Probability and Measure – Tutorial 5

1. Prove that the Lebesgue measure of the boundary of a closed rectangle in $\mathbb{R}^d$ is equal to zero.

2. Let $A \subset \mathbb{R}^d$. Show that if A is a set of measure 0 then the closure of $\mathbb{R}^d \setminus A$ equals $\mathbb{R}^d$. Is the converse statement valid?

3. Let $A \subset \mathbb{R}^d$. Show that the boundary ∂A is Borel measurable. Show that, if ∂A has Lebesgue measure 0, then A is Lebesgue measurable.

4. Suppose that $E \subset \mathbb{R}$ is a Lebesgue measurable set with finite measure $m(E)$. Prove that there is a measurable set $A \subseteq E$ such that $m(A) = m(E)/2$.

   Hint. Define $f(x) = m(E \cap (-x, x))$, $x \geq 0$, then f is uniformly continuous.

5. Show that if $A \in \mathcal{B}^d$, $x \in \mathbb{R}^d$ and $\alpha > 0$, then $A + x \in \mathcal{B}^d$ and $\alpha A \in \mathcal{B}^d$. Similarly, if $A \in \mathcal{M}^d$, then $A + x$, $\alpha A \in \mathcal{M}^d$. Also show that, if $A \in \mathcal{M}^d$, then $m(A) = m(A + x)$ and $m(\alpha A) = \alpha^d m(A)$.