# Stock Option Pricing Using Bayes Filters

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April 14, 2020

# OPTION & BLACK-SCHOLES

- Options are financial instruments that are derivatives based on the value of underlying asset such as stocks.
- An options contract offers the buyer the opportunity (or promise) to buy or sell—depending on the type of contract they hold—the underlying asset.

$$y_{ti} = p_t \Phi\left(d_+\right) - K_{ti} e^{-r_t T_{ti}} \Phi\left(d_-\right)$$

where

$$d_+ = \frac{\ln\left(\frac{p_t}{K_{ti}}\right) + \left(r_t + \frac{\theta_t}{2}\right) T_{ti}}{\sqrt{\theta_t T_{ti}}}$$

$$d_- = d_+ - \sqrt{\theta_t T_{ti}}$$

Where $p_t$ is asset price; $r_t$ is risk-free interest rate; $T_{ti}$ is exercise time; $K_{ti}$ is strike price; $\theta_t$ is volatility; $\Phi$ is CDF of normal.

# OPTION & BLACK-SCHOLES

- SPX 500 Option Data (04/01/2017 -03/31/2020)

| symbol | code | OptionType | expirydate | date | close | change | bid | ask | volume | openinterest | strike | spotclose | close2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| SPX | SPX1721D500 | call | 20170421 | 20170403 | 1761.1 | 0 | 1855.9 | 1860.2 | 0 | 14 | 500 | 2358.84 | 1761.1 |
| SPX | SPX1721D1000 | call | 20170421 | 20170403 | 1355 | 0 | 1356.2 | 1360.7 | 0 | 6857 | 1000 | 2358.84 | 1355 |
| SPX | SPX1721D1100 | call | 20170421 | 20170403 | 1279.8 | 0 | 1256.5 | 1260.7 | 0 | 20 | 1100 | 2358.84 | 1279.8 |
| SPX | SPX1721D1200 | call | 20170421 | 20170403 | 0 | 0 | 1156.3 | 1160.8 | 0 | 0 | 1200 | 2358.84 | 1158.55 |
| SPX | SPX1721D1225 | call | 20170421 | 20170403 | 0 | 0 | 1131.3 | 1135.6 | 0 | 0 | 1225 | 2358.84 | 1133.45 |
| SPX | SPX1721D1275 | call | 20170421 | 20170403 | 0 | 0 | 1081.6 | 1085.8 | 0 | 0 | 1275 | 2358.84 | 1083.7 |
| SPX | SPX1721D1300 | call | 20170421 | 20170403 | 0 | 0 | 1056.4 | 1060.9 | 0 | 0 | 1300 | 2358.84 | 1058.65 |
| SPX | SPX1721D1325 | call | 20170421 | 20170403 | 0 | 0 | 1031.4 | 1035.7 | 0 | 0 | 1325 | 2358.84 | 1033.55 |
| SPX | SPX1721D1350 | call | 20170421 | 20170403 | 0 | 0 | 1006.4 | 1010.9 | 0 | 0 | 1350 | 2358.84 | 1008.65 |
| SPX | SPX1721D1375 | call | 20170421 | 20170403 | 0 | 0 | 981.4 | 985.7 | 0 | 0 | 1375 | 2358.84 | 983.55 |
| SPX | SPX1721D1400 | call | 20170421 | 20170403 | 938 | 0 | 956.5 | 960.7 | 0 | 1 | 1400 | 2358.84 | 938 |
| SPX | SPX1721D1425 | call | 20170421 | 20170403 | 0 | 0 | 931.7 | 935.9 | 0 | 0 | 1425 | 2358.84 | 933.8 |
| SPX | SPX1721D1450 | call | 20170421 | 20170403 | 0 | 0 | 906.5 | 911 | 0 | 0 | 1450 | 2358.84 | 908.75 |
| SPX | SPX1721D1475 | call | 20170421 | 20170403 | 0 | 0 | 881.6 | 885.8 | 0 | 0 | 1475 | 2358.84 | 883.7 |
| SPX | SPX1721D1480 | call | 20170421 | 20170403 | 0 | 0 | 876.5 | 881 | 0 | 0 | 1480 | 2358.84 | 878.75 |
| SPX | SPX1721D1490 | call | 20170421 | 20170403 | 0 | 0 | 866.5 | 871 | 0 | 0 | 1490 | 2358.84 | 868.75 |

# EKF & GARCH(1, 1)

- $p_t$ Stock price on day $t$.
- $u_t$ Stock return on day $t$. $u_t = \frac{p_t - p_{t-1}}{p_{t-1}}$
- $\theta_t$ Stock volatility on day $t$
- $y_{it}$ the ith option price on day $t$

$$
\begin{aligned}
\theta_t &= \gamma V_L + \alpha u_t^2 + \beta \theta_{t-1} + w_t \\
w_t &\sim \mathcal{N}(0, W) \\
\gamma, \alpha, \beta &> 0 \\
\gamma + \alpha + \beta &= 1
\end{aligned}
$$

Where $V_L$ is the long-run average variance rate $V_L$. To better help our computation, we may relax our boundary constraint condition by treating $\gamma V_L$ as $\omega$, therefore the model becomes:

$$
\begin{aligned}
\theta_t &= \omega + \alpha u_t^2 + \beta \theta_{t-1} + w_t && (1) \\
w_t &\sim \mathcal{N}(0, W) && (2) \\
\omega, \alpha, \beta &> 0 && (3) \\
\alpha + \beta &< 1 && (4)
\end{aligned}
$$

$$
\begin{aligned}
\theta_t &= g_t(\theta_{t-1}, u_t, w_t) \\
y_{it} &= f_{ti}(\theta_t, p_t, K_{ti}, T_{ti}) + \nu_t \\
\nu_t &\sim N(0, v)
\end{aligned}
$$

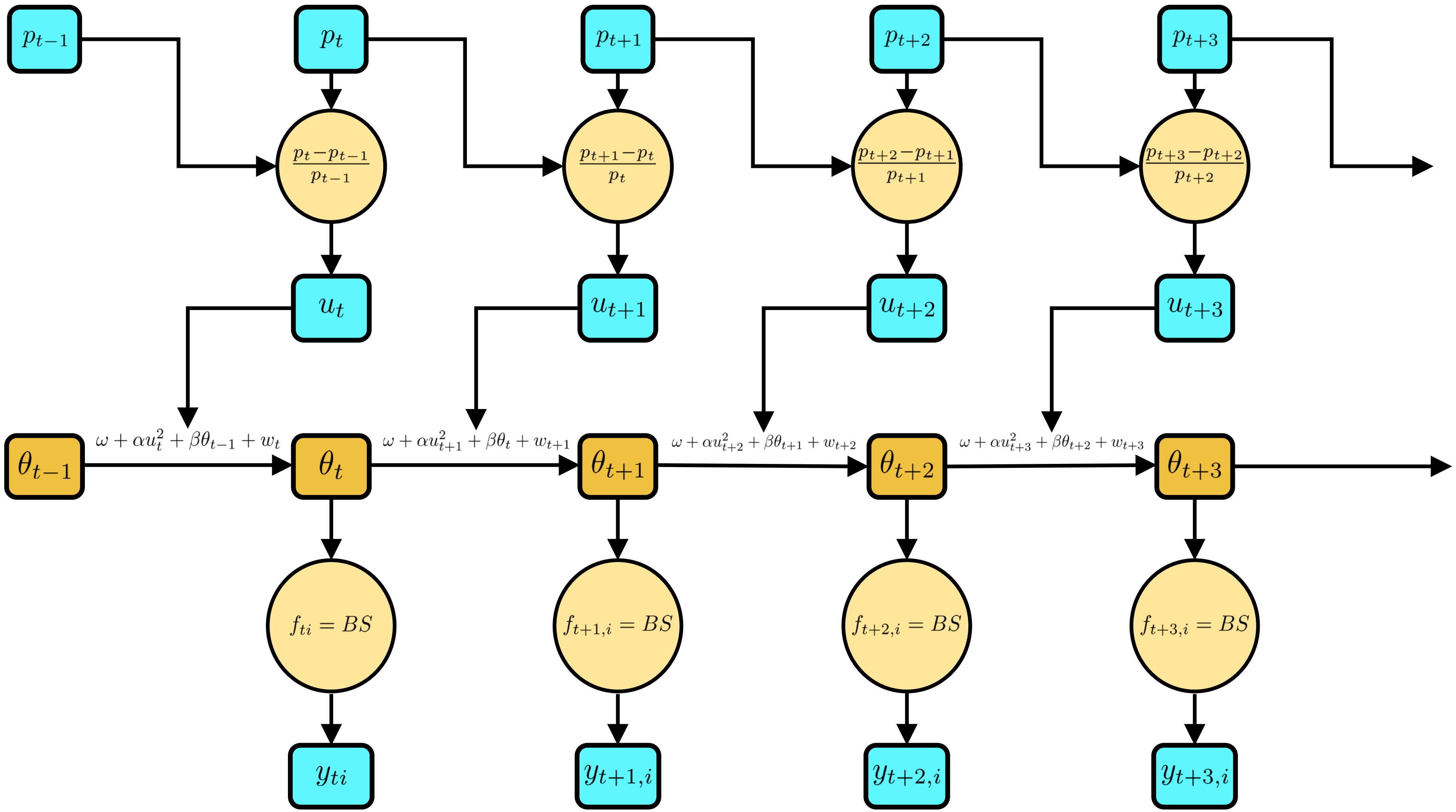
Where

$$g(\theta_{t-1}, u_t, w_t) = \omega + \alpha u_t^2 + \beta \theta_{t-1} + w_t$$

is the GARCH(1,1) process. In other word, the GARCH(1,1) is representing the dynamics inside the EKF. And

$$f_{ti}(\theta_t, p_t, K_{ti}, T_{ti}) = BS(\theta_t, p_t, K_{ti}, T_{ti})$$

# MODEL ILLUSTRATION



# PARAMETER INFERENCE

The Model requires statistical inference uppon the following parameters:

$$\{\omega, \alpha, \beta, W, v\}$$

We develop the MCMC algorithm that

1. Set priors $\mathbb{P}((\omega, \alpha, \beta)^\top), \mathbb{P}(W), \mathbb{P}(v)$. Random initialize.
2. Forward Filtering: Update all $m_t, C_t, \forall t \in 1:T$
3. Backward Smoothing: Sample $\theta_{1:T}$ from $\mathbb{P}(\theta_{1:T}|\mathcal{D}_T)$ by recursively sample $\theta_{t-1}$ from $\mathbb{P}(\theta_{t-1}|\theta_t, \mathcal{D}_T)$ using EKF smoothing.
4. Linear Regression with Constrains: Sample $(\omega, \alpha, \beta)^\top, W$ from $\mathbb{P}(\omega, \alpha, \beta, W|\theta_{1:T}, \mathcal{D}_T, v) = \mathbb{P}(\omega, \alpha, \beta, |W, \theta_{1:T})\mathbb{P}(W|\theta_{1:T})$ with rejection sampling for constraints of equation (3) and (4).
5. Inverse Gamma: Sample $v$ from $\mathbb{P}(v|\theta_{1:T}, \mathcal{D}_T, W, \omega, \alpha, \beta) = \mathbb{P}(v|\theta_{1:T}, \mathcal{D}_T)$
6. Repeat 2,3,4,5 until converges.

# PARAMETER INFERENCE