Probability and Measure – Tutorial 7

1. Let $(\Omega, \mathcal{A})$ be a measurable space. Assume we have functions

$$f_n : \Omega \to \overline{\mathbb{R}},\ n \in \mathbb{N}, \qquad J : \Omega \to \mathbb{N}.$$

   (a) Show that J is $(\mathcal{A}, P(\mathbb{N}))$-measurable if and only if $J^{-1}(k) \in \mathcal{A}$ for every $k \in \mathbb{N}$.

   (b) Assume that J is $(\mathcal{A}, P(\mathbb{N}))$-measurable and all the f_n are $(\mathcal{A}, \mathcal{B})$-measurable. Prove that the following function is measurable:

$$g(\omega) = f_{J(\omega)}(\omega).$$

2. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be $(\mathcal{B}, \mathcal{B})$-measurable. Assume $g : \mathbb{R} \to \mathbb{R}$ is equal to f except for countably many points in the domain. Show that g is $(\mathcal{B}, \mathcal{B})$-measurable.

   (b) Assume $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ are $(\mathcal{B}, \mathcal{B})$-measurable, $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all but countably many $x \in \mathbb{R}$. Show that f is $(\mathcal{B}, \mathcal{B})$-measurable. Hint: use part (a).

   (c) Prove that any right-continuous function $f : \mathbb{R} \to \mathbb{R}$ is $(\mathcal{B}, \mathcal{B})$-measurable. Hint: for each $n \in \mathbb{N}$ define

$$f_n = \sum_{z \in \mathbb{Z}} f((z+1)2^{-n}) \cdot \mathbb{1}_{(z2^{-n},(z+1)2^{-n})}$$

   and use part (b).

3. (a) Assume $(\Omega, \mathcal{A})$ is a measurable space and $f : \Omega \to \mathbb{R}^d$, $g : \Omega \to \mathbb{R}^k$ are measurable. Show that the function $h : \Omega \to \mathbb{R}^{d+k}$ given by

$$h(\omega) = (f(\omega), g(\omega))$$

   is measurable.

   (b) Prove that the function $\psi : \mathbb{R}^{2d} \to \mathbb{R}^d$ defined by

$$\psi(x_1, \dots, x_{2d}) = (x_1 + x_{d+1}, x_2 + x_{d+2}, \dots, x_d + x_{2d})$$

   is $(\mathcal{B}(\mathbb{R}^{2d}), \mathcal{B}(\mathbb{R}^d))$-measurable.

   (c) Using parts (a) and (b), give a new proof of the statement that if $f, g : \Omega \to \mathbb{R}^d$ are measurable, then $f + g$ is measurable.

4. Let $(\Omega, \mathcal{A}, \mu)$ be a complete measure space.

   (a) Show that, if $f, g : \Omega \to \overline{\mathbb{R}}$, f is measurable and $f = g$ almost everywhere, then g is measurable.

   (b) Show that, if $f_n : \Omega \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, $f : \Omega \to \overline{\mathbb{R}}$, the f_n are measurable and $f_n(\omega) \to f(\omega)$ for almost every ω, then f is measurable.