Probability and Measure – Tutorial 11

1. Assume $(\Omega_1, \mathcal{A}_1) = (\mathbb{R}, \mathcal{B})$ – the real line with Borel σ-algebra – and $(\Omega_2, \mathcal{A}_2) = (\mathbb{N}, P(\mathbb{N}))$ – the natural numbers with σ-algebra given by their power set. Show that, for $E \subset \Omega_1 \times \Omega_2$, we have that $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$ if and only if $E^{\omega_2} \in \mathcal{A}_1$ for all $\omega_2 \in \Omega_2$.

Some terminology for random variables. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \bar{\mathbb{R}}$ be a random variable. Recall that the distribution of X is the measure $\mathbb{P}_X = \mathbb{P} \circ X^{-1}$.

We say that X is *discrete* if it only attains countably many values. Let $\mathcal{X} \subset \bar{\mathbb{R}}$ be the set of values attained by X. The *probability mass function* of X is the function $p_X : \mathcal{X} \to \mathbb{R}$ defined by

$$p_X(x) = \mathbb{P}(X = x), \quad x \in \mathcal{X}.$$

We say that X is *absolutely continuous* if there exists a function $f_X : \mathbb{R} \to [0, \infty)$ such that

$$\mathbb{P}(X \in A) = \int_A f_X \, \mathrm{d}m \quad \text{for all } A \in \mathcal{B}.$$

Such a function f_X is called a *probability density function* of X (note that f_X is not uniquely defined: if we change it in a set of measure zero, the above property will still hold).

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \bar{\mathbb{R}}$ be a random variable. Let $g : \mathbb{R} \to \bar{\mathbb{R}}$ be a measurable function such that $g \circ X$ is integrable.

   (a) Prove that

   $$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g \, \mathrm{d}\mathbb{P}_X.$$

   (b) Prove that if X is discrete, attaining values on a countable set $\mathcal{X}$ and with probability mass function p_X, then

   $$\mathbb{E}[g(X)] = \sum_{x \in \mathcal{X}} g(x) \cdot p_X(x).$$

   (c) Prove that if X is absolutely continuous, with probability density function f_X, then

   $$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) \, \mathrm{d}x.$$

Review of the Riemann integral. Let $a, b \in \mathbb{R}$, $a \leq b$. Recall that a *partition* $\triangle$ of the interval $[a, b]$ is a finite set of intervals of the form

$$[x_0, x_1],\ [x_1, x_2], \dots [x_{n-1}, x_n]$$

with

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We say a partition $\triangle$ of $[a, b]$ is a *refinement* of another partition $\triangle'$ of $[a, b]$ if the endpoint of every interval in $\triangle'$ is also the endpoint of an interval of $\triangle$. The *norm* of a partition $\triangle$ (denoted $|\triangle|$) is defined as the largest among the lengths of the intervals of $\triangle$.
Given a function $f : [a, b] \to \mathbb{R}$ and a partition $\triangle$ of $[a, b]$, we define the *lower and upper sums*

$$S_*(f, \triangle) = \sum_{I \in \triangle} \left(\inf_{x \in I} f(x) \right) \cdot m(I), \qquad S^*(f, \triangle) = \sum_{I \in \triangle} \left(\sup_{x \in I} f(x) \right) \cdot m(I).$$

The function is said to be *Riemann integrable* if, given any sequence of partitions $\triangle_N$ of $[a, b]$ so that

$$\triangle_{N+1} \text{ is a refinement of } \triangle_N\ \forall N, \qquad \lim_{N \to \infty} |\triangle_N| = 0,$$

we have that

$$\lim_{N \to \infty} S_*(f, \triangle_N) = \lim_{N \to \infty} S^*(f, \triangle_N) \in \mathbb{R}.$$

In this case, the *Riemann integral* of f on $[a, b]$ is the common value of the above limits. It will be denoted here by $R(f, [a, b])$, to distinguish it from the Lebesgue integral.

3. Let $[a, b]$ be an interval and $f : [a, b] \to \mathbb{R}$ be Riemann integrable.

   (a) Prove that f is $(\mathcal{M}, \mathcal{B})$-measurable ($\mathcal{M}$ is the Lebesgue σ-algebra on $[a, b]$).

   (b) Prove that f is (Lebesgue) integrable and the values of its Lebesgue and Riemann integrals coincide.

4. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{\sin x}{x}$. Prove that the limit $\lim_{a \to \infty} \int_1^a f(x) \mathrm{d}x$ exists, but f is not Lebesgue integrable.