Problem Set 5

DUE: Wed. Feb 24th, 2020.

1 **Course mini-project proposal and development. Continue progressing on this. Talk/email to MW and TAs as convenient. The next homework will be based on (only) the summary proposal for your project.**

PROVE:

Done

2 **TVAR models. Next week we will start looking at another class of DLMs– revisiting AR models and then their key practical and widely-used extensions to time-varying autogressive (TVAR) models. Read PW Section 5.1 (skip over 5.1.3, or at least take it "very lightly" for now) and 5.2; and review the detailed course slides on TVAR models linked to the web page.**

There is code and examples on TVAR models in the course code base. Example code/scripts explore TVAR analyses of the quarterly changes in US inflation series and a subset of the EEG series from class discussion/slides.

PROVE:

Done

3 **A start in stochastic volatility (SV) modelling. Before we get into TVAR models, we will first look a bit more about the recurrent question of observational variances in DLMs that may/appear to change over time. We will focus in detail on the venerable and widely-used discount stochastic volatility model detailed in PW, Section 4.3.7, and in (very) detailed course slides on this SV model linked to the web page. Read and explore that in preparation, and in helping with Question 4 in this homework.**

Some/most of you will likely want to integrate SV into projects later in semester, and the basic ideas and methodology in the univariate model underlie a main initial class of multivariate volatility models– i.e., time-varying variance matrices– in one of our first multivariate DLM settings coming along (PW chapter 10). This basic model is also useful as a first

model for time-varying rates in time series models for integer counts (e.g., for flows in networks ....); again, we will see examples later on.

Prove:

Done

4 **The basic distribution theory in this question underlies the venerable and widely-used discount volatility model (P & W, Section 4.3.7). We will review the basic model, ideas and results in class; here, you will visit and work through some basic theory in advance to develop conceptual and technical understanding. We will also build on this later, for other kinds of models as noted above.**

The theory in this exercise concerns aspects of a bivariate distribution for two positive scalars ϕ_0 and ϕ_1. Mapping to the SV model in DLMs is made by noting the same setup arises there with, at any times $t-1, t,$ the match $\phi_0 \leftarrow v_{t-1}^{-1}$ and $\phi_1 \leftarrow v_t^{-1}$, and then the bivariate distribution relates to $p\,(v_{t-1}, v_t|\mathcal{D}_t)$

Two positive scalar random quantities ϕ_0 and ϕ_1 have a joint distribution defined by:

- **the margin $p\,(\phi_0)$ given by $\phi_0 \sim Ga(a, b)$ for some scalars $a > 0, b > 0$; and**
- **the conditional $p\,(\phi_1|\phi_0)$ that is implicitly defined by**

$$\phi_1 = \phi_0\eta/\beta, \quad \textbf{where} \quad \eta \sim Be(\beta a, (1-\beta)a) \quad \textbf{and} \quad \eta \perp\!\!\!\perp \phi_1$$

  and where $\beta \in (0, 1)$ is a known, constant discount factor.

- **(a)** **What is $E\,(\phi_1|\phi_0)$?**
- **(b)** **What are $E\,(\phi_0)$ and $E\,(\phi_1)$?**
- **(c)** **Starting with the joint density $p\,(\phi_0)\,p(\eta)$ (a product form since ϕ_0 and η are independent), make the bivariate transformation to (ϕ_0, ϕ_1) and show that**

$$p\,(\phi_0, \phi_1) = c\mathrm{e}^{-b\phi_0}\phi_1^{\beta a-1}\,(\phi_0 - \beta\phi_1)^{(1-\beta)a-1}, \quad \textbf{on } 0 < \phi_1 < \phi_0/\beta$$

  being zero otherwise. Here c is a normalizing constant that does not depend on the conditioning value of ϕ_0 (and we do not care about the value of c for the derivations here).

(d) Derive the p.d.f. $p\left(\phi_1\right)$ (up to a proportionality constant). Deduce that the marginal distribution of ϕ_1 is $\phi_1 \sim Ga(\beta a, \beta b)$

(e) Using the technical details of your derivations above (and without much more work), show that the reverse conditional $p\left(\phi_0 | \phi_1\right)$ is implicitly defined by

$$\phi_0 = \beta\phi_1 + \gamma \quad \textbf{where} \quad \gamma \sim Ga((1-\beta)a, b) \quad \textbf{with } \gamma \perp \phi_1$$

PROVE:

(a)

$$\mathbb{E}[\phi_1|\phi_0] = \frac{\phi_0}{\beta}\mathbb{E}(\eta) = \phi_0$$

(b)

$$\begin{aligned}
\mathbb{E}[\phi_0] &= \frac{a}{b} \\
\mathbb{E}[\phi_0] &= \mathbb{E}[\mathbb{E}[\phi_1|\phi_0]] \\
&= \mathbb{E}[\phi_0] \\
&= \frac{a}{b}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathbb{P}(\phi_0, \phi_1) &= \mathbb{P}(\phi_0)\mathbb{P}(\eta = \frac{\phi_1\beta}{\phi_0})\mathbb{I}_{[0<\phi_1<\phi_0/\beta]} \\
&= \frac{b^a}{\Gamma(a)}\phi_0^{a-1}e^{-b\phi_0}\frac{(\frac{\phi_1\beta}{\phi_0})^{\beta a-1}(1-\frac{\phi_1\beta}{\phi_0})^{(1-\beta)a-1}}{\mathrm{B}(\beta a, (1-\beta)a)}\mathbb{I}_{[0<\phi_1<\phi_0/\beta]} \\
&= c\phi_0^{a-1}e^{-b\phi_0}(\frac{\phi_1\beta}{\phi_0})^{\beta a-1}(1-\frac{\phi_1\beta}{\phi_0})^{(1-\beta)a-1}\mathbb{I}_{[0<\phi_1<\phi_0/\beta]} \\
&= c\phi_0^{a-1}e^{-b\phi_0}\phi_0^{1-\beta a}(\phi_1\beta)^{\beta a-1}\phi_0^{1-(1-\beta)a}(\phi_0-\phi_1\beta)^{(1-\beta)a-1}\mathbb{I}_{[0<\phi_1<\phi_0/\beta]} \\
&= ce^{-b\phi_0}\phi_1^{\beta a-1}\left(\phi_0-\beta\phi_1\right)^{(1-\beta)a-1}\mathbb{I}_{[0<\phi_1<\phi_0/\beta]}
\end{aligned}$$

(d)

$$
\begin{aligned}
\mathbb{P}(\phi_1) &= \int_{\mathbb{R}} \mathbb{P}(\phi_0, \phi_1) d\phi_0 \\
&= c \int_{\mathbb{R}} e^{-b\phi_0} \phi_1^{\beta a-1} \left(\phi_0 - \beta\phi_1\right)^{(1-\beta)a-1} \mathbb{I}_{[0<\phi_1<\phi_0/\beta]} d\phi_0 \\
&= c\phi_1^{\beta a-1} \int_{\phi_1\beta}^{\infty} e^{-b\phi_0} \left(\phi_0 - \beta\phi_1\right)^{(1-\beta)a-1} d\phi_0 \\
&= c\phi_1^{\beta a-1} e^{-b\beta\phi_1} \int_{\phi_1\beta}^{\infty} e^{-b(\phi_0-\beta\phi_1)} \left(\phi_0 - \beta\phi_1\right)^{(1-\beta)a-1} d(\phi_0 - \phi_1\beta) \\
&\propto c' \phi_1^{\beta a-1} e^{-b\beta\phi_1} \\
\phi_1 &\sim (\beta a, \beta b)
\end{aligned}
$$

(e)

$$
\begin{aligned}
\mathbb{P}(\phi_0|\phi_1) &= \frac{\mathbb{P}(\phi_0, \phi_1)}{\mathbb{P}(\phi_1)} \\
&\propto e^{-b\phi_0} \phi_1^{\beta a-1} \left(\phi_0 - \beta\phi_1\right)^{(1-\beta)a-1} \mathbb{I}_{[0<\phi_1<\phi_0/\beta]} \phi_1^{1-\beta a} e^{b\beta\phi_1} \\
&\propto e^{-b(\phi_0-\beta\phi_1)} \left(\phi_0 - \beta\phi_1\right)^{(1-\beta)a-1} \mathbb{I}_{[0<\phi_1<\phi_0/\beta]} \\
&\propto e^{-b\gamma} \gamma^{(1-\beta)a-1} \mathbb{I}_{[0<\phi_0-\gamma<\phi_0]} \\
\phi_0 &= \beta\phi_1 + \gamma \\
\gamma &\sim Ga((1-\beta)a, b) \\
\gamma &\perp \phi_1
\end{aligned}
$$

5 **P&W Chapter** 4, **Section 4.6: Problem 3.**

In the observational variance discount model of Section 4.3.7, **prove that the beta-gamma evolution model of Equation (4.17) yields the posterior to-prior gamma distributions of Equation (4.18)**

Use the results derived in Question 4 above to answer this. (Do not redevelop technical results already shown.)

Prove:

$$
\begin{aligned}
v_t &= \beta v_{t-1}/\gamma_t \\
\phi_t &= \phi_{t-1}\gamma_t/\beta \\
(\gamma_t|\mathcal{D}_{t-1}) &\sim \text{Be}\left(\beta n_{t-1}/2, (1-\beta)n_{t-1}/2\right) \\
(\phi_{t-1}|\mathcal{D}_{t-1}) &\sim G\left(n_{t-1}/2, d_{t-1}/2\right)
\end{aligned}
$$

We just need to take $\gamma_t = \eta$ in the last question. Because $\gamma_t \perp\!\!\!\perp \mathcal{D}_{t-1}$ (this doesn't have to be true for all the models, but we can see that $(\gamma_t|\mathcal{D}_{t-1})$ doesn't carry any information from the data. Therefore, they are independent) Therefore, we can just condition all distribution in previous questions upon $\mathcal{D}_{t-1}$ without affecting integration or any other algebra.

$$(\phi_t|\mathcal{D}_{t-1}) \sim G\left(\beta n_{t-1}/2, \beta d_{t-1}/2\right)$$

6 P&W Chapter 4, Section 4.6: Problem 4

Consider the observational variance discount model of Section 4.3.7

(a) Show that the time $t-1$ prior $(\phi_{t-1}|\mathcal{D}_{t-1}) \sim G\left(n_{t-1}/2, d_{t-1}/2\right)$ combined with the beta-gamma evolution model $\phi_t = \phi_{t-1}\gamma_t/\beta$ yields a conditional density $p\left(\phi_{t-1}|\phi_t, \mathcal{D}_{t-1}\right)$ that can be expressed as $\phi_{t-1} = \beta\phi_t + v^*_{t-1}$, where

$$\left(v^*_{t-1}|\mathcal{D}_{t-1}\right) \sim G\left((1-\beta)n_{t-1}/2, d_{t-1}/2\right)$$

is independent of ϕ_t

(b) Show further that $p\left(\phi_{t-1}|\phi_t, \mathcal{D}_T\right) \equiv p\left(\phi_{t-1}|\phi_t, \mathcal{D}_{t-1}\right)$ for all $T \geq t$

(c) Describe how this result can be used to recursively compute retrospective point estimates $E\left(\phi_t|\mathcal{D}_T\right)$ backwards in time, beginning at $t = T$

(d) Describe how this result can similarly be used to recursively simulate a full trajectory of values of $\phi_T, \phi_{T-1}, \ldots, \phi_1$ from the retrospective smoothed posterior conditional on $\mathcal{D}_T$

Again, use the results derived in Question 4 above to answer this. (Do not redevelop technical results already shown.)

PROVE:

(a) Observe question 4(e). Take $v^*_{t-1} = \gamma$ in question 4(e). We get the result.

(b) There exist measurable function that maps $\sigma(v^*_{t-1})$ back to $\sigma(\phi_{t-1})$ by:

$$\phi_{t-1} = \beta\phi_t + v^*_{t-1}$$

And we know that $\sigma(v_{t-1})*$ is independent from $\mathcal{D}_{t:\infty}$. Therefore,

$$\sigma([\phi_{t-1}|\phi_t, \mathcal{D}_T]) = \sigma([\phi_{t-1}|\phi_t, \mathcal{D}_{t-1}])$$
$$\Rightarrow \mathbb{P}(\phi_{t-1}|\phi_t, \mathcal{D}_T) \equiv \mathbb{P}(\phi_{t-1}|\phi_t, \mathcal{D}_{t-1})$$

And therefore the distribution is the same.

(c) For $\mathbb{E}[\phi_t|\mathcal{D}_T]$, we first get the first term $\mathbb{E}[\phi_t|\mathcal{D}_T]$. Notice that

$$\mathbb{E}[\phi_a|\mathcal{D}_T] = \mathbb{E}_{\phi_{a+1}|\mathcal{D}_T}[\mathbb{E}[\phi_a|\phi_{a+1}, \mathcal{D}_T]]$$

Each distribution is stationary. Therefore, we will end up with a chain of ϕ_a that is stationary in both distribution and expectation.

(d) Same as above.

[Last revised: February 25, 2020]