Probability and Measure – Tutorial 9

1. Let Ω be a set, $\mathcal{F}, \mathcal{G}$ be two σ-algebras of subsets of Ω with $\mathcal{F} \subset \mathcal{G}$ and μ be a measure on $\mathcal{G}$ (also denote the restriction of μ to $\mathcal{F}$ by μ). Let $f : \Omega \to \mathbb{R}$ be $(\mathcal{F}, \bar{\mathcal{B}})$.

   (a) Prove that f is $(\mathcal{G}, \bar{\mathcal{B}})$-measurable.

   (b) Prove if $f \geq 0$, then the integral of f defined in either measure space gives the same value.

   Suggestion: Adopt the notation

   $$\int_{(\Omega,\mathcal{F},\mu)} f \text{ and } \int_{(\Omega,\mathcal{G},\mu)} f$$

   to make the measure space where the integration is taking place explicit.

   (c) Prove that f is integrable in $(\Omega, \mathcal{F}, \mu)$ if and only if it is integrable in $(\Omega, \mathcal{G}, \mu)$, and in that case, the values of the integral of f taken in either space coincide.

2. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \bar{\mathbb{R}}$ be integrable. Prove that $|f| < \infty$ almost everywhere.

3. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \bar{\mathbb{R}}$ be integrable and nonnegative. Define

   $$E_n = \{\omega : f(\omega) > n\},\ n \in \mathbb{N}, \qquad E = \{\omega : f(\omega) = \infty\}.$$

   Show that $\lim_{n\to\infty} \mu(E_n) = \mu(E) = 0$.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and assume $f_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, are measurable real-valued functions. Assume that $f_n \to 0$ pointwise. In Exercise 4 of Tutorial 8, you have seen that, if the measure is a probability,

   $$\forall \varepsilon > 0,\ \lim_{n\to\infty} \mu(\{\omega : |f_n(\omega)| > \varepsilon\}) = 0$$

   (more generally, this implication holds if the measure space is finite). Show that this is not necessarily true in case the measure space is not finite.

5. Prove or give a counterexample to the following statement. If $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, are measurable functions with $f_1 \geq f_2 \geq \cdots \geq 0$ and such that $\int_{\mathbb{R}} f_n \, \mathrm{d}m \to 0$, then $f_n \to 0$ almost everywhere.