Probability and Measure – Tutorial 2

1. **(Conditional probability)** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $A, B \in \mathcal{A}$ be events; assume that $\mathbb{P}(B) > 0$. The **conditional probability of** A **given** B is defined as
$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$
   (a) Define $\mathbb{P}_B : \mathcal{A} \to \mathbb{R}$ by $\mathbb{P}_B(A) := \mathbb{P}(A|B)$ for each $A \in \mathcal{A}$. Prove that $\mathbb{P}_B$ is a probability measure.

   (b) Now assume $B_1, B_2 \in \mathcal{A}$ with $\mathbb{P}(B_1) > 0$ and $\mathbb{P}(B_2) > 0$. Define $(\mathbb{P}_{B_1})_{B_2} : \mathcal{A} \to \mathbb{R}$ by $((\mathbb{P}_{B_1})_{B_2})(A) = \mathbb{P}_{B_1}(A|B_2)$. Prove that $(\mathbb{P}_{B_1})_{B_2} = \mathbb{P}_{B_1 \cap B_2}$.

2. **(Independence of events)** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Two events A, B are called **independent** if
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$
More generally, events $\{A_\lambda\}_{\lambda \in \Lambda}$ (indexed by some set of indices Λ) are called **independent** if, for any finite set of distinct indices $\lambda_1, \ldots, \lambda_n \in \Lambda$, we have
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_{\lambda_i}\right) = \prod_{i=1}^{n} \mathbb{P}(A_{\lambda_i}).$$
   (a) Assume A, B are independent events. Prove that A, B^c are independent, that A^c, B are independent, and that A^c, B^c are independent.

   (b) Prove that if A, B are independent and $\mathbb{P}(B) > 0$, then $\mathbb{P}(A|B) = \mathbb{P}(A)$.

   (c) Prove that A is independent of itself if and only if $\mathbb{P}(A) \in \{0, 1\}$.

   (d) Consider the probability space given by $\Omega = \{(0,0), (0,1), (1,0), (1,1)\}$, $\mathcal{F} = P(\Omega)$, $\mathbb{P}(A) = \frac{\#A}{4}$ for all $A \subset \Omega$. Let $A = \{(0,0), (1,1)\}$, $B = \{(0,0), (0,1)\}$, $C = \{(0,1), (1,1)\}$. Show that A, B are independent, A, C are independent, B, C are independent, but A, B, C are not independent.

   (e) In a general probability space, given an event A, can we guarantee that the collection of events that are independent of A is a σ-algebra?

3. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $A_n \in \mathcal{A}$, $n \in \mathbb{N}$. Recall from Tutorial 1 that
$$\limsup_{n\to\infty} A_n = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many values of } n\}.$$
In Tutorial 1, you proved that this set is measurable.

   (a) Prove that, if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then
$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 0.$$
   This is known as the **first Borel-Cantelli lemma**.

   (b) Now assume that $B_n \in \mathbb{N}$ are *independent* events and $\sum_{n=1}^{\infty} \mathbb{P}(B_n) = \infty$. Prove that
$$\mathbb{P}\left(\limsup_{n\to\infty} B_n\right) = 1.$$
   (Hint: Use the inequality $1 - s \le e^{-s}$, for $s \in \mathbb{R}$). This is known as the **second Borel-Cantelli lemma**.