Probability and Measure – Tutorial 8

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $(\Omega', \mathcal{F}')$ be a measurable space and $f : \Omega \to \Omega'$ a measurable function. Show that
$$\mu_f(A') := \mu(f^{-1}(A')), \qquad A' \in \mathcal{F}'$$
is a measure on $\mathcal{F}'$.
Note. In case $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $X : \Omega \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$ is a random variable, then the probability measure $\mathbb{P}_X$ on $\bar{\mathcal{B}}$ is called the *distribution* of X. We write
$$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}(\{\omega : X(\omega) \in B\}), \quad B \in \bar{\mathcal{B}},$$
and the expression on the right-hand side is often abbreviated as $\mathbb{P}(X \in B)$. The same definition is given for a random vector, that is, a measurable function $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}^d)$; then, $\mathbb{P}_Y$ is a probability measure on $\mathcal{B}^d$, called the distribution of the random vector.

2. Let μ be a probability measure on $(\mathbb{R}^d, \mathcal{B}^d)$. Prove that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random vector $Y : \Omega \to \mathbb{R}^d$ such that μ is the distribution of Y.

3. For a measure ν on $(\mathbb{R}, \mathcal{B})$, an *atom* is a point $x \in \mathbb{R}$ such that $\nu(\{x\}) > 0$.
   (a) Let μ be a probability measure on $(\mathbb{R}, \mathcal{B})$. Prove that there exist measures μ_c, μ_a on $\mathcal{B}$ such that $\mu = \mu_c + \mu_a$, and moreover μ_c has no atoms and μ_a is purely atomic, meaning that there exist $x_1, x_2, \ldots \in \mathbb{R}$ and $\alpha_1, \alpha_2, \ldots > 0$ such that $\mu_a = \sum_{n=1}^{\infty} \alpha_n \cdot \delta_{\{x_n\}}$. Hint: Use Exercise 2 from Tutorial 3.
   (b) Prove that the measures μ_c, μ_a satisfying the properties of item (a) are uniquely determined.

4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probaiblity space where random variables $X_1, X_2, \ldots$ and X are defined.
   (a) Prove that $X_n \xrightarrow[\text{a.s.}]{n\to\infty} X$ (that is, X_n converges to X almost surely) if and only if
$$\text{for all } \varepsilon > 0, \quad \lim_{n_0 \to \infty} \mathbb{P}\left(\bigcup_{n \geq n_0} \{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\} \right) = 0.$$
   (b) We say X_n *converges to* X *in probability* if
$$\text{for all } \varepsilon > 0, \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$$
(this is sometimes denoted as $X_n \xrightarrow[\mathbb{P}]{n\to\infty} X$). Prove that if X_n converges to X almost surely, then X_n converges to X in probability.
   (c) Take as probability space the interval $[0, 1)$ with Borel σ-algebra $\mathcal{B}$ and Lebesgue measure m. Let $X \equiv 0$. Consider the collection of all intervals of the form
$$I = i2^{-k} + [0, 2^{-k}), \quad k \in \{1, 2, \ldots\},\ i \in \{0, \ldots, 2^k - 1\}.$$
This collection of intervals is countable, so we can enumerate them as $I_1, I_2, \ldots$. Then let $X_n = \mathbb{1}_{I_n}$. Prove that X_n converges to X in probability, but not almost surely.
   (d) Prove that if X_n converges to X in probability, then there exists a subsequence $X_{n_1}, X_{n_2}, \ldots$ converging to X almost surely.