Probability and Measure – Tutorial 13

1. **(Independence for random variables.)**
Definition. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $X_1, X_2, \ldots$ be a (finite or infinite) sequence of random variables defined on this space. We say these random variables are independent if the σ-algebras $\sigma(X_1), \sigma(X_2), \ldots$ are independent (recall that $\sigma(X_i)$ is the intersection of all σ-algebras $\mathcal{F}$ on Ω such that X_i is $(\mathcal{F}, \bar{\mathcal{B}})$-measurable).

For the following items, let (X, Y) be a two-dimensional random vector (so that X and Y are two random variables taking values in $\mathbb{R}$).

(a) Let $\mathbb{P}_{(X,Y)}$ denote the distribution of (X, Y), and $\mathbb{P}_X$ and $\mathbb{P}_Y$ denote the distributions of X and Y, respectively. Prove that X and Y are independent if and only if $\mathbb{P}_{(X,Y)} = \mathbb{P}_X \otimes \mathbb{P}_Y$.

(b) Prove that X and Y are independent if and only if

$$\mathbb{P}(X \in I_1,\ Y \in I_2) = \mathbb{P}(X \in I_1) \cdot \mathbb{P}(Y \in I_2) \quad \text{for any intervals } I_1, I_2.$$

(c) Prove that X and Y are independent if and only if, for any two measurable and bounded functions $g, h : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)] \cdot \mathbb{E}[h(Y)].$$

(d) Prove that if X and Y are independent and integrable, then XY is integrable and

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

2. Let $X_1, X_2, \ldots$ be a sequence of independent random variables defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $g : \mathbb{R}^m \to \bar{\mathbb{R}}$ and $h : \mathbb{R}^n \to \bar{\mathbb{R}}$ be measurable functions. Prove that, if $\{i_1, \ldots, i_m\}$ and $\{j_1, \ldots, j_n\}$ are disjoint sets of integers, then the random variables $g(X_{i_1}, \ldots, X_{i_m})$ and $h(X_{j_1}, \ldots, X_{j_n})$ are independent.

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X : \Omega \to \mathbb{R}$ be a non-negative random variable. Prove that

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(\{\omega \in \Omega : X(\omega) \geq x\}) m(\mathrm{d}x),$$

where m denotes the Lebesgue measure $\mathbb{R}$. (Recall that $\mathbb{E}[X] = \int_\Omega X \mathrm{d}\mathbb{P}$).

Hint. First prove that

$$\mathbb{E}[X] = \int_0^\infty x \,\mathrm{d}\mathbb{P}_X = \int_0^\infty \left(\int_0^\infty \mathbb{1}_{[0,x]}(t) m(\mathrm{d}t) \right) \mathbb{P}_X(\mathrm{d}x),$$

where $\mathbb{P}_X$ is the measure on the Borel σ-algebra defined by $\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B))$.