Probability and Measure – Tutorial 14

1. Consider the measure space $(\Omega, \mathcal{F}, \mu)$ with $\Omega = \mathbb{N}$, $\mathcal{F} = P(\mathbb{N})$ and μ the counting measure. Describe the space L^∞ in this setting.

2. (a) Consider the set $(0, \infty)$ with Borel σ-algebra and Lebesgue measure. Let $1 \leq p, q \leq \infty$, $p \neq q$. Show that $L^p(0, \infty)$ is not contained in $L^q(0, \infty)$. (Note there are several cases to consider:$[p = \infty, p < \infty], [p < \infty, q = \infty], [1 \leq p < q < \infty], [1 \leq q < p < \infty]$).

   (b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that, if $1 \leq p < q \leq \infty$, then $L^q(\Omega)$ is contained in $L^p(\Omega)$.

3. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p < \infty$. Let $f_n : \Omega \to \mathbb{R}$ be a sequence of measurable functions converging pointwise to $f : \Omega \to \mathbb{R}$. Assume that $|f_n| \leq g$ for some $g \in \mathcal{L}^p(\Omega)$ and all n. Show that $f \in \mathcal{L}^p(\Omega)$ and that $\|f_n - f\|_p \to 0$ as $n \to \infty$.

4. For this exercise, recall the definitions and statemens of Exercise 4 in Tutorial 8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X_n, $n \in \mathbb{N}$ and X be random variables defined on this space.

   (a) Let $p \in [1, \infty)$. Prove that if X_n converges to X in L^p, then X_n converges to X in probability.

   (b) Let $p \in [1, \infty)$. Prove that if X_n converges to X in L^p, then X_n has a subsequence converging to X almost surely. Also show that the sequence of random variables defined in Exercise 4c of Tutorial 8 converges in L^p, for all $p \in [1, \infty)$, but not almost surely.

   (c) Prove that if X_n converges to X in L^∞, then X_n converges to X almost surely (hence also in probability).

   (d) Construct an example of random variables X_n and X such that $X_n \to X$ almost surely (hence also in probability) but

   $$X_n \xrightarrow[L^p]{n\to\infty} X \text{ does not hold for any } p \in [1, \infty].$$

   Tip: Make a summary of the different modes of convergence in probability spaces and their relations (that is, which implies which). For the implications that do not hold, try also to remember the counterexamples.