Probability and Measure – Tutorial 1

1. Let Ω be a set and let

$$\mathcal{A} = \{\, A \subset \Omega : \, A \text{ or } A^c \text{ is countable}\,\}.$$

Show that $\mathcal{A}$ is a σ-algebra.

2. Let Ω be a nonempty set, let $\mathcal{A}$ be a σ-algebra, and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathcal{A}$. Show that the following sets belong to $\mathcal{A}$.

   (a) $\{\omega \in \Omega : \omega \in A_n \text{ for all } n \in \mathbb{N}\}$.

   (b) $\{\omega \in \Omega : \omega \notin A_n \text{ for all } n \in \mathbb{N}$.

   (c) $\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$; this set is denoted $\limsup\limits_{n\to\infty} A_n$.

   (d) $\{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many } n \in \mathbb{N}\}$; this set is denoted $\liminf\limits_{n\to\infty} A_n$.

   Describe the complements of these sets.

3. Show that the union of σ-algebras need not be a σ-algebra.

4. Let $\Omega = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\Omega)$ (the power set of Ω) and let p_n, $n \in \mathbb{N}$, be nonnegative numbers. Show that

$$\mu(A) = \sum_{n\in A} p_n, \quad A \subset \Omega,$$

defines a measure on Ω.

5. Let $\mathcal{A}$ be a collection of subsets of a set Ω with the following properties: (a) $\Omega \in \mathcal{A}$; (b) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$; (c) $A, B \in \mathcal{A}$ disjoint $\Rightarrow A \cup B \in \mathcal{A}$. Show that

$$A, B \in \mathcal{A}, \quad A \subset B \quad \Rightarrow \quad B \setminus A \in \mathcal{A}.$$