Probability and Measure – Tutorial 6

1. Let $(\Omega, \mathcal{A})$ be a measurable space and let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable. Show that f_a defined by
$$f_a(x) = \begin{cases} a, & \text{if } f(x) > a, \\ f(x), & \text{if } f(x) \leq a, \end{cases}$$
is a measurable function.

2. Let $(\Omega, \mathcal{A})$ be a measurable space and let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be a function. Show that the following statements are equivalent:
   (a) f is measurable;
   (b) f^2 is measurable and $\{\omega \in \Omega : f(\omega) > 0\}$ is measurable.

3. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable and assume that $\{\omega \in \Omega : f(\omega) > 0\}$ has positive measure. Show that for $\varepsilon > 0$ sufficiently small also $\{\omega \in \Omega : f(\omega) > \varepsilon\}$ has positive measure.

4. Let $(\Omega, \mathcal{A})$ be a measurable space and let $f, g : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be measurable functions. Show that the set
$$\{\omega \in \Omega : f(\omega) \neq g(\omega)\}$$
is measurable.

5. Let $(\Omega, \mathcal{A})$ be a measurable space and let $f_n : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be a sequence of measurable functions. Show that the set of all $\omega \in \Omega$ where $f_n(\omega)$ converges as $n \to \infty$ is measurable.

6. Let $(\Omega, \mathcal{A}, \mu)$ be a complete measure space and let $f_n, g_n : (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \overline{\mathcal{B}})$ be sequences of measurable functions such that
$$f_n = g_n \text{ a.e.}$$
If f_n and g_n converge almost everywhere to f and g, respectively, show that $f = g$ almost everywhere.

7. Let $f : \mathbb{R}^d \to \mathbb{R}$ be continuous. Show that f is measurable.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Show that f is measurable.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that f' is measurable.