Problem Set 7

DUE: Mon. Mar 23rd, 2020.

1 **Consider the Markov chain transition matrix:**

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

(a) Identify the communicating classes.

(b) Which states are recurrent, and which are transient?

(c) Find all the invariant (stationary) distributions.

PROVE:

(a)

$$\{3\}, \{2,4\}, \{1,5\},$$

(b)
- State 1: recurrent
- State 2: transient
- State 3: recurrent
- State 4: transient
- State 5: recurrent

(c) First, find the eigenvector with eigenvalue = 1, then normalizing sum of entry to 1. We end up with $(0.5, 0, 0, 0, 0.5), (0, 0, 1, 0, 0)$. This shows that

$$\pi_a = \begin{pmatrix} \frac{a}{2} \\ 0 \\ 1-a \\ 0 \\ \frac{a}{2} \end{pmatrix} \quad \forall a \in [0,1]$$

is a limiting distribution.

2 **Suppose $X_0, X_1, X_2, \ldots$ forms a Markov chain with initial distribution π_0 and transition matrix P. Show that $Y_t = X_{kt}$ forms a Markov chain with initial distribution π_0 and transition matrix P^k**

PROVE:

$$
\begin{aligned}
Y_0 &= X_0 \sim \pi_0 \\
Y_{t+1} &= X_{k(t+1)} = P^k X_{kt} = P^k Y_t
\end{aligned}
$$

3

(a) Show a finite state Markov chain is aperiodic if $\exists i$ such that $P_{ii} > 0$

(b) Show that for two communicating states i, j of a Markov chain, i is recurrent iff j is recurrent.

(c) Show that a finite state aperiodic irreducible Markov chain is regular.

PROVE:

(a) By definition of aperiocity, we know that as $\exists i$ such that $P_{ii} > 0$. Then the gcd of all states must be 1, leaving the chain to be aperiodic.

(b) Suppose $j \sim i$, define $\tau_{ij} = \inf\{t|X_t = j, X_0 = i\}$, and we know that $P(\tau_{ii} < \infty) = 1$; By Borel Cantelli, $P(\tau_{ij} < \infty) = 1$, and $P(\tau_{ji} < \infty) = 1$. Therefore, we have:

$$
\tau_{jj} = \inf\{t|X_t = j, X_0 = j\} \le \tau_{ji} + \tau_{ii} + \tau_{ij} \le \infty
$$

Therefore, $\tau_{jj} < \infty$ almost surely. And j is recurrent.

(c) Denote $t_{ij} := \inf\{t|P_{ij}^t > 0\}$. Because we have finite i and j, therefore, we have finite t_{ij}. Due to irreduciblility, $\forall i, j \in \mathcal{X}, t_{ij} < \infty$. Now take $T = \prod_{i,j\in\mathcal{X}}$, we find that $P_{ij}^T > 0 \forall i, j \in \mathcal{X}$.

4 **Each morning a student takes one of the three books he owns from his shelf. The books are chosen with probabilities $\alpha_i, i = 1, 2, 3,$ with choices independent across days. In the evening, he replaces the book at the left end of the shelf. Let p_n be the probability that on day n the student finds the books in order $1, 2, 3$ from left to right. Show that, regardless of the initial arrangement of the books, p_n converges as $n \to \infty$ and determine the limit.**

PROVE:

Solve $\pi = \pi P$. Note that we only need π_1 : from the first equation:

$$
\pi_1 = \frac{(\pi_3 + \pi_4)\,\alpha_1}{\alpha_2 + \alpha_3}
$$

	1,2,3	1,3,2	2,1,3	2,3,1	3,1,2	3,2,1
1,2,3	α_1	0	α_2	0	α_3	0
1,3,2	0	α_1	α_2	0	α_3	0
2,1,3	α_1	0	α_2	0	0	α_3
2,3,1	α_1	0	0	α_2	0	α_3
3,1,2	0	α_1	0	α_2	α_3	0
3,2,1	0	α_1	0	α_2	0	α_3

Add the third and the fourth equation:

$$\pi_3 + \pi_4 = \alpha_2$$

We get $\lim_{n\to\infty} p_n = \pi_1 = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_3}$

[Last revised: March 24, 2020]