Stock Option Pricing Using Bayes Filters

Author: Ziyang Bob Ding

March 24, 2020

Abstract

When using Black-Scholes formula to price options, the key is the estimation of the stochastic return variance. In this project, I wish to utilize EKF (extended Kalman Filter), which is an extension of Kalman Filter, a combination of the GARCH model and the implied volatilities, to model several assets and their volatility simultaneously. The technical difficulty of the project includes:

1. Using EM algorithm to calculate the empirical output for asset temporal return and their instantaneous volatility value, due to the relaxation of linear transformation from latent process to observable output

2. A derivation of modified Black-Scholes equation based on "scale of time" statement to break the original assumption, that volatility is constant, rather than rough volatility process.

Extended Kalman Filter

Original Kalman Filter take the following form:

$$\begin{aligned}
\boldsymbol{\theta}_t &= \mathbf{G}\boldsymbol{\theta}_{t-1} + \mathbf{B}\mathbf{u}_{t-1} + \mathbf{w}_t \\
x_t &= \mathbf{F}^\intercal \boldsymbol{\theta}_t + \nu_t \\
\mathbf{w}_t &\sim N(0, \mathbf{W}_t) \\
\nu_t &\sim N(0, v_t)
\end{aligned}$$

Where $\mathbf{u}_t$ is a custom control vector that we can design or eliminate.

The extended Kalman Filter takes the following form:

$$\begin{aligned}
\boldsymbol{\theta}_t &= g(\boldsymbol{\theta}_{t-1}, \mathbf{u}_{t-1}) + \mathbf{w}_t \\
x_t &= f(\boldsymbol{\theta}_t) + \nu_t \\
\mathbf{w}_t &\sim N(0, \mathbf{W}_t) \\
\nu_t &\sim N(0, v_t)
\end{aligned}$$

Where g, f are differentiable functions. Above are singular value DLMs that we have learned in class that can be used to model a single asset. This could be easily pushed into higher order (in tensor form, or summation) to model a group of assets at a time by also measuring their covariance. But the difference here is that EKF extends KF by relaxing the linear transformation from latent to observation; and evolution from time to time, into custom differentiable functions. In this way, a higher order of variability can be achieved.

Black-Schole's Formula

In this project, I will be dealing with European Option solely, considering it is easier for computation. Black-Schole's Formula is used to measure an European vanilla option's price based on the related assets (can be a portfolio of assets) return and volatility. The formula assumes the following:

1. Stock return follows quasi geometric Brownian Motion.
2. **The risk-free rate and volatility of the underlying are known and constant.**
3. **The returns on the underlying are normally distributed.**

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

The BS formula is the closed form solution of the above SPDE. It takes the following form:

$$C(S_0, t) = S_0\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$
$$d_1 = \frac{\ln\frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad d_2 = \frac{\ln\frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Where S_0 is asset price; r is risk-free interest rate; T is exercise time; K is strike price; σ is volatility; Φ is CDF of normal.

As we see that σ is required for the function, and it is constant. Having a varying volatility σ_t is a barbarous violation of assumption 3. Therefore, we'd apply **scale of time** property:

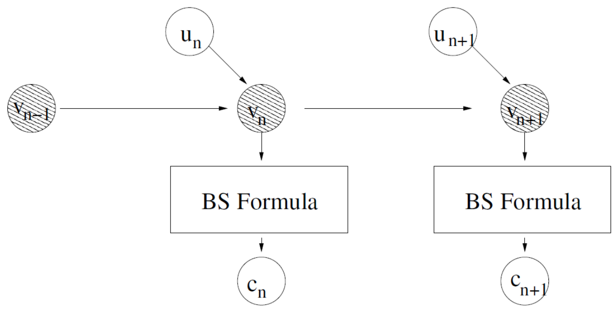
Donsker's Theorem: Let X_1 and X_2 be iid rv with $\langle X_i \rangle = 0$ and $\langle X_i^2 \rangle = 1$. Let $S(n) = \sum_{i=1}^n X_i$, and $S(t)$ be a linear interpolation of $S(n)$ for $t \geq 0$ then

$$\frac{1}{\sqrt{n}} S(nt) \Rightarrow B(t)$$

Therefore, we have

$$B(t) \sim N(0, t)$$
$$B(\alpha t) \sim N(0, \alpha t) \sim \sqrt{\alpha}N(0, t) \sim \sqrt{\alpha}B(t)$$

This is an important property, as we can "stretch" the each portion of the stock trajectory, and pretend each stretched portion has the same volatility. In this way, the Black-Scholes formula can be used.



Project Specifics

Attached is a representation of the project model. Notice the graph only represents the singular asset case, while the goal of the project is to model multiple assets at a time. Therefore, the multivariate Kalman Filter will be used.

- **Data Sources**: The data I will use are the SP 500 index option (or more) from year 1991 to 2002. The database is huge. For each day, it has option quotes for both call and put options. Then for each of them, there exist a number of different expiration dates. And for each given expiration date, there are a few possible strike prices. Then for the given expiration date and strike price, it includes the price quotes for ”ask”,”bid”,”high”,”low” and ”closing”.
- **Computation**: As direct closed form solution of updates might not be available, I will use EM algorithm for optimization. The derivation of a closed form M step is not guaranteed. If so, I will utilize ECIM in replace of EM algorithm. But generally, using EM algorithm will be the majorly considered algorithm for optimization.
- **Control**: The design of control vector $\mathbf{u}_t$ will highly likely to be eliminated in this project. However, if the project proceeds well, I will link this project to a Chaos Theory project that I've been working on, in which a robust control vector $\mathbf{u}_t$ will be designed.
- **Progress**: I've already coded up the DLM python object that I can use directly. In fact, I've used the class in another project I have finished. As the previous DLM python object is only for singular variable, I will make it multivariate.

References

[1] G. Rigatos *A Kalman filtering approach for detection of option mispricing in the Black-Scholes PDE model*. 2014 IEEE Conference on Computational Intelligence for Financial Engineering Economics (CIFEr), London, 2014, pp. 378-383.

[2] K. Liu and X. Wang *A Pragmatical Option Pricing Method Combining Black-Scholes Formula, Time Series Analysis and Artificial Neural Network*. 2013 Ninth International Conference on Computational Intelligence and Security, Leshan, 2013, pp. 149-153.

Last Revised: March 24, 2020