2}, \frac{1}{2}]$. We obtain

$$g'(u) = \frac{12u^2(1-u^2)(u^4+u^2+1)}{(2-3u^2+u^6)^2} > 0$$

when $u \in [-\frac{1}{2}, \frac{1}{2}]$ and so $g(u)$ is an increasing function in the interval $[-\frac{1}{2}, \frac{1}{2}]$. It follows now that $\max\limits_{u \in [-\frac{1}{2}, \frac{1}{2}]} g(u) = g(\frac{1}{2}) = \frac{32}{81}$ and $\min\limits_{u \in [-\frac{1}{2}, \frac{1}{2}]} g(u) = g(-\frac{1}{2}) = -\frac{32}{81}$.

Problem 11.2. A circle is tangent to the circumcircle of $\triangle ABC$ and to the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at points M and N, respectively. Prove that the excentre to side BC of $\triangle ABC$ lies on the segment MN.

Oleg Mushkarov, Nikolai Nikolov

Solution: Let O be the circumcentre of $\triangle ABC$ and L be the centre of the circle tangent to the circumcircle of $\triangle ABC$. First we shall find the radius ρ of this circle. For $\triangle OAL$ we get $AL = \dfrac{\rho}{\sin\frac{A}{2}}$, $AO = R$, $OL = R + \rho$, $\angle OAL = \dfrac{|B-C|}{2}$. From the Cosine Law we obtain $(R+\rho)^2 = R^2 + \dfrac{\rho^2}{\sin^2\frac{A}{2}} - \dfrac{2R\rho\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$. After simplification the above becomes

$$\rho\cos^2\frac{A}{2} = 2R\sin\frac{A}{2}(\sin\frac{A}{2} + \cos\frac{B-C}{2}) = 2R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Since $\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{p(p-b)}{ac}}$, $\cos\frac{C}{2} = \sqrt{\frac{p(p-c)}{ab}}$ and $abc = 4RS$, it follows from the previous equality that $\rho = \frac{r_a}{\cos^2\frac{A}{2}}$, where r_a is the exradius to side BC of $\triangle ABC$. Let $I = AL \cap MN$ and let T be the projection of I on the line AB. Since $AL \perp MN$, it follows that $IT = \frac{AI\cdot IM}{AM} = \frac{AI\cdot AM\cdot LM}{AM\cdot AL} = \frac{AI\cdot LM}{AL} = LM\cos^2\frac{A}{2} = r_a$. Since I lies on the bisector of $\angle A$, we conclude that I is the excentre to side BC of $\triangle ABC$.

Problem 11.3. Given an orthogonal coordinate system with origin O in the plane. Distinct real numbers are written at the points with integer coordinates. Let A be a nonempty finite set of integer points which is central-symmetric regarding O and $O \notin A$. Prove that there exists an integer point X such that if A_X is the image of A under translation defined by $\vec{OX}$, then at least half of the numbers written at the points of A_X are greater than the number written at X.

Avgustin Marinov

Solution: Let us denote the number of points in A, which is obviously even, by $2s$. Connect all integer points X with the points from A_X by arrows so that the arrow points to the smaller number. Suppose no point X with the required property exists. Then there are at least $s+1$ arrows pointing out of any point X. For every natural number n denote the square with vertices $(n,n),(-n,n),(-n,-n)$ and $(n,-n)$ by K_n.

Since A is a finite set, there exists a natural number d such that $A \subset K_d$. For every n denote the number of arrows within the square K_n by S_n. Since there are at least $s+1$ arrows pointing out of every integer point of K_n (and these arrows are within the square K_{n+d}), it follows that $(2n+1)^2(s+1) \leq S_{n+d}$. On the other hand, since A_X is a central-symmetric set, there are at most $s-1$ arrows pointing to every integer point of K_{n+d}. Therefore $S_{n+d} \leq (2n+2d+1)^2(s-1)$. Thus $(2n+1)^2(s+1) \leq (2n+2d+1)^2(s-1)$, so $s+1 \leq (1+\dfrac{2d}{2n+1})^2(s-1)$ for any n. When $n \to \infty$ one obtains $s+1 \leq s-1$, a contradiction. Therefore a point X with the required property does exist.

WINTER MATHEMATICAL COMPETITION

1995

Grade 8 — First Group.

Problem 1. Prove that for every positive integer n the following proposition holds:
"The number 7 is a divisor of $3^n + n^3$ if and only if 7 is a divisor of $3^n.n^3 + 1$."

Solution. If 7 is a divisor of n, then 7 is neither a divisor of $3^n + n^3$ nor a divisor of $3^n.n^3 + 1$.

Let 7 be not a divisor of n. In this case 7 divides $n^6 - 1 = (n^3 - 1)(n^3 + 1)$ and since 7 is a prime number, then 7 divides either $n^3 - 1$ or $n^3 + 1$. Now the above proposition follows from the equalities:

$$3^n.n^3 + 1 = (n^3 - 1)(3^n - 1) + (n^3 + 3^n)$$

and

$$3^n.n^3 + 1 = (n^3 + 1)(3^n + 1) - (n^3 + 3^n).$$

Problem 2. Let $ABCDE$ be a convex pentagon and let M, P, N, Q be the midpoints of the segments AB, BC, CD, DE respectively. If K and L are the midpoints of the segments MN and PQ respectively and the segment AE is of length a, find the length of the segment KL.

Figure 1.

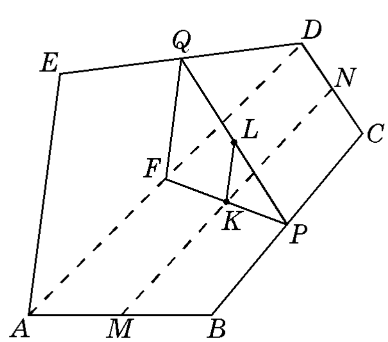
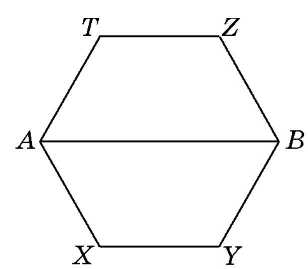


Figure 2.



Solution. Let F be the midpoint of the segment AD (Figure 1). Then the quadrilateral $MPNF$ is a parallelogram. Hence K is midpoint of the segment FP. It follows from here that $KL = \frac{1}{2}.FQ$. On the other hand $FQ = \frac{1}{2}.a$ (because F and Q are midpoints of ED and AD respectively).

Therefore $KL = \frac{1}{4}.a$.

Problem 3. Every point in the plane is colored either in black or in white. Prove that there exists a right angled triangle with hypotenuse of length 2 and an acute angle of 60°, which vertices are colored in one and the same colour.

Solution. First we shall show that there exist two points which are colored in one and the same colour and the distance between them is 2. Indeed let ABC be a equilateral triangle of side 2. Obviously two of its vertices (say A and B) are colored in one and the same colour (e.g. white). Let $AXYBZT$ be a regular hexagon with a big diagonal AB (Figure 2).

If one of the vertices X, Y, Z, T (e.g. X) is white, then the vertices of the triangle ABX ($\angle AXB = 90^\circ$, $\angle XAB = 60^\circ$) are colored in one and the same colour. Otherwise the vertices of the triangle XYT ($\angle YXT = 90^\circ$, $\angle XYT = 60^\circ$) are colored in one and the same colour.

Grade 8.

Problem 1. Let $A = \dfrac{1}{\sqrt{4x^2+4x+1}}$ and $B = \dfrac{2x-2}{\sqrt{x^2-2x+1}}$. Find all integer values of x, for which the number $C = \dfrac{2A+B}{3}$ is an integer.

Solution. We have

$$A = \frac{1}{\sqrt{(2x+1)^2}} = \frac{1}{|2x+1|}, \qquad B = \frac{2(x-1)}{\sqrt{(x-1)^2}} = \frac{2(x-1)}{|x-1|},$$

and

$$C = \frac{2}{3}\cdot\left(\frac{1}{|2x+1|} + \frac{x-1}{|x-1|}\right),$$

1. Let $x > 1$. Then

$$C = \frac{2}{3}\cdot\left(\frac{1}{2x+1}+1\right) = \frac{4(x+1)}{3(2x+1)} > 0$$

and

$$C - 1 = \frac{4(x+1)}{3(2x+1)} - 1 = \frac{1-2x}{3(2x+1)} < 0.$$

Hence $0 < C < 1$, i.e. C is not an integer for any $x > 1$.

2. Let $-\dfrac{1}{2} < x \le 1$. Then $x = 0$ (because x is an integer) and $C = 0$. Thus $x = 0$ is a solution of the problem.

3. Let $x < -\dfrac{1}{2}$. Then $x \le -1$ (because x is an integer). It is clear that

$$C = \frac{2}{3}\left(-\frac{1}{2x+1} - 1\right) = -\frac{4\,(x+1)}{3(2x+1)} \le 0$$

and

$$C + 1 = \frac{2}{3}\left(-\frac{1}{2x+1} - 1\right) + 1 = 1 - \frac{4(x+1)}{3(2x+1)} = \frac{2x-1}{2\,(2x+1)} > 0.$$

Hence $-1 < C \le 0$, i.e. $C = 0$ and $x = -1$.

Finally only $x = 0$ and $x = -1$ are solutions of the given problem.

Problem 2. Let M and N be the midpoints of the sides BC and AC of the triangle ABC, ($AB \ne AC$, $AB \ne BC$) and G be the intersection point of the lines AM and BN. The angle bisectors of $\angle BAC$ and $\angle ABC$ intersect BC and AC in the points D and E respectively. Prove that the quadrilateral $DEMN$ is inscribed in a circle if and only if there exists a circle, inscribed in the quadrilateral $CMGN$.

Figure 3.

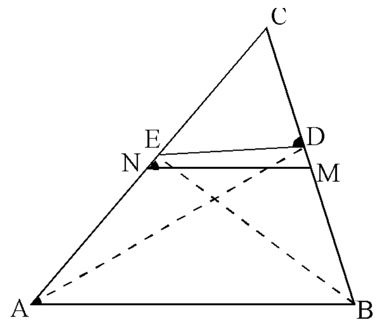
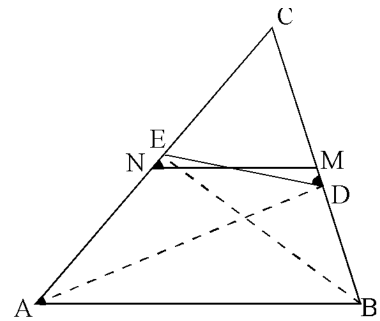


Figure 4.

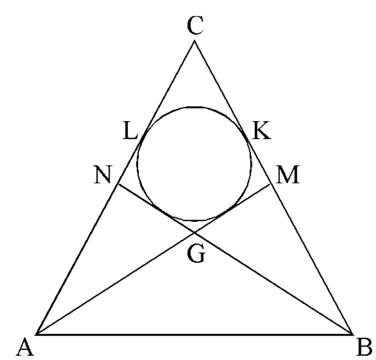


Solution. Without loss of generality we can suppose that N is between A and E. There are two possibilities for the points D and E which are shown in the Figure 3 and Figure 4.

The quadrilateral $DEMN$ is inscribed in a circle if and only if $\angle CNM = \angle CDE$, i.e. $\angle BAN + \angle BDE = 180^0$ (because $MN \| AB$ and $\angle CNM = \angle BAN$). Thus the quadrilateral $DEMN$ is inscribed in a circle iff the quadrilateral $ABDE$ is inscribed in a circle. This is equivalent to $\angle DAE = \angle DBE$, i.e. to $AC = BC$.

Therefore we should prove that there exists a circle, inscribed in the quadrilateral $CMGN$ if and only if $AC = BC$.

Figure 5.



Let $AC = BC$. Then $CM = CN$ and since G is the center of gravity of the triangle ABC, we have $GM = \frac{1}{3}AM = \frac{1}{3}BN = GN$. Hence $CM + GN = CN + GM$, i.e. there exists a circle, inscribed in the quadrilateral $CMGN$ (Figure 5).

Conversely if there exists a circle, inscribed in the quadrilateral $CMGN$, then $CM + GN = CN + GM$ and $\frac{1}{2}BC + \frac{1}{3}BN = \frac{1}{2}AC + \frac{1}{3}AM$. Hence $AM - BN = \frac{3}{2}(BC - AC)$. Let K and L be the points of contact of the circle and the sides CM and CN respectively. Obviously $CK = CL$.

On the other hand this circle is inscribed in the triangles ACM and BCN. Hence $CK = \frac{1}{2}(AC + CM - AM)$, $CL = \frac{1}{2}(BC + CN - BN)$. Thus $AM - BN = \frac{1}{2}(AC - BC)$. It follows from here that $\frac{3}{2}(BC - AC) = \frac{1}{2}(AC - BC)$, i.e. $AC = BC$.

Problem 3. Thirty points are given in the plane. Some of them are connected with segments as it is shown in the Figure 6. The points are labeled with different positive integers.

If a is a segment and p and q are the numbers, corresponding to its endpoints, we denote $\mu(a) = |p - q|$.

a) Construct an example of labeling of the points with the integers $1, 2, \ldots 30$, in which there exists exactly one segment a with $\mu(a) = 5$;

b) Prove that for every labeling there exists at least one segment a with $\mu(a) \geq 5$.

Figure 6.

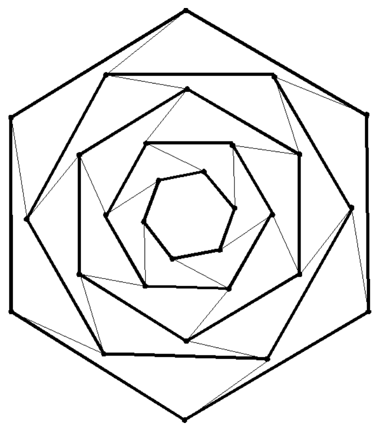
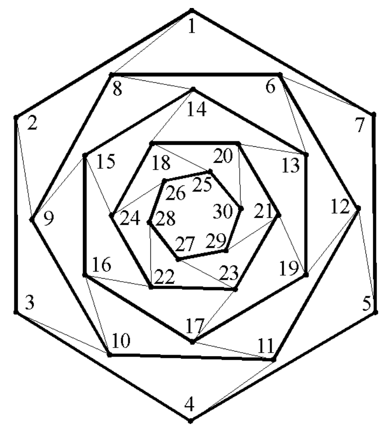


Figure 7.



Solution. a) A possible example is shown in the Figure 7.

b) Let the points be labeled with the positive integers $m = m_1 < m_2 < \ldots < m_{30} = M$. It is clear that $M \geq m + 29$.

Let A and B be the points labeled with m and M respectively and let $a_1 = AC_1$, $a_2 = C_1C_2$, $\ldots$, $a_{k-1} = C_{k-2}C_{k-1}$, $a_k = C_{k-1}B$ be the shortest path of segments, connecting A and B. It is not difficult to see that $k \leq 7$. If we assume that $\mu(a_i) < 5$ for $i = 1, 2, \ldots k$, then $m + 29 \leq M \leq m + 4k \leq m + 28$, which is a contradiction.

Grade 9.

Problem 1. Let m be a real number, such that the roots x_1 and x_2 of the equation

$$f(x) = x^2 + (m-4)x + m^2 - 3m + 3 = 0$$

are real numbers.

a) Find all values of m for which $x_1^2 + x_2^2 = 6$.

b) Prove that

$$1 < \frac{mx_1^2}{1-x_1} + \frac{mx_2^2}{1-x_2} + 8 \leq \frac{121}{9}.$$

Solution. a) Since x_1 and x_2 are real numbers, then

$$D(f) = (m-4)^2 - 4\left(m^2 - 3m + 3\right) = -3m^2 + 4m + 4 \geq 0.$$

Hence $-\dfrac{2}{3} \leq m \leq 2$. On the other hand

$$6 = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = -m^2 - 2m + 10.$$

We obtain from here that $m = -1 \pm \sqrt{5}$. But

$$-1 - \sqrt{5} < -\frac{2}{3} < -1 + \sqrt{5} < 2$$

and therefore only $m = \sqrt{5} - 1$ is a solution of the given problem.

b) We have

$$\begin{aligned}\frac{mx_1^2}{1-x_1} + \frac{mx_2^2}{1-x_2} &= \frac{m[x_1^2(1-x_2) + x_2^2(1-x_1)]}{f(1)} \\ &= \frac{x_1^2 + x_2^2 - x_1x_2(x_1+x_2)}{m-2} \\ &= \frac{m^3 - 8m^2 + 13m - 2}{m-2} = m^2 - 6m + 1\end{aligned}$$

Thus if $F = \dfrac{mx_1^2}{1-x_1} + \dfrac{mx_2^2}{1-x_2} + 8$, then $F = (m-3)^2$ and

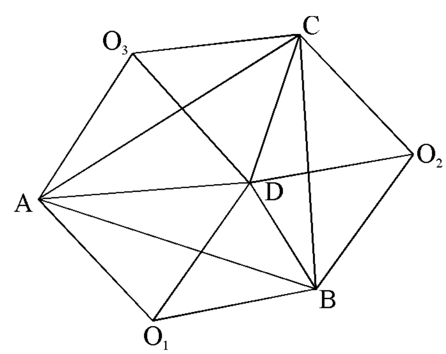
$$\frac{121}{9} = \left(-\frac{2}{3} - 3\right)^2 \geq F > (2-3)^2 = 1.$$

Problem 2. The point D lies inside the acute triangle ABC. Three of the circumscribed circles of the triangles ABC, ABD, BCD and CAD have equal radii. Prove that the fourth circle has the same radius.

Solution. There are two cases:

1. The radii of the circumscribed circles of the triangles ABD, BCD, CAD are equal.

Figure 8.



Let O_1, O_2 and O_3 be the centers of these circles (Figure 8). Obviously the quadrilaterals O_2CO_3D, O_3AO_1D and O_1BO_2D are rhombuses.

Hence $O_2C \| O_3D \| AO_1$ and $O_2C = AO_1$. Thus the quadrilateral CAO_1O_2 is a parallelogram. It follows from here that $AC \| O_1O_2$ and since $O_1O_2 \perp DB$, then $BD \perp AC$. Analogously $CD \perp AB$ and $AD \perp BC$. Therefore D is the altitude center of $\triangle ABC$.

Now it is easy to see that $\angle BDC = 180^0 - \angle BAC$, which implies that the circumscribed circles of $\triangle BCD$ and $\triangle ABC$ are symmetric according to the line BC.

Therefore their radii are equal.

2. The radii of the circumscribed circles of the triangles ABC, ACD and BCD are equal.

In this case the circumscribed circles of $\triangle ABC$ and $\triangle BCD$ are symmetric according the line BC. Hence $\angle BDC = 180^0 - \angle BAC$. Analogously $\angle ADC = 180^0 - \angle ABC$. Therefore $\angle ADB = 360^0 - \angle BDC - \angle ADC = 180^0 - \angle ACB$ and the circumscribed circles of $\triangle ABD$ and $\triangle ABC$ are symmetric according to the line AB. Thus they have equal radii.

Problem 3. Let A be a set with 8 elements. Find the maximal number of 3-element subsets of A, such that the intersection of any two of them is not a 2-element set.

Solution. Let $B_1, B_2, \ldots, B_n$ be subsets of A such that $|B_i| = 3$, $|B_i \cap B_j| \neq 2$ $(i,j = 1,\ldots,n)$. Assume that there exists an element $a \in A$, which belongs to four of the subsets $B_1, B_2, \ldots, B_n$ (e.g. $a \in B_1, B_2, B_3, B_4$). Then $|B_i \cap B_j| \geq 1$ $(i,j = 1,\ldots,4)$. But $B_i \neq B_j$ if $i \neq j$, i.e. $|B_i \cap B_j| \neq 3$. Thus $|B_i \cap B_j| = 1$ $(i,j = 1,2,3,4)$. It follows from here that $|A| \geq 1 + 4.2 = 9$, which is a contradiction. Therefore every element of A belongs to at most three of the subsets $B_1, B_2, \ldots, B_n$. Then $3n \leq 8.3$, i.e. $n \leq 8$.

If $A = \{a_1, a_2, \ldots, a_8\}$, then the subsets

$$\begin{array}{llll} B_1 = \{a_1, a_2, a_3\}, & B_2 = \{a_1, a_4, a_5\}, & B_3 = \{a_1, a_6, a_7\}, & B_4 = \{a_8, a_3, a_4\}, \\ B_5 = \{a_8, a_2, a_6\}, & B_6 = \{a_8, a_5, a_7\}, & B_7 = \{a_3, a_5, a_6\}, & B_8 = \{a_2, a_4, a_7\} \end{array}$$

provide an example of exactly eight 3-element subsets of A, such that $|B_i \cap B_j| \neq 2$.

Therefore the searched number is $n = 8$.

Grade 10.

Problem 1. Find all positive roots of the equation

$$\log_{(x+a-1)} \frac{4}{x+1} = \log_a 2,$$

where $a > 1$ is a real number.

Solution. It is clear that if $a > 1$ and $x > 0$, then $x + a - 1 > 0$ and $\frac{4}{x+1} > 0$. Hence in this case $\log_{(x+a-1)} \frac{4}{x+1}$ is well defined. Since

$$\log_{(x+a-1)} \frac{4}{x+1} = \frac{\log_a \frac{4}{x+1}}{\log_a (x+a-1)},$$

then the given equation is equivalent to

$$\frac{4}{x+1} = 2^{\log_a (x+a-1)}. \tag{1}$$

The function $\frac{4}{x+1}$ is strictly decreasing in the interval $(0, +\infty)$. The function $\log_a(x+a-1)$ is strictly increasing in the interval $(0, +\infty)$ and obviously the same is true for the function $2^{\log_a(x+a-1)}$. Therefore the equation (1) has no more than one root in the interval $(0, +\infty)$.

On the other hand it is easy to check that $x = 1$ is a root of this equation.

Problem 2. A circle k with center O and diameter AB is given. The points C and D are moving along the arc $\overset{\frown}{AB}$ so that C is between B and D and if $\angle BOC = 2\beta$ and $\angle AOD = 2\alpha$, then $\tan\alpha = \tan\beta + \frac{3}{2}$. Prove that the lines, which are perpendicular to CD and divide CD in ratio $1:4$ measured from C, pass through a fixed point of the given circle.

Solution. Let E be such a point on the arc $\overset{\frown}{AB}$, not containing C and D, that if $\angle BOE = 2\delta$, then $\tan\delta = 2$ (Figure 9).

We shall show that the point E satisfies the problem's conditions.

It is enough to prove that if F is the foot of the perpendicular from E to CD, then the point F is between C and D and $\frac{CF}{FD} = \frac{1}{4}$.

We have

$$\tan(\angle ECD) = \tan(\angle ECA + \angle ACD) = \tan(\frac{\pi}{2} - \delta + \alpha)$$

$$= \frac{\tan\left(\frac{\pi}{2}-\delta\right)+\tan\alpha}{1-\tan\left(\frac{\pi}{2}-\delta\right).\tan\alpha} = \frac{\frac{1}{2}+\tan\alpha}{1-\frac{\tan\alpha}{2}} = \frac{1+2\tan\alpha}{2-\tan\alpha}$$

$$= \frac{1+2\left(\tan\beta+\frac{3}{2}\right)}{2-\left(\tan\beta+\frac{3}{2}\right)} = \frac{4+2\tan\beta}{\frac{1}{2}-\tan\beta} = \frac{8+4\tan\beta}{1-2\tan\beta},$$

$$\tan(\angle EDC) = \tan(\angle EDB+\angle BDC) = \tan(\delta+\beta) = \frac{\tan\delta+\tan\beta}{1-\tan\delta.\tan\beta} = \frac{2+\tan\beta}{1-2\tan\beta}.$$

Thus

$$\tan(\angle ECD) = 4\tan(\angle EDC).$$

It follows from the last equality that either

$$\angle ECD \geq \frac{\pi}{2} \quad \text{and} \quad \angle EDC \geq \frac{\pi}{2}$$

or

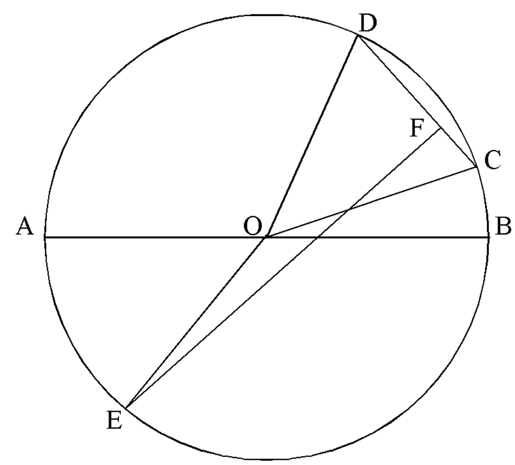
$$\angle ECD < \frac{\pi}{2} \quad \text{and} \quad \angle EDC < \frac{\pi}{2}.$$

But these angles belong to the triangle EDC. Hence they are acute angles. Therefore the point F is between the points C and D. Since

$$\tan(\angle ECD) = \frac{EF}{FC} \quad \text{and} \quad \tan(\angle EDC) = \frac{EF}{FD},$$

then $\frac{EF}{FC} = 4\frac{EF}{FD}$, i.e. $FD = 4.FC$.

Figure 9.



Problem 3. Find all prime numbers p, for which the number $p(2^{p-1}-1)$ is a k-th power $(k>1)$ of a positive integer.

Solution. Let $p(2^{p-1}-1) = x^k$ ($x>0$ is an integer). It is clear that $p \neq 2$, i.e. $p = 2q+1$ is an odd number. Since p/x, then $x = p.y$ (y is a positive integer) and $(2^q-1)(2^q+1) = p^{k-1}y^k$. At least one of the numbers 2^q-1 and 2^q+1 is a k-th power of an integer, because they are relatively prime numbers.

1. Let $2^q-1 = z^k$, i.e. $2^q = z^k+1$. If k is even, then z^k+1 is not divisible by 4 . Hence $q=1$, $p=3$ and $p(2^{p-1}-1) = 3^2$.

If $k = 2l+1$ then $2^q = (z+1)(z^{2l}-z^{2l-1}+\cdots-z+1)$, i.e. $z+1 = 2^\alpha$, where $0 \leq \alpha < q$. On the other hand

$$2^q = (2^\alpha-1)^{2l+1}+1 = 2^{2\alpha}.A + 2^\alpha(2l+1),$$

(A is an integer). The last equality contradicts with $\alpha < q$.

2. Let $2^q+1 = z^k$, i.e. $2^q = z^k-1$. If k is odd, we obtain a contradiction as in the previous case.

If $k = 2l$, then $(z^l-1)(z^l+1) = 2^q$ and since $\mathrm{GCD}(z^l-1, z^l+1) = 2$, we have $z^l-1 = 2$, i.e. $q=3$, $p=7$, $p(2^{p-1}-1) = 7.63 = 21^2$.

The sought numbers are $p=3$ and $p=7$.

Grade 11.

Problem 1. Find all the values of the real parameter p, for which the range of the function

$$f(x)=\frac{2(1-p)+\cos x}{p-\sin^2 x}.$$

contains the interval $[1,2]$.

Solution. Let $y=\cos x$. The problem is to find all values of p, such that for every $k\in[1,2]$ the equation

$$\frac{2(1-p)+y}{p-1+y^2}=k$$

has at least one root $y_0\in[-1,1]$, i.e. we should find all values of $q=1-p$ such that for every $k\in[1,2]$ the equation

$$ky^2-y-q(k+2)=0$$

has at least one root $y_0^2\neq q$, $y_0\in[-1,1]$.

If $y_0^2=q$, then $-y_0-2q=0$, $y_0=-2q$ and $4q^2=q$, i.e. $q_1=0$ or $q_2=\dfrac{1}{4}$.

If $q=0$ the roots of the equation $ky^2-y=0$ are $y_1=0$, $y_2=\dfrac{1}{k}$. But $y_2=\dfrac{1}{k}\in[-1,1]$ and $y_2^2\neq q=0$ for every $k\in[1,2]$. Thus $q=0$ satisfies the problem's conditions.

If $q=\dfrac{1}{4}$, then the roots of the equation $ky^2-y-\dfrac{1}{4}(k+2)=0$ are $y_1=-\dfrac{1}{2}$, $y_2=\dfrac{1}{2}+\dfrac{1}{k}$. But $y_2\in[-1,1]$ only if $k=2$. Thus $q=\dfrac{1}{4}$ doesn't satisfy the problems conditions.

Let $q\neq 0$ and $q\neq\dfrac{1}{4}$. The equation $ky^2-y-q(k+2)=0$ has real roots iff $D=1+4k(k+2)q\geq 0$, i.e. iff $q\geq-\dfrac{1}{4k(k+2)}$ for every $k\in[1,2]$. Hence $q\geq-\dfrac{1}{32}$. The vertex of the parabola $g(y)=ky^2-y-q(k+2)$ has as its first coordinate $y'=\dfrac{1}{2k}\in(0,1]\subset[-1,1]$. Therefore the equation $g(y)=0$ has at least one root $y_0\in[-1,1]$ iff at least one of the inequalities $g(-1)\geq 0$ and $g(1)\geq 0$ holds. It is easy to obtain from here that $q\leq\dfrac{k+1}{k+2}$ for every $k\in[1,2]$. Hence $q\leq\dfrac{2}{3}$, i.e. $q\in\left[-\dfrac{1}{32},\dfrac{2}{3}\right]$ and $q\neq\dfrac{1}{4}$.

Finally $p\in\left[\dfrac{1}{3},\dfrac{33}{32}\right]$, $p\neq\dfrac{3}{4}$.

Problem 2. The point O is on the edge AB of the tetrahedron $ABCD$. The circumscribed sphere of the tetrahedron $AOCD$ intersects the edges BC and BD in the points M and N ($M\neq C$, $N\neq D$) respectively. The circumscribed sphere of the tetrahedron $BOCD$ intersects the edges AC and AD in the points P and Q ($P\neq C$, $Q\neq D$) respectively. Prove that the triangles OMN and OQP are similar.

Solution. Since the points A, C, M and O are in one and the same plane and are lying on a sphere, then the quadrilateral $ACMO$ is inscribed in a circle (Figure 10). Then $\triangle BOM\sim\triangle BCA$, and $\dfrac{OM}{CA}=\dfrac{BM}{BA}$ (Figure 11). Analogously from $\triangle ABD$: $\dfrac{OQ}{BD}=\dfrac{AQ}{AB}$. Hence

$$\frac{OM}{OQ}=\frac{BM}{AQ}\cdot\frac{AC}{BD}. \tag{1}$$

From $\triangle BCD$ we have $\dfrac{MN}{CD} = \dfrac{BM}{BD}$, and from $\triangle ACD$: $\dfrac{PQ}{CD} = \dfrac{AQ}{AC}$. Thus

Figure 10.

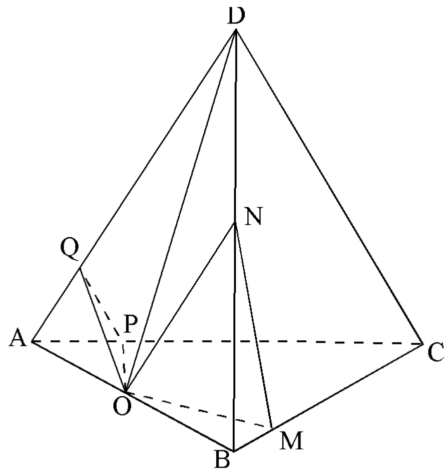
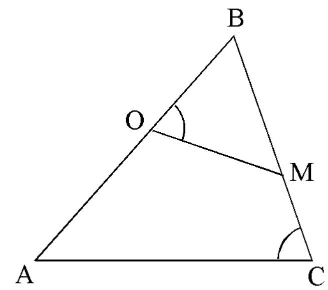


Figure 11.



$$\frac{MN}{PQ} = \frac{BM}{AQ} \cdot \frac{AC}{BD}. \tag{2}$$

It follows from (1) and (2) that $\dfrac{OM}{OQ} = \dfrac{MN}{PQ}$. Similarly $\dfrac{ON}{OP} = \dfrac{MN}{PQ}$.
Therefore $\triangle OMN \sim \triangle OQP$.

Problem 3. Solve in positive integers the equation:

$$1 + 5^x = 2^y + 2^z.5^t.$$

Solution. If $y \geq 2$ and $z \geq 2$, then the right side of the given equation is divisible by 4. But $1 + 5^x \equiv 2 \pmod 4$ and hence $\min(y, z) = 1$. On the other hand $2^y \equiv 1 \pmod 5$. Thus y is divisible by 4 (4 is the index of 2 modulo 5). It follows from here that $y \geq 4$, $z = \min(y, z) = 1$ and the equation is $1 + 5^x = 2^{4y_0} + 2.5^t$, where $y = 4y_0$ (y_0 is a positive integer).

If $t = 1$ then $5^x - 2^{4y_0} = 9$. Using congruence modulo 3 we obtain that x is even, i.e. $x = 2x_0$. Hence $(5^{x_0} - 2^{2y_0})(5^{x_0} + 2^{2y_0}) = 9$, from where $x_0 = y_0 = 1$, i.e. $x = 2$ and $y = 4$.

Let $t > 1$. Then $16^{y_0} \equiv 1 \pmod{25}$ and y_0 is divisible by 5 (5 is the index of 3 modulo 25). Hence $y_0 = 5y_1$ and $1 + 5^x = 2^{20y_1} + 2.5^t$. On the other hand $2^{10} \equiv 1 \pmod{11}$ and $5^x \equiv 2.5^t \pmod{11}$. Obviously $x > t$. Thus $5^{x-t} \equiv 2 \pmod{11}$, which is not true.

Therefore the given equation has a unique solution $(x, y, z, t) = (2, 4, 1, 1)$.

Grade 12.

Problem 1. For every real number x we denote by $f(x)$ the maximal value of the function $\sqrt{t^2 + 2t + 2}$ in the interval $[x - 2, x]$.

a) Prove that $f(x)$ is an even function and find its minimal value;

b) Prove that the function $f(x)$ is not differentiable for $x = 0$.

c) Prove that the sequence $a_n = \{f(n)\}$, $n = 1, 2, \ldots$ is convergent and find its limit.

(For every real number a we denote with $\{a\}$ the unique real number in the interval $[0, 1)$ for which the number $a - \{a\}$ is an integer.)

Solution. It is clear that the maximal value of the function $\sqrt{t^2+2t+2}$ in the interval $[x-2, x]$ is reached at the endpoints of this interval. Since the inequality $(x-2)^2+2(x-2)+2 \leq x^2+2x+2$ is equivalent to $x \geq 0$, then

$$f(x) = \begin{cases} \sqrt{x^2-2x+2} & \text{if} \quad x \leq 0 \\ \\ \sqrt{x^2+2x+2} & \text{if} \quad x \geq 0. \end{cases}$$

a) Obviously $f(x) = f(-x)$, i.e. $f(x)$ is an even function and its minimal value is $f(0) = 2$.

b) Since

$$\lim_{\substack{x\to 0\\ x<0}} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x<0}} \frac{\sqrt{x^2-2x+2}-\sqrt{2}}{x} = -\frac{\sqrt{2}}{2}$$

and

$$\lim_{\substack{x\to 0\\ x>0}} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{\sqrt{x^2+2x+2}-\sqrt{2}}{x} = \frac{\sqrt{2}}{2},$$

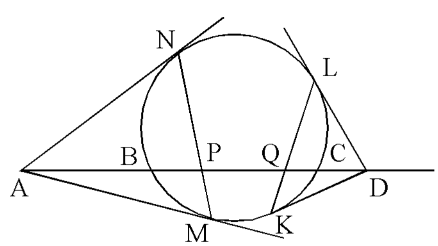
then $f'(0)$ doesn't exist.

c) It is clear that $a_n = \left\{\sqrt{n^2+2n+2}\right\}$. On the other hand $n+1 < \sqrt{n^2+2n+2} < n+2$, i.e. $a_n = \sqrt{n^2+2n+2}-(n+1)$.

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+2n+2}+n+1} = 0.$$

Figure 12.



Problem 2. The points A, B, C and D lie on a straight line in the given order. A circle k passes through the points B and C and AM, AN, DK and DL are tangents to k.

a) Prove that the points $P = MN \cap BC$ and $Q = KL \cap BC$ don't depend on the circle k.

b). If $AD = a$, $BC = b$ $(a > b)$ and the segment BC is moving along AD, find the minimal length of the segment PQ.

Solution. a) From the Steward's formula for the triangle AMN and the segment AP (Figure 12) we have:

$$\begin{aligned} AP^2.MN &= AM^2.NP + AN^2.MP - MN.MP.NP \\ &= (AM^2 - MP.NP).MN \end{aligned}$$

(here $AM = AN$ because they are tangents to a circle).

Hence

$$\begin{aligned} AP^2 = AM^2 - MP.NP &= AB.AC - BP.CP \\ &= AB.AC - (AC-AP)(AP-AB) \\ &= 2.AB.AC - AP(AB+AC) + AP^2, \end{aligned}$$

i.e. $AP = \dfrac{2AB.AC}{AB+AC}$. Analogously $DQ = \dfrac{2DB.DC}{DC+DB}$. These equalities show that the position of the points P and Q doesn't depend on the circle k.

b) Let us denote $AB = x$, $BC = y$ and $CD = z$. It follows from a) that

$$\begin{aligned} PQ = AD - AP - DQ &= x + y + z - \frac{2x(x+y)}{y+2x} - \frac{2z(y+z)}{y+2z} \\ &= (x+y)\left(1 - \frac{2x}{y+2x}\right) + z\left(1 - \frac{2(y+z)}{y+2z}\right) \\ &= \frac{y^2(x+y+z)}{(y+2x)(y+2z)}. \end{aligned}$$

Having in mind that $y = b$, $x+y+z = a$, we obtain

$$PQ = \frac{b^2 a}{(a+x-z)(a+z-x)} = \frac{b^2 a}{a^2-(x-z)^2} \geq \frac{b^2}{a}.$$

Therefore the minimal length of the segment PQ is $\dfrac{b^2}{a}$ and this length is reached iff $AB = CD = x = z = \dfrac{a-b}{2}$.

Problem 3. Find all prime numbers p and q, such that the number $2^p + 2^q$ is divisible by $p.q$.

Solution.

Lemma. If $k > 1$, then k doesn't divide $2^{k-1}+1$.

Proof. Assume that k divides $2^{k-1}+1$. Obviously k is odd. Let $k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $p_1, p_2, \dots, p_r$ are odd prime numbers ($p_i \neq p_j$ if $i \neq j$), $\alpha_1, \alpha_2, \dots, \alpha_r$ are positive integers and $r \geq 1$. Let $p_i - 1 = 2^{m_i}.t_i$, where t_i are odd integers ($i = 1, 2, \dots, r$). Let m_1 be the smallest number in the sequence $m_1, m_2, \dots, m_r$. It follows from $p_i \equiv 1 (\bmod (p_i - 1))$ that $p_i \equiv 1 \pmod{2^{m_i}}$ and $p_i^{\alpha_i} \equiv 1 \pmod{2^{m_i}}$, $i = 1, 2, \dots, r$. Hence $k - 1 = 2^{m_1}.u$ (u is an integer). If $2^{k-1} \equiv -1 \pmod{k}$, then $2^{2^{m_1}.u} \equiv -1 \pmod{k}$ and $2^{(p_1-1).u} \equiv -1 \pmod{p_1}$, because t_1 is odd. But $2^{p_1-1} \equiv 1 \pmod{p_1}$ — a contradiction.

Let $2^p + 2^q$ is divisible by $p.q$. We have three cases:

1. p and q are odd prime numbers. Then $2^p + 2^q \equiv 0 \pmod{p}$ and since $2^p \equiv 2 \pmod{p}$, then $2^q \equiv -2 (\bmod p)$ and $2^{pq} \equiv (-2)^p \equiv -2 \pmod{p}$. Similarly $2^{pq} \equiv -2 \pmod{q}$. Thus $2^{pq-1} \equiv -1 \pmod{pq}$, which is a contradiction with the lemma.

2. $p = 2$, $q > 2$. Then $4 + 2^q \equiv 0 (\bmod q)$ and it follows from $2^q \equiv 2 \pmod{q}$ that $6 \equiv 0 (\bmod q)$ and $q = 3$. It is clear that $2^2 + 2^3 = 12 \equiv 0 \pmod{2 \cdot 3}$.

3. $p = q = 2$. Then $2^2 + 2^2 = 8 \equiv 0 \pmod{2 \cdot 2}$.

Therefore the sought numbers are: $p = q = 2$; $p = 2$, $q = 3$; $p = 3$, $q = 2$.

WINTER MATHEMATICAL COMPETITION

1996

Grade 8

Problem 1. For which integer values of the parameter a the equation $|2x+1|+|x-2|=a$ has an integer solutions?

Solution. We shall consider the following cases:

I. $x>2$. Then $2x+1>0$, $x-2>0$ and the equation is equivalent to $2x+1+x-2=a$. Thus, $x=\frac{1+a}{3}$, which is a solution when $\frac{1+a}{3}>2$, i.e. when $a>5$.

II. $-\frac{1}{2}\le x\le 2$. Then $2x+1\ge 0$, $x-2\le 0$ and the equation has the form $2x+1-(x-2)=a$. Thus, $x=a-3$, which is a solution when $-\frac{1}{2}\le a-3\le 2$, i.e. $\frac{5}{2}\le a\le 5$. In this interval the integers are $a=3,4,5$ and respectively we get $x=0,1,2$, which are integer solutions.

III. $x<-\frac{1}{2}$. Then $2x+1<0$, $x-2<0$ and the equation is equivalent to $-(2x+1)-(x-2)=a$ or $-3x=a-1$. Thus $x=\frac{1-a}{3}$, which is a solution, when $\frac{1-a}{3}<-\frac{1}{2}$, i.e. when $a>\frac{5}{2}$.

According to the considered cases the given equation has solution when $a>\frac{5}{2}$. When $a=3$ the only integer solution is $x=0$. When $a=4$ there are two integer solutions $x=1$ and $x=-1$. When $a=5$, $x=2$ is the only integer solution. When $a>5$ the equation has two solutions $x_1=\frac{1+a}{3}$ and $x_2=\frac{1-a}{3}$. For these solutions we have: if $a=3k-1$, then x_1 is integer only; if $a=3k+1$, then x_2 is integer only; if $a=3k$, there is no integer solution.

As a result, we get integer solutions when $a=3$ or $a=3k\pm 1$, where k is a positive integer, greater than 1.

Problem 2. The bisector AD ($D\in BC$) of the acute isosceles triangle ABC divides it into two isosceles triangles. Let O and I be the incenter and the circumcenter of $\triangle ABC$, respectively. AO meets BC in point E, while F is the intersection point of the lines BI and DO. Prove that:

a) the quadrilaterals $ABEF$ and $ADCF$ are rhombi with equal side lengths;

b) If H is the altitude center of $\triangle ABE$, then the points A, D, E, F, H are concyclic.

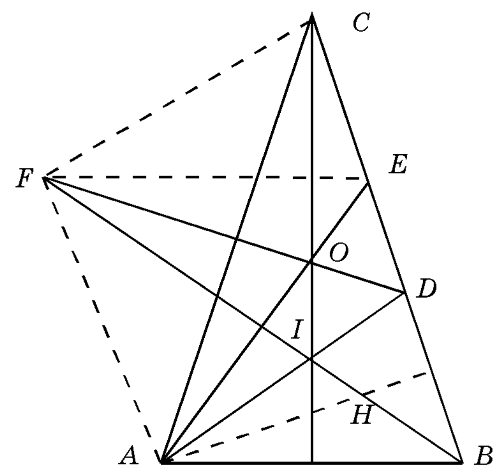
Solution. Firstly, let us justify the position of AD. If it is a bisector of the angle between the two equal sides, then it is perpendicular to BC, $\triangle ADB$ and $\triangle ADC$ are isosceles. Consequently $AD=BD=CD$ and $\angle BAD=\angle CAD=45^\circ$, i.e. $\angle BAC=90^\circ$, which contradicts to the condition that $\triangle ABC$ is acute. It follows that AD is bisector of the angle belonging to the base AB, while $\triangle ABD$ and $\triangle ACD$ are isosceles (Figure 1).

a) If $\angle BAD=\angle CAD=\alpha$, then $\angle ABC=2\alpha$. Since $\angle ADB>\angle CAD=\angle BAD$, the only possibility is $\angle ADB=\angle ABC=2\alpha$ and $AD=AB$. But $\angle ACD=\angle ADB-\angle CAD=2\alpha-\alpha=$

α, i.e. $\angle ACD = \angle CAD$ and $CD = AD$. From the equality $2\alpha+2\alpha+\alpha = 180°$ we define $\alpha = 36°$. Thus, $\angle ACB = 36°$, $\angle BAC = \angle ABC = 72°$. The points O and I lie on the bisector of $\angle ACB$, which is a segment bisector of AB. We have $OA = OC$ and $\angle OAC = \angle OCA = 18° = \angle OAD$, while $\angle OAB = 36° + 18° = 54°$, $\angle AEB = 180° - 72° - 54° = 54° = \angle BAE$. It follows that $BE = AB$ and the bisector BI of $\angle ABC$ is the segment bisector of AE. In the triangle ACD, which is isosceles, we have that AO and CO are bisectors and it follows that DO is the bisector of $\angle ADC$ and the segment bisector of AC. Then F is the intersection point of the segment bisectors of AE and AC, i.e. F is the circumcenter of $\triangle ACE$, the radius of the circumcircle is $AF = EF = CF$. We have $\angle AFE = 2\angle ACE = 72°$, while $\angle AFB = \angle BFE = 36° = \angle ABF = \angle EBF$ and consequently $AF = AB = BE = EF$, i.e. $ABEF$ is a rhombus. In the same way $CF = AF = AB = AD = CD$ and $ADCF$ is a rhombus, which side is $AD = AB$.

b) It is clear that $\angle AEF = \angle AEB = 54°$ and $\angle ADF = \frac{1}{2}\angle ADC =$ $frac12(180° - 72°) = 54°$, the segment AF is seen from the points D and E under $54°$. We draw a line through A, perpendicular to BC, which intersects BI in the point H — the altitude center of $\triangle ABE$, because $BI \perp AE$. But in $\triangle ABD$ the altitude AH is a bisector of $\angle BAD = 36°$. Thus, $\angle BAH = 18°$, $\angle AHF = 18° + 36° = 54° = \angle AEF = \angle AOF$. Note that the points D, E, H are in one and the same semiplane with respect to AF. Consequently, D, E, H together with A and F are concyclic.

Figure 1.



Problem 3. Every day a student preparing himself for the Winter competition in mathematics has been solving problems during a period of 5 weeks. He has been solving at least one problem daily but no more than 10 problems weekly.

a) Prove that during some consecutive days the student has solved 19 problems exactly.

b) If $1 \leq n \leq 34$ is a natural number, prove that during some consecutive days the student has solved n problems exactly.

Remark. Every week begins on Monday and ends on Sunday.

Solution. Since a) is a particular case of b), we shall solve b) only.

According to the condition the student has been solving problems during $5 \cdot 7 = 35$ days and has solved at most $5 \cdot 10 = 50$ problems. Let x_i be the number of the problems, solved during the i-th day ($i = 1, 2, \ldots, 35$).

Let $1 \leq n \leq 34$ be a fixed natural number. We want to prove that there exist such $k < l$, that $x_{k+1} + \cdots + x_l = n$. Denote $X_i = x_1 + \cdots + x_i$. Obviously,

$$1 \leq X_1 < X_2 < \ldots < X_{35} \leq 50$$

and the problem is to prove the existence of such $k < l$, that $X_l - X_k = n$.

Case 1. $1 \leq n \leq 19$. We consider the numbers

$$X_1 < X_2 < \ldots < X_{35}, \quad X_1 + n < X_2 + n < \ldots < X_{35} + n, \tag{1}$$

which are integers and their number is 70. Obviously they are in the interval $[1, 50 + n]$, in which there are $50 + n \leq 50 + 19 < 70$ integers. Consequently, among the numbers (1) there are at least two equal. The first 35 of them as well as the next 35 are different from each other. Therefore, there exist such k and l, for which $X_l = X_k + n$, i.e. for which $X_l - X_k = n$.

Case 2. $20 \le n \le 34$. Firstly, we shall prove the following

Lemma. If the integers $z_1, z_2, \ldots, z_m$ belong to the interval $[1, 2n]$ and if $m > n$, then among the numbers $z_1, z_2, \ldots, z_m$ there are two, the difference of which is equal to n exactly.

Proof. With the numbers from the interval $[1, 2n]$ we construct the following pairs:

$$(1, n+1), (2, n+2), \ldots, (n, 2n).$$

The number of these pairs is n and the difference of the numbers in each pair is equal to n. Since $m > n$, at least two of the numbers $z_1, z_2, \ldots, z_m$ belong to one and the same pair. Therefore their difference is equal to n.

Let us finish now the solution of the problem.

If $n \ge 25$, then $2n \ge 50$, and thus all the numbers $X_1, X_2, \ldots, X_{35}$ are in the interval $[1, 2n]$. On the other hand $n < 35$ and according to the lemma there are two numbers among $X_1, X_2, \ldots, X_{35}$ which difference is n.

If $20 \le n \le 24$, we represent the interval $[1, 50]$ as an union of the intervals $[1, 2n]$ $[2n+1, 50]$. In the second one there are $50 - (2n+1) + 1 = 50 - 2n$ integers. Then, the number of the integers among $X_1, X_2, \ldots, X_{35}$, which belong to the interval $[1, 2n]$, is at least $35 - (50 - 2n) = 2n - 15 \ge 40 - 15 = 25 > n$. Consequently, we can apply the lemma again.

Remark. The case 1 can be solved by the lemma proved above.

Grade 9

Problem 1. Let $f(x) = x^3 - (p+5)x^2 - 2(p-3)(p-1)x + 4p^2 - 24p + 36$, where p is a real parameter.

a) Prove that $f(3-p) = 0$.

b) Find all values of p, for which two of the roots of the equation $f(x) = 0$ are lengths of the cathetuses of a rectangle triangle which hypotenuse is equal to $4\sqrt{2}$.

Solution. a) We have $f(x) = (x + p - 3)(x^2 - 2(p+1)x + 4(p-3))$.

b) The roots of $f(x) = 0$ are $x_{1,2} = p + 1 \pm \sqrt{p^2 - 2p + 13}$ and $x_3 = 3 - p$. If $p > 3$, then $x_1 > 0$, $x_2 > 0$ and $x_3 < 0$. The equation $x_1^2 + x_2^2 = 32 = \left(4\sqrt{2}\right)^2$ gives $p = \pm 1$, which is impossible. If $p = 3$, then two of the roots are equal to 0 and this case gives no solution. If $p < 3$, then $x_1 > 0$, $x_3 > 0$ and $x_2 < 0$. Therefore, $32 = x_1^2 + x_3^2 = (p + 1 + \sqrt{p^2 - 2p + 13})^2 + (3 - p)^2$. This equation is equivalent to

$$(p+1)(3(p-3) + 2\sqrt{p^2 - 2p + 13}) = 0.$$

If $p \ne -1$, we get $2\sqrt{p^2 - 2p + 13} = 3(3 - p) > 0$, from where $5p^2 - 46p + 29 = 0$ and $p = \dfrac{23 \pm 8\sqrt{6}}{5}$. Since $\dfrac{23 + 8\sqrt{6}}{5} > 3$ and $\dfrac{23 - 8\sqrt{6}}{5} < 3$, we find $p_1 = -1$ and $p_2 = \dfrac{23 - 8\sqrt{6}}{5}$.

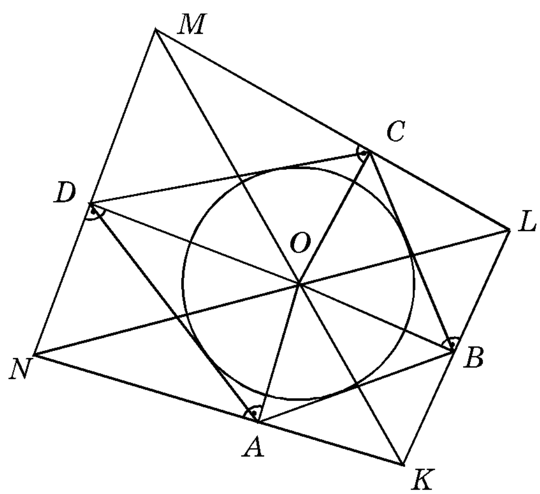
Problem 2. The incenter of the quadrilateral $ABCD$ is O. The lines $l_A \perp OA$, $l_B \perp OB$, $l_C \perp OC$ and $l_D \perp OD$ are drawn through the points A, B, C and D respectively. The lines l_A and l_B meet each other in the point K, l_B and l_C — in L, l_C and l_D — in M, l_D and l_A — in N.

a) Prove that the lines KM and LN meet each other in the point O.

b) If the lengths of the segments OK, OL and OM are p, q and r respectively, find the length of the segment ON.

Solution. a) We shall prove that the points N, O and L are colinear. Denote $\angle ABC = \angle B$, $\angle BCD = \angle C$, $\angle CDA = \angle D$ and $\angle DAC = \angle A$. Since O is incenter, the segments OA,OB,OC and OD are the bisectors of the corresponding angles of the quadrilateral $ABCD$. Note that each of the quadrilaterals $AKBO$, $BLCO$, $CMDO$ and $DNAO$ is inscribed.

Figure 2.



Consequently: $\angle NOK + \angle KOL = \pi - \angle ONA - \angle OKA + \pi - \angle OKB - \angle OLB = \pi - \angle ADO - \angle ABO + \pi - \angle BAO - \angle BCO = 2\pi - (\frac{\angle D}{2} + \frac{\angle B}{2} + \frac{\angle A}{2} + \frac{\angle C}{2}) = 2\pi - \pi = \pi$. It follows from here that the points N,O and L are colinear. Analogously, the points K,O and M are colinear. Therefore, O is the intersection point of the diagonals of the quadrilateral $KLMN$.

b) Firstly, we shall prove that the quadrilateral $KLMN$ is inscribed. Indeed,

$$\begin{aligned}
&\angle NKL + \angle NML \\
&= \angle AKO + \angle OKB + \angle DMO + \angle OMC \\
&= \frac{\angle B}{2} + \frac{\angle A}{2} + \frac{\angle C}{2} + \frac{\angle D}{2} = 2\pi.
\end{aligned}$$

Thus, $OK.OM = OL.ON$, from where

$$ON = \frac{OK.OM}{OL} = \frac{p.r}{q}.$$

Problem 3. A square with side length 5 is divided into unit squares by parallel to its sides lines. Let A be the set of the vertexes of the unit squares which are not on the sides of the given square. How many points from A can be chosen at most in a way that no three of them are vertexes of isosceles rectangle triangle?

Solution. We shall prove that the maximal number is 6. Let us enumerate the points in the way, shown on the table 1.

It is easy to be seen that no 3 of the points 1,2,3,8,12 and 16 are vertexes of a isosceles rectangle triangle. Assume that there exists a set of 7 points with the desired property. Note that if 4 points form a square, then at most 2 of them can be among the already chosen ones. The points 1,4,16 and 13; 2,8,15 and 9; 3,12,14 and 5 form squares.

13	14	15	16
9	10	11	12
5	6	7	8
1	2	3	4

Table 1:

Consequently, at most 6 of the chosen points lie on the contour. It follows from here that at least one of the points 6,7,10 and 11 is from the chosen ones. Due to the symmetry we may assume that this is the point 7. Since the points 7,16 and 9; 1,7 and 14 form isosceles rectangle triangles, then at most two of the points 1,9,14 and 16 are from the chosen ones. The points 5,7,13 and 15 form a square and therefore at most one of the points 5,13 and 15 is from the chosen ones. It follows from here that at least 3 points are chosen from 2,3,4,6,8,10,11 and 12. By the pigeonhole principle we deduce that at least two points are chosen in one of the sets 3,6,11,8 and 2,4,10,12. It is easy to see that if the two points are in the first set, then we have two possibilities — 3 and 11 or 6 and 8 (in both cases it is not possible to chose more points on the square which encounters 7). Analogously, if the two points are in the second set, then the possibilities are two again — 2 and 12 or 4 and 10 (in both cases it is not possible to chose more points on the square which encounters 7). The contradiction shows that the maximal number of points which can be chosen is equal to 6.

Grade 10

Problem 1. Let p and q be such integers that the roots x_1 and x_2 of the quadratic equation $x^2+px+q=0$ are real numbers. Prove that if the numbers 1, x_1, x_2 (in some order) form a geometric progression, then the number q is a perfect cube.

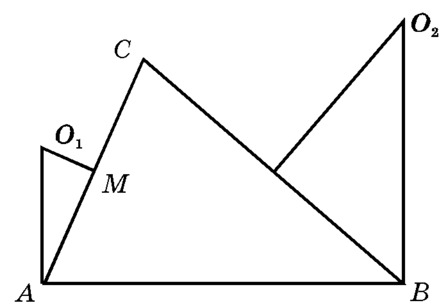
Solution. There are two possibilities for the order of the numbers in the geometric progression: $x_1, 1, x_2$ and $1, x_1, x_2$. In the first case we get $q = x_1x_2 = 1^2$, i.e. $q = 1$ is a perfect cube. Let now $x_2 = x_1^2$. We have $-p = x_1 + x_2 = x_1 + x_1^2$, i.e. x_1 satisfies the equation $x_1^2 + x_1 + p = 0$. On the other hand x_1 satisfies also $x_1^2 + px_1 + q = 0$. From these two equations we get $(p-1)x_1 + (q-p) = 0$. If $p \neq 1$, then x_1 is rational and $q = x_1x_2 = x_1x_1^2 = x_1^3$ is the cube of a rational number. Since q is integer, then it is a perfect cube. If $p = 1$, then $q = p$ and the quadratic equation becomes $x^2 + x + 1 = 0$. The last equation has no real root.

Problem 2. A triangle ABC with a radius R of the circumcircle is given. Let R_1 and R_2 be the radii of the circles k_1 and k_2, respectively, which pass through C and are tangent to the line AB in A and B, respectively.

a) Prove that the numbers R_1, R and R_2 form a geometric progression.

b) Find the angles of $\triangle ABC$, if the radius of the circle which is tangent to k_1, k_2 and the line AB, is equal to $\dfrac{R}{4}$.

Figure 3.



Solution. a) Let $AB = c$, $BC = a$ and $CA = b$. Denote by O_1 and O_2 the centers of the circles k_1 and k_2 (Figure 3). Then O_1 is the intersection point of the perpendicular from A to AB and the segment bisector of AC. Since $\angle MAO_1 = |90^\circ - \angle A|$ (M is the midpoint of AC), then $R_1 = AO_1 = \dfrac{AM}{\cos|90^\circ - \angle A|} = \dfrac{AC}{2\sin\angle A}$ and by the sine theorem it follows that

$$R_1 = R \cdot \frac{b}{a}. \tag{1}$$

Analogously,

$$R_2 = R \cdot \frac{a}{b}. \tag{2}$$

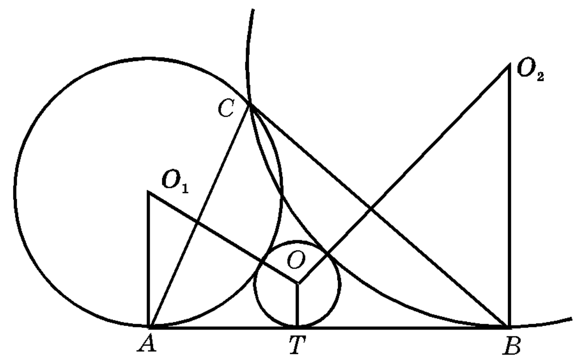
From here $R_1R_2 = R^2$ and the proposition is proved.

b) Let O be the center of the circle which touches k_1, k_2 and the line AB (Figure 4). Denote by T the tangent point of this circle with AB, and by r its radius. From the rectangle trapezoid $ATOO_1$ it follows that $AT = \sqrt{(R_1+r)^2 - (R_1-r)^2} = 2\sqrt{rR_1}$. Analogously, $BT = 2\sqrt{rR_2}$. Then $c = AT + TB = 2\sqrt{rR_1} + 2\sqrt{rR_2}$ and from here we find $r = \dfrac{c^2}{4(\sqrt{R_1}+\sqrt{R_2})^2}$. Using (1) and (2), we get

$$r = \frac{abc^2}{4R(a+b)^2}. \tag{3}$$

Since $\dfrac{ab}{(a+b)^2} \le \dfrac{1}{4}$ and $c^2 \le 4R^2$, it follows from (3) that $r \le \dfrac{R}{4}$. The equation is reached when $a = b$ and $c = 2R$. Now, it follows that $\angle A = \angle B = 45°$ and $\angle C = 90°$.

Figure 4.



Problem 3. A positive integer n and a real number φ are given in a way that $\cos\varphi = \dfrac{1}{n}$. Find all positive integers k, for which the number $\cos k\varphi$ is an integer.

Solution. **Case 1.** $n = 1$. Then $\cos\varphi = 1$ and $\varphi = 2m\pi$ $(m = 0, \pm 1, \ldots)$. For all $k \in \mathbb{N}$ the number $\cos k\varphi$ is an integer.

Case 2. $n = 2$. Then $\cos\varphi = \dfrac{1}{2}$ and $\varphi = \pm\dfrac{\pi}{3} + 2m\pi$ $(m = 0, \pm 1, \ldots)$. It is clear that for all $k \in \mathbb{N}$, which are divisible by 3, the number $\cos k\varphi$ is an integer.

Case 3. $n \ge 3$. We shall prove that for all $k \in \mathbb{N}$ the number $\cos k\varphi$ is not an integer. Let n be odd. We have $\cos\varphi = \dfrac{1}{n}$, $\cos 2\varphi = 2\cos^2\varphi - 1 = \dfrac{2-n^2}{n^2}$ and $(2-n^2, n) = 1$. We shall prove by induction that $\cos k\varphi = \dfrac{a}{n^k}$, where $(a, n) = 1$. Assume that the assertion is true for all integers from 1 to k. We shall check it for $k+1$. From

$$\cos(k+1)\varphi + \cos(k-1)\varphi = 2\cos k\varphi\cos\varphi$$

it follows that $\cos(k+1)\varphi = \dfrac{2}{n}\cdot\dfrac{a}{n^k} - \dfrac{b}{n^{k-1}} = \dfrac{2a - bn^2}{n^{k+1}}$, where $\cos k\varphi = \dfrac{a}{n^k}$ and $\cos(k-1)\varphi = \dfrac{b}{n^{k-1}}$. We have $(a, n) = 1$ and $(b, n) = 1$ according to the inductive assumption. It is clear that $(2a - bn^2, n) = 1$ and this ends the proof. The case when n is even is analogous. Now $\cos k\varphi$ is expressed by a fraction which denominator is equal to $2p^k$ with $n = 2p$, while the nominator has no common divisor with p.

Answer:

	if $n = 1$	$\forall k \in \mathbb{N}$;
	if $n = 2$	$k = 3q$, where $q \in \mathbb{N}$;
	if $n \ge 3$	there is no solution.

Grade 11

Problem 1. Find the values of the real parameter a, for which the function

$$f(x) = x^2 - 2x - |x - 1 - a| - |x - 2| + 4$$

has nonnegative values for all real x.

Solution. Firstly let $1 + a \le 2$, i.e. $a \le 1$. Then

$$f(x) = \begin{cases} x^2 + 1 - a, & x \le 1 + a \\ x^2 - 2x + 3 + a, & 1 + a \le x \le 2 \\ x^2 - 4x + 7 + a, & x \ge 2. \end{cases}$$

If $1+a \geq 2$, i.e. $a \geq 1$, we find that

$$f(x) = \begin{cases} x^2+1-a, & x \leq 2 \\ x^2-2x+5-a, & 2 \leq x \leq 1+a \\ x^2-4x+7+a, & x \geq 1+a. \end{cases}$$

Hence the smallest value of $f(x)$ is reached in one of the points 0, 1, 2, $1+a$.

We have $f(0) = 2-|a+1|$, $f(1) = 2-|a|$, $f(2) = 4-|1-a|$, $f(1+a) = a^2-|a-1|+3$. These four numbers must be nonnegative. We find that $a \in [-2, 1]$. Then $|a-1| = 1-a$ and $f(1+a) = a^2+a+2 > 0$ for all a.

Let now $a \notin [-2, 1]$. If $a < -2$, then $x^2-2x+3+a < 0$ when $x = 1$. Analogously, if $a > 1$, then $x^2+1-a < 0$ when $x = 0$.

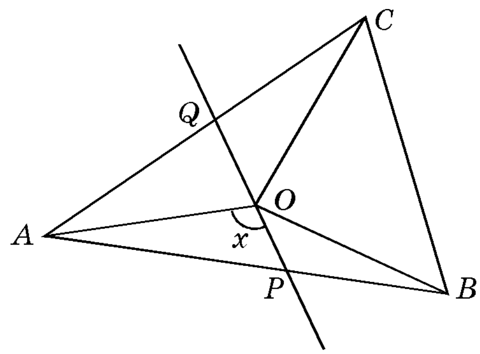
Finally, $a \in [-2, 1]$.

Problem 2. The point O is circumcenter of the acute triangle ABC. The points P and Q lie on the sides AB and AC respectively. Prove that O lies on the line PQ if and only if

$$\sin 2\alpha = \frac{PB}{PA}\sin 2\beta + \frac{QC}{QA}\sin 2\gamma,$$

where $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$.

Figure 5.



Solution. Firstly, let O be on PQ. Denote $x = \angle AOP$ (Figure 5). By the sine theorem for the triangles AOP, BOP, AOQ and COQ we find that

$$\frac{PA}{\sin x} = \frac{PO}{\sin(90^\circ-\gamma)},$$

$$\frac{PB}{\sin(2\gamma-x)} = \frac{PO}{\sin(90^\circ-\gamma)},$$

$$\frac{QA}{\sin(180^\circ-x)} = \frac{QO}{\sin(90^\circ-\beta)},$$

$$\frac{QC}{\sin(2\beta-180^\circ+x)} = \frac{QO}{\sin(90^\circ-\beta)}.$$

Then

$$\frac{PB}{PA}\sin 2\beta + \frac{QC}{QA}\sin 2\gamma = \frac{\sin(2\gamma-x)}{\sin x}\sin 2\beta - \frac{\sin(2\beta+x)}{\sin x}\sin 2\gamma =$$
$$\frac{1}{\sin x}((\sin 2\gamma\cos x - \sin x\cos 2\gamma)\sin 2\beta - (\sin 2\beta\cos x + \sin x\cos 2\beta)\sin 2\gamma)$$
$$= -\sin 2\beta\cos 2\gamma - \sin 2\gamma\cos 2\beta = -\sin 2(\beta+\gamma) = \sin 2\alpha.$$

Conversely, let the given equality be satisfied. Denote by Q' the intersection point of OP and AC. It follows from the above that $\sin 2\alpha = \dfrac{PB}{PA}\sin 2\beta + \dfrac{Q'C}{Q'A}\sin 2\gamma$. Then $\dfrac{Q'C}{Q'A} = \dfrac{QC}{QA}$. Since the points Q' and Q lie on the segment AC, then $Q' = Q$.

Problem 3. Find all functions $f(x)$ with integer values and defined in the set of the integers, such that

$$3f(x) - 2f(f(x)) = x$$

for all integers x.

Solution. The function $f(x) = x$ satisfies the condition of the problem.

Let $f(x)$ be a function which satisfies the condition. Let $g(x) = f(x) - x$. The condition can be written in the form

$$2f(f(x)) - 2f(x) = f(x) - x,$$

which is equivalent to

$$g(x) = 2g(f(x)).$$

From here we obtain

$$g(x) = 2g(f(x)) = 2^2 g(f(f(x))) = 2^3 g(f(f(f(x)))) = 2^4 g(f(f(f(f(x))))) = \dots .$$

Since the numbers $g(f(f \dots f(x)) \dots)$ are integer, then $g(x)$ is divisible by 2^n for all integers x and all natural numbers n. This is possible only if $g(x) = 0$. Thus, $f(x) = x$ is the only solution of the problem.

Winter mathematics competition—Burgas, 1997

Problem 8.1. Let F be the set of points with coordinates (x, y) such that $||x| - |y|| + |x| + |y| = 2$.

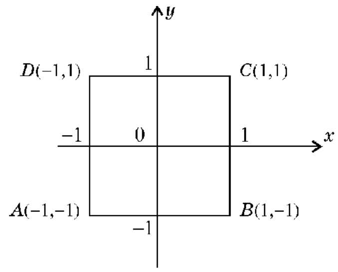
(a) Draw F.

(b) Find the number of points in F such that $2y = |2x - 1| - 3$.

Solution: (a) If $|x| \geq |y|$, then $||x| - |y|| + |x| + |y| = |x| - |y| + |x| + |y| = 2|x| = 2$, thus $|x| = 1$ and therefore $1 \geq |y|$, so $-1 \leq y \leq 1$. We conclude that the segments $-1 \leq y \leq 1$ on the lines $x = 1$ and $x = -1$ belong to F.

If $|x| \leq |y|$, then $||x| - |y|| + |x| + |y| = -|x| + |y| + |x| + |y| = 2|y| = 2$, thus $|y| = 1$ and therefore $1 \geq |x|$, so $-1 \leq x \leq 1$. We conclude that the segments $-1 \leq x \leq 1$ on the lines $y = 1$ and $y = -1$ also belong to F.

Thus we have determined that F consists of the sides of a square with vertices $A(-1, -1), B(1, -1), C(1, 1), D(-1, 1)$.



(b) We find the number of solutions of $2y = |2x-1|-3$ on each of the segments AB, BC, CD, DA.

The segment CD consists of all points (x, y) such that $-1 \leq x \leq 1$, $y = 1$. The equation $2 = |2x-1|-3$ has no solution x when $-1 \leq x \leq 1$. Therefore $2y = |2x-1|-3$ has no solution on CD.

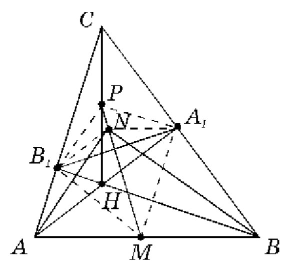
The segment AB consists of all points (x, y) such that $-1 \leq x \leq 1$, $y = -1$. The equation $-2 = |2x-1|-3$ has two solutions: $x = 0$ and $x = 1$. Therefore $2y = |2x-1|-3$ has two solutions on AB.

As above we get that $2y = |2x-1|-3$ has a unique solution on AD: $(x, y) = (-1, 0)$ and a unique solution on BC: $(x, y) = (1, -1)$. Note that the last one has already been obtained as a point on AB. Thus there are three solutions of $2y = |2x-1|-3$ in F: $(x, y) = (-1, 0), (0, -1), (1, -1)$.

Problem 8.2. Let H be the orthocentre of an acute triangle ABC. Prove that the midpoints of AB and CH and the intersecting point of the internal bisectors of $\angle CAH$ and $\angle CBH$ lie on a line.

Solution: Denote by M, P and N the midpoints of AB and CH and the intersecting point of the internal bisectors of $\angle CAH$ and $\angle CBH$ (fig. 1).

Let AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$) be altitudes in $\triangle ABC$. We show first that M, N and P lie on the axis of symmetry l of A_1B_1. From $\angle CA_1H = \angle CB_1H = 90°$ we get $PA_1 = PB_1 = \frac{CH}{2}$. Similarly, from $\angle AA_1B = \angle AB_1B = 90°$ we get $MA_1 = MB_1 = \frac{AB}{2}$. Therefore $M \in l$ and $P \in l$. We prove now that $\triangle NMA_1 \cong \triangle NMB_1$. Now $\angle BAA_1 = 90° - \beta$ and $\angle ABB_1 = 90° - \alpha$, thus $\angle NBB_1 = \angle NBA_1 = \frac{1}{2}\angle A_1BB_1 = 45° - \frac{\gamma}{2}$. By analogy $\angle NAA_1 = 45° - \frac{\gamma}{2}$, so $\angle ANB = 180° - \angle NAB - \angle NBA = 90°$. Therefore $\triangle ABN$ is a right triangle and $MN = MA_1 = MB_1 = \frac{AB}{2}$.



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Further $\angle NMA_1 = \angle NMB - \angle A_1MB = 90° - \gamma$. By analogy $\angle NMB_1 = 90° - \gamma$ and therefore the considered triangles are identical. It follows now that $NA_1 = NB_1$, so $N \in l$.

Problem 8.3. The n points $A_0, A_1, \ldots, A_{n-1}$ lie a circle in this order and divide it into equal arcs. Find an ordering $B_0, B_1, \ldots, B_{n-1}$

of the same points such that the length of $B_0B_1\dots B_{n-1}$ is maximal.

Solution: Let first $n = 2k+1$. Clearly a chord A_iA_j is of maximal length if $|i-j| = k$ or $k+1$. Cosider the following points:

$$A_0, A_k, A_{2k}, A_{k-1}, A_{2k-1}, A_{k-2}, A_{2k-2}, \dots, A_1, A_{k+1}.$$

Since each segment is of maximal length, it follows that the length of $A_0A_kA_{2k}A_{k-1}A_{2k-1}A_{k-2}A_{2k-2}\dots A_1A_{k+1}$ is maximal.

Let now $n = 2k$. A chord A_iA_j is of maximal length if $|i-j| = k$. There are k such segments: $A_0A_k, A_1A_{k+1}, \dots, A_{k-1}A_{2k-1}$. The second longest chord A_iA_j is obtained when $|i-j| = k-1$ or $k+1$. Cosider the following points:

$$A_0, A_k, A_{2k-1}, A_{k-1}, A_{2k-2}, A_{k-2}, \dots, A_{k+1}, A_1.$$

It is easy to see that there are k segments of maximal length and $k-1$ segments of the second greatest length. Trivially, this is the required ordering.

Problem 9.1. Let $\alpha \neq \beta$ be the roots of the equation $x^2+px+q = 0$. For any natural number n denote:

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

(a) Find p and q such that for any natural number n the following equality holds:

$$a_{n+1}a_{n+2} - a_na_{n+3} = (-1)^n.$$

(b) Prove that for these p and q it is true that

$$a_n + a_{n+1} = a_{n+2}$$

for any natural number n.

(c) Prove that for any natural number n, a_n is integer and if 3 divides n, then a_n is even.

Solution: (a) Since α and β are the roots of $x^2 + px + q = 0$, we know that $\alpha + \beta = -p$, $\alpha\beta = q$ and therefore:

$$\begin{aligned}
(-1)^n &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \cdot \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} - \frac{\alpha^n - \beta^n}{\alpha - \beta} \cdot \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} \\
&= \frac{1}{(\alpha - \beta)^2}[-(\alpha + \beta)(\alpha\beta)^{n+1} + (\alpha^3 + \beta^3)(\alpha\beta)^n] \\
&= \frac{1}{p^2 - 4q}(pq^{n+1} - p(p^2 - 3q)q^n) \\
&= \frac{q^n}{p^2 - 4q}(-p^3 + 4pq) = -pq^n.
\end{aligned}$$

Thus

$$pq^n = (-1)^{n+1}. \tag{1}$$

It follows from (1) for $n = 1$ and $n = 2$ that $pq = 1$ and $pq^2 = -1$ and so $p = -1$, $q = -1$. Direct verification shows that $p = -1$ and $q = -1$ satisfy (1) for any n. Also $\alpha \neq \beta$.

(b) Since α and β are the roots of $x^2 - x - 1 = 0$, we know that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. Therefore

$$a_n + a_{n+1} = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{\alpha^n(1 + \alpha) - \beta^n(1 + \beta)}{\alpha - \beta}$$

$$= \quad \frac{\alpha^n.\alpha^2 - \beta^n.\beta^2}{\alpha - \beta} = \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = a_{n+2},$$

which completes the proof of (b).

(c) Since $a_1 = \dfrac{\alpha - \beta}{\alpha - \beta} = 1$ and $a_2 = \dfrac{\alpha^2 - \beta^2}{\alpha - \beta} = \alpha + \beta = 1$, it follows by induction from

$$a_n + a_{n+1} = a_{n+2} \tag{2}$$

that a_n is integer for any n. From (2) we obtain:

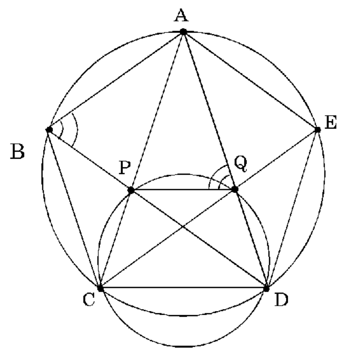
$$a_{n+3} = a_{n+2} + a_{n+1} = a_{n+1} + a_n + a_{n+1} = 2a_{n+1} + a_n.$$

Observe that $a_3 = a_1 + a_2 = 2$ is an even number. It is easy to see now (again by induction) that a_n is even for $n = 3k$.

Problem 9.2. A pentagon $ABCDE$ is inscribed in a circle. Let P be the intersecting point of AC and BD, and let Q be the intersecting point of AD and CE. Prove that if the triangles ABP, AEQ, CDP, CDQ and APQ have the same area, then $ABCDE$ is a regular pentagon.

Solution: It suffices to prove that the sides of $ABCDE$ are equal. Since $\triangle ABP$ and $\triangle CDP$ have equal areas, so do the triangles ACD and ADB. Therefore the quadrilateral $ABCD$ (fig. 1) is a trapezoid inscribed in a circle, so $AB = CD$. By analogy from $S_{AEQ} = S_{CDQ}$ we get $AC \| DE$ and $AE = CD$. Now, $\angle PCQ = \angle ACE = \dfrac{\overset{\frown}{AE}}{2} = \dfrac{\overset{\frown}{AB}}{2} = \angle ADB = \angle PDQ$. Therefore $CDQP$ is inscribed in a circle.

On the other hand it follows from $S_{CDP} = S_{CDQ}$ that $CDQP$ is a trapezoid. Thus $\angle CDP = \angle DCQ \Rightarrow \overset{\frown}{BC} = \overset{\frown}{DE}$ and therefore $BC = DE$. It remains to show that $AB = BC$. Consider the triangles ABP and APQ, whose areas are equal. They have AP as a common side and $\angle AQP = \angle APQ = \angle ACD = \angle ABD = \angle ABP$. It is easy to see now that $\triangle ABP \cong \triangle APQ$. Since $\angle APB = \angle ADP + \angle PAQ > \angle PAQ$,



черт. 1

it follows that $\angle APB = \angle APQ = \angle AQP = \angle ABP$. Therefore $\overset{\frown}{AB} + \overset{\frown}{CD} = \overset{\frown}{AE} + \overset{\frown}{DE}$, so $\overset{\frown}{AB} = \overset{\frown}{DE} = \overset{\frown}{BC}$ or $AB = BC$, which completes the proof.

Problem 9.3. Given a rectangular table of 100 rows and 1997 columns. The table is filled with zeroes and ones in such a way that there are at least 75 ones in any column. Prove that it is possible to remove 95 rows in such a way that there is at most one column consisting of zeroes in the remaining table (5 rows and 1997 columns).

Solution: We show first that there is a row with at least 1498 ones. Assume the contrary. Denote by a_i the number of ones in the i-th row ($i = 1, 2, \ldots, 100$) and by b_i the number of ones in the i-th column ($i = 1, 2, \ldots, 1997$). Now $\sum_{i=1}^{1997} b_i = \sum_{i=1}^{100} a_i$. Note that the sum on the left-hand side is at least $1997 \cdot 75$, whereas the sum on the right-hand side is at most $1497 \cdot 100$, a contradiction.

Without loss of generality assume that the first row begins by 1498 ones. Consider the table formed by the last 499 columns. As above we prove that there is a row of this table (not necessarily distinct from the first one) with at least 375 ones. Let that be the second row (if it is not the first one) and let it begin with 375 ones in the new table. Now consider the table formed by the last 124 columns. Analogously, there exists a row having at least 93 ones. Let that be the third row (if it is not the first or the second one) and let it begin with 94 ones in the new table. Consider next the table formed by the last 31 columns and note that there exists a row that originally has 24 ones. Finally consider the table formed by the last 7 rows and note that there exists a row that originally has 6 ones.

We now have 5 rows (if there are fewer of them, we add arbitrary rows). Remove the remaining 95 rows of the original table. Since $1498+375+93+24+6 = 1996$, there is at most one column consisting of zeroes.

Problem 10.1. Find all real numbers x such that $\tan\left(\frac{\pi}{12}-x\right)$, $\tan\frac{\pi}{12}$ and $\tan\left(\frac{\pi}{12}+x\right)$ form (in some order) a geometric progression.

Solution: Denote $a=\tan\frac{\pi}{12}$ and $y=\tan x$. There are three cases to consider:

1. $\tan\left(\frac{\pi}{12}-x\right)\cdot\tan\left(\frac{\pi}{12}+x\right)=\tan^2\frac{\pi}{12}$. Now $\frac{a-y}{1+ay}\cdot\frac{a+y}{1-ay}=a^2$. Therefore $a^2-y^2=a^2(1-a^2y^2)$, and so $(a^4-1)y^2=0$. Since $a\neq\pm1$ we get that $y=0$, so $\tan x=0$. Obviously all numbers x of the kind $x=k\pi$, $k\in\mathbb{Z}$ are solutions to the problem.

2. $\tan\frac{\pi}{12}\tan\left(\frac{\pi}{12}+x\right)=\tan^2\left(\frac{\pi}{12}-x\right)$. We obtain

$$a\frac{a+y}{1-ay}=\left(\frac{a-y}{1+ay}\right)^2\Longrightarrow(a^2+1)y[ay^2+(a^2-1)y+3a]=0.$$

The case of $y=0$ is settled in 1. Let y_1 and y_2 be the roots of the equation $ay^2+(a^2-1)y+3a=0$. Since $a=\tan 15\deg=\tan(45\deg-30\deg)=2-\sqrt{3}$ we get $y_1=y_2=\sqrt{3}$, so $\tan x=\sqrt{3}$. Obviously all x of the kind $x=\frac{\pi}{3}+k\pi$, $k\in\mathbb{Z}$ are solutions of the problem.

3. $\tan\frac{\pi}{12}\tan\left(\frac{\pi}{12}-x\right)=\tan^2\left(\frac{\pi}{12}+x\right)$. The substitution $z=-x$ transforms this case to the previous one. Therefore $x=-\frac{\pi}{3}+k\pi$, $k\in\mathbb{Z}$.

The required numbers are $x = k\pi$ and $x = \pm\dfrac{\pi}{3} + k\pi$, $k \in \mathbb{Z}$.

Problem 10.2. Two points C and M are given in the plane. Let H be the orthocentre of $\triangle ABC$ such that M is a midpoint of AB.

(a) Prove that $CH \cdot CD = |AM^2 - CM^2|$ where $D \in AB$ and $CD \perp AB$.

(b) Find the locus of points H when AB is of given length c.

Solution: (a) It is easy to see that the equality holds if $\triangle ABC$ is a right triangle. If $\triangle ABC$ is not a right triangle, then $\triangle BDH$ and $\triangle ADC$ exist and $\triangle BDH \sim \triangle ADC$. Therefore $\dfrac{DH}{AD} = \dfrac{BD}{CD}$ and so

$$CD \cdot DH = AD \cdot BD \tag{1}$$

There are three cases:

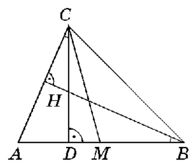
Case 1: $\triangle ABC$ is an acute triangle (fig. 1).

Case 2: $\triangle ABC$ is obtuse triangle and $\angle C > 90^\circ$ (fig. 2).

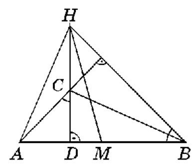
Case 3: $\triangle ABC$ is obtuse triangle and $\angle A > 90^\circ$ or $\angle B > 90^\circ$ (fig. 3).

In cases 1 and 2 it follows from (1) that

$$CD \cdot DH = (AM \mp DM)(AM \pm DM) = AM^2 - DM^2 = AM^2 - (CM^2 - CD^2),$$



черт. 1



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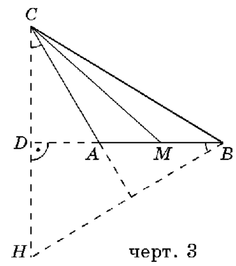


черт. 3

so

$$(2) \qquad AM^2 - CM^2 = CD(DH - CD).$$

In case 1 we have $CD = CH + DH$ and from (2) we get

$$AM^2 - CM^2 = CD(DH - CH - DH) = -CH \cdot CD.$$

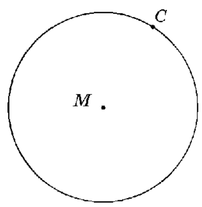
In case 2 we have $CD = DH - CH$ and from (2) we get

$$AM^2 - CM^2 = CD(DH - DH + CH) = CH \cdot CD.$$

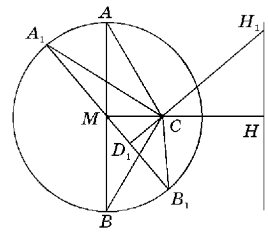
In case 3 it follows from (1) that $CD \cdot DH = (DM \mp AM)(DM \pm AM) = DM^2 - AM^2 = CM^2 - CD^2 - AM^2$ and therefore $CM^2 - AM^2 = CD(DH + CD)$. Now $DH = CH - CD$ and so $CM^2 - AM^2 = CD(CH - CD + CD) = CH \cdot CD$. Summarising all three cases considered (and the case of a right triangle $\triangle ABC$) we get

$$CH \cdot CD = |AM^2 - CM^2|.$$

(b) It follows from the conditions of the problem that A and B are diametrically opposite in the circle (k) of centre M and radius $\frac{c}{2}$. When the diameter AB varies in (k) we obtain all triangles ABC with the fixed vertex C. There are three cases:



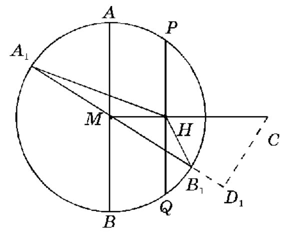
черт. 4



черт. 5

Case 1: $\frac{c}{2} = |CM|$ (fig. 4); Case 2: $\frac{c}{2} > |CM|$ (fig. 5); Case 3: $\frac{c}{2} < |CM|$ (fig. 6).

In case 1 C lies on (k) and triangles ABC are right triangles and their orthocentres coincide with C. In this case the locus consists of point C.



черт. 6

Consider now case 2. Let AB be a diameter in (k) perpendicular to CM. The orthocentre H of $\triangle ABC$ lies on line CM. Moreover H is an external point for (k) because $\triangle ABH$ is acute. The last follows from $\angle AHB = 180\deg - \angle ACB$ and $\angle ACB > 90\deg$.

Let A_1B_1 be a arbitrary diameter of (k) such that $C \notin A_1B_1$. Denote by H_1 the orthocentre of A_1B_1C and by D_1 the foot of the altitude from the vertex C. Since $\triangle A_1B_1C$ is obtuse, we get that $H_1 \neq C$. Consider $\triangle MD_1C$ and $\triangle CHH_1$ and use a). We are in case 2 of (a). For $\triangle ABC$ we obtain that $CH \cdot CM = \dfrac{c^2}{4} - CM^2$ and for $\triangle A_1B_1C$ that $CH_1 \cdot CD_1 = \dfrac{c^2}{4} - CM^2$. Therefore $\dfrac{CH}{CH_1} = \dfrac{CD_1}{CM}$, so $\triangle MD_1C \sim \triangle CHH_1$. Since $\triangle MD_1C$ is a right triangle, we know that $H_1H \perp CM$ and so H_1 lies on a line l through H which is perpendicular to CM. Let $H_1 \neq H$ be an arbitrary chosen point of l and let A_1B_1 be a diameter of (k) such that $AB \perp CH_1$.

Since H is an external point for (k) and $l \perp CM$ through H, it follows that all points of l are external for (k). In particular $H_1 \neq C$ and therefore A_1B_1 exists. The orthocentre of $\triangle A_1B_1C$ lies on CH_1

and it follows from the above that it lies on l. Since H_1 is the intersecting point of these two lines it is obvious that H_1 is the orthocentre of $\triangle A_1B_1C$.

Therefore the locus is a line l perpendicular to CM through a point H on the ray opposite to $\overrightarrow{CM}$ and of distance $\dfrac{c^2}{4|CM|} - |CM|$ from point C.

Consider case 3. Let AB be a diameter of (k) perpendicular to CM. The orthocentre H of $\triangle ABC$ lies on CM but now H is an internal point for (k). Consider an arbitrary diameter A_1B_1 of (k) such that $H \in A_1B_1$. Let H_1 be the orthocentre of $\triangle A_1B_1C$ and CD_1- the altitude. Consider $\triangle MD_1C$ and $\triangle CHH_1$. It is essential to show that $\triangle CHH_1$ is uniquely determined (for $\triangle MD_1C$ it is obvious). It suffices to prove that $H_1 \neq C$. If the contrary is true, then $\triangle A_1B_1C$ is a right triangle with its right angle at C and since A_1B_1 is a diameter, C must lie on (k), a contradiction. Since $\triangle ABC$ is acute, we apply case 1 of (a), so $CH \cdot CM = CM^2 - \dfrac{c^2}{4}$. There are three cases for $\triangle A_1B_1C$:

1. if it is an acute triangle, we apply case 1 of (a) and therefore $CH_1 \cdot CD_1 = CM^2 - \dfrac{c^2}{4}$;
2. if it is an obtuse triangle, we apply case 3 of (a) and we get the same equality (note that $\triangle A_1B_1C$ is not obtuse at C because C is an external point for (k));
3. if it is a right triangle (say at B_1) then $D_1 \equiv B_1 \equiv H_1$ and therefore $CH_1 \cdot CD_1 = CB_1^2 = CM^2 - \dfrac{c^2}{4}$.

The further considerations follow those from case 2. We conclude that the locus is a line perpendicular to CM and passing through $H \in CM^{\rightarrow}$ of distance $|CM| - \dfrac{c^2}{4|CM|}$ from C.

In particular it follows that $PQ \perp CM$, a well-known property.

Problem 10.3. How many natural numbers $\overline{a_1 a_2 \dots a_{2n}}$ exist such that:

(a) none of the digits a_i is zero;

(b) the sum $a_1a_2 + a_3a_4 + \dots + a_{2n-1}a_{2n}$ is an even number?

Solution: Denote the required number by A_n. The product $a_{2i-1}a_{2i}$ is even if at least one of the digits a_{2i-1} and a_{2i} is even. Therefore there are $5 \cdot 4 + 4 \cdot 5 + 4 \cdot 4 = 56$ choices for a_{2i-1} and a_{2i} such that $a_{2i-1}a_{2i}$ is even. Similarly, $a_{2i-1}a_{2i}$ is odd when both a_{2i-1} and a_{2i} are odd. There are $5 \cdot 5 = 25$ choices for a_{2i-1} and a_{2i} such that $a_{2i-1}a_{2i}$ is odd. The number of $\overline{a_1 a_2 \dots a_{2n}}$ such that i of the items $a_{2i-1}a_{2i}$ are odd is $\binom{n}{i}25^i 56^{n-i}$. Therefore $A_n = \sum_i \binom{n}{2i} 25^{2i} 56^{n-2i}$. Let $B_n = \sum_i \binom{n}{2i+1} 25^{2i+1} 56^{n-2i-1}$. Obviously

$$A_n + B_n = \sum_{i=0}^{n} \binom{n}{i} 25^i 56^{n-i} = (56+25)^n = 81^n,$$

$$A_n - B_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} 25^i 56^{n-i} = (56-25)^n = 31^n.$$

Thus $2A_n = 81^n + 31^n$ and $A_n = (81^n + 31^n)/2$.

Problem 11.1. The sequence $\{x_n\}_{n=1}^{\infty}$ is defined as:

$$x_1 = 3, \quad x_{n+1} = x_n^2 - 3x_n + 4, \quad n = 1, 2, 3, \dots$$

(a) Prove that $\{x_n\}_{n=1}^{\infty}$ is monotone increasing and unbounded.

(b) Prove that the sequence $\{y_n\}_{n=1}^{\infty}$ defined as $y_n = \dfrac{1}{x_1 - 1} + \dfrac{1}{x_2 - 1} + \cdots + \dfrac{1}{x_n - 1}$, $n = 1, 2, 3, \dots$, is convergent and find its limit.

Solution: (a) $x_{n+1} - x_n = (x_n - 2)^2 \geq 0$ and so $\{x_n\}_{n=1}^{\infty}$ is a monotone increasing function. We now prove by induction that $x_n \geq n + 2$. Obviously this equality holds for $n = 1$. Suppose it is true for $n = k \geq 1$. Then

$$x_{k+1} = x_k(x_k - 3) + 4 \geq (k + 2)(k - 1) + 4 \geq k + 3.$$

Therefore $x_n \geq n + 2$ when $n = 1, 2, 3, \dots$ and so the sequence is unbounded.

(b) It follows from the recursive definition of our sequence that $x_{k+1} - 2 = (x_k - 1)(x_k - 2)$. Hence

$$\frac{1}{x_{k+1} - 2} = \frac{1}{(x_k - 1)(x_k - 2)} = \frac{1}{x_k - 2} - \frac{1}{x_k - 1},$$

$$\text{so} \qquad \frac{1}{x_k - 1} = \frac{1}{x_k - 2} - \frac{1}{x_{k+1} - 2}.$$

By adding the above equalities for $k = 1, 2, \dots, n$ we get

$$y_n = \frac{1}{x_1 - 2} - \frac{1}{x_{n+1} - 2} = 1 - \frac{1}{x_{n+1} - 2}.$$

Since $0 \leq \dfrac{1}{x_{n+1}-2} \leq \dfrac{1}{n}$ it follows that $\lim\limits_{n\to\infty} \dfrac{1}{x_{n+1}-2} = 0$, so $\lim\limits_{n\to\infty} y_n = 1$.

Problem 11.2. Given $\triangle ABC$ such that $\angle ABC \geq 60\,\text{deg}$ and $\angle BAC \geq 60\,\text{deg}$. Let BL $(L \in AC)$ be the internal bisector of $\angle ABC$ and $AH, H \in BC$ be the altitude from A. Find $\angle AHL$ if $\angle BLC = 3\angle AHL$.

Solution: Denote $\theta = \angle AHL$, $\alpha = \angle BAC$, $\beta = \angle ABC$ and $\gamma = \angle ACB$. From the Sine Law for $\triangle AHL$ and $\triangle CHL$ we get $\dfrac{\sin\theta}{AL} = \dfrac{\sin\angle ALH}{AH}$ and $\dfrac{\sin(90\,\text{deg}-\theta)}{CL} = \dfrac{\sin\angle CLH}{CH}$. Since $\dfrac{CL}{AL} = \dfrac{BC}{BA} = \dfrac{\sin\alpha}{\sin\gamma}$ and $\dfrac{AH}{CH} = \tan\gamma$ (from $\angle ACB < 90\,\text{deg}$ it follows that H lies on a ray $\overrightarrow{CB}$), we get

$$\text{(1)} \qquad \frac{\cos\theta}{\sin\theta} = \frac{\sin\alpha}{\cos\gamma}.$$

It follows from the conditions of the problem that $3\theta = \alpha + \dfrac{\beta}{2}$, so $6\theta - \alpha = \alpha + \beta$. Thus $\cos\gamma = -\cos(6\theta - \alpha)$ and (1) is equivalent to $\cos\theta\cos(6\theta-\alpha) + \sin\theta\sin\alpha = 0 \iff \cos(7\theta-\alpha) + \cos(5\theta-\alpha) + \cos(\theta-\alpha) - \cos(\theta+\alpha) = 0 \iff -\sin 4\theta\sin(3\theta-\alpha) + \cos(3\theta-\alpha)\cos 2\theta = 0 \iff \cos 2\theta(2\sin 2\theta\sin(3\theta-\alpha) - \cos(3\theta-\alpha)) = 0$. There are two cases to consider:

1. $\cos 2\theta = 0$. From $0 < \theta < 90\,\text{deg}$ it follows that $\theta = 45\,\text{deg}$. (Working backwards we get that $135\,\text{deg} = \angle BLC = 3\angle AHL$

for any $\triangle ABC$ such that $\alpha + \dfrac{\beta}{2} = 135\,\deg$).

2. $\cos 2\theta \neq 0$. It follows from $3\theta - \alpha = \dfrac{\beta}{2}$ that

$$2\sin 2\theta = \operatorname{cotg}\frac{\beta}{2}. \tag{2}$$

But $60\,\deg \leq \beta < 180\,\deg$ and so

$$\operatorname{cotg}\frac{\beta}{2} < \operatorname{cotg}30\,\deg = \sqrt{3}. \tag{3}$$

On the other hand, also from the conditions of the problem we get $180\,\deg > 3\theta = \alpha + \dfrac{\beta}{2} \geq 90\,\deg$, so $30\,\deg \leq \theta < 60\,\deg$. Therefore

$$2\sin 2\theta \geq 2\sin 60\,\deg = \sqrt{3}. \tag{4}$$

The inequalities (3) and (4) show that (2) holds only if $\theta = 30\,\deg$. In this case $\alpha = \beta = \gamma = 60\,\deg$. (It is obviously true that $90\,\deg = \angle BLC = 3\angle AHL$ for any equilateral $\triangle ABC$).

Problem 11.3. Find all integer numbers $m, n \geq 2$ such that

$$\frac{1 + m^{3^n} + m^{2.3^n}}{n}$$

is integer.

Solution: Let m and n satisfy the conditions of the problem. Since n is an odd integer number, we get $(m, n) = 1$ and $n > 2$. When $n = 3$, all $m \geq 4$ such that $m \equiv 1 (\mathrm{mod}\ 3)$ are solutions, because if $m \equiv -1 (\mathrm{mod}\ 3)$, then $1 + m^{3^n} + m^{2 \cdot 3^n} \equiv 1 - 1 + 1 \equiv 1 (\mathrm{mod}\ 3)$. Let now $n > 3$. It follows that $m^{3^n} \not\equiv 1 (\mathrm{mod}\ n)$, because otherwise $1 + m^{3^n} + m^{2 \cdot 3^n} \equiv 3 (\mathrm{mod}\ n)$, i. e., $n/3$. On the other hand $1 + m^{3^n} + m^{2 \cdot 3^n} = \dfrac{m^{3^{n+1}} - 1}{m^{3^n} - 1}$ and therefore $m^{3^{n+1}} \equiv 1 (\mathrm{mod}\ n)$. Let k be the least natural number such that $m^k \equiv 1 (\mathrm{mod}\ n)$. Further, $k/3^{n+1}$ and $k \not/ \, 3^n$, so $k = 3^{n+1}$. Let $\varphi(n)$ be Euler's function. From $(m, n) = 1$ it follows that $m^{\varphi(n)} \equiv 1 (\mathrm{mod}\ n)$, so $k \leq \varphi(n)$. Therefore $3^{n+1} \leq \varphi(n) \leq n - 1$, which is impossible.

The required numbers are: $n = 3$ and all $m \geq 4$ such that $m \equiv 1 (\mathrm{mod}\ 3)$.

Winter mathematics competition—Pleven, 6–8 February 1998

Dedicated to the One Hundredth Anniversary of the UBM

МОН СМБ ЕВРИКА фондация

Problem 8.1. Let three numbers a, b and c be chosen so that $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$.

a.) Prove that $a = b = c$.

b.) Find the sum $x + y$ if $\dfrac{x}{3y} = \dfrac{y}{2x - 5y} = \dfrac{6x - 15y}{x}$ and the expression $-4x^2 + 36y - 8$ has its maximum value.

Solution:

a.) It is obvious that $a \neq 0$, $b \neq 0$, $c \neq 0$. The first equality gives $b^2 = ac$, whence by multiplying both sides by b we get $b^3 = abc$. Similarly $a^3 = abc$ and $c^3 = abc$. Hence $a^3 = b^3 = c^3$ and therefore $a = b = c$.

b.) By multiplying both the numerator and the denominator of the second fraction by 3 and using the result of a.) we obtain $x = 3y$. Thus $-4x^2 + 36y - 8 = -9(4y^2 - 4y + 1) + 1 = -9(2y - 1)^2 + 1$, and its maximum value is 1 when $2y - 1 = 0$. Therefore $y = \frac{1}{2}$ and $x = \frac{3}{2}$, i. e., $x + y = 2$.

Problem 8.2. In the acute triangle $\triangle ABC$ with $\angle BAC = 45$ deg, BE ($E \in AC$) and CF ($F \in AB$) are altitudes. Let H, M and K be the orthocentre of ABC and the midpoints of BC and AH, respectively.

a.) Prove that the quadrangle $MEKF$ is a square.

b.) Prove that the diagonals of the quadrangle $MEKF$ intersect at the midpoint of OH, where O is the circumcentre of $\triangle ABC$.

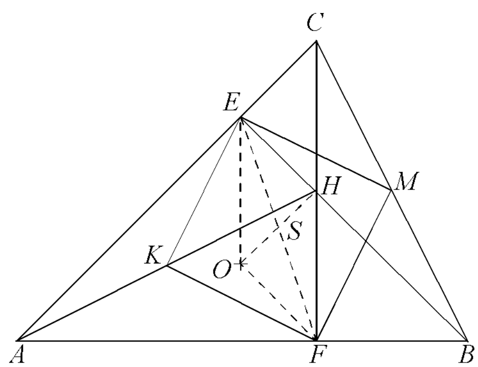
c.) Find the length of EF when the circumradius of $\triangle ABC$ is 1.

Solution:

a.) The segments EM and FM are medians to the hypotenuses of $\triangle BCE$ and $\triangle BCF$ and therefore $EM = FM = \frac{1}{2}BC$. Similarly, for $\triangle AHE$ and $\triangle AHF$ we get $EK = FK = \frac{1}{2}AH$. Since $\angle BAC = 45\,\deg$, we find that $\triangle AEB$ and $\triangle CEH$ are isosceles. Hence $AE = BE$ and $EC = EH$, i. e., $\triangle AHE \cong \triangle BCE$. Therefore $EK = EM$. Thus $MEKF$ is a rhombus. Furthermore,

$$\begin{aligned} \angle MEK &= \angle MEB + \angle HEK = \angle CBE + \angle HEK \\ &= \angle EAH + \angle HEK = \angle EAH + \angle AHE = 90\,\deg, \end{aligned}$$

i. e., the quadrangle is a square.



b.) It follows from a.) that the intersecting point S of the diagonals of the quadrangle $MEKF$ is the midpoint of both diagonals. Since $\triangle AEB$ is isosceles, E lies on the axis of symmetry of the segment AB and therefore $EO \perp AB$, i. e., $EO \| HF$. Similarly $FO \| EH$. Thus the quadrangle $EOFH$ is a parallelogram. From the above we conclude that S is the midpoint of OH.

c.) a.) implies that in the acute triangle $\triangle ABC$ with orthocentre H and $\angle BAC = 45\deg$ it is true that $AH = BC$. $\triangle AFE$ is of the same type and therefore $EF = AO = 1$. (It follows from b.) that O is orthocentre of this triangle.)

Problem 8.3. Let 1998 points be chosen on the plane so that out of any 17 it is possible to choose 11 that lie inside a circle of diameter 1. Find the smallest number of circles of diameter 2 sufficient to cover all 1998 points.

(We say that a circle covers a certain number of points if all points lie inside the circle or on its outline.)

Solution: Consider a regular hexagon with a side of length 3. Choose 1998 points as follows: the 6 vertices of the hexagon and 1992 points inside a circle of diameter 1 centred at the centre of the hexagon. It is clear that the above 1998 points satisfy the condition of the problem. Moreover any circle of radius 1 covers at most one of the vertices of the hexagon. Therefore the required number is *no less than 7* (in our case: 6 circles for each vertex and a single circle for the remaining points).

Now we shall prove that the required number is *no greater than 7.* Arbitrarily choose 8 points and add other 9, for a total of 17. It is clear that there is a circle of diameter 1 covering at least 11 of these 17 points. At most 6 points lie outside the circle and therefore at least 2 of the initially chosen 8 points lie inside the circle. The distance between these two points is no greater than 1. We have proved that among any 8 points there always exist 2 such that the distance between them is no greater than 1.

Now choose a circle of radius 1 centred in one of the points. If the remaining points lie inside the circle, the required number is 1 and thus no greater than 7. If this is not the case, take another point outside the first circle. If all points lie in the two circles, then the required number is 2 and thus no greater than 7. Continuing in this way we either obtain no more than 7 circles covering all points or have 7 circles and a point that lies outside all circles. Consider this point and the centres of the chosen circles. There exist 2 points among these 8 such that the distance between them is no greater than 1. But this is impossible because of the way we chose our points.

Together the two parts of the proof demonstrate that the required number is 7.

Problem 9.1. Find all quadratic functions $f(x) = x^2 - ax + b$ with integer coefficients such that there exist distinct integer numbers m, n, p in the interval $[1, 9]$ for which $|f(m)| = |f(n)| = |f(p)| = 7$.

Solution: Let $f(x)$ be a function satisfying the conditions of the problem. Such a function cannot take one and the same value for three different arguments (otherwise we would have a quadratic equation having three distinct roots). Therefore two of the numbers $f(m)$, $f(n)$ and $f(p)$ equal 7 (or -7) and the third one equals -7 (or 7).

Case 1. $f(m) = f(n) = 7$, $f(p) = -7$. Without loss of generality we may assume that $m > n$. Since m, n are roots of $x^2 - ax + b - 7 = 0$, we obtain that $a = m + n$, $b = mn + 7$.

Subtracting the two equalities

$$\begin{aligned} m^2 - am + b &= 7 \\ p^2 - ap + b &= -7, \end{aligned}$$

we find

$$14 = m^2 - p^2 - a(m-p) = (m-p)(m+p-a) = (m-p)(p-n).$$

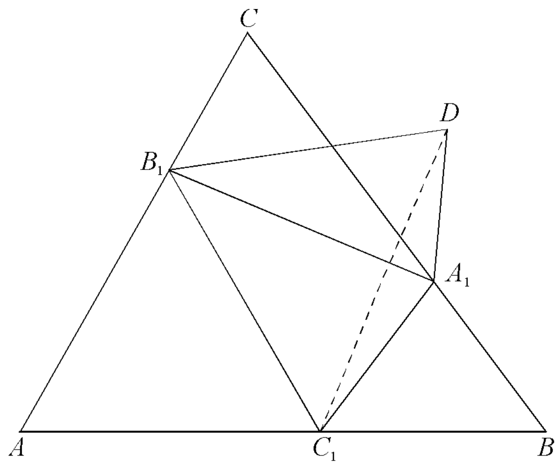
Thus the numbers $m-p$ and $p-n$ are either both positive or both negative and since $m > n$, they are positive. Moreover they are integer and therefore are equal to 1 and 14 or to 2 and 7. But since $m, n, p \in [1, 9]$ it follows that neither $m-p$ nor $p-n$ is 14. There are two cases to consider: $m-p = 2$, $p-n = 7$ and $m-p = 7$, $p-n = 2$, i. e., either $m = p+2$, $n = p-7$ or $m = p+7$, $n = p-2$. It is obvious that in both cases at least one of m, n, p lies outside the interval $[1, 9]$.

Case 2. $f(m) = f(n) = -7$, $f(p) = 7$. As in Case 1 we get $a = m+n$, $b = mn-7$ and $(m-p)(p-n) = -14$. Using similar arguments we obtain that either $m-p = 2$, $p-n = -7$ or $m-p = -2$, $p-n = 7$. (Without loss of generality we suppose that $|m-p| < |p-n|$.) Therefore the two options are $m = p+2$, $n = p+7$ and $m = p-2$, $n = p-7$. Simple calculations show that all triples (m, n, p) satisfying the conditions are $(3, 8, 1)$, $(4, 9, 2)$, $(6, 1, 8)$ and $(7, 2, 9)$. So the functions are $f(x) = x^2 - 11x + 17$, $f(x) = x^2 - 13x + 29$, $f(x) = x^2 - 7x - 1$ and $f(x) = x^2 - 9x + 7$.

Problem 9.2. Three points A_1, B_1 and C_1 lie on the sides BC, CA and AB of $\triangle ABC$ so that $AB_1 = C_1B_1$ and $BA_1 = C_1A_1$. Let D be the reflexion of C_1 in A_1B_1 ($D \neq C$). Prove that the line CD is perpendicular to the straight line through the circumcentres of $\triangle ABC$ and $\triangle A_1B_1C$.

Solution: It suffices to prove that D is the second intersecting point of the two circumcircles. We know that

$$\begin{aligned}\angle A_1DB_1 &= \angle A_1C_1B_1 = 180\deg - \angle BC_1A_1 - \angle AC_1B_1\\ &= 180\deg - \angle C_1BA_1 - \angle C_1AB_1 = \angle A_1CB_1.\end{aligned}$$



On the other hand, $A_1D = A_1C_1 = A_1B$ and $B_1D = B_1C_1 = B_1A$, which shows that A_1 and B_1 are the circumcentres of $\triangle BC_1D$ and $\triangle AC_1D$. Therefore $\angle ADB = \angle ADC_1 + \angle BDC_1 = \frac{1}{2}\angle AB_1C_1 + \frac{1}{2}\angle BA_1C_1 = 90\deg - \angle C_1AB_1 + 90\deg - \angle C_1BA_1 = \angle ACB$. Since C and D lie in one and the same semiplane in regard to both A_1B_1 and AB, it follows from $\angle A_1DB_1 = \angle A_1CB_1$ and $\angle ADB = \angle ACB$ that D is the second intersecting point of the circumcircles of $\triangle A_1B_1C$ and $\triangle ABC$. This completes the proof.

Problem 9.3. All natural numbers from 1 to 1998 inclusive are written 9 times (so that there are 9 ones, 9 twos and so on) in the cells of a rectangular table with 9 rows and 1998 collumns, so that the difference between any two elements lying in one and the same column is no greater than 3. Find the maximum possible value of the smallest sum amongst all 1998 sums of the elements lying in one and the same column.

Solution: Consider the placement of the ones in the columns. If they are all in a single column, then the minimum sum of elements lying in one column is 9. Let all ones lie in exactly 2 columns. Therefore there are at least 5 ones in a single column and thus the minimal sum is no greater that $5 \cdot 1 + 4 \cdot 4 = 21$. If all ones are placed in exactly 3 columns, then the sum of all numbers in these three columns is at most $9 \cdot 1 + 9 \cdot 4 + 9 \cdot 3 = 72$. Hence the minimal sum is at most $72 : 3 = 24$. If all ones are placed in exactly 4 columns, then the sum of all numbers in these columns is at most $9 \cdot 1 + 9 \cdot 4 + 9 \cdot 3 + 9 \cdot 2 = 90$ and therefore the minimal sum is at most $90 : 4$, i. e., 22. It is impossible to have ones in more than 4 columns, because in that case the total number of 2s, 3s and 4s does not suffice to fill the remaining cells. Therefore the required sum is at most 24.

The following example shows that this sum can be 24, consequently the answer is 24:

1	1	1	2	2	2	7	8	...	1998
1	1	1	2	2	2	7	8	...	1998
1	1	1	2	2	2	7	8	...	1998
3	3	3	5	5	5	7	8	...	1998
3	3	3	5	5	5	7	8	...	1998
3	3	3	5	5	5	7	8	...	1998
4	4	4	6	6	6	7	8	...	1998
4	4	4	6	6	6	7	8	...	1998
4	4	4	6	6	6	7	8	...	1998

Problem 10.1. Find all values of the real parameter a for which the equation $x^3-3x^2+(a^2+2)x-a^2=0$ has three distinct roots x_1, x_2 and x_3 such that $\sin\left(\frac{2\pi}{3}x_1\right)$, $\sin\left(\frac{2\pi}{3}x_2\right)$ and $\sin\left(\frac{2\pi}{3}x_3\right)$ form (in some order) an aritmetic progression.

Solution: Since $x^3-3x^2+(a^2+2)x-a^2=(x-1)(x^2-2x+a^2)$, in order for there to be three distinct real roots it is necessary that $D=1-a^2>0$. Therefore $a^2<1$ and thus $1\geq\sqrt{1-a^2}>0$. The roots of our equation are $x_1=1$, $x_2=1+\sqrt{1-a^2}$, $x_3=1-\sqrt{1-a^2}$. It follows now that $x_2+x_3=2$ and $2\geq x_2>1$ and $1>x_3\geq 0$.

There are two cases to consider:

1. The second term of the progression is $\sin\left(\frac{2\pi}{3}x_1\right)$. Then

$$\begin{aligned}\sin\left(\frac{2\pi}{3}x_2\right)+\sin\left(\frac{2\pi}{3}x_3\right) &= 2\sin\left(\frac{2\pi}{3}\right)\\ 2\sin\left(\frac{2\pi}{3}\left(\frac{x_2+x_3}{2}\right)\right)\cos\left(\frac{2\pi}{3}\left(\frac{x_2-x_3}{2}\right)\right) &= 2\sin\left(\frac{2\pi}{3}\right)\\ \cos\left(\frac{\pi}{3}(x_2-x_3)\right) &= 1.\end{aligned}$$

But $\frac{\pi}{3}|x_2-x_3|=\frac{2\pi}{3}\sqrt{1-a^2}\leq\frac{2\pi}{3}$, and hence $\frac{\pi}{3}(x_2-x_3)\in[-\frac{2\pi}{3},\frac{2\pi}{3}]$. Therefore $\cos\left(\frac{\pi}{3}(x_2-x_3)\right)=1$ when $x_2=x_3$, which is impossible, since the roots are distinct.

2. The first or the third term of the progression is $\sin\left(\frac{2\pi}{3}x_1\right)$. Then

$$\sin\left(\frac{2\pi}{3}\right)+\sin\left(\frac{2\pi}{3}x_i\right)=2\sin\left(\frac{2\pi}{3}(2-x_i)\right)$$

for $i = 2$ or 3. Hence

$$\sin\frac{2\pi}{3} + \sin\left(\frac{2\pi}{3}x_i\right) = 2\sin\frac{4\pi}{3}\cos\left(\frac{2\pi}{3}x_i\right) - 2\cos\frac{4\pi}{3}\sin\left(\frac{2\pi}{3}x_i\right).$$

After simple calculations we get $\cos\left(\frac{2\pi}{3}x_i\right) = -\frac{1}{2}$. From the restrictions for x_2 and x_3 we obtain $x_i = 1$ or $x_i = 2$. In the first case $a^2 = 1$, which is impossible, and in the second case $x_2 = 2, x_3 = 0$ and $a^2 = 0$.

Thus a has a unique value and it is $a = 0$.

Problem 10.2. A point C lies on the periphery of a circle. Two points A and B are chosen anticlockwise away from C such that if $\angle CAB = \alpha$ and $\angle CBA = \beta$, the following equality holds:

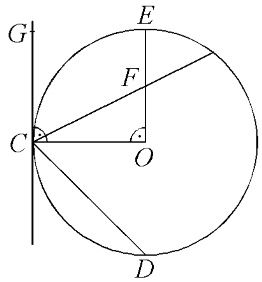
$$2\cos\left(\frac{\alpha}{2} + \beta\right) = \sin\left(\frac{\alpha}{2} - \beta\right).$$

Prove that the bisectors of $\angle CAB$ pass through a fixed point.

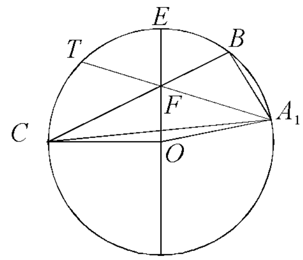
Solution: It is easy to see that $\alpha = 90\,\text{deg}$, $\beta = 45\,\text{deg}$ and $\tan\frac{\alpha}{2} = \frac{1}{2}$, $\beta = 90\,\text{deg}$ satisfy the condition for α and β. Therefore the required point is the midpoint of OE–F (fig. 1).

From the premises of the problem we obtain

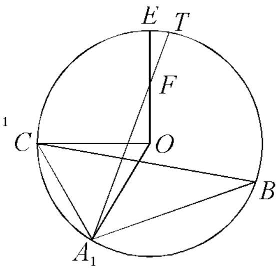
$$2\cos\frac{\alpha}{2}\cos\beta - 2\sin\frac{\alpha}{2}\sin\beta = \sin\frac{\alpha}{2}\cos\beta - \cos\frac{\alpha}{2}\sin\beta$$



черт. 1



черт. 2



черт. 3

and after dividing by $\cos\frac{\alpha}{2}\cos\beta$ ($\cos\frac{\alpha}{2} \neq 0$ (why?), and if $\cos\beta = 0$, i. e., $\beta = 90°$, we have one of the two cases already considered) we obtain

$$\tan\frac{\alpha}{2}(1 + 2\tan\beta) = 2 + \tan\beta$$

It follows in particular that if α is fixed, then β is uniquely determined.

Suppose that $\tan\frac{\alpha}{2} > 2$. Thus $\frac{\alpha}{2} > 45°$ and therefore $\alpha > 90°$. If $\tan\beta < 0$, we get $\beta > 90°$, which is impossible, since $\alpha+\beta < 180°$. If $\tan\beta > 0$, we get $2+\tan\beta > (1 + 2\tan\beta)\,2$ i. e., $\tan\beta < 0$, which is a contradiction.

Therefore $\tan\frac{\alpha}{2} \leq 2$ and B lies on CED where $\angle GCF = \angle FCD$ and $\tan\angle GCF = 2$ (fig. 1).

Fix the point B such that $\alpha < 90°$. Let T be the midpoint of the arc CB and let A_1 be the intersecting point of TF and the circle. We shall show that $A_1 \equiv A$. We obtain $\angle OA_1F = \frac{\alpha}{2} - (90° - \beta) = \beta + \frac{\alpha}{2} - 90°$ and $\angle A_1FO = 45° - \frac{\alpha}{2} + \beta - 45° = \beta - \frac{\alpha}{2}$. It follows from the Sine Theorem for $\triangle A_1FO$ that $\dfrac{\sin\left(\beta-\frac{\alpha}{2}\right)}{\sin\left(\beta+\frac{\alpha}{2}-90^\circ\right)}=2$, which is equivalent to $2\cos\left(\frac{\alpha}{2}+\beta\right)=\sin\left(\frac{\alpha}{2}-\beta\right)$. Therefore $A_1\equiv A$.

The case of $\alpha>90^\circ$ can be dealt with by analogy. The condition for B to lie on CED shows that A_1 lies between C and B (fig. 3).

Problem 10.3. Let n be a natural number. Find the number of sequences $a_1a_2\dots a_{2n}$, where $a_i=+1$ or $a_i=-1$ for $i=1,2\dots,2n$, such that

$$\left|\sum_{i=2k-1}^{2l}a_i\right|\le 2$$

for all k and l for which $1\le k\le l\le n$.

Solution: It is clear that a sequence having $a_{2k-1}+a_{2k}=0$ for $1\le k\le n$ satisfies the condition of the problem, because any sum of the form $\sum_{i=2k-1}^{2l}a_i$ equals zero. There are 2^n such sequences. Let us determine the number of sequences such that there exists a k for which $a_{2k-1}+a_{2k}\ne 0$. Let $k_1,k_2,\dots,k_s$ be all k with the above property. It is easily seen that if $a_{2k_i-1}+a_{2k_i}=2$ (-2), then $a_{2k_{i+1}-1}+a_{2k_{i+1}}=-2$ (2). Therefore all sums $a_{2k_i-1}+a_{2k_i}$ (and so also $a_{2k_i-1}a_{2k_i}$) are uniquely determined by $a_{2k_1-1}+a_{2k_1}$ (there are two possibilities for $a_{2k_1-1}a_{2k_1}$). There are two possibilities for any of the remaining $n-s$ pairs (for which $a_{2t-1}+a_{2t}=0$). Therefore there are

$$2^n+2\cdot2^{n-1}\binom{n}{1}+2\cdot2^{n-2}\binom{n}{2}+\cdots+2\cdot2^{n-k}\binom{n}{k}+\ldots+2\cdot2\binom{n}{n-1}+2\cdot\binom{n}{n}.$$

sequences with the required property. By adding and subtracting 2^n to and from the above expression we get:

$$2.\left(2^n\binom{n}{0}+2^{n-1}\binom{n}{1}+2^{n-2}\binom{n}{2}+\cdots+2\binom{n}{n-1}+\binom{n}{n}\right)-2^n=2\cdot 3^n-2^n$$

Thus there are $2\cdot 3^n-2^n$ sequences.

Problem 11.1. Consider the function $f(x)=\sqrt{x}+\sqrt{x-4}-\sqrt{x-1}-\sqrt{x-3}$, $x\geq 4$.

a.) Find $\lim\limits_{x\to\infty} f(x)$.

b.) Prove that $f(x)$ is an increasing function.

c.) Find the number of real roots of the equation $f(x)=a\sqrt{\frac{x-3}{x}}$, where a is a real parameter.

Solution: a.) By grouping the first and third radicals and the second and fourth radicals and rationalising we get that when $x>4$, $f(x)=\frac{1}{\sqrt{x}+\sqrt{x-1}}-\frac{1}{\sqrt{x-4}+\sqrt{x-3}}$. Therefore $\lim\limits_{x\to\infty} f(x)=0$.

b.) When $x>4$,

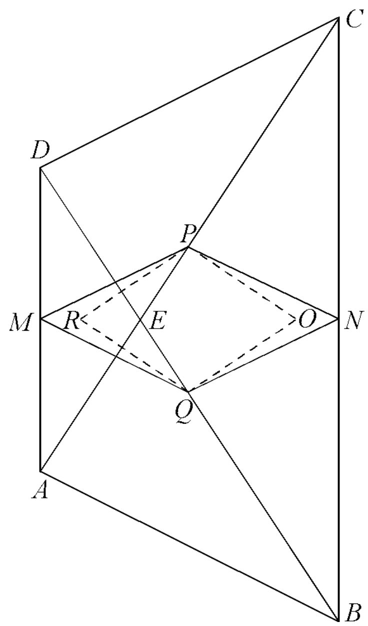
$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x-1}}-\frac{1}{2\sqrt{x-3}}+\frac{1}{2\sqrt{x-4}} \\ &= \frac{1}{2\sqrt{x-3}\sqrt{x-4}(\sqrt{x-3}+\sqrt{x-4})}+ \\ &\quad -\frac{1}{2\sqrt{x}\sqrt{x-1}(\sqrt{x}+\sqrt{x-1})}>0. \end{aligned}$$

Therefore $f(x)$ is an increasing function if $x \geq 4$.

c.) It follows from a.) and b.) that $f(x) < 0$ when $x \geq 4$, i.e., the equation could have a solution only if $a < 0$. Let $a < 0$. The function $g(x) = a\sqrt{\frac{x-3}{x}} = a\sqrt{1 - \frac{3}{x}}$ is decreasing and continuous and $\lim\limits_{x\to\infty} g(x) = a < 0$. Since $f(x)$ is an increasing and continuous function and $\lim\limits_{x\to\infty} f(x) = 0$, in accordance with the Bolzano–Weierstraß Theorem the equation $f(x) = g(x)$ has a solution (and it is a unique one) exactly when $f(4) \leq g(4)$, i.e., if $2(1 - \sqrt{3}) \leq a < 0$.

Problem 11.2. The convex quadrangle $ABCD$ is inscribed in a circle with centre O. Let E be the intersecting point of AC and BD. Prove that if the midpoints of AD, BC and OE lie on a straight line, then $AB = CD$ or $\angle AEB = 90\,\mathrm{deg}$.

Solution: It suffices to prove that if $\angle AEB \neq 90\,\mathrm{deg}$, then $AB = CD$. Let $\angle AEB \neq 90\,\mathrm{deg}$. If $O \equiv E$, then $ABCD$ is a rectangle and therefore $AB = CD$. Suppose $O \not\equiv E$. Let M, N, P, Q be the midpoints of AD, BC, AC, BD, respectively, and R be the intersecting point of the straight lines through P and Q perpendicular to BD and AC, respectively. It is clear that $MPNQ$ and $OPRQ$ are parallelograms. Therefore the midpoints of MN and OR coincide with the midpoint of PQ, and since the midpoint of OE lie on MN, we get that $RE \| MN$. On the other hand R is the orthocentre of $\triangle PQE$ and therefore $RE \perp PQ$. Hence $MN \perp PQ$, i.e., the parallelogram $MPNQ$ is a rhombus. It is easy to see now that $AB = 2PN = 2NQ = CD$, which solves the problem.



Note: The above solution shows that if O is the intersecting point of the axes of symmetry of AC and BD, then the assertion of the problem and its opposite are true for a quadrangle that is not inscribed in a circle. This could be demonstrated by using complex numbers or trigonometry.

Problem 11.3. Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of integer numbers such that their decimal representations consist of even digits ($a_1 = 2$, $a_2 = 4$, $a_3 = 6, \dots$). Find all integer numbers m such that $a_m = 12m$.

Solution: Let m be an integer number such that $m = b_0 + b_1 \cdot 5 + \cdots + b_n \cdot 5^n$. Denote $f(m) = 2b_0 + 2b_1 \cdot 10 + \cdots + 2b_n \cdot 10^n$. It is clear that $\{f(m) \mid m \in \mathbb{N}\}$ is the set of integer numbers with only even digits in their decimal representation. Since $f(m_1) < f(m_2) \iff m_1 < m_2$, it follows that $a_m = f(m)$ for any m. Therefore it suffices to find all m such that

$$12(b_0 + b_1 \cdot 5 + \cdots + b_n \cdot 5^n) = 2b_0 + 2b_1 \cdot 10 + \cdots + 2b_n \cdot 10^n,$$

i. e.,

$$6(b_0 + b_1 \cdot 5 + \cdots + b_n \cdot 5^n) = b_0 + b_1 \cdot 10 + \cdots + b_n \cdot 10^n. \tag{1}$$

Since $b_0+b_1\cdot5+\cdots+b_n\cdot5^n \leq 5^{n+1}-1$ and $b_0+b_1\cdot10+\cdots+b_n\cdot10^n \geq 10^n$, it follows from (1) that $6(5^{n+1} - 1) \geq 10^n$, i. e., $6 \cdot 5^{n+1} > 10^n$. Thus $2^n < 30$ and therefore $n \leq 4$. If $n = 4$, we get from (1) that $b_0 + 4b_1 + 10b_2 = 50b_3 + 1250b_4 \geq 1250$, which is imposible. In the same way it is easy to show that $n \geq 3$, i. e., $n = 3$. In this case $b_0+4b_1+10b_2 = 50b_3$. Obviously $b_3 = 1$ and $b_0 = b_1$, because b_0-b_1 is divisible by 5. As a result we have the equation $b_0+2b_2 = 10$, and its solutions are $b_0 = 2, b_2 = 4$ and $b_0 = 4, b_2 = 3$. Therefore all integer numbers m with the required property are $m = 2+2\cdot5+4\cdot5^2+5^3 = 237$ and $m = 4 + 4 \cdot 5 + 3 \cdot 5^2 + 5^3 = 224$.

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Problem 8.1. Find all natural numbers x and y such that:

a) $\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{3}$;

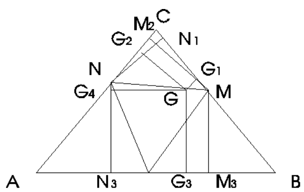
b) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{3} + \dfrac{1}{xy}$.

Solution: a) The equation is equivalent to $3y - 3x = xy$, so $x = \dfrac{3y}{y+3} = \dfrac{3y+9-9}{y+3} = 3 - \dfrac{9}{y+3}$. Therefore $y+3 = 9$ and thus $y = 6$. Hence there is a unique solution $x = 2, y = 6$.

b) Let $x \geq y$ be a solution of the problem. Now $\dfrac{1}{3} = \dfrac{1}{x} + \dfrac{1}{y} - \dfrac{1}{xy} \leq \dfrac{2}{y} - \dfrac{1}{xy} = \dfrac{2y-1}{xy} < \dfrac{2y}{xy} = \dfrac{2}{x}$, giving $x < 6$. When $x = 1$, $x = 2$ or $x = 3$, no solution exists. When $x = 4$, it follows that $y = 9$, and $x = 5$ implies $y = 6$. If $y \geq x$, we apply the same reasoning. The problem has four solutions:

$$x=4, y=9; x=5, y=6; x=6, y=5; x=9, y=4.$$

Problem 8.2. Given an acute $\triangle ABC$ with centroid G and bisectors $AM(M\in BC), BN(N\in AC), CK(K\in AB)$. Prove that one of the altitudes of $\triangle ABC$ equals the sum of the remaining two if and only if G lies on one of the sides of $\triangle MNK$.



Solution. We shall repeatedly use the following property: A segment connecting a vertex of a triangle with a point on the opposite side divides the triangle into two triangles such that the ratio of their areas equals the ratio of the parts into which the point divides the side.

Let $G\in MN$ and G_1, G_2, G_3 be the projections of G on BC, AC and AB, respectively. Further, denote the projections of N on BC and AB by N_1 and N_3 and those of M on AC and AB by M_2 and M_3. We shall prove that $GG_3 = GG_1 + GG_2$ and from the above property it will follow straightforwardly that the altitude from C is equal to the sum of the remaining two altitudes. We obtain

$$\frac{GG_1}{NN_1} = \frac{CM\cdot GG_1}{CM\cdot NN_1} = \frac{S_{GMC}}{S_{NMC}} = \frac{GM}{NM}.$$

By analogy $\dfrac{GG_2}{MM_2} = \dfrac{GN}{NM}$, implying that $\dfrac{GG_1}{NN_1} + \dfrac{GG_2}{MM_2} = 1$, so

(1) $$GG_1\cdot MM_2 + GG_2\cdot NN_1 = MM_2\cdot NN_1.$$

Let M_4 and G_4 be the projections of M and G on NN_3. It easily follows now that

$$\frac{NN_3 - GG_3}{NN_3 - MM_3} = \frac{GN}{NM} = \frac{GG_2}{MM_2}.$$

Further, using that $NN_3 = NN_1$ and $MM_3 = MM_2$, we obtain $GG_3 \cdot MM_2 = GG_1 \cdot MM_2 + GG_2 \cdot MM_2$ and therefore $GG_3 = GG_1 + GG_2$.

Conversely, let the altitude through C be the sum of the remaining two. Now $GG_3 = GG_1 + GG_2$. If $G^\star = GG_3 \cap MN$, then it follows straightforwardly that the sum of the distances from $G^\star$ to AC and BC equals to $G^\star G_3$. It is easy to check now that $G^\star \cong G$.

Problem 8.3. Let n be a natural number. Find all integer values of m such that $k = 2^{m-2}$ is integer and $A = 1999^k + 6$ is a sum of the squares of n integers (not necessarily distinct and different from zero).

Solution: It is sufficient to consider only nonnegative values of m.

1) $n = 1$ and $k = \dfrac{m}{2}$ is integer only if $m = 4p$ and $m = 4p + 2$. If $m = 4p$, then $A = (2 \cdot 1000 - 1)^{2p} + 1$ and it follows by induction that A is of the form $A = 4a + 7$, so A is congruent to 3 modulo 4. We conclude that A is not a perfect square. If $m = 4p + 2$, then $A = (25 \cdot 80 - 1)^{2p+1} + 6$ and it follows by induction that A is of the form $A = 25a + 5$. Therefore A is not a perfect square, because 5 divides A, but 25 does not.

2) $n = 2$ and $k = m$ is integer for any m. Now $A = 1999^m + 6$. When $m = 0$, we get $A = 7$, which is not a sum of two squares.

When $m = 1$, we obtain $A = 2005 = 41^2 + 18^2$. If $m \geq 2$, then $A = (2 \cdot 999 + 1)^m + 1$ and as above A is congruent to 3 modulo 4. On the other hand the sum of two perfect squares is congruent to 0, 1 or 2 modulo 4 and so no solution exists in this case.

3) $n = 2$ and $k = m$ is integer for any m. Now $A = (8 \cdot 250 - 1)^{2m} + 6$, which can be written in the form $A = 8a + 7$. Therefore A is congruent to 7 modulo 8, whereas a sum of three perfect squares is congruent to 0, 1, 2, 3, 4, 5 or 6 modulo 8. Thus no solution exists in this case.

4) $n \geq 4$ and k is integer for any m. Now $A = (1999^{m2^{n-3}})^2 + 2^2 + 1^2 + 1^2$ and if $a_1 = 1999^{m2^{n-3}}, a_2 = 2 \cdot a_3 = a_4 = 1, a_5 = a_6 = \cdots = a_n = 0$, then $A = a_1^2 + a_2^2 + \cdots + a_n^2$.

Answer: if $n = 1$, no solution exists;
if $n = 2$, there is an unique solution $m = 1$;
if $n = 3$, no solution exists;
if $n \geq 4$, any $m \geq 0$ is a solution.

Problem 9.1. Let p be a real parameter such that the equation $x^2 - 3px - p = 0$ has real and distinct roots x_1 and x_2.

a) Prove that $3px_1 + x_2^2 - p > 0$.

b) Find the least possible value of

$$A = \frac{p^2}{3px_1 + x_2^2 + 3p} + \frac{3px_2 + x_1^2 + 3p}{p^2}.$$

When does equality obtain?

Solution: a) It follows from the equation that $x_2^2 = 3px_2 + p$ and so $3px_1 + x_2^2 - p = 3p(x_1 + x_2) = 9p^2 > 0$. The inequality is strict because otherwise $x_1 = x_2 = 0$.

b) As in a), we obtain $3px_1 + x_2^2 + 3p = 3px_2 + x_1^2 + 3p = 9p^2 + 4p > 0$ (the last inequality follows from the conditions of the problem x_1 and x_2 to be distinct and real, giving $p \neq 0$). Therefore

$$A = \frac{p^2}{9p^2 + 4p} + \frac{9p^2 + 4p}{p^2} \geq 2$$

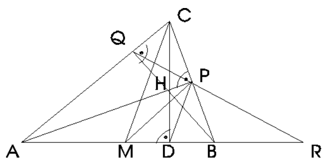
(from the Arithmetic–Geometric Mean Inequality) and equality obtains when $9p^2 + 4p = p^2$, i. e., when $p = -1/2$.

Problem 9.2. Given an acute $\triangle ABC$ such that $AC > BC$, let M be the midpoint of AB and let CD, AP and BQ be the altitudes. Denote the circumcircle of $\triangle PQC$ by k_1 and the circumcircle of $\triangle DRP$ by k_2, where R is the point of intersection of AB and PQ. Prove that:

a) MP is tangent to both k_1 and k_2.

b) $RH \perp CM$, where H is the orthocentre of $\triangle ABC$.

Solution: a) Note that $H \in k_1$. Since $\angle APM = \angle PAM = 90^\circ - \angle ABC = \angle BCD$, we obtain that MP is a tangent to k_1. On the other hand $\angle MPD = \angle MPB - \angle DPB = \angle MPB - \angle DPB = \angle MBP - \angle BPR\,(\triangle ABC \sim \triangle DBP)$, so $\angle ARP = \angle MBP - \angle BPR = \angle MBP - \angle QPC = \angle MBP - \angle BAC(\triangle ABC \sim \triangle PQC)$. Therefore $\angle MPD = \angle MBP$ and thus MP is a tangent to k_2.



b) Let $L = CM \cap k_1$. It follows from a) that $ML \cdot MC = MP^2 = MD \cdot MR$. We conclude that L lies on the circumcicrle of $\triangle DRC$ and therefore $RL \perp CM$. Further $HL \perp CM$, since HC is a diameter of k_1. Hence $RH \perp CM$.

Problem 9.3. A square table filled with nonnegative (not necessarily distinct) integer numbers is said to be a magic square with sum m if the sum of the numbers in each row and each column equals m. Prove that the number of magic squares 3×3 of sum m such that the minimal element among the elements on the main diagonal lies in the centre is $\binom{m+4}{4}$.

Solution: It is evident that knowing the elements of main diagonal and the element in the cell $(1, 2)$ (see fig. 1) one can determine all elements in the table. Indeed, there is a unique choice for all remaining cells (see fig. 2). Therefore it suffices to see when all elements are nonnegative and b is the minimal element among the elements on the main diagonal.

a	d	
	b	
		c

Fig. 1

a	d	$m-a-d$
$m+c-a-b-d$	b	$a+d-c$
$b+d-c$	$m-b-d$	c

Fig. 2

It is clear from fig. 2 that the following inequalities hold:

(1) $a + d \leq m$;
(2) $b + d \leq m$;

(3) $c \le a + d$;
(4) $c \le b + d$;
(5) $a + b + d - c \le m$.

The conditions of the problem imply $b \le a$ and $b \le c$. It is clear now that (3) follows from (4) and (2) and (5) follow from (1). Therefore we can consider only (1) and (4).

Consider the following chain of inequalities

$$b \le 2b + d - c \le a + b + d - c \le a + d \le m$$

(the first follows from (4), the second from $b \le a$, the third from $b \le c$, and the fourth is equivalent to (1)). It is easy to see that knowing the quadruple $(b, 2b+d-c, a+b+d-c, a+d)$ we can uniquely determine a, b, c and d and so find a magic square. Therefore the required number equals the number of quadruples, which is $\binom{m+4}{4}$.

Problem 10.1. Find all values of the real positive parameter a such that the inequality $a^{\cos 2x} + a^{2\sin^2 x} \le 2$ holds for any real x.

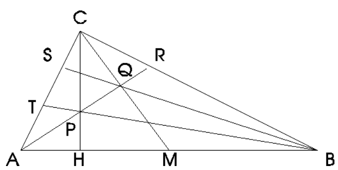
Solution: We know that $a^{\cos 2x} + a^{2\sin^2 x} = a^{1-2\sin^2 x} + a^{2\sin^2 x} = \dfrac{a}{a^{2\sin^2 x}} + a^{2\sin^2 x}$. Substitute $t = a^{2\sin^2 x}$. Since $0 \le \sin^2 x \le 1$, we obtain that t is between 1 and a^2. Our inequality now becomes $\dfrac{a}{t} + t \le 2 \iff t^2 - 2t + a \le 0$. Since it holds true for any x (i. e., for any t between 1 and a^2), it follows that the roots of $f(t) = t^2 - 2t + a = 0$ lie outside the open interval determined by 1 and a^2. Therefore $f(1) \le 0$ and $f(a^2) \le 0$. The first inequality gives $a \le 1$ and the second one implies $a^4 - 2a^2 + a \le 0 \iff a^3 - 2a + 1 \le 0 \iff (a-1)(a^2+a-1) \le 0$. Since $a \le 1$, we obtain $a^2+a-1 \ge 0$.

The solution of this inequality is $a \in \left[\dfrac{-1-\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2}\right]$. So we obtain $a \in \left[1, \dfrac{-1+\sqrt{5}}{2}\right]$.

Problem 10.2. Let CH and CM be an altitude and a median in a non-obtuse $\triangle ABC$. Let the bisector of angle BAC meet CH and CM at points P and Q, respectively. If $\angle ABP = \angle PBQ = \angle QBC$, prove that:

a) $\triangle ABC$ is a right triangle;

b) $BP = 2CH$.



Solution: a) Let $R = BC \cap AP$, $T = AC \cap BP$ and $S = AC \cap BQ$. Denote $AB = c$, $BC = a$, $CA = b$. It is easy to see that P lies between A and Q (otherwise $\angle ABP > \angle PBQ$). It follows from Ceva's Theorem for point P that:

$$(1) \qquad \frac{AH}{HB} \cdot \frac{BR}{RC} \cdot \frac{CT}{TA} = 1 \iff \frac{b\cos\alpha}{a\cos\beta} \cdot \frac{c}{b} \cdot \frac{S_{BTC}}{S_{ABT}} = 1$$

$$\iff \frac{b\cos\alpha}{a\cos\beta} \cdot \frac{c}{b} \cdot \frac{BT \cdot a \cdot \sin\frac{2\beta}{3}}{BT \cdot c\sin\frac{\beta}{3}} = 1 \iff \frac{\cos\alpha}{\cos\beta} \cdot \frac{\sin\frac{2\beta}{3}}{\sin\frac{\beta}{3}} = 1$$

$$\iff \frac{\cos\alpha}{\cos\beta} \cdot \frac{2\sin\frac{\beta}{3}\cos\frac{beta}{3}}{\sin\frac{\beta}{3}} = 1 \iff \frac{\cos\alpha}{\cos\beta} = \frac{1}{2\cos\frac{\beta}{3}}.$$

It follows from Ceva's Theorem for point Q that:

(2)

$$\frac{AM}{MB}\cdot\frac{BR}{RC}\cdot\frac{CS}{SA}=1 \iff \frac{c}{b}\cdot\frac{S_{BSC}}{S_{ABS}}=1 \iff \frac{c}{b}\cdot\frac{BS\cdot a\cdot\sin\frac{\beta}{3}}{BS\cdot c\sin\frac{2\beta}{3}}=1$$

$$\iff \frac{c}{b}\cdot\frac{a\sin\frac{\beta}{3}}{c\sin\frac{2\beta}{3}}=1 \iff \frac{a\sin\frac{\beta}{3}}{2b\sin\frac{\beta}{3}\cos\frac{\beta}{3}}=1 \iff \frac{a}{b}=2\cos\frac{\beta}{3}.$$

Now (1) and (2) imply $\dfrac{\cos\alpha}{\cos\beta}=\dfrac{b}{a}$. From the Sine Law we obtain $\dfrac{\sin\beta}{\sin\alpha}=\dfrac{b}{a}$, so $\dfrac{\cos\alpha}{\cos\beta}=\dfrac{\sin\beta}{\sin\alpha} \iff \sin 2\alpha=\sin 2\beta$. If $\alpha=\beta$, then the triangle is isosceles and therefore $P\equiv Q$, implying that $\angle ABP=\angle PBQ=\angle QBC=0^\circ$, which is impossible. Thus $\alpha+\beta=90^\circ$ and therefore $\angle ACB=90^\circ$.

b) It follows from $\triangle BCS$ that $\cos\dfrac{\beta}{3}=\dfrac{a}{BS}$. Combining the above with (2) gives $2b=BS$. Note that $\triangle ABC\sim\triangle BHC$, which implies $\dfrac{BP}{CH}=\dfrac{BS}{AC}=2$. Therefore $BP=2CH$.

Problem 10.3. Let A be a set of natural numbers with no zeroes in their decimal representation. It is known that if $a=\overline{a_1a_2\dots a_k}\in A$, then $b=\overline{b_1b_2\dots b_k}$, where $b_j, 1\le j\le k$ is the remainder of $3a_j$ modulo 10, belongs to A and the sum of the digits of b equals the sum of the digits of a.

a) Prove that the sum of the digits of a k-digit number in A equals $5k$.

b) Find the smallest k-digit number which could be an element of A.

Solution: a) Let $a = \overline{a_1a_2\dots a_k}$ be a k-digit number from A, the sum of whose digits is S.

Consider the following numbers: $b = \overline{b_1b_2\dots b_k}, c = \overline{c_1c_2\dots c_k}$ and $d = \overline{d_1d_2\dots d_k}$, where $b_j, 1 \le j \le k$ is the remainder of $3a_j$ modulo 10, $c_j, 1 \le j \le k$ is the remainder of $3b_j$ modulo 10 and $d_j, 1 \le j \le k$ is the remainder of $3c_j$ modulo 10.

By the conditions of the problem all b, c and d belong to A. Further

$$(1) \qquad S = \sum_{j=1}^{k} a_j = \sum_{j=1}^{k} b_j = \sum_{j=1}^{k} c_j = \sum_{j=1}^{k} d_j.$$

Direct verification shows that for fixed j the sum $a_j + b_j + c_j + d_j$ is equal to 20 (e. g., if $a_j = 3$, then $b_j = 9, c_j = 7, d_j = 1$ and therefore $a_j + b_j + c_j + d_j = 20$). It follows now from (1) that $4S = \sum_{j=1}^{k} a_j + \sum_{j=1}^{k} b_j + \sum_{j=1}^{k} c_j + \sum_{j=1}^{k} d_j = \sum_{j=1}^{k}(a_j + b_j + c_j + d_j) = 20k$. Therefore $S = 5k$, *Q. E. D.*

b) We shall prove that the required number is $a = \overline{a_1a_2\dots a_{2t}}$, where $a_1 = 1, a_2 = 1, \dots, a_t = 1, a_{t+1} = 9, a_{t+2} = 9, \dots, a_{2t} = 9$ if $k = 2t$ and $b = \overline{b_1b_2\dots b_{2t+1}}$, where $b_1 = 1, b_2 = 1, \dots, b_t = 1, b_{t+1} = 5, b_{t+2} = 9, \dots, b_{2t+1} = 9$ if $k = 2t + 1$. It is easy to see that a and b could be elements of a set having the required property.

Let $k = 2t$ and suppose there exists $c = \overline{c_1c_2\dots c_{2t}} \in A$ such that $c < a$. Since there are no zeroes among the digits of c, we obtain $c_1 = c_2 = \dots = c_t = 1$. But it follows from a) that the sum of the digits of c is $5k = 10t$. The last is possible only if

$c_{t+1} = c_{t+2} = \ldots = c_{2t} = 9$. Hence $c = a$, a contradiction with the choice of c.

Similarly, suppose $k = 2t + 1$ and there exists $c = \overline{c_1 c_2 \ldots c_{2t+1}} \in A$ such that $c < b$. Since there are no zeroes among the digits of c we obtain $c_1 = c_2 = \ldots = c_t = 1$. But it follows from a) that the sum of the digits of c is $5k = 10t + 5$. The latter is possible only if $c_{t+1} \geq 5$ and since $c < b$, it follows that $c_{t+1} = 5$. It is easy to see now that $c_{t+1} = c_{t+2} = \ldots = c_{2t} = 9$. Hence $c = b$, a contradiction with the choice of c.

Problem 11.1. Given the sequence $a_n = n + a\sqrt{n^2 + 1}$, $n = 1, 2, \ldots$, where a is a real number:

a) Find the values of a such that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

b) Find the values of a such that the sequence $\{a_n\}_{n=1}^{\infty}$ is monotone increasing.

Solution: a) If $a = -1$, the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent because $a_n = n - \sqrt{n^2+1} = \dfrac{-1}{n + \sqrt{n^2+1}} = -\dfrac{1}{n}\dfrac{1}{1 + \frac{1}{n^2}} \to 0$ when $n \to \infty$. Conversely, let the sequence $\{a_n\}_{n=1}^{\infty}$ be convergent. Since $a_n = n - \sqrt{n^2+1} + (a+1)\sqrt{n^2+1}$, we get that the sequence $(a+1)\sqrt{n^2+1}$ is also convergent. Since $\sqrt{n^2+1} \to \infty$ when $n \to \infty$, it follows that $a + 1 \neq 0$, so $a = -1$.

b) Let $\{a_n\}_{n=1}^{\infty}$ be a monotone increasing sequence, i. e., $a_{n+1} \geq$

a_n for any n. This inequality is equivalent to

$$(\star) \qquad \frac{a(2n+1)}{\sqrt{(n+1)^2+1}+\sqrt{n^2+1}} \geq -1.$$

Since

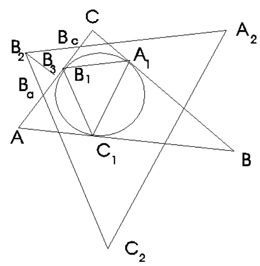
$$\lim_{n\to\infty} \frac{2n+1}{\sqrt{(n+1)^2+1}+\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{(1+\frac{1}{n})^2+\frac{1}{n^2}}+\sqrt{1+\frac{1}{n^2}}} = 1$$

it follows from $(\star)$ that $a \geq -1$.

Conversely, let $a \geq -1$. It follows from $\frac{2n+1}{\sqrt{(n+1)^2+1}+\sqrt{n^2+1}} < \frac{2n+1}{n+1+n} = 1$ that $(\star)$ holds true so the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing. The required values of a are $a \in [-1, +\infty)$.

Problem 11.2. Given a $\triangle ABC$ with circumcentre O and circumradius R. The incircle of $\triangle ABC$ is of radius r and touches the sides AB, BC and CA in the points C_1, A_1 and B_1. Let the lines determined by the midpoints of the segments AB_1 and AC_1, BA_1 and BC_1, CA_1 and CB_1 meet at points C_2, A_2 and B_2. Prove that the circumcircle of $\triangle A_2B_2C_2$ is of centre O and radius $R + \frac{r}{2}$.

Solution: We show first that the projection B_3 of B_2 on AC is the midpoint of AC. Let B_a and B_c be the midpoints of AB_1 and CB_1. We shall use the standard notation for the elements of $\triangle ABC$.



We obtain $\dfrac{B_aB_3}{B_2B_3} = \operatorname{tg}\dfrac{\alpha}{2} = \dfrac{r}{p-a}$ and $\dfrac{B_cB_3}{B_2B_3} = \operatorname{tg}\dfrac{\gamma}{2} = \dfrac{r}{p-c}$, so $\dfrac{B_aB_3}{B_cB_3} = \dfrac{p-c}{p-a}$. Since $B_aB_c = \dfrac{b}{2}$, it follows that $B_aB_3 = \dfrac{p-c}{2} = \dfrac{CB_c}{2}$ and $B_cB_3 = \dfrac{p-a}{2} = \dfrac{AB_a}{2}$, which gives $AB_3 = CB_3$. Therefore $B_2O = B_2B_3 + B_3O = \dfrac{(p-c)(p-a)}{2r} + R\cos\beta$.

We shall show that the above expression equals $R+\dfrac{r}{2}$ and by analogy $A_2O = C_2O = R + \dfrac{r}{2}$, which will complete the proof.

We obtain that $\dfrac{(p-c)(p-a)}{2r} + R\cos\beta = \dfrac{r}{2} + R \iff \dfrac{S}{2(p-b)} - \dfrac{S}{2p} = R(1-\cos\beta) \iff \dfrac{Sb}{2p(p-b)} = 2R\sin^2\dfrac{\beta}{2} \iff \dfrac{r}{p-b} = \dfrac{4R}{b}\sin^2\dfrac{\beta}{2} \iff \operatorname{tg}\dfrac{\beta}{2} = \dfrac{2\sin^2\frac{\beta}{2}}{\sin\beta} \iff \sin\beta = 2\sin\dfrac{\beta}{2}\cos\dfrac{\beta}{2}$, which is a true equality.

Problem 11.3. Find the smallest natural number n such that the sum of the squares of its divisors (including 1 and n) equals $(n+3)^2$.

Solution: It is clear that n has at least three divisors and let $1 < d_1 < d_2 < \cdots < d_k < n$ be those different from 1 and n. The conditions of the problem imply

$$(\star) \qquad d_1^2 + d_2^2 + \cdots + d_k^2 = 6n + 8.$$

Let $n = p^\alpha$, where p is a prime number. It follows now from $(\star)$ that $p^2 + p^4 + \cdots + p^{2\alpha-2} = 6p^\alpha + 8$, so $p\backslash 8$ and therefore $p = 2$. The above equality implies $1 + p^2 + p^4 + \cdots + p^{2\alpha-4} = 6p^{\alpha-2} + 2$, which is impossible.

Therefore $k \neq 1, 3, 5$, because otherwise the number of divisors of n equals 3, 5, 7, i. e., $n = p^2$, $n = p^4$ or $n = p^6$, where p is a prime number. Suppose that $k \geq 6$. Since $d_i d_{k-i} = n$, it follows from $(\star)$ that $(d_{k-1} - d_1)^2 + (d_{k-2} - d_2)^2 + (d_{k-3} - d_3)^2 \leq 8$. The last inequality is impossible, since the numbers $d_{k-1} - d_1, d_{k-2} - d_2$ and $d_{k-3} - d_3$ are distinct (if for example $d_{k-1} - d_1 = d_{k-2} - d_2 = A$, then $d_1(A + d_1) = d_2(A + d_2)$, so $d_1 = d_2$). We conclude now that $k = 2$ or $k = 4$.

Assume $k = 4$. Then n has 6 divisors and thus n is of the form $n = p \cdot q^2$, where p and q are distinct prime numbers. (n is not of the form $n = p^5$). It follows from $(\star)$ that

$$(\star\star) \qquad p^2 + q^2 + q^4 + p^2 q^2 = 6pq^2 + 8.$$

If $q \geq 5$, then $q^4 + p^2 q^2 \geq 2pq^3 \geq 10pq^2 > 6pq + 8$ and therefore $q = 2$ or $q = 3$. Direct verification shows that inequality $(\star\star)$ is impossible. Thus $k = 2$ and hence $n = pq$, where p and q are distinct prime numbers such that

$$p^2 + q^2 = 6pq + 8.$$

Since $q\backslash p^2 - 8$, it is easy to see that if $p \leq 17$ then $p = 7$, $q = 41$ and $n = 287$. Since $17^2 = 289 > 287$, we conclude that the smallest n with the required property is $n = 287$.