Probability and Measure – Tutorial 3

1. Let Ω be a set. Prove that the σ-algebra generated by all sets of the form $\{\omega\}$, with $\omega \in \Omega$, is equal to the collection

$$\{A \subset \Omega : \text{ either } A \text{ is countable or } A^c \text{ is countable}\}.$$

2. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space; assume that $\{x\} \in \mathcal{A}$ for any $x \in \Omega$. Prove that the set $\{x \in \Omega : \mu(\{x\}) > 0\}$ is countable.

3. Let Ω be a set and let $\mathscr{F} = \{A_1, \ldots, A_k\}$ be a finite collection of subsets of Ω. Assume that the sets in $\mathscr{F}$ form a *partition of* Ω, meaning that they are pairwise disjoint and that their union is equal to Ω.

   (a) Prove that the collection

   $$\left\{ \bigcup_{i \in I} A_i : I \subset \{1, \ldots, k\} \right\}$$

   is a σ-algebra of subsets of Ω.

   (b) Prove that the collection of item (a) is equal to $\sigma(\mathscr{F})$.

4. Let Ω be a set and let $\mathscr{G}$ be a finite collection of subsets of Ω. For any $\omega, \omega' \in \Omega$, write $\omega \sim \omega'$ if

$$\text{for all } A \in \mathscr{G}, \text{ we either have } \{\omega, \omega'\} \subset A \text{ or } \{\omega, \omega'\} \cap A = \varnothing.$$

   For each $\omega \in \Omega$, let $M_\omega := \{\omega' \in \Omega : \omega \sim \omega'\}$.

   (a) Prove that for any $\omega, \omega' \in \Omega$, the sets M_ω and $M_{\omega'}$ are either equal or disjoint.

   (b) Prove that

   $$\bigcup_{\omega \in \Omega} M_\omega = \Omega.$$

   (c) Let $\mathscr{G}'$ denote the collection of all sets B such that either $B \in \mathscr{G}$ or $B^c \in \mathscr{G}$. Prove that, for any $\omega \in \Omega$, we have

   $$M_\omega = \bigcap_{B \in \mathscr{G}' : \omega \in B} B.$$

   (d) Define the collection $\mathscr{F} := \{M_\omega : \omega \in \Omega\}$. Prove that $\mathscr{F}$ is finite. Given an arbitrary enumeration of $\mathscr{F}$, $\mathscr{F} = \{A_1, \ldots, A_k\}$, prove that

   $$\sigma(\mathscr{G}) = \left\{ \bigcup_{i \in I} A_i : I \subset \{1, \ldots, k\} \right\}.$$