Lecture 12

Linear Control of Manipulators

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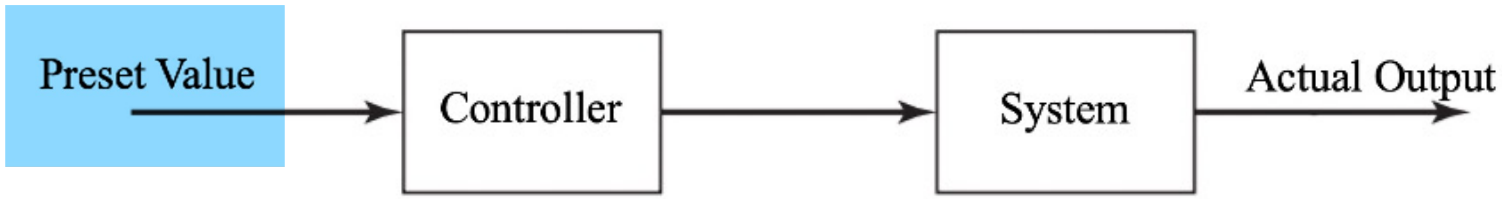
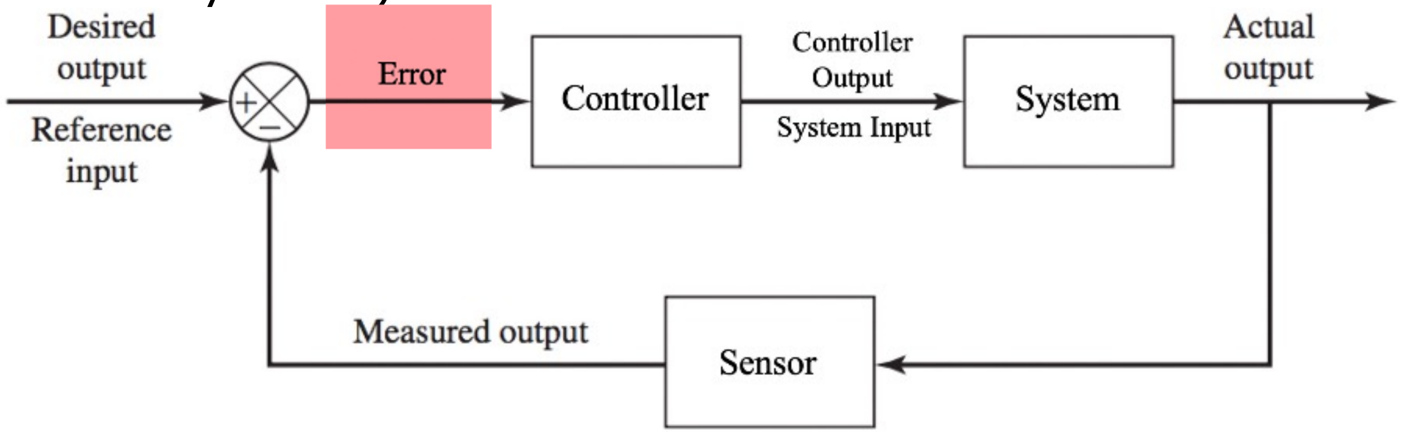
Control concepts

- **What is a system?**
  - A system gives you an output (response) when you apply an input.
  - How a system behaves/responds to an input depends on the dynamic model of the system (nature of the system).
- **Can we change the dynamics (nature) of a system?**
  - Never!
- **Can we change the output (response) of a system?**
  - Only by changing the input to the system. Different inputs to a system result in different outputs.
- **What is a Controller?**
  - Anything that play with the system's input to result in a desired output (response).

Control concepts

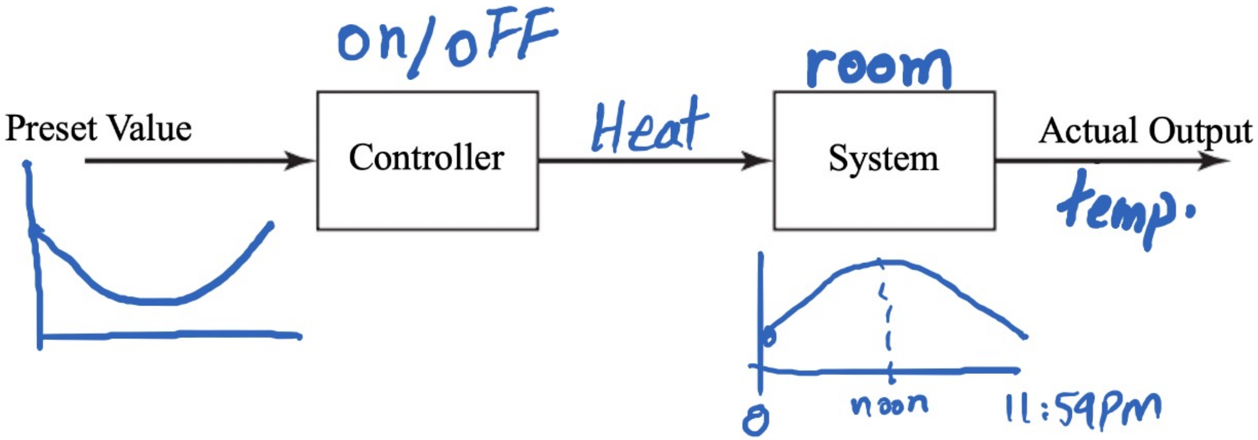


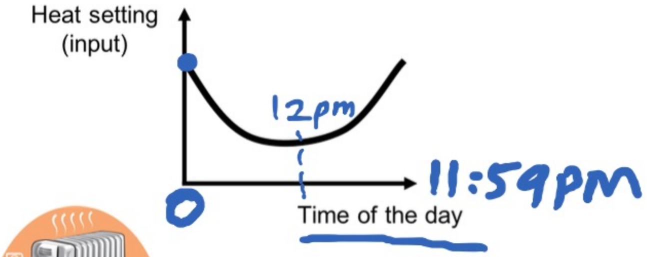
- **How does a controller work?**
  - A controller has its own input and output.
  - A controller sits right before the system in the block diagram.
  - Output of a controller is the input to the system.
  - Input of a controller is:
    - A **preset value** (in <u>feedforward (open-loop)</u> control systems)
    - **Error** (in <u>feedback (closed-loop)</u> control systems)



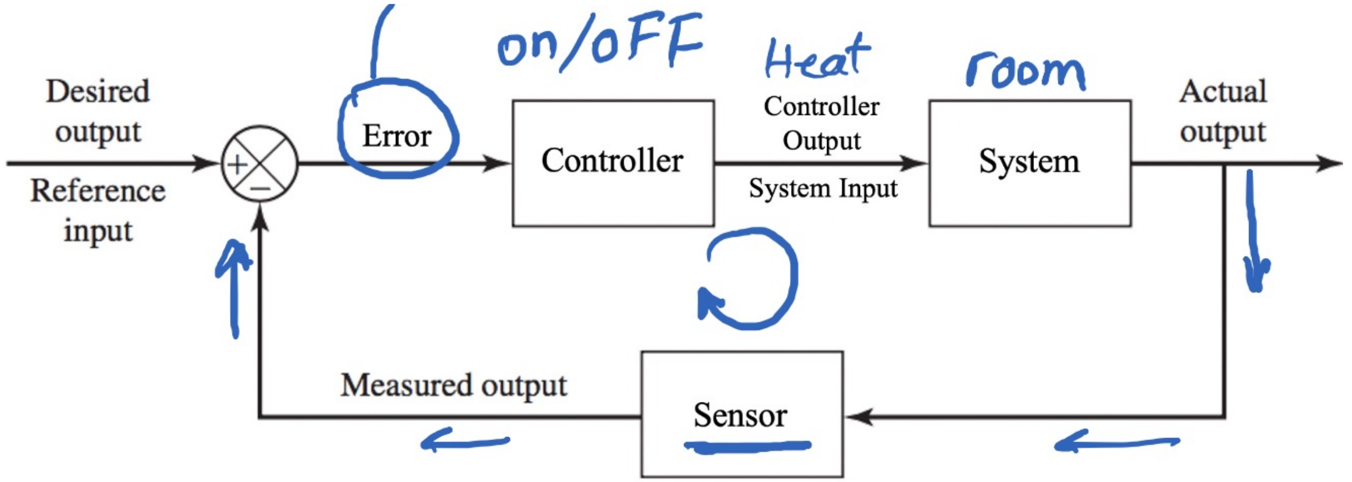
https://www.electronics-tutorials.ws/systems/open-loop-system.html

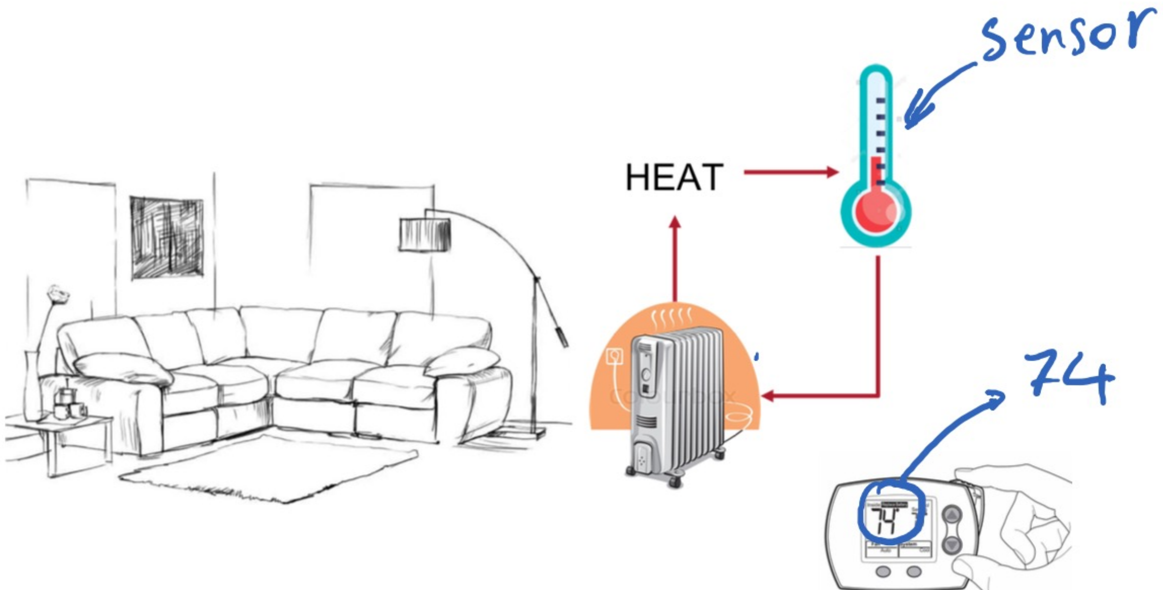
Open-Loop Control



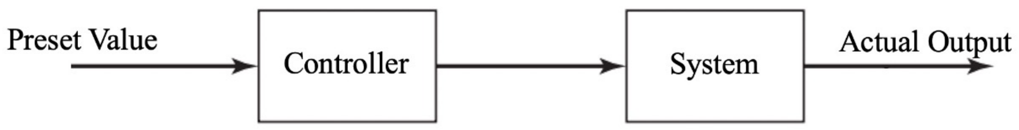


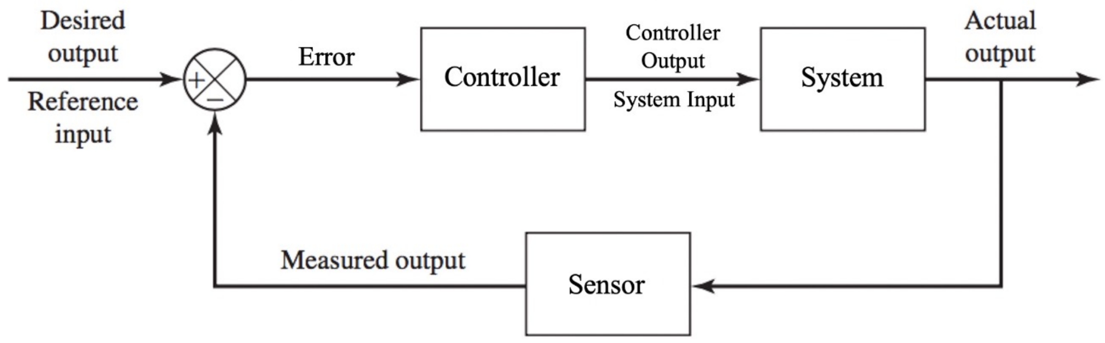
Closed-Loop Control





Open-Loop vs. Closed-Loop Control systems

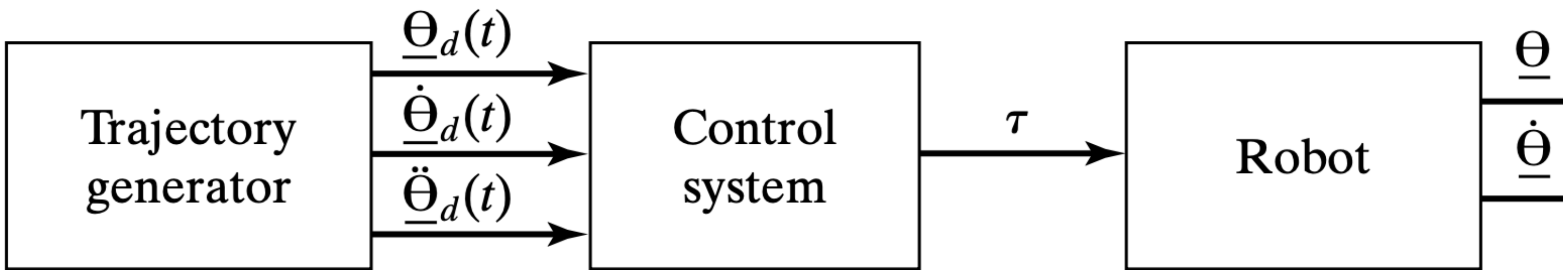




Open-Loop	Closed Loop
Not accurate if the exact model is not known	Accurate
Not robust to changes, disturbances, etc.	A degree of robustness to changes
High-performance is not essential	High-performance can be achieved
Very simple, does not need sensors	More complex, needs sensors

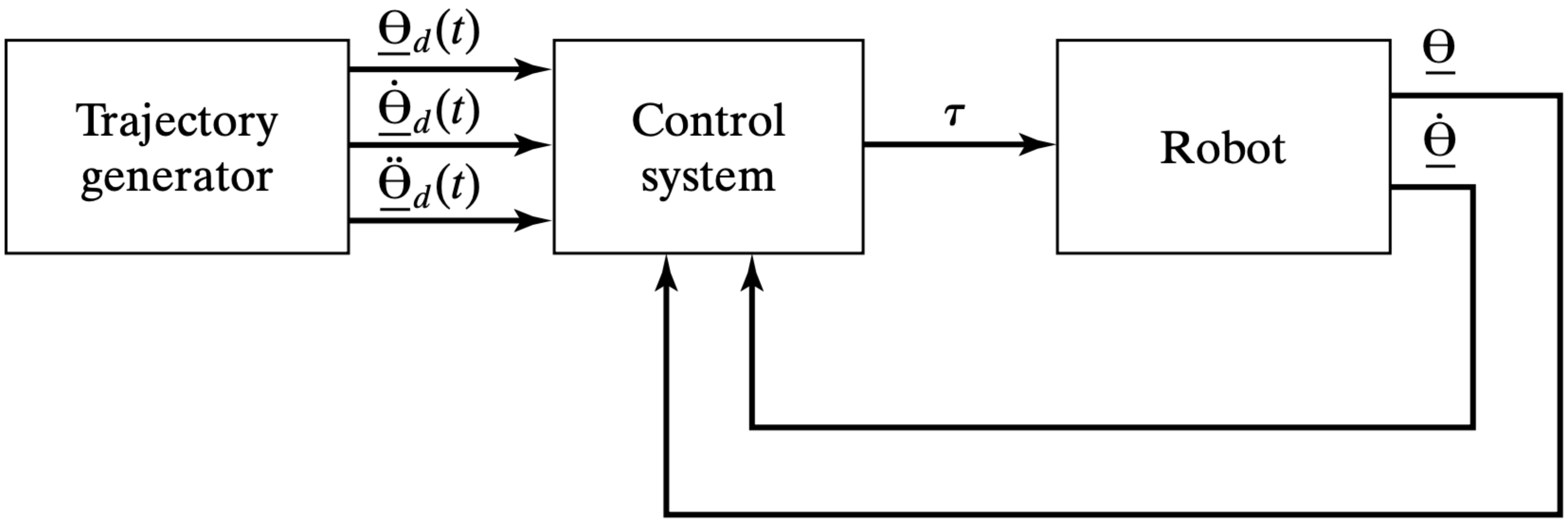
In Robotics, we are interested in Closed-loop Control of systems

Control System



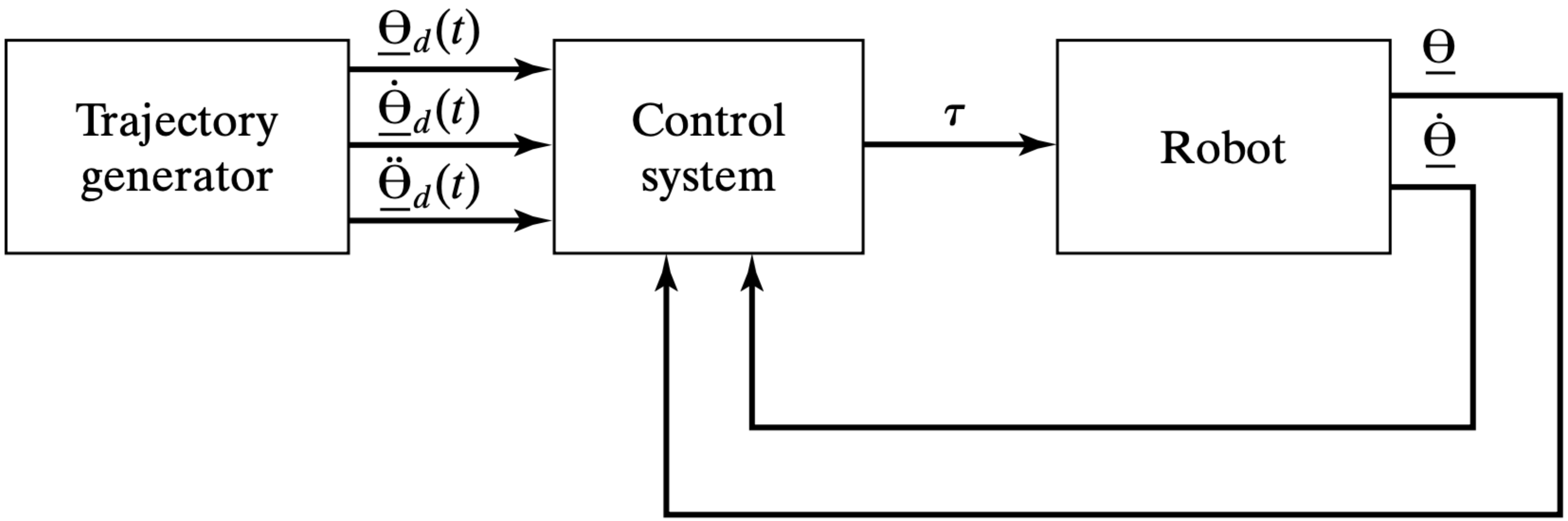
- **Torque τ**:
  the amount of rotational force that the motor develops
- **Control system**:
  - We wish to cause the manipulator joints to follow prescribed position trajectories, but the actuators are commanded in terms of torque.
  - Compute appropriate actuator commands that will realize this desired motion.

Closed-Loop Control



- **Feedback**:
  – To build a high-performance control system is to make use of feedback from joint sensors

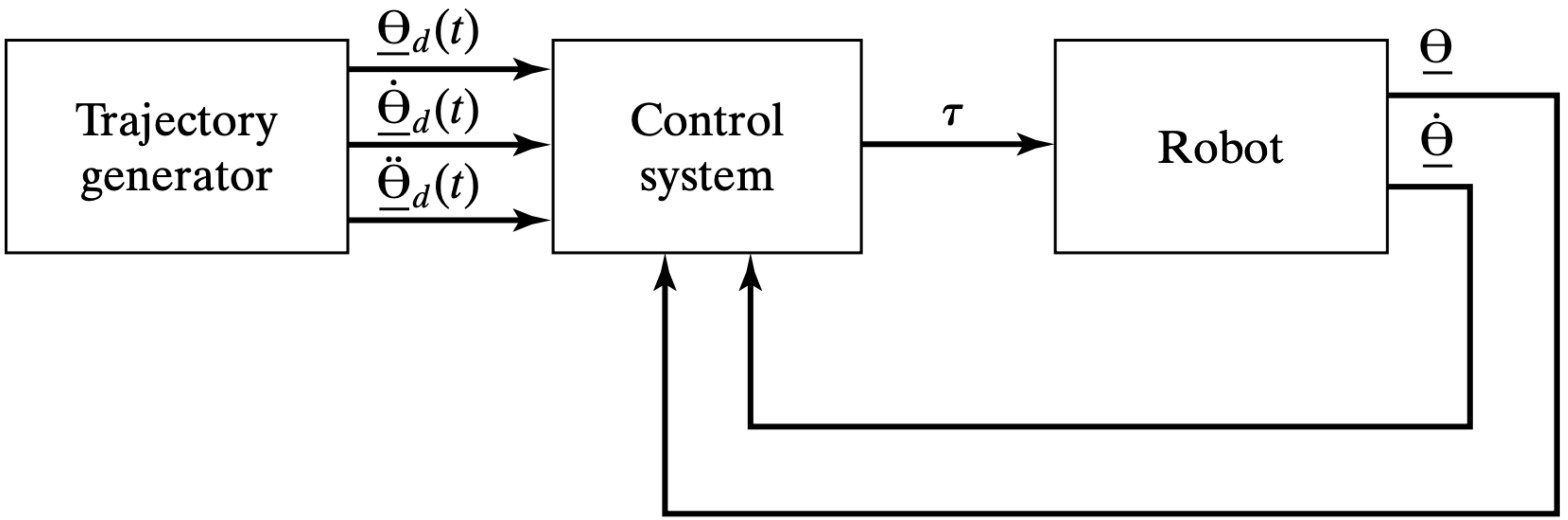
Closed-Loop Control



- **Dynamic equation**:

$$\tau = M(\Theta_d)\ddot{\Theta}_d + V(\Theta_d, \dot{\Theta}_d) + G(\Theta_d).$$

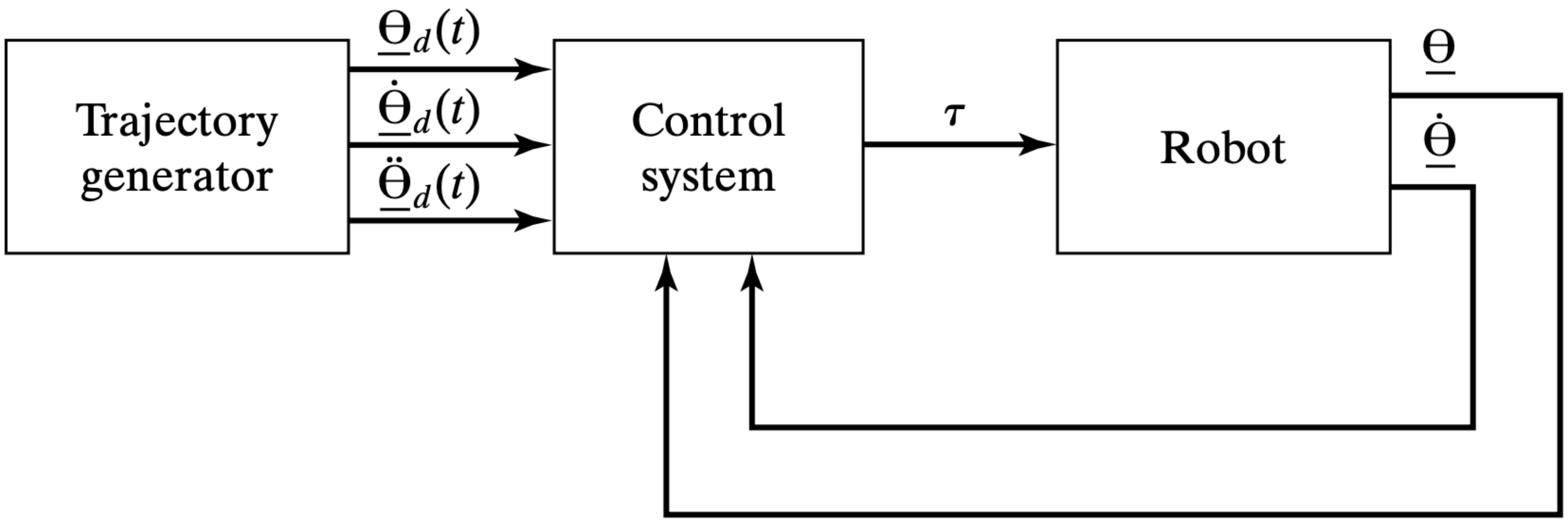
Closed-Loop Control



- **Error**:

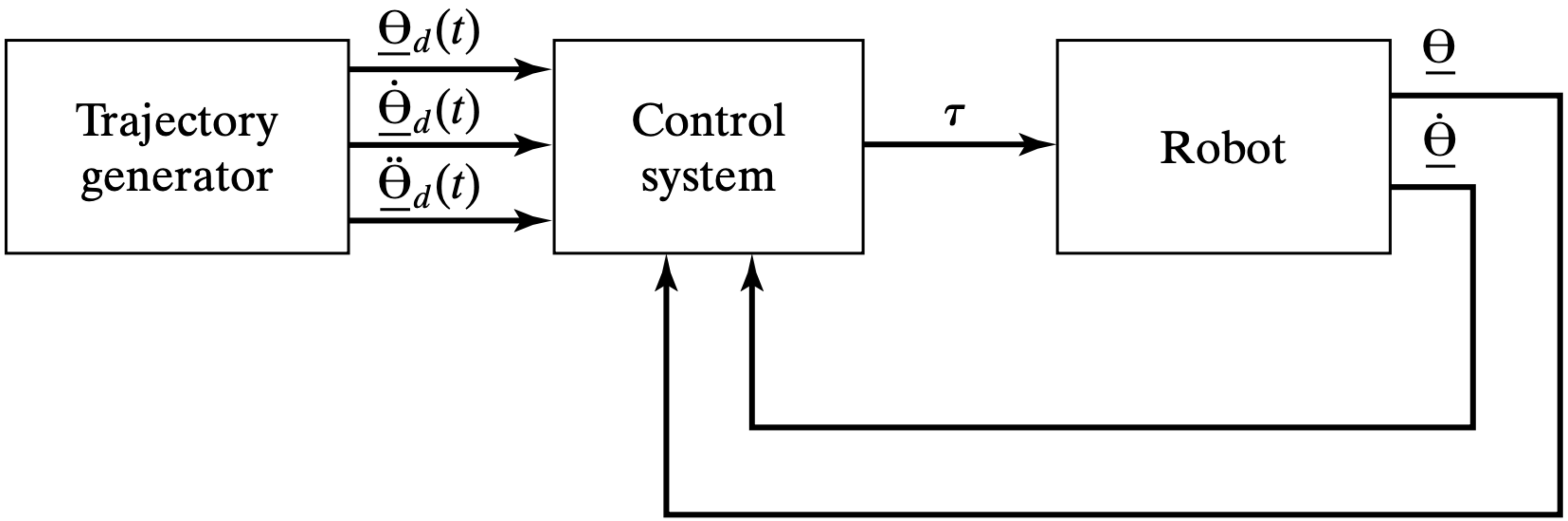
$$E = \Theta_d - \Theta,$$
$$\dot{E} = \dot{\Theta}_d - \dot{\Theta}.$$

Closed-Loop System



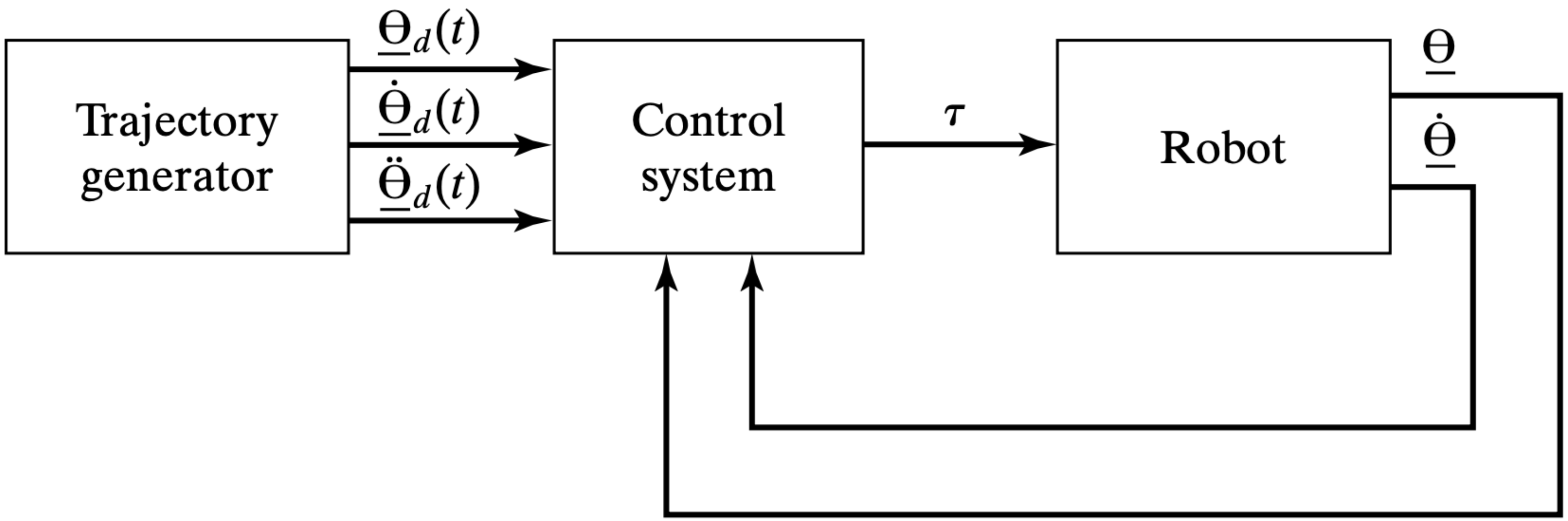
- **Closed-Loop System**:
  - The control system can compute how much torque to require of the actuators as some function of the error.
- **Basic idea:** to compute actuator torques that would tend to reduce the error.

Closed-Loop System - Requirements



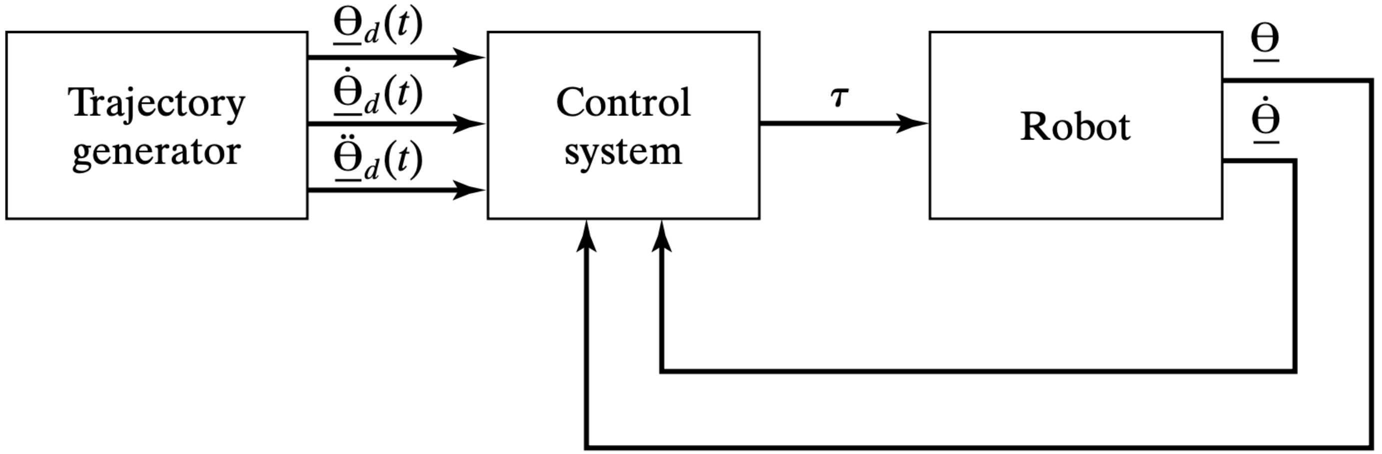
- The central problem in designing a control system is to ensure that the resulting closed-loop control system meets certain requirements.
- **Stability**:
  - A system **is stable** if the errors remain “small” when executing various desired trajectories, even in the presence of some “moderate” disturbances.
  - An improperly designed control system can sometimes result in **unstable** performance, in which errors are increased instead of reduced.

Closed-Loop System - Requirements

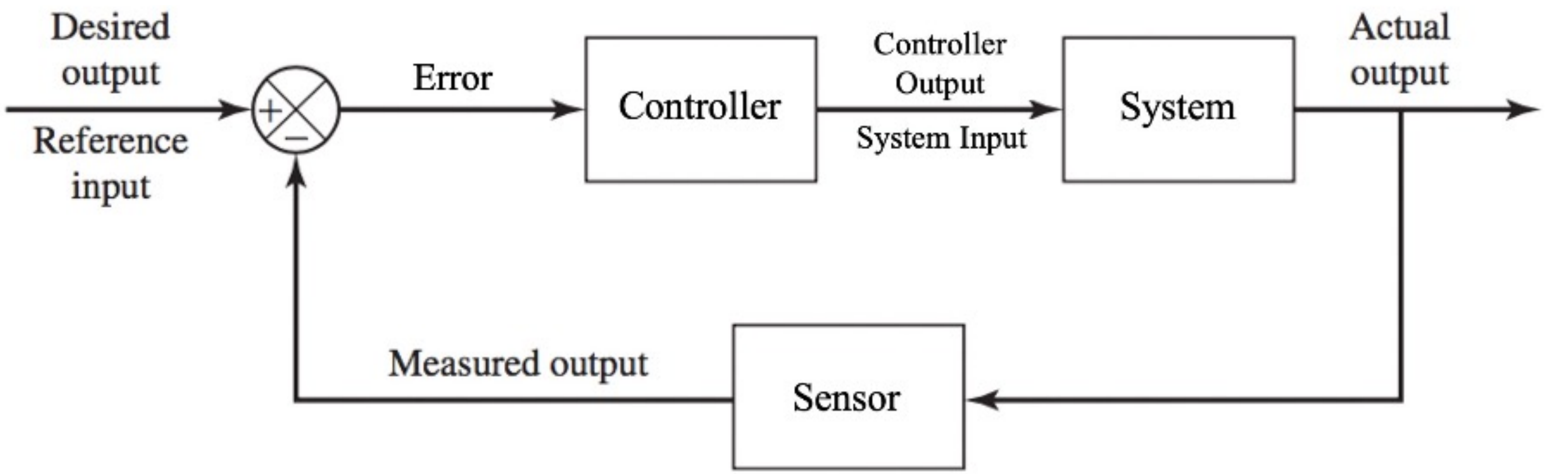


- Tasks of a Control Engineer:
  - To prove that the control design yields a **stable** system
  - To prove that the closed-loop performance of the system is **satisfactory**.

MIMO vs. SISO



MIMO: multi-input, multi-output

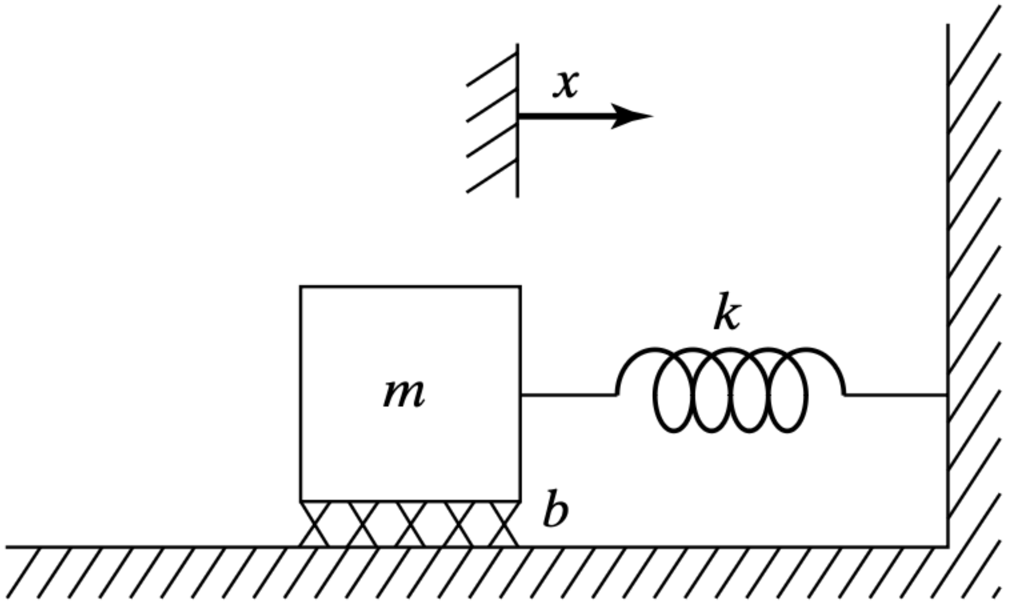


SISO: single-input, single-output

Second-order Linear System

- Spring-Mass system with Friction
  - Mass ***m***
  - spring of stiffness ***k***
  - Friction of coefficient ***b***
  - Equation of motion:

$$m\ddot{x} + b\dot{x} + kx = 0.$$



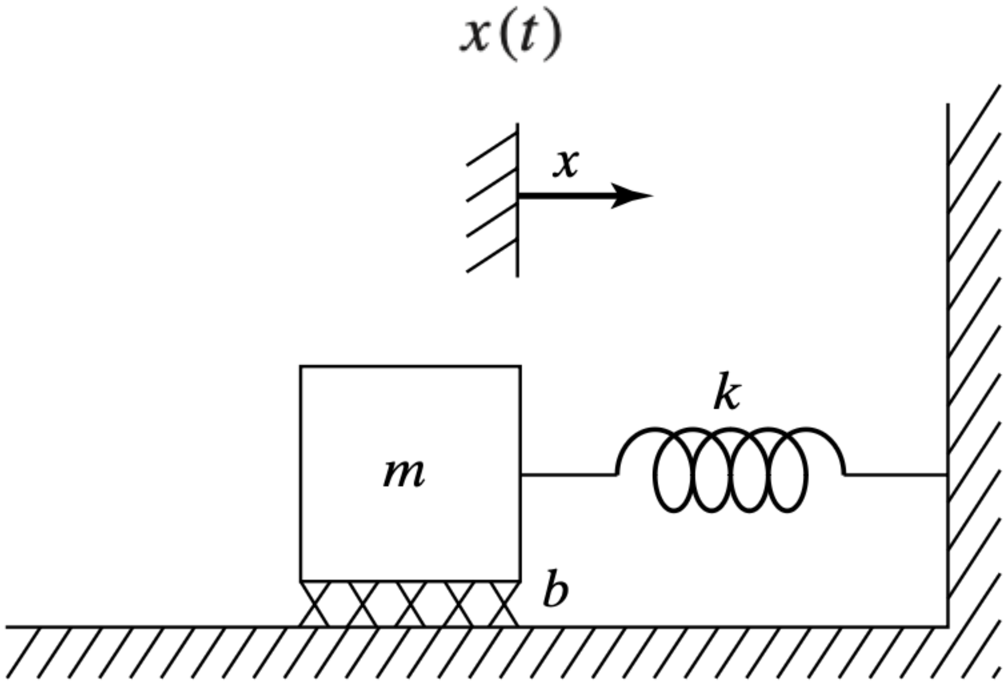
Second-order Linear System - Laplace transform

- Equation of motion: $m\ddot{x} + b\dot{x} + kx = 0.$

- Characteristic equation:

$$ms^2 + bs + k = 0.$$

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m},$$
$$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}.$$



Second-order Linear System - Response

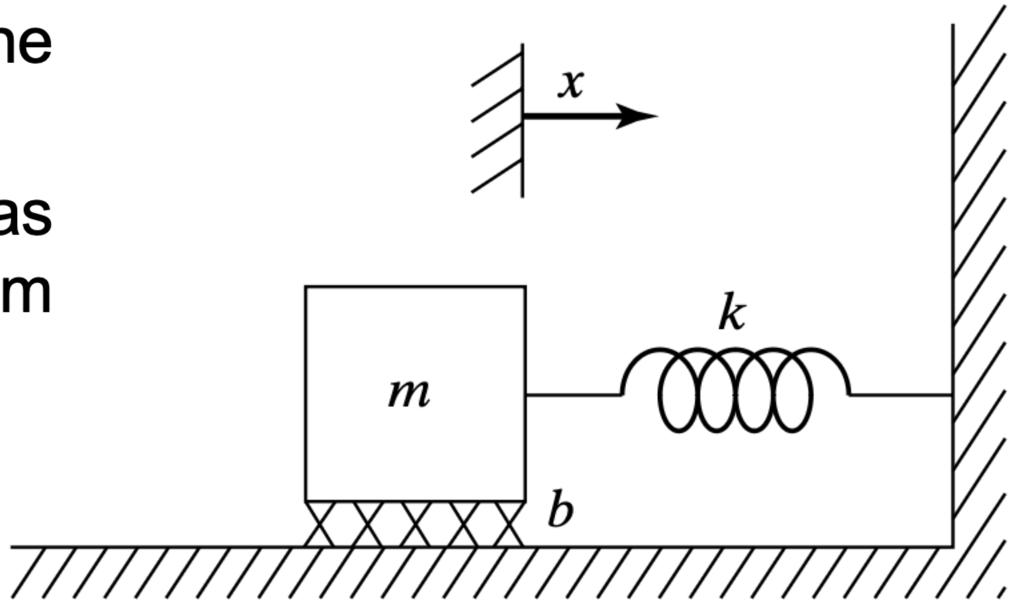
1. **Real and Unequal Roots**. This is the case when $b^2 > 4\,mk$; that is, friction dominates, and sluggish behavior results. This response is called **overdamped**.
2. **Complex Roots**. This is the case when $b^2 < 4\,mk$; that is, stiffness dominates, and oscillatory behavior results. This response is called **underdamped**.
3. **Real and Equal Roots**. This is the special case when $b^2 = 4\,mk$; that is, friction and stiffness are "balanced," yielding the fastest possible nonoscillatory response. This response is called **critically damped**.

$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m},$$

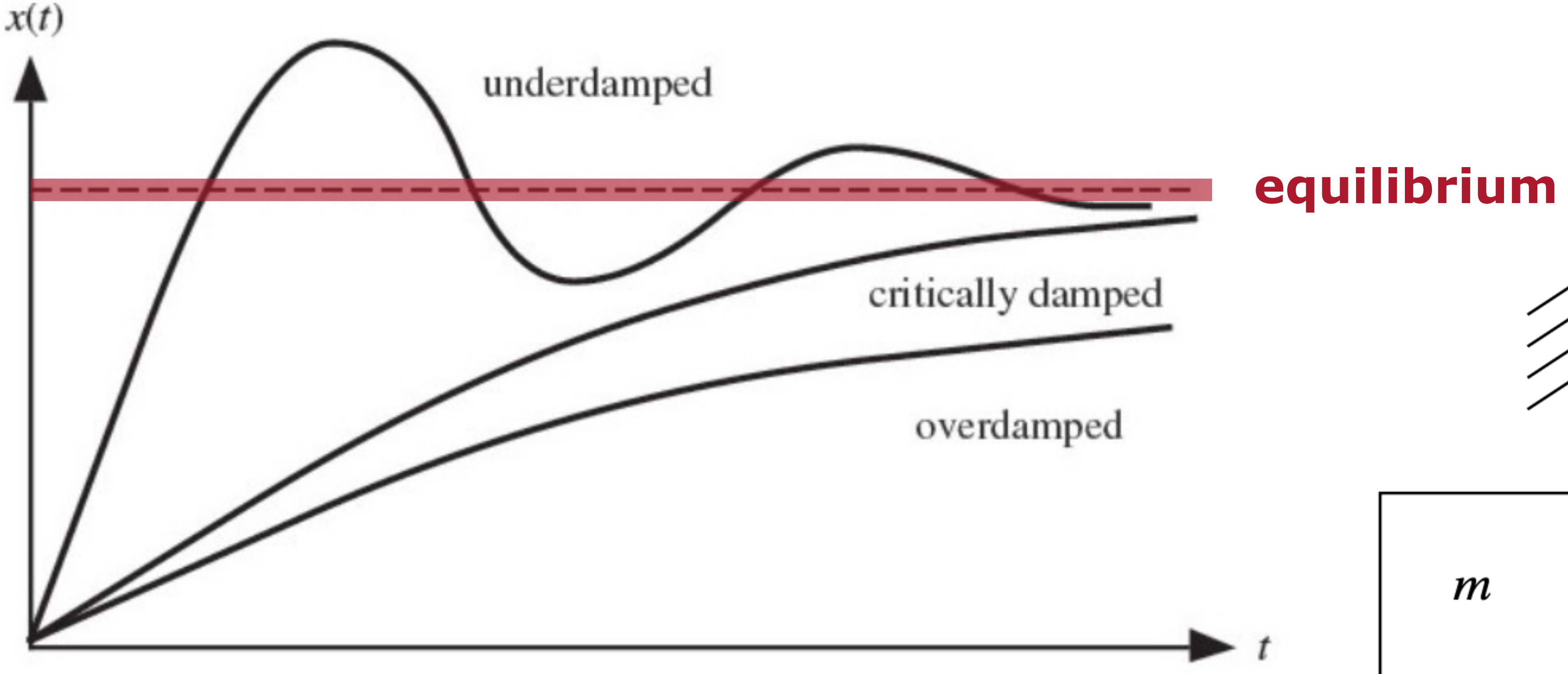
$$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}.$$

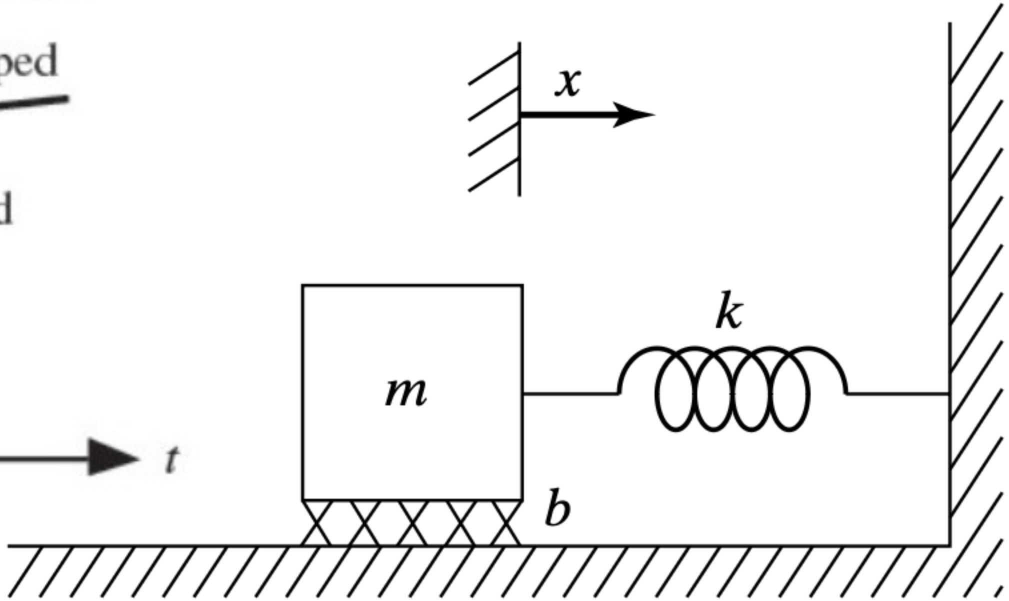
Overdamped, Underdamped, Critically damped Systems

- An **overdamped system** moves slowly toward equilibrium.
- An **underdamped system** moves quickly to equilibrium, but will oscillate about the equilibrium point as it does so.
- A **critically damped system** moves as quickly as possible toward equilibrium without oscillating about the equilibrium.



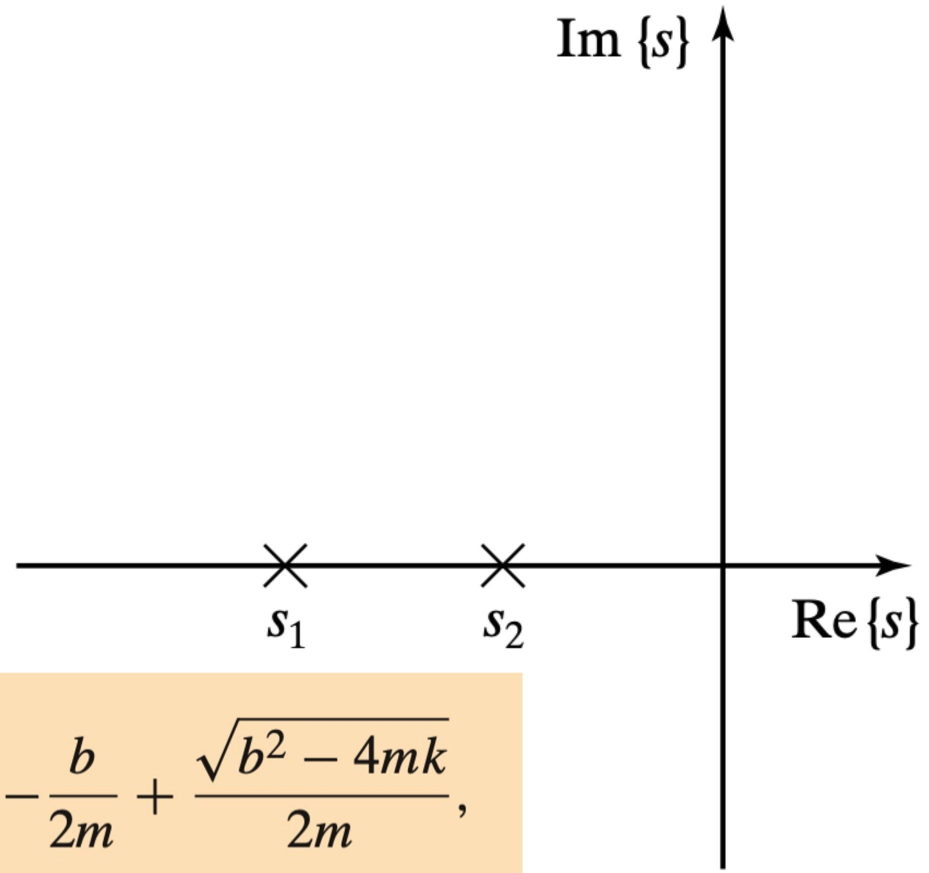
Overdamped, Underdamped, Critically damped Systems

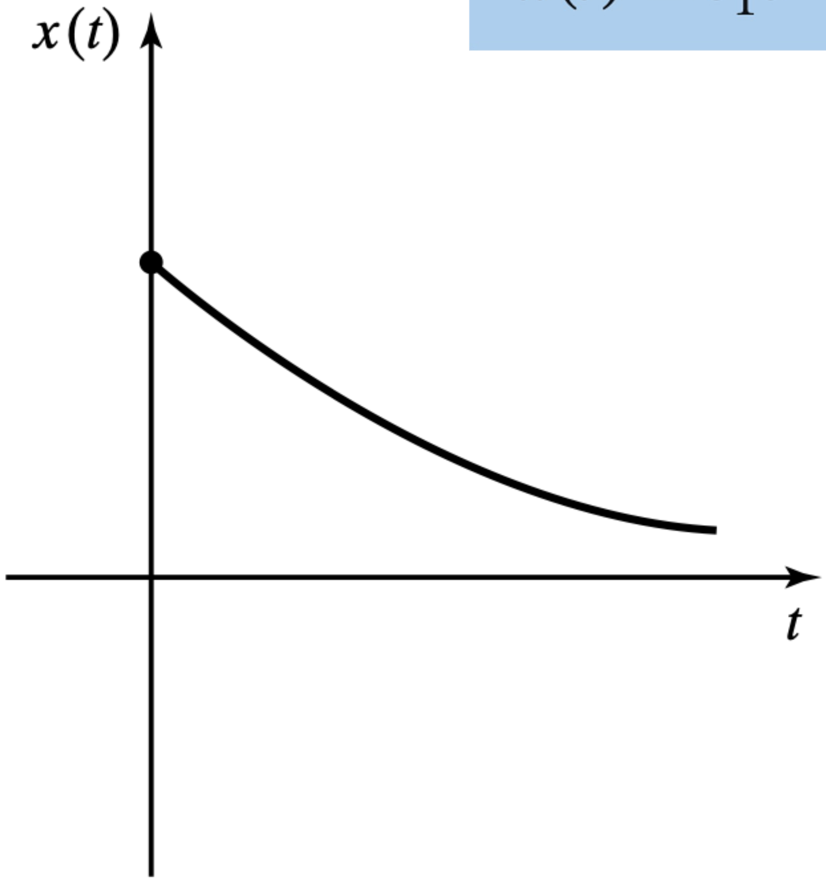




1. Overdamped (Real and Unequal Roots)

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t},$$



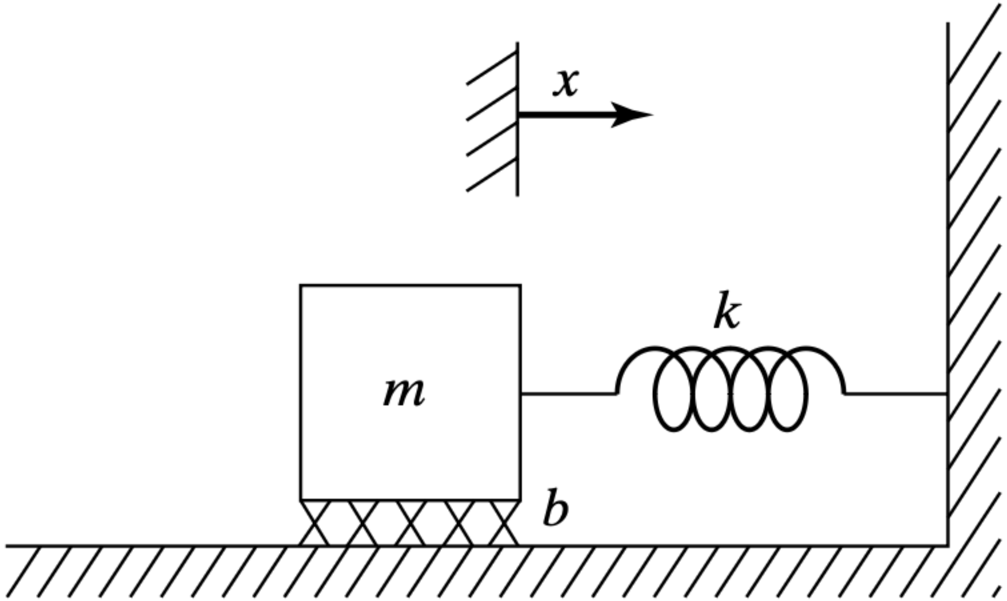


$$s_1 = -\frac{b}{2m} + \frac{\sqrt{b^2 - 4mk}}{2m},$$

$$s_2 = -\frac{b}{2m} - \frac{\sqrt{b^2 - 4mk}}{2m}.$$

1. Overdamped - Example 1

- Determine the motion of the system below if the parameter values are *m*=1, *b*=5, and *k*=6, and the mass is initially at rest and is released from the position x = -1.



1. Overdamped - Example 1 - Solution

$$s^2 + 5s + 6 = 0,$$

$$s_1 = -2 \text{ and } s_2 = -3.$$

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

$$x(0) = -1 \text{ and } \dot{x}(0) = 0$$

$$t = 0.$$

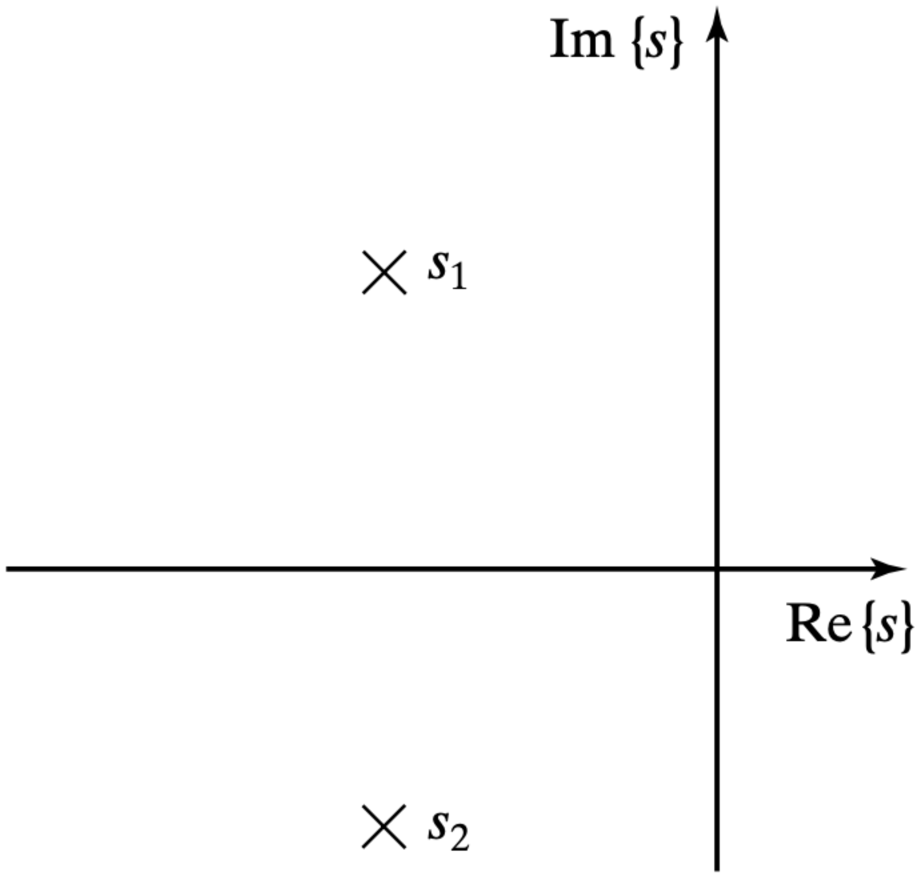
$$c_1 + c_2 = -1$$

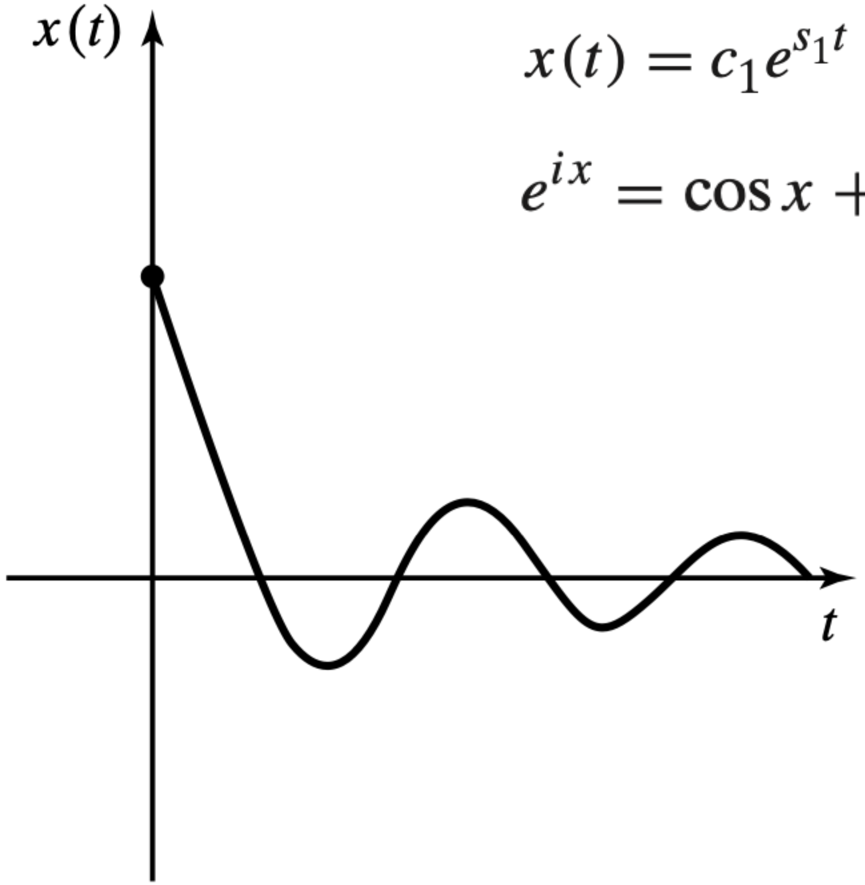
$$-2c_1 - 3c_2 = 0,$$

$$c_1 = -3 \text{ and } c_2 = 2.$$

$$x(t) = -3e^{-2t} + 2e^{-3t}.$$

2. Underdamped (Complex Roots)





$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t},$$

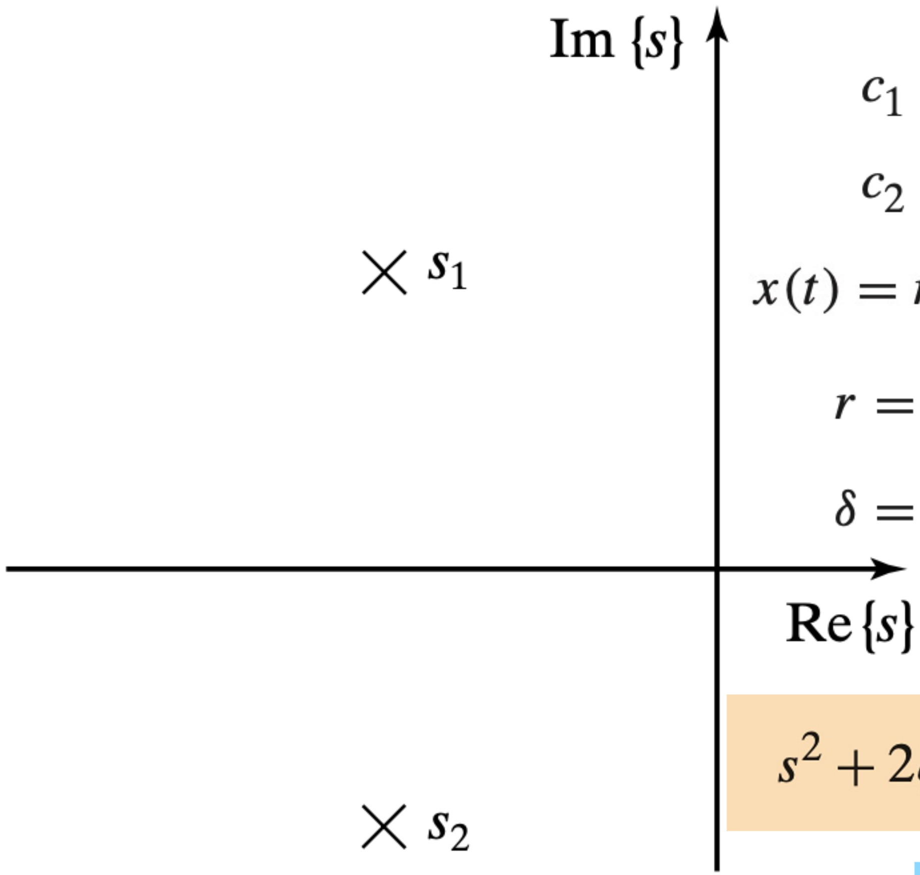
$$e^{ix} = \cos x + i \sin x,$$

$$s_1 = \lambda + \mu i,$$

$$s_2 = \lambda - \mu i,$$

$$x(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t).$$

2. Underdamped (Complex Roots)



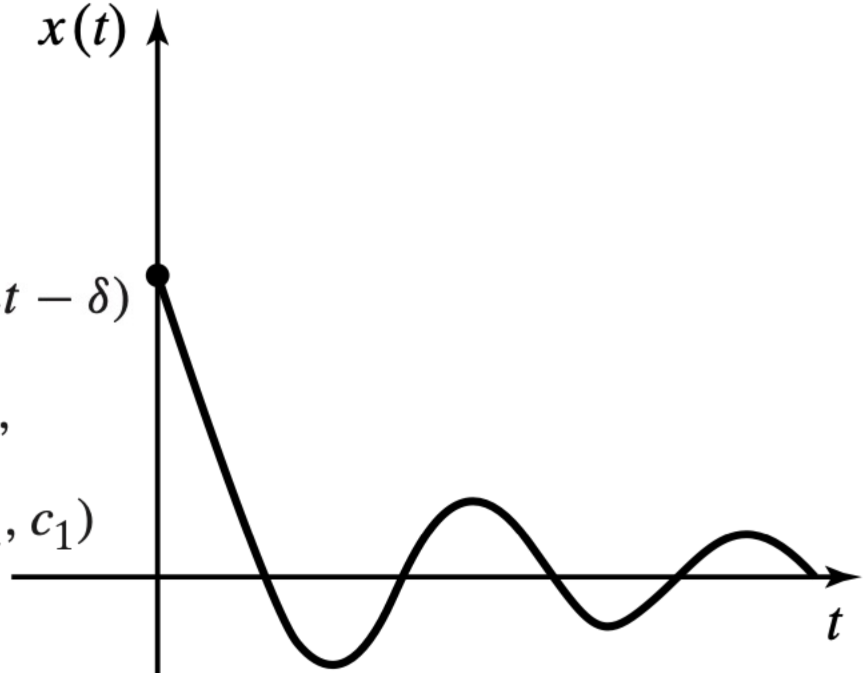
$$c_1 = r\cos\delta,$$

$$c_2 = r\sin\delta,$$

$$x(t) = re^{\lambda t}\cos(\mu t - \delta)$$

$$r = \sqrt{c_1^2 + c_2^2},$$

$$\delta = \text{Atan2}(c_2, c_1)$$



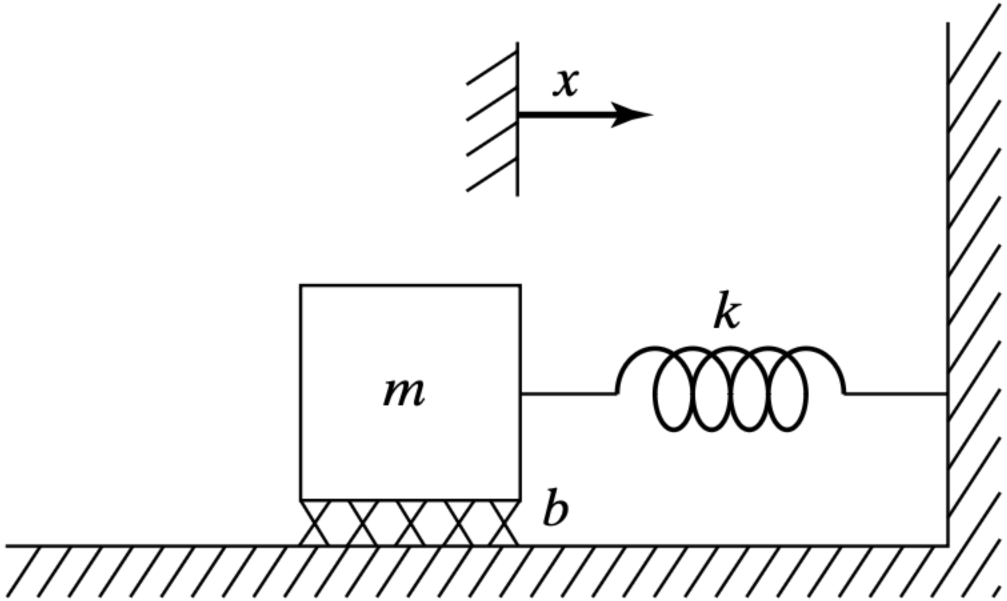
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0,$$

damping ratio → $\zeta = \dfrac{b}{2\sqrt{km}},$

natural frequency → $\omega_n = \sqrt{k/m}.$

2. Underdamped - Example 2

- Determine the motion of the system below if the parameter values are *m=1*, *b=1*, and *k=1*, and the mass is initially at rest and is released from the position x = -1.



2. Underdamped - Example 2 - Solution

$$s^2 + s + 1 = 0,$$

$$s_i = -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} i.$$

$$x(t) = e^{-\frac{t}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2} t + c_2 \sin \frac{\sqrt{3}}{2} t \right).$$

$$x(0) = -1 \text{ and } \dot{x}(0) = 0,$$

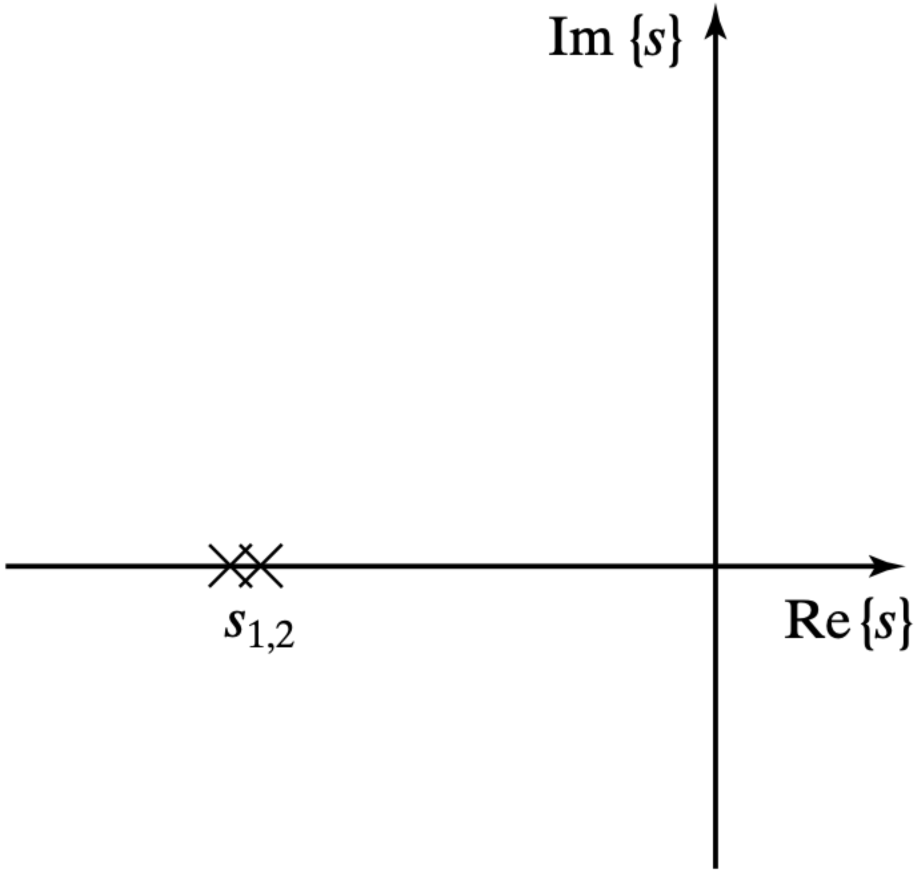
$$c_1 = -1$$

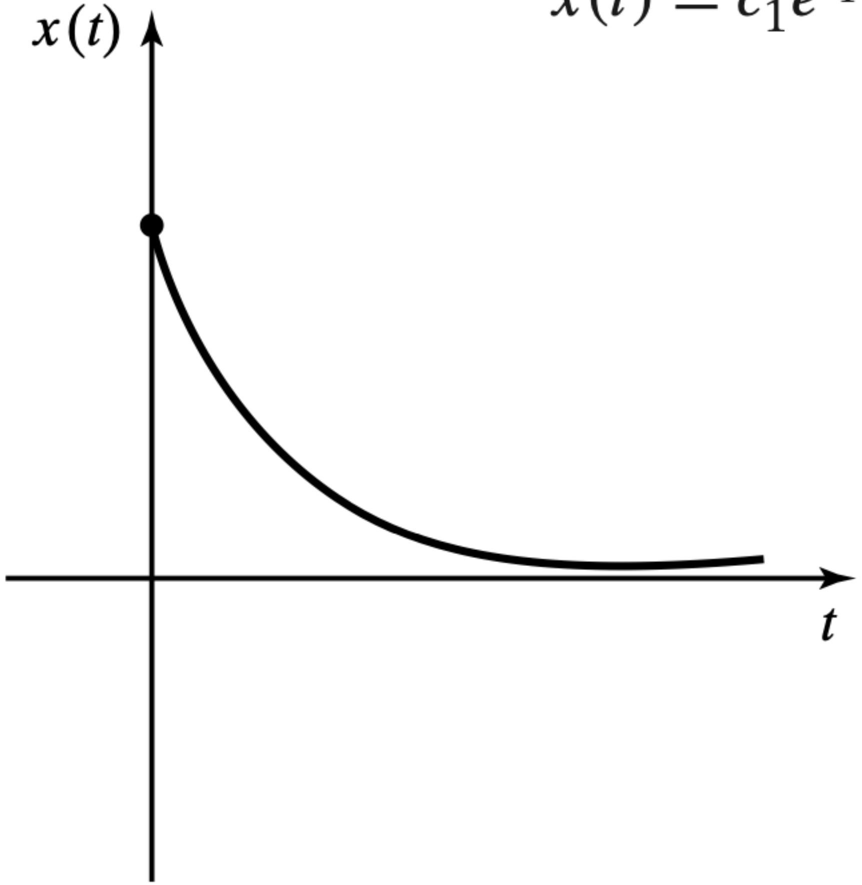
$$-\frac{1}{2} c_1 - \frac{\sqrt{3}}{2} c_2 = 0,$$

$$c_1 = -1 \text{ and } c_2 = \tfrac{\sqrt{3}}{3}.$$

$$x(t) = e^{-\frac{t}{2}} \left(-\cos \frac{\sqrt{3}}{2} t - \frac{\sqrt{3}}{3} \sin \frac{\sqrt{3}}{2} t \right).$$

3. Critically damped (Real and Equal Roots)





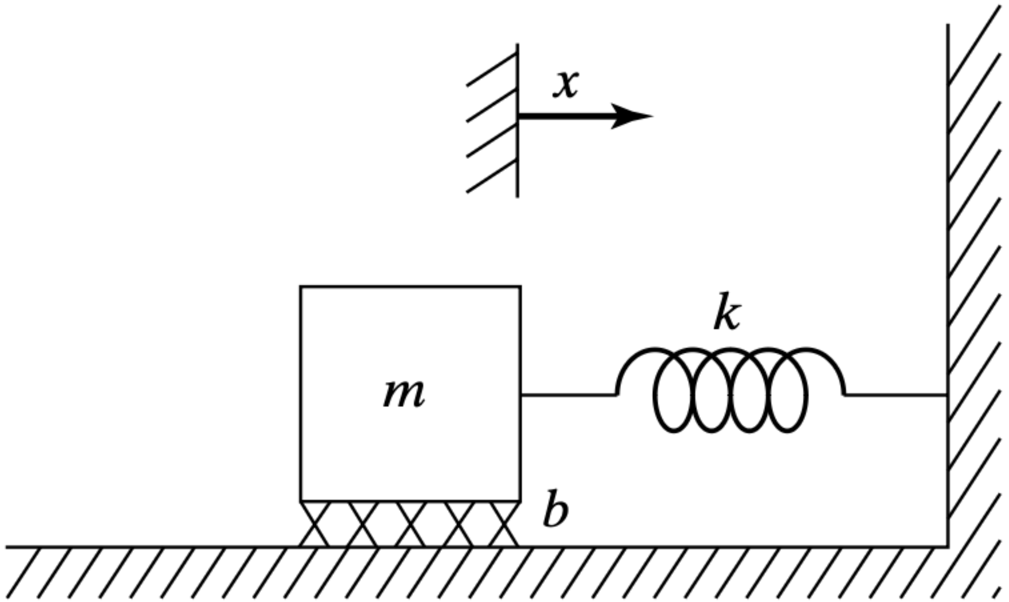
$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t},$$

$$s_1 = s_2 = -\frac{b}{2m}$$

$$x(t) = (c_1 + c_2 t)e^{-\frac{b}{2m}t}.$$

3. Critically damped - Example 3

- Determine the motion of the system below if the parameter values are $m=1$, $b=4$, and $k=4$, and the mass is initially at rest and is released from the position x = -1.



3. Critically damped - Example 3 - Solution

$$s^2 + 4s + 4 = 0,$$

$$s_1 = s_2 = -2.$$

$$x(t) = (c_1 + c_2 t)e^{-2t}.$$

$$x(0) = -1 \text{ and } \dot{x}(0) = 0,$$

$$c_1 = -1$$

$$-2c_1 + c_2 = 0,$$

$$c_1 = -1 \text{ and } c_2 = -2.$$

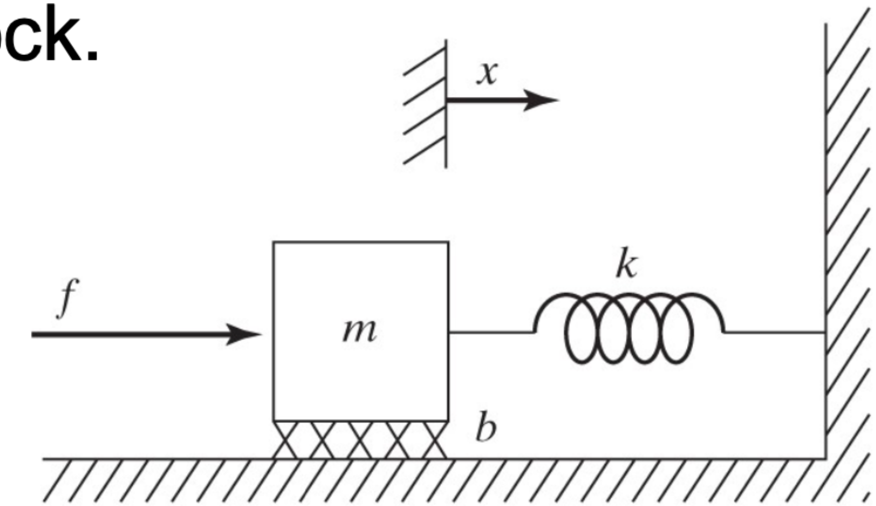
$$x(t) = (-1 - 2t)e^{-2t}.$$

Control of Second-Order Systems

- What if our second-order mechanical system is not what we wish it to be?
  - Maybe it is underdamped and oscillatory and we would like it to be critically damped.
  - Maybe the spring is missing ($k=0$) and the system never returns to x=0 if disturbed.

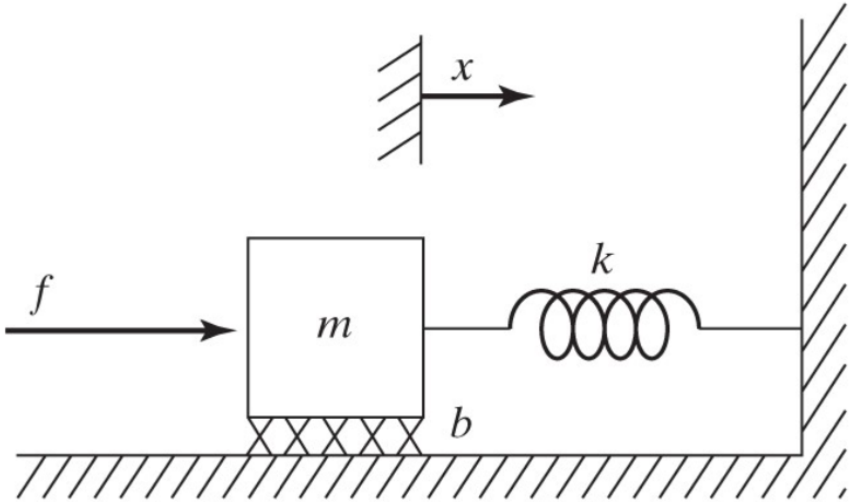
Control of Second-Order Systems

- Let us add an actuator that applies a force f in the block.
- Equation of motion: $m\ddot{x} + b\dot{x} + kx = f$
- Let us assume that we have sensors capable of detecting the position and the velocity of the block.
- $k, b, m > 0$



Control of Second-Order Systems

- Control law – to compute the force that should be applied by the actuator as a function of the sensed feedback:
- $f = -k_u\dot{x} - k_p\, x$



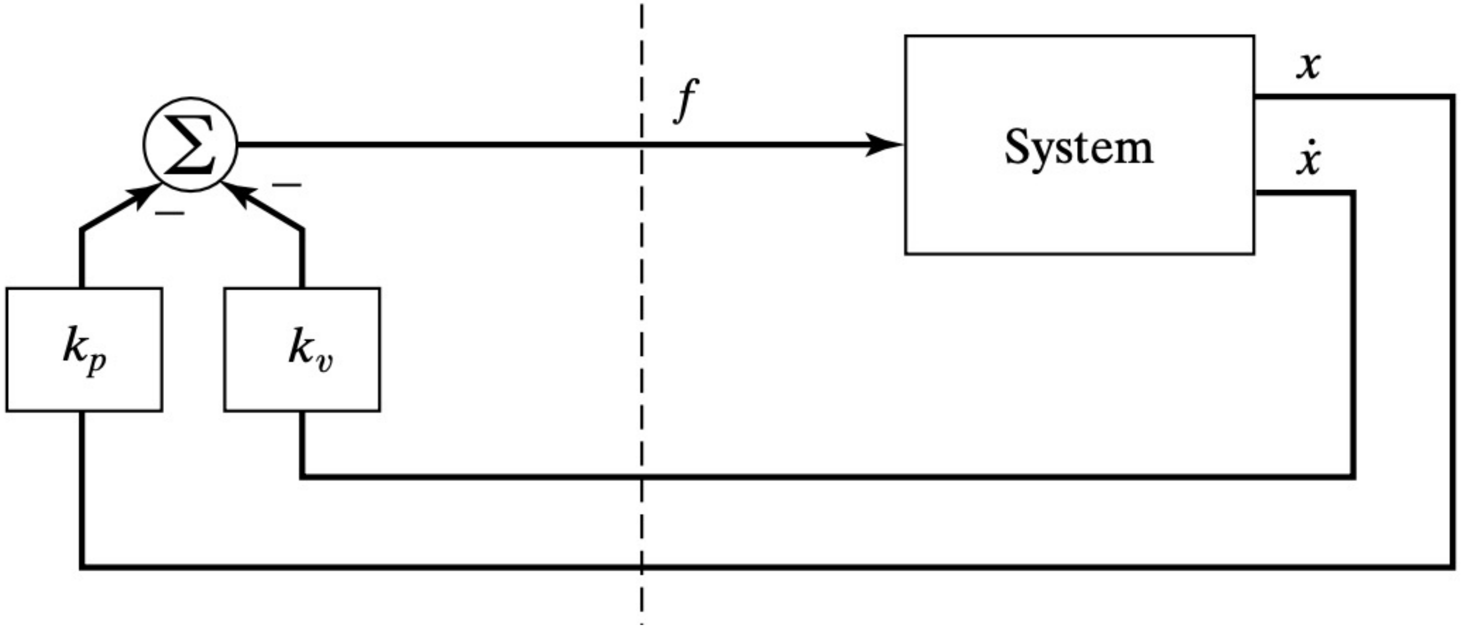
Control of Second-Order Systems

- **Position-regulation system:** It simply attempts to maintain the position of the block in one fixed place regardless of disturbance forces applied in the block.

$$m\ddot{x} + b'\dot{x} + k'x = 0,$$

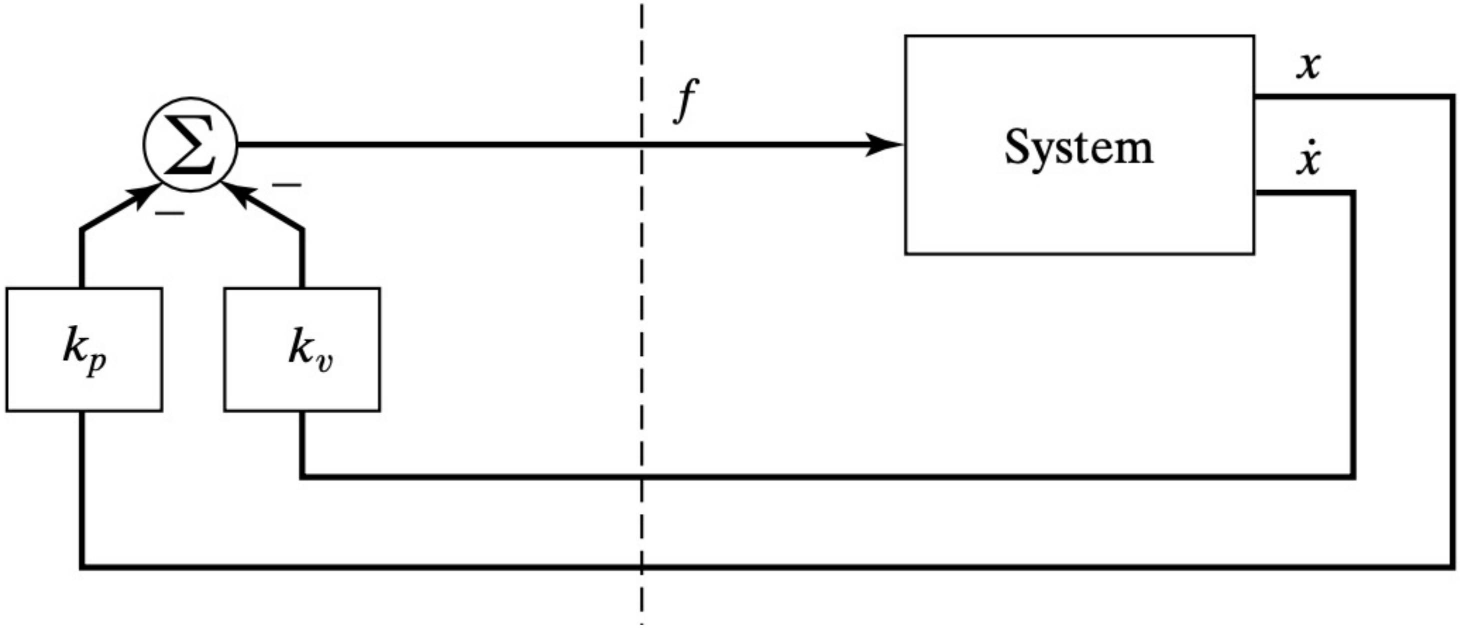
$$b' = b + k_v$$

$$k' = k + k_p.$$



Control of Second-Order Systems - Example 4

If the parameter of the system are *m=1*, *b=1*, and *k=1*, find gains k_u, k_p for a position-regulation control law that results in the system's being critically damped with a closed-loop stiffness of 16.

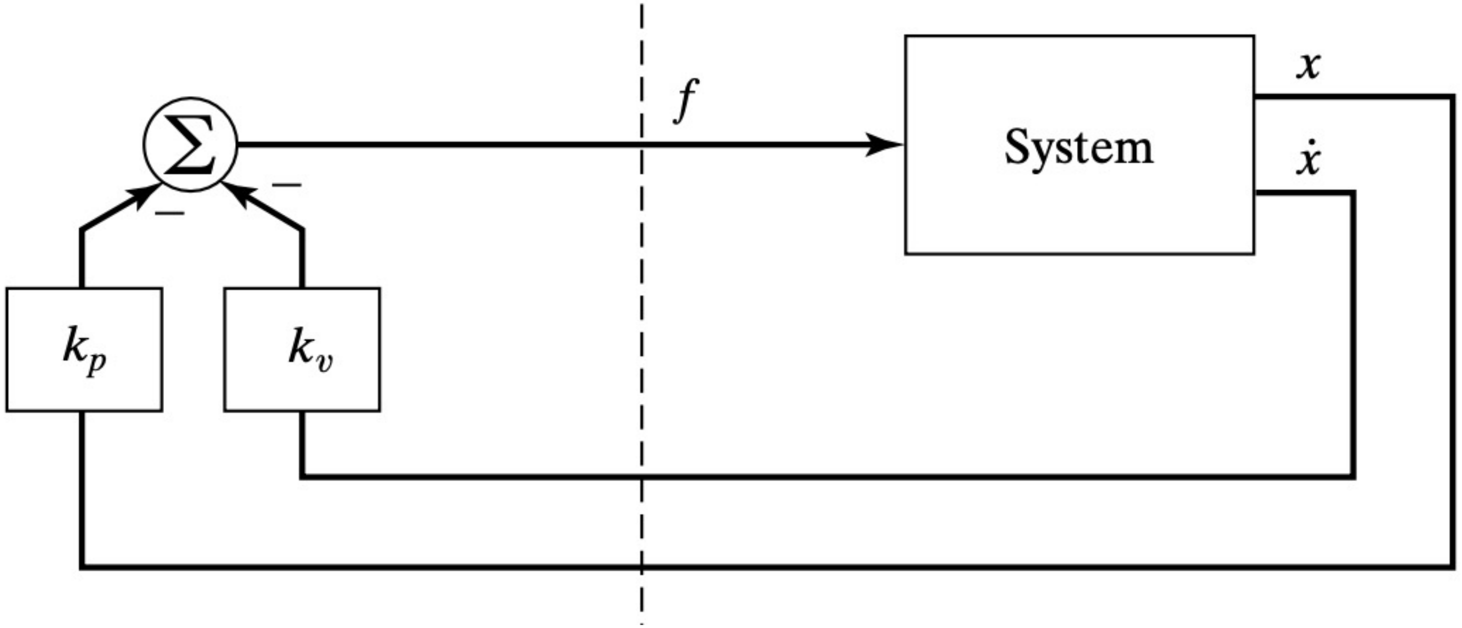


Control of Second-Order Systems - Example 4 - Solution

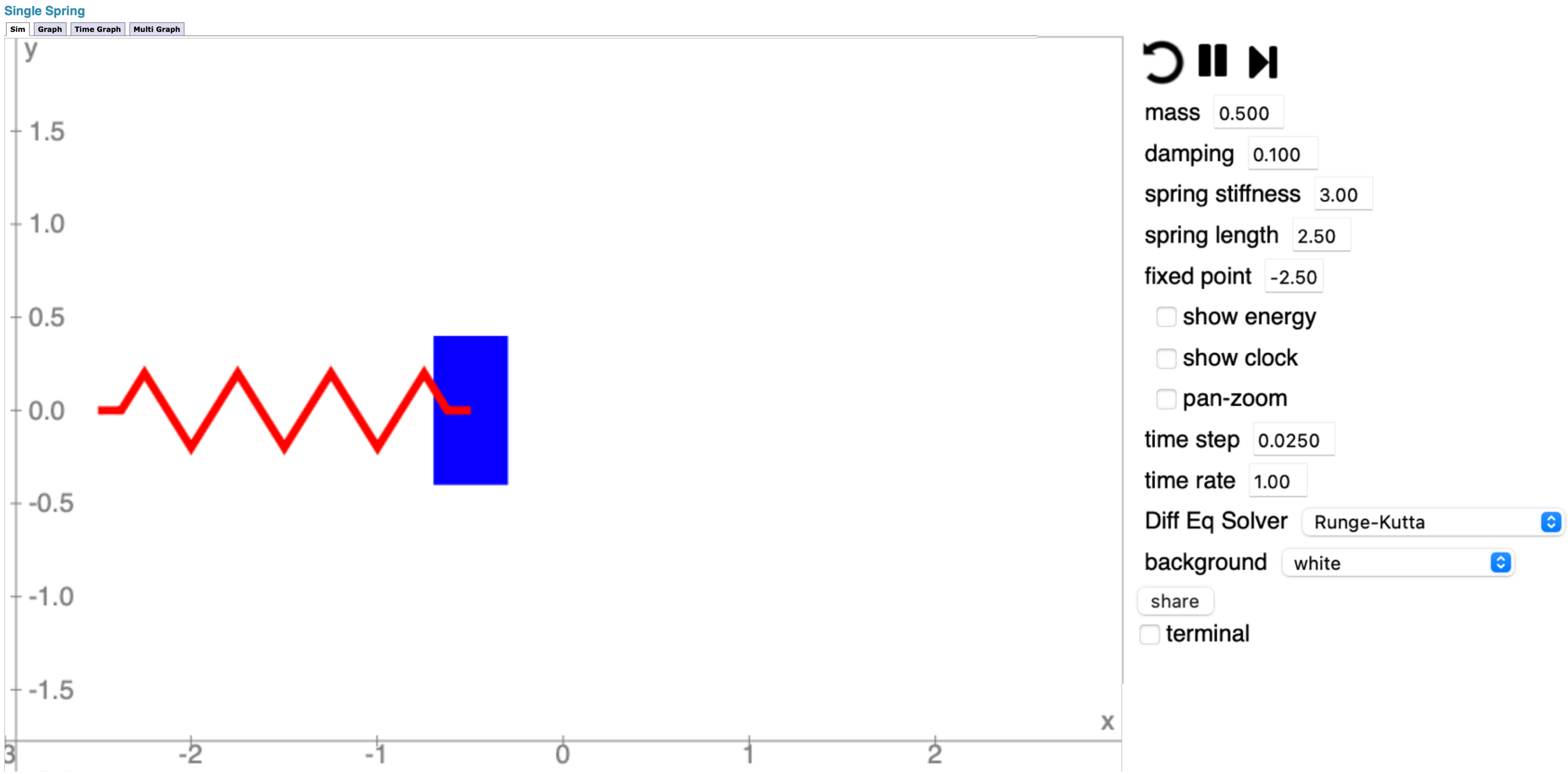
If we wish k' to be 16.0, then, for critical damping, we require that $b' = 2\sqrt{mk'} = 8.0$. Now, $k = 1$ and $b = 1$, so we need

$$k_p = 15.0,$$

$$k_v = 7.0.$$

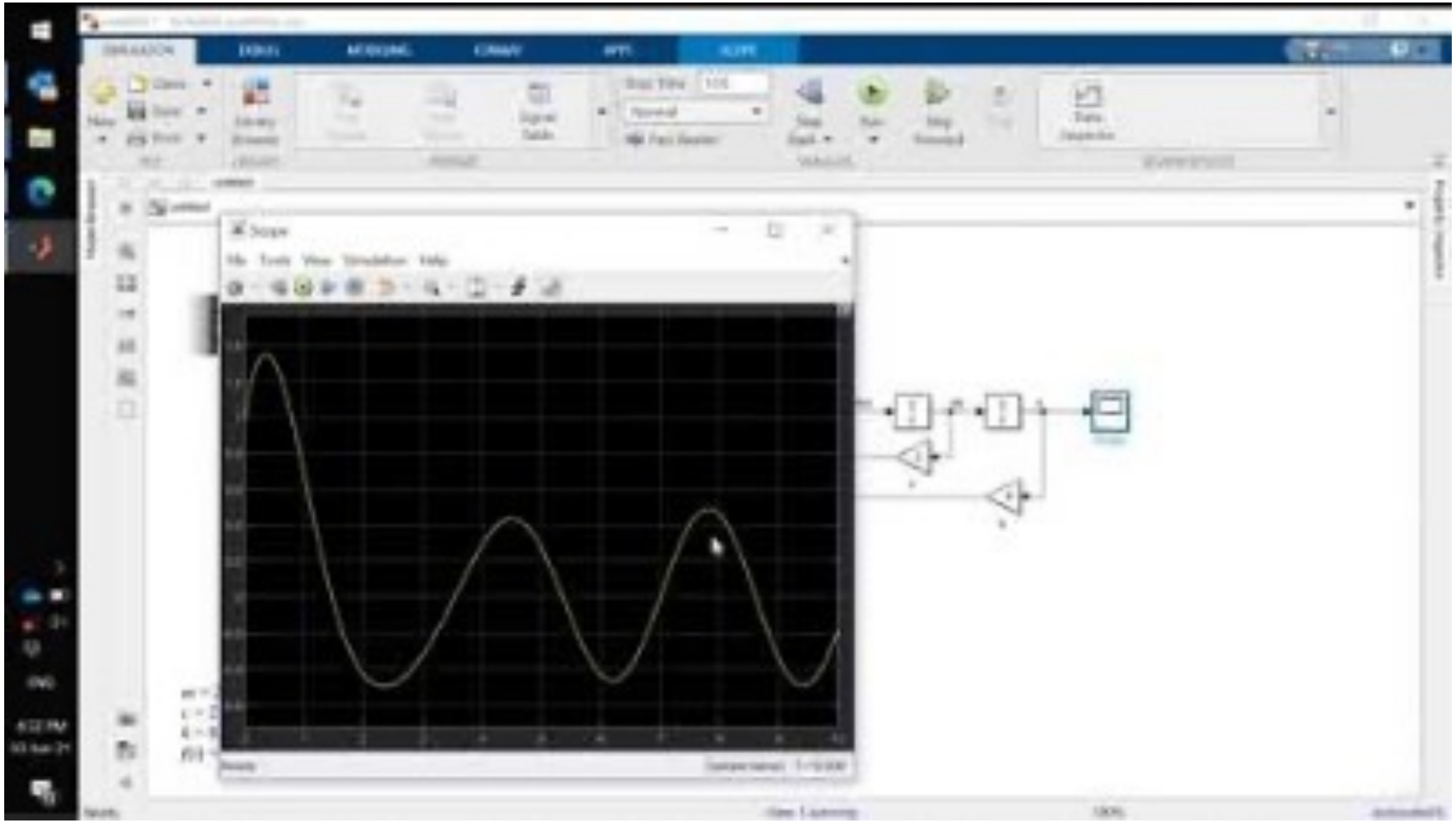


Spring-Mass-Damper System - Simulator



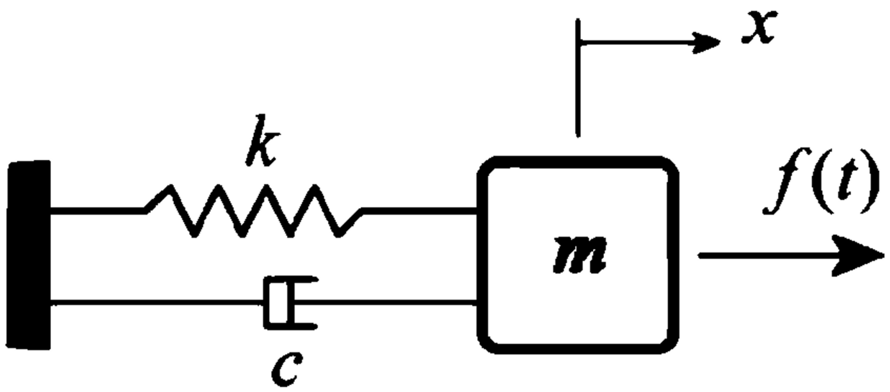
https://www.myphysicslab.com/springs/single-spring-en.html

Spring-Mass-Damper System - in Simulink



https://www.youtube.com/watch?v=pcRJerq6AqA

Spring-Mass-Damper System - in Simulink



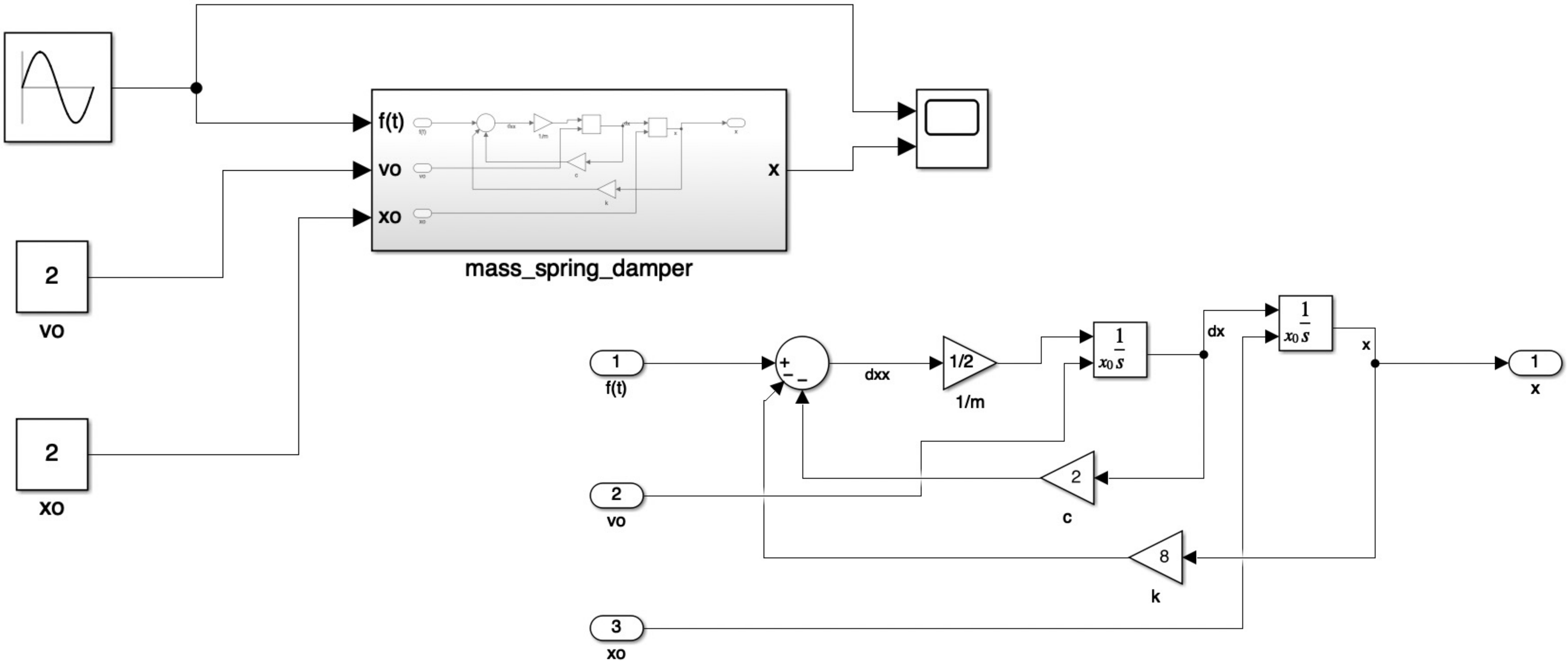
$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$\ddot{x} = \frac{1}{m}(f(t) - c\dot{x} - kx)$$

$m = 2$ kg
$c = 2$ N.s/m
$k = 8$ N/m
$f(t) = 2\sin(2t)$

https://www.youtube.com/watch?v=pcRJerq6AqA

Spring-Mass-Damper System - in Simulink



https://www.youtube.com/watch?v=pcRJerq6AqA

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