Probability and Measure – Tutorial 12

1. Let $f(x,y) = e^{-xy} - 2e^{-2xy}$.
   (a) Show that $\int_0^1 \left(\int_1^\infty (e^{-xy} - 2e^{-2xy})\mathrm{d}x\right) \mathrm{d}y > 0$.
   (b) Show that $\int_1^\infty \left(\int_0^1 (e^{-xy} - 2e^{-2xy})\mathrm{d}y\right) \mathrm{d}x < 0$.
   (c) Determine $\int_0^1 \left(\int_1^\infty |e^{-xy} - 2e^{-2xy}|\mathrm{d}x\right) \mathrm{d}y$.

2. Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces and $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$ be the product space. Let $f : \Omega_1 \to \mathbb{R}$ and $g : \Omega_2 \to \mathbb{R}$ be integrable functions. Define $h : \Omega_1 \times \Omega_2 \to \mathbb{R}$ by
$$h(\omega_1, \omega_2) = f(\omega_1) \cdot g(\omega_2).$$
   Show that h is integrable and
$$\int_{\Omega_1 \times \Omega_2} h \, \mathrm{d}(\mu_1 \otimes \mu_2) = \int_{\Omega_1} f \, \mathrm{d}\mu_1 \cdot \int_{\Omega_2} g \, \mathrm{d}\mu_2.$$
   Hint. Proceed in steps. First assume that f and g are simple functions. Next, assume that they are both measurable and nonnegative; take sequences f_n and g_n of simple functions with $f_n \nearrow f$ and $g_n \nearrow g$. Finally treat f and g general integrable functions.

3. **(Independence for collections of events.)**
   Definition. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $\mathcal{F}_1, \mathcal{F}_2, \ldots \subset \mathcal{A}$ be a (finite or infinite) sequence of collections of sets. We say $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are independent if, for any $i_1, \ldots, i_n$ with $1 \le i_1 < \cdots < i_n$ and any $A_1 \in \mathcal{F}_{i_1}, \ldots, A_n \in \mathcal{F}_{i_n}$ we have $\mathbb{P}(\cap_{i=1}^n A_i) = \prod_{i=1}^n \mathbb{P}(A_i)$.
   (Note: In Tutorial 2 we saw the definition of independence for a sequence of sets; here we are talking about independence for a sequence of *collections* of sets).
   (a) Assume $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are independent and are all closed under intersections (that is, if $A, B \in \mathcal{F}_i$, then $A \cap B \in \mathcal{F}_i$). Prove that $\sigma(\mathcal{F}_1), \sigma(\mathcal{F}_2), \ldots$ are independent.
   Hint. This will follow once you prove that $\sigma(\mathcal{F}_1), \mathcal{F}_2, \mathcal{F}_3, \ldots$ are independent. To prove this, start by showing that the collection of $B \in \mathcal{F}_1$ with the property that, for any $i_1, \ldots, i_n$ with $2 \le i_1 < \cdots < i_n$ and any $A_1 \in \mathcal{F}_{i_1}, \ldots, A_n \in \mathcal{F}_{i_n}$, we have
$$\mathbb{P}(B \cap A_1 \cap \cdots \cap A_n) = \mathbb{P}(B) \cdot \mathbb{P}(A_1) \cdots \mathbb{P}(A_n).$$
   Show that this collection is a Dynkin system and contains $\mathcal{F}_1$.
   (b) Now assume that $\mathcal{G}_1, \mathcal{G}_2, \ldots \subset \mathcal{A}$ is a (finite or infinite) sequence of independent σ-algebras. Let $I_1, I_2, \ldots$ be disjoint sets of natural numbers, and for each i define $\mathcal{H}_i = \sigma(\mathcal{G}_m : m \in I_i)$. Prove that $\mathcal{H}_1, \mathcal{H}_2, \ldots$ are independent.
   Hint. For each i, let $\mathcal{D}_i$ denote the collection of all sets of the form
$$A_1 \cap \cdots \cap A_n : \ A_j \in \mathcal{G}_{m_j} \text{ for each } j, \text{ where } m_1, \ldots, m_n \in I_i, \ m_1 < \cdots < m_n.$$
   Prove that $\mathcal{D}_1, \mathcal{D}_2, \ldots$ are independent. Also prove that each $\mathcal{D}_i$ is closed under intersections. Then use part (a).