Probability and Measure – Tutorial 15

1. Let $(\Omega, \mathcal{F})$ be a measurable space. Given two measures τ, κ on $\mathcal{F}$, we write $\tau \sim \kappa$ if $\tau \ll \kappa$ and $\kappa \ll \tau$.

   (a) Assume that μ, ν are σ-finite measures on $\mathcal{F}$ with $\nu \ll \mu$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Prove that $\mu \ll \nu$ if and only if $f > 0$ μ-a.e., and that in this case, $\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = \frac{1}{f}$.

   (b) Assume λ, ν_1, ν_2 are σ-finite measures on $\mathcal{F}$ with $\nu_i \sim \lambda$ and $\frac{\mathrm{d}\nu_i}{\mathrm{d}\lambda} = f_i$, for $i = 1, 2$. Prove that $\nu_1 \sim \nu_2$ and that $\frac{\mathrm{d}\nu_2}{\mathrm{d}\nu_1} = \frac{f_2}{f_1}$.

   (c) Assume that α, β, γ are σ-finite measures with $\alpha \ll \beta$ and $\beta \ll \gamma$. Prove that $\alpha \ll \gamma$ and that $\frac{\mathrm{d}\alpha}{\mathrm{d}\gamma} = \frac{\mathrm{d}\alpha}{\mathrm{d}\beta} \cdot \frac{\mathrm{d}\beta}{\mathrm{d}\gamma}$.

2. Give an example of two measures μ_1 and μ_2 on $(\mathbb{R}, \mathcal{B})$ that are mutually singular and both absolutely continuous with respect to Lebesgue measure.

3. Reread Exercise 3 from Tutorial 8. Prove that any probability measure μ on $(\mathbb{R}, \mathcal{B})$ can be decomposed as $\mu = \mu_1 + \mu_2 + \mu_3$, where:

   - μ_1 is purely atomic (in particular, it is singular with respect to Lebesgue measure);
   - μ_2 is absolutely continuous with respect to Lebesgue measure;
   - μ_3 has no atoms and is singular with respect to Lebesgue measure.

   Moreover, prove that this decomposition is unique.

   Remark. Assume the probability measure μ is the distribution of a random variable X (that is, $\mu(A) = \mathbb{P}(X^{-1}(A))$ for any $A \in \mathcal{B}$), and that F is the associated distribution function (that is, $F(x) = \mathbb{P}(X \leq x)$ for all x). You can now have a deeper understanding of some definitions given in your first Probability Theory course:

   - In case μ has only the atomic part (that is, if $\mu = \mu_1$), then X is called discrete. In this case, let $\Lambda = \{x : \mu(\{x\}) > 0\}$. It is not hard to see that this set is countable. The function $f : \Lambda \to \mathbb{R}$ given by $f(x) = \mu(\{x\})$ is the probability mass function of X.
   - In case μ has only the part that is absolutely continuous with respect to Lebesgue measure (that is, if $\mu = \mu_2$), then it has a Radon-Nykodim derivative with respect to Lebesgue measure, $f = \frac{\mathrm{d}\mu}{\mathrm{d}m}$. The function f is called the probability density function of X, and we have $\mathbb{P}(X \in A) = \mu(A) = \int_A f \, \mathrm{d}m$.
     In elementary courses, such a random variable is called "continuous". This nomenclature is somewhat imprecise. It would be better to say X is continuous if $\mu = \mu_2 + \mu_3$ and *absolutely continuous* if $\mu = \mu_2$. The reason is that the condition $\mu = \mu_2 + \mu_3$ already implies that (and in fact is equivalent to) F is a continuous function.

4. Let $0 \leq x_1 < x_2 < \cdots < x_n$ and $a_1, \ldots, a_n < 0$ be real numbers. Define $F : \mathbb{R} \to \mathbb{R}$ by

   $$F(x) = (1 - e^{-x}) \cdot \mathbb{1}_{(0,\infty)}(x) + \sum_{i=1}^{n} a_i \cdot \mathbb{1}_{[x_i,\infty)}(x), \quad x \in \mathbb{R}.$$

   Let μ_F be the Lebesgue-Stieltjes measure associated to F (recall that μ_F is the unique measure on $\mathbb{R}$ satisfying $\mu_F((a, b]) = F(b) - F(a)$ for all a, b with $a < b$). Give the Lebesgue decomposition of μ_F with respect to Lebesgue measure, and find the Radon-Nikodym derivative of the absolutely continuous part of μ_F with respect to Lebesgue measure.