Probability and Measure – Tutorial 10

1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ be a nonnegative measurable function. Show that $\nu : \mathcal{A} \to \mathbb{R}$ defined by
$$\nu(A) = \int_A f \mathrm{d}\mu, \quad A \in \mathcal{A}$$
is a measure.

2. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \overline{\mathbb{R}}$ be integrable and nonnegative. Define
$$E_n = \{\omega : f(\omega) \geq n\}, \ n \in \mathbb{N}, \qquad E = \{\omega : f(\omega) = \infty\}.$$
Show that $\lim_{n\to\infty} n \cdot \mu(E_n) = 0$.

3. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ be an integrable function. Assume that for every $E \in \mathcal{A}$ we have $\int_E f \, \mathrm{d}\mu \geq 0$. Show that $f \geq 0$ almost everywhere.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $f, f_1, f_2, \cdots : \Omega \to \overline{\mathbb{R}}$ be measurable functions such that
$$f_n(\omega) \to f(\omega), \quad f_n(\omega) \geq f_{n+1}(\omega) \geq 0, \ \ \omega \in \Omega.$$
Show that, if f_1 is integrable, then
$$\lim_{n\to\infty} \int_\Omega f_n \mathrm{d}\mu = \int_\Omega f \mathrm{d}\mu.$$
Show that without the assumption that f_1 is integrable, the result need not be true.

5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $E \in \mathcal{A}$. Define
$$f_n = \begin{cases} \mathbb{1}_E, & n \text{ odd}, \\ 1 - \mathbb{1}_E, & n \text{ even} \end{cases}$$
Applying Fatou's Lemma to this sequence, what do you observe?

6. Determine the limit
$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} \mathrm{d}x.$$
Hint: use $(1 + x/n)^n \uparrow e^x$.

7. Define the functions f_n by
$$f_n(x) = ne^{-nx}, \ \ x \in \mathbb{R}.$$
Are the functions f_n dominated by an integrable function on the interval $[0, 1]$? How about on $[1, \infty)$.