Probability and Measure (24/25 2b)
Dr. Gilles Bonnet
Deadline: **Monday 5 May** 23:59 on Brightspace

Homework 1

- Write your solutions **by hand** in a clear and readable way (not with tons of crossed words or paragraph).
- **Justify your results**, in a clear way that is easy to follow.
- Try to be concise. Avoid lengthy explanations when a few words suffice.

The following table gives an overview of the intended point distribution of the homework.

Question:	1	2	3	4	Total
Points:	44	30	26	0	100

Consider a random trial with finitely many possible outcomes, and let E be the set of these outcomes. For each $e \in E$, let p_e be the probability of observing outcome e. We assume that $0 < p_e < 1$ for all $e \in E$.
Now, suppose this trial is repeated infinitely many times, independently of each other. Let $\Omega = E^{\mathbb{N}}$ be the set of all possible infinite sequences of outcomes arising from these repeated trials. The aim of this homework is to show that there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ that models this situation under the assumption that all trials are independent.
For this we will use the outer measure approach, as we did to construct the Lebesgue measure in the lecture.
We have seen in the very first hour of the very first lecture that it is in general not a great idea to take the power set of Ω as the σ-algebra $\mathcal{A}$ we want to work with. On the other hand, we do not want the sigma algebra to be too small either as would be $\{\emptyset, \Omega\}$ for instance.
The minimal wish of your lecturer is that, for any $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, and any $e_1, \dots, e_n \in E$, the event that the first n trials yield the outcomes $e_1, \dots, e_n$, in this order, is measurable (your lecturer is a reasonable person). This means that we will work with the σ-algebra $\mathcal{A} = \sigma(\mathcal{C})$ generated by

$$\mathcal{C} = \{C(e_1, \dots, e_n) : n \in \mathbb{N},\ e_1, \dots, e_n \in E\} \cup \{\Omega\},$$

where

$$C(e_1, \dots, e_n) = \{\omega \in \Omega : \omega_1 = e_1, \dots, \omega_n = e_n\}.$$

We might use the following slight abuse of notation $C(e_1, \dots, e_n) = \Omega$ if $n = 0$, which can occasionally allow us to consider the cases $C = \Omega$ and $C = C(e_1, \dots, e_n) \in \mathcal{C}$ simultaneously.

1. **(Warming up)**
   (a) (5 points) Given basic knowledge from the Probability Theory course, for any $C \in \mathcal{C}$, which value p_C should we aim for $\mathbb{P}(C)$ in our construction?
   Remark: Distinguish the cases $C = \Omega$ and $C = C(e_1, \dots, e_n)$. Give your answer and a very brief explanation.

(b) (5 points) Let $C_1 = C(e_1, \dots, e_{n_1}) \in \mathcal{C}$ and $C_2 = C(e'_1, \dots, e'_{n_2}) \in \mathcal{C}$ be two cylinder sets, with $n_1 \le n_2$. Show that if $C_1 \cap C_2 \neq \emptyset$, then $C_2 \subset C_1$.

(c) (12 points) Show that for any $C_1, \dots, C_M \in C$ there exists $I \subset \{1, \dots, M\}$ such that $\cup_{i=1}^M C_i = \cup_{i \in I} C_i$ and the C_i are pairwise disjoint.
Hint: You might want to consider the case where one of the set is Ω separately.

(d) (10 points) Let $0 \le m \le n < \infty$ be integers. Show that, for any $e_1, \dots, e_m \in E$, we have

$$p_{C(e_1,\dots,e_m)} = \sum_{e_{m+1},\dots,e_n \in E} p_{C(e_1,\dots,e_n)}.$$

Remark: In case $m = 0$, we interpret the left hand side as $p_\Omega = 1$ following the convention presented in the introduction of this homework.

(e) (12 points) Let $0 \le n \le n_1 \le \dots \le n_M < \infty$ be integers and consider cylinders of the form

$$C_i = C(e^i_1, \dots, e^i_{n_i}) \in \mathcal{C}, \quad i = 1, \dots, M.$$

Assume that they are pairwise disjoint and that their union is a cylinder set of the form $C = C(e_1, \dots, e_n) \in \mathcal{C}$.
Show that $p_C = \sum_{i=1}^M p_{C_i}$.

2. **(Outer measure)** Analogously as in the Definition 2.2.1 of m^*, we set

$$\mathbb{P}^*(A) = \inf \left\{ \sum_{m=1}^{\infty} p_{C_m} \mid C_1, C_2, \dots \in \mathcal{C} \text{ and } A \subset \bigcup_{m=1}^{\infty} C_m \right\} \in [0, \infty],$$

where p_{C_m} is as in Question 1.

(a) (5 points) Show that, for any $\epsilon > 0$, there exists $C \in \mathcal{C}$ such that $p_C < \epsilon$.

(b) (5 points) Show that for any $\epsilon > 0$ there exists $C_1, C_2, \dots \in \mathcal{C}$ such that $\sum_{i \ge 1} p_{C_i} < \epsilon$.

(c) (20 points) Show that $\mathbb{P}^*$ is an outer measure.

3. **(Outer measure of cylinder sets)** Next we want to show that $\mathbb{P}^*(C) = p_C$ for all $C \in \mathcal{C}$.

(a) (6 points) Show that $\mathbb{P}^*(C) \le p_C$ for all $C \in \mathcal{C}$.
Remark: If you answer correctly the following parts, you'll get the equality and therefore could get the upper bound for free. But you can of course, prove it directly with simple arguments.

To show that we have equality (and not only a lower bound), we take inspiration from the proof of Lemma 2.2.2. This requires to consider a topology on Ω. We consider the topology generated by $\mathcal{C}$, that is the coarsest topology such that the sets in $\mathcal{C}$ are open. This topology has a few properties that are useful to us:

- Any $C \in \mathcal{C}$ is both open and closed (also sometimes called with the funny term *clopen*).
- The space Ω is compact.
- In particular any $C \in \mathcal{C}$ is compact (since a closed subsets of a compact set is compact).

You might use the above facts without proving them.

(b) (10 points) Show that for any cylinder $C \in \mathcal{C}$, we have

$$\mathbb{P}^*(C) = \inf \left\{ \sum_{m=1}^{M} p_{C_m} \mid C_1, \dots, C_M \in \mathcal{C} \text{ and } C \subset \bigcup_{m=1}^{M} C_m \right\}$$

(c) (5 points) Show that for any cylinder $C \in \mathcal{C}$, we have

$$\mathbb{P}^*(C) = \inf \left\{ \sum_{m=1}^{M} p_{C_m} \mid C_1, \dots, C_M \in \mathcal{C} \text{ pairwise disjoint, and } C \subset \bigcup_{m=1}^{\infty} C_m \right\}$$

(d) (5 points) Show that for any cylinder $C \in \mathcal{C}$, we have

$$\mathbb{P}^*(C) = p_C$$

4. **(Bonus. Not graded)** Below are a few questions that are not graded and will not even be corrected. They are here only to give food for thoughts to those with a lot of appetite. Do not include your solutions to these questions in your homework. In any case they will not be read (Your fantastic TA has enough work already!)

   (a) Can you show that cylinders are measurable sets with respect to $\mathbb{P}^*$?

   (b) In this construction we started with a probability measure on a finite set E (c.f. the assumption $\sum p_e = 1$) and we constructed a probability measure on the space $\Omega = E^{\mathbb{N}}$ of infinite sequences of outcomes. What happens if we start with a measure on E that is not a probability measure? Does anything even make sense?

   (c) The topology we considered has more interesting properties:

   - It is the topology of pointwise convergence. That means that this is the coarsest topology such that for any sequence $(\omega_n)_{n\in\mathbb{N}}$ which converge pointwise to some $\omega \in \Omega$ and for any neighborhood U of ω, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\omega_n \in U$.
   - This topology is metrizable. This means that there exists a distance d on Ω such that the topology induced by d is the same as the topology we considered. One possibility is to consider the weighted Hamming distance:

   $$d(\omega, \omega') = \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbf{1}_{\{\omega_i \neq \omega'_i\}}.$$

   Can you make sense of these statements? Can you prove them? What does a ball of radius r centered at ω look like with this metric?