Tutorial Exercises: Probability and Measure

Richard Harnisch (S5238366)

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Tutorial 1

Exercise 1

To show that $\mathcal{A}$ is a σ-algebra, we need to verify three properties:

- Non-empty: $\Omega \in \mathcal{A}$
  Ω is a subset of itself. Additionally, $\Omega^c = \emptyset$, which is countable. Thus $\Omega \in \mathcal{A}$.

- Closed under complement: $A \in \mathcal{A} \implies A^c \in \mathcal{A}$
  First note that $A^{cc} = A$. If $A \in \mathcal{A}$, then either A or A^c is countable. Now consider A^c: It is in $\mathcal{A}$ if A^c or A^{cc}, which is A, is countable. We know this must be true. Therefore, $A \in \mathcal{A} \implies A^c \in \mathcal{A}$.

- Closed under countable unions: $A_1, A_2, \ldots, A_n \in \mathcal{A} \implies \bigcup_{i=1}^{n} A_i \in \mathcal{A}$
  For the purpose of notation, $\bigcup_{i=1}^{n} A_i := U$. We will prove this by examining two exhaustive cases.
  Case 1: There exists an A_k^c that is countable.
  By DeMorgan's law, $U^c = \bigcap_{i=1}^{n} A_i^c$. Obviously, $\bigcap_{i=1}^{n} A_i^c \subset A_k^c$. Since A_k^c is countable, U^c is too and is in $\mathcal{A}$.
  Case 2: There is no A_k^c that is countable.
  Then, every A^c must be countable to be in $\mathcal{A}$. Thus U is the union of a countable collection of countable sets and is therefore itself countable. Therefore $U \in \mathcal{A}$.

Exercise 2

(a)

This set is the intersection of all A_n. Consider the set of all A_n. Each A_n^c is in $\mathcal{A}$ by property 2 of σ-algebras, and so is their union by property three. The complement of this union must thus also be by property two, which is equal to the intersection of all A_n by applying DeMorgan's law. $\qquad\square$

(b)

This set is the intersection of all A_n^c. Consider the set of all A_n^c. Each $A_n^{c\,c} = A$ is in $\mathcal{A}$, and so is their union by property three of σ-algebras. The complement of this union must thus also be by property two, which is equal to the intersection of all A_n^c. $\qquad\square$

(c)

This is the intersection of an infinite set of A_n. The same reasoning as for (a) holds.

(d)

This is the intersection of an infinite set of A_n. As in (c), the same reasoning as for (a) holds.

Exercise 3

$$\Omega = \mathbb{N}$$
$$A = \{\emptyset, \{1\}, \{1\}^c, \mathbb{N}\}$$
$$B = \{\emptyset, \{2\}, \{2\}^c, \mathbb{N}\}$$
$$\{1\} \cup \{2\} = \{1, 2\} \notin A \cup B$$

A and B are σ-algebras, but $A \cup B$ is not by violating clusure under countable unions.

Exercise 4

To show μ is a measure on Ω, we need to verify three properties:

- **Nonnegativity:** Any p_n is a nonnegative number. Thus the sum of any collection of such p_n will also be nonnegative.

- **Null empty set:** $\mu(\emptyset) = 0$ because it is a sum over no summands.

- **Countable additivity:** $A_1, A_2, \dots \subset \mathbb{N}$ are pairwise disjoint. Since $\mu(A_i)$ is the sum of all $p_n, n \in A_i$, and all A_n are disjoint, if we take the μ of a union of As, then we are simply concatenating sequences of nonnegative numbers. Taking the sum of the whole sequence or taking it piecewise and then the sum of the pieces does not change the result. Therefore μ satisfies countable additivity.

Exercise 5

- $\Omega \in \mathcal{A}$

- $A \in \mathcal{A} \implies A^c \in \mathcal{A}$

- $A, B \in \mathcal{A}$ disjoint $\implies A \cup B \in \mathcal{A}$

Show:
$$A, B \in \mathcal{A}, A \subset B \implies B \backslash A \in A$$

$$B^c \in \mathcal{A}$$
$$B^c \cap A = \emptyset$$
$$B^c \cup A \in \mathcal{A}$$
$$(B^c \cup A)^c = B \backslash A$$
$$B \backslash A \in \mathcal{A}$$
$$\square$$

Tutorial 2

Exercise 1

$$\mathbb{P}_B = \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

(a)

To show $\mathbb{P}_B$ is a probability measure, we need to show non-negativity, countable additivity and that the total mass $\mathbb{P}(\Omega) = 1$.

1. **Non-negativity:** Take any $A \in \mathcal{A}$. Consider $\mathbb{P}_B(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$. Since σ-algebras are closed under intersections, $A \cap B \in \mathcal{A}$, so $\mathbb{P}(A \cap B) \geq 0$ since $\mathbb{P}$ is a probability measure. Since $\mathbb{P}(B) \geq 0$, both the numerator and the denominator of $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ are ≥ 0 and so the whole fraction is too. Therefore $\forall A \in \mathcal{A} : \mathbb{P}_B(A) \geq 0$.

2. **Countable additivity:** Let $(A_i)_{i \geq 1} \subset \mathcal{A}$ be pairwise disjoint, e.g. they share no elements. We need to show:

$$\mathbb{P}_B\left(\bigcup_{i \geq 1} A_i\right) = \sum_{i \geq 1} \mathbb{P}_B(A_i)$$

We can consider the left-hand side of this equation:

$$\mathbb{P}_B\left(\bigcup_{i \geq 1} A_i\right) = \frac{\mathbb{P}((\bigcup_i A_i) \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(\bigcup_i (A_i \cap B))}{\mathbb{P}(B)}$$

Recall that $\mathbb{P}$ is countably additive. Therefore:

$$\frac{\mathbb{P}(\bigcup_i (A_i \cap B))}{\mathbb{P}(B)} = \frac{\sum_{i \geq 1} \mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_{i \geq 1} \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \sum_{i \geq 1} \mathbb{P}_B(A_i)$$

This is exactly the right-hand side. Thus $\mathbb{P}_B$ is countably additive.

3. **Normalization ($\mathbb{P}(\Omega) = 1$):**

$$\mathbb{P}(\Omega) = \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$$

(b)

$$((\mathbb{P}_{B_1})_{B_2})(A) = \mathbb{P}_{B_1}(A|B_2) = \frac{\mathbb{P}_{B_1}(A \cap B_2)}{\mathbb{P}_{B_1}(B_2)} = \frac{\mathbb{P}((A \cap B_2)|B_1)}{\mathbb{P}(B_2|B_1)}$$

$$= \frac{\frac{\mathbb{P}((A \cap B_2) \cap B_1)}{\mathbb{P}(B_1)}}{\frac{\mathbb{P}(B_2 \cap B_1)}{\mathbb{P}(B_1)}}$$

$$= \frac{\mathbb{P}((A \cap B_2) \cap B_1)}{\mathbb{P}(B_2 \cap B_1)} = \frac{\mathbb{P}(A \cap (B_2 \cap B_1))}{\mathbb{P}(B_2 \cap B_1)} = \mathbb{P}_{B_2 \cap B_1}(A) = \mathbb{P}_{B_1 \cap B_2}(A)$$

Tutorial 3

Exercise 1

The σ-algebra generated by the $\{\omega\}$ sets is the smallest possible σ-algebra containing all of the $\{\omega\}$ sets. By the properties of σ-algebras, this means it contains all countable subsets of Ω and their complements. It includes all countable subsets of Ω because $\{\omega\}$ are the singleton sets of all elements in Ω. Using property 3 of σ-algebras, we can construct arbitrarily large countable sets containing any number of elements from Ω, which must be in the algebra. Using property 2, their complements must also be in it. This is precisely the definition of the collection, so the algebra is equal to the collection.

Exercise 2