Periodicity of sinusoidal frequencies as a basis for the analysis of Baroque and Classical harmony: a computer based study

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Abstract

The thesis of this dissertation is that tonality is derived from very precise tonal relationships involving the first three primes. It is specifically asserted that within any piece of tonal music, with tonic being given as an octave equivalent of 1/1, the relative frequency for any note can be represented in the form $\left(\frac{2}{1}\right)^x * \left(\frac{3}{2}\right)^y * \left(\frac{5}{4}\right)^z$, $\{x, y, z\} \in Z$. A database of chord progressions is electronically created and catalogued. Within this database, a higher rate of occurrence for the simplest chords of 5-limit just intonation is demonstrated. Listening experiments based upon the most commonly occurring chord progressions lend further support to the assertion that 5-limit just intonation is in fact the origin of Baroque tonality. Finally, a rule-based system that prioritises important tonal relationships is demonstrated by tuning several entire chorales according to the principles of 5-limit just intonation.

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Chapter 1
Introduction

> *"And there will be no person but does not acknowledge these consonances to be very exact, and that the ratios correspond perfectly to experience."* [1]

1.1 General Statement of Thesis

Every tuning system used in traditional Western music is based upon an initial set of frequency relationships, each of which can be expressed as the ratio of two integers. Fixed 12-note tuning systems that predominate Western music, most notably equal temperament, compromise individual frequencies in the attempt to improve overall efficiency and tonal flexibility.

Theoretical tuning systems restricted neither by the number of frequencies per octave nor by the accuracy of sound production may be systematically explored using digital sound synthesis techniques. Such tuning systems can target the human auditory system directly, taking into account the diverse interdependent physical, physiological and psychological factors affecting the perception of musical pitch.

The approach of tuning by ratios cannot be discounted out of hand. Just intonation can duplicate equal temperament arbitrarily beyond the discrimination of the human auditory system. For example, to increase A-440 by a semitone in equal temperament, one would multiply 440 Hz by $\sqrt[12]{2}$. To an accuracy of seven decimal places (well beyond discrimination thresholds), this value would be A 440 x 1.0594631 = B♭ 466.1637640 Hz.

To duplicate this feat in just intonation, simply select a fraction whose numerator X and denominator Y provide the same approximation for $\sqrt[12]{2}$. In this case, X/Y = 10,594,631/10,000,000 = 1.0594631 which, when multiplied by 440 Hz, will yield exactly the same desired frequency of 466.1637640 Hz. One cent, approximately 1.0005778, can

[1] Marin Mersenne, *Harmonie Universelle* (The Hague: Martinus Nijhoff, 1957, English translation by Roger E. Chapman), pp. 28-29.

similarly be approximated as the fraction 10,005,778/10,000,000, and one hundredth of a cent, 1.0000058, as 10,000,058/10,000,000. Just intonation can similarly approximate every interval from any system to an arbitrary level of accuracy; thus, for all practical purposes, equal temperament, and indeed any tuning system, is a subset of just intonation. Therefore, it is not a question of whether just intonation, defined as a system in which all frequencies are related to each other as integer ratios, can provide the correct frequencies, but rather of which frequencies its arbitrarily precise ratios should produce.

There is some evidence that the ability to identify tonal relations is not entirely learned. According to Lola L. Cuddy:

> Infants less than one year of age can discriminate intervals of the tonal system and can discriminate the major from the augmented triad (for reviews see Trehub 1993; Trehub & Trainor, 1993). Evidence of tonal knowledge has been obtained from very young children, and the full properties of the tonal hierarchy by early school years (Cuddy & Badertscher, 1987; Hargreaves, 1986; Krumhansl & Keil, 1982; Zenatti, 1993). Moreover, the music to which both children and adults are exposed in our culture consistently provides instances of tonal structure. For example, the frequency with which tones are sounded in tonal-harmonic music is correlated with relative stability of the tones in the tonal hierarchy (Kumhansl, 1987, 1990). Thus, in day-to-day musical encounters the internal schema for the tonal hierarchy is regularly activated, and becomes highly overlearned.[2]

The phenomenon of overlearning tonal relations is important when considering that the twelve tones traditionally employed in equal temperament are also highly overlearned. We are all indoctrinated in a musical culture whose keyboards use, for example, sharp major thirds. This intractable connection between conditioned learning and expectation in musical listening somewhat clouds the issue of determining which intonation system is optimal.

[2] Irene Deliege and John Sloboda (editors), *Perception and Cognition of Music* (East Sussex: Psychology Press, Ltd., 1997), p. 338-39.

Misunderstandings resulting from such overlearning of tonal relations can be found in the writings of no less luminary a figure than Christian Huygens (1629-1695), a Dutch astronomer and physicist who was the first to propose the wave theory of light. In spite of his many accomplishments, including discovering the rings of Saturn and inventing the pendular clock, Huygens came to some astonishingly unscientific conclusions regarding tuning.

Having spent much of his spare time studying both music and tuning, Huygens asserted that the musical scale was founded in universal principles. He reasoned further that such musical scales "could be either developed or understood by the inhabitants of other planets".[3] This concept is startling, especially considering the time in which it was written.

Even more startling are the criteria Huygens considered as universal to music theory. According to Huygens, the principles that supposed music theorists of another world would understand included just intonation, or the use of simple ratios between tones, and temperament, because it was necessary to temper tones to avoid commas. The use of ratios seems reasonable enough, but tempering commas is not a scientific basis for discovering the underlying principles of intonation. In fact, temperament of commas, to which so many tuning specialists over the centuries have devoted their study, is not really an area of scientific inquiry, but rather a question of how to tune a harpsichord, piano or organ. In this more specific context, the question can be recast as, “How do we readjust the fifths and thirds so that our keyboards, based on a twelve-note octave, sound best? In doing so, how do we redistribute the Pythagorean comma?”

[3] Christiaan Huygens, *Le cycle harmonique* (Rotterdam, 1691; second edition with English and Dutch translations, Rudolf Rasch, ed., Utrecht: Diapason Press, 1986), p. 38.

Many systems, papers and books propose different means by which to accomplish this redistribution, none of them definitively so. In fact, it has been mathematically proven that no such system can ever be devised that is key-independent. The point Huygens and most other tuning specialists missed and continue to miss is this: *If the original Pythagorean system is the correct foundation for tuning, why does it immediately present us with a comma that we must avoid?*

The goal of attaining theoretical perfection with twelve notes has already been proven an impossibility by Hall (1974):

> We want to adjust 36 connections independently (even more if we specify "just" sevenths or tritones) when there are only twelve things being connected; in fact, since the starting point is arbitrary, there are only 11 degrees of freedom. In these terms it is clear that there is absolutely no hope of our ever devising a truly just scale, one in which all intervals are in tune; in mathematical terms, it is a badly over-determined system, a non-trivial optimisation problem. Since the people dealing with it have usually not been equipped with this viewpoint, it is no wonder that the problem has often been attacked intuitively rather than systematically, and that such a wide array of tunings and temperaments have been proposed as context-independent 'solutions'.[4]

The solution to finding a system, assuming it exists, in which any piece of tonal music can be made in tune, must then rely on using more than twelve tones. A truly modern method of tuning, which can only be implemented electronically, must determine the number and relative frequencies of pitches according to the capabilities of the human auditory system. The rules governing such a tuning system must be clear and their implementation consistent.

[4] Donald E. Hall, 'Quantitative Evaluation of Musical Scale Tunings,' *American Journal of Physics*, vol. 42 no. 7 (1974), p. 547.

1.2 Goals of this Dissertation

As this dissertation will demonstrate, the question of how to make the fifths and thirds sound best hints at a more complete theory begun by the music theorists Didymus, Ptolemy and Zarlino; understood and championed by Mersenne and Rameau; refined by the eminent nineteenth century scientist Hermann Helmholtz; and supported by experimental data from a wide range of sources during the twentieth century. In order to elucidate the central issues of intonation, this dissertation will include many interrelated approaches: tracing the evolution of tuning and tonality; compiling and analysing a database of chord progressions from the chorales of J.S. Bach; performing listening experiments based upon these chord progressions; reviewing literature of tuning, music theory and especially of psychoacoustics; and analysing digital recordings of actual performances. It is hoped that through a synthesis of these techniques, a more descriptive model of intonation and tonal theory in general can be developed, especially as it pertains to digital sound synthesis.

1.3 Organisation of this Dissertation

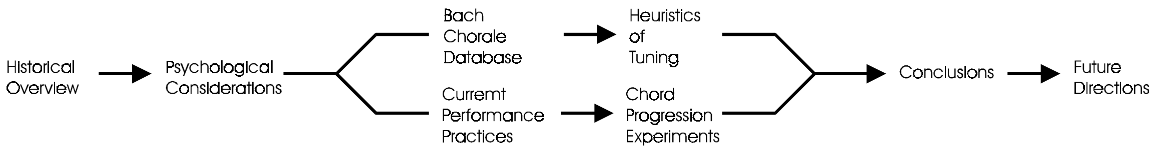

Figure 1. Organization of this dissertation

As illustrated in Figure 1, this dissertation begins with an historical overview that will present the origins and traditions of Western tuning systems, with emphasis on the theories that point the way to the future. Next, physiological and psychological considerations pertaining to tuning, especially the cochlea and its function as a frequency analyser, will be reviewed.

The text will then branch into two parallel paths, the upper being associated with theory, programming and data collection, and the lower with verification through the analysis of performers' and listeners' tuning preferences. The centrepiece of this dissertation, the "Bach Chorale Tuning Database", details the computer-based cataloguing of chord progressions contained in the chorales of J. S. Bach. "Automated Tuning by Ratios" provides a strict set of heuristics by which to tune chords and chord progressions, and from which entire Bach chorales are analysed in "Appendix V: Heuristic Analysis of Bach Chorales".

In the chapter, "Intonation Analysis of Four Professional String Quartets", performers' preferences are detailed by applying Fourier analysis software to CD recordings.[5] In the following chapter, "Tuning Database Chord Progression Experiment", synthesised examples of chord progressions, chosen solely by rate of occurrence through an automated selection process, are used for listening experiments that determine tuning preferences associated with varying timbres and durations. Conclusions are drawn from the cumulative data.

Finally, this dissertation will focus attention on the central role tuning will play in music as the computer's immense potential to precisely express complex psychoacoustic data is gradually realised. Extensions discussed include timbre optimisation, as well as the use of chord progression databases to assist in and to automate the process of analysis and composition.

[5] David Johnston, *Cool Edit* (Phoenix: Syntrillium Software Corp., 2000).

Chapter 2
Historical Overview

> *"In order to understand the relationship between sounds, investigators took a string, stretched so that it could produce a sound, and divided it with movable bridges into several parts. They discovered that all the sounds or intervals that harmonize were contained in the first five divisions of the string, the lengths resulting from these divisions being compared with the original length."* [6]

2.1 The Greek Tuning Tradition

2.1.1 Pythagoras (c. 580 – 500 BC)

Around the middle of the sixth century BC, Pythagoras founded a brotherhood that was based on religion, philosophy and science. Pythagorean doctrines that emerged from this group influenced Plato and Aristotle, as well as laid the foundations for mathematics and Western philosophy. By virtue of being the first person known to have applied mathematical principles and observational methods to the study of sound, Pythagoras may be considered the earliest acoustician.

None of Pythagoras' writings have survived, making it impossible to determine exactly which doctrines originated with and which were posthumously attributed to him by his pupils and later followers. It is important to realize that Pythagoras and his pupils were not the originators of Greek mathematics, and that many discoveries associated with Pythagoras' name were actually made at a later time. The original brotherhood of Pythagoras had already disappeared by the early fourth century BC.

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[6] Jean-Philippe Rameau, *Treatise on Harmony* (translated by Philip Gossett, Dover Publications, New York, 1971), p. 4.

According to tradition, Pythagoras performed a series of experiments in which he slipped a wedge underneath a tightened string. He discovered that when the portion of the string to one side of the wedge bears a simple relationship to the portion on the other side, the interval between them is consonant.

When such a wedge is placed against the string so that the length of one side of the string is twice as long as the other, the relationship between the frequencies is 1:2, corresponding to the octave. Similarly, applying the wedge so that the corresponding lengths are 2:3 and 3:4 produce the respective intervals we now recognize as the perfect fifth and perfect fourth.

Pythagoreans held the concept of decad, which is the perfection of the number ten, as a guiding principle. The tetraktys signified the importance of the first four numbers, which add up to the decad, i.e., $1 + 2 + 3 + 4 = 10$. Intervals made out of these first four numbers were preferred over those that were not. This may explain, in part, why the ratio of the major third, 5:4, was avoided by the Pythagoreans, but was taken up later by Ptolemy and others. In a Pythagorean scale there are only two kinds of small intervals, namely, the whole tone 9:8 and the limma 256:243.

A prominent feature of Pythagorean scales and Greek scales in general is that they contain fixed intervals. Every scale is made up of two tetrachords, each spanning a perfect fourth (4:3) and divided by a tone (9:8). Thus, although the tetrachord itself could be divided in various ways, the division of the octave into two fourths and a major second was considered invariable.

2.1.2 Archytas (c. 428 – 350 BC)

Contemporary and friend of Plato, Archytas was the most distinguished among the second generation disciples of Pythagoras. In the area of acoustics, Archytas proposed the concept that sound is produced by pulsations of air. In mathematics, his many accomplishments include an original three-dimensional solution for the “doubling of the cube” problem.

The theory of Archytas states that there are three means, or methods of determining the ratio of a note that divides a larger interval: arithmetic, geometric and harmonic. Each of these methods can be defined by using the three variables x, y and z, representing the lowest note, the highest note, and the mean note, respectively.[7]

The arithmetic mean is a simple average in which y exceeds z by the same amount as z exceeds x. Take, for example, 12:9:6. The first term, 12, exceeds 9 by the same amount that 9 exceeds 6. String lengths 12, 9 and 6 would correspond to a middle C ascending a fourth to F, which ascends in turn to C one octave above middle C.

The geometric mean occurs when y is to z exactly as z is to x. For example, the geometric mean of two octaves is an octave; the geometric mean of a major ninth (9:4) is a perfect fifth (3:2).

The harmonic mean z of the smaller value x and the larger value y occurs when 1/z is the arithmetic mean of 1/x and 1/y. More concisely,

$$z = \frac{2xy}{x + y}$$

For example, if x were represented by string length 3 and y by string length 2, then z would equal string length 12/5. The corresponding relative frequencies for x, y and z could be expressed as C4, G6 and E5, forming a C major triad in closed root position. As another example, if x and y were the respective string lengths 4 and 3, then z would be string length 24/7. The corresponding frequencies for x, y and z in this case could be expressed as G6, C8 and B♭7 which, when played together, would form a second inversion dominant seventh chord minus third, with the seventh being more than 31 cents flat to its equivalent in equal temperament.

[7] George M Phillips, ‘Mean’ (WWW: Department of Mathematical and Computational Sciences, University of St. Andrews), http://www-maths.mcs.st-andrews.ac.uk/~gmp/gmpmean.html.

Three genera of scales served as theoretical models in the writings of Archytas and subsequent ancient Greek theorists. In all three genera, the first tetrachord, spanning a perfect fourth (4:3), serves as a generator that contains all the information required to complete the scale.[8] The second tetrachord, like the subject of a baroque fugue, is simply the first tetrachord transposed up a perfect fifth by multiplying all its ratios by 3/2. The two tetrachords are separated by the remaining interval between a perfect fourth and a perfect fifth, namely, the tone, 9:8.

The generator for the diatonic scale by Archytas given in Figure 2 contains successive upward intervals of approximately 1/3 of a tone (63 cents), a stretched whole tone (231 cents) and a whole tone (204 cents). Another example of Archytas' scales is the chromatic (Figure 3), whose generator employs the approximate ascending intervals of 1/3 tone (63 cents), a stretched semitone (141 cents) and a Pythagorean minor third (294 cents). A final example is the enharmonic, shown in Figure 4, whose generator ascends by the approximate intervals of 1/3 tone (63 cents), quarter tone (49 cents), and a standard just major third (386 cents).

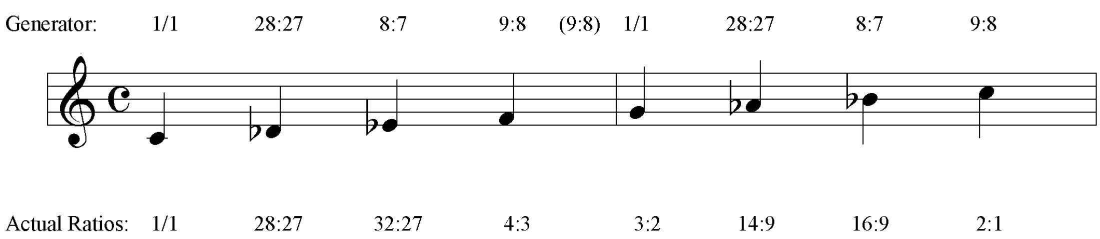

Figure 2. Diatonic scale of Archytas (adapted from Barker)

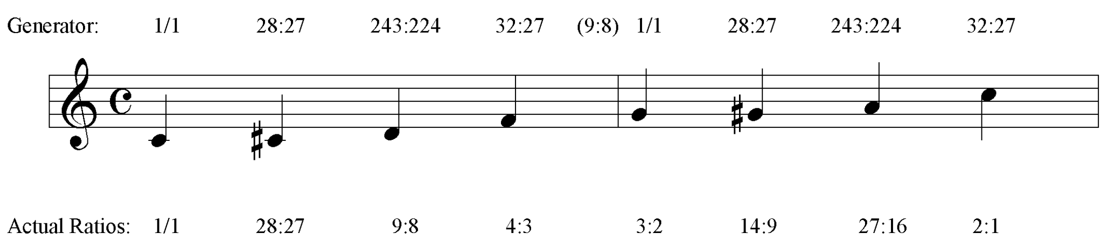

Figure 3. Chromatic scale of Archytas (adapted from Barker)

[8] Andrew Barker, ed., *Greek Musical Writings. Vol. 2 Harmonic and Acoustic Theory* (Cambridge: Cambridge Univ. Press, 1989), p. 46.

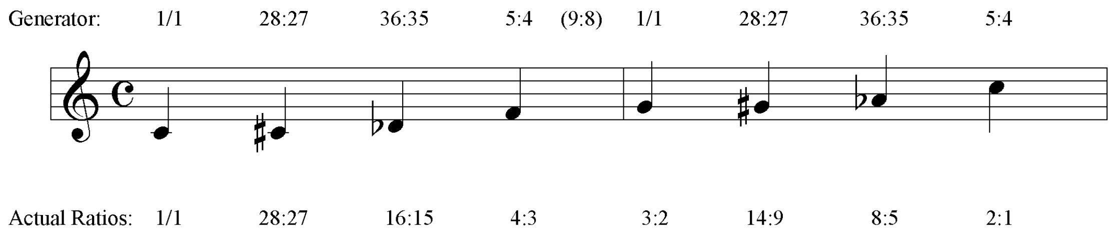

Figure 4. Enharmonic scale of Archytas (adapted from Barker)

One observation that can be made with regard to these scales is that the ratios between the notes of the generators are usually epimoric, meaning that the numerator of the fraction is greater by one than the denominator. An exception to this is found in Archytas' Chromatic Scale (Figure 3), in which the value for the third note is the value of the second multiplied by 243/224.

Another observation is that the actual ratios produced in the generators, in relation to the first note, also tend to be epimoric. An exception is found for this rule also, in the third note of Archytas' Diatonic Scale (Figure 2), where the ratio is 32/27. In Archytas' Enharmonic Scale (Figure 4), every ratio after the first note (1/1) of the generator, in relation to its previous note as well in relation to the generator's first note, is epimoric.

Archytas discovered that there exists no geometric mean for any epimoric ratio. For example, the octave (2:1) cannot be divided into two equal parts using fractions. *One need look no further than this basic axiom to notice that equal temperament cannot possibly be implemented with complete precision using strings*. Other intervals important to Greek music that obey this rule are the perfect fifth (3:2) and perfect fourth (4:3). An important interval for the later theory of just intonation is the major third (5:4), which by virtue of being epimoric cannot be broken into two equal tones.

All of this cleverness involving the three means and epimoric ratios underscores the fact that geometric principles were more important to Archytas than simplicity of ratios or consonance, which is completely understandable, considering that tonality had not yet been discovered. Such principles did, however, provide a point of departure from strictly Pythagorean tuning.

2.1.3 Aristoxenus (born c. 350 BC)

Shortly before Archytas' death around 350 BC, Aristoxenus was born. Around 330 BC Aristoxenus began his studies with Aristotle, and most likely left the Lyceum around the time of Aristotle's death in 322 BC. Of the 453 'books' attributed by Suda to Aristoxenus, none has survived intact.

Aristoxenus' approach to tuning was markedly different than that of Archytas. Whereas Archytas and the Pythagoreans emphasized that intervals could be properly be measured only as mathematical ratios, Aristoxenus and his followers paid more attention to the human perceptions of the listener. In Aristoxenus' way of thinking, musical intervals were not necessarily bound to strict definition by ratios, but were rather a collection of intervals to be added and subtracted. This approach is very similar to modern musical theory practices, where 12 semitones always add to an octave and intervals in general are added and subtracted.

Elementa Harmonica, Aristoxenus' most important work on music, is a treatise in three parts. The first two of these books pertain directly to tuning systems. In Book One, Aristoxenus carefully defines intervals:

> An interval is that which is bounded by two notes which do not have the same pitch, since an interval appears, roughly speaking, to be a difference between pitches, and a space capable of receiving notes higher than the lower of the pitches which bound it, and lower than the higher of them. Difference between pitches lies in their having been subjected to greater or lesser tension.[9]

[9] Aristoxenus, *Elementa Harmonica* (translation by Andrew Barker, ed.), *Greek Musical Writings. Vol. 2 Harmonic and Acoustic Theory* (Cambridge: Cambridge Univ. Press, 1989), p. 136.

> Now that we have explained these points, we must try to define the interval of a tone. The tone is the difference in magnitude between the first two concords. It is to be divided in three ways, since the half, the third and the quarter of it should be considered melodic. All intervals smaller than these are to be treated as unmelodic. Let the smallest of these be called the least enharmonic dieses, the next the least chromatic diesis, and the greatest the semitone.[10]

So, from the preceding statement, it is clear that Aristoxenus has defined a tone as 9:8, being the difference between the first two concords, namely, the fifth (3:2) and the fourth (4:3). This is proven by the fact that 4/3 x 9/8 = 3/2. It is, however, not clear how he intended to break this interval into two, three or four. It would certainly not be accomplished by using a geometric mean for either two or four parts.

Aristoxenus, according to Lindley, "conceived the gamut of pitch as a continuous line which could be divided into simple fractions, so that the octave could be divided into 6 tones; the tone into semitones or quarter tones; the fourth into two tones and a semitone, and so on."[11] Lindley disputes the notion that Aristoxenus was actually proposing a sort of equal temperament, arguing he had maintained that the ear grasped fourths and fifths. Further, Aristoxenus generally recorded his scales in ratios rather than in parts, as shown in Figure 5 and Figure 6.[12]

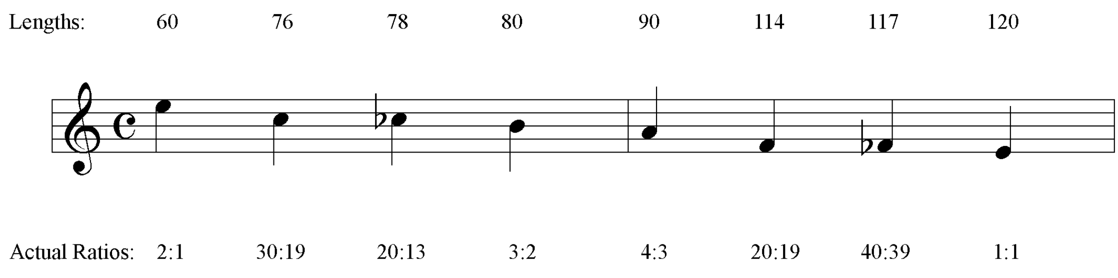

Figure 5. Enharmonic scale of Aristoxenus (adapted from Barbour)

[10] Aristoxenus, op. cit., p. 139.

[11] Mark Lindley, *The New Grove Dictionary of Music and Musicians* (London: Macmillan Publishers Ltd., 1995), vol.1, p. 591.

[12] J. Murray Barbour, *Tuning and Temperament* (New York: Da Capo Press, 1972), pp. 16-17.

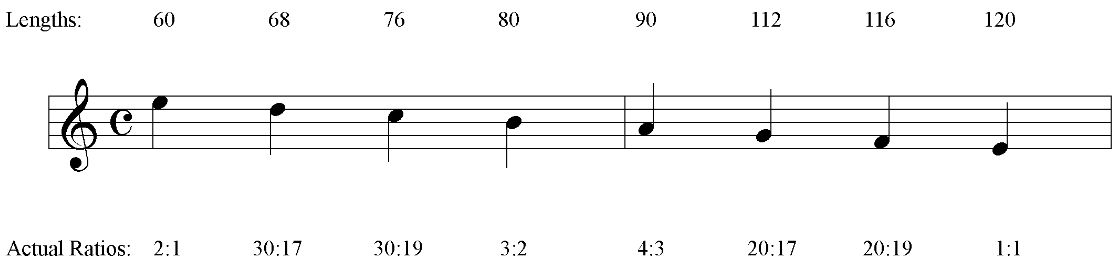

Figure 6. Syntonon scale of Aristoxenus (adapted from Barbour)

In Book 2 of Elementa Harmonica, Aristoxenus attempted to define musical terms within a musical framework. He concluded there were eight magnitudes of concords, the smallest being the perfect fourth. Any interval smaller was thought to be discordant. The next consonant interval was the perfect fifth, with every magnitude between the perfect fourth and fifth deemed discordant. The interval of the octave was given as the next consonant interval.

It is unclear from Book Two what the other five consonances to which Aristoxenus referred were; however, the following passage from Book One provides three more, namely, octave plus fourth, octave plus fifth, and double octave, bringing the total to six.

> So far as the nature of melody is concerned, the concordant appears to increase without limit, as does the discordant: for when any concordant interval is added to the octave, whether it is greater than the octave or smaller or equal to it, the resulting whole is concordant.

The seventh concord is not given, but the eighth is, when Aristoxenus states, " . . . there does appear to be a concord which is greatest. It is the double octave and a fifth, since our compass does not extend as far as three octaves."[13] It is clear, however, that the seventh concord, being less than two and one half octaves, and not being the fourth, fifth, octave, octave plus fourth, octave plus fifth, double octave, or double octave plus fifth, is the only equivalent of a fourth, fifth, or octave left, namely, the double octave plus fourth. In short, fourths, fifths, octaves and equivalents were considered by Aristoxenus to be consonant, up to two octaves and a fifth, whilst all others were not.

[13] Aristoxenus, op. cit., p. 140.

2.1.4 Euclid (born c. 300 BC)

Of the two musical texts that bear his name, only the *Sectio Canonis* (Divisions of the Octave) is considered to have been authored by Euclid, the famed mathematician who proved the infinitude of primes. Sectio Canonis reads very much like a geometry textbook, which, in part, it actually is.

Notable in Euclid's system, "proven" using twenty propositions, is the absence of intervals based on the number five, in contrast to the scales of Archytas, Didymus and Aristoxenus. All of the intervals of Euclid's system specifically relate to the octave, perfect fifth and perfect fourth, i.e., they are generated from the initial set of primes 2 and 3.

It is worth mentioning that Euclid immediately qualifies, without proof, that a consonant interval must be restricted to multiple or epimoric ratios, describing concordant intervals as:

> . . . making a single blend out of the two, while the discordant do not. In view of this, it is to be expected that the concordant notes, since they make a single blend of sound out of the two, are among those numbers which are spoken of under a single name in relation to one another, being either multiple or epimoric.[14]

Propositions 3 and 16, in which it is proven that no epimoric interval can be divided by another, are important because they illustrate, in conjunction with proposition 11, that octaves, perfect fifths and perfect fourths (and had Euclid included 5:4 and 6:5, major and minor thirds as well) are all intervals that cannot be equally divided in two using fractions. Proposition 8 is especially significant, as it illustrates that a perfect fourth taken from a perfect fifth equals a whole tone whose ratio is 9:8. Propositions 9 and 14 prove that six such whole tones exceed the octave.

[14] Euclid, *Sectio Canonis* (translation by Andrew Barker, ed.), *Greek Musical Writings. Vol. 2 Harmonic and Acoustic Theory* (Cambridge: Cambridge Univ. Press, 1989), p. 193.

2.1.5 Euclid's Immutable Systema

Figure 7, Table 1 and Figure 8 illustrate Euclid's Immutable Systema as representations of a divided string, as a table of ratios with their corresponding Greek names, and as a musical scale, respectively. These divisions of the string are found in proposition 19 of the Sectio Canonis.

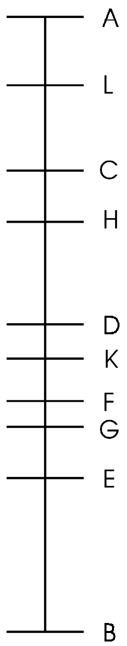

Figure 7. Euclid's Immutable Systema (adapted from Barker)

String	Length Ratio	Frequency Ratio	Interval Name
AB	1:1	1:1	Proslambanomenos
LB	8:9	9:8	Hypate Hypaton
CB	3:4	4:3	Diatonos Hypaton
HB	2:3	3:2	Hypate Meson
DB	1:2	2:1	Mese
KB	4:9	9:4	Paramese
FB	3:8	8:3	Nete Synemmenon
GB	1:3	3:1	Nete Diezeugmenon
EB	1:4	4:1	Nete Hyperbolaion

Table 1. Euclid's Immutable Systema

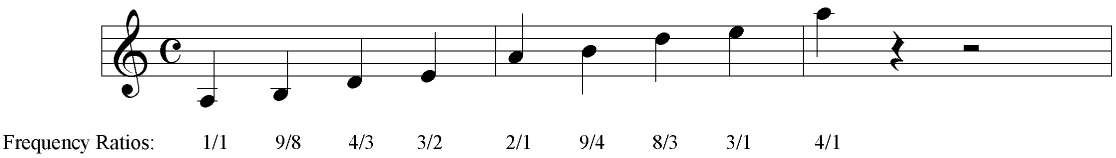

Figure 8. Euclid's Immutable Systema

From the immutable systema is derived the immutable systema with moveable notes, which is the basis from which Euclid derives his version of the pure minor scale, and is again illustrated as representations of a divided string, as a table of ratios with their corresponding Greek names, and as a musical scale, respectively, in Figure 9, Table 2 and Figure 10. Most noticeable in terms of tuning is that every note is derived from only two prime generators, 2 and 3, making it a purely Pythagorean scale. It is worth mentioning that, whilst this scale does not sound terribly dissonant when played melodically, it would not serve well as the basis for harmonization of, say, four-part harmony. The reason for this is that the ratios would be too complex for the simple chords they were representing. For instance, instead of having the ratio 4:5:6, the C major triad would have the ratios of 32/27:3/2:16/3. This is one of many examples illustrating that early Greek tuning specialists paid more attention to their complicated heuristics than to vertical harmony, which would be expected at a time when harmony would probably have been almost completely undeveloped.

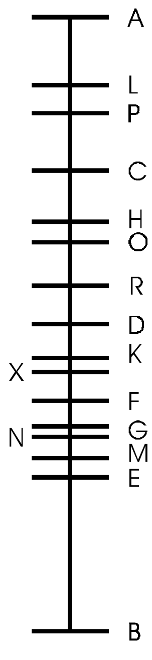

Figure 9. Euclid's Immutable Systema with moveable notes

Table 2. Euclid's Immutable Systema with moveable notes

String	Length Ratio	Frequency Ratio	Interval Name
AB	1:1	1:1	Proslambanomenos
LB	8:9	9:8	Hypate hypaton
PB	27:32	32:27	Parhypate hypaton
CB	3:4	4:3	Diatonos hypaton
HB	2:3	3:2	Hypate meson
OB	81:128	128:81	Parhypate neson
RB	3:16	16:3	Diatonos meson
DB	1:2	2:1	Mese
KB	4:9	9:4	Paramese
EB	1:4	4:1	Nete hyperbolaion
XB	27:64	64:27	Trite diezeugmenon
GB	1:3	3:1	Nete diezeugmenon
NB	81:256	256:81	Trite hyperbolaion
MB	9:32	32:9	Diatonos hyperbolaion
FB	3:8	8:3	Nete synemmenon

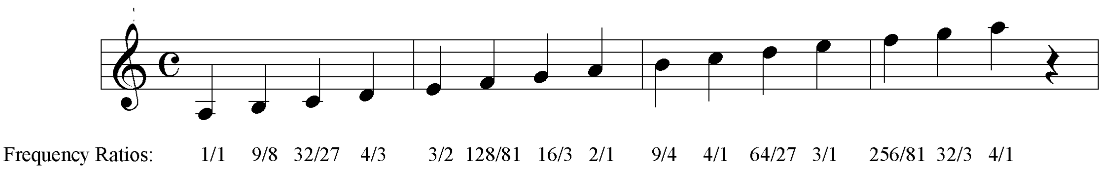

Figure 10. Euclid's Immutable Systema with moveable notes

2.1.6 Didymus (Second half of first century BC)

Didymus is an important figure in tuning theory for stressing the importance of the major third (5/4) and other intervals containing the factor five into his scales. This innovation allowed him to break the tetrachord (4:3) into two separate whole tone ratios, 9:8 and 10:9, with a remaining semitone of 16:15. This is in sharp contrast to what at the time was the standard practice of dividing the tetrachord into two equal whole tones of ratio 9:8, with a remaining limma of 256:243 for the semitone. An example of Didymus' tuning can be seen in Figure 11.

Didymus is frequently cited among tuning specialists as having been the first person to use the ratios currently employed in the standard just intonation scale. This claim is actually less than accurate. Whilst it is true that the values he chose between successive tones are indeed the same values Rameau (please see Figure 16 on p. 35) and Helmholtz employed approximately 2,000 years later,[15] it is also true that his order of tones is different, and therefore the actual ratios between all notes of the scale are radically altered. Further, Didymus delved into tunings that would be very awkward in just intonation, such a the chromatic with a generator of 6:5, 25:24, 16:15, and the enharmonic scale, with a generator of 5:4, 31:30, 32:31.[16]

The differences are not trivial. For example, the interval C to D in Didymus' system is associated with a ratio of 10:9 (also used by Rameau and justifiable for certain chord progressions) instead of 9:8. This means that the D, if it were the dominant of the C, would no longer have the relationship 3:2 with the dominant, G, of C. Other notable differences are the strange minor third (E-G) tuned as 32:27 instead of 6:5, and the perfect fourth between G and C, which is now 27:20 instead of 4:3. Comparing these two scales illustrates that it is not the relationship between two consecutive tones which determines overall consonance or dissonance, but rather the relationships between all tones within the tonal construct.

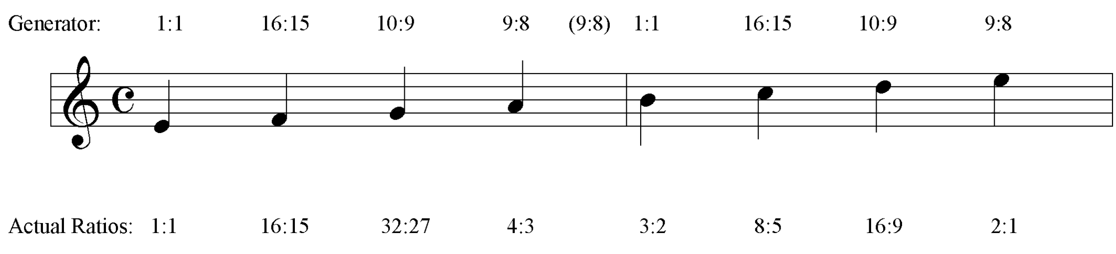

Figure 11. The diatonic scale of Didymus

[15] Hermann L.F. von Helmholtz, *On the Sensations of Tone as a Psychological basis for the Theory of Music* (New York: Dover, 1954), p. 15.

[16] Lukas Richter, *The New Grove Dictionary of Music and Musicians* (London: Macmillan Publishers Ltd., 2001), vol. 5, pp. 326-27.

2.1.7 Ptolemy (c. 83 – 161)

In his three-volume text *Harmonika* (Harmonics), mathematician, astronomer, astrologer and geographer Ptolemy included detailed tuning systems invented by himself and others, including Archytas, Aristoxenus, Eratosthenes and Didymus. In Harmonika, Ptolemy denigrated the Pythagorean practice of adhering strictly to established rules for generating new tones. He particularly criticized, as evidenced in the following fragment of his writing, the Pythagoreans for excluding the ratio 8:3 from consonances on the basis that it was neither multiple nor superparticular:

> Another crucial problem is made for them (the Pythagoreans) by the fact that they associate the concords only with those epimoric and multiple ratios and not with other – I mean such ratios as the epitetartics [5:4] and the five-times multiple [5:1], though when they are compared with the others there is a single form – and further by the fact that they make their selection of the concords in whatever way suits their fancy. From each of the first numbers that make up their ratios they subtract a unit, on behalf of the similarity arising from both, and the remaining numbers they posit as belonging to the dissimilarities, and the smaller these turn out to be, the more concordant they say they are. This procedure is utterly ludicrous. For the ratio is not characteristic only of the first numbers that constitute it, but belongs absolutely to all those that are related to one another in the same way; and hence the same result will occur in their case too, the combined 'dissimilarities' of the very same ratios turning out sometimes to be the least, sometimes the greatest.[17]

It is commendable that Ptolemy grasped the concept of each note gaining its tonal context not only from neighbouring notes, but from all of them. However, by his own logic, he could be criticized for claiming that melodic notes must fall into epimoric ratios.

Ptolemy also disapproved of the Aristoxenian practice of designating intervals by means of diastema (distance apart). This custom involved adding together a series of intervals whose precise ratios were either not firmly established, or else did not sum to the correct values for

[17] Ptolemy, *Harmonika*, (translation by Andrew Barker, ed.), *Greek Musical Writings. Vol. 2 Harmonic and Acoustic Theory* (Cambridge: Cambridge Univ. Press, 1989), p. 278.

the larger intervals of which they were a part. For instance, the Aristoxenian step was established as 9:8, but 6 steps did not add up to an octave. Similarly, Ptolemy proved that two and a half Aristoxenian (9:8) tones do not comprise the fourth (4:3).[18]

Ptolemy described interval types in terms of the simplicity of their ratios. In doing so, he placed intervals in terms of three categories of consonance:

1) Homophonoi (homophones), which are octaves and equivalents

2) Symphonoi (concordant), which are fifths, fourths and their octave equivalents, including 8:3

3) Emmeleis (melodic), which are those closest to the concords, and the epimoric ratios that are smaller than the epitritic (4:3), such as the major third (5:4), and tone (9:8)

It is interesting to note the detail into which Ptolemy goes to justify the use of specific intervals. For example, he supports the use of fourths, fifths and their octave equivalents because they are:

> . . . those that divide the octave most nearly into halves, that is, the fifth and the fourth, so that we can again opposite the fifth as being in hemiolic ratio and the fourth in epitritic; and second are those formed by putting each of the first concords with the first of the homophones, the octave and a fifth in the ratio put together from the duple and the hemiolic, which is the triple, and the octave and a fourth in the ratio put together from the duple and the epitritic, which is that of 8 to 3. For the fact that this ratio is neither epimoric nor multiple will now be no embarrassment to us, since we have adopted no preliminary postulate of that sort.

According to Barbour, “Ptolemy’s syntonic diatonic has special importance to the modern world because it coincides with just intonation, a tuning system founded on the first five intervals of the harmonic series - octave, fifth, fourth, major third, minor third.”

[18] Ptolemy, op. cit., pp. 295-96.

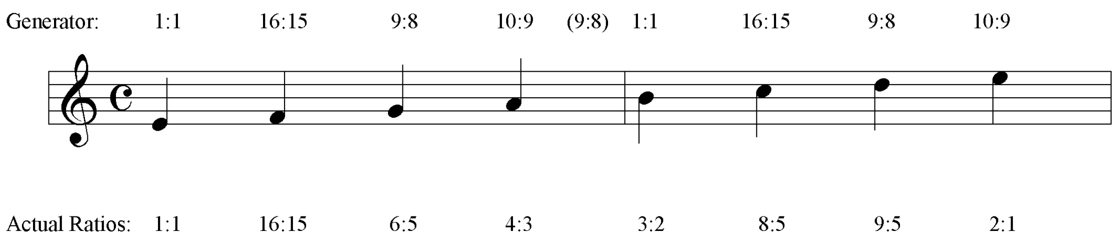

Figure 12. Ptolemy's diatonic syntonon (adapted from Barbour)

As pointed out by Barbour, Didymus' diatonic scale (Figure 11) used the same intervals as Ptolemy's diatonic syntonon, shown in Figure 12, but in slightly different order.[19] Actually, the intervals sound very different, which is not surprising considering that the order of Ptolemy's Diatonic Syntonon corresponds to a justly tuned C major scale starting on E, whilst Didymus' Diatonic scale does not. Of course, neither of these tunings was derived from polyphonic music, but Ptolemy's Diatonic Syntonon would certainly sound better than Didymus' diatonic scale when used as the basis to tune a piece of four-part tonal music, such as a chorale by J. S. Bach. As with the diatonic scale of Didymus, this single scale tuning is not an indication of standardised tuning, as can be seen by contrasting it with another version of his diatonic scale, shown in Figure 13.

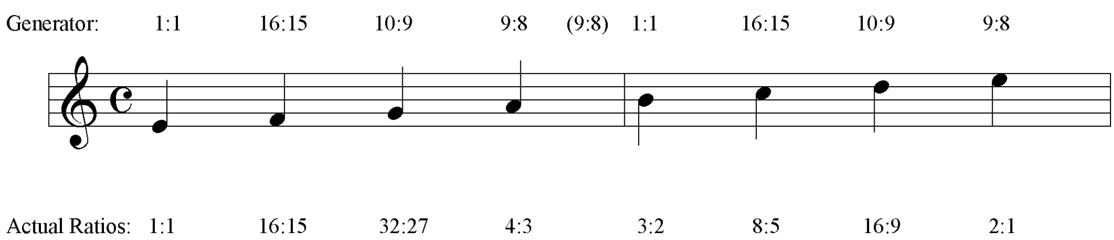

Figure 13. Ptolemy's diatonic (adapted from Barbour)

[19] Barbour, op. cit., p. 20.

2.2 Important Figures of the Renaissance

2.2.1 Gioseffo Zarlino (1517 – 1590)

Notable composer, conductor and student of Adrian Willaert, Gioseffo Zarlino of Venice was a key figure in the development of music theory during the Renaissance. Zarlino's first and most significant work, Le Institutioni Harmoniche (1558), builds upon the theory of the Ancient Greeks and the Middle Ages, as well as introduces new theories of tuning and composition, including counterpoint. His views on tetrachords emphasize the structure of major and minor triads, anticipating the theory of Rameau.

Zarlino's two other noted papers on music theory are the Dimostrationi Harmoniche (1571) and the Sopplimenti (1588), both of which are mostly extensions of Institutioni Harmoniche. The latter of these papers is a response to Vincenzo Galilei's critique of his earlier writings.

Most notable in terms of theory is Zarlino's endorsement of the ratio 5:4 as the basis of the major third, a view that put him at odds with many of his contemporaries. For this reason, Zarlino may be thought of as having been an early crusader for the cause of just intonation, but he also embraced meantone temperament for the keyboard,[20] as shown in Table 3, as well as tempered tuning for the lute.

Note	C 0	C# -2	D –4/7	E♭ +6/7	E -8/7	F +2/7	F# -12/7	G -2/7	G# -16/7	A –6/7	B♭ +4/7	B –10/7	C 0
Cents	0	70	191	313	383	504	574	696	817	887	1008	1078	1200

M.D. 25.0 S.D 25.3

Table 3. Zarlino's comma temperament (2/7 comma)

[20] Barbour, op. cit., p. 33.

2.2.2 Vincenso Galilei (c. 1520 – 1591)

Father of the renowned scientist Galileo, as well as a student of Zarlino, Vincenso Galilei was a composer, lutenist and theoretician. His most important work in music theory was the Dialogo Della Musica Antica et della Moderna (1581). In the field of acoustics, he made three important observations about strings and the pitches they produce:

1) The frequencies of two strings having the same tension are inversely proportional to their lengths.

2) The frequencies of two strings having the same length are proportional to the square of their respective tensions. Thus, two identical strings with identical lengths will produce an octave when weighted with one and four pounds, respectively.

3) The frequency of a pipe is inversely proportional to the cube of the volume of air it contains.

2.2.3 Christian Huygens (1629 – 1695)

A scientific giant of his time and the first to propose the wave theory of light, the Dutch scientist Huygens also dabbled in tuning. His many investigations into systems of intonation, mostly those of twelve-note scales, were influenced by the writings of Mersenne and Zarlino, among many others.

Huygens' most noted tuning system divides the octave into 31 logarithmically equal parts. This concept of using additional pitches to approximate important intervals of meantone tuning anticipates the organ designs of Helmholtz, who tried to take advantage of the human auditory system's limitations in frequency discrimination to provide the illusion of achieving modulations within 5-limit just intonation.

Huygens' 12-note tuning systems all revolve around the essential principle of meantone tuning, namely, of tempering the perfect fifths in order to make the major thirds in tune. Once the thirds are so obtained, they are broken into two equal tones by taking the square root of the major third.[21] Of course, neither perfect fifths nor major thirds are made truly in tune, and the resulting compromise becomes the tuning system that can be implemented on a limited set of fixed strings.

2.2.4 Marin Mersenne (1588 – 1648)

An eminent French mathematician renowned for his studies in prime and perfect numbers, Mersenne is an important figure in music for several reasons. First, he observed that a period of a string's waveform over time is determined by length, tension and density (a discovery made independently by Galileo). Mersenne is also credited for having been the first to understand the relationship between partials and their fundamentals. In terms of tuning, he is best known for describing in 1635 a theoretical system of tuning known as equal temperament that would allow all of the semitones on the keyboard to produce precisely the same interval.

It is more of historical than theoretical significance that Mersenne systematized equal temperament, a tuning system that his own detailed theory refutes. Although Mersenne was first and foremost a mathematician, he was also arguably the most advanced tuning theorist of his time. Whilst it is true that he performed the mathematically trivial task of formalizing equal temperament, Mersenne neither endorsed nor favoured it as a theoretical model. His views on just intonation are clearly expressed throughout his masterpiece, "Harmonie Universelle", as in the following passage:

> If the string EG which I transfer onto LM is divided so that LK has three parts while LM has four, it will produce the fourth. If LK has two parts of the string and LM three, one will have the fifth; if one gives eight parts to LM and three to LK, one will have the eleventh. If LM has three parts and LK one, one will have the twelfth; and if one divides this string in such a way the LM has four parts and LK a single part, one will have the fifteenth. And there will be no person but does not acknowledge these consonances to be very exact, and that the ratios correspond perfectly to experience.[22]

[21] Huygens, op. cit., pp. 46-47.
[22] Mersenne, op. cit., pp. 28-29.

Proposition VI from Harmonie Universelle is extremely important, as it asserts that a single string divided into eight parts, shown in Figure 14, contains all of the consonances, given by Mersenne to be the octave, fifth, fourth, major and minor thirds, and major and minor sixths. Notice in the following quote how Mersenne rejects intervals based on the prime factor seven because they create too much complexity when mixed with simpler intervals:

> Let the string AI be divided into eight equal parts. I say first that EA makes the unison with EI, secondly that the octave is as AI to EI; thirdly that the fifth is AI to GI; and the twenty-second, AI to HI. The twelfth is AG to EG; the nineteenth GA to FG; and the fifth, AG to CG.
>
> The major seventeenth is AF to EF. The major tenth AF to DF, and the major third AF to BF. The fourth is AI to AG; the minor third is AG to BG or AF. The major sixth is AF to AD, the minor sixth is AI to AF, and the eleventh is AI to AD. And if we wish to use numbers, we will find first in the first six parts of the monochord which represent the string of a monochord divided into six parts that the octave is at one to two, the twelfth at 1 to 3, the fifteenth 1 to 4, the major seventeenth 1 to 5, the nineteenth 1 to 6, the fifth 2 to 3, the fourth 3 to 4, the major third 4 to 5, and the minor third 5 to 6. If to this is added the seventh and eight parts of the preceding monochord, one will have as many dissonances as of comparisons which can be made of each number with seven, which is so unsatisfactory in the harmony that it can make nothing but dissonances with other numbers from one to thirteen.[23]

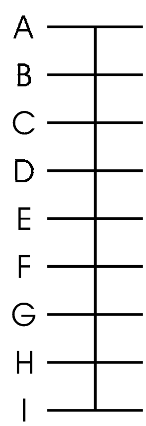

Figure 14. Mersenne’s eight-part string of consonances

[23] Mersenne, op. cit., pp. 32-33.

Foreshadowing the theory of Rameau and Helmholtz, Mersenne excludes from his list of consonances, derived from eight divisions of the string, any interval whose ratio contains a prime factor greater than five in the numerator or denominator. Note in Table 4 that all of the numerators and denominators for Mersenne's named intervals (unnamed intervals not being shown here) can be broken into prime factors of 2, 3 and 5.

Table 4. Mersenne's consonances and dissonances

Interval	Ratio
Octave	2/1
Major seventh	15/8
Minor seventh	9/5
Minor seventh	16/9
Major sixth	5/3
Minor sixth	8/5
Fifth	3/2
Tritone	45/32
Fourth	4/3
Major third	5/4
Major tone	9/8
Minor tone	10/9
Major semitone	16/15
Minor semitone	25/24

Mersenne notes that when two or more intervals are to be played on a single string, the minimum number of divisions for that string to accomplish this task can be determined by summing the numerator and denominator for each interval, then multiplying all such sums together. The following excerpt demonstrates Mersenne's simple formula to determine the overall complexity for a group of intervals:

> For example, if one wishes to find the octave, the fifth, the fourth, and the major third, it is necessary to multiply 5 by 3 to have 15, and 15 by 7 to have 105, and finally 195 <105, sic> by 9 to have 945, which signifies that the string must be divided into 945 equal parts to serve as monochord for the four abovementioned consonances, of which the octave will be found on 626 <630, sic>, which is two-thirds of 945. The finger being placed on 567 will give the fifth, if one places it on 540, one will have the fourth; and finally if one places it on 525 he will have the major third.[24]

[24] Mersenne, op. cit., p. 51.

This simple formula is not optimally efficient. As an example, for the three intervals octave, fifth and double octave, 75 divisions would be indicated, when in reality only 15 are required. A better approach would be to reduce each fraction; sum its numerator and denominator; take all such sums and remove duplicates; and take the least common multiple of all remaining sums.

2.3 The Pythagorean (Ditonic) Comma

All systems developed subsequent to Pythagorean tuning and prior to the general adoption of equal temperament had to address the problem of the Pythagorean comma, which is the difference between twelve successive perfect fifths (3:2) and seven successive octaves (2:1). As it turns out, 3/2 raised to the power of 12 is roughly, but only roughly, an octave equivalent of the original note. This discrepancy, which amounts to 23.4 cents or about a quarter of a semitone, is known as the Pythagorean or ditonic comma.

It so happens that no matter how many times a given frequency is multiplied by 3/2, it will never equal an octave equivalent. It is readily apparent that no power of two can ever coincide with a power of three. For one thing, all powers of three are odd, whilst all those of two are even. More generally, “no power of one prime can ever equal some power of any other prime”.[25] Thus, the “circle” of fifths is really a spiral that never rejoins at its original point.

2.4 Overview of Meantone Temperaments

Already in use at the beginning of the sixteenth century, meantone temperaments employ a compromise to alleviate the problems inherent in the Pythagorean twelve-note system. Meantone temperament can generally be characterized by the adjustment of the perfect fifths so as to bring the major thirds (frequency ration 5/4) more closely into tune. To achieve this, the tuner must temper the fifths and fourths by making the fifths smaller and the fourths

[25] Hall, op. cit., p. 544.

larger.[26] The major third in meantone tuning is generally broken into two equal whole-tones, even though such a division was demonstrated impossible if the major third is tuned as 5:4 or as any epimoric ratio. Strictly, meantone tuning is a system of tuning with flattened fifths (the fourth root of 5) and pure major thirds (5:4).

Meantone temperaments have severe drawbacks. By tuning a keyboard in this manner for the key of C major, for instance, the key of E Major might sound very out of tune in comparison. There are numerous systems defined under the general heading ‘Meantone’, but they all share the same defects that no key is perfectly in tune and that some keys are quite out of tune, necessitating retuning of the keyboard for different compositions. In addition to these limitations, modulation is restricted due to the dissonance of certain keys. For example, Barbour refers to a table entitled “Rousseau’s Monochord”, duplicated in Table 5:

Names	C^{0}	C#$^{-2}$	D^{0}	E♭$^{+1}$	E^{-1}	F^{0}	F#$^{-2}$	G^{0}	A♭$^{+1}$	A^{-1}	B♭$^{+1}$	B^{-1}	C^{0}
Cents	0	70	204	316	386	498	568	702	814	884	954	1088	1200

M.D. 25.0; S.D. 26.7

Table 5. Rousseau’s monochord (from Barbour)

Presumably we are analysing a keyboard tuned exactly as prescribed in the table. But what key are we in? Why should a twelve-note keyboard ever be tuned outside of equal temperament if the key is not known in advance? Such basic questions are ignored among the statistical analyses that yield no practical insights whatsoever.

For example, the interval from C to D will have a very different sound when played in the key of B♭ than when played in the key of C, unless equal temperament or some similar logarithmic division using smaller intervals is used. Flexible just intonation, for instance, would implement the interval 10/9 to the former, and 9/8 to the latter version of this major second, or the reverse depending on the chord progression. Because of such uncertainties, equal temperament is ideal for keyboards when the key is unknown.

[26] Mark Lindley, *The New Grove Dictionary of Music and Musicians*, Vol. 2, (London: Macmillan Publishers Ltd., 1980), p. 875.

Perhaps the most important lesson to learn from meantone temperaments is that they can never provide the theoretical basis for a comprehensive theory of tuning. It is intuitively obvious that any twelve-note tuning system other than equal temperament will sound better in some keys than in others. Hall, who gave a mathematical proof that no twelve-note meantone system can be ideal, summarizes his achievement:

> First, we can show quite simply that an ideal just scale of twelve pitches is mathematically impossible, and that we must choose which of many out-of-tune scales will best serve our needs. Second, there may not be any one best choice; when we examine our criteria for this choice carefully, we shall find that we choose different tunings depending on which piece of music is to be played.[27]

2.5 Equal Temperament

2.5.1 Origins of Equal Temperament

According to Jorgensen, Ling Lun (twenty-seventh century BC) of China is the first known person to have formulated equal temperament.[28] If this is true, it would be interesting to determine what influences ancient Chinese instruments might have had, perhaps as a result of undocumented exchange between the two cultures in ancient times, on Greek instruments and, by extension, musical thinking. In the event that Chinese musical traditions were completely or very nearly distinct from Western music during this period, the implication would be that twelve tone tuning systems evolved independently in two different cultures separated by two thousand years, therefore implying an underlying set of principles leading musicians to adopt its practice.

[27] Hall, op. cit., pp. 543-552.

[28] Owen Henry Jorgensen, *Tuning* (East Lansing: Michigan State University Press, 1991) p. 15.

Long before Mersenne developed a precise mathematical description of equal temperament, its approximation was already being implemented. According to Barbour:

> From the middle of the sixteenth century, all the theorists agreed that the fretted instruments, lutes and viols, were tuned in equal temperament. Vicentino made the first known reference to this fact, going so far as to state that both types of instrument had been so tuned from their invention. If we may believe pictorial evidence, especially that of the Flemish painters, so meticulous about detail, frets were adjusted to equal temperament as early as 1500, although there is not complete agreement on this point.[29]

Whether Barbour could really tell if an instrument is tuned in equal temperament by simply looking at its picture, no matter how accurately painted, is surely questionable. However, Barbour's claim is shared by others such as Lindley, who states, "Just Intonation fretting schemes (with a distinction for two sizes of whole-tone) remained always a matter of theory and experiment, never of common practice."[30] Barbour maintains that equal temperament had, for practical purposes, been used as far back as the lute, arguing, "Vicentino stated that the fretted instruments had always been in equal temperament. As for the keyboard instruments, Zarlino declared that temperament was as old as the complete chromatic keyboard."

He continues, "However that may be, Riemann discovered the first mention of temperament in a passage from Gafurius' Practica music (1496). There, among the eight rules of counterpoint, Gafurius said that organists assert that fifths undergo a small, indefinite amount of diminution called temperament (participata)."[31]

[29] Barbour, op. cit., pp. 8-9.
[30] Lindley, *Lutes, Viols and Temperaments* (Cambridge University Press, 1984), p. 94.
[31] Barbour, op. cit., pp. 25.

2.5.2 Mersenne's Formalization of Equal Temperament

What is known with certainty is that Mersenne, the very same early devotee of just intonation, demonstrated in 1635 a formalized method of flattening each of the Pythagorean fifths by approximately 1/50 of a semitone in order to make the Pythagorean comma disappear. The now obvious solution he adopted was to merely take the appropriate root, in the case of twelve notes, the twelfth root, that gives the value of the octave, 2/1, when raised to the power of twelve.[32] The result, $\sqrt[12]{2}$ or approximately 1.0594631, is an irrational number, and thus cannot be calculated in complete detail. The semitones generated by using powers of this result, except for octave equivalents, have irrational vales. Figure 15, which generates a chromatic scale based on A–440, illustrates the concept of equal temperament.

$$\begin{aligned}
A &= 440 \times (\sqrt[12]{2})^{0} = 440.000cps \\
A\# &= 440 \times (\sqrt[12]{2})^{1} = 466.164cps \\
B &= 440 \times (\sqrt[12]{2})^{2} = 493.883cps \\
C &= 440 \times (\sqrt[12]{2})^{3} = 523.883cps \\
C\# &= 440 \times (\sqrt[12]{2})^{4} = 554.365cps \\
D &= 440 \times (\sqrt[12]{2})^{5} = 587.329cps \\
D\# &= 440 \times (\sqrt[12]{2})^{6} = 622.253cps \\
E &= 440 \times (\sqrt[12]{2})^{7} = 659.255cps \\
F &= 440 \times (\sqrt[12]{2})^{8} = 698.456cps \\
F\# &= 440 \times (\sqrt[12]{2})^{9} = 739.988cps \\
G &= 440 \times (\sqrt[12]{2})^{10} = 783.990cps \\
G\# &= 440 \times (\sqrt[12]{2})^{11} = 830.609cps \\
A &= 440 \times (\sqrt[12]{2})^{12} = 880.000cps
\end{aligned}$$

Figure 15. Equal temperament based on A-440

In equal temperament, as with its ancestor, Pythagorean tuning, major and minor thirds are noticeably out of tune, especially as they stand out so clearly in a tonic triad, but are considerably more in tune than Pythagorean thirds. Specifically, the Pythagorean and equally

[32] Arthur Benade, *Fundamentals of Musical Acoustics* (New York: Oxford University Press, 1976), pp. 295–296.

tempered versions of the major third are sharp to that used in just intonation by approximately 21 cents and 14 cents, respectively, showing equal temperament to be the better match. Equal temperament's minor third is also more nearly matched to the just minor third (the type found in a minor triad) than is the Pythagorean minor third, with the equally tempered and Pythagorean versions being flat to the 6/5 ratio by 16 and 22 cents, respectively.

There are three important points to consider with regard to equal temperament. First, it is the optimum system when using twelve pitches and when the key is unknown, therefore making it ideal for modulation to any key within a twelve-note system. Second, it closely approximates important intervals in just intonation, being a vast improvement in this regard over Pythagorean tuning. Finally, the only intervals that are truly in tune according to integer ratios when using equal temperament are the octave and its equivalents.

2.5.3 Complete Solution for 12-Note System Impossible

Hall (1974) provided a detailed mathematical proof that it is impossible to find an ultimate tuning system using twelve notes. He summarizes as follows:

> The point to which this all leads is that there is no way to provide a full complement of pure intervals with only twelve notes per octave; that is, there does not exist any arrangement which constitutes a 'tuning' for both fifths and thirds. We are always forced to have tempering of one or the other, and the only question is how to distribute this temperament.[33]

2.6 The Continued Development of Just Intonation

2.6.1 Jean-Philippe Rameau (1683 – 1764)

Continuing the theoretical traditions of Zarlino, Mersenne and others, Rameau, France's greatest composer and theoretician of the time, sought to classify chords as collections of discrete frequencies derived from the acoustical properties of strings. In agreement with Descartes, whose empirical methodology he adopted, Rameau believed that the ear could

[33] Hall, op. cit., p. 547.

distinguish only the first six partials, with the seventh partial falling outside the realm of functional harmony.[34] As will be seen many times in subsequent chapters, this view in a slightly modified form is supported by a wealth of modern experimental data.

In classifying chords, Rameau rightly points out that of the first six partials, “only the octave, the fifth, and the major third are directly generated by the fundamental sound.” For instance, E♭, related to C as 6:5, does not occur in the harmonic series of C, whereas E, related to C as 5:4, does.

After deducing that a string cannot directly generate the minor third, Rameau includes the minor third in the primary consonances anyway, as shown in Table 6.[35] Rameau authoritatively refutes the observation of Descartes that a minor sixth can be justified as being an inversion of a major third, when Rameau himself allowed the minor third in a similar manner. Perhaps Rameau sought to simplify harmonic theory by allowing a minor third to be a primary consonance; otherwise, the underpinnings of his most recognized contribution to music theory, that of the fundamental bass in which every chord is identified as a collection of major and minor thirds, would have seemed less self evident, which perhaps they should have.

Primary Consonances	Secondary Consonances
Perfect fifth	Perfect fourth
Major third	Major sixth
Minor third	Minor sixth

Table 6. Rameau’s primary and secondary consonances

Rameau treats major and minor triads, which he calls “perfect”, in the traditional fashion that the former be tuned in the ratios of 1:3:5 and the latter as 10:12:15. Other chords are generated by multiplying the numerators of both intervals with both the numerators and denominators of the other. For instance, a minor third 5:6 and a perfect fifth 2:3 produce 10, 18, 15, and 12. Hence, the new chord, the minor seventh, is tuned as 10, 12, 15 and 18.

[34] Rameau, op. cit., pp. 4-6.
[35] Rameau, op. cit., p. 15.

Although this method is not justified, it does ensure that all numerators and denominators can be broken into only the prime factors 2, 3 and 5 that Rameau allows in his theory, and is in complete agreement with the principles of Mersenne. A cursory tuning experiment validates the use of this tuning, which is really just a minor triad with a seventh that is related to the minor triad's third as 3:2.

Rameau's tuning of the dominant seventh, as is the case in general, does not attempt to relate the seventh of the chord to the chord that follows. Thus his use of 20:25:30:36 for a dominant seventh is logical only within itself and is not in tune to the ear. Perhaps most importantly, its seventh is too sharp to serve as a dominant, denying the important relationship between the seventh of a dominant and the root of tonic, namely, an octave equivalent of 1/3 to 1, as opposed to his own octave equivalent of 27/15 to 1, vastly less direct.

Rameau probably would not have adopted tunings that would have been considered unconventional from the time of Helmholtz onwards had he possessed equipment that could reliably tune chord progressions. His tuning for the major scale, shown in Figure 16,[36] would not sound suitable as a melody if its underlying chord were functioning as a dominant. For example, if A is 1 and E is 3:2, then the fifth of E would be B 9:8, which does not equal 10:9.

However, in the less likely event that the B, the second note in an accompanied major scale, is the root of a B minor ii chord, the tuning would be correct. This assumes that the B minor ii chord's D maintains a very important relationship to tonic A of 4:3. Such an assumption, along with one that a minor triad is tuned in the familiar 10:12:15 relationship, would be consistent with Rameau's tuning of 10:9 for a B.

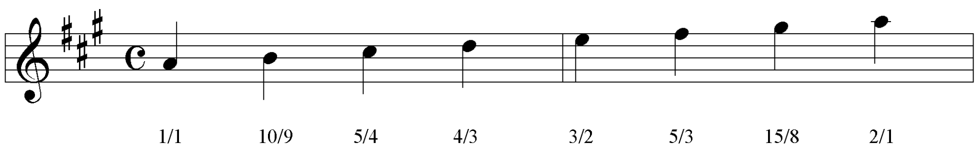

Figure 16. Rameau's perfect diatonic system

[36] Rameau, op. cit, p. 28.

Rameau trivializes the difference between 10:9 and 9:8 in another context whilst explaining the comma. This difference of 81:80 is actually nearly 22 cents, and is therefore very audible in nearly any musical context.

> The two different tones arise because of the difference between the major tone and the minor, both of which are present in the diatonic system; this difference is a comma whose ratio is 80:81. Although the ear cannot perceive this difference, particularly in intervals suitable for harmony and melody, it is nevertheless appropriate for us to explain the relationship of this comma to the different notes of the system from which an interval can be formed.[37]

Perhaps Rameau's greatest contribution to music theory is the concept of the fundamental bass, in which each chord is projected by a single root, regardless of the inversion of the chord. Although the principle upon which this concept was based, namely that each chord could be generated by a single string, was greatly simplified because, for instance, the string that generates an E minor triad does not correspond with the note E, but rather with an unheard lower C, the concept itself is central to the issue of tuning and harmony.

2.6.2 Hermann von Helmholtz (1821 – 1894)

Helmholtz was the consummate Renaissance man of the nineteenth century. Trained first as a physician, Helmholtz is best known in the disciplines of physics and mathematics, in particular for extending and mathematically formulating the law of the conservation of energy. In addition, he made notable contributions in the field of anatomy, being the first to measure the velocity of nerve impulses. In the area of optics, he invented the ophthalmoscope and authored the three-volume treatise Physiological Optics, which remained for decades the authoritative work on the physiology of vision. Even to list his more notable achievements in nearly every area of science would be beyond the scope of this brief introduction to tuning's most celebrated figure.

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[37] Rameau, op. cit., p. 32.

Helmholtz described resonance not in terms of string lengths or tensions, but rather in physiological terms. According to his resonance theory, hair cells of the inner ear perform the function of tuned resonators, a concept that is still the main topic of discussion among those studying tuning from the vantage point of acoustics. This view that incoming sound is filtered by the inner ear into discrete signals that are later reintegrated by the brain is discussed at length in Chapter 3, “Psychological Considerations”, beginning on p. 41.

2.6.3 Harry Partch (1901 – 1974)

Harry Partch, perhaps the most well-known tuning specialist of the twentieth century, believed that integer ratios form the foundation of pitch classes. He proposed that, "The faculty, the prime faculty, of the ear is the perception of small number intervals, and the ear cares not a whit whether this intervals are in or out of the overtone series." This being or not being the case, as it may, Partch seemed to embrace many intervals that would be at odds with the literature, as shown in Table 7. For example, Partch made common use of the prime generator 7 in both the numerator and denominator.[38]

Ratio	Interval Name
7/6	Septimal minor third
12/7	Septimal major sixth
8/7	Septimal major second
7/4	Septimal minor seventh

Table 7. Septimal intervals of Harry Partch

Partch is never at a loss to provide colourful names for his intervals, such as “diapente, sesquialterate interval” for perfect fifth (3/2) or “sesquiocteran interval” for a major second (9/8). Whether or not the number seven appears in the numerator or denominator, Partch assigns these intervals the name “septimal”.

[38] Harry Partch, *Genesis of a Music,* Second enlarged edition (New York: Da Capo Press, 1974), p. 68.

He fails, however, to provide a unifying set of principles for using less consonant intervals, such as those in Table 8. More importantly, he does not perform or refer to any listening tests that validate these particular values, nor does he specify the tonal context in which they should be used. It is probably because of his experimental compositions that Partch's name is more closely associated with tuning than perhaps any other person of the twentieth century.

Ratio	Interval Name
25/24	Small semitone or minor second)
48/25	Large just major seventh
45/32, 64/45	Just "tritones"
7/5, 10/7	Septimal "tritones"

Table 8. "Non-classical" intervals of Harry Partch

Partch's absolute consonance scale, (1/1, 2/1, 3/2, 4/3, 5/4, 5/3, 6/5, 7/6, 7/5, 7/4, 8/7), appears to be both arbitrary and unsupported either by experimental evidence or by a cohesive theory. He either fails to describe or to understand that 9/8 is vastly more consonant than 7/4 because 9 and 8 are both composites, whereas although 4 is also composite, 7 is not. That 8/7 presumably precedes 8/5, which is not even given, illustrates Partch's asystematic approach. Possibly Partch is excluding these consonant intervals for some unexplained reason, but careful reading reveals no clear structure to his theory.

The theorist should strive to understand what the limitations of the human auditory system are, and thus direct the music theory to remain within this limited perceptual framework. Partch seems to prefer his theory to the perceptual reality that the listener will not readily understand chords built on overly dissonant intervals.

As many people do, Partch believed that 7:1, the ratio of the seventh partial to the first, is the derivation of the seventh in a dominant seventh chord. If this is the correct tuning for a dominant seventh, as so many claim it is, why did the twelve-note system that has survived so many centuries allow this note to be near the middle of a minor second, where it could never be used, when it is ostensibly a member of the dominant seventh chord, the fourth most important chord function (after major tonic, minor tonic and dominant triad) in tonality? Why, for instance, do performers of brass instruments systematically use fingerings that avoid the seventh harmonic? For example, a C trumpet playing the seventh of a C dominant seventh chord leading to F major could play the note with no valves pressed at all instead of the first valve, the choice of which is associated with the root of the subdominant. One explanation might be that the vast majority of trumpeters prefer not to have the conductor stop in the middle of the phrase and point out the offensive flatness of their theoretical experiments. Where is the experimental evidence that this tuning for the seventh of a dominant seventh chord, 21:16 in relation to tonic, is preferred to the vastly more simple octave equivalent of 4:3 that is so conveniently also the root of the subdominant?

With similar disregard for tradition, Partch dismisses 5-limit theory, as if it were arbitrarily selected throughout the history of instrument making, with the proclamation:

> Among them there was no agreement that a 5-limit should prevail, but because it was expedient in the building and tuning of fretted instruments and because its demands on notation were less complex, and for no other primary reason, it prevailed.[39]

With this somewhat confused (as it would seem they were really arranging their frets in accordance with something like equal temperament) and unsupported claim, Partch, as with so many of his followers, ignores the likelihood that five-limit just intonation, along with its closely associated approximation equal temperament, prevailed specifically because it already pushes human psychoacoustic frequency resolution near to its limit. If there were some major benefit to using pitches derived from 7/5, 8/7, 13/12 or similarly complex ratios, both historical and experimental evidence would support their use. Such intervals will almost certainly still be perceived as mistuned intervals generated from small primes. (Please see section 3.4.4 on p. 61.)

[39] Partch, op. cit., p. 119.

2.7 Summary

The octave, perfect fifth and perfect fourth were associated with the respective string lengths of 2:1, 3:2 and 4:3 by the Pythagoreans around 500 B.C. These basic consonances remained at the heart of Greek music theory until Didymus proposed in the second half of first century B.C. that the interval 5:4 was also consonant. By the turn of the first century A.D., Ptolemy was clearly using what we now call 5-limit just intonation, as all of the ratios in many of his scales could be generated from the first three primes 2, 3 and 5.

Ptolemy's use of 5-limit just intonation remained a guiding principle for theorists of the early Renaissance, especially Zarlino, who began to apply the theory to the tuning of chords. The culmination of Renaissance tuning theory is found in Mersenne's Harmonie Universelle, which states that a single string divided into eight parts contains all of the consonances. He further asserted that the number seven was too dissonant to the ear to be used as a generator for consonant intervals.

France's most celebrated classical composer and theorist, Rameau, adopted the empirical methodology of Descartes in his study of chords and chord progressions. Rameau's belief that the ear could distinguish only the first six partials, with the seventh partial falling outside the realm of functional harmony, is in complete agreement with Mersenne's theory.

A pivotal figure in tuning history is the nineteenth century physician and scientist Helmholtz, whose detailed investigations into the workings of the ear laid the foundations for all of the important developments in tuning that would follow in the twentieth century. The theory of Helmholtz, in agreement with Mersenne and Rameau, that the human auditory system's limitations with regard to frequency determine, to a large extent, what we perceive as consonant or dissonant will be a key topic of discussion in the following and subsequent chapters.

Chapter 3
Psychological Aspects of Tuning

> *"Consonance is the blending of a higher with a lower tone. Dissonance is incapacity to mix, when two tones cannot blend, but appear rough to the ear."* [40]

The degree to which a given tuning of an interval, chord or tonal passage is perceived as being in tune is determined by numerous interrelated parameters that include register, timbre, duration, amplitude, voice spacing and many others. Such parameters can be combined in an endless array of permutations to form tonal structures and their corresponding perceptual entities, each associated with its own unique quality and degree of consonance/dissonance. For example, in the lowest octave of the piano, the normally consonant interval of a major third, no matter how its frequencies are set, will be dissonant, especially when its timbre is rich in partials; it is held for a long duration; and when it played with a high amplitude.

This chapter will explore three categories of research pertaining to tuning:

1) The reprocessing of complex physical signals by the cochlea into a collection of discrete nerve impulses

2) Other psychological factors that affect the perception of pitch, such as pitch discrimination and the dominance region

3) The most prominent theories of intonation

From these diverse perspectives, it is possible to piece together a model upon which to base a more detailed and complete theory of tuning. Of special interest are theories and their corroborating evidence that indicate only specific combinations of partials are perceived by the auditory system. The fact that several competing theories are incomplete in themselves indicates that auditory events are filtered and recombined in a way that is not yet entirely

[40] Euclid, quoted by Hermann L.F. von Helmholtz, *On the Sensations of Tone as a Psychological basis for the Theory of Music* (New York: Dover, 1954), p. 226.

understood. From these incomplete theories, it is nevertheless possible to gain a better understanding of the underlying mechanisms affecting tuning.

3.1 Tuning Considerations Regarding the Auditory System

3.1.1 Outer Ear

Sound waves first reach the auditory system at the visible part of the ear, the pinna, which provides clues as to the direction of the sound. These vibrations of sound continue past the pinna through the ear canal, where they are amplified, especially for a certain frequency range. Helmholtz stated that the ear favours notes between 2640 and 3168 Hz "by its own resonance" and that partials in this range are rendered consonant.[41] This is corroborated by Mathews:

> The ratio of sound pressure at the eardrum to sound pressure at the outer opening of the canal will be greatest at a frequency around 3000 Hz. From this alone, we can deduce that the maximum sensitivity of the ear to sound should be around 3000 Hz.[42]

It is curious that the ear canal has evolved so that it maximally amplifies frequencies centred around 3000 Hz, whereas the frequency to which the sensitivity and resolving power of the cochlea are greatest is approximately 1000 Hz. The cochlea could be focusing, via outer hair cells thought to boost selected frequencies, upon fundamental frequencies associated with upper partials that are, on the average, three times the frequency of the fundamental. This would be consistent with the dominance region theory, as described in section 3.3.5 on p. 55.

[41] Hermann L.F. von Helmholtz, *On the Sensations of Tone as a Psychological basis for the Theory of Music* (New York: Dover, 1954), p. 116.

[42] Max Mathews, 'The Ear and How it works,' (Perry R. Cook, Editor) *Music, Cognition and Computerized Sound* (Cambridge: MIT Press, 1999), p. 4.

3.1.2 Middle Ear

Sound vibrations are transferred from the eardrum to 3 ossicles, or small bones, called the malleus, incus and stapes. Forming a lever system, these three small bones amplify the sound vibrations stimulating the oval window that interfaces with the cochlea. By the time the vibrations reach the inner ear through the oval window, they are magnified as much as 800-fold due to the shape of the ear canal; the lever system of the ossicles; and the pinpointing arrangement of the eardrum and the oval window.

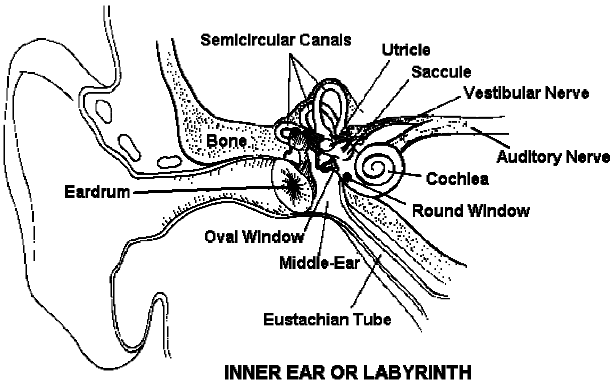

Figure 17. The mechanism of hearing (http://www.teleport.com/~veda/gallery.html)

3.1.3 Cochlea (Inner Ear)

Mechanical energy from the stapes enters the inner ear through the oval window, and travels along a wave front from the base to the apical end of the helicotrema. This creates vibrations in the basilar membrane that are transferred to inner hair cells. More than 23,500 overlapping cilia ensure that for each frequency, energy is dispersed between many inner hair cells.

Hair cells within the organ of corti are tonotopically ordered. (See Figure 18.) Higher frequencies are absorbed first by inner hair cells near the base; mid range frequencies are absorbed by their associated inner hair cells between the base and the apex; finally, the low frequency energy stimulates inner hair cells near the apex.

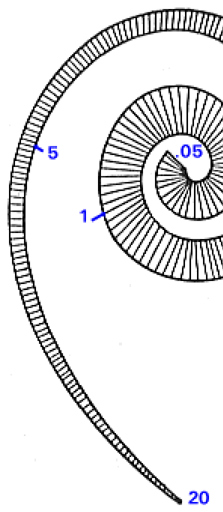

Figure 18. Distribution of frequencies (KHz) along basilar membrane according to G. von Bekesy (http://www.iurc.montp.inserm.fr/cric/audition/english/ear/cochlea/cochlea.htm)

Information is sent to the auditory cortex through the eighth cranial nerve, a bundle of approximately 30,000 individual fibres, which sends a coded electrical signal corresponding to the amplified vibrations of sound it has received. Each auditory nerve fibre uses discrete action potentials to transmit sound-induced signals to the brain. Included in an auditory nerve signal sent to the brain is information regarding which nerve fibres are activated, as well as the rate and time pattern of the spikes in each fibre. For the study of intonation, it is important to note that the approximately 30,000 nerve fibres leaving the cochlea to form the auditory nerve are grouped according to the frequency of sound being represented.

The nerve fibres comprising the auditory nerve lead to different parts of the auditory cortex depending on the frequencies they carry. The auditory cortex lies in a deep furrow called the Sylvian fossa. The high tones terminate deep within the Sylvian fossa, whilst the low tones end near the outer surface.[43]

[43] David Worrall, 'Course notes for Physics and Psychophysics of Music' (WWW: The Australian Centre for the Arts and Technology, Australian National University), http://www.anu.edu.au/ITA/ACAT/drw/PPofM/hearing/hearing5.html.

From the fact that nerve fibres terminate in different locations according to frequency, it can be inferred that a subsequent process must occur in which frequency information is recombined to form complex auditory events. It is significant that unprocessed “residual” frequencies are sent along the eighth cranial nerve, as demonstrated by Houtsma and Smurzynski (1976).[44] (Please see section 3.4.2 on p. 58.)

3.2 Critical Bandwidth Theory

3.2.1 Introduction to Critical Bandwidth

When two sinusoids are adequately separated in frequency, they are processed by the cochlea as distinct entities, with the perceived loudness of one sinusoid being unaffected by the other. In the event that their frequencies are sufficiently close to one another, the two sinusoidal signals interact, thus masking, or diminishing the amplitude of, one another within the cochlea.[45]

For each auditory neuron within the cochlea, a tuning curve, or graph of minimum sound pressure level for neural response versus frequency, can be drawn. The characteristic frequency (CF) is the frequency at which such a neuron can be stimulated with a minimum of amplitude. Deviating from the CF either above or below will cause a corresponding decrease in the output from the neuron being examined.

The psychophysical tuning curve (PTC), similar is shape to the neural tuning curve, is obtained through tests in which the listener hears first a tone whose frequency and amplitude are predetermined, and then a second tone, or masker. The level at which the masker masks the tone is then recorded in order to determine the overall shape of the auditory filter.

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[44] A.J.M. Houtsma and J. Smurzynski, ‘Pitch identification and discrimination for complex tones with many harmonics,’ *Journal of the Acoustical Society of America*, vol. 87 (1990), pp. 304-10.

[45] Mathews, op. cit., p. 9.

Results indicate that the inner ear is a bank of overlapping, linear bandpass filters. A combination of two or three filters, known collectively as a Roex filter, can be used to model the bandwidth of the cochlea for a given frequency.[46] For a single sinusoid, the shape of the auditory filter, as indicated by threshold curves at various frequencies, is exponential.[47] Moore characterizes the shape of the auditory filter as a rounded exponential function with a passband whose skirts are close to exponential but whose top is flattened:

> For a young normal listener, a moderate level, and a 1.0 KHz centre frequency, the equivalent rectangular bandwidth of the filter is about 130 Hz and it is approximately symmetrical on a linear frequency scale. The filter applies an attenuation of about 25dB 300 Hz above or below the signal frequency.[48]

In musical terms, the size of the tonal window in which the sine components of one or more tones interact is approximately a minor third. The size of a critical band will increase as it descends to the frequencies of the lower keys of the piano. Additionally, amplitude causes the size of the critical band to augment, especially in the lower register.

3.2.2 Helmholtz Theory of Dissonance

The predecessor of critical bandwidth theory is Helmholtz' theory of dissonance, which states that dissonance is caused by the beating of partials contained in two or more primary tones.[49] Conversely, the theory claims that consonance results when two tones occur simultaneously without beating, as when two sine waves form the intervals of octaves, perfect fifths, perfect fourths, and, under certain circumstances, smaller intervals. According to Helmholtz:

> Combinational tones are the most general cause of beats. They are the sole cause of beats for simple tones which lie as much as, or more than, a minor third apart."[50]

[46] Patterson and Moore, 'Auditory filters and excitation patterns as representations of frequency resolution,' (Brian C. J. Moore, Editor) *Frequency Selectivity in Hearing* (London: Academic Press, 1986), p. 136-7.
[47] Patterson and Moore, Op. Cit, p. 123.
[48] Patterson and Moore, Op. Cit, p. 173.
[49] Helmholtz, op. cit., pp. 166-85.
[50] Helmholtz, op. cit., p. 204.

Helmholtz postulated that maximum dissonance occurs when the rate of beating between the two tones is about 33 per second, independent of frequency. When the difference in frequency becomes sufficiently large, beating disappears, at which time the interval is consonant. Against musical intuition, such a relationship is consonant, regardless of the ratio between the participating tones, at least when they are pure sine waves.

An important observation made by Helmholtz is that the rapidity of beats obscures dissonance. He argued that for octaves, fourths, and fifths, a relatively large change in frequency difference was necessary to produce slow beating. For sixths and thirds, he claimed that avoiding slow beats between partials required much more precision in tuning. At the same time, with less consonant intervals such as major and minor thirds and sixths, variations in beating become more extreme in relation to the frequency differences between tones. For this reason, as in the following quote, it is possible to play a major third very sharp without much dissonance, not because beats have been avoided, but because they are occurring too quickly to be perceived.[51]

> Hence it is not, or at least not solely, the large number of beats that renders them inaudible. The magnitude of the interval is a factor in the result, and consequently we are able with high tones to produce more rapid audible beats than with low tones.
>
> . . . The beats of a whole tone, which in deep positions are very distinct and powerful, are scarcely audible at the upper limit of the thrice-accented octave [say at 2000 vib.]. The major and minor Third, on the other hand, which in the middle of the scale [264 to 528 vib.] may be regarded as consonances, and when justly intoned scarcely show any roughness, are decidedly rough in the lower octaves and produce distinct beats.
>
> . . . The roughness arising from sounding two tones together depends, then, in a compound manner on the magnitude of the interval and the number of beats produced in a second.[52]

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[51] Helmholtz, op. cit., p. 185.
[52] Helmholtz, op. cit., p. 171-72.

Helmholtz stated that although the number of beats increases as the interval widens, the roughness itself decreases. Roughness, he claimed, remains approximately the same upon octave transposition. It also depends to a lesser extent on amplitude, one of several interdependent parameters examined in the following sections.

3.2.3 Plomp and Levelt

In an often cited study, Plomp and Levelt refined Helmholtz' theory of dissonance by proposing that the dissonance of an interval is determined by the interaction of partials within a critical band.[53] In this study, experimental subjects rated the consonance/dissonance of various randomly generated intervals ranging from unison to octave. Lower tones in the interval pairs were selected from 125, 250, 500, 1000 and 2000 Hz, with both lower and upper tones being simple sine waves.

Consonance/dissonance curves for all registers were approximately the same. However, it was discovered that below about 500 cps, intervals become more dissonant, due to an increase in the size of the critical bandwidth, suggesting one reason that composers tend to use larger intervals in the low register. (Another reason would, of course, be the mimicking of the spacing within the harmonic series itself.) For sine waves, the consonance/dissonance curves show a sharp decrease in consonance that is lowest for intervals equal to approximately one-quarter of a critical bandwidth. (See Figure 19.)

[53] R. Plomp and W.J.M. Levelt, 'Tonal Consonance and Critical Bandwidth,' *Journal of the Acoustical Society of America*, vol. 38 (1965), pp. 548-60.

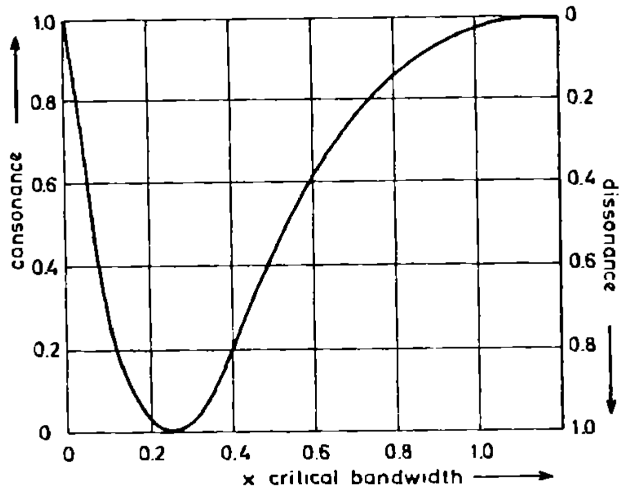

Figure 19. Standard consonance curve for two simple tones (from Plomp and Levelt, 1965)

Plomp and Levelt suggested that consonance/dissonance curves could be simulated for complex waves by summing the consonance/dissonance curves for the partials contained in the complex waves. In these graphs, sharper peaks represent more basic frequency ratios. Note how their computed dissonance for intervals, shown in Figure 20, locates the same peaks, but at quite different relative levels, found in the 1909 listening study by Kaestner, given in Figure 21. It is also noteworthy that all of the peaks predicted for this tone correspond to the ratios between the partial numbers used, implying that timbre and consonance of intervals are deeply related.

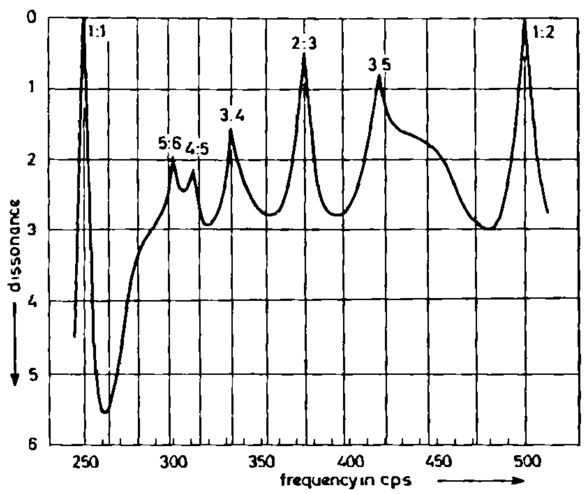

Figure 20. Computed dissonance for intervals between two six-harmonic tones, the lower of which is 250 Hz (from Plomp and Levelt, 1965)

Based on their findings, Plomp and Levelt concluded that frequency distance is more decisive than ratio for simple waves. Further, simple tones separated by more than a critical bandwidth are optimally consonant, whereas the most dissonant intervals are separated by approximately a quarter of a critical bandwidth.

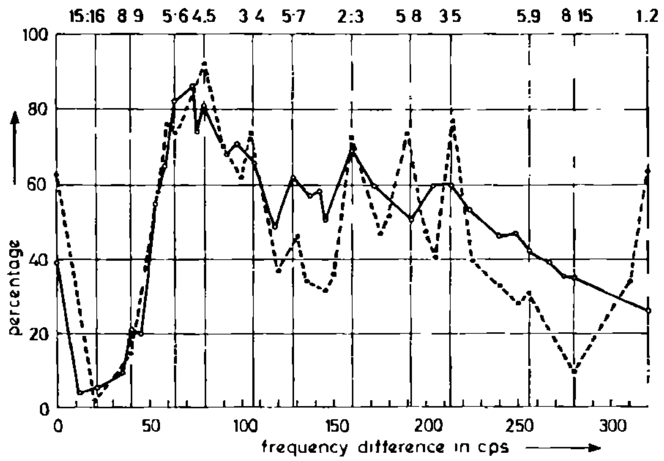

Figure 21. Percentage of cases in which a tone interval was judged more pleasant than others. Solid line is for simple waves, broken for complex. (Plomp and Levelt, 1965, from Kaestner, 1909)

Perhaps the most significant finding in this study, in relation to tuning, is an affirmation of Helmholtz' claim that only the first six partials are consonant with each other:

> Apart from the range below 100 cps, the dissonance value is 0 for the octave. This means that, for up to 6 harmonics, all frequency differences between adjacent harmonics exceed critical bandwidth. It appears that this does not apply for tones with higher partials. This fact explains why complex tones with strong higher harmonics sound much sharper than tones consisting of only 6 harmonics.[54]

The finding by Plomp and Levelt that the first six partials are generally free of shared critical bands might also explain why so many theorists, including Zarlino, Mersenne, Rameau and Helmholtz, base their theories upon intervals derived from the first six partials.

[54] Plomp and Levelt, op. cit., p. 556.

3.3 Other Factors Affecting Consonance/Dissonance

3.3.1 Ranges of Pitch Discrimination

For sine waves ranging from 1 to 2 KHz, which is the range of frequencies most easily distinguished by a normal human ear, the difference limen is in the vicinity of 0.1% to 0.2%. For example, the interval from 1000 Hz to 1002 Hz, corresponding to a musical interval of about three one-hundredths of a semitone, or three cents, would be the approximate threshold of discrimination for an average listener. Increasing or decreasing from the 1-2 KHz range decreases the ability of the listener to differentiate frequencies of successive tones, as shown in Figure 22.[55]

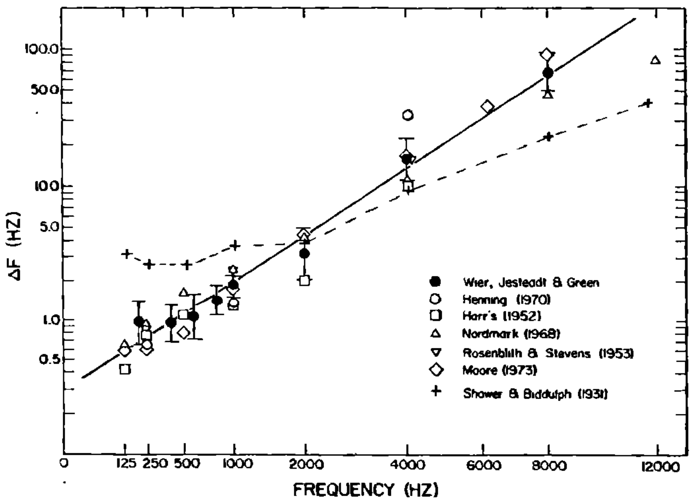

Figure 22. Frequency difference limen data from several studies in 30 to 50 dB SL (from Wier, Jesteadt and Green, 1977)

Amplitude and duration also play their respective roles in the differentiation of pitch. When the amplitude descends to around 10 dB, pitches become more difficult to discriminate. Similarly for duration, as tones are made increasingly shorter than 50 to 100 ms., there is a corresponding increase in the difference limen.

[55] C. C. Wier, W. Jesteadt and D. M. Green, 'Frequency discrimination as a function of frequency and sensation level,' *Journal of the Acoustical Society of America,* vol. 61 no. 1 (1977), pp. 179-80.

3.3.2 Harmonic Coincidence

Harmonic coincidence refers to the percentage of harmonics of a tone that coincide with harmonics from another tone. For two frequencies related as integers p and q, p and q being mutually prime, the coinciding harmonics will be multiples of pq. As an example, for the interval of an octave, p would be 1 and q would be 2. The coinciding harmonics for a sawtooth wave would be every other harmonic for p, and every harmonic of q. For a perfect fifth using the sawtooth timbre, with p = 2 and q = 3, the coinciding harmonics, having relative frequencies of {6, 12, 18, . . .}, would include every third harmonic of p and every other harmonic of q.

A study of mistuned consonances by Vos (1984) indicates the dominantly perceived beat frequency is equal to that of the first pair of nearly coinciding harmonics. A later study by Vos and van Vianen used Difference Threshold (DT) experiments to determine listeners' ability to distinguish pure from tempered intervals.[56] Level-variation depth (amplitude) of the beating harmonics was manipulated by attenuation of tone 1 or tone 2 by a variable amount. Independent variables were musical interval; attenuation of tone 1 versus attenuation of tone 2; and beat frequency of the first pair of nearly coinciding harmonics (2, 4 and 8 Hz). Results indicated that "correlation coefficients r were found between the DTs and measures of frequency-ratio complexity such as $\log_{10}(pq)$, p + q, and $(pq)^{1/2}$, corresponding values of r being -0.99, -0.98 and –0.98, respectively." Changing the timbre from sawtooth to a timbre having equal amplitude harmonics did not appreciably change the results. However, when nearly coinciding harmonics from one of the two tones were removed, DTs became much higher, implying that interference between harmonics played a major role in the identification of tempering.

[56] Joos Vos and Ben G. van Vianen, 'Thresholds for discrimination between pure and tempered intervals: the relevance of nearly coinciding harmonics,' *Journal of the Acoustical Society of America*, vol. 77 (1985), pp. 176-87.

Another interesting conclusion of this research is that the presence of harmonics from one tone just above or below a given upper harmonic of another causes mutual masking of the given harmonics. For this reason, more complex intervals, such as 4:7, are heard as dissonant, as the second partial of four is interacting with the first partial of seven. Conversely, the intervals of the octave and perfect fifth, whose notes contain lower harmonics that do not so nearly coincide, are heard as consonant.

As shown in Figure 23, the formula of p+q as a rough indicator of complexity, an approach formulated in detail by Smith in the eighteenth century,[57] does not hold true for specific points of data. Going unnoticed in this study, for instance, 5:7 has a higher DT than 5:8, suggesting that primeness could play a role. Similarly, 4:7 has a higher DT than 5:6.

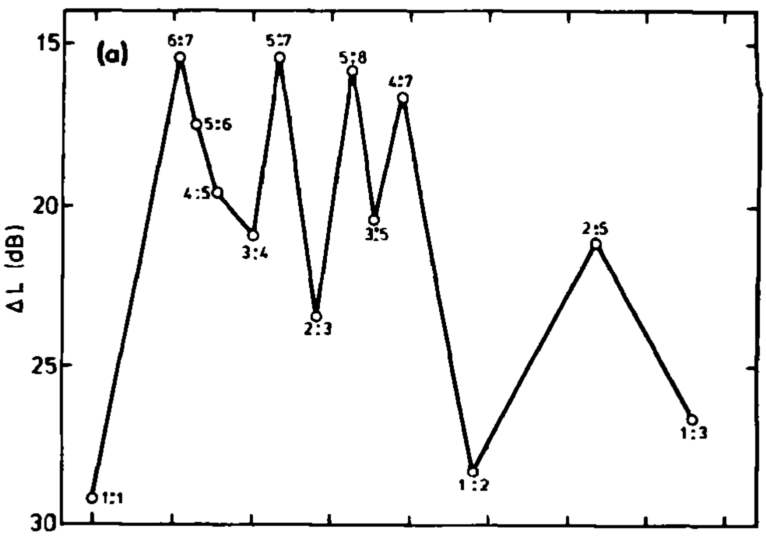

Figure 23. Thresholds for discrimination between pure and tempered intervals using beat frequencies of 2, 4, and 8 Hz. Interval size in cents (from Vos and van Vianen, 1985)

[57] Robert Smith, *Harmonics* (New York: Da Capo Press, 1966), pp. 17-30.

3.3.3 Steven's Rule

Steven's rule states that increasing the intensity of a low tone causes its perceived pitch to descend, whereas increasing the intensity of a high tone causes it to be perceived as ascending. Figure 24 demonstrates this effect using Terhardt's equation. Utilising tones of 3 KHz to 5 KHz, Hartmann verified Terhardt's equation by finding that the percentage of perceived upward change in frequency increases linearly with its pitch. As predicted by the equation, he found that the reverse effect takes place for low pitches, increasing the rate of change downward in perceived frequency for values of 300, 250 and 200 Hz.[58]

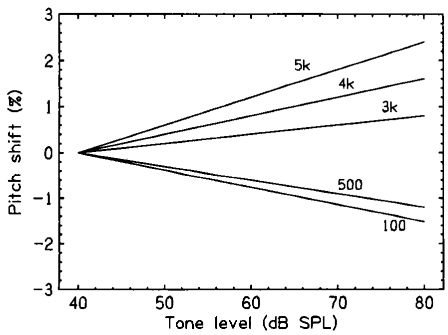

Figure 24. Steven's rule (from Hartmann, 1996, and based on Terhardt *et al.*, 1982)

3.3.4 Inharmonicity

Due to the thickness of the strings, some instruments, most notably piano, produce an overtone series whose partials are stretched. A theoretical string having no width will also have nodes having no width. Under and only under such theoretical constraints, each vibrating segment, corresponding to a partial, will have a length whose proportion is exactly the inverse of the partial number.

With a real string having thickness, the node will have the width of the string. This will result in vibrating segments that are smaller than those of the idealised string. As a consequence, the string, whose segments are in effect smaller than ideal, will have partials that are higher in

[58] William M. Hartmann, 'Pitch, Periodicity, and Auditory Organization,' *Journal of the Acoustical Society of America*, vol. 100 no. 6 (1996), p. 3495.

pitch than those of the idealised string with no width.[59] Take, for example, a middle C whose partials are slightly sharper than those produced by an idealised sawtooth with a lowpass filter emphasizing the first six partials. To compensate for the fact that the second partial of middle C, one octave above middle C, is sharper than the piano tone one octave above middle C, the tuner will sharpen the higher tone. This results not only in the consonance between the second partial of the lower tone and the fundamental partial of the higher one, it also increases the overall consonance between all partials of the two tones. The inharmonicity of the piano strings is highest for the lower tones. The fifteenth partial can be as much as 16 times the frequency of the fundamental.[60]

More extreme forms of inharmonicity have been extensively explored by William A. Sethares, whose studies emphasize tuning systems with consonance/dissonance scales that are based solely on the beating of partials and not on the ratios of the frequencies themselves. He concludes that the tuning system, whether or not the notes involved are based on a strict harmonic series, should be contingent upon the spectral content, and should emphasize avoiding the beating between partials.

3.3.5 Dominance Region

The dominance region refers to the region of the spectrum where harmonics exert the most influence on the perception of pitch. Ritsma and Engel (1967) showed that the region of the third, fourth and fifth harmonics is dominant in determining the pitch of a harmonic complex. They accomplished this through listening experiments in which subjects were asked to differentiate between two complex tones, the second distinguished from the first by a small shift in frequency.[61]

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[59] Jorgensen, op. cit., p. 741.

[60] Claude Risset and David L. Wessel, 'Exploration of Timbre by Analysis and Synthesis,' (Diana Deutsch, Editor) *The Psychology of Music* (New York: Academic Press, 1982), pp. 241-269.

[61] R. Ritsma and F. Engel, 'Frequencies dominant in the perception of the pitch of complex sounds,' Journal of the Acoustical Society of America, vol. 42 (1967), pp. 191-98.

The spectral content of the two tone complexes were varied by means of lowpass and highpass filters, respectively. When the two filtered tone complexes were sounded simultaneously, the tone complex containing more spectral energy in the first five partials consistently predicted the listeners' responses to changes in frequency more accurately than did the tone complex rich in upper partials, indicating that the lower partials imparted more information to the auditory system than did the higher ones.

For tuning, the main implication of the dominance region is that in any register the first five or six partials are imparting information that is decisive in the determination of pitch. When taken in combination with the fact that partials higher than six are sharing critical bands, thereby causing dissonance, there is reason to believe that timbres using only the first six partials are both optimally efficient in conveying pitch and minimally dissonant.

3.4 Theories of Pitch

3.4.1 Place Theory

Ohm and Helmholtz initially believed that the final sensation of pitch could effectively be modelled by a Fourier transform, in which the signal remains a set of discrete frequency components.[62] This view was refuted by their contemporary, a physicist named August Seebeck, causing Ohm and Helmholtz to retract their erroneous conclusions.

The approach that Seebeck employed in his refutation of Ohm's Acoustical Law was to create sounds from their most basic components, sine waves. The method involved first drilling into a disk circular holes whose number corresponded to the relative frequency of the partial to be included in the final complex wave. Such a disk would then be spun so that its holes would pass over a single source of high-speed air, producing something close to a sine wave for each set of holes. A close approximation of a sine wave could be played using a set of holes with even spacing; a major triad in closed root position could be accurately produced with sets of

[62] Brian C.J. Moore, *An Introduction to the Psychology of Hearing* (London: Academic Press, 1997), p. 4.

evenly spaced holes numbering, respectively, 4, 5 and 6. This meticulous and painstaking approach was required to achieve the proper safeguards absent in Helmholtz' experiments employing glass resonators.

The surprising result was that even when the set of holes corresponding to the fundamental frequency was missing, the fundamental was still perceived.[63] This implies that the pattern of partials itself is being processed as a single entity when the tones bear certain integer relationships to each other. Modern experiments show that even when difference tones are masked with noise, fundamental tones are still recognisable in notes with missing fundamentals. Further support of Seebeck's conclusion was obtained from experiments in which harmonics were played simultaneously in separate ears. Again, a low pitch similar to a missing fundamental was elicited.[64] This corroborates the notion previously proven from a physiological point of view that that the processing of frequency information from discrete partials into complex tones takes place after the auditory nerve impulses leave the cochlea.

The modern version of the place theory maintains that spikes are transmitted along the auditory nerve as neural impulses to higher auditory centres in brain. As noted previously, specific locations along the basilar membrane cause excitation of corresponding neurons in the auditory nerve. As described by Hartmann, "All physiological studies show that the tonotopic organization of neurons is maintained throughout the ascending auditory system – all the way up to the auditory cortex."[65] Such a correspondence does not, however, take into account post-cochlear processing, as in the studies of Terhardt (1974) and Hartmann (1996).

[63] R. S. Turner, 'The Ohm-Seebeck dispute, Hermann von Helmholtz, and the origins of physiological acoustics', *British J. Hist. Sci.*, Vol. 10 no. 34 (1977), pp. 1-24.
[64] R. A. Rasch and R. Plomp, "The Perception of Musical Tones" (Diana Deutsch ed.), *The Psychology of Music* (New York: Academic Press, 1982), pp 8-19.
[65] Hartmann, op. cit., p. 3493.

3.4.2 Residue Theory

The residue theory, first proposed by Schouten (1940), stems from the observation that the fundamental of a complex tone can be perceived in the absence of the fundamental itself. According to the residue theory, unresolved components, due to the limited frequency resolution of the cochlea, form a periodic or quasi-periodic waveform that is transmitted as a tone to the central auditory system via the auditory nerve.

This theory was ultimately proven to be inadequate, or at least incomplete, by systematically gathered empirical evidence, and was subsequently replaced by central, neural signal processing models. (See "Virtual Pitch/Central Processing Theory" on p. 61.) For complex tones, in particular those whose lower partials were present, harmonics presented dichotically were indistinguishable from those presented either monotically or diotically, completely refuting the notion that the final encoding of pitch takes place before leaving the ear.

The residue theory is actually a valid explanation of pitch identification under certain conditions when tones are composed entirely of higher partials. Houtsma and Smurzynski (1976) performed a series of experiments using tones based on a missing fundamental ranging in frequency from around 200 – 300 Hz.[66] Melodic identification experiments using 11 harmonics indicated that when the lowest of these harmonics was the seventh, identification was 100 percent for the musically experienced subjects. When the lowest harmonic was 13, however, identification fell to below 60 percent. As expected, either increasing the value for the lowest harmonic or decreasing the number of harmonics produced lower identification scores by the subjects. Further experiments explored just noticeable differences for tones made of 11 consecutive harmonics, with the lowest harmonic ranging from the seventh through the twenty-fifth. Results based on two types of tones, in-phase and Schroeder phase, indicate that somewhere between the tenth and thirteenth harmonic is a shift to another auditory process. The fact that under these conditions, in-phase tones are more easily distinguished that out-of-phase ones suggests that, for sufficiently high partials, identification

[66] Houtsma and Smurzynski, op. cit., pp. 304-10.

through residue processing actually takes place. It should be noted, however, that these results are for just noticeable differences (JNDs). Identification of intervals, even when the lowest harmonic number is 13, is below 60 percent, suggesting musical tone identification is taking place with harmonics below this value. The experimenters did not mention the possibility that subjects could have been identifying tones based upon the difference tones they generate, which might have been strong enough to mimic the sounds of one or more of the first few harmonics. This possibility could of course have been eliminated by masking such difference tones with noise.

3.4.3 Temporal Theory

The theory that the brain detects and prefers small integer frequency ratios by means of neural-firing patterns containing a common periodicity is examined by Boomsliter and Creel (1961).[67] According to their research, the auditory system must operate on pitch at a higher level than individual frequency transmissions from hair cells. Support for this initial assumption is given through the research of Bekesy, who showed that frequency discrimination is finer than accounted for by the input from individual cochlear hair cells.

Boomsliter and Creel provide evidence for their theory that the auditory system is performing analyses on “long wave forms”. Each long waveform has a duration corresponding to the common wave period of the given note’s partials. Their detailed analysis suggests that a hierarchical network of circuits in the nervous system determines the true pitch of a note by finding the intersections corresponding to higher order temporal relationships, which in turn cause a synapse to fire. The higher order synaptic patterns are, in turn, processed until there is no higher order pattern found. The highest order pattern found is perceived as the fundamental. This approach of multiple-level analyses on partials is the topic of “Virtual Pitch/Central Processing Theory” on p. 61.

[67] Paul Boomsliter and Warren Creel, ‘The Long Pattern Hypothesis in Harmony and Hearing,’ *Journal of Music Theory,* Vol. 5 No. 2 (1961), pp. 2-30.

There is some physiological data to support the Long Wave Theory proposed by Boomsliter and Creel. Nerve firings tend to occur at a particular phase of the stimulating waveform, a process known as phase locking, which breaks down above 4-5 KHz. Javel (1980) investigated the responses of single auditory neurones to stimuli consisting of three successive high harmonics of a complex tone.[68] He showed that a portion of the neural activity in the neurones responding to the component frequencies was phase locked to the overall repetition rate of the stimulus equal to the absent fundamental frequency.

As the level of a pitch increases, neural spikes become synchronized with the signal's period, increasing with the level. The extent to which the spikes are synchronized with the stimulus increases to a peak of around 90%, varying with frequency. Synchrony vanishes rapidly as the frequency increases from 2 to 5 KHz due to timing jitter intrinsic to neural firing, which is in agreement with experiments that determine difference limens across different registers. Phase locking occurs in complex tones such that "dominant" tones are phase locked, but not weak nearby neighbours. According to Moore:

> In general, the temporal patterns of response are dominated by the most prominent frequency components in the complex stimulus, with the result that there may be little or no phase locking to weak components which are close in frequency to stronger ones. Thus is seems reasonable to suggest that a tone (with a frequency below about 5 KHz) will be masked when the subject cannot detect its effect on the time pattern of nerve impulses evoked by the stimulus as a whole.[69]

[68] Moore, op. cit., pp. 42-47.
[69] Moore, op. cit., p. 110.

Although the timing theory fits many facts better than the placement theory, such as repetition pitch phenomenon, in which a signal is delayed and added to itself; the conclusion by Vos and Vianen that changing the timbre from sawtooth to a timbre having equal amplitude harmonics did not appreciably change the results of experiments involving mistuned intervals; and the perception of pitch in response to sine-wave amplitude-modulated (SAM) noise, it is shown to be incompatible with other data. One such incompatibility is the pitch-shift effect, in which partials are stretched but the period of the envelope remains stable.

3.4.4 Virtual Pitch/Central Processing Theory

The theory of virtual pitch introduced by Terhardt (1974) starts from the assumption that the theory of dissonance, first introduced by Helmholtz and verified through the experiments of Plomp-Levelt and Kameoka-Kuriyagawa, is correct but incomplete.[70] The theory proposed by Helmholtz, Terhardt asserts, is “psychoacoustic consonance”, which is only part of a more comprehensive theory.

According to the virtual pitch theory, musical consonance is related to psychoacoustic consonance, but differs in many ways. Simple musical intervals, such as a perfect fifth (3:2), are considered consonant even in the low register, where much beating occurs. Conversely, a dissonant interval, such as a tritone, will be identified as being musically dissonant even when, as is the case when it is produced with sine waves in the middle register, it produces no beating. Significantly, single tone complexes whose partials are integer related are always considered consonant, even when beating is present. Virtual pitch theory attempts to account for these and other aspects of musical perception unexplained by either periodicity detection theory or Helmholtz’ theory of dissonance.

[70] Ernst Terhardt, ‘Pitch, Consonance and Harmony,’ *Journal of the Acoustical Society of America*, vol. 55 no. 5 (1974), pp. 1061-69.

Spectral pitch cues are derived in “analytic mode“ from pure tones. According to the model, a pitch cue is generated if a frequency is present in the sound stimulus and if it can be separated from adjacent ones. The perceived pitch is the actual pitch multiplied by a factor determined by the sound pressure. Notable in Terhardt’s theory is that partials 3-6 are considered dominant in terms of pitch perception.

Virtual pitch cues, learned and recognized in “synthetic mode”, result from recognized patterns of complex tones whose ratios are integer related. Such learning, according to the virtual pitch theory, takes place during speech acquisition. Analysis of the voice by Brown (1996) lends unequivocal support to this theory:

> The voice, along with the bowed strings, is an instrument where harmonic components are expected since sound production by periodic glottal pulses is definitely phase locked. The amplitudes of the harmonics dropped off very rapidly so clean frequency measurements were only possible up to the third harmonic, but results are near perfect and support the expectation of exact integer ratios of harmonics.[71]

The model includes a learning matrix of roughly 1,000 x 1,000. Rows represent spectral pitch, which are merely sine waves adjusted for amplitude, whilst columns transmit lowest pitch clues, representing the perceived fundamental of the tone. Columns can also represent virtual pitch clues from traces that are retained in the learning matrix, being already formed during speech acquisition.

Cohen, Steven and Wyse (1995) extend virtual pitch with their Spatial Network model, which employs pattern matching to transform a spectral representation into pitch.[72] Each harmonic can be thought of as belonging to numerous virtual pitches. Based on the strength and number of coinciding harmonics, a single virtual pitch with the highest level of activation is selected as the best fitting fundamental and consequently the perceived pitch. Paramount in the model

[71] Judith C. Brown, ‘Frequency ratios of spectral components of musical sounds,’ *Journal of the Acoustical Society of America*, vol. 99 no. 2 (1996), p. 1214.

[72] Michael A. Cohen, Steven Grossberg, and Lonce L. Wyse, ‘A Spectral Model of Pitch Perception’, *Journal of the Acoustical Society of America*, vol. 98 no. 2 (1995), pp. 862-879.

is the concept of the harmonic sieve, which is a template based upon the harmonic series, but whose "holes" through which the harmonics enter allow for a certain amount of error, as supported by pitch shift experiments such as those of Patterson and Wightman (1976). A component contributes to a pitch if it is close enough to a harmonic of the pitch to fall through that harmonic's hole. The concept of a tonal sieve is supported by Moore *et al.* (1985), who discovered that mistuning a given harmonic beyond 3% of its original frequency causes that harmonic's impact on overall pitch to diminish. As the component is stretched beyond 3%, the perceived pitch gradually returns to that of the fundamental.

Virtual pitch is refined further by Hartmann (1996), who concludes that given practically any sound source, the auditory system will attempt to apply, through a series of pattern matching procedures, a template based upon based on consecutive lower harmonics.[73] The pattern matching procedure will produce a fundamental whether or not it actually exists. Harmonics themselves are input as perceived pitches that can be influenced by amplitude. Supporting evidence includes the fact that pitches arise from separate components in two ears (Houtsma and Goldstein, 1972 and Houtgast, 1976). Further evidence is provided by experimental data indicating that the missing fundamental can even be created with successive partials 3-5 using 40ms for each partial separated by 10ms silence (Hall and Peters, 1981).

3.5 Closing Remarks

It would appear that our auditory system includes both temporal and place mechanisms in the early stages of auditory processing. According to Hartman (1996), there is a possibility that timing of neural pulses and cochlear place mechanisms dominate, respectively, for low and high frequencies.[74] Virtual pitch theory asserts that these mechanisms can be thought of as input devices to a central processor that a) performs multiple-level analyses of incoming signals originating as discrete partials, and b) groups the incoming signals into higher order

[73] Hartmann, op. cit., pp. 3491-3502.
[74] Hartmann, op. cit., pp. 3496-97.

units we perceive as pitches. It is not a huge leap of intuition to hypothesize that the same central processor is grouping these pitches into chords and chord progressions.

From several sources there is ample evidence for the claims by Mersenne, Rameau and Helmholtz that consonance, both in terms of timbre and in terms of intervals, is derived from the first six partials. First, Plomp and Levelt (1965), citing their own research on critical bandwidth theory, demonstrate that peaks of consonance corresponding to the simplest intervals of just intonation are found in comparisons of two-tone complexes containing the first six partials. According to their critical bandwidth theory, adding partials higher than the sixth within a sawtooth wave will create dissonance even within a single tone.[75] Second, the theory of harmonic coincidence points out that complex intervals, such as 4:7, accent critical bandwidth effects. Third, according to Moore, weak components are not phase-locked in temporal coding, thereby increasing the probability that higher partials are discarded before being sent to the central processor. Fourth and perhaps most significantly, the theory of the dominance region indicates that the third, fourth and fifth harmonics are the optimum conveyors of pitch information, and easily mask their higher partial neighbours. Fifth and finally, the widely adopted theory of virtual pitch is based upon the multiple-level analysis of pitch clues derived from harmonics 3-6.

The remaining chapters will explore the concept that the first 5-6 partials form the basis not only of timbral consonance, but also of tuning as it relates to timbre. In the process, rules can be ascertained to explain and regulate the ways in which chords are grouped together to form tonal progressions.

[75] Plomp and Levelt, op. cit., p. 556.

Chapter 4
Bach Chorale Tuning Database

4.1 Specification and Aims

The purpose of this study is to create and investigate a database of chords and chord progressions contained in the chorales of J.S. Bach. Key among the questions to be addressed is whether the chords and progressions most commonly employed within this collection are derived from the simplest chords of 5-limit just intonation.

The first task of this research is to create a database in which the rate of occurrence for every chord and chord progression is determined. For completeness of data, all chords and repeating chord progressions, up to a specified length of 7 chords, are examined for 375 chorales. The lengthy process of creating this database includes converting Csound scores into arrays of distinct sonorities; converting these arrays of sonorities into arrays of chord indices; and comparing every set of chord indices with every other set. This automated process is discussed at length in section 4.4 beginning on p. 70.

The second undertaking is to examine the most commonly occurring chords and progressions within this database to decide if they are consistent with chords constructed using the simple frequency ratios of 5-limit just intonation. An affirmative result would indicate that Bach is composing music based upon the integer relationships between notes and their partials. Analysis of data begins in section 4.5 on p. 107.

The database of chords and progressions provides a wealth of data with which other topics of tuning may be addressed. One such use of the data is to perform listening experiments based on the most commonly occurring chords and progressions, as is done in Chapter 7 beginning on p. 188.

4.2 Previous Related Studies

4.2.1 Bach Chorales

The chorales of J. S. Bach are frequently investigated in computer based studies. As compositional miniatures, chorales are relatively easy to analyse in detail, especially as they tend to centre around a single key, although comparatively few tend to lack a clear tonal centre. Totalling around 400 in number, depending on how they are counted, Bach's chorales provide an abundant source of statistical data. Perhaps most significantly, the chorales of J.S. Bach comprise a traditional cornerstone of music theory pedagogy.

Ferková (1992) wrote a set of programs that perform chordal, functional and tonal analysis.[76] The key was detected both globally and locally, with every chord being assigned a certain function on the basis of the current key area. Many of the problems inherent in harmonic identification are also present in tuning, as chords must first be identified before they can be tuned if a rule-based system is employed. As in the current study, Ferková mentions that chords present in transitional sections are problematic. An interesting concept introduced in Ferková's study is that a chord can be categorized as having a certain chordal dynamic potential based upon its type. This model is expanded to include local functional harmony and tonality as higher levels of structure.

For his Ph.D. dissertation, K. Ebcioglu (1988) wrote "An Expert System for Harmonizing Four-part Chorales" which employs a "last chord first" harmonization. In a similar vein, Dominik Hoernel has used both neural networks and rule-based systems to generate chorale harmonisations and variations.[77]

[76] Eva Ferková, 'Computer Analysis of Classic Harmonic Structures,' *Computing in Musicology*, vol. 8 (1992), p. 85.

[77] Dominik Hoernel, 'A Multiscale Neural-Network Model for Learning and Reproducing Chorale Variations,' *Computing in Musicology*, vol. 11 (1998), pp. 141-58.

Steven Rasmussen developed a program called chord.z, written in ibex, with which he processed 185 four-part chorales BWV 253-438 from the CCARH MuseData database to determine mode.[78] The approach of identifying mode is not employed in the present study, in which mode, provided in the input Csound score, is assumed to be either major or minor.

Rasmussen determined mode by comparing the final chord of the chorale with its key signature. He explains, "Since the concluding chord of each chorale is, by definition, the tonic of the mode, I omitted it from this and the following stages of analysis." Identification of mode was based on the total number of major and minor triads found; the combined totals of these for each scale degree; the percentage of the grand total each figure represents; and the percentage of time each scale degree was used as the root of an opening or cadential chord.

It is presumed that many of the chorales were excluded from Rasmussen's study on the basis that they do not end on tonic, as more than five percent of the chorales fall into this category. Notable among these is "Das alte Jahr vergangen ist", Riemenschneider listing #162, which has a final E major cadence that can only be taken as a dominant of dominant when the chorale is analysed in its true key of D minor (although the key signature has no sharps or flats).

4.2.2 Tonal Analysis

Most recently, Heinrich Taube has created an automated music analyst that is not based in heuristics, but instead on "forward chaining" rule sets which employ numerous logical "if/then" conditionals.[79] Musical data are pattern matched against existing models of chordal patterns. Disregarding key, the program instead searches for local dominants, regardless of the mode. Local points of certainty are determined, and a bottom up analysis is created.

[78] Steven C. Rasmussen, 'Modality vs. Tonality in Bach's Chorale Harmonisations,' *Computing in Musicology*, vol. 10 (1995-96), pp. 49-58.

[79] Heinrich Taube, 'Automatic Tonal Analysis: Toward the Implementation of a Music Theory Workbench,' *Computer Music Journal*, vol. 23, no. 4 (1999), p. 27.

A very interesting feature of Taube's program is that all chords are identified as triads or sevenths, an approach similar to that of Rameau. Another subtlety is that, as in Schenkerian analysis, a detail at one level of analysis can be explained or subsumed by a more general or a higher-level structure.

The technique employed by Taube of parsing chord progressions into discrete sonorities is very similar to the CONVERT program used in the current study. In addition to providing discrete snapshots of tonality whenever the slightest harmonic motion occurs, Taube's program tracks metric position and stress level. Many such multi-level analytic techniques presented by Taube are beyond the scope of the current study; however, cross-pollination between detailed tuning of the present study and multi-level analysis of the type presented by Taube, whilst certain to prove intellectually challenging, would unquestionably produce new analytical tools with which to study music theory and composition.

4.3 Verification of Keys for Chorales

4.3.1 Chorales whose key signatures do not match perceived key

Many of the chorales have a modal character, but the vast majority of these can be confidently assigned a key signature that is compatible with the perceived key. Whilst key identification can be a subjective matter, especially when the key is intentionally blurred with much dramatic effect, it is argued that in nearly every case a key can be stated as the actual tonal centre.

The approach taken was to aurally and visually analyse every chorale to identify key. A more objective approach would be to utilize computer programs that are able, by finding a preponderance of dominant chords, to automate the process of finding keys. However, the approach taken was useful in understanding the depth and variety of Bach's tonal language.

Table 9 represents the MIDI files for chorales whose keys signatures are incompatible with the key signatures actually perceived. It is not at all unexpected that so many of these chorales are written in a different key than would normally correspond with the modern major and minor modes. This is due to the fact that Bach often used key signatures corresponding to the traditional church modes, notably Dorian and Phrygian.

Table 9. Chorales whose key signatures do not match actual key

Music ID	Riemenschneider	Kalmus	Key Given	Key Analysed
000206B_.MID	262	7	D min	G min
000707B_.MID		44	B min	E min
001805B_.MID	100	73	G min	C min
004008B_.MID	8	105	C min	F min
004606BS.MID			D min	G min
005605B_.MID	87	72	G min	C min
006402B_.MID	160	108	A min	G maj
007706B_.MID	253	6	D min	G min
008305B_.MID	325	250	A min	D min
008506B_.MID	122	216	G min	C min
010207B_.MID	110	320	G min	C min
014505B_.MID	17	84	B min	F# min
015301B_.MID	3	5	E min	A min
017606B_.MID	119	45	G min	F min
017705B_.MID	71	183	G min	F min
024537B_.MID	113	50	C min	B♭ min
025400B_.MID	186	2	A min	D min
026500B_.MID	180	21	A min	D min
027500B_.MID	210	35	C maj	D min
027600B_.MID	197	36	C maj	D min
027700B_.MID	15	38	A min	D min
028000B_.MID	66	43	A min	D min
028800B_.MID	162	55	A min	D min
029600B_.MID	231	64	C maj	G maj
029700B_.MID	232	65	A min	D min
030900B_.MID	166	93	D min	G min
034100B_.MID	168	170	D min	G min
034300B_.MID	302, 199	172	D min	G min
035100B_.MID	19	182	D min	G min
035400B_.MID	369	187	F min	B♭ min
035700B_.MID	244	191	G min	C min
037200B_.MID	218	226	D min	G min
037400B_.MID	227	232	D min	G min
038200B_.MID	49	249	A min	D min
038700B_.MID	185	260	C maj	D min
040300B_.MID	203	287	D min	G min
040800B_.MID	171	303	D min	G min
041200B_.MID	206	310	D min	G min
042300B_.MID	237	336	D min	G min
042500B_.MID	241	349	A min	D min
043700B_.MID	133	382	A min	D min

4.3.2 Further Correction of Key Signatures

In addition to the key signatures of Bach that do not match the perceived key, several of the original midi files were assigned the incorrect mode, such as minor instead of major, and vice-versa, during the process of creating the MIDI files. All of these corrections can be reviewed in “Appendix II: Listing of All Chorales” on p. 246.

4.4 Description and Evaluation of Programs and File Formats

4.4.1 Preliminary Conversion

The original data is in the form of MIDI files. Chorales stored in standard MIDI format are first converted into Csound score (*.sco) files using the public domain program MIDI2CS. To facilitate processing, a batch (BAT) file is used to convert all the MIDI files within a directory. The resulting score files are then placed in a directory where they can be processed collectively.

4.4.2 Csound Score Explained

Csound scores serve as the starting point for a program set that eventually outputs a catalogue of chord progressions. Musical scores in Csound seem very technical at first, but they are in fact quite simple. The first two bars from a Bach chorale in A major illustrate the correspondence between the standard score and a Csound score.

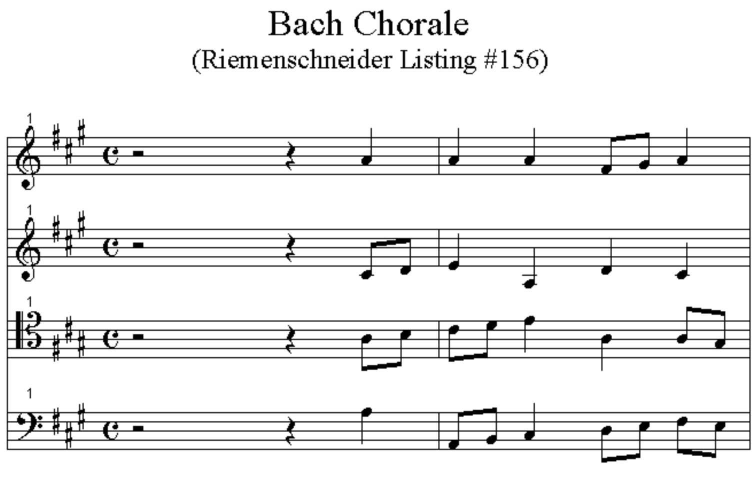


Figure 25. Two bars from a Bach Chorale

The following Csound score is given for the soprano voice. Omitted alto, tenor and bass voices are scored similarly. Compilation begins with the function table beginning “f100”. Lines that begin with the `;` character are ignored by the Csound compiler.

```
t 0 96.0000
;--------------------------------------------
;     miditrack 1     instrument 1
;--------------------------------------------
; p1 = instr
; p2 = start
; p3 = duration

; simple sine function
f100 0 2048 10 1

i01    3.00000   1.0000  1760.00000  16192 ;  A3 ch01
; bar 1
i01    4.00000   1.0000  1760.00000  16192 ;  A3 ch01
i01    5.00000   1.0000  1760.00000  16192 ;  A3 ch01
i01    6.00000   0.5000  1479.97769  16192 ; F#3 ch01
i01    6.50000   0.5000  1661.21879  16192 ; G#3 ch01
i01    7.00000   1.0000  1760.00000  16192 ;  A3 ch01
; bar 2
```

For the purposes of this program set, the important aspect of the Csound score is the event, which provides information about each particular instrument and the notes that it plays. The very first event for Instrument 1 is the following:

```
i01    3.00000   1.0000  1760.00000  16192 ;  A3 ch01
```

With regard to the parameters in each event, as shown in Figure 26, they are:

1) Instrument Number
2) Start Time
3) Duration
4) Frequency in Hz (skipped)
5) Amplitude (skipped)
6) MIDI key, given as note and octave designation
7) Channel (skipped)

Of these parameters, only instrument number, start time, duration and MIDI key are used in subsequently described programs that process Csound events. An additional parameter, block_end, is used during processing to indicate the last event within a blocked sonority.

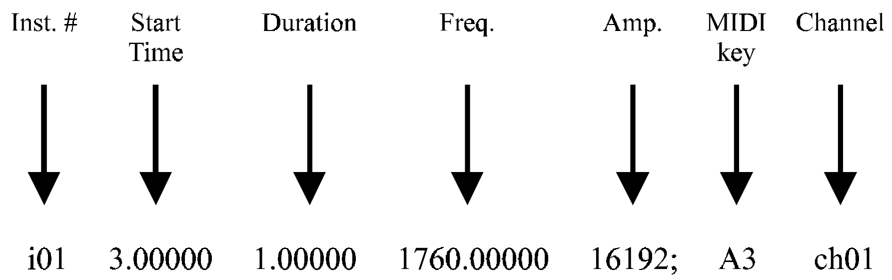


Figure 26. A Csound event

4.4.3 Brief Synopsis of Program Set

Three modules, CONVERT, MAKELIST and STORE, process the data originating as a collection of Csound score files. As shown in Figure 27, CONVERT translates each Csound score into a text file of note arrays. For every such text file, MAKELIST creates an equivalent list file representing each chorale as an array of chord indices. Finally, STORE outputs a single text file that lists and enumerates every chord and chord progression contained in the collection.

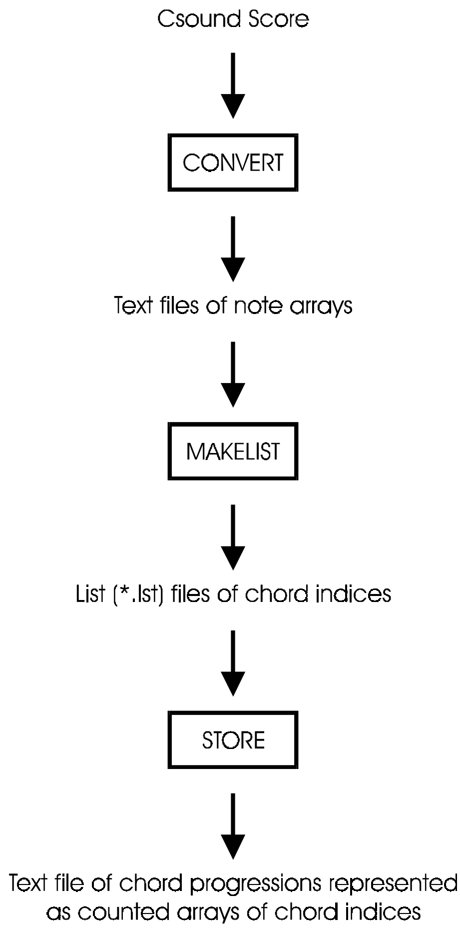


Figure 27. Overview of Bach Chorale Database Program Set

4.4.4 CONVERT

Overview of CONVERT

CONVERT takes as input a Csound score file, already converted from a MIDI file, of an entire chorale or other piece of music and outputs a text file containing a sequential collection of chords. Whenever any voice or group of voices moves, enters or exits, a discrete chord “block”, or “snapshot”, is stored. This approach, called “pseudo-figured bass”, a more detailed version of the traditional system known as figured bass, assumes that every note in every chord has a harmonic function. For the purposes of this study, rhythmic content is discarded.

Chords are next sorted according to MIDI number. The entire piece is transposed into the key of either C major or C minor, as in a previous study by Krumhansl.[80] Chords are then classified according to a) the bass note and b) the chord type, defined by the notes present. Taken together, bass note and chord type provide all of the information needed to represent chords in pseudo-figured bass.

4.4.5 Sample Run of CONVERT

For the present study, CONVERT was used to convert each of 375 complete chorales into a series of discrete chord snapshots, or pseudo-figured bass. Figure 28, a V-V^7-I progression that happens to be the most commonly occurring three-chord pseudo-figured bass pattern in Bach's chorales, is used throughout this description of the CONVERT program.

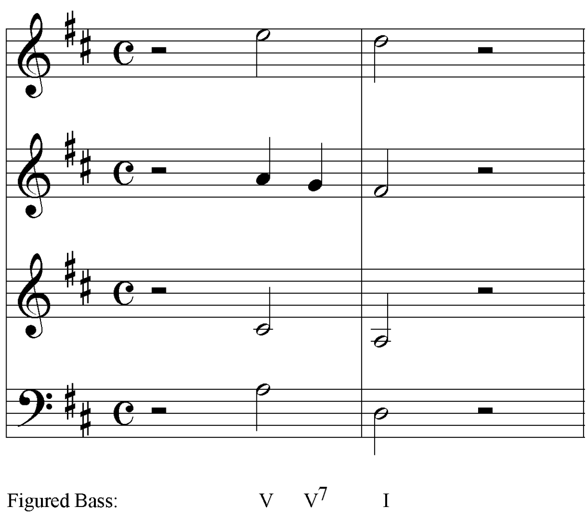


Figure 28. V-V^7-I Example

[80] Krumhansl, op. cit., p. 180.

Store Each Csound Event into Structure Event

The first step is to create a structure into which the parameters for each Csound event are stored. This structure, called *event*, contains the following parameters:

1) Instrument number (not used in present study, but retained for future studies)

2) Start Time

3) Duration

4) MIDI Key

5) End of Block, indicating the last note of a chord snapshot during which no voice enters, exits or changes

Create Array of Event Structures, Array[]

Array[], an array of events, holds as many event structures as there are individual notes within a chorale or other piece of music. Some of the larger chorales require more than 2,000 events each. Table 10 illustrates the first five rows of the unused array, initialised to zeros.

Instrument Number	Start Time	Duration	MIDI Key	Block_end
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

Table 10. An empty array of events

Csound Score Input

Figure 29 provides the designated parameters in raw data format from each line of the Csound score of this V-V^7-I progression. Extraneous information contained in the score has been omitted. The Csound Score's sharp-flat value, sf = 2, specifies that there are two sharps, whilst the mode index, mi = 0, indicates the major mode. Therefore, according to the MIDI information received from the input file, the key of the score is D major. An annotated Table 11 provides the same score in a friendlier format.

Note that the starting times and durations are out of sequence within the bar. This is of no consequence either for the Csound compiler for which the score was intended, or for the current program set, which employs its own sorting module for these parameters.

```
;key signature sf=2 mi=0
i01   2.00000   1.0000      1760.00000   16192   ;   A3    ch01
i01   2.00000   2.0000      2637.02046   16192   ;   E4    ch01
i01   3.00000   1.0000      1567.98174   16192   ;   G3    ch01
i01   2.00000   2.0000      1108.73052   16192   ;   C#3   ch01
i01   2.00000   2.0000      880.00000    16192   ;   A2    ch01
;     bar       1
i01   4.00000   2.0000      2349.31814   16192   ;   D4    ch01
i01   4.00000   2.0000      1479.97769   16192   ;   F#3   ch01
i01   4.00000   2.0000      880.00000    16192   ;   A2    ch01
i01   4.00000   2.0000      587.32954    16192   ;   D2    ch01
```

Figure 29. Raw data from Csound Score of V-V^7-I example

Instrument Number	Start Time	Duration	Frequency (ignored)	Amplitude (ignored)	Note (MIDI value)	Channel (ignored)
i01	2.00000	1.0000	1760.00000	16192	A3	ch01
i01	2.00000	2.0000	2637.02046	16192	E4	ch01
i01	3.00000	1.0000	1567.98174	16192	G3	ch01
i01	2.00000	2.0000	1108.73052	16192	C#3	ch01
i01	2.00000	2.0000	880.00000	16192	A2	ch01
i01	4.00000	2.0000	2349.31814	16192	D4	ch01
i01	4.00000	2.0000	1479.97769	16192	F#3	ch01
i01	4.00000	2.0000	880.00000	16192	A2	ch01
i01	4.00000	2.0000	587.32954	16192	D2	ch01

Table 11. Representation of Csound score for V-V^7-I example

Transfer Parameters from Csound Score into Array

The program now locates and reads each of the five desired parameter values (Instrument number, Start Time, Duration, MIDI Key and End of Block) from every event line of the Csound score into the designated parameters of array[]. Of particular interest is the MIDI key value, which is first read in from the Csound score as a letter and number representing note and octave designation, respectively. Each MIDI key value is read into an array of characters, Note[], shown in Table 12. Note[] is then sent, along with key and mode information, to the convert module, which examines it character by character.

Note[0]	Note[1]	Note[2]
F	#	3

Table 12. Encoding for note parameters

The first character in Note[], Note[0], is the letter number corresponding to one of the white keys of the piano (A-G). C is designated as zero, and the other keys by the number of semitones between C and the upper note. For example, the variable MIDI key is assigned the initial value of 6 for the note F#. The next member in Note[], Note[1], in this case is a sharp (#) symbol. (The MIDI to Csound conversion program used to create the initial Csound score does not use flats for individual notes, only for key signatures.) If Note[1] is a sharp, the MIDI key value is incremented by one.

The remaining character, Note[1] if no sharp is present or Note[2] if one is, will represent the octave designation. An octave adjustment variable, octave_adj, is assigned the value of the octave designation times 12, providing the number of semitones by which to adjust the MIDI key value. For example, F#3 would correspond to a MIDI value of 42, resulting from the initial value of 6 for F# plus 36 semitones derived from the three octaves indicated in the octave designation.

Each MIDI key value is transposed into the key of C major or C minor through the implementation of a lookup table, shown in Table 13, which compares the mode index (MI) and sharp-flat values (SF) already obtained from the Csound score. The MIDI key is then incremented the number of semitones indicated by the adjustment value.

The centre value in Table 13 corresponds to the adjustment for C, which is zero. Adding one sharp will necessitate an adjustment upward of 5 semitones, as shown in the value to the right of zero. Adding one flat will call for the adjustment upward of 7 semitones, as indicated in the value to the left of the zero. For convenience, all adjustments are made upward. This is of no significance, as octave transpositions will later be discarded.

Sharp-Flat (SF) Value	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
MIDI key increment value	1	6	11	4	9	2	7	0	5	10	3	8	1	6	11

Table 13. Table lookup for key signatures

In the current example, we are in the key of D with an SF value of 2, representing two sharps. To transpose to the key of C, we start at the middle value, zero, and take the second value to the right, which is 10; thus, 10 is added to every MIDI key value in the piece. For example, 42, representing F#, becomes 52, representing E. This transposition is performed for every MIDI value; thus, the entire chorale is transposed from the key of D to that of C, as shown in Table 14.

If the Csound score's mode index (MI) value is 1, indicating minor mode, every MIDI key value in the piece is adjusted upward by 3 semitones. For example, if the SF value is 0, indicating no sharps or flats, but the MI value is 1, indicating minor mode, then 3 is added to the MIDI key value, transposing the piece from the key of A minor to that of C minor.

Instrument Number	Start Time	Duration	MIDI Key	Block_end
1	2	1	55	0
1	2	2	62	0
1	3	1	53	0
1	2	2	47	0
1	2	2	43	0
1	4	2	60	0
1	4	2	52	0
1	4	2	43	0
1	4	2	36	0

Table 14. Array of events, transposed into C, for V-V^7-I example

Sort events in array according to starting time and duration

The array of events, array[], is now sorted using a simple quick-sort algorithm, first according to starting time, from lowest to highest; then according to duration, from longest to shortest. Durations are sorted from longest to shortest so that a particular event whose duration is longer than another event commencing at the same time will later be automatically truncated, resulting in the two being the same length, with residue from the truncation being treated separately. Sorted output for the entire V-V^7-I example is given in Table 15.

Instrument Number	Start Time	Duration	MIDI Key	Block_end
1	2	2	62	0
1	2	2	47	0
1	2	2	43	0
1	2	1	55	0
1	3	1	53	0
1	4	2	60	0
1	4	2	52	0
1	4	2	43	0
1	4	2	36	0

Table 15. Sorted output for V-V^7-I example

Block discrete chords

Once event array[] has been sorted, it is processed by the function block_array(). Block_array() detects rhythmic motion in any voice. If all voices move together, block_array() will leave the events unchanged. In the case, however, of any voice entering, leaving, changing pitch or merely repeating, a new event is created for every active voice (except at the end, when all voices become inactive). In effect, every change in harmony or rhythm, no matter how superficial, necessitates a new event. Each resulting event in the output file from block_array() is therefore a snapshot of time during which no harmonic or rhythmic changes take place. The input array received by the function block_array() for the V-V^7-I example would, if its events were placed back inside a Csound score and compiled, produce a binary output file that would sound like the example given in Figure 30.

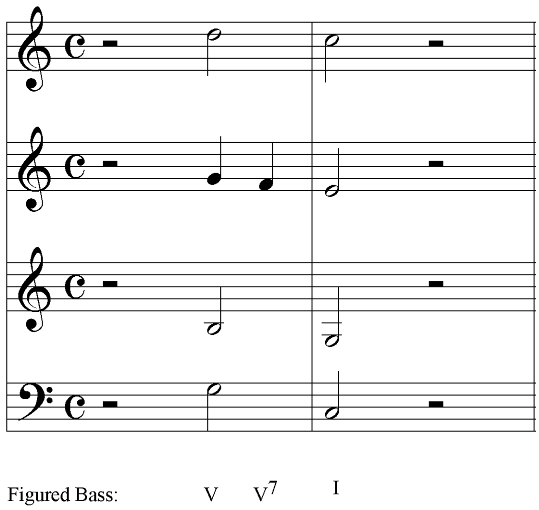


Figure 30. Musical representation of input array for function block_array()

If the array of events from the output file of block_array were placed inside a Csound score and compiled, the resulting binary output file would now sound like Figure 31. In essence, a chord block starts and ends with the rhythmic change of any voice or group of voices.

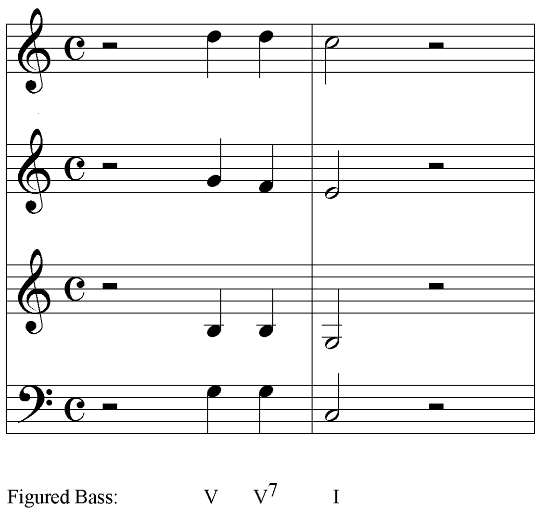


Figure 31. Musical representation of output array for function block_array()

As can be seen in Table 16, the number of events has grown from 9 for the input array to 12 for the output array. This is due to the three minims in measure one for the soprano, tenor and bass voices being broken into six crotchets so that all four voices move at the pace of the rhythmically most active voice, the alto.

Instrument Number	Start Time	Duration	MIDI Key	Block_end
1	2	1	62	0
1	2	1	47	0
1	2	1	43	0
1	2	1	55	0
1	3	1	43	0
1	3	1	47	0
1	3	1	62	0
1	3	1	53	0
1	4	2	60	0
1	4	2	52	0
1	4	2	43	0
1	4	2	36	0

Table 16. Representation of output from module block_array for V-V^7-I example

Algorithm for function block_array

The block_array() function is the most complex subroutine within the CONVERT program. The following algorithm for this function assumes that events are already sorted by ascending starting time and descending duration.

1. Initialise counter row to 0.
2. For each event
    3. current_event is array[row]
    4. nxt_new_start is starting time of next event with starting time different than array[row].start
    5. current_event_start_time <- array[row].start
    6. diff <- nxt_new_start – current_event_start_time
    7. truncate1 <- array[row].duration - diff
    8. nxt_dur_sss is duration of next event with simultaneous starting time, but a smaller duration, than array[row].duration
    9. truncate2 is array[row].duration - nxt_dur_sss

```
10. if (truncate1 > truncate2)
        truncate <- truncate1
11. if (truncate2 > truncate1)
        truncate <- truncate2
12. if (nxt_new_start is 0)
        row++
13. if truncate > 0
        temp_event <- array[row]
        temp.start <- temp.start + temp.duration - truncate
        temp.duration <- truncate
        array[row].duration <- array[row].duration - truncate
        shift all array members (events) after current_event down by one
        insert temp_event into the newly created space
        i <- row+1
        while (array[i].start > array[i+1].start)
                swap array[i] and array[i+1]
                i++
        while (array[i].start equals array[i+1].start and array[i].duration <
        array[i+1].duration)
                swap events array[i] and array[i+1]
                i++
        row++
end of for loop
```

Insert End of Block Marker

Function insert_block_end() receives as input the output, event array[], from the function block_array(). Insert_block_end identifies groupings sharing the same starting time. When such a block of notes, representing a “snapshot” or “steady state” of a chord, is found, the block_end parameter in its last event is assigned the value 1. All events sharing the same starting time, which also have the same durations, are thus identified as a collection of MIDI keys.

The algorithm is relatively simple. For every event in array[], if the starting time added to the duration of the present event is less than or equal to the starting time of the subsequent event, or if the last event member of the array is found, a 1 is placed in the block_end member of the event to signify that it is the last member of a blocked chord. The resulting output file of insert_block_end() for the V-V^7-I example is given in Table 17.

Instrument Number	Start Time	Duration	MIDI Key	Block_end
1	2	1	62	0
1	2	1	47	0
1	2	1	43	0
1	2	1	55	1
1	3	1	43	0
1	3	1	47	0
1	3	1	62	0
1	3	1	53	1
1	4	2	60	0
1	4	2	52	0
1	4	2	43	0
1	4	2	36	1

Table 17. Output from module insert_block_end() for V-V^7-I example

Create array to compare MIDI values

The function make_midikey_array() receives as input the output array of events, event array[], from the previous function, insert_block_end(). Make_midikey_array() also takes as input a new array, comparison_array[], which will be used throughout the rest of the conversion process. Make_midikey_array(), which is illustrated in Figure 32, receives the MIDI values from event array[] for each set of events blocked according to the motion of voices. In the first four events of Table 17 are contained the MIDI values 62, 47, 43, and 55. These are the four values that appear in the first row of comparison_array[], which is the output file of make_midikey_array().

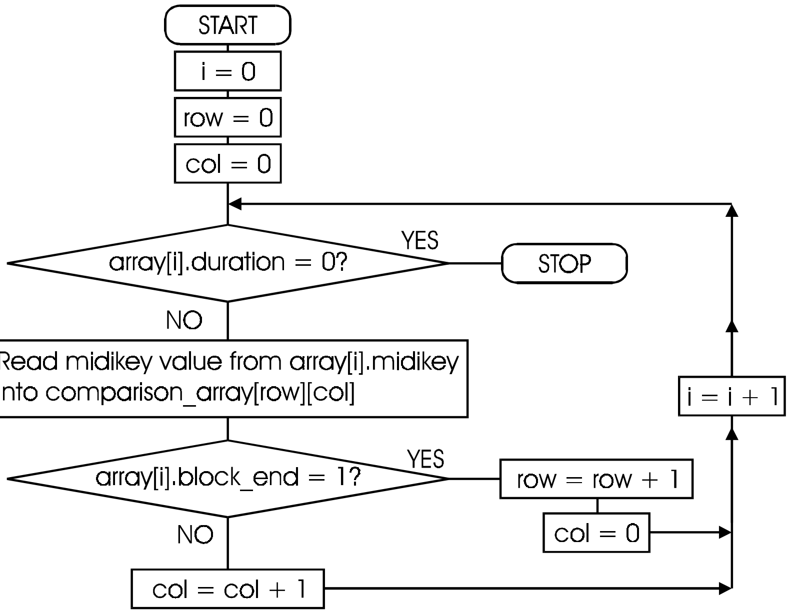


Figure 32. Algorithm for make_midikey_array()

The output array comparison_array[], Table 18, contains zeros in the first column as a result of the print function, show_array2(). The first column is used by subsequent functions to hold the MIDI values for the bass notes.

Bass	MIDI Values											
0	62	47	43	55	0	0	0	0	0	0	0	0
0	43	47	62	53	0	0	0	0	0	0	0	0
0	60	52	43	36	0	0	0	0	0	0	0	0

Table 18. Output from module make_midikey_array() for V-V^7-I example

Find and store the bass note for each sonority

The function get_min() receives as input comparison_array[] after it has been processed by make_midikey_array(). Get_min() locates the MIDI key corresponding to the lowest frequency for each chord and stores it in the column reserved for bass notes. The simple algorithm can be described as follows:

1) For every row, representing the MIDI values of the blocked chord, of comparison_array[]
    a. Initialise variable min as the value contained in the first column of the row
    b. For every column within this row containing a non-zero value
        i. If variable min > value of currently examined column
            1. Vaule of min is assigned value of currently examined column
    c. Place value of min in Bass column.

The bass MIDI value shown in Table 19, corresponding to the lowest MIDI key, is then placed in the rightmost column of the current row (but is printed to file in the leftmost column). The other MIDI values remain the same.

Bass	MIDI Values											
43	62	47	43	55	0	0	0	0	0	0	0	0
43	43	47	62	53	0	0	0	0	0	0	0	0
36	60	52	43	36	0	0	0	0	0	0	0	0

Table 19. Output from module get_min() for V-V^7-I example

Convert values to modulus 12

The function convert_mod12() takes as input the output array, comparison_array[], from module get_min(). Its single purpose is to place all notes from each chord into same octave using the modulus operator. This is accomplished by replacing any non-zero column values in every row of comparison_array[] with the twelve modulus of that value. If the remainder of that value is zero, 12 is added to it.

The output from convert_mod12() shown in Table 20 contains only MIDI key values that are equal to or less than 12. This approach assumes that, other than the bass note, spacing and order of the voices do not affect the function of the chord. Such an approach is consistent with Rameau's principles of figured bass.

Bass	MIDI Values											
7	2	11	7	7	0	0	0	0	0	0	0	0
7	7	11	2	5	0	0	0	0	0	0	0	0
12	12	4	7	12	0	0	0	0	0	0	0	0

Table 20. Output from module convert_mod12() for V-V^7-I example

Sort MIDI keys into ascending order

The function sort_midikeys() receives as input the output array, comparison_array[], from the function convert_mod12(). The purpose of sort_midikeys() is to sort every MIDI key, except for the bass, from every row of comparison_array[] into ascending numerical sequence, as illustrated in Table 21. This is accomplished with the following algorithm:

For each chord member of comparison_array[]

1. For each column containing a non-zero value in that chord, except for the bass note
    a. copy the chord value contained in that column into a 1-D array, small_array[]
    b. sort small_array[] into ascending order
    c. replace the values of the current chord member with the values from small_array[]

Bass	MIDI Values											
7	2	7	7	11	0	0	0	0	0	0	0	0
7	2	5	7	11	0	0	0	0	0	0	0	0
12	4	7	12	12	0	0	0	0	0	0	0	0

Table 21. Output from module sort_midikeys() for V-V^7-I example

Remove duplicate MIDI keys

Because only the bass value and chord type are necessary for determining a chord designation, duplicate MIDI keys representing notes are discarded. The function remove_dups() receives as input the output array, comparison_array[], from the function sort_midikeys() and deletes duplicate MIDI keys, other than the MIDI key representing the bass, from each chord contained in comparison_array[], as shown in Table 22. The algorithm used to remove the duplicates is as follows:

For each chord member of comparison_array[]

1. for every MIDI value in that chord, except for the bass
   a. compare it with the next value
   b. if the two are the same value
      i. move all members after current value to the left, thus removing the duplicate, and leave a zero in place of the last non-zero value
   c. return to step a, because more than one consecutive duplicate MIDI value is possible

Bass	MIDI Values											
7	2	7	11	0	0	0	0	0	0	0	0	0
7	2	5	7	11	0	0	0	0	0	0	0	0
12	4	7	12	0	0	0	0	0	0	0	0	0

Table 22. Output from module remove_dups() for V-V^7-I example

Select the optimally compact inversion

The purpose of the function compare_inversions() is to determine which inversion of each chord spans the smallest number of semitones. The input received by compare_inversions() is the output array comparison_array[] from function remove_dups(). The MIDI values, starting in column two, are already free of duplicates, fit within the same octave and are in ascending sequence, due to the previous functions that have operated on this array. All that is needed is to generate each inversion through upward octave transposition of the lowest member; find the interval between its leftmost member with its rightmost; and determine which of the inversions spans the smallest distance.

As given in Table 23, a D minor triad would have the values 2, 5, and 9, corresponding to D, F and A. Three intervals are needed for comparison: The smallest interval is 7, corresponding to the interval D to A; therefore, the optimally compact version of this chord is D, F, A.

	Notes (MIDI Keys)	Interval
Inversion 1	D(2) F(5) A(9)	9 – 2 = 7
Inversion 2	F(5) A(9) D(14)	14 – 5 = 9
Inversion 3	A(9) D(14) F(17)	17 – 9 = 8

Table 23. Three inversions of a D minor triad

Algorithm for compare_inversions()

Every time a chord is encountered, a 1-D array, initialised to zeros, is created. From left to right, this 1-D array is filled with the MIDI values of the chord. The dominant seventh chord taken from the V-V^7-I example will be used to illustrate the process. The initial inversion is shown in Table 24, with white text on black background used to reference notes by their conventional labels. The values on the bottom row, already transposed to the key of C, are those actually used by the compare_inversions() program. The compare_inversions() program now counts the number of semitones between 2 and 11 to provide the smallest interval found so far, which is 9.

D	F	G	B					
2	5	7	11	0	0	0	0	0

Table 24. Sorted MIDI key representation of a V^7 chord

In Table 25, the 2 from the first column, representing D, has been transposed upward twelve semitones to the value 14 and moved to the location of the first available zero to the right. The difference between lowest and highest values is now 9, the same as before. (Were the final value to be a tie between two chords, the first of the two would be selected.) So far, 9 is the lowest value, and is therefore the candidate interval for the most compact inversion.

	F	G	B	D				
	5	7	11	14	0	0	0	0

Table 25. "First inversion" of MIDI key representation for a V^7 chord

Table 26 illustrates that after transposing the F, represented by MIDI key 5, up an octave to the value 17, the routine again compares the difference between the outer notes to determine the interval of 10. Since this is not the smallest value so far, the candidate interval remains 9.

		G	B	D	F			
		7	11	14	17	0	0	0

Table 26. "Second inversion" of MIDI key representation for a V^7 chord

The lowest MIDI key is once again transposed up an octave, becoming the highest note of the new inversion, as shown in Table 27. The resulting interval for the chord, 8, supplants the previous value of 9 for the smallest interval, so the optimally compact inversion, B D F G, is used to identify the chord type.

			B	D	F	G		
			11	14	17	19	0	0

Table 27. "Third inversion" of MIDI key representation for a V^7 chord

All three chords of the V - V^7 - I progression are similarly catalogued in Table 28. For example, the first chord, a dominant triad in root position, can be unambiguously described as the chord whose bass note is G (MIDI key 7) and whose most compact inversion is root position G, B, D, corresponding to MIDI keys 7, 11 and 14. The chords are now ready to be indexed in the following program set, MAKELIST. Thus, the process of converting a MIDI score into pseudo-figured bass is complete.

Bass	MIDI Values											
7	7	11	14	0	0	0	0	0	0	0	0	0
7	11	14	17	19	0	0	0	0	0	0	0	0
12	12	16	19	0	0	0	0	0	0	0	0	0

Table 28. Output from module compare_inversions() for V-V^7-I example

4.4.6 MAKELIST

The purpose of MAKELIST is to convert a set of text (*.txt) files generated by CONVERT into a corresponding set of output list (*.lst) files. Every resulting output list file will contain a series of integers, each of which is an index value representing a unique chord.

MAKELIST first looks at each line of the input text file and scans its fields into a member (class) of an array. Every field in the input text file that precedes a zero is read in as a member of an optimally compact chord. The approach of determining the optimally compact inversion for the purposes of cataloguing is similar, but not identical to, that used by Allen Forte,[81] differing only in the respect that the currently considered computer program, when forced to

[81] Allen Forte, *The Structure of Atonal Music* (London: Yale University Press, 1973), pp. 3-5.

select between two or more inversions whose distance between lowest and highest notes are the same, simply selects the first one. Forte's analytical procedure, to which he refers as "normal order", selects the inversion with the least difference between first and second integers, or, if they are the same, between the next two integers that differ. In the present study, the bass note is stored separately so that the information of the traditional figured bass is retained, in keeping with the practices of Rameau. Such a classification system is referred to in this dissertation as "pseudo-figured bass".

Text files are processed one at a time. During the first pass, every condensed chord is compared with an array of chords to see if it is already present. If so, it is skipped; otherwise, it is sorted into its proper place in the array. Chords are sorted by digit left to right. Thus, chords with smaller leftmost digits, such as C# (index 1), are stored toward the beginning of the file, those with the larger ones, such as A natural (index 9), are stored toward the end.

During the second pass, each text file is pattern matched against the previously stored and sorted array. It is then indexed according to that chord's position within the array of chords, chord_array[]. Within this second loop, each chord is now stored in a list (*.lst) file containing only the indices that uniquely identify its chord type. Each resulting list file is a series of chord indices separated by the newline (RETURN) character.

After a text file is examined and closed, each chord in the chord_array[] is sorted according to the MIDI key values going left to right; thus, chords with the smallest leftmost values are stored at the top. For example, an unusual chord found in this study of 375 chorales is assigned the lowest index value of 1, as shown in Table 29. It is assigned the lowest chord index because its note index values from left to right are lower than any other. This method of indexing will be described in detail.

Bass	Optimally Compact MIDI Key Inversion			
C#	C#	D	E	F
1	1	2	4	5

Table 29. Pseudo-figured bass corresponding to a low index value

It should be mentioned that an indexing scheme based on every permutation of 1 to 12 notes could have been employed, allowing for every possible chord type. Such a system is unnecessary for a single study, but could be deemed useful if several researchers were to undertake similar studies and compare their results.

4.4.7 Sample Run of MAKELIST

To illustrate the steps executed by the MAKELIST program, the first bar from two separate chorales will be used. During the execution of MAKELIST in the present study, every bar was processed for 375 chorales.

Example1.txt, shown in Figure 33, is the input text file for the first bar of a Bach Chorale (Riemenschneider listing 233, Kalmus listing 365). This progression, originally in A major, has already been transposed to C major and converted into pseudo-figured bass by CONVERT, described in the previous section. For ease of reference, Figure 34 and Table 30 provide a musical score and schematic representation, respectively, of this input file.

12	12	16	19	0	0	0	0	0	0	0	0	0
9	5	9	12	0	0	0	0	0	0	0	0	0
4	4	7	11	0	0	0	0	0	0	0	0	0
4	4	5	7	11	0	0	0	0	0	0	0	0
9	4	7	9	12	0	0	0	0	0	0	0	0

Figure 33. Input text file for Example1.txt

Figure 34. Musical Representation of Example1.txt

Chord	Bass	Optimally Compact MIDI Key Inversion											
1 (I)	12	12	16	19	0	0	0	0	0	0	0	0	0
2 (iv^6)	9	5	9	12	0	0	0	0	0	0	0	0	0
3 (iii)	4	4	7	11	0	0	0	0	0	0	0	0	0
4 (iii)	4	4	5	7	11	0	0	0	0	0	0	0	0
5 (vi^7)	9	4	7	9	12	0	0	0	0	0	0	0	0

Table 30. Schematic representation of Example1.txt

Example2.txt, shown as an input text file (Figure 35), as a musical score (Figure 36) and as a schematic representation (Table 31), is the first bar from another Bach Chorale (Riemenschneider listing 96, Kalmus listing 201). This passage, as with Example1.txt, has already been transposed, in this case from D minor to C minor, as well as converted into pseudo-figured bass.

12	12	15	19	0	0	0	0	0	0	0	0	0
2	12	14	15	19	0	0	0	0	0	0	0	0
3	12	15	19	0	0	0	0	0	0	0	0	0
12	12	15	19	0	0	0	0	0	0	0	0	0
8	12	14	17	20	0	0	0	0	0	0	0	0
8	2	5	8	11	0	0	0	0	0	0	0	0
7	12	15	19	0	0	0	0	0	0	0	0	0
5	12	15	17	0	0	0	0	0	0	0	0	0

Figure 35. Input file for Example2.txt

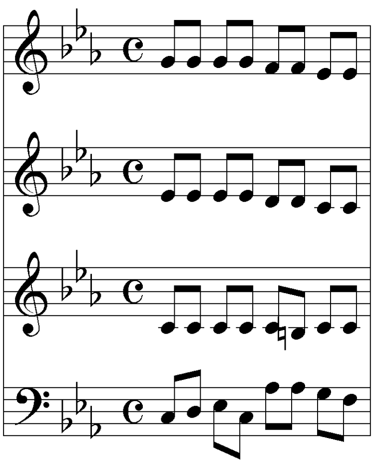


Figure 36. Musical representation of Example2.txt

Chord	Bass	Optimally Compact MIDI Key Inversion											
1 (i)	12	12	15	19	0	0	0	0	0	0	0	0	0
2 (i)	2	12	14	15	19	0	0	0	0	0	0	0	0
3 (i^{6})	3	12	15	19	0	0	0	0	0	0	0	0	0
4 (i)	12	12	15	19	0	0	0	0	0	0	0	0	0
5 (vii$^{o4/2}$)	8	12	14	17	20	0	0	0	0	0	0	0	0
6 (vii$^{o4/2}$)	8	2	5	8	11	0	0	0	0	0	0	0	0
7 (i$^{6/4}$)	7	12	15	19	0	0	0	0	0	0	0	0	0
8 (iv^{7})	5	12	15	17	0	0	0	0	0	0	0	0	0

Table 31. Schematic representation of Example2.txt

MAKELIST Loop 1

Text files are processed one at a time. During the first pass of MAKELIST, every chord is compared with each member of chord_array[], an array of chords that is initially empty, to see if that chord is already present. If it is present, it is skipped; otherwise, it is appended to chord_array[] as its last member.

Each optimally compact MIDI key inversion is now stored, including bass, as a single character string, which greatly facilitates sorting. After it has incorporated the data collected from input text file Example1.txt, chord_array[] will contain the information shown in Table 32. Because every chord so far is unique, the input file and chord_array[] still closely resemble each other.

Pseudo-figured Bass	Index Value
12 12 16 19	0
9 5 9 12	0
4 4 7 11	0
4 4 5 7 11	0
9 4 7 9 12	0

Table 32. Chord_array[] after Example1.txt is processed

After the data has been collected for input text file Example2.txt, chord_array[] contains the information given in Table 33. Although the first and fourth chords of Example2.txt are identical, the character string in the sixth row “12 12 15 19” from Table 33 is listed only once, as only one copy is needed in the index.

Pseudo-figured Bass	Index Value
12 12 16 19	0
9 5 9 12	0
4 4 7 11	0
4 4 5 7 11	0
9 4 7 9 12	0
12 12 15 19	0
2 12 14 15 19	0
3 12 15 19	0
8 12 14 17 20	0
8 2 5 8 11	0
7 12 15 19	0
5 12 15 17	0

Table 33. Chord_array[] after Example1.txt and Example2.txt are processed

When all files have been processed through Loop 1, all chords in chord_array[] are sorted in character order, as illustrated in Table 34. That is to say, even though 2 precedes 12, 12 will occur first because its first character, 1, precedes 2.

Pseudo-figured Bass	Index Value
12 12 15 19	1
12 12 16 19	2
2 12 14 15 19	3
3 12 15 19	4
4 4 5 7 11	5
4 4 7 11	6
5 12 15 17	7
7 12 15 19	8
8 12 14 17 20	9
8 2 5 8 11	10
9 4 7 9 12	11
9 5 9 12	12

Table 34. Sorted Chord_array[] at end of MAKELIST Loop 1

As shown in Table 34, MAKELIST enumerates each chord representation from top to bottom, starting with 1, and prints a copy of the indexed chords to file for reference. After thirteen chords were processed, one duplicate was omitted, and twelve chords were indexed. The index, along with loop 1 of MAKELIST, illustrated in Figure 37, is now complete and is stored as a text file, as well as in chord_array[].

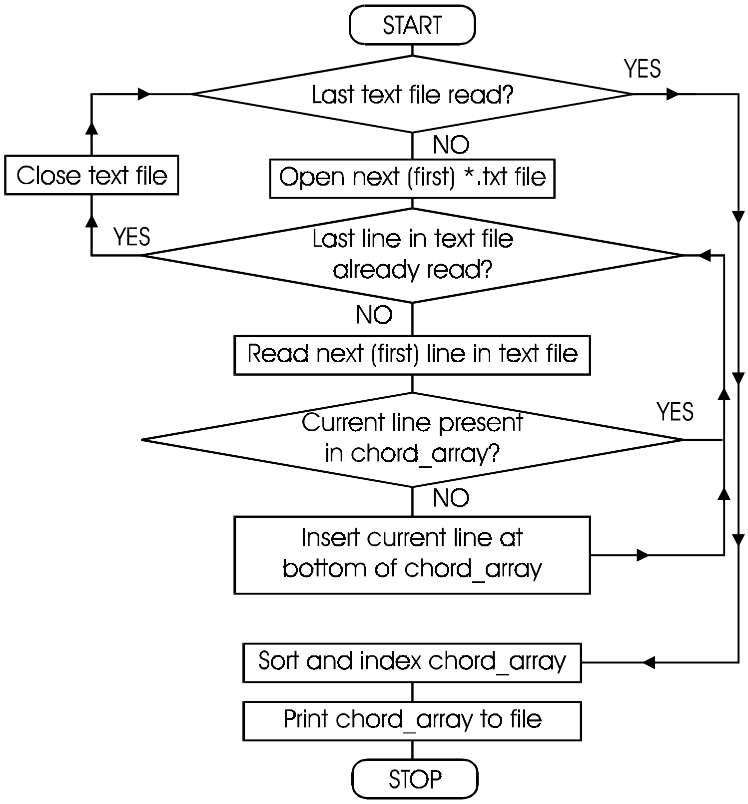


Figure 37. Flow chart for MAKELIST program, loop 1

MAKELIST Loop 2

Now that a catalogue for all chords in the examined set of chorales has been created, every text file stored as sequences of pseudo-figured bass can now be rewritten as a series of chords indices. This greatly facilitates the comparison and cataloguing of chord progressions, as they can now be stored as sequences of integers rather than sequences of integer arrays.

In the second loop of MAKELIST each text input file is examined separately. After a file is opened, it is examined one line at a time. For example, the first line in the Example1.txt text file is:

12 12 16 19

This set of integers is converted to a character string and compared with each line of the sorted and indexed chord_array[], shown in Table 34. This particular string, representing a C major triad in root position, is found in the second row of chord_array[], and is therefore assigned the corresponding index value of 2. Thus, in this collection of chords, derived from these two very short passages of music, a C major triad in root position will always be stored as the index value 2.

The output list file for Example1.txt, when processed with Example2.txt, is the output list file Example1.lst, shown in Figure 38. Translated into English, this amounts to, “Here is the list containing the second indexed chord, followed by the twelfth, the sixth, the fifth and finally the eleventh”.

```
 2
12
 6
 5
11
```

Figure 38. Example1.lst

Similarly, the output list file for Example2.txt, when processed with Example1.txt, is the output list file Example2.lst, shown in Figure 39.

```
 1
 3
 4
 1
 9
10
 8
 7
```

Figure 39. Example2.lst

Loop 2 of MAKELIST, a flow chart of which is given in Figure 40, is complete. Now that each chorale has been reduced to a sequence of integers representing chords, chorales can be compared against each other. Cataloguing every chord progression, along with its frequency of occurrence, contained in the entire collection of chorales is by far the most difficult task of the Bach Chorale Database Program Set. This final step in the cataloguing process is performed by STORE, which is described at length in the next section.

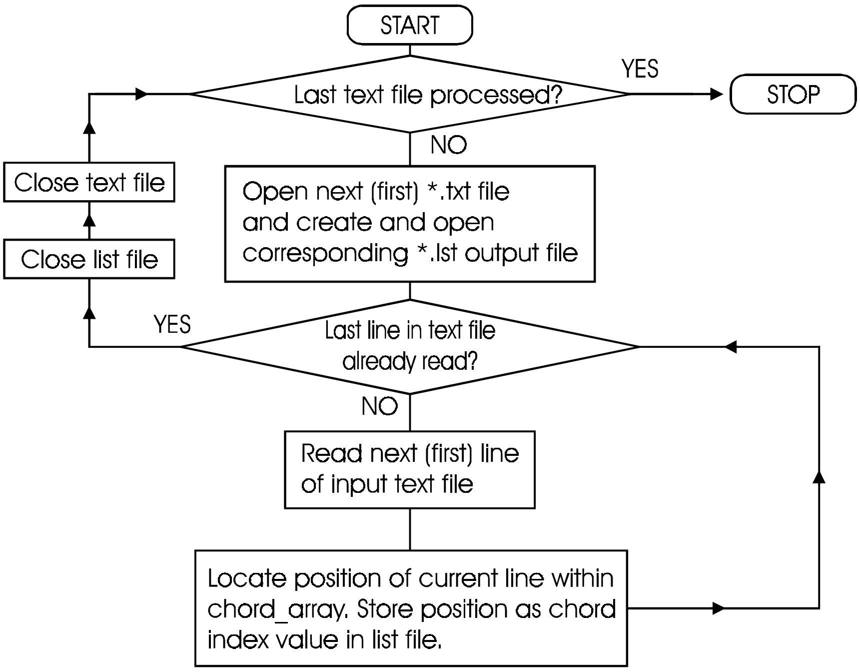


Figure 40. Flow chart for MAKELIST, loop 2

4.4.8 STORE

Overview

The purpose of STORE is to locate, catalogue and count every chord progression occurring in more than one chorale, from length 1 to n, within a collection of list files. STORE takes as input a directory containing (*.lst) files already processed by MAKELIST. The output file is a single text file containing a separate line for every chord progression of length 1 to n. At the end of each such line is an integer indicating the number of occurrences.

Algorithm's limitations

It is impractical to store all repetitions within a single chorale owing to the fact that if every chorale were compared to itself, there would simply be too many entries, as every chord progression would now be matched with itself. Even if the algorithm were to use a separate module that compared a file with itself by using a separator between chords already examined and not yet examined within a single piece, it would also need to perform a structural analysis to determine when the phrases repeat in order to avoid repetitions that occur in almost every, if not actually every, chorale. For all of this trouble, the valuable information, namely, the chord progressions that are most numerous, would be made less, not more, clear.

To avoid cataloguing an overabundance of chords and progressions, files are not compared with themselves. Only when a chord progression is found in more than one chorale is it stored and counted. Once a matching chord or progression has been found between two chorales, every occurrence of that chord or progression is located for the entire group of list files being processed.

4.4.9 Sample Run of STORE

Throughout the following description of STORE, five brief files, File_1.lst through File_5.lst, shown in Figure 41 through Figure 45, are used to illustrate the cataloguing process. Extremely simplified input is used so that each step may be shown in detail.

Create array of file names

First, an array of file names is created for every list (*.lst) file in the directory, as illustrated in Figure 48. From this array of file names, two names are selected at a time. The list file corresponding to the first unprocessed name is opened and its contents are read by function MAKE_ARRAYS into array1[], an array of integers representing indices (created by MAKELIST). The next unread file is similarly read into array2[]. For every ith and jth file name, an i^{th} (array1[]) and j^{th} (array2[]) array are created and the two files are compared.

Index value: 4 3 1

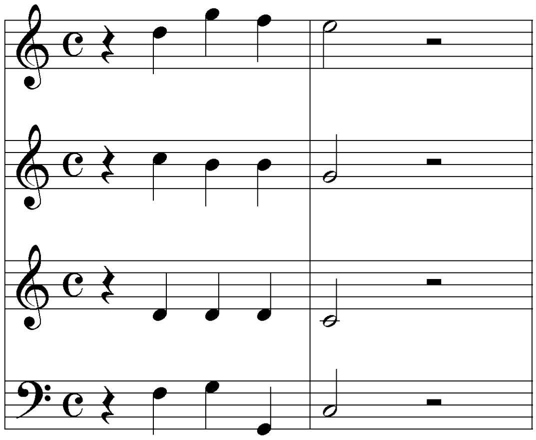
Figure 41. Musical representation of File_1.lst

Index value: 4 3 5

Figure 42. Musical representation of File_2.lst

Index Value: 5 4 1

Figure 43. Musical representation of File_3.lst



Index Value: 2 4 3 1

Figure 44. Musical representation of File_4.lst

Index Value: 2 4 3 1

Figure 45. Musical representation of File_5.lst

Array of List Files
File_1.lst
File_2.lst
File_3.lst
File_4.lst
File_5.lst

Table 35. Array of file names used in comparisons

In this sample run using the five abbreviated files listed in Table 35, File_1.lst is first compared with File_2.lst through File_5.lst; File_2 is then compared with File_3.lst through File_5.lst; File_3.lst is compared with File_4.lst through File_5.lst; and finally File_4.lst is compared with File_5.lst. Thus, every file is compared with every other file and an output txt file is continually built up from matching chord progressions. The rest of this section will then focus upon the comparisons between two files.

Compare chord progressions for equality

Comparison for equality between text strings is managed and performed by a pattern matching program, COMPARE_INT_ARRAY, illustrated in flow chart Figure 49 on p. 106. Starting positions for array1[] and array2[] are established initially as zero. All chord progressions ranging in chord length from length 1 to 7 are compared from these two starting positions. Then the starting position for array2[] is incremented by one and the comparisons for 1-7 chords are repeated. This continues until the starting points for array2[] have been exhausted, at which time the starting point for array1[] is incremented and the starting point for array2[] is reset to 0.

All starting points for array1[] are thus systematically exhausted as well, at which time the next two files in cue are compared. When, during the comparisons, a matching chord progression is found, as shown in the flow diagram of Figure 50 on p. 107, it is compared with a text file, out_file.txt, which is initially empty. If the matching chord is already present in out_file.txt, the value of its corresponding position in an integer array, increment_array[], is incremented and stored until all files have been processed. If the matching chord is not already present, it is stored on a separate line in out_file.txt.

Take, for example, the list files shown in Figure 41 and Figure 42 containing the chord indices (4, 3, 1) and (4, 3, 5), respectively. Detailed comparisons made by STORE for these two files are given in Table 36.

Table 36. Comparisons by STORE between File_1.lst and File_2.lst

File_1.lst	File_2.lst	Match?
4	4	YES
4	3	NO
4	5	NO
3	4	NO
3	3	YES
3	5	NO
1	4	NO
1	3	NO
1	5	NO
4, 3	4, 3	YES
4, 3	3, 5	NO
3, 1	4, 3	NO
3, 1	3, 5	NO
4, 3, 1	4, 3, 5	NO

If only these two files were processed, the output file would look like Figure 46. The output file is in agreement with Table 36, as the first member, the second member and the first two members from File 1 are all present in File 2. Sequences containing the indices 1 and 5, as well as the chords they represent, are not present in the output file, as they do not occur in more than one input list file.

```
4   *   2
3   *   2
4   3   *   2
```

Figure 46. STORE output file for File_1.lst and File_2.lst

Write to output file

When all list (*.lst) files have been thus processed, each chord progression (each row of integers representing index values of pseudo-figured bass) is rewritten to a new text file, out_fil2.txt. The asterisk (*) character is placed to the right of the last chord index value on every line. The last value on every line, to the right of the asterisk, is the value in increment_array[] corresponding to the position of that chord progression's row, indicating the number of times the matching chord is found in the entire collection. Figure 47 gives the STORE output file for all five of the small input files File_1.lst through File_5.lst.

```
4   *   5
1   *   4
5   *   2
3   *   4
4   3   *   4
3   1   *   3
4   3   1   *   3
2   *   2
2   4   *   2
2   4   3   *   2
2   4   3   1   *   2
```

Figure 47. STORE output file for File_1.lst through File_5.lst

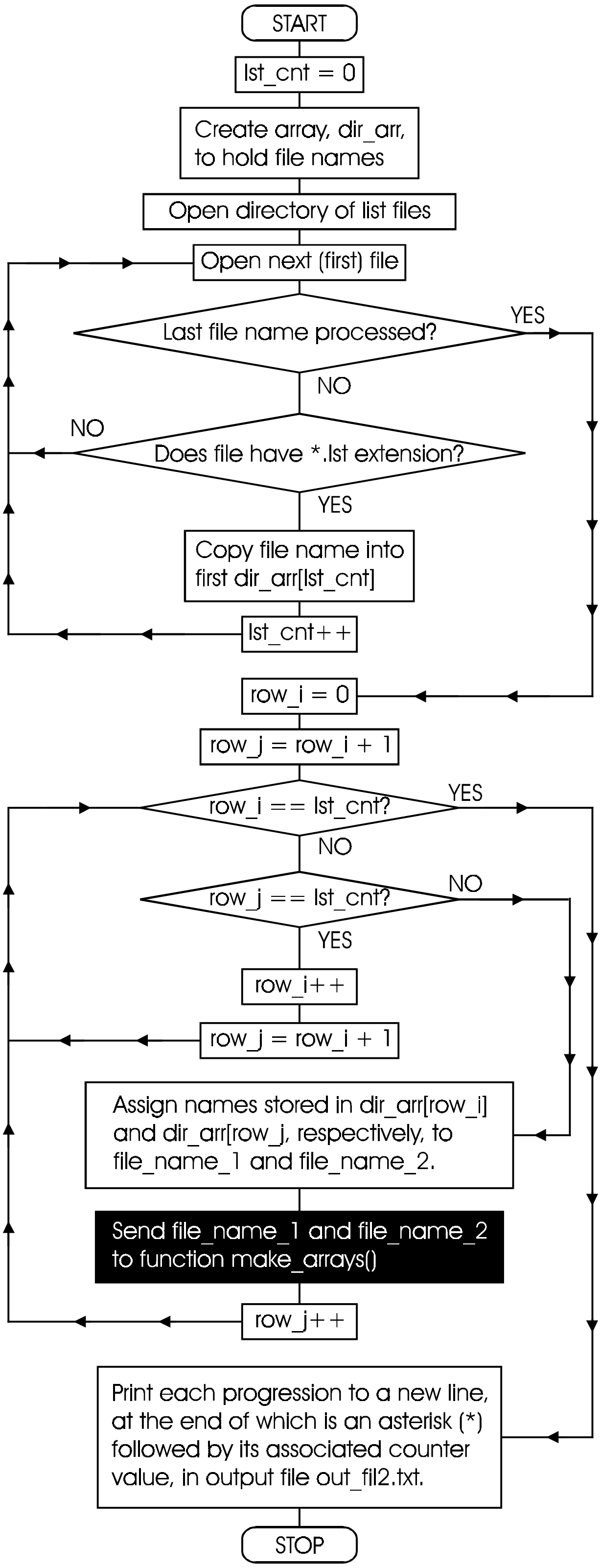


Figure 48. Flow diagram for program STORE

In creating the actual chord catalogue for the present study, 375 chorales were compared by pattern matching every chord progression from length 1 to 7 between every two files. The number of times two list files are compared for 375 list files is (375 * 374)/2 = 70,125. The number of comparisons of chord progressions between two files for length 1 through 7 compared containing 50 (some are much larger than this) chord indices each would be $50^2 + 49^2 + 48^2 + 47^2 + 46^2 + 45^2 + 44^2 = 15,491$ chord progressions. If 15,491 were the number of chord progressions to be compared between every two files, then for 375 chorales the number of chord progression evaluations with a maximum search length of seven chords would be 1,086,306,375, assuming every evaluation is necessary. Actually, because of duplicate chord progressions within a given primary list file, some redundant comparisons can be avoided. This estimate does not take into account the number of times the matching progressions are compared with each line of the output text file on the hard drive to see if such progressions are already present.

Algorithm for Function MAKE_ARRAYS

The function MAKE_ARRAYS serves only two functions, namely, to copy the contents of two list files into integer arrays and pass them onto the function COMPARE_INT_ARRAY.

1. Open list_file_1 and list_file_2
2. Create integer arrays array1[] and array2[]
3. Copy contents of list_file_1 and list_file_2 into array1[] and array2[], respectively
4. Send array1[] and array2[] to function COMPARE_INT_ARRAY

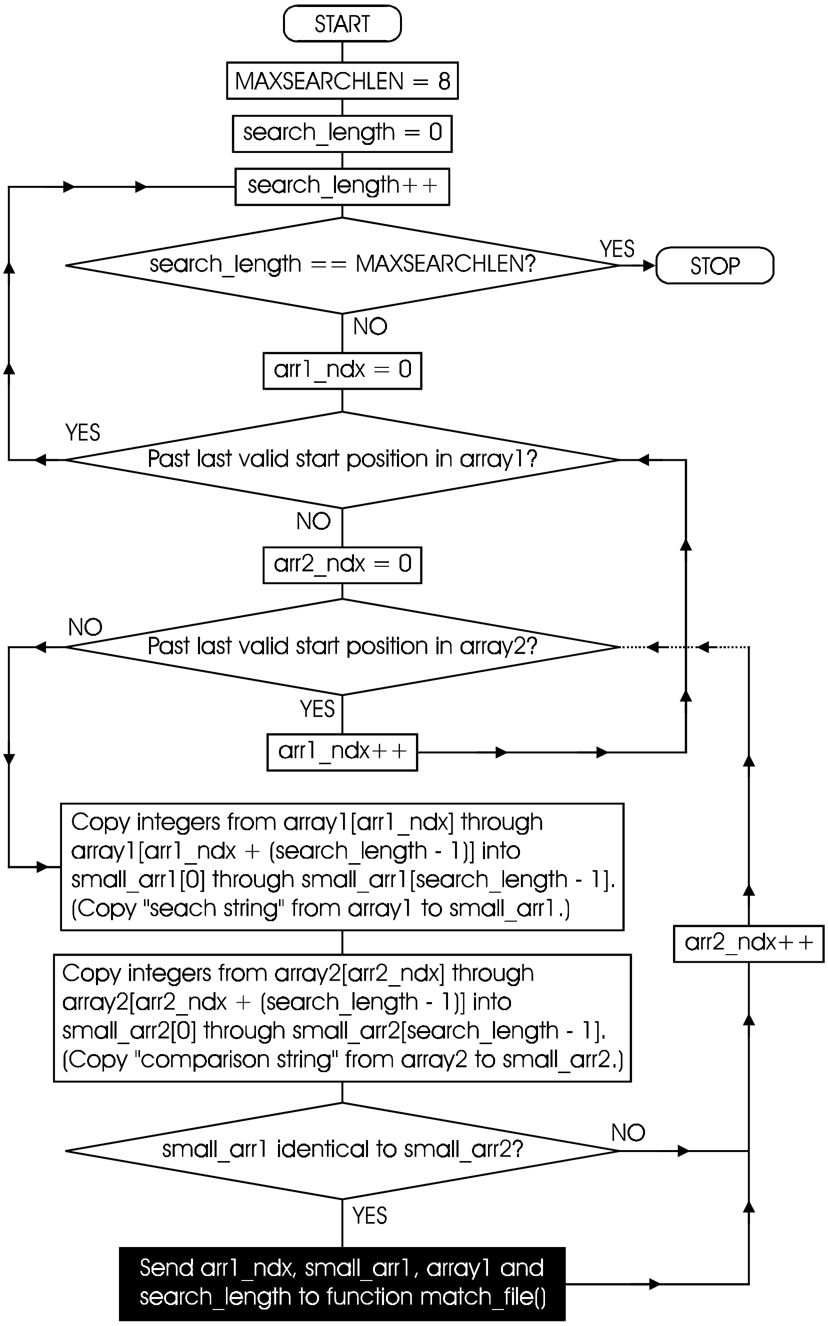


Figure 49. Flow diagram for function COMPARE_INT_ARRAY

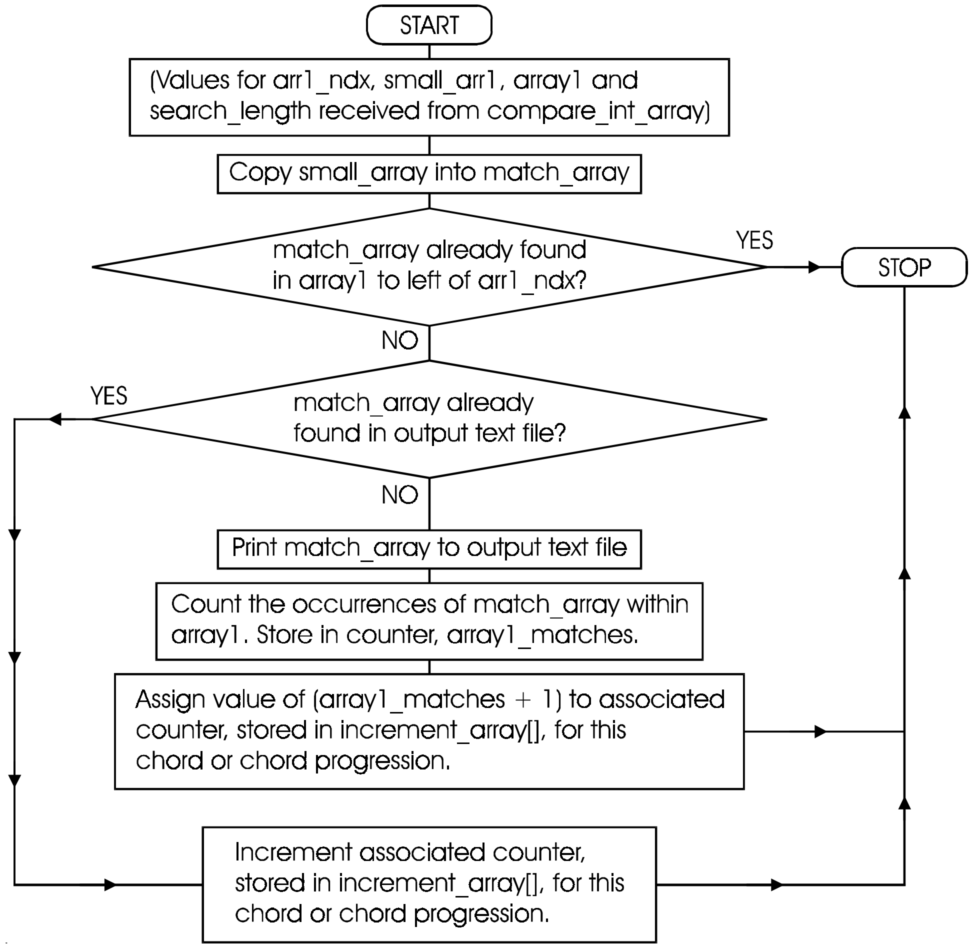


Figure 50. Flow diagram for function MATCH_FILE

4.5 Data Output and Analysis

4.5.1 Description of Output

The output of the STORE is a 249-page file containing 14,175 lines of text. Each such line represents a unique chord or chord progression containing 1-7 chords in the format <chord progression index array> * <counter>. Figure 51 represents the chord progression I IV IV^7 V IV^6 vii^{ø7} I, which occurs 9 times in the 375 chorales examined.

208 626 596 775 911 162 208 * 9

Figure 51. A single line of output representing an enumerated array of chord indices

All told, 14,175 unique chords and chord progressions of length 1-7 occur in two or more chorales a total of 105,813 times in 375 chorales. Chords and progressions occurring in only one chorale are skipped by STORE. For a complete set of individual chord indices, please refer to “Appendix III: Chord Indices for the Bach Chorale Tuning Database” on p. 253.

4.5.2 Choice of Analysis Affects Results

The method adopted in this study was to relate any chorale, whether major or minor, to tonic C. Thus, for instance, in the present analysis, V^7/I in the major key is counted exactly the same as V^7/i in the minor key because they both share exactly the same chord index. There are benefits to adopting this methodology. Krumhansl (1990), for instance, noted that in music of the eighteenth and nineteenth centuries, major and minor modes are treated equivalently in terms of their tonal functions.[82] Another advantage of this approach is that all chord progressions can be analysed at the same time.

It should be noted, however, that analysing in this manner has its weak as well as strong points. First of all, the number of dominant chords will be twice as numerous as when analysing major and minor chorales separately. For example, if there are two chord progressions, I-V and i-V, major tonic will be counted once, minor tonic will be counted once, and dominant will be counted twice. For this and many other reasons, it would be useful to reanalyse this data using the two separate groups major and minor.

Another logical approach yet would be to analyse the minor chorales as if their key centres were actually the sixth scale degree of the chorales in the major mode. This method would clearly illustrate the relationship between major and relative minor that permeates so many of the chorales. Each approach has its advantages, and each is worthy of numerous separate studies.

[82] Carol L. Krumhansl, *The Cognitive Foundations of Musical Pitch* (Oxford: Oxford University Press, 1990), p. 180.

4.5.3 Distribution of Chords

Overview

In all 375 chorales included in this study, there are 955 separately catalogued chords, with each chord representing a unique tonal “snapshot”. (Please see Appendix III starting on p. 253.) Many such chords occur only once, and are often juxtapositions of two common chords.

Perhaps the most striking feature of single chords taken as a group is the tendency of simple chords to occur with far more frequency than the others. An important question from the perspective of tuning is that of the relative frequency of chords derived from 5-limit just intonation.

4.5.4 Ten Most Common Chords

The ten most frequently occurring chords, detailed in Table 37, account for over 35% of chords used, with 945 chords accounting for the remaining 65%, as illustrated in Figure 52. Data for Table 37 was extracted using the computer program PRGBYLEN, which returns only those progressions containing the number of chord indices specified by the user. The same program was employed to retrieve each chord progression found in section 4.5.6, starting on p. 114.

Table 37. Ten most common chords in Bach Chorale Tuning Database

Rank	Occurrences	Chord Index Array	Chord
1	2670	208	I
2	2655	775	V
3	1694	202	i
4	968	482	I^6
5	863	145	V^6
6	836	951	vi
7	811	680	V^7
8	808	432	III
9	734	626	IV
10	663	43	V/III

It is immediately obvious that every chord in Table 37 is either a major or minor triad, or else a dominant seventh. The two simplest chord types in just intonation, the major (derived from 1:3:5) and minor (derived from 1:1/3:1/5), are clearly the most favoured, with the one exception, V^7, taking seventh place. The major triad, minor triad and dominant seventh are considered by Rameau to be the three most important types of chords.

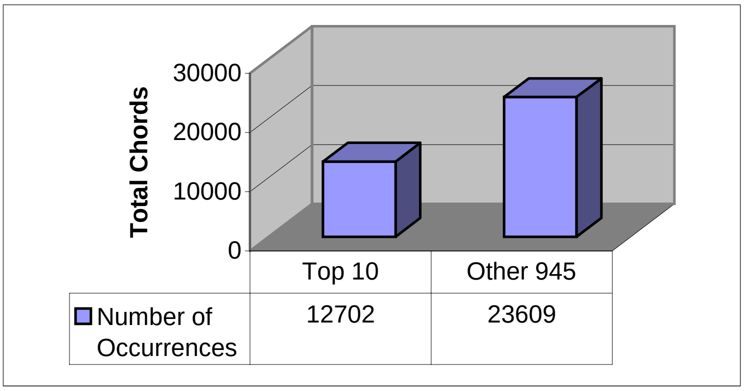


Figure 52. Total occurrences for 10 most common chords compared to all others

4.5.5 100 Most Common Chords

Clearly the most common chords rise to the top, as the 100 most frequently used chords comprise only slightly more than 10% of those available, but account for approximately 78% of those employed. The remaining 855 chords account for only approximately 22% of the total, as shown in Figure 53.

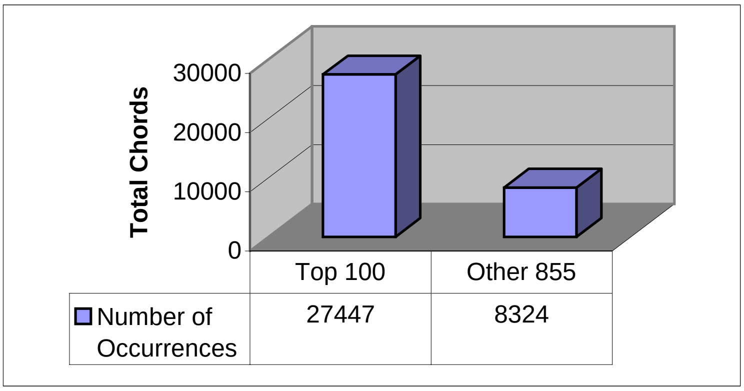


Figure 53. Total occurrences for 100 most common chords compared to all others

The 100 most common chords, listed according to frequency of occurrence, from these 375 chorales are detailed in Table 38. Each row in the table identifies a particular chord, from left to right, by its index number, its number of occurrences within the collection, its pseudo-figured bass representation, its figured bass representation, its ranking from 1-100 and its chord type. Please refer to "Appendix IV: Top 100 Chords by Tonal Function" on p. 262 for a detailed account of the 100 most frequently employed chords, where they are broken down by both tonal function and inversion.

Table 38. 100 most common chords in Bach Chorale Tuning Database

Index Number	Total Occurrences	Pseudo-figured Bass (Bass + notes)	Chord	Rank	Chord Type
208	2670	C C E G	I	1	Major
775	2655	G G B D	V	2	Major
202	1694	C C E♭ G	i	3	Minor
482	968	E C E G	I^{6}	4	Major
145	863	B G B D	V^{6}	5	Major
951	836	A A C E	vi	6	Minor
680	811	G B D F G	V^{7}	7	Dom. 7
432	808	E♭ E♭ G B♭	III (/i)	8	Major
626	734	F F A C	IV	9	Major
43	663	B♭ B♭ D F	V/III	10	Major
911	588	A F A C	IV^{6}	11	Major
400	541	E♭ C E♭ G	i^{6}	12	Minor
340	493	D D F A	ii	13	Minor
297	462	D B D F	vii^{o6}	14	Dim.
778	440	G G C D	V(susp. 4), $V^{6/5}$(-5, susp. 4)/V	15	Other
835	384	A♭ F A♭ C	iv^{6}	16	Minor
720	382	G E♭ G B♭	III^{6}	17	Major
345	378	D D F# A	V/V	18	Major
112	373	B B D F G	$V^{6/5}$	19	Dom. 7
860	352	A♭ A♭ C E♭	VI	20-21	Major
770	352	G G B♭ D	iii/III	20-21	Minor
546	316	F B D F G	$V^{4/2}$	22	Dom. 7
619	309	F F A♭ C	iv	23	Minor
285	289	D B♭ D F	V^{6}/III	24	Major
514	288	E E G B	iii	25	Minor
576	263	F D F A	ii^{6}	26	Minor
630	245	F A C D F	$ii^{6/5}$	27	Min. 7
553	233	F C D F A♭	$ii^{\phi 6/5}$(Consistent), $vii^{\phi 6/5}$/III(Contradictory)	28	Half-dim. 7
573	228	F D F A♭	ii^{o6}	29	Dim.
357	203	D F# A C D	V^{7}/V	30	Dom. 7
670	199	F# F# A C D	$V^{6/5}$/V	31	Dom. 7
692	188	G C E G	$I^{6/4}$	32	Major
689	181	G C E♭ G	$i^{6/4}$	33	Minor
309	179	D C D F	vii^{o6}(susp. 2)	34	Other
127	176	B E G B C	$I^{4/2}$	35	Major
82	174	B♭ G B♭ D	iii^{6}/III	36	Minor
275	164	C A C E	vi^{6}	37	Minor
915	159	A F# A C	vii^{o6}/V	38-41	Dim.
596	159	F E F A C	IV^{7}	38-41	Maj. 7
79	159	B♭ G B♭ C E♭	$i^{4/2}$	38-41	Min. 7
745	159	G F G B	V^{7} (-5)	38-41	Other

Index Number	Total Occurrences	Pseudo-figured Bass (Bass + notes)	Chord	Rank	Chord Type
49	150	B♭ D F A♭ B♭	V^7/III	42	Dom. 7
226	148	C F G C	I(susp. 4), $V^{6/5}$(-5, susp. 4)	43	Other
265	144	C A♭ C E♭	VI^6	44	Major
733	143	G E G B	iii^6	45-46	Minor
190	143	C C D E G	I(+2)	45-46	Other
740	141	G E G A C	$vi^{4/2}$	47	Min. 7
648	134	F# D F# A	V^6/V	48-49	Major
559	134	F C E F G	I(+4, 4 in bass)	48-49	Other
310	131	D C D F G	$V^{4/3}$ (susp. 4)	50	Dom. 7
801	130	A♭ C E♭ F A♭	$iv^{6/5}$	51	Min. 7
220	129	C E G A C	$vi^{6/5}$	52	Min. 7
192	126	C C D F G	$V^{6/5}$(susp. 4)	53	Other
273	120	C A C D#/E♭	AMB	54-55	Other
771	120	G G B	AMB	54-55	Other
250	119	C G C D	V(susp. 4, 4 bass), $V^{4/2}$(-5, susp. 4)/V	56	Other
186	117	C C D E♭ G	i(+2)	57	Other
522	113	E E G# B	V/vi	58	Major
800	108	A♭ C D F A♭	$ii^{\phi 4/3}$ (Consistent) $vii^{\phi 4/3}$/III (Contradictory)	59	Half-dim. 7
555	106	F C E♭ F G	i(+4, 4 in bass)	60-61	Other
923	106	A G A C	vi^7 (-5), vii^{o6}(susp. 2)/V	60-61	Other
825	103	G# E G# B	V^6/vi	62	Major
298	101	D B D F G	$V^{4/3}$	63-64	Dom. 7
950	101	A A C E♭ F	$V^{6/5}$/V/III	63-64	Dom. 7
954	100	A A C# E	V/ii	65-66	Major
203	100	C C E	AMB	65-66	Other
557	99	F C E♭ F A♭	iv^7	67	Min. 7
513	98	E E G B♭ C	$V^{6/5}$/IV, $V^{6/5}$/iv	68	Dom. 7
897	97	A E G A C	vi^7	69	Min. 7
361	94	D G B D	$V^{6/4}$	70	Major
308	92	D C D E G	I (+2, 2 bass)	71	Other
471	91	E B D E G	iii^7	72	Min. 7
56	90	B♭ E♭ G B♭	$III^{6/4}$	73-74	Major
196	90	C C E♭	AMB	73-74	Other
349	89	D D G A	V(susp. 4)/V, $V^{6/5}$(-5, susp. 4)/V/V	75	Other
380	87	D A C D F	ii^7	76	Min. 7
111	86	B B D F	vii^o	77-78	Dim.
217	86	C E G B C	I^7	77-78	Maj. 7
654	80	F#/G♭ D#/E♭ F#/G♭ A C	AMB (Contradictory)	79	Other
628	78	F A B D F	$vii^{\phi 4/3}$ (Contradictory), $ii^{\phi 4/3}$/vi (Consistent)	80-81	Half-dim. 7
245	78	C G B♭ C E♭	i^7	80-81	Min. 7

Index Number	Total Occurrences	Pseudo-figured Bass (Bass + notes)	Chord	Rank	Chord Type
240	77	C F# A C D	$V^{4/2}$/V	82-83	Dom. 7
607	77	F F G C	I^{6} (susp. 2), $V^{4/2}$(-5, susp. 4)	82-83	Other
234	76	C F A C	$IV^{6/4}$	84-85	Major
303	76	D C D E♭ G	i(+2, 2 in bass)	84-85	Other
808	75	A♭ D F A♭ B♭	$V^{4/2}$/III	86-87	Dom. 7
53	75	B♭ E♭ F B♭	V (susp. 4)/III, $V^{6/5}$(-5, susp. 4)/V/III	86-87	Other
9	73	C# C# E G A	$V^{6/5}$/ii	88	Dom. 7
61	70	B♭ E G B♭ C	$V^{4/2}$/IV, $V^{4/2}$/iv	89-90	Dom. 7
216	70	C E G B♭ C	V^{7}/IV, V^{7}/iv	89-90	Dom. 7
338	69	D D F A♭ B♭	$V^{6/5}$/III	91-92	Dom. 7
948	69	A A C D F	$ii^{4/3}$	91-92	Min. 7
847	68	A♭ G A♭ C E♭	VI^{7}	93-95	Maj. 7
363	68	D G C D	$V^{6/4}$(susp. 4), V^{7}(-5, susp. 4)/V	93-95	Other
612	68	F F G A C	IV (+2)	93-95	Other
312	67	D C D F#	V^{7}(-5)/V	96	Other
855	64	G# G# B D E	$V^{6/5}$/vi	97	Dom. 7
505	62	E E F A C	$IV^{4/2}$	98-99	Major
191	62	C C D F	vii^{o}(susp. 2)	98-99	Other
702	61	G D F G B♭	iii^{7}/III	100	Min. 7
	Total: 27447				

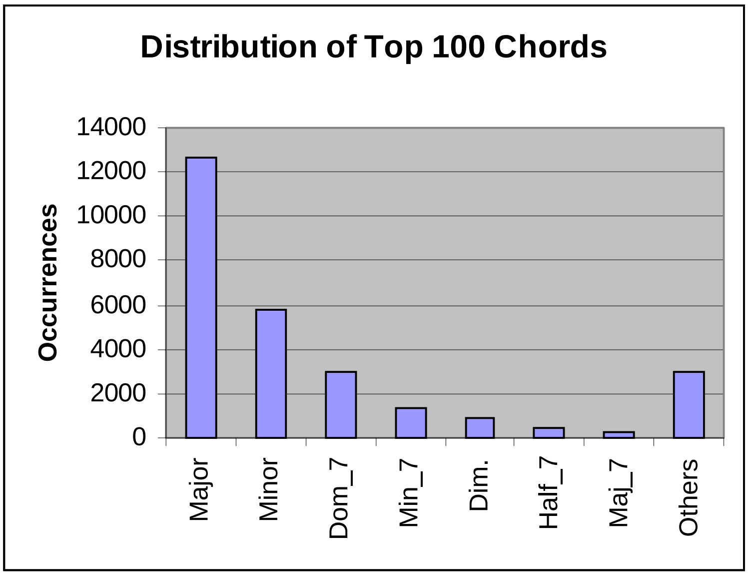
A full 96 of these top 100 chords are easily tuned according to the heuristics given in Chapter 5 starting on p. 119. Three of the remaining 4 chords, respectively ranked 28, 59, and 80-81 (index 628), are all half-diminished seventh chords, which, when serving as supertonic seventh leading to dominant, are consistent with 5-limit theory, whereas, when used as the leading tone half-diminished seventh, are not, as discussed in section 5.10.3 on p. 144. The fully diminished seventh chord ranked 79 is the only member of the top 100 chords that is problematic in 5-limit intonation no matter what function it might serve.

Table 39 and Figure 54 illustrate the extreme preference by Bach for the simplest possible chord types in 5-limit just intonation, with the major triad alone accounting for 46% of the total. Tonic and primary dominant function chords account for over 42% of the top 100 chords, and more than 32% of all chords found in this collection.

Table 39. Top 100 chords by type

Chord Type	Occurrences
Major	12630
Minor	5822
Dom. 7	2981
Min. 7	1386
Dim.	935
Half-dim. 7	419
Maj. 7	313
All Others	2961
Total:	27447

Figure 54. Top 100 chords by type



4.5.6 Most Common Chord Progressions from Length 1 – 7

The most commonly occurring chord progressions containing from one to seven chords are given in Table 40. It is upon these chord progressions that the experiments of Chapter 7, beginning on p. 188, are based. Progressions containing chords that are directly repeated, such as I-I-I, or I-V-V, are removed from consideration. Data was extracted from an output text file using the computer program PRGBYLEN.

Table 40. Most common chord progression from length 1 to 7 chords

Number of Chords	Chord Index Array	Chord Progression	Occurrences
1	208	I	2670
2	775 680	V V^{7}	524
3	775 680 208	V V^{7} I	252
4	778 775 680 208	V(susp. 4) V V^{7} I	45
5A	208 217 340 297 482	I I^{7} ii vii^{o6} I^{6}	13
5B	340 297 482 559 775	ii vii^{o6} I^{6} (F C E G) V	13
5C	208 626 596 775 911	I IV IV^{7} V IV^{6}	13
6A	208 626 596 775 911 162	I IV IV^{7} V IV^{6} $vii^{\phi7}$	9
6B	626 596 775 911 162 208	IV IV^{7} V IV^{6} $vii^{\phi7}$ I	9
7	208 626 596 775 911 162 208	I IV IV^{7} V IV^{6} $vii^{\phi7}$ I	9

4.5.7 Ten Most Common Chord Progressions for 2 – 7 Chords

Table 41 through Table 46 provide details of the chord progressions ranging from two to seven chords that occur most commonly in the 375 chorales examined. Only five of these 61 most elite chord progressions are, by virtue of containing a half-diminished leading tone seventh chord ($vii^{\phi7}$ * 4, $vii^{\phi4/3}$ * 1), as discussed in section 5.10.3 on p. 144, contradictory according to the rules found in section 5.11 starting on p. 145.

When examining the chord progressions that occur most commonly, it is again clear that simple chords are statistically rising to the top. With 955 chords from which to select, a total of 51 chords are employed a total of 284 times. Of these 51 chords, only the half-diminished $vii^{\phi7}$ is self-contradictory (as discussed in section 5.10.3 on p. 144) when its leading tone is treated as the third of the dominant. The most common chord progressions are constructed almost entirely from chords consistent with 5-limit just intonation, yielding further evidence that this tuning system is indeed the origin of Bach's tonal language.

Table 41. Ten most common 2-chord progressions

Rank	Occurrences	Chord Index Array	Chord Progression
1	524	775 680	V V^{7}
2	408	775 208	V I
3	364	680 208	V^{7} I
4	242	778 775	V(susp. 4) V
5	228	775 202	V i
6	181	775 546	V $V^{4/2}$
7	165	208 127	I $I^{4/2}$
8	160	208 775	I V
9	154	208 626	I IV
10	144	202 79	i $i^{4/2}$

Table 42. Ten most common 3-chord progressions

Rank	Occurrences	Chord Index Array	Chord Progression
1	252	775 680 208	V V^{7} I
2	117	775 680 202	V V^{7} i
3	93	345 357 775	V/V V^{7}/V V
4	87	775 546 482	V $V^{4/2}$ I^{6}
5	80	778 775 680	V(susp. 4) V V^{7}
6	76	553 775 680	$ii^{\phi 6/5}$ V V^{7}
7	66	208 127 951	I $I^{4/2}$ vi
8	58	208 127 911	I $I^{4/2}$ IV^{6}
9	56	775 546 400	V $V^{4/2}$ i^{6}
10	54	778 775 208	V(susp. 4) V I

Table 43. Eleven most common 4-chord progressions

Rank	Occurrences	Chord Index Array	Chord Progression
1	45	778 775 680 208	V(susp. 4) V V^{7} I
2	37	553 775 680 202	$ii^{\phi 6/5}$ V V^{7} i
3	33	630 775 680 208	$ii^{6/5}$ V V^{7} I
4	24	220 345 357 775	$vi^{6/5}$ V/V V^{7}/V V
5	24	202 553 775 680	i $ii^{\phi 6/5}$ V V^{7}
6	23	297 482 559 775	$vii^{\circ 6}$ I^{6} I(+4, 4 in bass) V
7	20	202 303 400 555	i i (+2, 2 in bass) i^{6} i(+4, 4 in bass)
8	20	190 208 778 775	I(+2) I V(susp. 4) V
9	20	208 626 596 775	I IV IV^{7} V
10	20	553 775 680 208	$ii^{\phi 6/5}$ V V^{7} I
11	20	778 775 680 202	V(susp. 4) V V^{7} i

Table 44. Ten most common 5-chord progressions

Rank	Occurrences	Chord Index Array	Chord Progression
1	13	208 217 340 297 482	I I^{7} ii $vii^{\circ 6}$ I^{6}
2	13	340 297 482 559 775	ii $vii^{\circ 6}$ I^{6} I(+4, 4 in bass) V
3	13	208 626 596 775 911	I IV IV^{7} V IV^{6}
4	12	112 190 208 778 775	$V^{6/5}$ I(+2) I V(susp. 4) V
5	12	202 553 775 680 202	i $ii^{\phi 6/5}$ V V^{7} i
6	12	630 376 775 680 208	$ii^{6/5}$ ii^{7}(-3) V V^{7} I
7	11	208 778 775 680 208	I V(susp. 4) V V^{7} I
8	10	596 775 911 162 208	IV^{7} V IV^{6} $vii^{\phi 7}$ I
9	10	297 482 559 778 775	$vii^{\circ 6}$ I^{6} I(+4, 4 in bass) V(susp. 4) V
10	9	202 192 202 145 202	i $V^{6/5}$(susp. 4, 4 in bass) I V^{6} i

Table 45. Ten most common 6-chord progressions

Rank	Occurrences	Chord Index Array	Chord Progression
1	9	208 626 596 775 911 162	I IV IV^{7} V IV^{6} vii$^{\phi 7}$
2	9	626 596 775 911 162 208	IV IV^{7} V IV^{6} vii$^{\phi 7}$ I
3	8	897 309 297 482 559 775	vi^{7} vii$^{\circ 6}$(susp. 2) vii$^{\circ 6}$ I^{6} I(+4, 4 in bass) V
4	8	860 432 587 720 801 56	VI III V$^{4/3}$(susp. 4)/III III^{6} iv$^{6/5}$ III$^{6/4}$
5	8	835 689 553 775 680 208	iv^{6} i$^{6/4}$ ii$^{\phi 6/5}$ V V^{7} I
6	8	720 860 432 587 720 801	III^{6} VI III V$^{4/3}$(susp. 4)/III III^{6} iv$^{6/5}$
7	7	482 361 220 345 357 775	I^{6} V$^{6/4}$ vi$^{6/5}$ V/V V^{7}/V V
8	7	482 630 376 775 680 208	I^{6} ii$^{6/5}$ ii^{7}(-3) V V^{7} I
9	7	309 297 482 559 778 775	vii$^{\circ 6}$(susp. 2) vii$^{\circ 6}$ I^{6} I(+4, 4 in bass) V(susp. 4) V
10	7	630 628 576 775 680 208	ii$^{6/5}$ vii$^{\phi 4/3}$ ii^{6} V V^{7} I

Table 46. Ten most common 7-chord progressions

Rank	Occurrences	Chord Index Array	Chord Progression
1	9	208 626 596 775 911 162 208	I IV IV^{7} V IV^{6} vii$^{\phi 7}$ I
2	8	720 860 432 587 720 801 56	III^{6} VI III V$^{4/3}$(susp. 4)/III III^{6} iv$^{6/5}$ III$^{6/4}$
3	6	432 587 720 801 56 59 43	III V$^{4/3}$(susp. 4)/III III^{6} iv$^{6/5}$ III$^{6/4}$ III$^{6/4}$(susp. 4) V/III
4	6	587 720 801 56 59 43 432	V$^{4/3}$(susp. 4)/III III^{6} iv$^{6/5}$ III$^{6/4}$ III$^{6/4}$(susp. 4) V/III III
5	6	208 309 297 482 559 778 775	I vii$^{\circ 6}$(susp. 2) vii$^{\circ 6}$ I^{6} I(+4, 4 in bass) V(susp. 4) V
6	6	860 432 587 720 801 56 59	VI III V$^{4/3}$(susp. 4)/III III^{6} iv$^{6/5}$ III$^{6/4}$ III$^{6/4}$(susp. 4)
7	6	213 145 126 911 905 740 596	IV$^{4/3}$ V^{6} iii$^{6/4}$ IV^{6} IV^{6}(+2) vi$^{4/2}$ IV^{7}
8	6	145 126 911 905 740 596 778	V^{6} iii$^{6/4}$ IV^{6} IV^{6}(+2) vi$^{4/2}$ IV^{7} V(susp. 4)
9	6	905 740 596 778 684 680 208	IV^{6}(+2) vi$^{4/2}$ IV^{7} V(susp. 4) I$^{6/4}$(+2) V^{7} I
10	6	126 911 905 740 596 778 684	iii$^{6/4}$ IV^{6} IV^{6}(+2) vi$^{4/2}$ IV^{7} V(susp. 4) I$^{6/4}$(+2)

4.6 Concluding Remarks

Automation has permitted the speed and precision necessary to accurately enumerate the chords and chord progressions contained within a large collection of Bach's chorales. Hundreds of millions of comparisons between chord progressions have produced a wealth of data with which to study Bach's tonal habits.

In making the case that 5-limit just intonation is the origin of Bach's tonal language, it is extremely significant that 96% of the 100 most frequently used chords by Bach, as well as over 98% of the 27,447 instances of these chords, can be easily tuned according to simple heuristics outlined in Chapter 5, starting on p. 119. Although many of the 955 chords catalogued in this study cannot be tuned within this framework, such as the fully diminished seventh, the half-diminished vii$^{\phi7}$ or the dominant ninth, the ones that rise to the top in terms of frequency of occurrence fit the 5-limit paradigm perfectly.

The wealth of data provided by automated cataloguing of chords is extremely useful in studying tuning from an experimental vantage point, as is done in the listening experiments of Chapter 7, starting on p. 188. Perhaps more importantly, automated chord cataloguing points the way to new methods of tonal analysis and composition.

Chapter 5
Automated Tuning by Ratios

> *"If it be asked why no more primes than 1, 2, 3, 5 are admitted to musical ratios; one reason is, that consonances whose vibrations are in ratios whose terms involve 7, 11, 13, &c, caeteris paribus would be less simple and harmonious than those whose ratios involve the lesser primes only."* [83]

5.1 Overview

Transcribing traditionally notated music, with its rigidly enforced set of twelve tones, into the precise ratios of just intonation, whose number of possible notes is open-ended, is not accomplished by means of a simple one-to-one correspondence. Careful examination of each note is required to determine its implied frequency relationships not only within a single chord, but also within the greater hierarchical tonal structure of the progression, section and movement to which it belongs.

Adherence to the basic principle that the frequency of each note must be related to tonic as the ratio of two integers is generally easy to achieve when tuning a single chord or scale, and often when tuning several chords in succession; however, certain progressions and modulations occur in which notes that bear the same letter and accidental designations must be assigned distinct frequency values. Such a note with more than one tuning is known as a mutable tone,[84] the frequencies for which can be obtained through a chain of heuristics.

Often when transcribing pieces of music from traditional notation into just intonation, there are instances when no such mutable tone will satisfy all of the heuristics. Such an instance will constitute a contradiction. Surprisingly, commonly employed tonal techniques and even musical idioms often result in contradiction within the framework of just intonation. Two such tonal contradictions occurring within 5-limit just intonation will be examined in detail.

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[83] Smith, op. cit., p. 44.

[84] Llewelyn Southworth Lloyd, *Intervals, Scales and Temperaments* (London: St. Martin's Press, 1978), pp. 224-25.

Once the methodology and heuristics for assigning detailed frequencies to the most commonly occurring chords have been established, it is then possible to tune entire short pieces of music as collections of ratios. In order to demonstrate that tuning by ratios is actually practical in tonal music, five Bach chorales selected for their brevity are tuned in complete detail. (Please see "Appendix V: Heuristic Analysis of Bach Chorales" on p. 265.)

5.2 Related Works

As far as is known, no previous study has undertaken the task of tuning entire pieces of tonal music in just intonation. It is quite possible that Rameau was the last theorist to provide original and detailed rules for tuning chord progressions in tonal music. Several recent studies, however, have adopted a "bottom-up" approach, attempting to use avoidance of closely spaced non-coinciding partials as a basis for tuning chords and progressions.

5.2.1 Automated Tuning and Critical Bandwidth

Sethares' method of adaptive tuning is based on the premise that a tuning system should be flexibly applied in response to timbres of individual notes; to the chords themselves; and to the sequential arrangement of chords in chord progressions.[85] Founded upon the earlier work of Plomp and Levelt, adaptive tuning is built around the assumption that overall dissonance is merely the sum of the beating between all partials weighted individually within all critical bandwidths.

Sethares' program elects to change values for major and minor thirds according to how many partials are present. Using 4 partials, for instance, C major and C minor triads share the same tuning. However, for 5-16 partials, approximately correct values for major and minor triads are obtained. Adaptive tuning appears most efficient "when played with harmonic timbres of sufficient complexity."

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[85] William A. Sethares, 'Adaptive tunings for musical scales,' *Journal of the Acoustical Society of America*, vol. 96 no. 1 (1994), pp. 10-18.

One immediately obvious shortcoming of Sethares' study is that the tunings derived through automation are not compared with other tunings through listening experiments. Furthermore, the approach of merely summing the partials' relative amplitudes is not consistent with a study by Vos and van Vianen (1985), who determined that changing the timbre from sawtooth to a series of partials having equal amplitude did not appreciably change the subjects' ability to discriminate between pure and tempered intervals.[86] The concept of the dominance region, in which the auditory system is predisposed to weight the first 5-6 partials, with emphasis on the partials 3-5, is similarly not taken into account. Most importantly, as will be shown, adaptive tuning ignores simple tonal hierarchies that constitute the backbone of tonality.

In Figure 55 and Table 47, borrowed from Hall (1974), Sethares has found it necessary to end on a different tuning for tonic than the one from which he began. The main difference between Table 47 and the solution of the author (Figure 56 and Table 48) is that the D in Table 47 is treated as an unchanging note; therefore, the strict rule of maintaining frequencies of held notes (a rule exploited deliberately in this example) necessitates that chords 4 and 5 are tuned so that the dominant and tonic roots are now mutable tones.

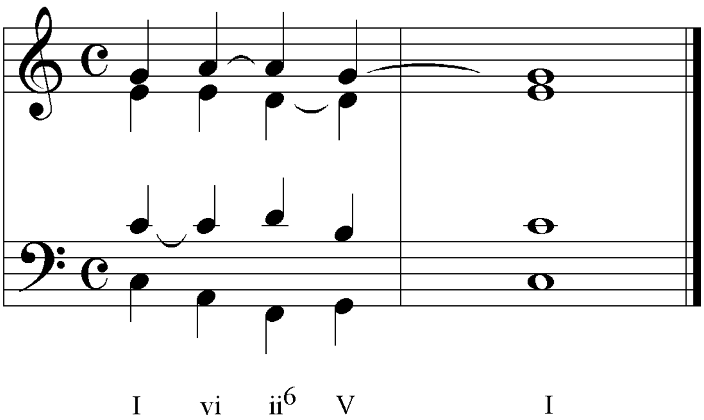

Figure 55. Common tone example (Sethares 1994, taken from Hall 1974)

[86] Vos and van Vianen, op. cit., p. 181.

Chord 1	Chord 2	Chord 3	Chord 4	Chord 5
G 392.5	A 436	A 436	G 387.5	G 387.5
E 327	E 327	D 290.5	D 290.5	E 323
C 261.5	C 261.5	D 290.5	B 290.5	C 258.5
C 131	A 109	F 87	G 96.5	C 129

Table 47. Sethares' just intonation for common tone example

With all due respect to the subtle technical and acoustical considerations of Sethares' study, making tonic and dominant roots mutable in this very common and brief chord progression defies all musical logic. As structural pillars, tonic C and dominant G roots are decidedly more important than D as the root of supertonic or as the fifth of dominant G. Therefore, the held D should be retuned, not only preserving the most important tonal relationships of tonic and dominant, but at the same time necessitating only one instead of four mutable tones.

The values in Figure 56 and Table 47, discovered independently by Blackwood and the present author, require only one mutable tone, D, to reflect the different roles it plays in two very different chords, namely, the ii^6 and the V chords. All of the other notes in this tuning can be directly traced to the Helmholtz tuning for the major scale.

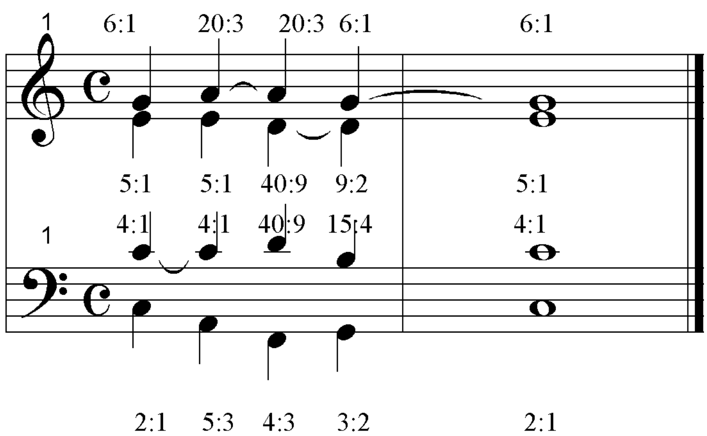

Figure 56. Author's just intonation for common tone example (All notes derived from C = 1/1)

Chord 1	Chord 2	Chord 3	Chord 4	Chord 5
G 392.439	A 436.043	A 436.043	G 392.439	G 392.439
E 327.032	E 327.032	D 290.695	D 294.329	E 327.032
C 261.625	C 261.626	D 290.695	B 245.274	C 261.626
C 130.812	A 109.010	F 87.209	G 98.110	C 130.812

Table 48. Author's just intonation for common tone example (C = 65.406)

As for the triviality of the held note becoming mutable, this is because the passage has been badly written for the express purpose of pointing out a valuable contradiction. It is a simple matter to rewrite the passage so that the two consecutive D's are in different octaves and voices to avoid the held mutable tone. Future listening experiments could determine whether held mutable tones played in various chord progressions and tuning systems are even noticeable as such.

Clearly, Sethares' adaptive tuning, shown in Table 49, is deficient in terms of adhering to basic tonal relationships. Tonic C, for example, is treated as a mutable tone in the second chord instead of being the tonal centre for the entire passage. The principle of minimizing the overall number of mutable tones would require maintaining the tunings of the two common tones and adjusting the new note if necessary which, according to the simple major scale used by Helmholtz, it is not.

Chord 1	Chord 2	Chord 3	Chord 4	Chord 5
G 392.5	A 440	A 438.5	G 391	G 392.5
E 327	E 330	D 292	D 294	E 327
C 261.5	C 264	D 292	B 245	C 261.5
C 131	A 110	F 87.5	G 98	C 131

Table 49. Sethares' adaptive tuning for common tone example

Sethares, whether or not intentionally, hides his actual tunings by rounding off the frequencies to the nearest half of a cycle per second. First, this is an arbitrary way to round these particular frequencies, especially as it affects the lower frequencies more than the upper ones. Second, it obscures the accuracy of the tuning. More importantly, when Sethares actually provides detailed ratios, as in Table 50, "Adaptive and Just Intonation", only the relative relationships within individual chords are given. For example, in this table the bass note value of the first chord is not given, but implied as 1/1. The tenor's C above that note is given as 2/1. The alto's E is indicated as 5/4, which is the tenor's C 2/1 raised by a major third. The soprano's G 6/5 is the alto's E 5/4 raised by a minor third. It is much clearer to relate all ratios to a single tonal centre, such as C 1/1.

Chord 1	Chord 2	Chord 3	Chord 4	Chord 5
G 6/5	A 4/3	A 3/2	G 4/3	G 6/5
E 5/4	E 5/4	D 1/1	D 6/5	E 5/4
C 2/1	C 6/5	D 5/3	B 5/4	C 2/1
C 1/1	A 1/1	F 1/1	G 1/1	C 1/1

Table 50. Sethares' adaptive and just intonation for common tone example

The lack of clarity resulting from assigning the root of each chord the value of 1/1 is apparent in Table 50. For example, the bass note in chord two is an A whose value, as with the C in the first chord, is again given as 1/1. The only clue as to its relation to the previous note C is the rounded off frequencies, given in Hertz, for the bass notes in Sethares' Adaptive table given in Table 49 which, as pointed out, lack sufficient precision for clarity.

Adaptive tuning is provocative in that it attempts to actually systematize tonality according to critical bandwidth theory. However, the results from the Hall example show that the tonal centre is being completely ignored as early as chord two. This approach is reminiscent of early chess computer programs that played material tactics perfectly within the scope of three or four moves, but failed to forecast the loss of the game in five or six.

If adaptive tuning could actually accomplish its task of minimizing dissonance between partials whilst preserving important tonal relationships according to hierarchical rules, significant strides in automating the process of tuning could be made. It must be pointed out that the timbral analysis phase should place less emphasis on amplitude of partials and more upon partial number.

Horner and Ayers (1996) elaborated further on the ideas of Hall (1974) and Sethares (1994). The use of very sophisticated techniques, such as a genetic algorithm that creates permutations based on selection (propagation of the best individuals of a population through succeeding generations, weeding out those less fit), crossover (mating pairs of candidate solutions in search of superior combinations) and mutation (random modification of bits within randomly selected individuals), is impressive, but the heuristics employed are illogical from a musical point of view. The failure to prioritise such important relationships as tonic and dominant, for instance, is simply unmusical, and only begs the following questions:

1) What is the most important chord in the passage? Of course, this would have to be tonic.
2) Which chords are most important in relation to tonic, i.e., what are their positions within the tonal hierarchy? Second only to tonic would surely be dominant function chords, such as V, V^7 and vii^o, followed by predominant functions such as ii^6, IV and V/V.

Strangely, after solving the Hall problem in a manner that preserves tonic and dominant functions, as was done independently by Blackwood and the present author by creating a single mutable tone between supertonic and dominant,[87] Horner and Ayers discard the solution in favour of another that retains common tones. It is commendable to dissect a musical object to see what is inside, but unnecessary to do so if the initial assumption that common tones take precedence over everything else, including tonality itself, is steadfastly held.

There is absolutely no reason common tones should be taken as the main principle of tonality. In all three of the cited papers, the researchers focused on the contradiction between common tones and tonal stability using badly written passages designed for the purpose of pointing out such flaws. As noted by Blackwood, a better approach would be to rewrite the passage so that tonal logic and proper interval spacing are maintained.

[87] Easley Blackwood, *The Structure of Recognizable Diatonic Tunings* (Princeton: Princeton University Press, 1985), p. 74.

5.2.2 Cope's Automated Composer

David Cope, whose computer programs have been used to mimic the compositional techniques of Bach and Mozart, among others, states that three chords are the "key players" in tonality, namely, tonic, dominant and subdominant.[88] According to his theory, tonality, which is based on these three tonal functions, occurs not only at the most detailed level of the chord, but also manifests itself at the higher levels of phrase, section and movement.

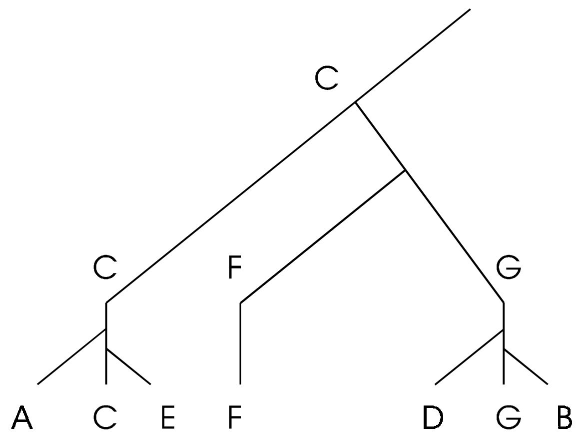

Figure 57. A parse of the C major Scale (from Cope, 1991)

Cope's contributions to artificial intelligence in music, breathtaking in detail though they often are, fail to address the fine points associated with just intonation, such as mutable and contradictory tones. The oversimplification of this paradigm can be seen in the D of the G major triad on the right in Figure 57, which would be tuned differently if it were part of a D minor triad that includes the F in the middle. The A on the far left, in contrast, could just as easily be derived from the same F in the middle, as all of them fit comfortably together in the same major seventh chord à la Rameau. To state that the tonic, dominant and subdominant chords are the key players in tonality is perhaps at best an unproven statistical fact. Certainly it could be argued, for instance, that supertonic or even submediant is more important functionally, at the detailed level of chords anyway, than subdominant. A self-consistent

[88] David Cope, *Computers and Musical Style* (Oxford: Oxford University Press, 1991), pp. 30-41.

theory of tonality that provides specific results with which to confirm or refute theories must be meticulous and comprehensive, delving into the smallest detail of any vertical sonority. Applied, such a theory could lead to automated compositions that sound more nearly like, and perhaps in some ways even surpass, those of human composers creatively exploring tonality through the human auditory central processor.

5.3 Shenkerian theory as a paradigm

5.3.1 Top-down approach necessary when tuning entire pieces

According to the theory of Heinrich Schenker, a piece of music may be thought of as existing at several different levels simultaneously. At the broadest level, for example, an entire movement might be thought of as a tonic-dominant-tonic structure.

Applying heuristics of tuning becomes complicated when the issue of modulation comes into play. Even within a single chorale by J. S. Bach, where modulations are not so abundant, brief excursions into other key areas are common. When analysing a piano sonata by Mozart, numerous and commonplace modulations present problems that can only be solved using a multi-level approach. Especially when tuning a phrase within a development section from such a sonata, a tuning algorithm must examine the piece first at the levels of piece, movement and section to establish the global tonal framework. It must then investigate the level of subsection to determine the key of the phrase, which in turn provides the information required to determine the tonal orientation of each sub-phrase. Only then is the algorithm able to accurately select specific values for the chords themselves.

To eliminate the possibility of tuning a passage based upon the wrong key centre, it is important to establish the key at the levels of movement, section, subsection, phrase, sub-phrase, and downward until the level of the note. It is not uncommon for the root of a dominant or even tonic to become mutable within a single movement; however, they will generally take place in separate phrases and, more likely, in separate sections.

5.4 Introduction to Mutable Tones

When musical notation is translated into ratios, even for simple chord progressions, there quickly arise notes that must be assigned more than one tuning in order to accommodate the rules used for vertical harmony. Even when prime generators for just intonation are limited to 2, 3, and 5, there is no limit to the number of possible mutable tones. The two most common mutable tones within a single key are illustrated in the following subsections.

5.4.1 The Mutable Second Scale Degree

The second scale degree in a major key, when used as the fifth of the dominant, is tuned as an octave equivalent of 9/8. This is readily apparent, as tonic is 1/1 and tonic's fifth is an octave equivalent of 3/2. The fifth of this fifth is 3/2 x 3/2 = 9/4, whose octave equivalent is 9/8.

When used as the root of a ii chord, the second scale degree in a major key is tuned as an octave equivalent of 10/9. The derivation for this tuning is far less obvious. Take, for example, tonic C. The supertonic will be a D minor triad, the third of which is an F. F is a cornerstone for this key, meaning that it is rarely mutable unless some unusual tonal excursion takes place, and even then, will generally take place outside of the local key area and phrase. Therefore, the F is related to tonic as 4/3. The other two notes can be found by simply observing that 4/3 is the third of a minor ii triad. The relative frequency values for a minor triad are 10:12:15 (for closed root position). Therefore, if F is 4/3 and the ratio of F to D is 12/10, then D is 4/3 x 10/12 = 10/9.

5.4.2 The Mutable Sixth Scale Degree

The frequency value for the sixth scale degree in the major key (or a major key area) when it is functioning as the fifth of a V/V (or the root of a V/V/V), is an octave equivalent of 27/16. As shown above, the fifth degree of dominant is tuned as 9/8. The fifth degree of this value is simply 9/8 x 3/2 = 27/16.

The sixth scale degree in the major key (or a major key area), when it is functioning as the root of a vi chord, is tuned as an octave equivalent of 5/3. The third and fifth of the minor vi triad can be thought of as common tones of the tonic major triad's root and third. For example, an A minor vi triad has C and E in common with a C major tonic triad. Since perfect fifths are always tuned as 3/2, and a closed root position major tonic triad is tuned as 4:5:6, then E5 divided by 3 equals A, the root of a minor vi chord.

The sixth scale degree in the major key (or a major key area), when it is functioning as the third of a subdominant, is tuned as an octave equivalent of 5/3. This tuning is derived by multiplying the root of the subdominant, related to tonic as 4/3, by 5/4 to obtain the final relative frequency of 5/3.

5.5 Formal Description of Intervals Used in 5-limit Theory

The formal language used to describe intervals is borrowed from Blackwood.[89] Only three basic intervals a, $\bar{v}$ and $\bar{t}$, corresponding respectively to the octave, perfect fifth and major third, are needed to generate all the intervals of tonal music according to 5-limit theory. The mathematical symbols used are explained by Blackwood as follows:

> The superior bars are to distinguish the pure intervals $\bar{v}$ and $\bar{t}$ from the more general versions, which will be written v and t. The symbol for the octave will always appear as a, it being generally agreed that this interval cannot be tuned any other way than 2/1.[90]

$$a = \log 2$$

$$\bar{v} = \log \frac{3}{2}$$

$$\bar{t} = \log \frac{5}{4}$$

[89] Blackwood, op. cit., pp. 12-21.
[90] Blackwood, op. cit., p. 12.

As illustrated in Equation 1, adding one interval to another is equivalent to multiplying their respective ratios. Similarly, subtracting an interval *b* from another interval *a* is equivalent to *a* divided by *b*.

If the size of an interval is $ma + n\bar{v} + q\bar{t}$, then its ratio is $\left(\frac{2}{1}\right)^m \left(\frac{3}{2}\right)^n \left(\frac{5}{4}\right)^q$.

From the definitions of a, $\bar{v}$ and $\bar{t}$, we have:

$$\begin{aligned} ma + n\bar{v} + q\bar{t} &= m\log 2 + n\log\frac{3}{2} + q\log\frac{5}{4} \\ &= \log\left(\frac{2}{1}\right)^m + \log\left(\frac{3}{2}\right)^n + \log\left(\frac{5}{4}\right)^q \\ &= \log\left(\frac{2}{1}\right)^m \left(\frac{3}{2}\right)^n \left(\frac{5}{4}\right)^q \end{aligned}$$

Equation 1. Blackwood's method of representing intervals in 5-limit just intonation

5.6 Given Axioms of Tuning

5.6.1 Octave Equivalence

The octave is a unique interval in that it does not affect pitch. There is no satisfactory explanation of why this is so. It is mentioned only in passing that the long pattern hypothesis proposed by Boomsliter and Creel, which states that the overall pattern of the waveform is what is actually perceived, might shed some light on this phenomenon. (Please refer to "Temporal Theory" on p. 59.)

5.6.2 Equivalence of Intervals

An interval type is defined by the ratio of its frequency components. Therefore, an interval may be transposed to any other register by multiplying its two frequencies by the same value. As stated by Blackwood:

> In order for two intervals to have the same sound other than a difference in register, it is both necessary and sufficient that the ratios of the frequencies forming them should be equal. There is no known reason or proof for this

statement; it is simply a fact which may be observed, and is to be regarded as an axiom. Now let there be two intervals, I_1 and I_2, the first composed of frequencies f_1 and f_2, and the second of frequencies f_3 and f_4. Let r_1 and r_2 be numbers such that $\frac{f_2}{f_1} = r_1$ and $\frac{f_4}{f_3} = r_2$. The axiom may now be stated as follows: intervals I_1 and I_2 are identical except for a difference in register if, and only if, $r_1 = r_2$.[91]

5.7 Proven Axioms of Tuning

5.7.1 Integral powers of mutually prime intervals cannot coincide

As noted independently by Benade and Blackwood, no integral power of one prime can ever equal the integral power of another prime.[92] From this principle, it can be deduced that no matter what the specific values for prime generators a, $\bar{v}$ and $\bar{t}$,

1) $a^n \neq \bar{v}^n$
   (No integral power of 2 will ever equal an integral power of 3.)

2) $a^n \neq \bar{t}^n$
   (No integral power of 2 will ever equal an integral power of 5.)

3) $\bar{v}^n \neq \bar{t}^n$
   (No integral power of 3 will ever equal an integral power of 5.)

4) A fraction raised to an integral power cannot be an integer.

5) An integral root of an integer cannot be a fraction.

[91] Blackwood, op. cit., p. 4.

[92] Arthur Benade, *Fundamentals of Musical Acoustics* (New York: Oxford University Press, 1976), pp. 295–296.

5.8 Propositions of Five Limit Tuning

Along with the axioms given in the previous section, this section provides a list of general propositions upon which a series of heuristics, some of which are absolute and others that depend upon varying parameters, can be based. An illustration of an absolute heuristic is that of always tuning the perfect fifth as 3:2. Any close approximation will be perceived as the pure interval 3:2; conversely, any attempt at using the perfect fifth to fit some other relationship will sound too dissonant to be included in Baroque or Classical harmony. The following list of propositions can be used to ensure that the principles of just intonation are maintained for most simple chord progressions.

5.8.1 Intervals

1. *Within a single set of simultaneously occurring frequencies, an octave equivalent always bears a relationship of* $2^n:1$.

This is taken as a given. All traditional Western tuning systems, including just intonation, Pythagorean tuning, and equal temperament, steadfastly obey this rule.

2. *Any tone in any chord can be transposed by any number of octaves, and the tuning, except for octave transposition, of its individual notes will remain unchanged.*

This is slightly different than the axiom of octave equivalence. As with the axiom of octave equivalence, there is no proof, but common practice shows that tunings are stable upon the octave transposition of individual tones. Any chord can be inverted to produce any other inversion, and the resulting tuning for this inversion, except for octave transposition of the affected notes, will remain unchanged.

The basis for proposition 2 is that an inverting an interval or chord does not change its basic quality or harmonic function. Although this is not entirely true, as substituting a second inversion tonic for the final cadence will clearly illustrate, it does hold true in the vast majority of cases. Even when the function of the interval or chord is changed through inversion, it is tacitly assumed that no change in the note's frequency, other than the octave transposition, is necessary, as stated by Rameau:

> If one sound forms a perfect consonance with the fundamental sound, it will also form a perfect consonance with its octave; if another forms an imperfect consonance or a dissonance on the one hand, it will also form an imperfect consonance or a dissonance on the other; if another has to ascend or descend on the one hand, it will ascend or descend on the other, finally, everything that harmonizes on the one hand will also harmonize on the other.[93]

3. *Any valid chord can be transformed into another valid chord by dividing 1 by each of the chord's relative frequencies.*

For example, the minor triad is 1 divided by the major triad. The $\text{ii}^{\phi 7}$ variety of the half-diminished seventh chord (135:160:192:240) is 1 divided by the dominant seventh chord.

4. *Within a single chord, the perfect fifth (or its octave equivalent) is always tuned in the proportion of 3:2 (or its octave equivalent).*

The perfect fifth can be derived from the third partial of the harmonic series. It is further the basis of all ancient Greek tuning systems, including Pythagorean tuning and its descendent, equal temperament. Rameau pointed out that nearly every chord in tonal music contains a perfect fifth (or its inverse, the perfect fourth, or one of their octave equivalents), and believed it to have only one tuning.

5. *Within a single chord, the perfect fourth (or its octave equivalent) is always tuned in the proportion of 4:3 (or its octave equivalent).*

The perfect fourth, as with its inversion the perfect fifth, has only one tuning, and any other tuning will sound mistuned. It is such a basic tonal cornerstone that there should not be any exceptions within a single chord to this simple rule.

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[93] Rameau, op. cit., p. 11.

6. *Within a single chord, the major third (or its octave equivalent) is always tuned in the proportion of 5:4 (or its octave equivalent).*

As with the perfect fifth (3:2) and the perfect fourth (4:3), the major third, no matter how it is tuned, will be perceived as a single interval, in this case, 5:4. Mersenne, Rameau, Helmholtz and a host of other notable tuning specialists list only one tuning for this interval. By combining this rule with rules 4 and 5, many chords can quickly either be tuned or shown to be contradictory.

5.8.2 Chords

7. *Any major triad must be tuned so that its relative frequencies are octave equivalents of 1:3:5.*

This is the fundamental tuning for the simplest and most common chord type. The components of the major triad are found in the first five partials of the harmonic series, with relative frequencies of 1, 2, 3, 4 and 5. When octave equivalents are removed from these first five partials, the values 1, 3 and 5 are left. (Closed root position would be 4:5:6.) For example, in the Table of Intervals for the Major Triad (Table 51), the first C in the left hand column, C1, is multiplied by 5:4 to obtain an E 5:4. This ratio corresponds to the interval of a major third.

	C (1)	E (5:4)	G (3:2)
C (1)	1	5:4	3:2
E (5:4)	4:5	1	6:5
G (3:2)	2:3	5:6	1

Table 51. Table of intervals for the major triad

8. *Any minor triad must be tuned so that its relative frequencies are octave equivalents of 1:2, 1:3 and 1:5.*

The minor triad is regarded as a mirror image of the major triad. A detailed comparison between the intervals found in the major triad (Table 51) and the corresponding intervals found in the minor triad (Table 52) reveals that perfect fifths, major thirds and minor thirds are identical for both triads. The minor triad can thus be tuned as 1:1/3:1/5, or equivalently, 3:5:15, or in closed root position, 10:12:15. If root C of a minor triad is taken as 1, then E♭ = 6:5 and G = 3:2.

	C (1)	E♭ (6:5)	G (3:2)
C (1)	1	6:5	3:2
E♭ (6:5)	5:6	1	5:4
G (3:2)	2:3	4:5	1

Table 52. Table of intervals for the minor triad

9. *Any dominant seventh chord must be tuned so that the frequency for each of its notes is related to its root as an octave equivalent of 2, 3, 5 and 1:9. (In closed root position, this will be 36:45:54:64.)*

This rule is not a familiar one in the literature. It is completely avoided by Rameau, for instance, who favours the proportions 20:25:30:36. There is much to say for the seventh being tuned as an octave equivalent of 1:9. In the key of C, for example, given the high incidence of progressions involving a common tone F between in either a ii^6 – V^7 – I or a IV – V^7 – I progression, and with no particular reason to change the tuning of the common tone F, it is unlikely 20:25:30:36 should be used as a tuning for a dominant seventh chord (with similar logic applying to Rameau’s tuning of the fully diminished seventh chord). Simple ratios, forming the foundation of Rameau’s tuning theory, should express themselves between, not only within, chords.

With the assignment of simple ratios to such tunings, chords are capable of being intermediaries to more chords and progressions than if they used complex ratios. Had Rameau extended his own principle of the pre-eminent perfect fifth (and its inversion the perfect fourth) to not only chords but to chord progressions as well, he would have found this much better tuning for the dominant seventh chord.

10. *Any major seventh chord must be tuned so that the frequency for each of its notes is related to its root as an octave equivalent of 2, 3, 5 and 15.*

This is in keeping with Rameau's practice. Contained within this chord is a major triad, to which is added the major seventh. This major seventh acts as both the fifth of the major triad's major third, and the major third of the major triad's fifth.

11. *The function of chord x may be applied to another chord y by simply multiplying the relative frequencies of chord x by the fundamental frequency of chord y.*

For example, if the frequency values for chord x, the dominant triad (12:15:18 in relation to tonic), are applied to chord y, in this case, tonic, then the values remain the same (12:15:18). If they are applied to a chord y in the case of a dominant, then each frequency of chord x is multiplied by the root of the dominant chord, an octave equivalent of 3:1, resulting in the values for the tonal function V/V, 36:45:54. The dominant of V/V, which is V/V/V, would similarly be tuned by multiplying the values for x (12:15:18), by the root of y, 36, resulting in octave equivalents of 108:135:162.

12. *The supertonic minor ii chord in a major key is tuned as octave equivalents of 10/9:4/3:5/3.*

For example, take tonic C as 1. We can assume the F, which is a cornerstone, being a fifth below tonic, to be related to C as 2:3. If a supertonic D ii chord, which serves the same pre-dominant function as IV, were to use the same subdominant tunings for both F and A, then the tuning for the ii chord would be D 10:9, F 4:3, and A 5:3.

> 13. *The major IV chord is always tuned in relation to its local tonic as octave equivalents of 4:3, 5:3 and 2.*

This tuning for major subdominant is rather self-evident, as by proposition 5 the root is known to be 4:3, and by proposition 7, the fixed tuning for the major triad is also known.

> 14. *The dominant of a dominant of tonic (V/V/I) is tuned as octave equivalents of 9:8, 45:32 and 27:16.*

In other words, the dominant of C 1 is G 3:2; the dominant of G 3:2, in turn, is 3:2 x 3:2 = D 9:4; the fifth of this secondary dominant D chord is 3:2 x 3:2 x 3:2 = 27:16. Therefore, the tuning for this secondary dominant in relation to tonic C would be D 9:8, F# 45:32 and A 27:16.

> 15. *The minor seventh chord is always tuned in the proportions 10:12:15:18.*

This is in agreement with Rameau's axiom that any perfect fifth must have the relationship of 3:2, as is the case for the intervals 10:15 and 12:18. Additionally, the tuning for the minor triad 10:12:15 contained within the minor seventh maintains proposition 8. Similarly, the tuning for major triad 12:15:18, also found within the minor seventh chord, maintains proposition 7.

> 16. *The major seventh chord is always tuned as relative frequencies of 8:10:12:15.*

This tuning is again in agreement with Rameau's rule of tuning all perfect fifths in the ratio of 3:2. As with the minor seventh, this tuning for the major seventh maintains the tunings of embedded major and minor triads within the larger chord.

5.9 Reasons for Adopting 5-Limit Just Intonation

5.9.1 Evidence from Historical Trends and Experimental Evidence

As noted in Chapter 2, starting on p. 7, and Chapter 3, beginning on p. 41, there is a substantial body of evidence both from historical trends and from experimental evidence that 5-limit just intonation is in fact the theoretical ideal. Aside from these, there are purely theoretical considerations that support the use of a system whose intervals are based upon ratios generated from the first three prime numbers.

5.9.2 Five-Limit Just Intonation Explains Most Common Chords

The choice of intervals used in 5-limit just intonation is not an arbitrary one. Only the intervals $a = \log 2$ (the octave), $\bar{v} = \log \frac{3}{2}$ (the perfect fifth) and $\bar{t} = \log \frac{5}{4}$ (the major third), and combinations thereof are allowed in 5-limit just intonation. (See Formal Description of Intervals Used in 5-limit Theory, p. 129.) The first of these, the octave, is beyond debate; the octave is used as 2:1 in every Western tuning system. Similarly, the perfect fifth being tuned as 3:2 is a given in every major tuning system, from Pythagorean, based entirely on perfect fifths, to equal temperament, whose fifths are tempered but nevertheless based on the interval 3:2. Intervals whose prime factors include $\bar{t}$ are the only ones that can be argued to be unfounded in steadfastly held tradition.

As discussed in Chapter 4, beginning on p. 65, the most commonly found intervals in the chorales of J.S. Bach are the major, minor and dominant seventh chords. The tunings for these chords are easily shown to be simplest when tuned in accordance with just intonation. For example, given that C has the relative frequency of 1, a Pythagorean major third would be C first raised by four consecutive perfect fifths as 1 * 3/2 * 3/2 * 3/2 * 3/2 = E 81/16, adjusted downward by two octaves for a value of 81:64. The derivation would be represented as $4\bar{v} - 2a$. A just major third tuned as 5:4, on the other hand, would be represented, simply, as $\bar{t}$.

Additional support for the theory of 5-limit just intonation is to be found in the listening experiments of Chapter 7, beginning on p. 188. Just intonation is shown to easily outperform Pythagorean tuning in every broad category, is competitive with equal temperament in every category, and even surpasses the other two in half the conditions investigated.

5.9.3 The Misconceived Generator Seven

If the long wave hypothesis of Boomsliter and Creel is given credence, periods of chords become much more complex and therefore less consonant when the seventh partial is introduced. Although Plomp and Levelt indicate that the minor second is more dissonant than the extremely dissonant interval 5:7 when pure sine waves are used,[94] there are no instances when the interval 5:7 sounds anything short of horrid within any vertical sonority from the Baroque and Classical periods, whereas the minor second, which is traditionally tuned as 15:16, is used extensively in just intonation.

It should be pointed out not only that there is a strong historical trend against the interval 5:7, but also that no keyboard or fretted instrument approximates it. This might in part be explained by harmonic coincidence between small ratios, such as 2:3 or 4:5, being high for their lower partials, whereas in the case of 5:7, the first coinciding harmonic for the string of length 5 is the seventh, whilst the corresponding lowest harmonic for string length 7 is the fifth. It should additionally be mentioned in passing that 7:4, commonly misconceived as a theoretical tuning for a minor seventh, is in fact not even in contention for the tunings employed by any of the quartets analysed in Chapter 6 commencing on p. 152.

[94] Plomp and Levelt, op. cit., p. 551.

5.10 Ambiguities, Illusions and Contradictions

Perhaps most pieces of music in the Baroque and Classical styles cannot be tuned in just intonation without points of contradiction, no matter which frequency values are chosen. One could argue, from a theoretical standpoint, that such pieces are paradoxical and therefore not suitable as theoretical models, unless paradox is deemed an acceptable attribute. The acceptance of paradox does not lead, however, to a clear and unambiguous theory that can be tested.

5.10.1 Augmented triads and diminished sevenths

The not so uncommonly employed augmented triad poses an obvious problem to just intonation, as the major third, when tuned as 5:4, will fail to reach an octave equivalent upon three repetitions. This is but a single instance of the proven axiom, given in section 5.7.1 on p. 131, that integral powers of mutually prime intervals cannot coincide, as noted by Blackwood:

> It is clearly impossible to tune CE, EG#, and G#C so that each of these intervals will be entirely free of beats, since at least one of them must exceed a pure major third by an amount not less than one third of a diesis, or 13.686 cents. Hence a completely smooth tuning for an augmented triad cannot be found.[95]

This is not the only conclusion that may be reached regarding the augmented triad. From Blackwood's axiom, given in 5.7.1 on p. 131, "An integral root of an integer cannot be a fraction," it can be concluded that no justly tuned chord whatsoever corresponds to three equal logarithmic divisions of the octave.

From the same axiom, it is clear that interval of the augmented fourth and the diminished seventh chord cannot be made of equal ratios. This conclusion can similarly be applied to the whole-tone and chromatic scales. Therefore, whenever augmented and diminished triads, or whole tone and chromatic scales are encountered, it can conclusively be proven that just

[95] Blackwood, op. cit., p. 107.

intonation, in spite of the fact that it can generate any simple fraction whatsoever, cannot produce corresponding intervals of uniform size with which to tune the chord or scale. In general and without exception, equal logarithmic divisions of the octave are impossible in just intonation.

5.10.2 A Contradiction with the Doubly Diminished Seventh Chord

Musical structures written in equal temperament are often difficult or impossible to translate note for note into simple fractional values of just intonation. The reason for this difficulty in translation is never the fault of just intonation, which can provide any interval x/y to any precision desired. Therefore, when contradiction arises, the structure itself must then be laid on shaky theoretical ground.

As noted in section 5.10.1 on p. 140, the diminished seventh is theoretically questionable from the outset. It is worth mentioning again that the octave, whose ratio 2/1 is epimoric, was known to ancient Greek theorists to be among the many ratios that cannot be evenly divided using ratios. It follows naturally that if the octave cannot be evenly divided by two, it certainly cannot by four. The law of epimoric ratios holds equally true for the major third, 5:4, the perfect fourth, 4:3 and the perfect fifth, 3:2. According to Helmholtz, " . . . the intensity of the harsh dissonances is much increased by their contrast with perfect chords. The chord of the diminished seventh, for example, which is much used in modern music, borders upon the insupportable, when the other chords are tuned justly."

An often employed device in tonal music is the doubly diminished seventh chord used as a pivot point from which to travel to any of four dominant sevenths. This is accomplished by lowering any of the four pitches in the doubly diminished seventh by a semitone. Two of the four possible resolutions of the diminished seventh to the dominant seventh chords are outlined in Table 53 and Table 54.

vii^{o7}	V^7
F	F
D	D
B	B
A♭	G

Table 53. Diminished 7th to dominant 7th Case 1

vii^{o7}	V^7
F	F
D	D
B	B♭
A♭	A♭

Table 54. Diminished 7th to dominant 7th Case 2

Because the tuning for a dominant seventh is known, the right side of Case 1 may be filled in based on C = 1. Under the assumption that F, B and D are common tones, their values obtained from the dominant seventh at right are transferred to the doubly diminished seventh at left, as shown in Table 55.

vii^{o7}	V^7
F 16/3	F 16/3
D 9/2	D 9/2
B 15/4	B 15/4
A♭	G 3

Table 55. Diminished 7th to dominant 7th Case 1

The premise is that the values determined for the doubly diminished seventh chord in Case 1, other than the pitch that changes, can be reinterpreted so that they resolve appropriately to the other dominant chords given. Under this assumption, the first three values obtained for the doubly diminished seventh chord in Case 1 may be transferred to the same chord in Case 2. Therefore, the values for common tones F and D are assigned positions within the new dominant seventh, as shown in Table 56.

vii^{o7}	V^7
F 16/3	F 16/3
D 9/2	D 9/2
B 15/4	B♭
A♭	A♭

Table 56. Diminished 7th to dominant 7th Case 2

B♭, the root of the B♭ dominant seventh chord, can be derived from F16/3 by multiplying it by 2/3, resulting in B♭ 32/9. An updated dominant seventh, Case 2, is shown in Table 57.

vii^{o7}	V^7
F 16/3	F 16/3
D 9/2	D 9/2
B 15/4	B♭ 32/9
A♭	A♭

Table 57. Diminished 7th to dominant 7th Case 2

When B♭ 32/9 is multiplied by 5/4, an octave equivalent of the previously obtained value D9/2 should result, assuming that a doubly diminished seventh chord can resolve to more than one dominant seventh chord. As it turns out, B♭ 32/9 multiplied by 5/4 = 40/9, which is not an octave equivalent of D9/2. This incongruity indicates that the attempt of resolving a diminished seventh chord to more than one dominant seventh chord results in contradiction. Therefore, using the standard values (or indeed, any rational values) as the numerators and denominators for the dominant seventh chord, there is no chord that can resolve to four or even two dominant sevenths using three common tones when the frequencies of the notes are all integer related.

The previously illustrated contradiction clarifies the problem encountered when attempting to translate equal temperament, or any fixed exponential system, into just intonation. Because it is a fixed system with only twelve intervals and their octave equivalents, equal temperament can only approximate integer ratios other than octave equivalents. For this reason, the same equally tempered chord has been improperly used to imply more than one chord, acting as a tonal pivot chord to two or more chords that have no such tonal pivot chord in common.

5.10.3 Half-Diminished $\text{vii}^{\phi 7}$ Chord

At first glance, the half-diminished seventh resolving to tonic, shown in Figure 58, would seem an easy chord to tune, as it contains two embedded triads: the diminished (B-D-F) and the minor (D-F-A). The natural approach, introduced by Rameau, would be to superimpose them so that all relationships in the embedded chords are maintained in the larger one.

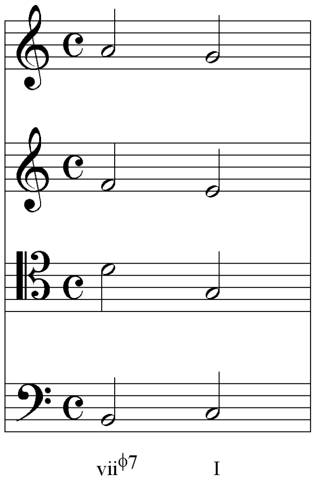

Figure 58. Half-diminished $\text{vii}^{\phi 7}$ to I progression in key of C

Table 58 assumes that tonic is C1 two octaves below middle C. The tuning for F is obtained through its subdominant relationship with tonic, which from the time of the early Greeks has been tuned in the ratio of 2:3. Tuning the other members of the minor triad is straightforward, as A is a major third, which is always an octave equivalent of 5:4, above F. Similarly, D will be a perfect fifth, which is always an octave equivalent of 3:2, below A. As can be observed, removing the B in the bass leaves the traditional tuning for the minor triad, which poses no difficulty for 5-limit tuning.

	Half-diminished $\text{vii}^{\phi 7}$	Tonic
Soprano	A 20/3	G 6/1
Alto	F 16/3	E 5/1
Tenor	D 40/9	G 3/1
Bass	B (to be determined)	C 2/1

Table 58. Half-diminished $\text{vii}^{\phi 7}$ tuned from embedded minor triad

The diminished triad made of B, D, F, the lower three members of Table 59, is equally amenable to being tuned in 5-limit just intonation, as its members are merely the third, fifth and seventh of a dominant seventh chord, whose tonal function it shares. Notice that the D in Table 59 has a tuning that is incompatible with the tuning given in Table 58.

	Half-diminished $vii^{\phi 7}$	Tonic
Soprano	A (to be determined)	G 6/1
Alto	F 16/3	E 5/1
Tenor	D 9/2	G 3/1
Bass	B 15/8	C 2/1

Table 59. Half-diminished $vii^{\phi 7}$ tuned from embedded diminished triad

There is no way around this dilemma using the 5-limit principles embraced by Zarlino, Mersenne, Rameau and Helmholtz. This chord not only contains two embedded triads, but an embedded syntonic comma (81:80) as well. This is due to the fact that the minor ii triad contains F a perfect fifth below, whilst the diminished vii^{o} triad is generated from an unheard G a perfect fifth above, tonic C. The D's generated by these two chords are therefore a syntonic comma apart. (Another half-diminished chord, the $ii^{\phi 7}$ resolving to V or V^7, is legitimate, as the root of $ii^{\phi 7}$ becomes the fifth of V^7, with the third, fifth and seventh of $ii^{\phi 7}$ serving as an embedded subdominant iv, which performs the same pre-dominant role as $ii^{\phi 7}$.)

5.11 Heuristics for Tuning Chords

Although these heuristics will actually account for the majority of chords encountered within a clearly defined key area, they are only the beginning to a much broader picture of tonality. In full-scale classical compositions, for instance, chord progressions should be identified at a higher level than when simply analysing the current sonority.[96] If a progression that is stripped of its passing tones, for instance, is compared to another that never had them, the two progressions could still be shown, in many cases, to be the same. This approach would necessitate analysis either by the user or by a highly sophisticated computer program.

[96] Taube, op. cit., pp. 20-27.

5.11.1 Detailed Heuristics

The following heuristics were based upon Given Axioms of Tuning, p. 130, Proven Axioms of Tuning, p. 131, and Propositions of Five Limit Tuning, p. 132. Frequency values are based on tonic being an octave equivalent of 1. For detailed implementation using entire chorales, please refer to “Appendix V: Heuristic Analysis of Bach Chorales” on p. 265.

1. Tonic (I) major triad? If yes,
    a. The frequency of the root will be an octave equivalent of 2.
    b. The frequency of the third will be an octave equivalent of 5.
    c. The frequency of the fifth will be an octave equivalent of 3.
2. Tonic (i) minor triad? If yes,
    a. The frequency of the root will be an octave equivalent of 2.
    b. The frequency of the third will be an octave equivalent of 3/5.
    c. The frequency of the fifth will be an octave equivalent of 3.
3. Tonic major (I^7) chord? If yes,
    a. The frequency of the root will be an octave equivalent of 2.
    b. The frequency of the third will be an octave equivalent of 5.
    c. The frequency of the fifth will be an octave equivalent of 3.
    d. The frequency of the seventh will be an octave equivalent of 15.
4. Tonic minor (i^7) chord? If yes,
    a. The frequency of the root will be an octave equivalent of 2.
    b. The frequency of the third will be an octave equivalent of 3/5.
    c. The frequency of the fifth will be an octave equivalent of 3.
    d. The frequency of the seventh will be an octave equivalent of 9/5.
5. Supertonic (ii) minor triad? If yes,
    a. The frequency of the root will be an octave equivalent of 5/9.
    b. The frequency of the third will be an octave equivalent of 2/3.
    c. The frequency of the fifth will be an octave equivalent of 5/3.

6. Supertonic ii^7 minor seventh chord? If yes,
    a. The frequency of the root will be an octave equivalent of 5/9.
    b. The frequency of the third will be an octave equivalent of 2/3.
    c. The frequency of the fifth will be an octave equivalent of 5/3.
    d. The frequency of the seventh will be an octave equivalent of 2.
7. Supertonic half-diminished $ii^{\o 7}$ chord? If yes,
    a. The frequency of the root will be an octave equivalent of 9/2.
    b. The frequency of the third will be an octave equivalent of 2/3.
    c. The frequency of the fifth will be an octave equivalent of 2/5.
    d. The frequency of the seventh will be an octave equivalent of 2.
8. Mediant (III/i) major triad? If yes,
    a. The frequency of the root will be an octave equivalent of 3/5.
    b. The frequency of the third will be an octave equivalent of 3.
    c. The frequency of the fifth will be an octave equivalent of 9/5.
9. Mediant (iii/I) minor triad? If yes,
    a. The frequency of the root will be an octave equivalent of 5.
    b. The frequency of the third will be an octave equivalent of 3.
    c. The frequency of the fifth will be an octave equivalent of 15.
10. Mediant (iii^7/I) minor seventh chord?
    a. The frequency of the root will be an octave equivalent of 5.
    b. The frequency of the third will be an octave equivalent of 3.
    c. The frequency of the fifth will be an octave equivalent of 15.
    d. The frequency of the seventh will be an octave equivalent of 9.
11. Subdominant (IV) major triad? If yes,
    a. The frequency of the root will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a subdominant.
    b. The frequency of the third will be an octave equivalent of 5/3 multiplied by the frequency of the root of the chord to which it is a subdominant.
    c. The frequency of the fifth will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a subdominant.

12. IV^7 major seventh chord? If yes,
    a. The frequency of the root will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a subdominant.
    b. The frequency of the third will be an octave equivalent of 5/3 multiplied by the frequency of the root of the chord to which it is a subdominant.
    c. The frequency of the fifth will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a subdominant.
    d. The frequency of the seventh will be an octave equivalent of 5 multiplied by the frequency of the root of the chord to which it is a subdominant.
13. Subdominant iv minor triad? If yes,
    a. The frequency of the root will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a subdominant.
    b. The frequency of the third will be an octave equivalent of 2/5 multiplied by the frequency of the root of the chord to which it is a subdominant.
    c. The frequency of the fifth will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a subdominant.
14. iv^7 minor seventh chord? If yes,
    a. The frequency of the root will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a subdominant.
    b. The frequency of the third will be an octave equivalent of 2/5 multiplied by the frequency of the root of the chord to which it is a subdominant.
    c. The frequency of the fifth will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a subdominant.
    d. The frequency of the seventh will be an octave equivalent of 3/5 multiplied by the frequency of the root of the chord to which it is a subdominant.
15. Dominant (V) triad? If yes,
    a. The frequency of the root will be an octave equivalent of 3 multiplied by the frequency of the root of the chord to which it is a dominant.
    b. The frequency of the third will be an octave equivalent of 15 multiplied by the frequency of the root of the chord to which it is a dominant.
    c. The frequency of the fifth will be an octave equivalent of 9 multiplied by the frequency of the root of the chord to which it is a dominant.

16. Dominant seventh (V^7) chord? If yes,

    a. The frequency of the root will be an octave equivalent of 3 multiplied by the frequency of the root of the chord to which it is a dominant.

    b. The frequency of the third will be an octave equivalent of 15 multiplied by the frequency of the root of the chord to which it is a dominant.

    c. The frequency of the fifth will be an octave equivalent of 9 multiplied by the frequency of the root of the chord to which it is a dominant.

    d. The frequency of the seventh will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a dominant.

17. Submediant (VI/i) major triad? If yes,

    a. The frequency of the root will be an octave equivalent of 2/5 multiplied by the frequency of the root of the chord to which it is a submediant.

    b. The frequency of the third will be an octave equivalent of 2/1 multiplied by the frequency of the root of the chord to which it is a submediant.

    c. The frequency of the fifth will be an octave equivalent of 3/5 multiplied by the frequency of the root of the chord to which it is a submediant.

18. Submediant (vi/I) minor triad? If yes,

    a. The frequency of the root will be an octave equivalent of 5/3 multiplied by the frequency of the root of the chord to which it is a submediant.

    b. The frequency of the third will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a submediant.

    c. The frequency of the fifth will be an octave equivalent of 5 multiplied by the frequency of the root of the chord to which it is a submediant.

19. Submediant (vi^7) minor seventh chord? If yes,

    a. The frequency of the root will be an octave equivalent of 5/3 multiplied by the frequency of the root of the chord to which it is a submediant.

    b. The frequency of the third will be an octave equivalent of 2 multiplied by the frequency of the root of the chord to which it is a submediant.

    c. The frequency of the fifth will be an octave equivalent of 5 multiplied by the frequency of the root of the chord to which it is a submediant.

    d. The frequency of the seventh will be an octave equivalent of 3 multiplied by the frequency of the root of the chord to which it is a submediant.

20. Diminished leading vii^{o} triad? If yes,
    a. The frequency of the root will be an octave equivalent of 15 multiplied by the frequency of the root of the chord to which it is a leading vii^{o}.
    b. The frequency of the third will be an octave equivalent of 9 multiplied by the frequency of the root of the chord to which it is a leading vii^{o}.
    c. The frequency of the fifth will be an octave equivalent of 2/3 multiplied by the frequency of the root of the chord to which it is a leading vii^{o}.
21. Half-diminished vii$^{\phi 7}$ chord? If yes,
    a. CONTRADICTION. Please see "Half-Diminished viiϕ7 Chord" on p. 144.
22. For any octave equivalent of any perfect fifth within a single sonority, the lower tone will be related to the upper as an octave equivalent of 2:3.
23. For any octave equivalent for any perfect fourth within a single sonority, the frequency of the lower tone will be related to the frequency of the upper tone as 3:4.
24. For any octave equivalent of any major third within a single sonority, the lower tone will be related to the upper one as an octave equivalent of 4:5.
25. For any octave equivalent of a minor third serving as the third and fifth of a major triad, the third and fifth will be tuned, relative to each other, as octave equivalents of 5 and 3.
26. For any octave equivalent of a minor third serving as the root and third of a minor triad, the root and third will be tuned, relative to each other, as octave equivalents of 5 and 3.
27. For any octave equivalent of a minor third serving as the fifth and seventh of a dominant seventh chord, the fifth and seventh will be tuned, relative to each other, as octave equivalents of 27 and 32.
28. For any octave equivalent of any minor second within a single sonority, the lower tone will be related to the upper as an octave equivalent of 15:16.
29. For any octave equivalent of any major seventh within a single sonority, the lower tone will be related to the upper as an octave equivalent of 8:15.

5.12 Extensions of Automated Tuning

The challenges in automating the tuning process for an entire piece of music are numerous. Before embarking upon such a venture, it is necessary to first select a piece that is tonal. A very small percentage of works from the Baroque and Classical Periods, much less the Romantic, will lend themselves completely to being tuned as a collection of precise and simple ratios. Assuming the piece is reasonably tonal, there are many obstacles that must be faced, mostly related to local tonality.

In order to identify the key centre within a multi-level musical structure, an automated tuning program must unequivocally determine when a passage has modulated from one key to another. When such a modulation has taken place, the program must select a precise ratio corresponding to the relationship between the two keys. A simple example would be when the key of the first section is C major and that of the second is G. The ratio indicated to identify the second section would then be an octave equivalent of 3/2.

A much more complex example would be the development section in a sonata form written by Mozart. Even if it the movement is free of tonal contradiction, its deliberate excursions from tonic and, indeed, from the sense of tonal stability itself will result in complexity not found in a simple chorale of Bach. The ratios that are used to describe the relationships between the pitch of tonic and those found in the development section can become quite complex, resulting in myriad mutable tones. Being able to keep clear track of such complexities is one of the key remaining challenges of automated tuning.

Chapter 6
Intonation Analysis of Four Professional String Quartets

6.1 Overview

The purpose of this study is to investigate the tuning preferences of four separate professional string quartets, as evidenced by their performances of the opening theme to the Adagio movement of Mozart's last string quartet (K. 590). Within this brief passage, selected for its tonal simplicity and repetition of chords, special attention is drawn to four specific chord types: major, minor, dominant seventh and minor seventh. Through statistical methods, determination is made as to which of three traditional tuning systems – Pythagorean, just intonation or equal temperament – most nearly accounts for the tunings actually employed by the performers.

6.2 Previous Studies

In his intonation study of Haydn's Emperor Quartet, Nickerson investigated the tuning practices of six quartets as they pertained to melody.[97] The material chosen was ideally suited for this purpose, as the melody is passed to each instrument whilst the harmony remains the same. Intervals size was "reckoned upward from the root," whilst root values were obtained through averaging the most "musically prominent" roots. Given the technology of the time, it is perhaps understandable that a more comprehensive approach was not adopted.

Values for intonation systems are not given in this study. The 27-cent discrepancy between just intonation and Pythagorean frequency values detected for the perfect fourth is almost certainly a publishing error, as they should be identical. Not noted by Nickerson is the fact that data indicated sharpening of not only thirds in relation to both just intonation and equal temperament, but also fourths, fifths and sixths for both solo and ensemble performance.

[97] James F. Nickerson, 'Intonation of Solo and Ensemble Performance of the Same Melody,' *Journal of the Acoustical Society of America,* vol. 21 no. 6, 1949, pp. 593-595.

Results from this study implied that tuning for major thirds differed according to whether played in a solo or ensemble setting. Significantly, just intonation was found to be more nearly achieved in an ensemble than in a solo setting. In this study, Pythagorean tuning most nearly accounted for the performance practices of the string quartets.

Loosen's study (1993) involving intonation practices of professional violinists compared results with idealized tunings based on three models of intonation.[98] As with previous studies, the intonation systems compared were Pythagorean tuning, equal temperament and just intonation. Each of eight professional violinists played an ascending and descending 3-octave C major scale, with each note held for 3 seconds. The passage played was a C major scale, without vibrato, as in tune as possible. Tunings were compared both between adjacent notes and with tonic. Values stored were based on the average of each performer's tunings, but the definition for average was not clearly given and is presumed the average in cents.

The results indicated that Pythagorean tuning and equal temperament were found to be almost equally reliable as a predictor of how violinists tuned their scales monophonically. The direction of the scale, in general, made no difference. Mention is made of the violinists' tendency to depart from models more in the uppermost octave. Loosen states, ". . . the ability to discriminate between different frequencies is quasi-constant in an interval 262 – 2986 Hz." As a partial explanation, it is suggested by the present author that the higher partials would tend to become gradually less audible at higher ranges, thus making pitch discrimination for fundamentals correspondingly more difficult.

[98] Franz Loosen, 'Intonation of solo violin performance with reference to equal-tempered, Pythagorean and just intonations,' *Journal of the Acoustical Society of America*, vol. 93 (1993), pp. 525-539.

6.3 Tuning Properties of Stringed Instruments

6.3.1 Pythagorean Tuning of Open Strings

The instruments used in a string quartet possess unique properties with regard to tuning. Their strings are arranged as a series of perfect fifths corresponding to Pythagorean tuning. Take, for example, the violin, whose strings are tuned, in ascending order, to the notes G, D, A and E. Assuming that A is tuned to 440 Hz and that all fifths are pure, the resulting pitches for the strings are G-195.556, D-293.333, A-440 and E-660. Similarly, pitches for the cello would be C-65.185, G-97.778, D-146.667, and A-220.

Blackwood notes that such an arrangement of open strings will produce intervals that are incompatible with just intonation.[99] For instance, with a cello playing an open C-65.185 and the two violins playing G-195.556 and E-660, a major triad whose major third is extremely sharp according to 5-limit just intonation will result. Of course, string players do not have to play notes in their open positions, but the positions are nevertheless used as points of reference during performance.

6.3.2 Inharmonicity of Stringed Instruments

In her paper on the spectral properties of musical instruments, Brown (1996) discovered that stringed instruments have a certain type of inharmonicity arising when vibrato is used.[100] It turns out that all partials above the fundamental have frequencies very nearly corresponding to the idealized harmonic series. The fundamental, however, is off by from 5 to 25 cents for violin and up to 40 cents for cello on either side of the vibrato. This holds true for violin, viola and cello when they are played with vibrato.

[99] Blackwood, op. cit., p. 76
[100] Brown, op. cit., pp. 1210-19.

Although no results are cited for violin or viola played without vibrato, the single experiment with cello playing an open string G2, played by Yo Yo Ma, indicated precise harmonicity within ± 3 cents up to the twenty-fifth partial. Plucked viola sounds strongly indicated inharmonicity.

6.4 Other Performance Practices Affecting Tuning

6.4.1 Bending of Intervals

Burns and Ward state that observers tend to hear intervals less than a fourth as perceptually wider, and to hear intervals greater than a fourth as perceptually narrower, than they are. Accordingly, they claim, performers tend to adjust intervals so that they are smaller for narrow intervals and larger for wide intervals.[101] This actually does not explain the fact that major thirds tend to be played sharp, unless one examines not the interval of the major third itself, but instead the minor second resolving from that major third, say, serving as the third of a dominant, to the root of tonic. In this case, the leading tone minor second, as a horizontal interval, would be narrowed to increase tonal tension through closer proximity and greater dissonance between partials.

In a paper about Mozart's intonation practices, Chestnut cites "Several nineteenth century violin teachers, including one of Joachim's teachers," as having made some interesting observations regarding intonation practices of string players. Although the historical accuracy, especially in view of the fact that the source is not specifically given, is questionable, the concept itself is clear and easily tested:

> Several nineteenth-century violin teachers, however, including one of Joachim's teachers, explicitly recommended shading pitches in the direction of their "attachment," as defined by harmonic context, in order to achieve an "animated" performance. In other words, they explained this practice in terms of the structure of music, not in terms of the structure of stringed instruments. We

[101] Edward M. Burns and W. Dixon Ward, 'Intervals, Scales and Tuning,' Deutsch, Diana (ed.) *The Psychology of Music* (San Diego: Academic Press, 1982), pp. 247-250.

> might call such a tuning system "functional intonation." In such a system, tendency tones, such as the leading tone and the sevenths of the dominant-seventh chord, are inflected in the direction of their tendency to resolve, diatonic half steps are smaller than chromatic half steps, and sharped notes are higher than enharmonically equivalent flatted notes. Present-day string intonation seems to follow this nineteenth-century practice.[102]

If Chestnut is correct in his assertion that one of Joachim's teachers taught the method of bending thirds in the direction of the notes to which they resolve, then it would appear this particular teacher's method of intonation was at least partially ignored by Joachim who, when tested, according to Helmholtz, tuned thirds in just intonation.[103] With equal temperament as a reference, tuning in just intonation would imply substantial flattening, not sharpening, of major thirds in dominant major triads and dominant seventh chords, whilst sevenths in justly tuned dominant seventh chords would be identical to those of Pythagorean tuning and only very slightly flatter than the sevenths used in equal temperament.

6.4.2 Similarities and Differences between Tuning Systems

It is interesting to note that equal temperament, when given the task of producing a major third from tonic by means of generating four consecutive flattened fifths (e.g., C-G-D-A-E), produces a major third that more nearly approximates the tuning for just intonation than does Pythagorean tuning. One possible reason for the wide adoption of equal temperament would be that it approximates the important major and minor thirds of just intonation whilst simultaneously closely approximating the perfect fifths and fourths of Pythagorean tuning by "falling in between" the two tuning systems.

All relationships created exclusively through perfect fifths and fourths are identical in just intonation and Pythagorean tuning. Therefore, for most chords near the tonal centre, the only difference between just intonation and Pythagorean tuning is the tuning of the major or minor

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[102] John Hind Chestnut, 'Mozart's teaching of intonation," *Journal of the American Musicological Society*, vol. 30 no. 2 (1977), pp. 254-271.
[103] Helmholtz, op. cit., p. 325.

third. Thus, comparisons between common tunings in just intonation and Pythagorean tuning will simultaneously reflect the absolute sameness of their perfect fifths, and the often wide gaps between their major and minor thirds. This underscores the need to evaluate individual chords separately for each tuning system considered.

6.4.3 Equal Temperament Avoids Overall Dissonance in Practice

Equal temperament (reviewed in section 2.5 beginning on p. 30) is not only a mathematically certain approach to avoiding dissonance in a fixed 12-note system, it is also a way of performing that avoids dissonance with other performers. This approach is valid when trying to play in tune with a piano or other keyboard instrument, as well as when performing with other instruments such as from the woodwind and brass families.

Notably exempt from most intonation restrictions are the violin, viola, cello and bass. It should be kept in mind, however, that the strings themselves are points of tuning reference; that these strings are tuned roughly in Pythagorean tuning before the performance commences; and that Pythagorean tuning and equal temperament, especially for the most common chords, are very similar.

Perhaps an even greater influence upon string quartet performance is that of adaptation to a culturally adopted tuning system. Although their instruments offer nearly complete control over intonation, string players will still tend to perform in a manner that minimizes potential dissonance within a twelve note system, especially taking into account that the other performers have in all likelihood developed the same dissonance-avoiding approach. Take, for example, a major triad whose root and fifth are being played in equal temperament. It is quite impossible to play the third in a manner that makes it free of beating. Consequently, playing the major third in equal temperament is guaranteed to make the chord acceptably in tune. It is not suggested that playing in equal temperament is deliberate or even conscious; rather, it is a reliable and efficient means of avoiding dissonance, within twelve note systems, that can be unconsciously learned through listening and practice.

6.5 Description of Tuning Systems

For a description of the three tuning systems used, please refer to Chapter 2, starting on p. 7. For details on how individual chords were tuned in just intonation, please refer to Chapter 5, beginning on p. 119.

6.6 Methodology

6.6.1 Collection of Data

CD recordings were first obtained for four different professional string quartets playing Mozart's String Quartet K. 590. The ensembles chosen for this study were the Berg, Quator, Salomon and Vienna quartets. The CD tracks corresponding to the Andante movement were then converted into WAV files using CDex version 1.2, public domain software from ALFA Technologies.

The remainder of the sound analysis was performed using Cool Edit 2000, from Syntrillium Software. Each of the wave files was resampled from 41K samples per second to 6K, decreasing the upper part of the frequency spectrum available for analysis in order to increase the resolution of the remaining lower part of the spectrum. Three of the files, originally in stereo format, were converted to monophonic so that all four files would be using an identical 6K monophonic format.

The settings employed were discovered through trial and error using a Csound score and orchestra for this eight bar passage whose generated pure sine wave output was stored in a WAV file. Csound produced incredibly accurate renditions of the equally tempered values, as measured by Cool Edit's Hanning filters, with error for the most out of tune note being less than 0.08 cents. Of the many settings explored, two Hanning filters used in tandem were found to be the most accurate. Larger FFT sizes in the second of the Hanning filters tended to cause the signal to completely jump out of the specified frequency range, whilst lower ones tended to give less accurate results.

Figure 59. First eight bars from Adagio movement of Mozart's last string quartet, K. 590

6.6.2 Data Acquisition

Once these settings with which to analyse the four wave files were chosen, data acquisition could begin. The overriding principle adopted was to keep the chords as pure as possible, in general sacrificing length to accomplish purity. It was sometimes necessary to do the reverse, especially in the case of short notes for the cello, which were very weak in the low register, and which were more easily detected in longer notes. In many cases for the cello's lowest notes, it was necessary to use the second and third partials to determine the fundamental. This is a sound tactic with steady tones not employing vibrato, but when vibrato or pizzicato is used, the fundamental can vary in relation to its partials.

Each chord was isolated manually using the mouse for rough and arrow clicks for fine control. The finest gradients of control were in increments of 1 millisecond, far smaller than necessary. In selecting the sample of sound representing the chord, there was a trade-off between obtaining sufficiently long tones and avoiding unwanted contamination of sound. These contaminants, often present in combination, were typically:

a) Notes that tended to bend or smear in the direction of their resolution

b) Reverberation of previous chords that were noticeable in the present chord, especially so at the end of a phrase and on a short note

c) Mixing of chords, as when one or more voices moves before the others

For each note in every chord played by each of the four ensembles, the selected chord was processed through a Hanning filter using an FFT size of 24000. A window of 20 Hz was applied on either side of the note, given in equal temperament, being examined. For example, when the filter is applied for an A-440 played by Violin I, only frequencies falling within the range of 420 – 460 Hz were left for further analysis.

Output from this band-pass filter was then analysed using a Hanning filter with an FFT size of 4096. Precision of output for this Hanning filter is two decimal places for notes above 100 Hz, three for notes below 100 Hz. This accounts, in part, for the varying level of precision for the data at this and subsequent stages.

Thus, for every chord, four frequency values were extracted through Hanning filters and stored in Table 82 of “Appendix I: Tables for Mozart Quartet Experiments” starting on p. 225. Each line of the table represents an individual note of Mozart’s score for this passage. Parameters for each note are, from left to right, a) instrument name, b) starting time, c) duration, d) frequency in equal temperament, e) frequency in Pythagorean tuning, f) frequency in flexible just intonation, g) the note name, and h-k) the raw frequency values for each of the four professional quartets.

6.7 Creating a frequency baseline for analysis

The particular frequency of the tonal centre, which is largely a function of how the performers tune up their instruments before commencing the movement, is an important parameter in the analysis of intonation. In order to compare the tunings for each of these professional ensembles against one another, or against tunings derived from intonation systems, it is necessary to establish a baseline tonal centre from which the frequencies for all notes played by all ensembles can be compared.

It is important to establish that the note representing the tonal centre, in this case C, is not itself mutable in just intonation. Although other notes within this passage are mutable according to the heuristics of Chapter 5 starting on p. 119, the note C in this passage is fortunately used only as the root of the C major tonic chord and as the seventh of the D dominant seventh (V^7/V) and D minor seventh (ii^7) chords, with C remaining the same for these different tonal functions; thus, all C’s in this passage can be used in an unbiased scaling process. Only the note C can be used without risk of prejudice to the data, as any other note tuned in the three systems considered would have at least two tunings.

6.7.1 Determining middle C equivalents

In order to facilitate comparisons, all Cs in all voices and all octaves are transposed to middle C. The middle C equivalents given in Table 60 for each of the quartets are compared one at a time with C-261.62557 Hz.

Table 60. Raw frequency data for all Cs transposed within closest proximity to middle C

Start	Note	Berg	Quator	Salomon	Vienna
0	C4	261.77	255.42	255.7	264.54
0	C2	263.08	254.42	255.224	264.68
1	C4	262.83	254.4	255.83	265.22
1	C2	264.596	254.964	255.252	264.7
1.5	C4	260.91	255.39	255.6	264.79
1.5	C2	262.4	253.988	253.756	261.88
2	C4	262.65	255.63	255.57	264.8
2	C2	262.2	255.012	255.42	265
2.5	C4	261.73	255.67	255.6	264.73
2.5	C2	265.14	254.968	256.04	260.856
3	C4	261.52	255.56	255.86	264.04
13	C3	267.42	253.08	256.34	267.26
14.5	C4	261.7	257.4	257.2	264.58
14.5	C3	260.38	259.14	256	262.92
16	C3	256.12	255.58	256.36	262.66
17.5	C4	261.77	256.22	257.09	266.31
17.5	C3	259.88	256.74	254.32	267.12
20.5	C4	261.77	255.73	256.05	266.85
21	C4	262.14	255.83	255.92	264.26

A simple program, CENTS, determines the differences in cents between middle C and each of these middle C equivalents representing performed notes. The output of CENTS is given in the last column of Table 61, the adjustment table for the Salomon Quartet. The average of these differences in cents is used to obtain the adjustment value.

Table 61. Middle C adjustment table for Salomon Quartet

Instr	Start	End	Note	Salomon Middle C Equiv.	Middle C ± Cents
Vl 2	0.889	1.996	C4	255.7	-39.6
Cel	0.889	1.996	C2	255.22	-42.8
Vl 2	2.019	2.518	C4	255.83	-38.7
Cel	2.019	2.518	C2	255.25	-42.6
Vl 2	2.555	2.988	C4	255.6	-40.3
Cel	2.555	2.988	C2	253.76	-52.8
Vl 2	2.922	3.458	C4	255.57	-40.5
Cel	2.922	3.458	C2	255.42	-41.5
Vl 2	3.462	3.938	C4	255.6	-40.3
Cel	3.462	3.938	C2	256.04	-37.3
Vl 2	3.968	5.598	C4	255.86	-38.5
Cel	14.702	15.252	C3	256.34	-35.3
Vl 2	16.177	16.714	C4	257.2	-29.5
Cel	16.177	16.714	C3	256	-37.6
Cel	17.699	18.212	C3	256.36	-35.1
Vl 2	19.199	19.701	C4	257.09	-30.2
Cel	19.199	19.701	C3	254.32	-49
Vl 2	22.164	22.733	C4	256.05	-37.2
Vla	22.164	22.733	C4	255.92	-38.1
				Average Difference	-39.3105

6.7.2 Normalization Factors

The final adjustment value is obtained by multiplying the average difference by –1 to compensate for the discrepancy. As shown in Table 62, the discrepancies in cents for each quartet are converted into normalization factors by raising the cent, defined as the $\sqrt[1200]{2}$, to the power of the adjustment value. The values obtained from each professional quartet are now shifted so that they centre on a standardized value for middle C by multiplying every performed frequency by the normalization factor associated with that quartet.

Table 62. Normalization values of middle C equivalents for the four string quartets

Quartet	Average discrepancy in cents	Final Adjustment Value	Normalization Factor
Berg	Sharp by 3.084211 cents	-3.084211	$(\sqrt[1200]{2})^{-3.084211} = 0.99822$
Quator	Flat by 40.7526 cents	40.7526	$(\sqrt[1200]{2})^{40.7526} = 1.0238189$
Salomon	Flat by 39.3105 cents	39.3105	$(\sqrt[1200]{2})^{39.3105} = 1.0229664$
Vienna	Sharp by 19.42632 cents	-19.42632	$(\sqrt[1200]{2})^{-19.42632} = 0.9888416$

6.8 Mutable Tones Treated Separately

In total, nine of the chromatic scale's twelve notes and their octave equivalents are used in this passage, namely, A, B, C, C#, D, E, F, F#, and G. For either equal temperament or Pythagorean tuning, there is only one possible tuning for each note in the respective system; however, in just intonation, this brief tonal example, although selected for its simplicity, contains two mutable tones and their octave equivalents.

The fact that A is serving two separate and incompatible tonal functions results in the first of these mutable tones. (Please see "Introduction to Mutable Tones", p. 128.) Table 63 illustrates the two different tonal functions served by the note A, as well as the dual frequencies necessitated by the incompatible functions.

The first region where the note A occurs commences at time value 6, meaning that the equivalent time of six crotchets have already taken place, corresponding to beat one of measure three. This region extends to the third quaver of measure 6. Within this region, all A's are related to tonic C as the third (5/4) of the subdominant (4/3), or equivalently as the subdominant of the third, which in either case equals 5/4 x 4/3 = 5/3 or an octave equivalent. Therefore, if C = 261.62557, then A = 5/3 x 261.62557 = 436.0427 or its octave equivalent.

Similarly, the second region contains A's that are all of the secondary dominant (V/V) function. Since in just intonation (or, for that matter, Pythagorean tuning), the relationship between the root of a chord and its dominant is 3/2, then A is the fifth (3/2) of a D major triad, which is the dominant (3/2) of a (G) chord, which in turn is the dominant (3/2) of tonic C. Therefore, if C = 261.62557, then A = 3/2 x 3/2 x 3/2 x 261.62557, which is 27/8 x 261.62557 = 882.98628 or one of its octave equivalents.

Table 63. Mutable tone ‘A’ from Mozart example

Type I: Fifth of D Minor Triad (ii) or D Minor Seventh (ii^7)			Type II: Fifth of D Major Triad (V/V) or D Dominant Seventh (V^7/V)		
Region Begins	Region Ends	Frequencies used	Region Begins	Region Ends	Frequencies used
6.0	16.5	109.0107 218.0213 436.0427	19.5	22.5	220.7466 441.4932

Similar information is provided in Table 64 for the mutable tone D. When tuned as a member of a G dominant chord, its frequency is derived as the fifth (3/2) of the dominant (3/2) of tonic. The tuning for D in this case is therefore 3/2 x 3/2 x 261.62557 = 588.65752 or an octave equivalent. When tuned as a member of a supertonic ii chord, its tuning is the subdominant (4/3) of the subdominant (4/3) of the third (5/4), providing a tuning of 80/36 x C 261.62557 = 581.39014 or an octave equivalent.

Table 64. Mutable tone ‘D’ from Mozart Example

Type I: Root of D Minor Triad (ii) or D Minor Seventh (ii^7)			Type II: Fifth of G Dominant (V) Triad or G Dominant Seventh (V^7)		
Region Begins	Region Ends	Frequencies used	Region Begins	Region Ends	Frequencies used
6.0	13.5	72.6738 145.34754 290.695	4.5	5.0	294.329
15.0	16.5	145.34754 290.695	13.5	14.0	294.329
			16.5	23.0	73.5822 147.164 294.329

6.9 Analysis I: Evaluating the Individual Notes

Analysis 1 explores both the tunings and ranges of tunings for every note type as it is played by each of the four professional quartets. In all, 27 notes ranging from C2 to A4 are examined separately, with emphasis given to those that occur most frequently.

In this analysis, a standard deviation is first given for the most commonly performed pitches in order to demonstrate the wide range of frequencies actually employed for each note, both within each ensemble and in general. Comparisons are then drawn between the pitches actually performed and those predicted by the three standard models: equal temperament, Pythagorean tuning and just intonation. Finally, regression analysis is performed note-by-note on the entire passage under consideration for each of the four quartets in order to determine the tuning system most nearly adopted in practice.

6.9.1 Converting Normalized Values to Cents

Before performing various statistical analyses on the data, it is convenient to first sort all of the events representing normalized frequencies chromatically from bottom to top. The data thus processed is stored in the text file sort_by_note.txt. The first nine notes of the normalized data for the viola are shown, in chromatic order, in Table 65. All such normalized data can be viewed in Table 83 of “Appendix I: Tables for Mozart Quartet Experiments” on p. 227.

Table 65. Sample of normalized data for viola

Inst	Start	End	E.T.	Pyth	Just	Note	Berg	Quator	Salomon	Vienna
Vla	15	0.5	174.614	174.417	174.417	F3	176.246	176.148	174.62	175.994
Vla	19.5	0.5	184.997	186.255	183.956	F#3	186.038	183.755	184.717	185.774
Vla	0	0.5	195.998	196.219	196.219	G3	196.949	195.099	196.011	195.899
Vla	1	0.5	195.998	196.219	196.219	G3	198.077	195.662	195.826	195.83
Vla	1.5	0.5	195.998	196.219	196.219	G3	198.077	195.519	195.387	196.374
Vla	2	0.5	195.998	196.219	196.219	G3	195.322	195.437	196.916	196.305
Vla	2.5	0.5	195.998	196.219	196.219	G3	196.5	197.085	196.041	196.582
Vla	3	2	195.998	196.219	196.219	G3	195.412	195.498	196.696	196.394
Vla	4.5	0.5	195.998	196.219	196.219	G3	196.09	197.341	194.2	195.108

This text file is processed in turn by the program MIDC2CNT, which converts every frequency value into its distance, in cents, from middle C and stores the new data in the output file midc2cent.txt. A sample of this data, in which the same data from Table 65 has been converted into distance in cents from middle C, is shown in Table 66. (Please refer to Appendix I: Tables for Mozart Quartet Experiments, Table 84 on p. 229.)

Table 66. Sample of normalized data converted to distance, in cents, from middle C

Inst	Start	End	E.T.	Pyth	Just	Note	Berg	Quator	Salomon	Vienna
Vla	15.00	0.50	-700.00	-701.95	-701.95	F3	-683.89	-684.85	-699.94	-686.37
Vla	19.50	0.50	-600.00	-588.26	-609.77	F#3	-590.28	-611.66	-602.62	-592.74
Vla	0.00	0.50	-499.99	-498.04	-498.04	G3	-491.61	-507.95	-499.88	-500.87
Vla	1.00	0.50	-499.99	-498.04	-498.04	G3	-481.73	-502.96	-501.51	-501.48
Vla	1.50	0.50	-499.99	-498.04	-498.04	G3	-481.73	-504.23	-505.40	-496.67
Vla	2.00	0.50	-499.99	-498.04	-498.04	G3	-505.97	-504.95	-491.90	-497.28
Vla	2.50	0.50	-499.99	-498.04	-498.04	G3	-495.56	-490.42	-499.61	-494.84
Vla	3.00	2.00	-499.99	-498.04	-498.04	G3	-505.18	-504.41	-493.84	-496.50
Vla	4.50	0.50	-499.99	-498.04	-498.04	G3	-499.18	-488.17	-515.95	-507.87

6.9.2 Mutable Tones A and D are Treated Separately

As described in section 6.8, both notes A and D are mutable in this passage. Therefore, values for D2, A2, D3, A3, D4 and A4 are analysed according to the tonal functions in which they are used. Thus, for example, D2 is broken into two categories, D2I and D2II, corresponding to root of a supertonic and fifth of a dominant chord, respectively.

6.9.3 Standard Deviation of Performed Pitches

In a normal distribution, about 68% of the frequencies will fall within one standard deviation of the mean, whilst around 95% will fall within two standard deviations. Generally speaking, the larger the number of data points to analyse, the more reliable is the data. Figure 60 and Table 67 present the standard deviations for the ten most commonly occurring notes played, along with the number of data points analysed. One of the striking results shown in Figure 60 and Table 67 is the rather wide variation of pitch. For example, the first entry, G3, is played 16 times by all four quartets. The Berg Quartet uses the most variable intonation for this note, with approximately 68% of the occurrences of G3 falling within 9.303595 cents of the mean; conversely, roughly 32% of the occurrences of G3 are played either sharper or flatter than the average by at least 9.303595 cents. The Quator Quartet, having the least variability for this most reliable set of points, is still playing roughly 32% of its G3's sharper or flatter than the average by at least 5.81611551 cents.

The most extreme variation in pitch for these top ten most numerous notes comes from the second violin of the Vienna Quartet when playing B3. The viola also plays this note once, but this single value is very nearly the mean. With seven data points, the extreme variation would be expected to occur more often than with G3, which has more than twice as many data points. A standard deviation of 14.92489 indicates that pitches represented by the data points are more than a seventh of a tone sharper or flatter than the mean approximately one-third of the time.

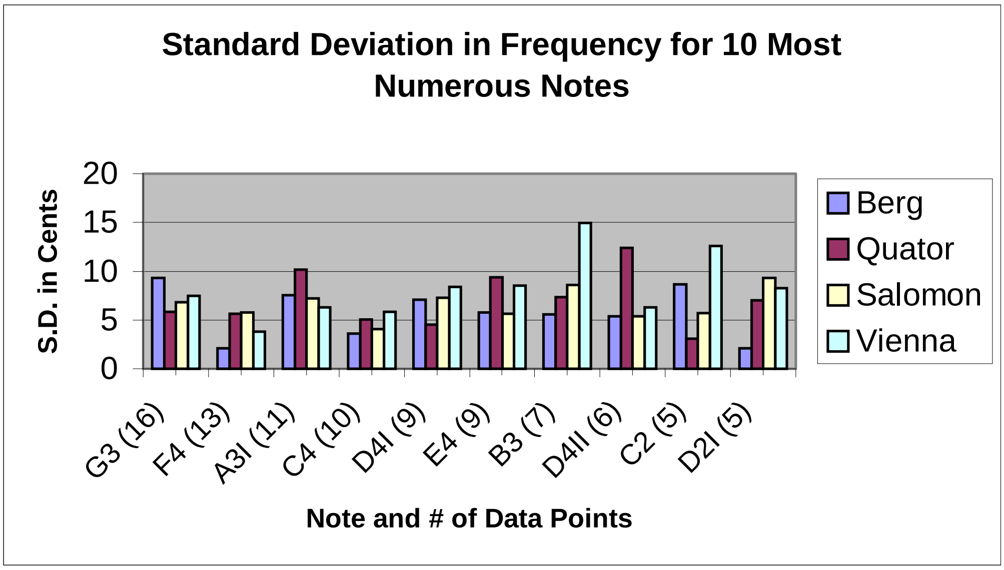

Figure 60. Standard deviations in frequency for 10 most frequently occurring notes

Table 67. Standard deviations in frequency for 10 most common notes

Note	Data Points	Berg	Quator	Salomon	Vienna
G3	16	9.303595	5.81611551	6.830954	7.495792
F4	13	2.0992007	5.65860316	5.744187	3.821292
A3I	11	7.51819127	10.1719065	7.221442	6.266093
C4	10	3.63085236	5.06998313	4.060728	5.830022
D4I	9	7.05661057	4.55548692	7.271607	8.380454
E4	9	5.75727173	9.37178271	5.658956	8.519413
B3	7	5.5863635	7.34976125	8.570582	14.92489
D4II	6	5.35065136	12.3994333	5.370831	6.294009
C2	5	8.66419529	3.07581209	5.708218	12.60551
D2I	5	2.103425777	6.995244813	9.2993484	8.2392312

6.9.4 Comparisons between Performed and Predicted Frequencies

By normalising the data from a performance of this passage, it is possible to compare the frequencies with the predicted values of one or more tuning systems. For each professional quartet, all of the 27 pitch types performed, from C2 to A4, were enumerated and compared with the values predicted by equal temperament, Pythagorean tuning and just intonation. The method for determining the values used in just intonation is explained in Chapter 5 starting on p. 119.

The wide fluctuations in performed frequencies, as previously illustrated in Figure 60 and Table 67, are again apparent in Figure 61 through Figure 64, which give the difference in cents between each frequency actually performed and the corresponding values predicted by the three tuning systems employed. Take, for example, the chart for the Berg Quartet in Figure 61. Near the middle of the chart is the note G3, in which all three bars are very nearly centred with the performed values. This is to be expected, as G3 is the fifth of tonic; the fifth of tonic is nearly identical in all three systems; and there are 16 points of data, minimizing the effects of anomalous fluctuation. To the left of G3 is F#3, which contains only one data point. The lightest shaded bar for F#3 is well above centre, indicating it is being performed much sharper than predicted by just intonation.

Although it is clear even upon visual inspection that equal temperament is doing the best job at estimating the recorded values, and that just intonation is the least predictive, the available data for this short example is, as mentioned before, rather restricted for many of the notes. Another point to address is the fact that these notes are not being compared with the other notes in the same chord, but rather with the tonal centre, C.

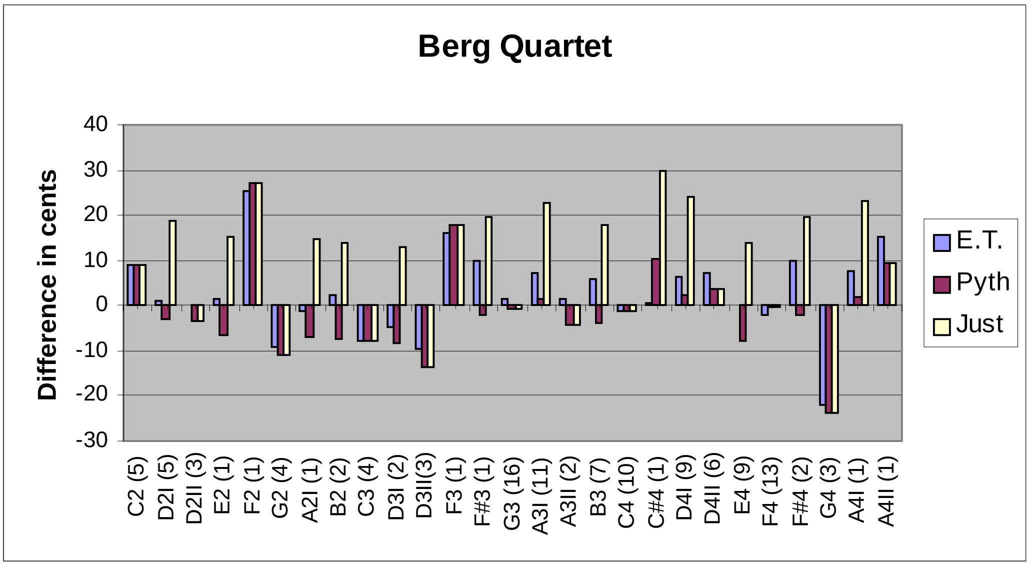

Figure 61. Differences in cents between frequencies performed by Berg Quartet and calculated values for three tuning systems

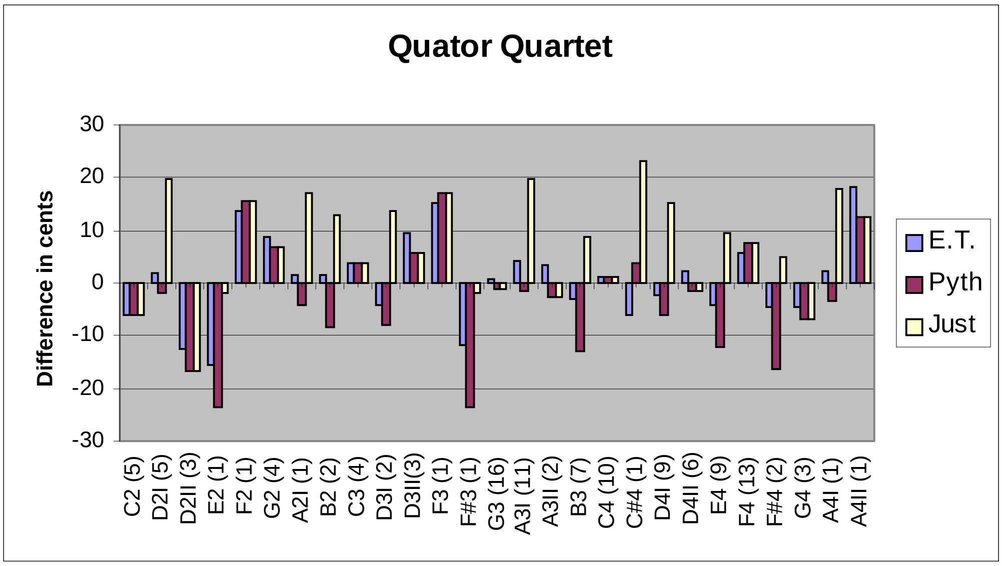

Figure 62. Differences in cents between frequencies performed by Quator Quartet and calculated values for three tuning systems

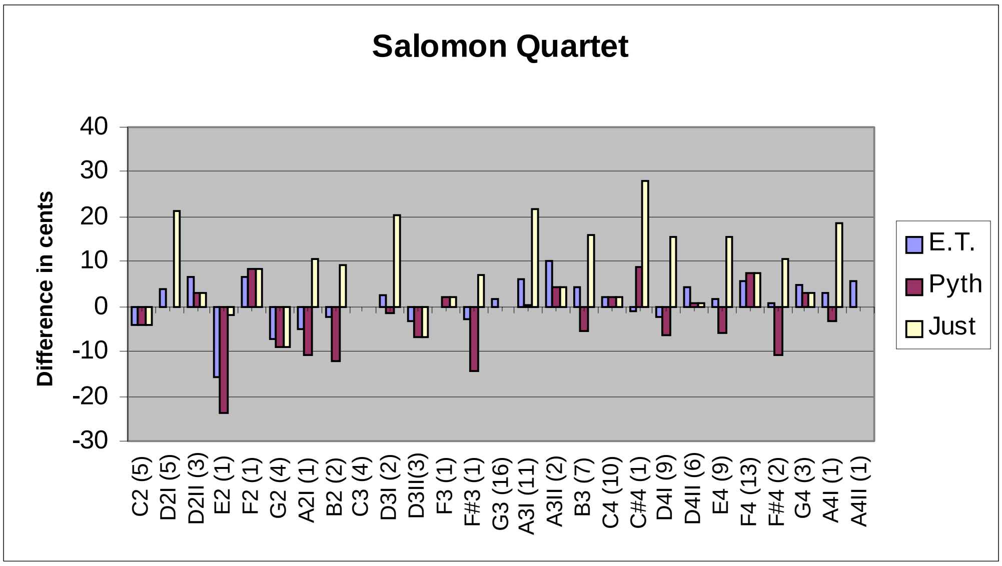

Figure 63. Differences in cents between frequencies performed by Salomon Quartet and calculated values for three tuning systems

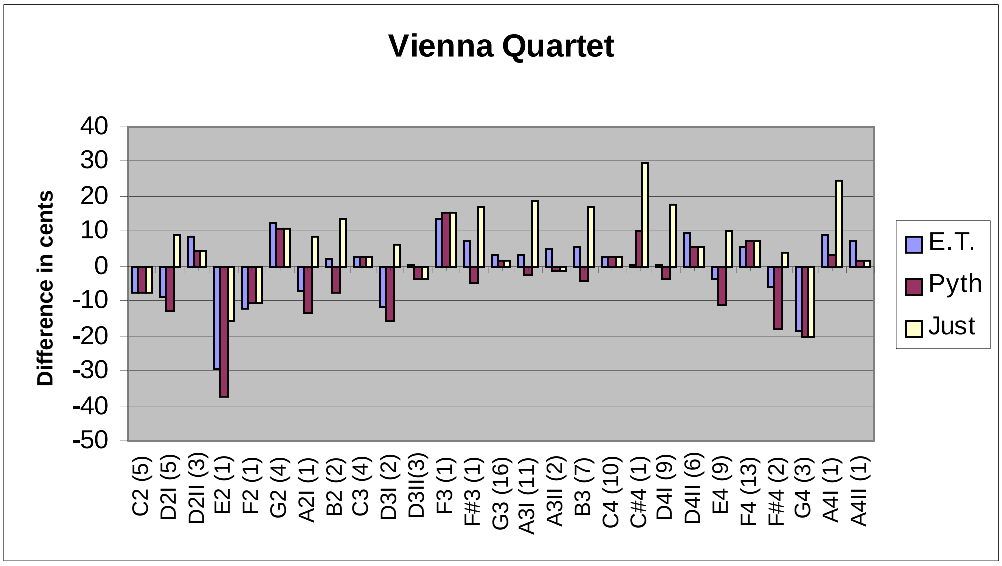

Figure 64. Differences in cents between frequencies performed by Vienna Quartet and calculated values for three tuning systems

6.9.5 Regression Analysis of Entire Passage

In order to avoid over-analysing averaged data, especially when some note types are represented too infrequently to provide accurate data, regression analysis is used to determine exactly which system most nearly predicts the tunings, in relation to a standardized tonic, actually performed. Table 68 through Table 71 provide results of regression analysis for each of the four recordings.

Clearly, in all four cases, just intonation is the least predictive system of performed frequency values. Of the remaining two systems, equal temperament narrowly but consistently outperforms Pythagorean tuning.

Table 68. Berg Quartet regression statistics

	Equal Temperament	Pythagorean Tuning	Just Intonation
Multiple R	0.99993	0.999929	0.999836
R Square	0.99986	0.999858	0.999672
Adjusted R Square	0.991729	0.991728	0.991542
Standard Error	10.59245	10.65231	16.17939
Observations	124	124	124

Table 69. Quator Quartet regression statistics

	Equal Temperament	Pythagorean Tuning	Just Intonation
Multiple R	0.999952	0.999932	0.999897
R Square	0.999903	0.999864	0.999795
Adjusted R Square	0.991773	0.991733	0.991664
Standard Error	8.790315	10.44843	12.8213
Observations	124	124	124

Table 70. Salomon Quartet regression statistics

	Equal Temperament	Pythagorean Tuning	Just Intonation
Multiple R	0.999967	0.999961	0.999891
R Square	0.999934	0.999923	0.999782
Adjusted R Square	0.991804	0.991793	0.991651
Standard Error	7.276146	7.876927	13.23619
Observations	124	124	124

Table 71. Vienna Quartet regression statistics

	Equal Temperament	Pythagorean Tuning	Just Intonation
Multiple R	0.999936	0.999925	0.999885
R Square	0.999872	0.999849	0.999769
Adjusted R Square	0.991742	0.991719	0.991639
Standard Error	10.13373	10.98894	13.60741
Observations	124	124	124

So far, all data from each of the four professional quartets has been processed in relation to three fixed sets of data points calculated according to the three tuning systems examined. It is important to bear in mind that the performers might well be using just intonation or Pythagorean tuning over short stretches of time; however, under the constraints given that all points of the models are fixed, even taking into account experimental error, most notably the effects of vibrato upon the fundamental, it is clear that the quartets are all playing this passage somewhere in between equal temperament and Pythagorean tuning, more nearly the former than the latter.

6.10 Analysis II: Analysis of Four Chord Types

In Analysis I, all notes were related to tonic C, which is the only note in the selected passage that can be used as an unbiased and fixed point of reference. One question that cannot be addressed when using such a fixed reference point is that of how the performers are tuning individual chords in relation to a given model. Consider, for example, the A major triad in bar four, which is serving as the dominant to the D minor triad in both the previous and following bars. If the D minor triad is not tuned in just intonation, then the A major dominant triad, even if its individual frequencies are tuned in relation to each other as the ideal 4:5:6 ratio, will not be considered in tune according to just intonation, as its tuning centre itself is not in tune to tonic.

One easy way around this problem is to simply conduct experiments on the tunings of single chords and short chord progressions, an approach that is highly recommended. If, however, existing CD recordings are to be used, another approach must be devised that permits analysis of chords without using a single point of reference.

6.10.1 Chord Normalization

In Analysis I, all notes were multiplied by a normalization value obtained by comparing all stored values for the note C with the equal temperament value for middle C-261.62557 Hz. This approach is valid when addressing the issue of how nearly the values for an entire passage, as calculated for a particular tuning model such as equal temperament, can predict the actual tunings adopted by a particular ensemble.

It is a very different question to ask how nearly the values for a particular chord or chord type, as predicted by such a theoretical model, correspond to those actually performed by an ensemble. It is not immediately apparent that the two questions are different. As an illustration, take an amateur church choir, perhaps one that has just begun practicing after a long vacation. The choir is given starting pitches on the piano from the choir director. The singers proceed to sing the first stanza of a hymn, after which the director has them stop and recommence. When the starting pitches are played by the director for the second time, it is apparent that the choir has already gone flat an entire half step. This provides an extreme example of why individual chords and chord types must not be compared to a fixed centre, as the tonal centre itself can gravitate, for various reasons, away from the idealized centre. One such gravitational attractor of tuning for a string quartet is the Pythagorean arrangement of the strings themselves, which serve as reference points of intonation to the performers.

6.10.2 Creating a Frequency Baseline for Individual Chords

In order to compare the frequency values contained in each chord with the theoretical models calculated from equal temperament, Pythagorean tuning and just intonation, it is first necessary to normalize the chord in question against each model. The method of normalization employed in this study is to find the average discrepancy between each of the notes contained in the chord and its corresponding note predicted by the model.

As an example, take the frequency values for a C major triad shown in Table 72. The recorded values played, in this case by the Quator Quartet, are subtracted from those calculated using the equal temperament model. The average of these discrepancies is then used to offset the differences between predicted and observed data so that the overall distances from corresponding notes between the two are minimized. The purpose for all of this adjusting is to permit the observed data to be compared note at a time to its chord's centre, as its centre relates to the model, thus permitting any chord of the same type at any pitch level to be compared with the model or the collective of that type.

Table 72. A single normalized chord, as performed by the Quator Quartet

Instrument	Note	Equal Temperament (Cents above/below middle C)	Actual Values (Cents above/below middle C)	ET – Quator	Adj ET-Quator (Normalized)
Violin 1	E4	400	388.73	11.27	4.3625
Violin 2	C4	0	-0.8	0.80	-6.1075
Viola	G3	-499.99	-507.95	7.96	1.0525
Cello	C2	-2399.99	-2407.59	7.60	0.6925
			AVE: 6.9075		

6.10.3 Chord Types Analysed

There are precisely four chord types, not counting their inversions, in this eight bar passage, which is remarkable in itself, considering the fact that this tonal segment contains 31 chords. Such variety from so few chords! Every single chord can be unambiguously categorized as either major, minor, dominant seventh or minor seventh. Within each chord, every note has the same duration, and there are no passing tones, suspensions or any other tones that do not serve a harmonic function. All of these qualities facilitate analysis of tuning for individual chords and chord types.

Although there are different inversions for major, minor and dominant seventh chords, it is assumed that inverting a chord does not affect its theoretical tuning, although this issue would make an interesting topic of study. In the present study, however, every chord type, regardless of inversion or function, will be assumed to use the same relative tunings, allowing for octave equivalence, upon octave transposition of any tone or combination of tones.

A complete set of normalization tables, broken down by chord type, inversion and chord name, is given in "Appendix I: Tables for Mozart Quartet Experiments", Table 85 through Table 100 beginning on p. 231. Averaged data for these tables are presented in Table 73 through Table 76, along with their associated graphs in Figure 65 through Figure 80.

6.10.4 Major Triad

Several observations regarding averaged data for the major triad normalized against three tuning systems can be made by examining Table 73 and its accompanying graphs in Figure 65 through Figure 68. First of all, nearly all of the difference in tuning is being accounted for by the major third, as the perfect fifths of just intonation and Pythagorean tuning in this passage are identical. A perfect fifth in equal temperament is less than two cents flat to either of these, which is hardly significant, as the standard deviation for the most consistently played perfect fifth in a major triad, no matter to which system it is compared, is more than twice this amount.

With 26 data points for the root, 14 for the third and 16 for the fifth, the major triad is statistically more reliable than the other three chords, much more so than the two seventh chords examined. Whilst the Berg Quartet seems to be playing the major triad very nearly in Pythagorean tuning, the Salomon and Vienna Quartets seem to settle between equal temperament and Pythagorean tuning, without strong preference for either. The one ensemble clearly going against the trend is the Quator Quartet, which is playing the major third only 2.65 cents sharp, averaged over 14 data points, to the value predicted by just intonation.

Table 73. Average values and standard deviations for major triad normalized against three systems

		Diff. E.T.	Diff. Pyth	Diff. Just	S.D. E.T.	S.D. Pyth	S.D. Just
BERG							
	Root (26)	2.674712	0.682692	5.609038	8.400532	8.513479	8.47056
	Third (14)	-3.27071	0.77875	-14.0966	9.555527	10.82531	9.471263
	Fifth (16)	-1.48453	-1.79078	3.219844	8.163993	8.472391	8.082026
QUATOR							
	Root (26)	-2.70894	-4.70096	0.225385	5.557304	6.020142	5.568693
	Third (14)	8.172679	12.22214	-2.65321	4.652008	6.916661	4.622831
	Fifth (16)	-2.74906	-3.05531	1.955313	4.88718	4.822182	4.913128
SALOMON							
	Root (26)	1.41125	-0.58077	4.345577	5.499048	5.81447	5.498491
	Third (14)	1.767679	5.817143	-9.05821	5.388912	6.656722	5.284739
	Fifth (16)	-3.84	-4.14625	0.864375	4.815997	4.633353	4.805846
VIENNA							
	Root (26)	-0.3174	-2.30942	2.616923	8.902001	9.134601	8.97792
	Third (14)	3.356786	7.40625	-7.46911	9.253927	10.75252	9.114689
	Fifth (16)	-2.42141	-2.72766	2.282969	7.811325	8.083801	7.792771

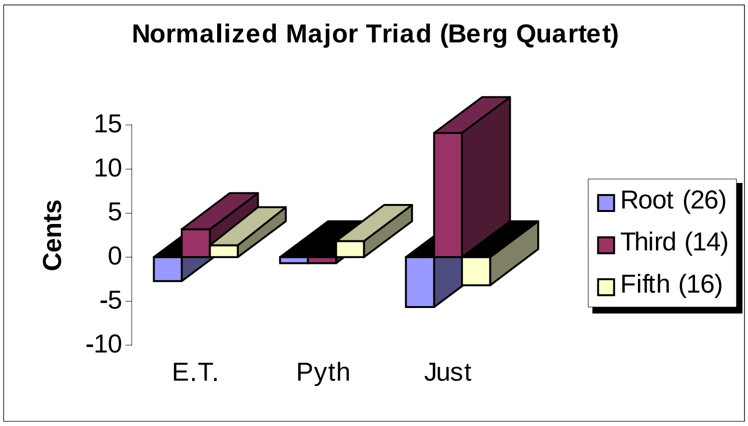

Figure 65. Average values for major triad normalized against three systems (Berg Quartet)

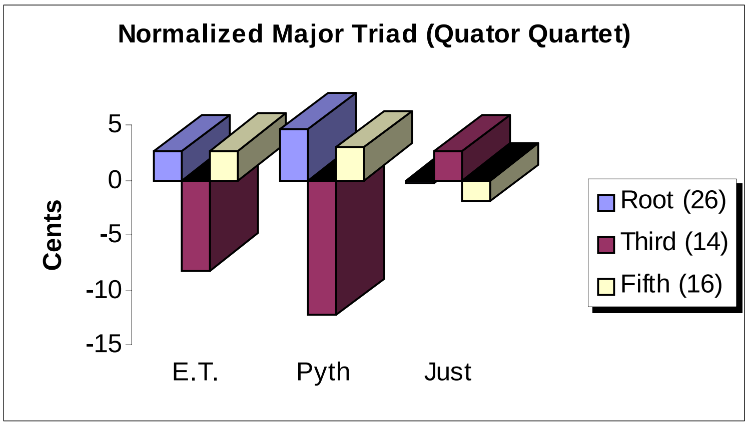

Figure 66. Average values for major triad normalized against three systems (Quator Quartet)

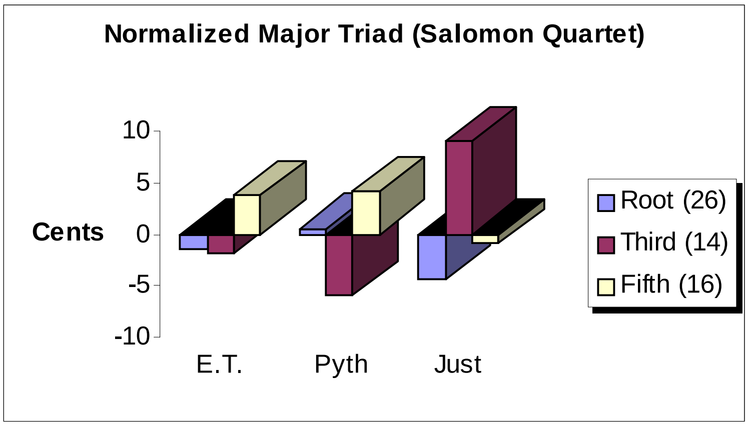

Figure 67. Average values for major triad normalized against three systems (Salomon Quartet)

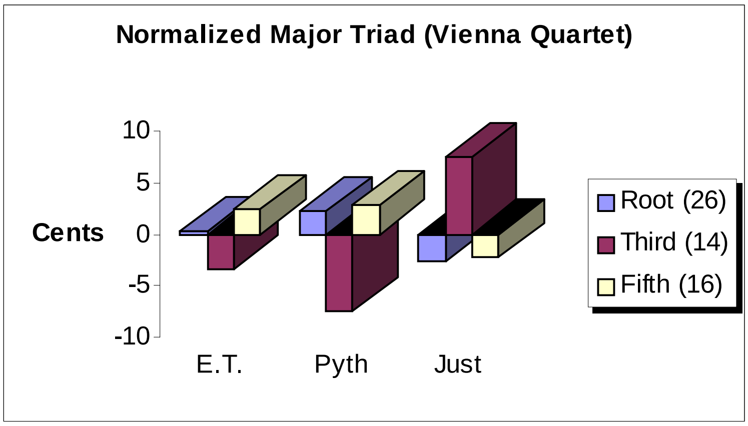

Figure 68. Average values for major triad normalized against three systems (Vienna Quartet)

6.10.5 Minor Triad

Close inspection of the averaged data for the minor triad normalized against the same three tuning systems, shown in Table 74 and the accompanying graphs of Figure 69 through Figure 72, reveals that the individual ensembles are maintaining their preferences for the previous major triad. With regard to the minor triad, 14 points of data are processed for the root, as well as 9 each for the third and fifth.

The Berg Quartet narrowly prefers Pythagorean tuning, and the Salomon and Vienna Quartets continue to ever so slightly favour equal temperament over Pythagorean tuning. All three of these quartets are definitely avoiding the sharpened minor thirds of just intonation. The renegade champion of just intonation for this triad, as with the major triad, turns out again to be the Quator Quartet, which is playing the minor third slightly flatter, on the average, than predicted by just intonation.

Table 74. Average values and standard deviations for minor triad normalized against three systems

		Diff. E.T.	Diff. Pyth	Diff. Just	S.D. E.T.	S.D. Pyth	S.D. Just
BERG							
	Root (14)	1.2775	2.322321	-3.43857	6.534784	6.475784	6.607028
	Third (9)	1.602222	-3.0125	11.92389	8.832498	8.84415	9.445813
	Fifth (9)	-3.58944	-0.6	-6.575	6.025742	6.135195	5.503603
QUATOR							
	Root (14)	3.826964	4.871786	-0.88911	3.064395	3.032486	3.479927
	Third (9)	-7.14139	-11.7561	3.180278	4.009007	4.165261	4.251185
	Fifth (9)	1.188333	4.177778	-1.79722	5.286933	5.60145	4.318942
SALOMON							
	Root (14)	3.383036	4.427857	-1.33304	8.227715	8.206778	8.401373
	Third (9)	-3.35583	-7.97056	6.965833	6.129657	6.24744	6.132962
	Fifth (9)	-1.90667	1.082778	-4.89222	6.590703	6.715088	6.277732
VIENNA							
	Root (14)	4.834821	5.879643	0.11875	8.860467	8.95827	8.704773
	Third (9)	-3.81639	-8.43111	6.505278	7.250684	7.043842	7.55574
	Fifth (9)	-3.70444	-0.715	-6.69	4.385638	4.692438	4.278573

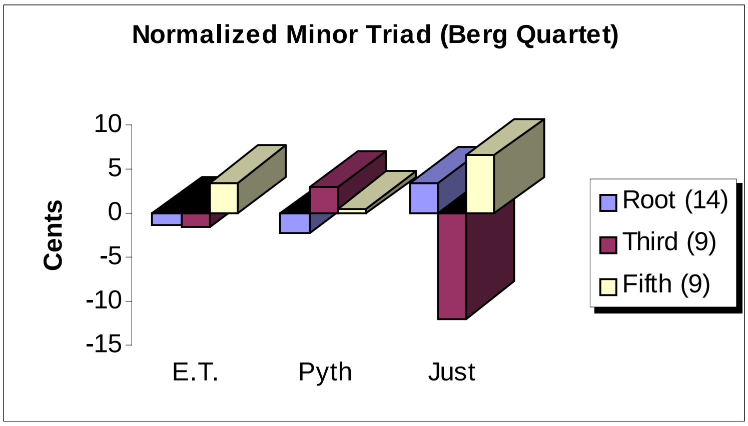

Figure 69. Average values for minor triad normalized against three systems (Berg Quartet)

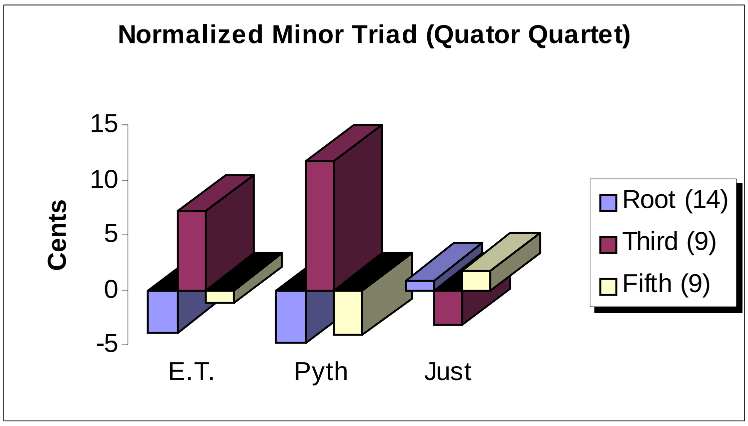

Figure 70. Average values for minor triad normalized against three systems (Quator Quartet)

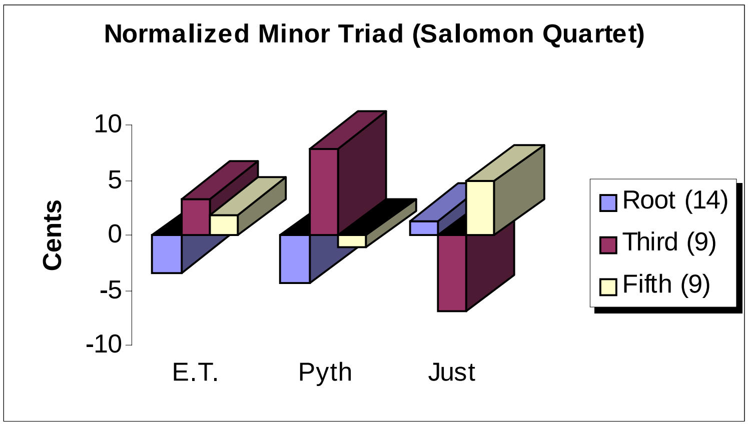

Figure 71. Average values for minor triad normalized against three systems (Salomon Quartet)

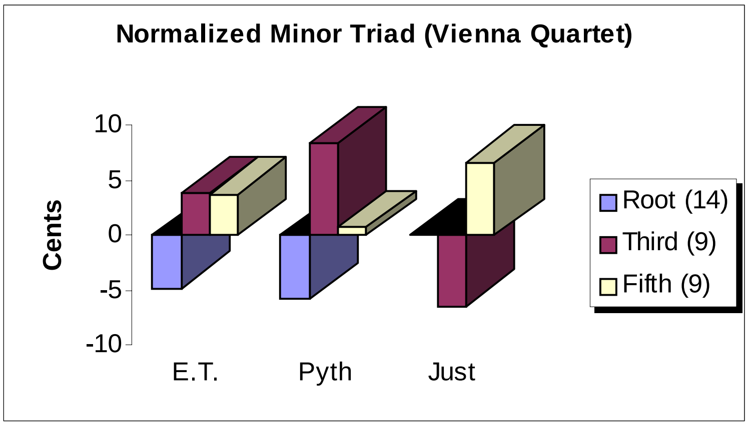

Figure 72. Average values for minor triad normalized against three systems (Vienna Quartet)

6.10.6 Dominant Seventh Chord

The averaged data for the dominant seventh chord is given in Table 75 and the associated charts of Figure 73 through Figure 76. Except for the Berg Quartet, which seems fairly evenly divided between equal temperament and Pythagorean, the preference seems to be consistently for equal temperament. In every case, the major third is played flat according to Pythagorean and sharp according to just intonation. This avoidance of either extreme could be a significant reason equal temperament is outperforming the other two systems in this context.

It should be noted that the number of data points available for this chord, 8 for the root, 6 for the third and seventh, and only four for the fifth (because two of the dominant seventh chords have no fifth), is substantially less than for the major or minor triads. The results are therefore less conclusive.

Table 75. Average values and standard deviations for dominant seventh normalized against three systems

		Diff. E.T.	Diff. Pyth	Diff. Just	S.D. E.T.	S.D. Pyth	S.D. Just
BERG							
	Root (8)	4.158125	2.935625	8.310625	5.694718	5.673125	5.673073
	Third (6)	-4.20708	2.299583	-13.8229	10.67032	10.63048	10.63428
	Fifth (4)	-6.2025	-5.7075	-0.335	3.147249	3.14328	3.147249
	Seventh(6)	2.797917	-2.40875	2.965417	4.541009	4.385229	4.384059
QUATOR							
	Root (8)	1.56	0.3375	5.7125	4.793394	4.767844	4.770632
	Third (6)	-0.23292	6.27375	-9.84875	8.620773	8.5838	8.580157
	Fifth (4)	-0.00125	0.49375	5.86625	12.78575	12.7905	12.78575
	Seventh(6)	-1.84625	-7.05292	-1.67875	2.837425	2.982115	2.984678
SALOMON							
	Root (8)	-0.44281	-1.66531	3.709688	5.901975	6.056661	6.058835
	Third (6)	0.347083	6.85375	-9.26875	6.966277	6.746062	6.753185
	Fifth (4)	-3.30813	-2.81313	2.559375	5.054286	5.055619	5.054286
	Seventh(6)	2.44875	-2.75792	2.61625	1.051702	0.949623	0.94973
VIENNA							
	Root (8)	-1.95219	-3.17469	2.200313	6.674099	6.802104	6.803978
	Third (6)	2.619167	9.125833	-6.99667	8.763148	8.56836	8.572323
	Fifth (4)	-1.74063	-1.24563	4.126875	4.827341	4.825745	4.827341
	Seventh(6)	1.144167	-4.0625	1.311667	6.746863	6.838197	6.836942

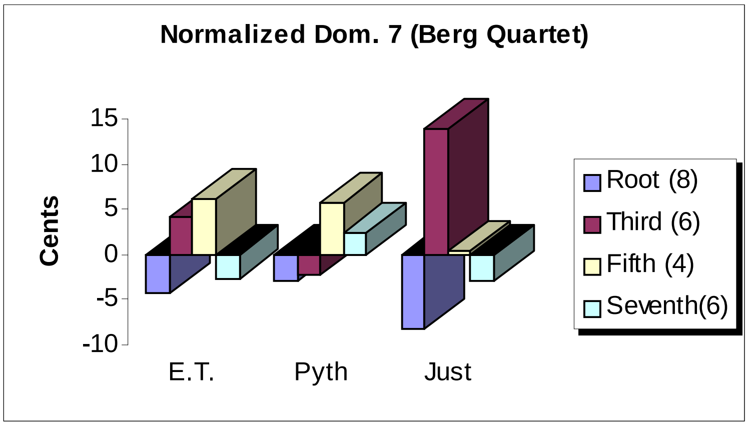

Figure 73. Average values for dominant seventh normalized against three systems (Berg Quartet)

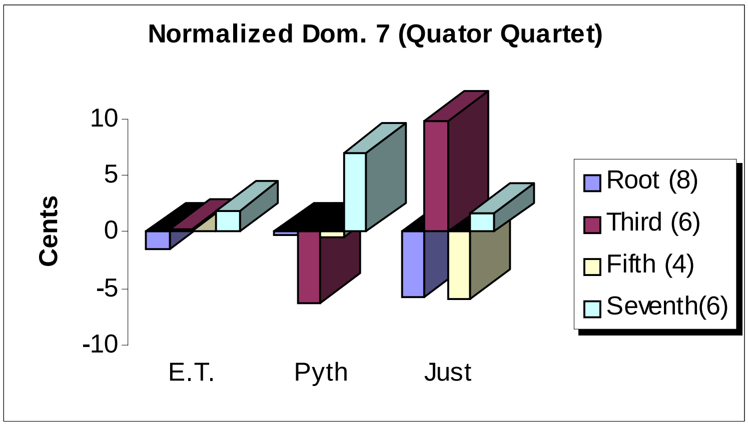

Figure 74. Average values for dominant seventh normalized against three systems (Quator Quartet)

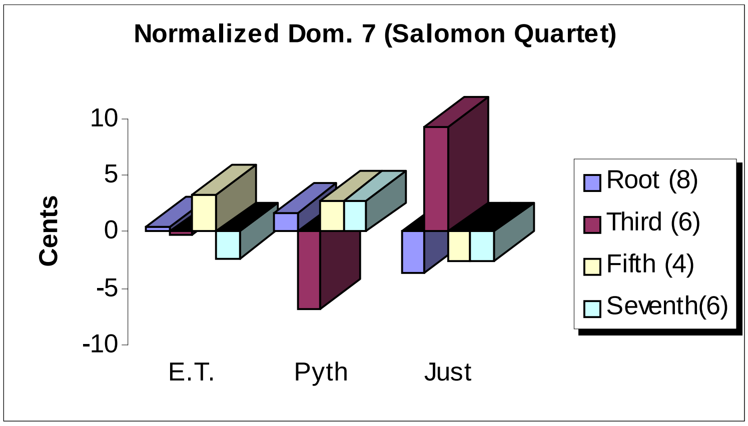

Figure 75. Average values for dominant seventh normalized against three systems (Salomon Quartet)

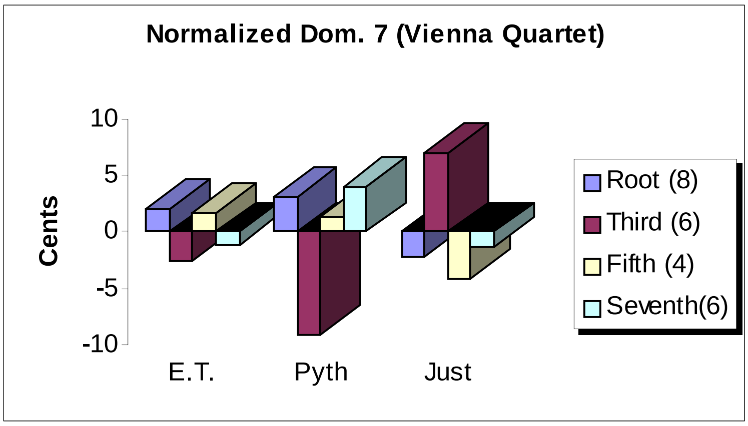

Figure 76. Average values for dominant seventh normalized against three systems (Vienna Quartet)

6.10.7 Minor Seventh Chord

For completeness, the averaged data, found in Table 76 and its related charts in Figure 77 through Figure 80, is given for the minor seventh chord. Data is noticeably sparse for this chord, with only 3 data points representing each of the root, third, fifth and seventh. As with the other three chord types, the Berg Quartet's tuning preferences were best predicted, however narrowly over equal temperament, by Pythagorean tuning. Another observation to note is that none of the quartets is performing this chord type in just intonation. Finally, when the chord varies significantly from the model, the tonic/fifth and the third/seventh tend to be sharp or flat together, indicating a binding of the notes forming the either of the minor seventh chord's two perfect fifths.

Table 76. Average values and standard deviations for minor seventh normalized against three systems

		Diff. E.T.	Diff. Pyth	Diff. Just	S.D. E.T.	S.D. Pyth	S.D. Just
BERG							
	Root (3)	-4.80167	-2.8475	-13.6025	14.40548	14.40542	14.40542
	Third (3)	5.055	1.145833	11.90083	-3.405	-7.3125	3.4425
	Fifth (3)	-5.005	-1.09417	-11.8492	5.445771	5.445081	5.445081
	Seventh(3)	4.751667	2.795833	13.55083	26.99093	26.99057	26.99057
QUATOR							
	Root (3)	-1.485	0.469167	-10.2858	10.53422	10.5316	10.5316
	Third (3)	-0.92167	-4.83083	5.924167	7.102351	7.103849	7.103849
	Fifth (3)	-10.535	-6.62417	-17.3792	10.97763	10.97534	10.97534
	Seventh(3)	12.94167	10.98583	21.74083	5.923201	5.925423	5.925423
SALOMON	E.T.						
	Root (3)	2.3925	4.346667	-6.40833	3.908276	3.906866	3.906866
	Third (3)	-0.3675	-4.27667	6.478333	4.319342	4.317911	4.317911
	Fifth (3)	-4.04417	-0.13333	-10.8883	2.287127	2.288941	2.288941
	Seventh(3)	2.019167	0.063333	10.81833	1.875954	1.877805	1.877805
VIENNA							
	Root (3)	2.655	4.609167	-6.14583	14.02306	14.02356	14.02356
	Third (3)	2.431667	-1.4775	9.2775	7.211835	7.209466	7.209466
	Fifth (3)	-1.72167	2.189167	-8.56583	6.371967	6.373369	6.373369
	Seventh(3)	-3.365	-5.32083	5.434167	13.43954	13.4384	13.4384

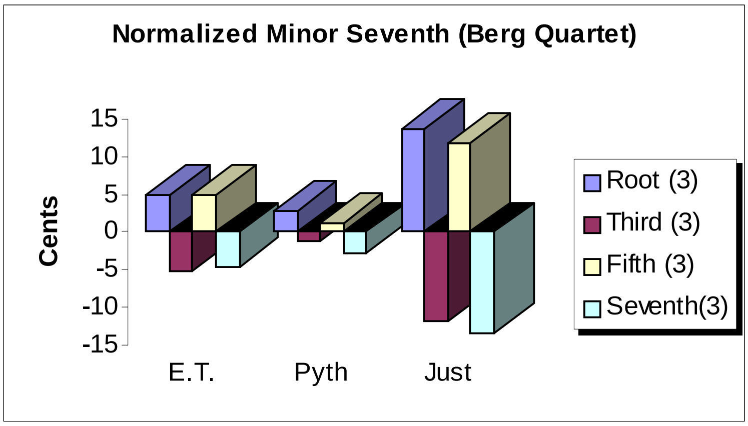

Figure 77. Average values for minor seventh normalized against three systems (Berg Quartet)

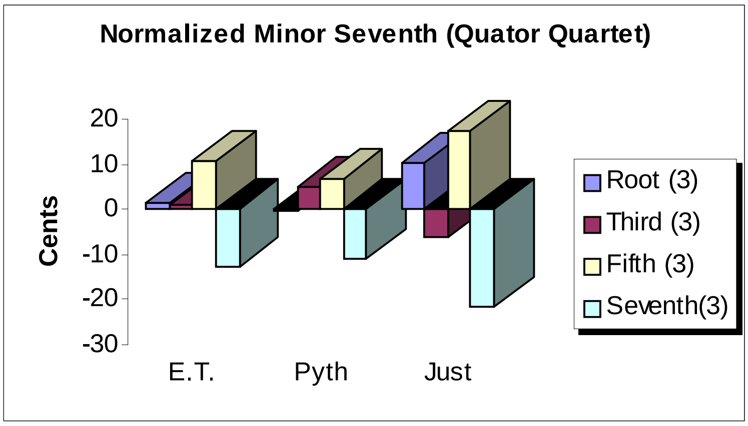

Figure 78. Average values for minor seventh normalized against three systems (Quator Quartet)

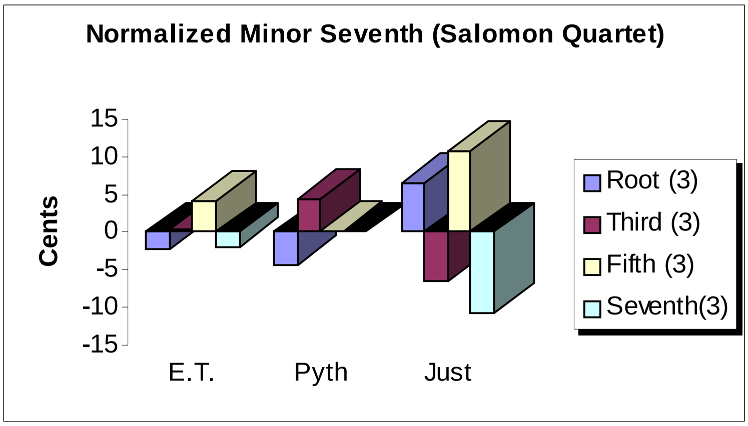

Figure 79. Average values for minor seventh normalized against three systems (Salomon Quartet)

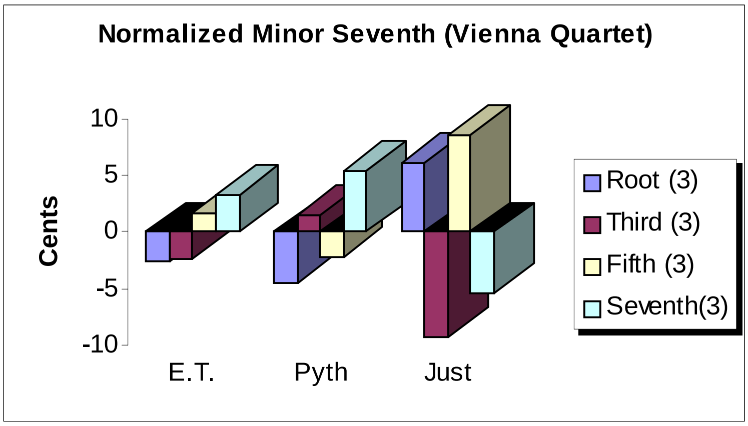

Figure 80. Average values for minor seventh normalized against three systems (Vienna Quartet)

6.10.8 Additional Observations on Analysis II

One reason that major and minor triads, in addition to having more data points available by far than the two seventh chords, are more amenable to analysis is that they are present in clusters, allowing the ensemble to zero in on the tuning gradually, as if one long continuous chord were being played. In contrast, the dominant seventh chord, although twice coming in pairs, changes inversions when going from first inversion to root position, whilst the minor seventh chord occurs only in isolation.

Another reason that the data for the major and minor triads is more reliable is that, on the average, these chords contain notes of longer duration. For example, both major and minor triads appear once apiece as sustained dotted quarter notes, permitting the performers ample time to settle on the desired tuning, whereas both the dominant seventh and minor seventh chords occur only as brief eighth notes.

6.11 Conclusions

Clearly, on the average, when comparisons between observed and predicted data were made in relation to a fixed tonic centre, all of the ensembles were found to be playing in equal temperament. However, as has been demonstrated by analysis of major and minor triads for the Quator Quartet, it is quite possible for an ensemble to play a chord or group of chords in just intonation and still play the overall passage, on the average, in equal temperament. Similarly, the Berg Quartet was found to be playing in equal temperament overall when analysis was based on a fixed tonal centre, but in Pythagorean tuning when the focus was on the individual chords.

The Quator Quartet plays the major third of the dominant seventh almost exactly in equal temperament instead of very nearly in just intonation for the major triad. Each of the other three quartets is playing the thirds very nearly identically for both major and dominant chord types. It would seem that the Quator Quartet is the only ensemble noticeably changing tuning between the two chord types, sharpening the third toward the note of resolution from its previous tuning (taking into account only three of the 26 major triads are themselves dominant function). This ensemble also happens to be the only quartet playing either of these chord types significantly outside of equal temperament, as well as the only one not employing substantial vibrato. A further point of interest regarding major thirds in both major triads and dominant seventh chords would be that, on average, none of the quartets is flattening the major third as low as just intonation or raising it as high as Pythagorean. It would seem that the Quator is the only quartet providing averaged data that lends support to the notion that performers tend to play major third more sharply in dominant function than in major triads. Nowhere in this averaged data is there indication that major thirds are being systematically flattened as a result of being intervals smaller than a perfect fourth, as predicted by Burns and Ward.

Evidence from this experiment is inconclusive for several reasons. Perhaps the most important of these is that the sound has been analysed on the basis of the fundamental frequencies only, which are prone to the effects of inharmonicity when vibrato is used. Because, for instance, vibrato tends to be employed when longer notes are held, which are the same notes that would more likely be played in just intonation, such inharmonicity could substantially contaminate the data in favour of another tuning system. This premise is supported by the observation that the only ensemble shown to play major and minor triads in just intonation is also using less vibrato, although this is an aural and not a formal observation.

More experiments should be conducted on sustained chords and chord progressions, in all inversions, played in a variety of keys and registers, especially without vibrato. This data should be contrasted with data obtained from parallel experiments that employ precisely the same experimental parameters, but with shorter chord durations, to verify or refute the notion that performers tend to play equal temperament when time does not permit fine tuning, and tend to play in just intonation when sustained tones cause audible beating.

Chapter 7
Tuning Database Chord Progression Experiment

7.1 Overview

The Tuning Database Chord Progression Experiment was designed to determine the extent to which listeners' tuning preferences for certain chords and progressions are influenced by selected variable parameters. Parameters to be examined in this study are:

1) Intonation system
2) Duration
3) Timbre

Chord progressions selected and synthesized for this experiment were first electronically catalogued and enumerated from 375 of Bach's chorales. (Please refer to Chapter 4, beginning on p. 65.) For each chord length, ranging from one to seven chords, the most frequently occurring chord or chord progression containing non-repeating chords was chosen from this set of chorales.

7.2 Previous Related Studies

Several previous studies have undertaken the task of determining the effects of certain parameters upon the perception of tuning. In an early experiment by Corso, performers were asked to tune their instruments in unison with five different reference tones: square wave generator, sawtooth oscillator, sine wave oscillator, half wave rectifier and piano.[104] It was determined from this experiment that the harmonic structure of the reference tone had no significant effect on the tuning accuracy. It was further shown that the method of presentation, whether by having the performer tune after the reference tone or simultaneously with it, had no significant effect upon the final tuning given by the performer.

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[104] Corso, J.F. 'Unison Tuning of Musical Instruments,' *Journal of the Acoustical Society of America*, vol. 26 no. 5 (1954), pp. 746-50.

In his Ph.D. thesis, Sisson (1969) performed listening experiments to determine tuning preferences for the following seven intervals: perfect fifth, major third, minor third, major triad, minor triad, dominant seventh and major seventh. In addition to these chords, one chord progression, I-IV-I-V^7-I, was used in the experiments. Results indicated that bending thirds is perceived as less offensive than bending fifths or octaves,[105] which is in agreement with the predictions of Helmholtz, who claimed, as in the following quote, that dissonance due to rapidity of beats is less noticeable in sixths and thirds:

> Hence for Sixths and Thirds the pitch numbers of the notes must be much more nearly in the normal ratio, if we wish to avoid slow beats, than for octaves and Unisons. On the other hand a slight imperfection in the tuning of Thirds brings us much sooner to the limit where the beats become too rapid to be distinctly separable.[106]

Also in agreement with the theory of Helmholtz, as well as with Plomp/Levelt, is Sisson's finding that preference was given more often to just intonation when the material was either harmonic, rich in partials, or both; conversely, equal temperament and Pythagorean tuning were preferred in melodic passages and when simple tones were used.[107]

Roberts and Mathews (1984) performed similar listening experiments based on major (4:5:6) and minor (10:12:15) chords, as well as two non-traditional chords (3:5:7) and (5:7:9).[108] The middle tone was varied by -30, -15, 0, 15 or 30 cents, with the other tones remaining constant. These scaling values were selected in part because the equally tempered major third is 14 cents larger than the just major third, whilst the equally tempered minor third is 16 cents smaller than the minor triad variety of the just minor third.

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[105] Jack Ulness Sisson, *A Comparative Study of Tuning Preferences of Musicians from the Major Performing Areas with Reference to Just Intonation, Pythagorean Tuning, and Equal Temperament*. Ph.D. Thesis, Norman, OK (1969), pp. 18-37.

[106] Helmholtz, op. cit., p. 185.

[107] Sisson, op. cit., p. 77.

[108] Linda A. Roberts and Max V. Mathews, 'Intonation sensitivity for traditional and non-traditional chords', *Journal of the Acoustical Society of America*, vol. 75 (1984), pp. 952-59.

Noted by Roberts and Mathews was the emergence of two distinct groups of subjects having opposing preferences, as illustrated in Figure 81. Results of the first experiment, for instance, in which subjects of varying musical experience were studied, revealed that Group 1, the "rich" group made of nine members (Figure 82), preferred tuning that was off-centre by about 15 cents. Group 2, the "pure" group (Figure 83) in the minority of four members, preferred just intonation.

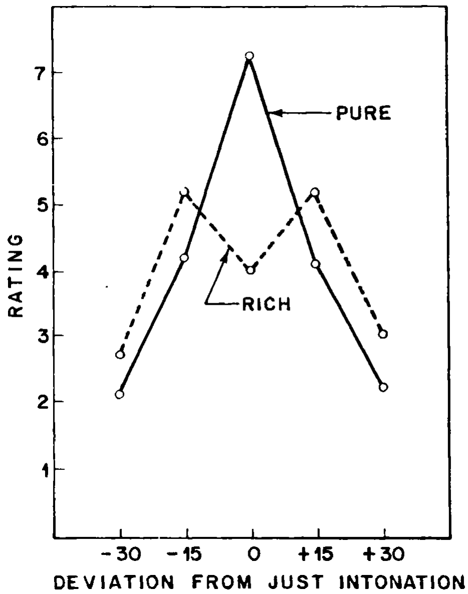


Figure 81. Ratings for the two groups of listeners in experiment 1, averaged across chord type (from Roberts and Mathews, 1984)

In experiment 2, 6 pianists and 6 string players were asked to rate the same chords used in experiment 1. Although the same two groups emerged as in experiment 1, this time the majority (66%) favoured just intonation. Whilst pianists were evenly divided, string players preferred just intonation 5 to 1. It must be emphasized that the non-traditional chords are not part of traditional tonality, and that within the realm of traditional tonality, the only intervals of substance being measured are the major and minor thirds, respectively, of the major and minor triads.

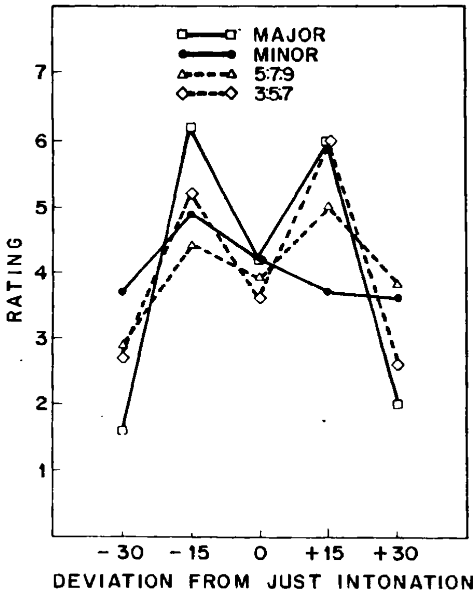


Figure 82. Ratings for the "rich" listeners in experiment 1 (from Roberts and Mathews, 1984)

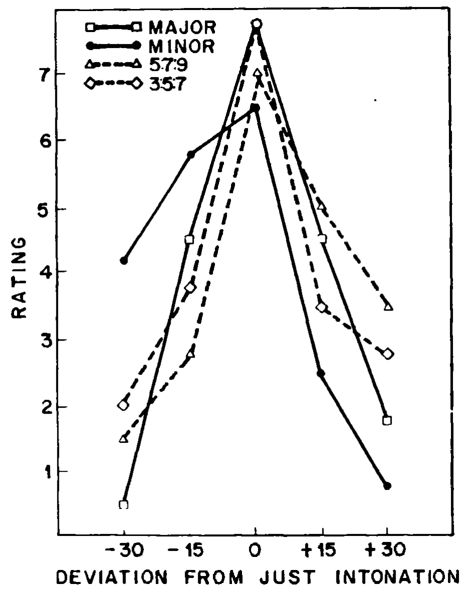


Figure 83. Ratings for the "pure" listeners in experiment 1 (from Roberts and Mathews, 1984)

Two possible reasons cited that rich listeners, such as the professional musicians of experiment 2, chose for the middle note to be tempered an average of 15 cents are the favouring of slow beats similar to vibrato rates and the preference for equal temperament. The latter explanation is consistent with the preferred flatness of the third, relative to just intonation, for the minor triad in both the experiments involving professional musicians, as well as with the preferred sharpness of the third of the major triad by the rich listeners.

The divergence between pure and rich subjects was maintained in experiment 3, in which subjects evaluated individual chords within a progression. Both traditional and non-traditional chords were used, which brings into question the legitimacy of the "chord progressions" used, as they would surely not be expected to elicit the traditional tonal vocabulary of the subjects. Nevertheless, in this experiment, professional pianists and string players both preferred just intonation to equally tempered by 3 to 2 and 4 to 1, respectively.

In a study by Platt and Racine (1985), the effects of both timbre and learning upon tuning were measured by means of listening experiments.[109] Subjects compared an initial tone lasting 1/2 second with a comparison tone of 1 second. Timbres of both initial tones and comparison tones were varied between sine and sawtooth waves. Subjects then adjusted the frequency of the comparison tone until a perceived match with the initial tone was achieved.

In a follow-up experiment, subjects were broken into two groups to compare the effects of learning, with the control group getting no feedback, whilst the other group was told after each tone pair matching what the correct sound should be. The effect of learning was shown to be significant, with overall error being reduced from 20 to 16 cents. Results indicated that complex pitches are perceived as being higher than sine wave tones. As would be expected, matching was more accurate when the two tones were of the same timbre.

[109] John R Platt and Ronald Racine, 'Effect of frequency, timbre, experience, and feedback on musical tuning skills'. *Perception and Psychophysics*, vol. 38 no. 6 (1985), pp. 543-53.

Krumhansl (1990) performed a set of experiments that demonstrated a clear tonal hierarchy of pitches and chords. Experiments were all performed using equal temperament under the assumption that there were 12 possible pitch classes and 24 possible keys. Major and minor keys were treated equivalently, as is the present study.

In Krumhansl's tone completion tests, a tonal hierarchy was demonstrated that was nearly equivalent for major and minor. The circle of fifths was confirmed as a strong predictor of tonal distance. Nearly identical with the values specified by Helmholtz and confirmed by Roberts/Mathews are those given by Krumhansl in her scale of dissonance.[110] Gestalt principles of proximity, similarity and good continuation cited by Krumhansl are proposed predictors of tonal preference. In Krumhansl's chordal studies, listeners much more easily identified the differences between pairs of chord progressions when the single differing chord of the pair was non-diatonic. When pairs of atonal passages were played, subjects were far less able to identify differences between them.

A recent study by Loosen (1995) on violinists, pianists and non-musicians examined musical experience as a predictor of tuning preference.[111] The stimulus consisted of a diatonic C major scale, from C3 to C4, played either in ascending or descending sequence. Timbre was composed of a single generated sawtooth wave that was carefully constructed to avoid over-mechanization. Subjects were asked to state if and which of a pair of scales was more accurately tuned, with pairs being either of the same or a different type. Results from Loosen's study indicated that both violinists and pianists favoured equally tempered versions of the scales overall, but in direct comparisons between Pythagorean and equal temperament, violinists favoured Pythagorean, whereas pianists showed preference for equal temperament. Notable was that violinists rarely failed to find a difference between a pair of examples when a difference in fact existed, whereas pianists were much more often prone to such an error. Non-musicians were shown to be unable to distinguish between the three tuning systems.

[110] Krumhansl, op. cit., pp. 56-57.

[111] Loosen. 'The Effect of Musical Experience on the Conception of Accurate Tuning', *Music Perception,* vol. 12 no. 3 (1995), pp. 291-306.

The present study is most similar to that performed by Sisson inasmuch as it involves chords and chord progressions. As with Sisson's experiments, the interplay between timbre, duration and three traditional tuning systems is explored. The present experiment also shares similarities with the study by Roberts and Mathews, as the tones are very precisely generated for different chord types in order to explore specific listener preferences.

Unlike any of the previous studies, the Bach Chord Progression Experiment is based upon actual chord progressions taken from a musical collection, in this case, the chorales of J.S. Bach. Selected purely on their rate of occurrence, the sound examples employed represent the most common chords and progressions from this vast musical collection.

7.3 Methodology for Tuning Database Experiment

7.3.1 Selection of Chord Progression Examples

For each progression length, ranging in number from 1 - 7 chords, as presented in Table 77, the most commonly occurring chord progression from 375 Bach chorales was electronically selected, as described in Chapter 4 beginning on p. 65. Chord progressions with immediately repeated chords, such as I-I-V-I or I-I-I-V-V, were removed from consideration. For chord progressions containing five chords, there was a three-way tie for the most numerous chord. Similarly, two chords tied for first place when six-chord progressions were examined. In both cases, only one of the candidates ended on tonic, and was selected on that basis.

Table 77. Chord progressions used in listening experiments

Number of Chords	Chord Index Array	Chord Progression	Occurrences
1	208	I	2670
2	775 680	V V^7	524
3	775 680 208	V V^7 I	252
4	778 775 680 208	V(susp. 4) V V^7 I	45
5	208 217 340 297 482	I I^7 ii vii^{o6} I^6	13
6	626 596 775 911 162 208	IV IV^7 V IV^6 $vii^{\phi7}$ I	9
7	208 626 596 775 911 162 208	I IV IV^7 V IV^6 $vii^{\phi7}$ I	9

7.3.2 Choosing a Specific Chord Progression Example

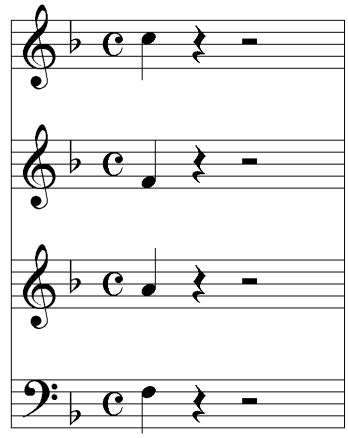
Once the most commonly occurring chord progression was found for each chord length, shown in the "Chord Progression" column of Table 77, the task remained to select an actual example from the database of chorales. This was accomplished by writing two brief programs:

1) FINDPROG, which examines every list file (detailed in Chapter 4, beginning on p. 65) to locate the starting position and file name for every occurrence of the sought chord progression and stores that information as a record in an output text file

2) QRSELECT, which randomly chooses one such specific record to obtain a specific starting location within one of the 375 list files

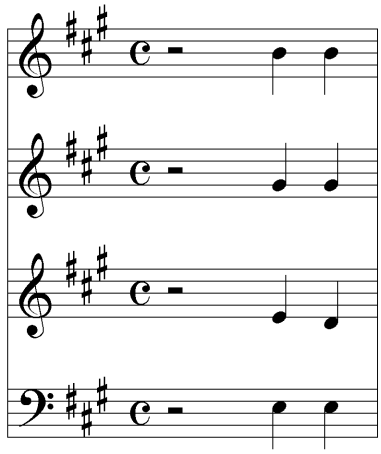
These simple programs ensure that the selection process for chord progressions remains completely unbiased, as well as provide a means of verification that the chords and chord progressions have been enumerated correctly. (Extensive testing revealed no disparity between data collected using the two separate algorithms.)

The chord progressions chosen for the experiment are shown from Figure 84 through Figure 90. The actual keys, pitches and registration used in the experiments are unchanged from the original MIDI files. Rhythm, on the other hand, is discarded in favour of steady blocked chords of equal duration. Except for Figure 85, which is a V-V^7, all of the examples end on a tonic major triad. This is due, save for progressions containing five and six chords, to the fact that the most common progression for that number happen to end on tonic. As mentioned previously, for 5 and 6-chord progressions, tonic cadence was given preference for selection over other candidates having the same number of occurrences. The problematic $vii^{\phi 7}$ chord found in the 6-chord and 7-chord progressions was tuned with its root relating to tonic as a leading tone (octave equivalent of 15/16), and with its third, fifth and seventh tuned, respectively, as root, third and fifth of a minor ii triad. (Please see section 5.10.3, p. 144.)



Figured Bass: I

Figure 84. Most common single chord



Figured Bass: V V^7

Figure 85. Most common two-chord progression

Figured Bass: V V^7 I

Figure 86. Most common three-chord progression

Figured Bass: V$^{(\text{susp 4})}$ V V^{7} I

Figure 87. Most common four-chord progression

Figured Bass: I I^{7} ii vii^{o6} I^{6}

Figure 88. Most common five-chord progression

Figured Bass: IV IV^{7} V IV^{6} vii$^{\phi 7}$ I

Figure 89. Most common six-chord progression

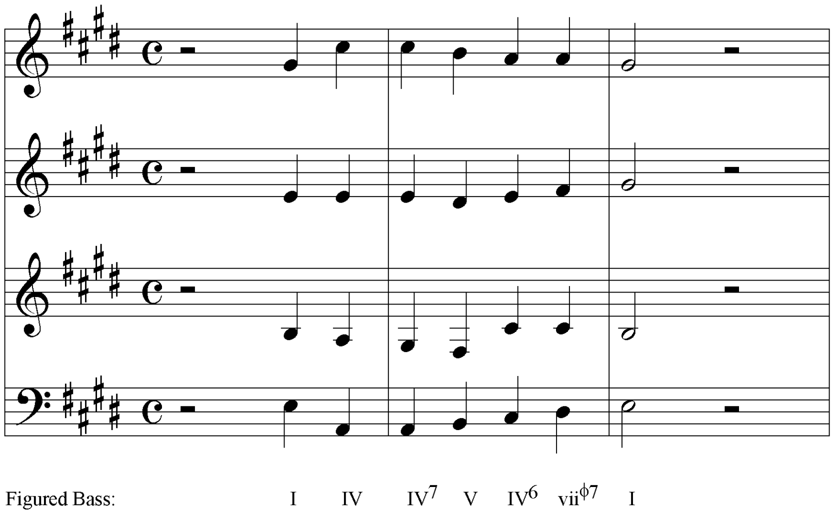


Figure 90. Most common seven-chord progression

7.3.3 Parameters Examined

Each of the seven chord progressions was tuned according to three systems of intonation:

1) Equal Temperament

2) Pythagorean

3) Just Intonation

For each tuning of a chord progression, four versions were played:

1) Sawtooth/Slow

2) Sawtooth/Fast

3) Sine wave/Slow

4) Sine wave/Fast

As shown in Table 78, for each of the seven chord progressions selected, twelve versions were synthesized, bringing the total number of examples to 84.

Table 78. Twelve different example types for each chord progression

	Sawtooth/ Slow	Sawtooth/ Fast	Sine Wave/ Slow	Sine Wave/ Fast
Equal Temperament				
Pythagorean				
Just Intonation				

7.3.4 Cadence

For chord progressions ending on a cadence, which is the case for all but the two shortest examples, the final chord was held for twice the duration of the other chords in the example; thus, progressions whose individual chords had durations of one second were given a cadence lasting two seconds, whereas progressions whose individual chords lasted 0.5 seconds had a cadence lasting one second. The envelope of the cadence was given the same rise time of 10 milliseconds and a decay time of 32 milliseconds, with attack and decay remaining linear.

7.3.5 Apparatus

Chords and chord progressions were first generated by Csound Digital Audio Software to produce wave (WAV) files. These files were stored on a CD and presented both using a CD player and headphones for individual subjects, as well as using speakers for small groups. The sampling rate employed was 44,100 samples per second, with monophonic format being used.

7.3.6 Stimuli

Sounds were generated using Csound's Generation Routine 10, which performs additive synthesis in phase. Sine wave examples were produced using only the first partial, whereas the frequency spectrum for the sawtooth examples was generated using the first 8 partials. Amplitudes of the individual partials for a sawtooth wave are the precise inverses of their partial numbers; thus, partials 1, 2, 3, 4, 5, 6, 7 and 8 were synthesized with the respective relative amplitudes of 1/2, 1/3, 1/4, 1/5, 1/6, 1/7 and 1/8.

Each chord has a rise time of 10 milliseconds and a decay time of 32 milliseconds, with both attack and decay being linear. For shorter chords, the overall duration, including rise time and decay, is 1/2 sec for both sine and sawtooth waves. For longer chords, the overall duration, including rise time and decay, is 1 sec for both sine and sawtooth waves. There are no breaks of silence between chords, so in effect, the chord progressions with short durations are played in steady quavers, whilst the longer ones are steady crotchets.

7.4 Examination

7.4.1 Subjects

Seventeen university students ranging in age from 17 to 50, with an average age of 21.76 years, participated in this experiment. Fourteen of the seventeen subjects were paid for their participation. All subjects were active musicians, with twelve being declared majors in music or a music related field.

7.4.2 Examination Procedure

A brief questionnaire was given at the beginning of the experiment to determine age, departmental studies, musical instrument and number of years playing the instrument. A brief training session with six practice examples was given to ensure that subjects thoroughly understood the instructions.

For the main part of the examination, subjects were asked to listen to a quasi-randomly ordered sequence of chord progressions. They were instructed to rate each chord progression on a scale from 1 - 7, corresponding to the choices Worst, Very Bad, Bad, Average, Good, Very Good and Best. Subjects were not required to remember any previously played chord progression.

7.5 Analysis

The degree to which a chord or chord progression is perceived as consonant or dissonant is contingent not only upon the tuning system in which it is played, but also upon other parameters affecting intonation. No such factor, including intonation system, timbre, duration and musical experience, is an island unto itself, but rather is correlated with the other factors in varying degrees. As this analysis will prove, tuning system preference varies according to these interrelated parameters, most notably to timbre and duration, which will be examined separately and in combination.

7.5.1 Method of Statistical Analysis

Ordinal logistic regression was performed to determine the extent to which the following parameters were able to predict the preferences of the subjects:

1) Tuning system
2) Chord progression
3) Timbre
4) Duration
5) Timbre and duration

The p-values given were obtained by performing the chi-squares test for independence, taking as input two parameters:

1) The value from the chi-squared distribution for the statistic
2) The appropriate degrees of freedom

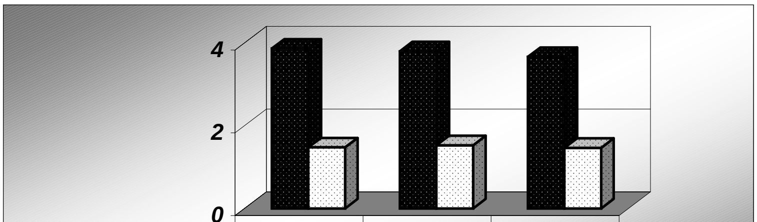
A p-value of 0.05 or lower permits the rejection of the null hypothesis. As an example, Table 79 provides the number of times each rating is given to every tuning system when a long sine wave is employed. The resulting p-value of 0.365 for this particular test indicates that the null hypothesis cannot be rejected.

Rating	E.T.	Just	Pyth	Total
1	2	3	1	6
2	13	8	10	31
3	27	17	29	73
4	24	38	28	90
5	36	28	30	94
6	10	16	16	42
7	7	9	5	21
Total	119	119	119	357
Chi-Square = 13.053, DF = 12, P-Value = 0.365				

Table 79. Observed values for examples using sine wave and long duration

7.5.2 Tuning System as a Predictor of Consonance/Dissonance

The first observation that can be made regarding averaged ratings, given in Figure 91, is that all three tuning systems are similar in overall performance. The difference, on a scale of one to seven, between the most favoured system, equal temperament, and the least favoured, Pythagorean, is only 0.1995, which amounts to less than one seventh of a standard deviation. Much less is the discrepancy between equal temperament and just intonation, which, at 0.0651 on a 1-7 scale, amounts to less than 5% of a standard deviation.



	Equal	Just	Pyth
Mean	3.8634	3.7983	3.6639
St. Dev.	1.4769	1.525	1.4641

Figure 91. Overall rating of each tuning system as predictor of consonance/dissonance

7.5.3 Chord Progression as a Predictor of Consonance/Dissonance

One question of interest is the degree to which the individual chord progressions, whose overall ratings can be seen in Figure 92 and Table 80, and whose musical representations can be seen in Figure 84 through Figure 90, are associated with consonance and dissonance. It can be observed that five of the examples fall into two perceptual groups. For instance, except for the fact it is missing the last chord, the three-chord progression in Figure 86 is the same as the four-chord progression in Figure 87. Similarly, Figure 85 is, except for not having the final chord, the same as Figure 86. Thus, Figure 85 through Figure 87 are all the same progression in various levels of truncation. This truncation effect holds as well for the six and seven-chord progressions in Figure 89 and Figure 90. It is extremely interesting that this phenomenon of embedded chords is taking place, especially that the most common chord progression containing four chords contains embedded versions of the most common progressions containing 1, 2 and 3 chords.

The only chord progression not being a member of a larger group is the five-chord progression, which receives the lowest average rating, and is found in Figure 88. This chord is also notable in that it is one of two chord progression examples, the other being the two-chord progression found in Figure 85, that do not end on a root-position tonic major triad.

It is somewhat inexplicable that the 7-chord progression found in Figure 90 receives the highest average rating, whilst its neighbour, an embedded version of itself found in Figure 89, comes in sixth place. This could be partly due to the fact that the 7-chord progression both begins and ends on tonic; however, this idea is refuted by the 4-chord progression found in Figure 87, which also begins on a non-tonic and comes in second place. The most obvious alternate explanation would be that Figure 89 and Figure 90, whilst having identical figured bass when the first chord of Figure 90 is removed, uses quite different chord spacings, with corresponding changes in the directions of the independent voices. Spacing of voices is almost certainly a factor in dissonance, and would provide a fascinating area of future research.

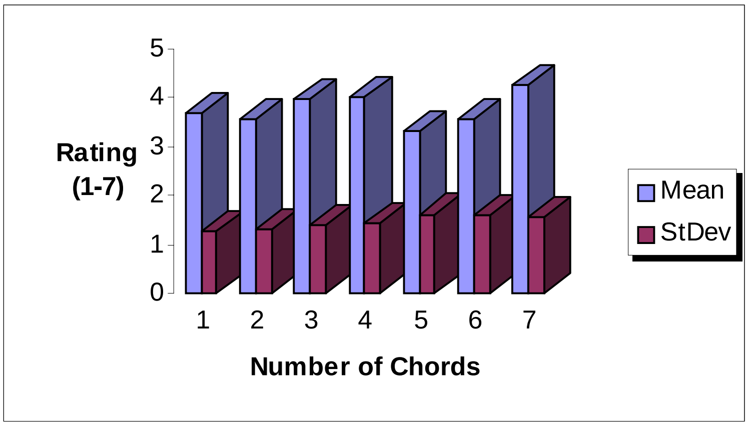


Figure 92. Overall rating of each chord progression as a predictor of consonance/dissonance

Table 80. Overall rating of each chord progression as a predictor of consonance/dissonance

Number of Chords	Mean	StDev
5 Chords	3.314	1.619
6 Chords	3.564	1.616
2 Chords	3.5686	1.2942
1 Chord	3.7059	1.2679
3 Chords	3.9902	1.4002
4 Chords	4.0294	1.4174
7 Chords	4.255	1.574
($p < 0.005$)		

7.5.4 Timbre as a Predictor of Consonance/Dissonance

Of the four parameters under discussion, the one having the most clearly measurable effect on ratings of dissonance is timbre. As seen in Figure 93, there is a strong preference of nearly 4:3 in favour of chord progressions played with a sine wave as opposed to a sawtooth timbre. The p-value is very nearly zero, well below 0.005, and thus the results are highly reliable.

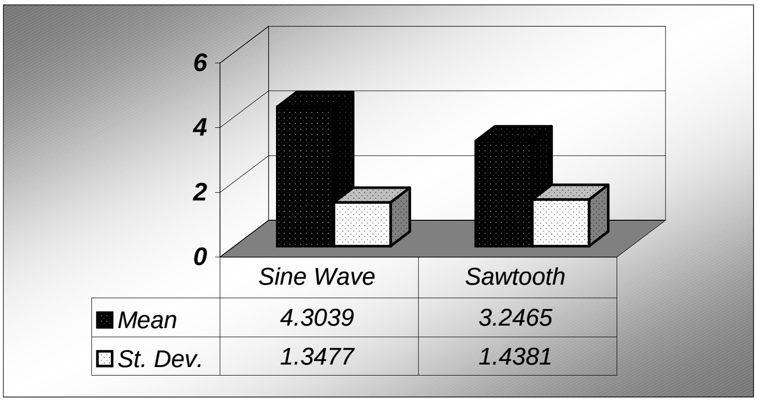


Figure 93. Overall rating of each timbre as a predictor of consonance/dissonance

7.5.5 Duration as a Predictor of Consonance/Dissonance

It is somewhat unexpected that by itself, unlike timbre, duration seems to play a minor role in the subjects' ratings of chord progressions. Figure 94 shows almost no difference when the examples are played with long instead of short durations. The low p-value of 0.012, allowing the rejection of the null hypothesis, would indicate that repeating this experiment would yield the same results. As will be demonstrated, duration combines with timbre to affect the tuning system preference in this experiment.

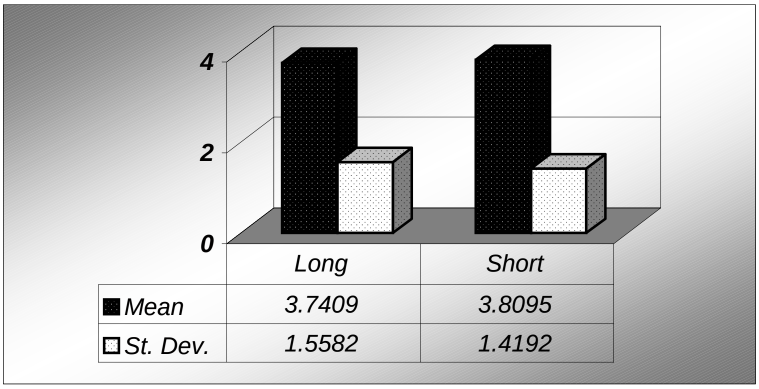


Figure 94. Overall rating of long and short durations as predictors of dissonance

7.5.6 Effect of Timbre on Preference of Tuning System

It is not unexpected that timbre, the single strongest predictor of tuning preference, is the parameter having the largest impact on tuning system preference. What is unexpected, as can be seen in Figure 95 and Figure 96, is that the sine wave timbre appears to inconclusively ($p = 0.63$) favour just intonation, whereas the sawtooth timbre assuredly ($p < 0.005$) favours equal temperament. This trend is contrary to the predictions of Helmholtz, Sisson and Plomp/Levelt, all of whom state that non-justly tuned chords performed using a rich timbre should prove more dissonant than their justly tuned counterparts; however, their predictions were made about individual chords, not chord progressions.

Much more important still is that the precise timbre used is a sawtooth using eight partials, which could very possibly be one of the more dissonant forms of the sawtooth wave. This is especially true when taking into account that the partials are in phase and that the attacks and decays of all of the partials are identical for every note in every chord.

Another somewhat unexpected result is that just intonation is outperforming equal temperament when sine waves are used. Additionally, Pythagorean tuning is not preferred in either of these broad timbral categories. Also notable is that when the sawtooth wave is used, equal temperament is the clear favourite, and Pythagorean tuning is its clear counterpart. There is again, however, a scarcity of information to date regarding tuning experiments that involve chord progressions, as opposed to notes, scales or chords.

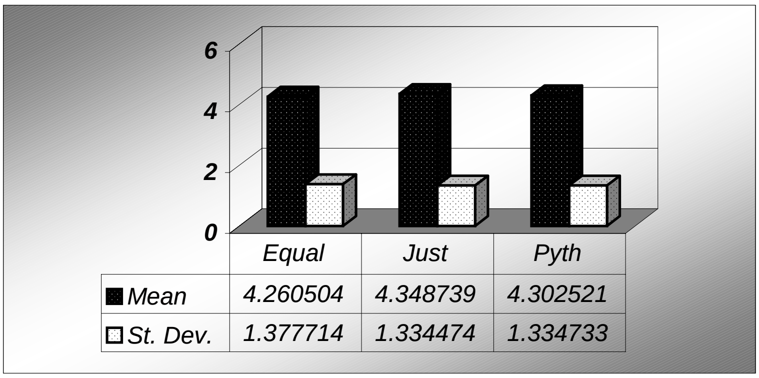


Figure 95. Overall rating of each tuning system when using sine wave timbre

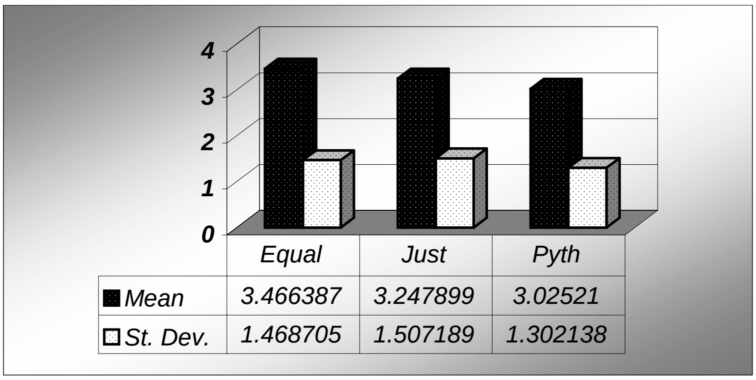


Figure 96. Overall rating of each tuning system when using sawtooth timbre

7.5.7 Effect of Duration on Preference of Tuning System

Consistent with the predictions of Sisson, increasing duration appears to be favourable to the ratings of just intonation, the only system of the three that has a better rating for long than short durations. As can be seen in Figure 97, when the longer tones are employed, just intonation performs marginally better than equal temperament, and substantially better than Pythagorean tuning. The p-value of 0.02 confers confidence in these results. Data obtained for the short duration, shown in Figure 98, inconclusively ($p = 0.56$) places equal temperament ahead of just intonation and Pythagorean tuning.

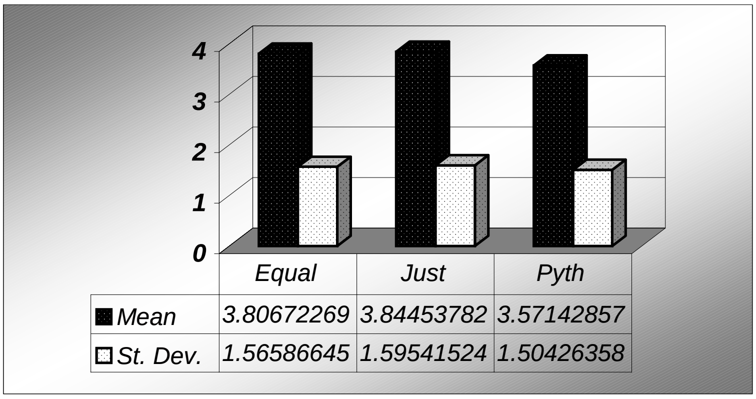


Figure 97. Overall rating of each tuning system when using long duration

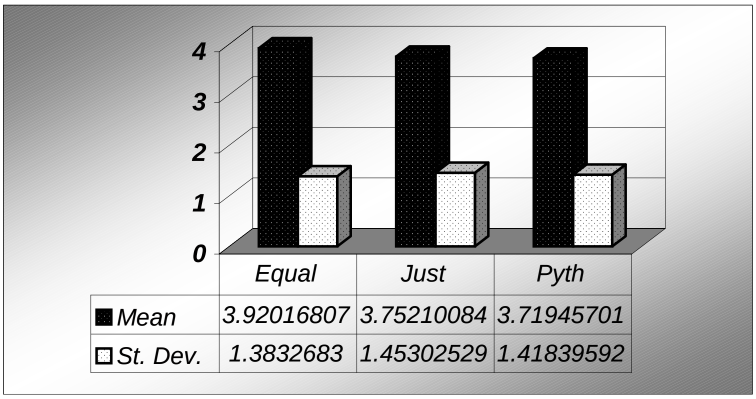
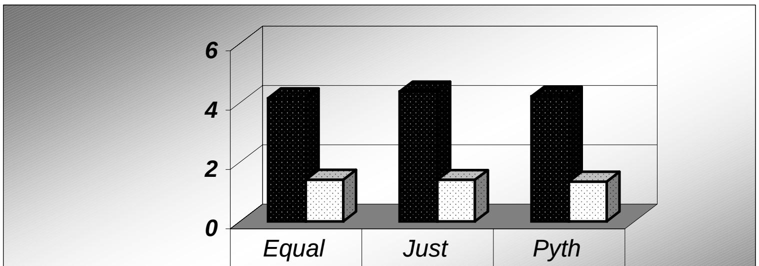


Figure 98. Overall rating of each tuning system when using short duration

Shorter tones fare better than longer ones in the case of Pythagorean tuning, which nevertheless is ranked again in third place when the notes are thus shortened (Figure 98). To a lesser extent equal temperament's favourability increases with shorter durations, whilst just intonation's decreases a similar amount, giving equal temperament a slight edge in this category. The p-value of 0.56 indicates that the parameters examined are not decisively accounting for these results.

7.5.8 Effect of Timbre and Duration on Preference of Tuning System

It can be seen in Figure 99 that just intonation has found its optimal setting when the long sine wave tone is employed. In contrast, equal temperament comes in last place with its lowest relative ranking. A p-value of 0.365 indicates it is not conclusive that the given parameters are actually accounting for the ratings.



	Equal	Just	Pyth
Mean	4.151261	4.378151	4.210084
St. Dev.	1.406032	1.402127	1.33958

Figure 99. Overall rating of each tuning system when using sine wave timbre and long duration

It would seem that there is not enough information for the listeners to make a clear determination when the short sine wave, a chart for which is given in Figure 100, is employed. The p-value of 0.749 points to the strong possibility that the sine and short parameters do not account for the ratings given, which are very nearly flat. It is of interest that the high p-value is associated with the only conditions under which Pythagorean tuning is ever so slightly favoured over the other two systems. This would indicate that there is no experimental condition under which Pythagorean tuning is conclusively preferred over the other two systems.

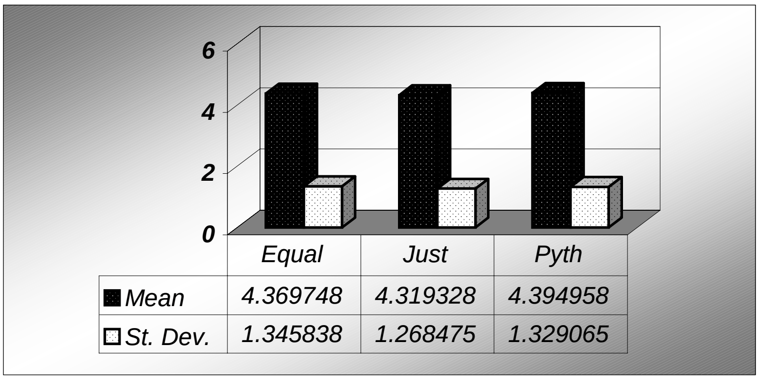


Figure 100. Overall rating of each tuning system when using sine wave timbre and short duration

The most interesting set of parameters under consideration is undoubtedly the long sawtooth wave, as it is not only associated with preference for equal temperament instead of the presumed system just intonation, but also garners the p-value associated with very high confidence, 0.006, indicating that the parameters are indeed accounting for the ratings given. As can be seen in Figure 101, equal temperament is doing a slightly better job than just intonation under these circumstances, and Pythagorean tuning is falling into its most distant third place.

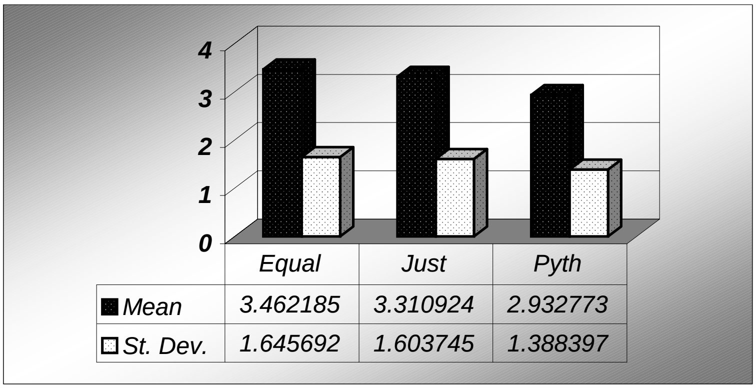


Figure 101. Overall rating of each tuning system when using sawtooth timbre and long duration

It is apparent in Figure 102 that equal temperament easily outperforms just intonation when short sawtooth waves are used, whilst the shortcomings previously seen with Pythagorean tuning involving the long sawtooth wave are less apparent with the shorter sawtooth timbre. A p-value of 0.054 is very close to conclusive that the parameters actually are accounting for the ratings obtained.

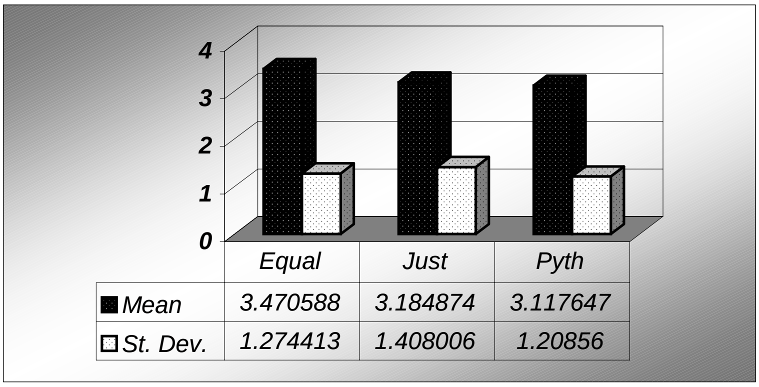


Figure 102. Overall rating of each tuning system when using sawtooth timbre and short duration

7.5.9 Tuning System Preferences of Pianists/non-Pianists

One might assume that pianists would be more likely to prefer equal temperament over other systems, and might even tend to least favour just intonation. Loosen's experiments in melodic listening,[112] as well as in performance,[113] indicate a decided preference for equal temperament by pianists, with Pythagorean tuning ranking a close second.

To determine if playing piano had an effect on the ratings given for each of the tuning systems, the data was reprocessed for the two groups corresponding to those who listed piano as their primary instrument and those with a primary instrument other than piano. Six of the seventeen subjects participating in this experiment listed piano as their primary instrument, as shown in Table 81. One subject, number 17, listed piano as a secondary instrument. Of the six

[112] Loosen. 'The Effect of Musical Experience on the Conception of Accurate Tuning', *Music Perception*, vol. 12 no. 3 (1995), pp. 291-306.

[113] Loosen, 'Intonation of solo violin performance with reference to equal-tempered, Pythagorean and just intonations,' *Journal of the Acoustical Society of America*, vol. 93 (1993), pp. 525-539.

pianists included in the piano group, it should be noted that only three are music majors, and of these, one is a music education major.

Whilst on the surface, it would seem a contradiction that pianists prefer just intonation and non-pianists prefer equal temperament, it should be taken into account that only six are included in the pianist category, decreasing reliability. Another possible explanation is provided by Loosen (1995), who discovered that pianists were much more likely to find no difference between scales tuned in equal temperament and Pythagorean tuning, which is not so surprising considering intonation does not directly affect the performance habits of pianists. Finally, the p-value of 0.119 obtained from the results for pianists does not allow rejection within that group of the null hypothesis for independence; in contrast, the p-value of 0.009 obtained from the results for non-pianists does confer confidence in their results.

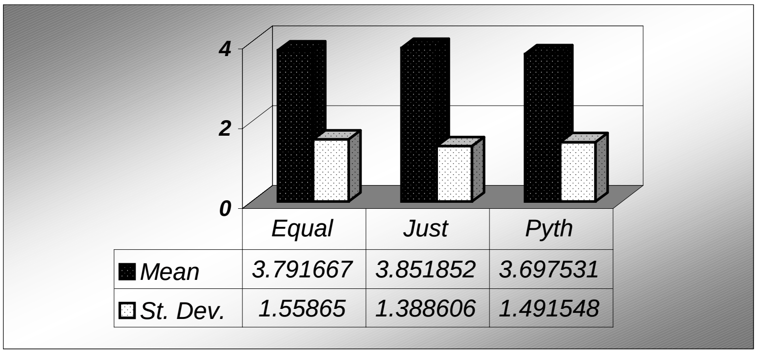


Figure 103. Overall ranking of each tuning system by pianists (5)

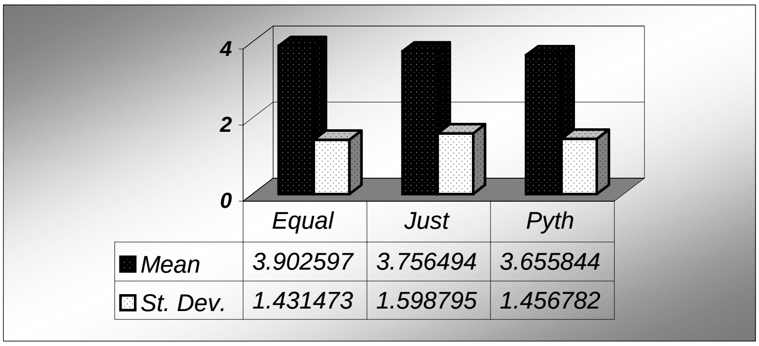


Figure 104. Overall rating of each tuning system by non-pianists (12)

Table 81. Stated instruments of seventeen subjects

Subject	Instrument	Departmental Studies
1	Piano	Music
2	Oboe	Music
3	Guitar	Music
4	French Horn	Music
5	Trumpet	Biology and English
6	Guitar	Music/Business
7	Piano/Clarinet	Pre-Music Therapy
8	Piano, Voice	Undecided
9	Violin	Psychology
10	Voice/Sax	Music (Vocal Performance)
11	Voice	Music (Vocal Performance)
12	Piano	Music Education
13	Piano, Violin, Viola	Music Performance
14	Piano, Accordion	Computer Science, French, Math
15	Flute	Undecided
16	Voice	Music Education
17	Voice/Violin/Piano	Music Performance

7.5.10 Tuning System Preferences of Music Majors/Non-Majors

For completeness, the results for music majors, numbering eleven, are contrasted with those obtained for the six non-majors. As shown in Figure 105 and Figure 106, both groups favour equal temperament, but the music majors only by the narrowest of margins. The p-values for majors and non-majors were 0.09 and 0.03, respectively, with the former being just outside and the latter just inside the value of confidence (0.05). As in all but the single insignificant case of the short sine wave, Pythagorean tuning was the least favoured of the tuning systems.

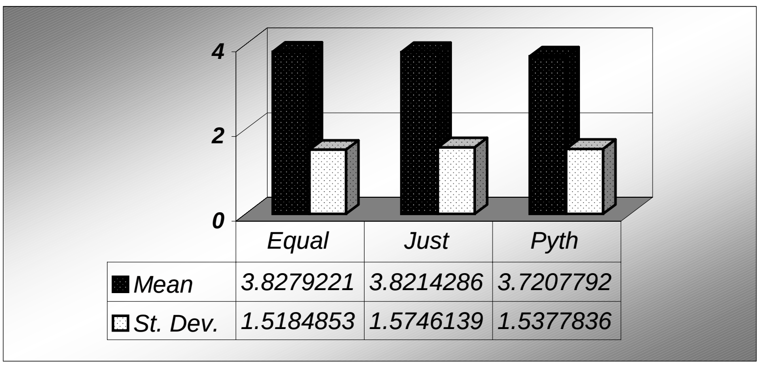


Figure 105. Tuning system preferences of music majors

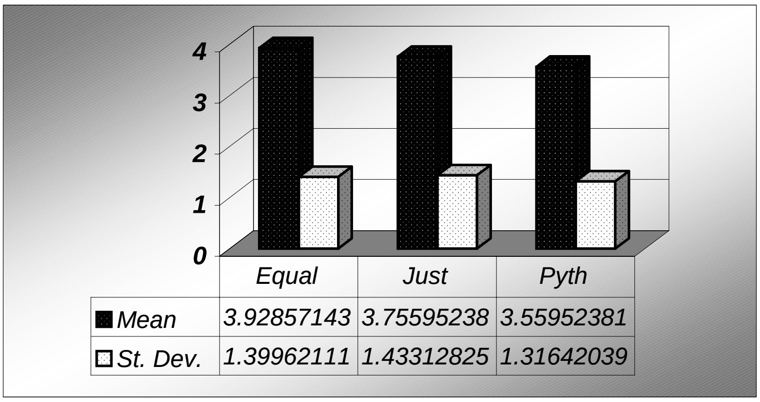


Figure 106. Tuning system preferences of non-music majors

7.6 Possible Improvements to Experiments

The sounds used in these experiments are quite mechanical and uniform. For example, the onset of attack is precisely 10 milliseconds and the decay exactly 32 milliseconds, which will ensure a very mechanical and somewhat unmusical sound. Furthermore, only partial series in phase and with identical amplitudes are used, in contrast to musical tones that are composed of dynamically changing partials not necessarily in phase. Finally, there is no vibrato, reverberation or other fluctuation in frequency typical of musical sounds.

Several remedies are available for the overly mechanized sounds that were noted by some of the subjects, and which might have accounted for the fact that all but two subjects rated the average chord progression as falling short of the "average" rating. First, the attacks and decays could be randomly varied, as could the starting times and durations. As for counteracting the uniformity of phase and pitch, one approach would be to employ an FM signal with built-in random pitch variation.

One further possible refinement to these experiments became apparent after they were already in progress, namely, that the seventh partial contained in the sawtooth examples could have been omitted. According to theorists ranging from Zarlino to Rameau, and from Helmholtz to Plomp and Levelt, the seventh partial itself is a source of dissonance. According to Helmholtz, for instance, the construction of the piano is heavily influenced by the principle of minimizing partials higher than the sixth. It is quite possible that the differences between tuning systems, especially if varied durations and timbres were used, would be more apparent and that the corresponding preferences would be different if the sawtooth examples were synthesized using only the first six partials.

7.7 Considerations for Future Experiments

It is a virtual certainty that the sawtooth wave used in these experiments is more favourable to the relative ratings of equal temperament than of just intonation or Pythagorean tuning. It is also apparent that when the simple sine wave is used instead, both just intonation and Pythagorean tuning are more often preferred.

A central question that should be addressed is that of the effects obtained by calibrating the number of partials in the sawtooth so that they fall in between the values of 1 and 8 used in this experiment. Two specific numbers of partials that would be interesting to explore for the sawtooth wave are 3 and 5, corresponding to the respective maximum allowed prime generators of Pythagorean tuning and just intonation.

The connection between the number of partials to use in a sawtooth wave and the highest prime generator of the tuning system is not a superficial one. Take, for example, a major triad tuned in Pythagorean tuning using the sawtooth wave. There is a strong clash between the first partial of the triad's third and the fifth partial of its root. Similarly for a dominant seventh chord using the same timbre, the seventh partial of the chord's root very much clashes with the first partial of its seventh (octave equivalent of 16/9 in relation to its root), whether with just intonation or Pythagorean tuning. Because the equivalent seventh of equal temperament is approximately four cents sharper than in either of the other two systems, it will be further separated from the seventh partial and, as a result, possibly less dissonant.

It should be noted that the seventh partial should clearly be causing critical bandwidth effects with the other partials of its tone, as well as with fundamentals of other tones, in both just intonation and Pythagorean tuning. It is highly possible that the ideally periodic relationships between notes contained in, say, a justly-tuned major triad provide the perfect background against which the extreme dissonance of the seventh partial can express itself, whereas the notes in the equally tempered version of this chord, being out of phase and obscuring the tuning of the major third in particular, provide no such optimal background.

It is clear that further experiments need to be performed on chords and chord progressions, with special emphasis on variation in timbre. It might well turn out that the wave design given by Helmholtz involving only the first five or six partials would be optimal for just intonation. It is almost certain that the sine wave is not the only waveform that can outperform the 8-partial sawtooth wave when just intonation and/or Pythagorean tuning are used, and it is quite possible that the sine wave is not the waveform most conducive to achieving these objectives.

Using the central processor as a guiding principle does not exclude tones having more than six partials from being considered consonant. Including partials that are octave equivalents of the first six partials, for example, should not add dissonance, as the separation between partials of at least a critical band will be maintained. It is possible that avoiding only partials whose numbers are either primes greater than five, or multiples of such primes, will permit the resulting sonority to be identified as a consonant entity by the central processor.

7.8 Concluding Remarks

This experiment yields significant results as to the authenticity of 5-limit just intonation being the true basis of tonality. First of all, just intonation outperforms Pythagorean in nearly every category, most notably when the long sawtooth timbre is employed. Further, in every category, just intonation is competitive with equal temperament, particularly when simple waves and long waves are used.

Throughout this study, it is found that lengthening the duration or increasing the number of partials, especially in combination, decreases the p-value, thus increasing the reliability of the results. It must be admitted that the timbre associated with the highest level of confidence, the 8-partial sawtooth wave, favours equal temperament. With regard to this statistical fact, there are three significant factors at work that fall outside the realm of this experiment:

1) The seventh partial is creating dissonance that destroys the otherwise harmonious timbre, thereby clouding the effect of harmony produced by the precise ratios of just intonation.

2) The seventh partial has a significant though lesser effect on equal temperament, a tuning system that simultaneously approximates the most important intervals whilst blurring all intervals, rendering the more dissonant intervals less noticeable.

3) Equal temperament has an edge at the outset, as it is the system with which all of the subjects are most familiar in musical settings. If this effect of musical indoctrination is significant at all and could somehow be eliminated, just intonation would almost certainly be the most widely chosen under all, not just half, of the conditions tested.

Considering that equal temperament is historically derived from Pythagorean tuning, it is curious that under experimental conditions equal temperament performs much more similarly to just intonation than to Pythagorean tuning. One possible explanation as to why equal temperament and just intonation are so very nearly matched under all conditions is that the former is mimicking the latter, or, put more directly, they are perceived as roughly the same system under the conditions given. In any event, Pythagorean tuning, even though it more nearly predicts the tunings used by string players than does just intonation, is simply not the system preferred overall by listeners in this experiment.

Chapter 8
Conclusions

Clearly, the assertion that ratios are at the heart of traditional Western tuning systems is an irrefutable one. Even the modern standard, equal temperament, must be recognised as a tuning system originally derived from Pythagorean tuning, itself a subset of just intonation.

The history of tuning provides many clues as to which frequency relationships determine consonance and dissonance. During the second century A.D., Ptolemy, the final and authoritative ancient Greek theorist of note, was systematically employing 5-limit just intonation in scales whose tunings are compatible with those used in the present study. In the early seventeenth century, Mersenne documented the most important intervals of 5-limit theory. A century later, his countryman Rameau was describing individual chords, as well as scales, in terms of ratios. These three key figures all derived their consonant intervals and chords strictly from ratios whose numerators and denominators can be factorised into the first three prime numbers 2, 3 and 5.

A pivotal figure in tuning is Helmholtz, a nineteenth-century scientist and physician who saw the connection between 5-limit just intonation and the limitations of the cochlea. His dissonance theory, based upon the beating of partials, paved the way both for scientific investigations of the auditory system and for modern tuning theory.

A key breakthrough by Plomp and Levelt in 1965 demonstrated beyond doubt that partials within a critical band interfere with each other. An important conclusion of their work is that, in general, the highest two consecutive partials not sharing a critical band are the fifth and sixth. Their calculations revealed that when one six-partial sawtooth wave is gradually separated in frequency from the other, the only consonant intervals within the first octave are 5:6, 4:5, 3:4, 2:3, and 3:5, all basic ratios of 5-limit theory.

A wealth of additional psychoacoustical data points to the first five to six partials as the spectral region of a periodic tone most closely associated with pitch perception. The study of harmonic coincidence indicates that consonant intervals are those that share common lower harmonics. Research by Ritsma and Engel on the dominance region reveals that partials 3-5 are the principal determinants of pitch for a harmonic complex. Moore indicates that there may be little or no phase locking to weak components which are close in frequency to stronger ones, and that such weak components are likely masked from the overall time pattern of nerve impulses. The most recent and comprehensive theory of pitch perception, virtual pitch, is based on the concept that a learning matrix of integer relationships is formed during the early stages of speech acquisition. Terhardt, the originator of virtual pitch, proposes that the third through sixth harmonics are dominant in the pitch identification process.

Further support of 5-limit just intonation is found through a survey of the chords used in the Bach Chorale Tuning Database, the results of which indicate Bach is composing music based upon the integer relationships between notes and their partials. Within this collection of chorales, extreme preference is demonstrated by Bach for major and minor triads, with the dominant seventh also playing a key role. Major and minor triads happen to be the simplest chords in 5-limit just intonation, and are anything but the simplest in Pythagorean tuning. For example, the closed root position major triad in just intonation is tuned as 4:5:6; in Pythagorean tuning, the proportions are 64:81:96, which are obviously not the relationships predicted by any model of a central processor.

Equal temperament's equivalent tuning for the major triad, which cannot be described in terms of integers, is $1:\sqrt[12]{2}^{4}:\sqrt[12]{2}^{7}$, also not likely to be the relationships identified by a central processor, certainly not under the universal assumption that such a processor identifies simple periodic relationships. It is not necessary to even invoke psychoacoustic theory to dismiss the tuning of this or any other chord in equal temperament, as not a single period of a chord tuned in such a manner can be heard from beginning to end in a finite length of time.

Evidence from the intonation analysis of four professional string quartets is not decisive. Although on the average equal temperament and Pythagorean tuning are being used more often than just intonation, the quartet using the least vibrato was shown to actually play in just intonation for major and minor triads when a method of analysis focusing on specific chords was employed. There are at least four reasons string ensembles seem to avoid just intonation during performance:

1) The instruments' strings, tuned in Pythagorean tuning, are reference points of intonation.

2) The timbres produced are similar to the sawtooth waves used in the listening experiments, encouraging the adoption of a system that blurs dissonance.

3) When vibrato is used, inharmonicity increases for the fundamental, affecting the analysis of the fundamental frequency.

4) The performers have adapted to equal temperament, both from musical indoctrination and from acquired performance practices that avoid overall dissonance in a wide range of settings.

Listening experiments involving short chord progressions confirm equal temperament's extreme similarity to 5-limit just intonation. This is expected in spite of the fact that equal temperament is derived from Pythagorean or 3-limit just intonation, as the "bent" intervals of equal temperament approximate not only 5-limit just intonation's perfect fifth (3:2) and its inverse the perfect fourth (4:3), but also its major and minor thirds (5:4 and 6:5),[114] along with their respective inverses the minor and major sixths (8:5 and 5:3). While equal temperament and just intonation perform almost identically in listening experiments, Pythagorean tuning,

[114] There are actually two types of minor thirds, 6:5 and 32:27, used in 5-limit just intonation. The former represents the relationship between the fifth and the third of a major triad, whilst the latter corresponds to the interval between the seventh and the fifth of a dominant seventh chord.

from which equal temperament is derived, is simply not favoured when compared with the other two systems, particularly when complex tones are used.

The first conclusion that may be drawn from the Tuning Database Chord Progression Experiment is that just intonation performs almost as well as equal temperament under nearly all combinations of parameters. This alone is enough to validate 5-limit just intonation, based as it is on the simplest possible ratios that model the most commonly employed chords, as the true origin of tonality. A somewhat surprising result of this experiment is that equal temperament appears not to be favoured when simple tones are used, as predicted by conventional wisdom, but when complex tones are employed. Equally surprising is the preference for just intonation in the broad categories of simple tones and long tones, again completely contrary to expectation.

The timing theory of Boomsliter and Creel can account for some of these results if one considers that in the case of the simple timbre, just intonation is providing the least amount of noticeable interference, as well as the simplest overall pattern. In contrast, the specific complex timbre used, an 8-partial sawtooth wave, creates overly complex long patterns in the tones themselves that can be blurred by equal temperament, but are accented by just intonation and especially Pythagorean tuning. Such conclusions might seem to overestimate the amount of dissonance generated by the relatively weak seventh partial in a sawtooth wave, but according to Vos and Vianen, changing the timbre from sawtooth to a timbre having equal amplitude harmonics does not appreciably change the perception of mistuned intervals.[115]

The most practical information gained through the listening experiments, in agreement with the work of Sethares, is that timbre and tuning system are intimately interrelated. Additional experimentation might prove that when the sawtooth wave is used, just intonation will perform optimally, relative to other systems, with 5-6 partials, whereas Pythagorean tuning, using the sawtooth wave type, should perform at its peak using three.

[115] Vos and van Vianen, op. cit., pp. 176-87.

Recent attempts to automate tuning have been shown to be flawed. Automation of tuning, as with analysis, must take place at multiple levels. Finding combinations of chords that mutate in order to avoid beating of partials in the next chord, whilst technically complex, is musically over-simplistic. Certain functions within the tonal hierarchy, most notably tonic and dominant, must be given priority over all others. The precise rules that can be recursively applied to achieve prioritised tunings at every level of tonal structure are yet unknown. Even if they were known in complete detail and could be applied unambiguously, they could not be used to simply tune an entire repertoire of even a single composer, certainly not of Bach, whose chorales contain identifiable contradictions within 5-limit just intonation, and would certainly contain such contradictions no matter which ratios were chosen.

Finally, it has been demonstrated that a relatively small number of heuristics, based upon the use of simple ratios derived from the first three prime numbers, can be used to tune entire pieces of music. In order for these heuristics to succeed, the pieces must be highly tonal; the tonal centre must be preserved by prioritising tonal functions; and the timbre must be compatible with the chords being played. Extending these heuristics and coupling them with multi-level analysis programs will no doubt provide invaluable tools with which to explore tonal theory and composition.

Chapter 9
Future Directions

The immediate future of tuning is extremely exciting, as tuning systems can be implemented with complete precision by means of digital audio processing. Tonality is a truly open-ended structure even when its frequencies are derived purely from the first three prime numbers.

9.1 Detailed Listening Experiments

9.1.1 Exclusion of Prime Generators Greater than Five

Can small prime numbers greater than five be categorically excluded as generators of tonal tuning systems? The presumed answer, from both historical and experimental evidence, is that they can. Detailed listening experiments should be performed to compare 5-, 7- and 11-limit just intonation in order to confirm what many celebrated tuning specialists have been stating for hundreds of years, namely, that only the first three primes generate consonant intervals.

Based upon the results obtained from the Chord Progression Experiment in the present study, Pythagorean tuning can be eliminated completely when using complex tones in chord progression experiments. Equal temperament should continue to be used as a standard of comparison, especially as it has interesting properties related to blurring, and therefore avoiding the perception of, beating between upper partials of two or more tones. With regard to using equal temperament in such experiments, two points should always be kept in mind:

1) The results will generally be skewed in favour of equal temperament, the system in which Western musicians are indoctrinated.

2) On purely theoretical grounds, equal temperament can be excluded from consideration as being the ideal.

9.1.2 Timbre Optimisation

Once, as predicted by nearly all related psychoacoustic research, tuning systems based on prime generators greater than five are eliminated from practical consideration, a key topic to address will be that of determining how various timbral properties, including partial strength and distribution, duration, amplitude and register, interact to increase or decrease consonance. Such specific knowledge could be used by composers and arrangers to design electronic instruments that are optimal for conveying a sense of harmony within a given passage.

Because major and minor triads, as well as dominant seventh chords, have been shown to play key roles in tonality, they would serve as ideal sound examples for future listening experiments involving individual chords and chord progressions. Keeping the listening examples brief, perhaps using only one or two chords, would permit several different timbres, especially those involving combinations of partials 1-6, to be explored in detail.

9.1.3 Varying Timbre According to Chord Type, Inversion and Spacing

Some very interesting and uncharted territory in tuning pertains to varying timbre according to the specific chord being played. Helmholtz briefly mentioned that voice spacing affects the level of consonance, a view supported by the casual observation that densely packed chords are more dissonant than those with more widely spaced notes. No doubt the related issues of chord type and chord inversion will also affect the level of dissonance. To explore these interrelated issues, several chords should be prepared for experiment by varying the parameters of inversion, register, spacing of voices and number of partials. It is quite possible the results from listening experiments would indicate that when tightly packed notes are present, simpler waves will be generally indicated, whilst open spacings will be more often be associated with rich timbres. According to experimental results, timbre could thus be electronically adjusted to the specific chord or group of chords at hand.

9.2 Automated Tuning

The rule based system given in "Heuristics for Tuning Chords", p. 145, for tuning chords according to tonal function can be used as a paradigm for a larger set of heuristics that operate on tonal passages. As noted by Ferková, chords found in transitional sections are problematic in terms of tonal analysis,[116] and are equally problematic for a program that tunes them.

Applying these heuristics at a higher level requires a multi-level tonal analyser, such as that of Taube,[117] that can accurately identify each chord type as it fits within the progression, passage, subsection, section and key of the piece. This is an enormous task, and quite impossible to implement with 100% accuracy, as the passages themselves will contain perhaps a large number of contradictions, themselves occurring at more than one level.

No matter how automated tuning is implemented, it must take into account that certain relationships, such as tonic and dominant, are more important than others, and rely on a set of principles that orders the tonal hierarchy in an unambiguous manner. It must also strictly know the tonal function of every chord, as well as the key of every higher-level function in which the chord resides.

[116] Ferková, op. cit., p. 85.
[117] Taube, op. cit., p. 27.

Appendix I: Tables for Mozart Quartet Experiments

Table 82. Raw frequency data from four string quartets playing opening of Mozart's K. 590, II

Instrument	Start	Dur	Freq (E.T)	Freq (Pyth	Freq (Just)	Note	Berg	Quator	Salomon	Vienna
Vl 1	0.0	0.5	329.62756	331.1199	327.032	E4	330.21	319.87	321.59	330.56
Vl 2	0.0	0.5	261.62556	261.62556	261.62556	C4	261.77	255.42	255.70	264.54
Vla	0.0	0.5	195.99772	196.2192	196.2192	G3	197.30	190.56	191.61	198.11
Cel	0.0	0.5	65.40640	65.40640	65.40640	C2	65.77	63.605	63.806	66.17
Vl 1	1.0	0.5	329.62756	331.1199	327.032	E4	330.33	319.62	322.94	331.43
Vl 2	1.0	0.5	261.62556	261.62556	261.62556	C4	262.83	254.40	255.83	265.22
Vla	1.0	0.5	195.99772	196.2192	196.2192	G3	198.43	191.11	191.43	198.04
Cel	1.0	0.5	65.40640	65.40640	65.40640	C2	66.149	63.741	63.813	66.175
Vl 1	1.5	0.5	329.62756	331.1199	327.032	E4	328.67	319.58	321.52	332.17
Vl 2	1.5	0.5	261.62556	261.62556	261.62556	C4	260.91	255.39	255.60	264.79
Vla	1.5	0.5	195.99772	196.2192	196.2192	G3	198.43	190.97	191.43	198.59
Cel	1.5	0.5	65.40640	65.40640	65.40640	C2	65.600	63.497	63.813	65.47
Vl 1	2.0	0.5	329.62756	331.1199	327.032	E4	330.12	319.62	322.49	331.61
Vl 2	2.0	0.5	261.62556	261.62556	261.62556	C4	262.65	255.63	255.57	264.80
Vla	2.0	0.5	195.99772	196.2192	196.2192	G3	195.67	190.89	188.31	198.52
Cel	2.0	0.5	65.40640	65.40640	65.40640	C2	65.55	63.753	63.855	66.25
Vl 1	2.5	0.5	329.62756	331.1199	327.032	E4	330.11	320.00	321.87	331.96
Vl 2	2.5	0.5	261.62556	261.62556	261.62556	C4	261.73	255.67	255.6	264.73
Vla	2.5	0.5	195.99772	196.2192	196.2192	G3	196.85	192.50	191.64	198.80
Cel	2.5	0.5	65.40640	65.40640	65.40640	C2	66.285	63.742	64.01	65.214
Vl 1	3.0	1.5	391.99544	392.4384	392.4384	G4	391.7	381.36	384.21	390.17
Vl 2	3.0	1.5	261.62556	261.62556	261.62556	C4	261.52	255.56	255.86	264.04
Vla	3.0	2.0	195.99772	196.2192	196.2192	G3	195.76	190.95	192.28	198.61
Cel	3.0	1.5	82.40689	82.7800	81.7580	E2	82.624	79.769	79.824	81.932
Vl 1	4.5	0.5	293.66477	294.3288	294.3288	D4	295.81	289.13	287.92	298.15
Vl 2	4.5	0.5	246.94165	248.3399	245.274	B3	248.09	239.44	241.30	246.70
Vla	4.5	0.5	195.99772	196.2192	196.2192	G3	196.44	192.75	189.84	197.31
Cel	4.5	0.5	97.99886	98.1096	98.1096	G2	97.494	96.111	94.561	99.055
Measure 2										
Vl 1	6.0	0.5	349.22823	348.8341	348.8341	F4	349.38	342.18	340.62	354.25
Vl 2	6.0	0.5	293.66477	294.3288	290.6951	D4	296.32	285.95	285.19	295.39
Vla	6.0	0.5	220.00000	220.7466	218.0213	A3	220.37	214.82	217.12	221.76
Cel	6.0	0.5	73.41619	73.5822	72.6738	D2	73.486	71.453	72.378	74.159
Vl 1	7.0	0.5	349.22823	348.8341	348.8341	F4	349.28	342.49	342.33	353.71
Vl 2	7.0	0.5	293.66477	294.3288	290.6951	D4	296.63	285.94	286.01	298.15
Vla	7.0	0.5	220.00000	220.7466	218.0213	A3	221.86	215.76	217.21	221.69
Cel	7.0	0.5	73.41619	73.5822	72.6738	D2	73.657	71.518	71.897	73.495
Vl 1	7.5	0.5	349.22823	348.8341	348.8341	F4	349.65	343.59	344.02	354.78
Vl 2	7.5	0.5	293.66477	294.3288	290.6951	D4	294.98	286.52	288.76	298.26
Vla	7.5	0.5	220.00000	220.7466	218.0213	A3	221.95	215.96	215.88	223.23
Cel	7.5	0.5	73.41619	73.5822	72.6738	D2	73.532	71.889	71.315	73.585
Vl 1	8.0	0.5	349.22823	348.8341	348.8341	F4	349.16	343.75	343.94	355.10
Vl 2	8.0	0.5	293.66477	294.3288	290.6951	D4	296.5	286.38	286.34	295.60
Vla	8.0	0.5	220.00000	220.7466	218.0213	A3	221.89	215.69	216.05	223.60
Cel	8.0	0.5	73.41619	73.5822	72.6738	D2	73.6	72.096	72.033	74.301
Vl 1	8.5	0.5	349.22823	348.8341	348.8341	F4	349.81	344.56	342.11	355.50
Vl 2	8.5	0.5	293.66477	294.3288	290.6951	D4	293.84	287.79	286.49	296.85
Vla	8.5	0.5	220.00000	220.7466	218.0213	A3	221.96	215.76	214.19	233.38
Cel	8.5	0.5	73.41619	73.5822	72.6738	D2	73.706	72.003	71.997	73.811

Instrument	Start	Dur	Freq (E.T)	Freq (Pyth	Freq (Just)	Note	Berg	Quator	Salomon	Vienna
Vl 1	9.0	1.5	440.00000	441.4932	436.0427	A4	442.72	430.33	430.82	447.32
Vl 2	9.0	1.5	293.66477	294.3288	290.6951	D4	294.09	287.23	287.36	296.41
Vla	9.0	2.0	220.00000	220.7466	218.0213	A3	221.57	214.53	215.49	223.49
Cel	9.0	1.5	87.30706	87.2086	87.2086	F2	88.752	85.953	85.662	87.66
Vl 1	10.5	0.5	329.62756	331.1199	327.0320	E4	328.56	323.30	324.38	333.17
Vl 2	10.5	0.5	277.18263	275.6220	272.52667	C#4	277.75	269.78	270.78	280.34
Vla	10.5	0.5	220.00000	220.7466	218.0213	A3	219.1	214.74	215.74	223.31
Cel	10.5	0.5	110.00000	110.3733	109.0107	A2	110.13	107.54	107.21	110.78
Measure 3										
Vl 1	12.0	0.5	349.22823	348.8341	348.8341	F4	349.19	341.55	343.96	352.61
Vl 2	12.0	0.5	293.66477	294.3288	290.6951	D4	293.72	285.18	285.38	298.17
Vla	12.0	0.5	220.00000	220.7466	218.0213	A3	221.81	214.53	215.10	223.20
Cel	12.0	0.5	146.83239	147.1644	147.1644	D3	145.95	142.79	143.75	147.17
Vl 1	13.0	0.5	349.22823	348.8341	348.8341	F4	350.66	341.54	343.21	353.51
Vl 2	13.0	0.5	293.66477	294.3288	290.6951	D4	295.04	286.46	286.57	295.23
Vla	13.0	0.5	220.00000	220.7466	218.0213	A3	222.34	215.65	215.72	223.81
Cel	13.0	0.5	130.81279	130.81279	130.81279	C3	133.71	126.54	128.17	133.63
Vl 1	13.5	0.5	349.22823	348.8341	348.8341	F4	349.3	341.08	340.87	353.94
Vl 2	13.5	0.5	293.66477	294.3288	294.3288	D4	295.37	283.79	286.45	299.19
Vla	13.5	0.5	195.99772	196.2192	196.2192	G3	195.98	190.92	192.57	199.63
Cel	13.5	0.5	123.47082	124.1670	122.637	B2	129.36	126.43	120.41	124.74
Vl 1	14.0	0.5	349.22823	348.8341	348.8341	F4	349.47	342.03	341.76	354.94
Vl 2	14.0	0.5	246.94165	248.3399	245.274	B3	248.51	242.11	242.44	254.04
Vla	14.0	0.5	195.99772	196.2192	196.2192	G3	195.73	191.28	192.85	198.85
Cel	14.0	0.5	97.99886	98.1096	98.1096	G2	97.685	95.875	95.744	100.13
Vl 1	14.5	0.5	329.62756	331.1199	327.032	E4	330.77	322.64	321.61	335.79
Vl 2	14.5	0.5	261.62556	261.62556	261.62556	C4	261.70	257.40	257.20	264.58
Vla	14.5	0.5	195.99772	196.2192	196.2192	G3	194.96	191.61	192.45	197.26
Cel	14.5	0.5	130.81279	130.81279	130.81279	C3	130.19	129.57	128.00	131.46
Vl 1	15.0	0.5	349.22823	348.8341	348.8341	F4	349.15	341.06	343.38	354.12
Vl 2	15.0	0.5	220.00000	220.7466	218.0213	A3	220.63	213.56	215.04	222.92
Vla	15.0	0.5	174.61412	174.41707	174.41707	F3	176.56	172.05	170.70	177.98
Cel	15.0	0.5	146.83239	147.1644	147.1644	D3	147.47	143.38	143.75	147.81
Vl 1	16.0	0.5	349.22823	348.8341	348.8341	F4	349.24	341.01	342.32	354.10
Vl 2	16.0	0.5	293.66477	294.3288	290.6951	D4	296.34	286.56	288.20	299.05
Vla	16.0	0.5	220.00000	220.7466	218.0213	A3	220.98	218.52	216.54	221.69
Cel	16.0	0.5	130.81279	130.81279	130.81279	C3	128.06	127.79	128.18	131.33
Vl 1	16.5	0.5	349.22823	348.8341	348.8341	F4	349.30	342.03	341.77	354.08
Vl 2	16.5	0.5	293.66477	294.3288	294.3288	D4	296.28	286.44	289.21	298.37
Vla	16.5	0.5	195.99772	196.2192	196.2192	G3	196.04	192.30	192.19	199.95
Cel	16.5	0.5	123.47082	124.1670	122.637	B2	123.60	120.00	120.67	125.27
Vl 1	17.0	0.5	349.22823	348.8341	348.8341	F4	349.04	341.78	342.03	355.03
Vl 2	17.0	0.5	246.94165	248.3399	245.274	B3	246.96	242.10	243.63	250.93
Vla	17.0	0.5	195.99772	196.2192	196.2192	G3	195.71	192.12	191.28	198.73
Cel	17.0	0.5	97.99886	98.1096	98.1096	G2	97.607	96.009	95.735	100.22
Vl 1	17.5	0.5	329.62756	331.1199	327.032	E4	331.44	322.64	322.81	334.57
Vl 2	17.5	0.5	261.62556	261.62556	261.62556	C4	261.77	256.22	257.09	266.31
Vla	17.5	0.5	195.99772	196.2192	196.2192	G3	198.01	191.98	191.45	198.39
Cel	17.5	0.5	130.81279	130.81279	130.81279	C3	129.94	128.37	127.16	133.56
Measure 4										
Vl 1	18.0	0.5	293.66477	294.3288	294.3288	D4	296.31	286.90	288.09	299.76
Vl 2	18.0	0.5	246.94165	248.3399	245.274	B3	247.95	240.49	241.07	250.13
Vla	18.0	0.5	195.99772	196.2192	196.2192	G3	195.95	191.68	191.51	199.66
Cel	18.0	0.5	146.83239	147.1644	147.1644	D3	146.94	143.78	144.09	148.42

Instrument	Start	Dur	Freq (E.T)	Freq (Pyth	Freq (Just)	Note	Berg	Quator	Salomon	Vienna
Vl 1	19.0	0.5	293.66477	294.3288	294.3288	D4	293.88	287.75	287.71	299.54
Vl 2	19.0	0.5	246.94165	248.3399	245.274	B3	248.22	239.86	240.79	250.04
Vla	19.0	0.5	195.99772	196.2192	196.2192	G3	197.01	191.44	191.61	199.92
Cel	19.0	0.5	146.83239	147.1644	147.1644	D3	145.87	144.43	142.84	148.37
Vl 1	19.5	0.5	293.66477	294.3288	294.3288	D4	295.09	289.38	287.53	296.85
Vl 2	19.5	0.5	220.00000	220.7466	220.7466	A3	220.53	215.82	216.07	223.92
Vla	19.5	0.5	184.99721	186.2549*	183.9555	F#3	186.37	179.48	180.57	187.87
Cel	19.5	0.5	146.83239	147.1644	147.1644	D3	146.05	144.43	142.92	148.83
Vl 1	20.0	0.5	329.62756	331.1199	327.03200	E4	331.85	323.40	323.97	332.91
Vl 2	20.0	0.5	246.94165	248.3400	245.2740	B3	248.37	240.60	243.53	250.96
Vla	20.0	0.5	195.99772	196.2192	196.2192	G3	195.85	191.48	192.42	197.52
Cel	20.0	0.5	73.41619	73.5822	73.5822	D2	73.656	70.762	72.022	74.792
Vl 1	20.5	0.5	369.99442	372.5099	367.9110	F#4	368.00	360.60	361.42	372.89
Vl 2	20.5	0.5	261.62556	261.62556	261.62556	C4	261.77	255.73	256.05	266.85
Vla	20.5	0.5	220.00000	220.7466	220.7466	A3	220.62	214.76	216.56	222.29
Cel	20.5	0.5	73.41619	73.5822	73.5822	D2	73.655	71.209	72.157	74.386
Vl 1	21.0	1.5	440.00000	441.4932	441.4932	A4	444.66	434.32	431.55	446.86
Vl 2	21.0	1.5	369.99442	372.5099	367.9110	F#4	377.58	360.21	362.25	372.87
Vla	21.0	1.5	261.62556	261.62556	261.62556	C4	262.14	255.83	255.92	264.26
Cel	21.0	1.5	73.41619	73.5822	73.5822	D2	73.373	71.591	71.969	74.627
Vl 1	22.5	0.5	391.99544	392.4384	392.4384	G4	385.83	382.08	384.32	391.82
Vl 2	22.5	0.5	391.99544	392.4384	392.4384	G4	385.83	382.08	384.32	391.82
Vla	22.5	0.5	246.94165	245.2740	245.2740	B3	249.66	240.83	241.21	250.97
Cel	22.5	0.5	97.99886	98.1096	98.1096	G2	97.843	96.832	95.554	99.944

Table 83. Normalized frequency data from four string quartets playing opening of Mozart's K. 590, II

Inst	Start	Dur	E.T.	Pyth	Just	Note	Berg	Quator	Salomon	Vienna
Vl1	0.0	0.5	329.62756	331.1199	327.032	E4	329.622	327.489	328.976	326.871
Vl2	0.0	0.5	261.62556	261.62556	261.62556	C4	261.304	261.504	261.573	261.588
Vla	0.0	0.5	195.99772	196.2192	196.2192	G3	196.949	195.099	196.011	195.899
Cel	0.0	0.5	65.40640	65.40640	65.40640	C2	65.6529	65.12	65.2714	65.4316
Vl1	1.0	0.5	329.62756	331.1199	327.032	E4	329.742	327.233	330.357	327.732
Vl2	1.0	0.5	261.62556	261.62556	261.62556	C4	262.362	260.46	261.705	262.261
Vla	1.0	0.5	195.99772	196.2192	196.2192	G3	198.077	195.662	195.826	195.83
Cel	1.0	0.5	65.40640	65.40640	65.40640	C2	66.0313	65.2592	65.2786	65.4366
Vl1	1.5	0.5	329.62756	331.1199	327.032	E4	328.085	327.192	328.904	328.464
Vl2	1.5	0.5	261.62556	261.62556	261.62556	C4	260.446	261.473	261.47	261.835
Vla	1.5	0.5	195.99772	196.2192	196.2192	G3	198.077	195.519	195.387	196.374
Cel	1.5	0.5	65.40640	65.40640	65.40640	C2	65.4832	65.0094	64.896	64.7395
Vl1	2.0	0.5	329.62756	331.1199	327.032	E4	329.532	327.233	329.896	327.91
Vl2	2.0	0.5	261.62556	261.62556	261.62556	C4	262.182	261.719	261.44	261.845
Vla	2.0	0.5	195.99772	196.2192	196.2192	G3	195.322	195.437	196.916	196.305
Cel	2.0	0.5	65.40640	65.40640	65.40640	C2	65.4333	65.2715	65.3215	65.5108
Vl1	2.5	0.5	329.62756	331.1199	327.032	E4	329.522	327.622	329.262	328.256
Vl2	2.5	0.5	261.62556	261.62556	261.62556	C4	261.264	261.76	261.47	261.776
Vla	2.5	0.5	195.99772	196.2192	196.2192	G3	196.5	197.085	196.041	196.582
Cel	2.5	0.5	65.40640	65.40640	65.40640	C2	66.167	65.2603	65.4801	64.4863
Vl1	3.0	1.5	391.99544	392.4384	392.4384	G4	391.003	390.444	393.034	388.699
Vl2	3.0	1.5	261.62556	261.62556	261.62556	C4	261.054	261.647	261.736	261.094
Vla	3.0	2.0	195.99772	196.2192	196.2192	G3	195.412	195.498	196.696	196.394
Cel	3.0	1.5	82.40689	82.7800	81.7580	E2	82.4769	81.669	81.6573	81.0178

* This value could also be calculated as 183.7480 by following the opposite chain of fifths. To consider either of these, $(3^6/2^8 = 729/256)$ or $(2^{11}/3^6 = 2048/729)$, to be the real derivation of the third in a secondary dominant is to have very much faith in Pythagorean tuning, especially when the 5-limit alternative, $(32*5)/24 = 45/16$, is available. It seems far more conceivable that the presumed central processor in the auditory system would be capable of identifying the relatively small and simple ratios, such as 45/16, provided by 5-limit theory.

Inst	Start	Dur	E.T.	Pyth	Just	Note	Berg	Quator	Salomon	Vienna
Vl1	4.5	0.5	293.66477	294.3288	294.3288	D4	295.283	296.017	294.532	294.823
Vl2	4.5	0.5	246.94165	248.3399	245.274	B3	247.648	245.143	246.842	243.947
Vla	4.5	0.5	195.99772	196.2192	196.2192	G3	196.09	197.341	194.2	195.108
Cel	4.5	0.5	97.99886	98.1096	98.1096	G2	97.3205	98.4003	96.7327	97.9497
Vl1	6.0	0.5	349.22823	348.8341	348.8341	F4	348.758	350.33	348.443	350.297
Vl2	6.0	0.5	293.66477	294.3288	290.6951	D4	295.793	292.761	291.74	292.094
Vla	6.0	0.5	220.00000	220.7466	218.0213	A3	219.978	219.937	222.106	219.286
Cel	6.0	0.5	73.41619	73.5822	72.6738	D2	73.3552	73.1549	74.0403	73.3315
Vl1	7.0	0.5	349.22823	348.8341	348.8341	F4	348.658	350.648	350.192	349.763
Vl2	7.0	0.5	293.66477	294.3288	290.6951	D4	296.102	292.751	292.579	294.823
Vla	7.0	0.5	220.00000	220.7466	218.0213	A3	221.465	220.899	222.199	219.216
Cel	7.0	0.5	73.41619	73.5822	72.6738	D2	73.5259	73.2215	73.5482	72.6749
Vl1	7.5	0.5	349.22823	348.8341	348.8341	F4	349.028	351.774	351.921	350.821
Vl2	7.5	0.5	293.66477	294.3288	290.6951	D4	294.455	293.345	295.392	294.932
Vla	7.5	0.5	220.00000	220.7466	218.0213	A3	221.555	221.104	220.838	220.739
Cel	7.5	0.5	73.41619	73.5822	72.6738	D2	73.4011	73.6013	72.9528	72.7639
Vl1	8.0	0.5	349.22823	348.8341	348.8341	F4	348.538	351.938	351.839	351.138
Vl2	8.0	0.5	293.66477	294.3288	290.6951	D4	295.972	293.201	292.916	292.302
Vla	8.0	0.5	220.00000	220.7466	218.0213	A3	221.495	220.827	221.012	221.105
Cel	8.0	0.5	73.41619	73.5822	72.6738	D2	73.469	73.8132	73.6873	73.4719
Vl1	8.5	0.5	349.22823	348.8341	348.8341	F4	349.187	352.767	349.967	351.533
Vl2	8.5	0.5	293.66477	294.3288	290.6951	D4	293.317	294.645	293.07	293.538
Vla	8.5	0.5	220.00000	220.7466	218.0213	A3	221.565	220.899	219.109	220.887
Cel	8.5	0.5	73.41619	73.5822	72.6738	D2	73.5748	73.718	73.6505	72.9874
Vl1	9.0	1.5	440.00000	441.4932	436.0427	A4	441.932	440.58	440.714	442.329
Vl2	9.0	1.5	293.66477	294.3288	290.6951	D4	293.567	294.072	293.96	293.103
Vla	9.0	2.0	220.00000	220.7466	218.0213	A3	221.176	219.64	220.439	220.996
Cel	9.0	1.5	87.30706	87.2086	87.2086	F2	88.594	88.0003	87.6293	86.6819
Vl1	10.5	0.5	329.62756	331.1199	327.0320	E4	327.975	331.001	331.83	329.452
Vl2	10.5	0.5	277.18263	275.6220	272.52667	C#4	277.256	276.206	276.999	277.212
Vla	10.5	0.5	220.00000	220.7466	218.0213	A3	218.71	219.855	220.695	220.818
Cel	10.5	0.5	110.00000	110.3733	109.0107	A2	109.934	110.101	109.672	109.544
Vl1	12.0	0.5	349.22823	348.8341	348.8341	F4	348.568	349.685	351.86	348.675
Vl2	12.0	0.5	293.66477	294.3288	290.6951	D4	293.197	291.973	291.934	294.843
Vla	12.0	0.5	220.00000	220.7466	218.0213	A3	221.415	219.64	220.04	220.709
Cel	12.0	0.5	146.83239	147.1644	145.34754	D3	145.69	146.191	147.051	145.528
Vl1	13.0	0.5	349.22823	348.8341	348.8341	F4	350.036	349.675	351.092	349.565
Vl2	13.0	0.5	293.66477	294.3288	290.6951	D4	294.515	293.283	293.151	291.936
Vla	13.0	0.5	220.00000	220.7466	218.0213	A3	221.944	220.787	220.674	221.313
Cel	13.0	0.5	130.81279	130.81279	130.81279	C3	133.472	129.554	131.114	132.139
Vl1	13.5	0.5	349.22823	348.8341	348.8341	F4	348.678	349.204	348.699	349.991
Vl2	13.5	0.5	293.66477	294.3288	294.3288	D4	294.844	290.55	293.029	295.852
Vla	13.5	0.5	195.99772	196.2192	196.2192	G3	195.631	195.468	196.993	197.402
Cel	13.5	0.5	123.47082	124.1670	122.637	B2	123.874	124.281	123.175	123.348
Vl1	14.0	0.5	349.22823	348.8341	348.8341	F4	348.848	350.177	349.609	350.979
Vl2	14.0	0.5	246.94165	248.3399	245.274	B3	248.068	247.877	248.008	251.205
Vla	14.0	0.5	195.99772	196.2192	196.2192	G3	195.382	195.836	197.279	196.631
Cel	14.0	0.5	97.99886	98.1096	98.1096	G2	97.5111	98.1586	97.9429	99.0127
Vl1	14.5	0.5	329.62756	331.1199	327.032	E4	330.181	330.325	328.996	332.043
Vl2	14.5	0.5	261.62556	261.62556	261.62556	C4	261.234	263.531	263.107	261.628
Vla	14.5	0.5	195.99772	196.2192	196.2192	G3	194.613	196.174	196.87	195.059
Cel	14.5	0.5	130.81279	130.81279	130.81279	C3	129.958	132.656	130.94	129.993
Vl1	15.0	0.5	349.22823	348.8341	348.8341	F4	348.529	349.184	351.266	350.169
Vl2	15.0	0.5	220.00000	220.7466	218.0213	A3	220.237	218.647	219.979	220.433
Vla	15.0	0.5	174.61412	174.41707	174.41707	F3	176.246	176.148	174.62	175.994
Cel	15.0	0.5	146.83239	147.1644	145.34754	D3	147.208	146.795	147.051	146.161
Vl1	16.0	0.5	349.22823	348.8341	348.8341	F4	348.618	349.132	350.182	350.149
Vl2	16.0	0.5	293.66477	294.3288	290.6951	D4	295.813	293.386	294.819	295.713
Vla	16.0	0.5	220.00000	220.7466	218.0213	A3	220.587	223.725	221.513	219.216
Cel	16.0	0.5	130.81279	130.81279	130.81279	C3	127.832	130.834	131.124	129.865
Vl1	16.5	0.5	349.22823	348.8341	348.8341	F4	348.678	350.177	349.619	350.129
Vl2	16.5	0.5	293.66477	294.3288	294.3288	D4	295.753	293.263	295.852	295.041
Vla	16.5	0.5	195.99772	196.2192	196.2192	G3	195.691	196.88	196.604	197.719
Cel	16.5	0.5	123.47082	124.1670	122.637	B2	123.38	122.858	123.441	123.872
Vl1	17.0	0.5	349.22823	348.8341	348.8341	F4	348.419	349.921	349.885	351.068
Vl2	17.0	0.5	246.94165	248.3399	245.274	B3	246.52	247.867	249.225	248.13
Vla	17.0	0.5	195.99772	196.2192	196.2192	G3	195.362	196.696	195.673	196.512
Cel	17.0	0.5	97.99886	98.1096	98.1096	G2	97.4333	98.2958	97.9337	99.1017
Vl1	17.5	0.5	329.62756	331.1199	327.032	E4	330.85	330.325	330.224	330.837

Inst	Start	Dur	E.T.	Pyth	Just	Note	Berg	Quator	Salomon	Vienna
Vl2	17.5	0.5	261.62556	261.62556	261.62556	C4	261.304	262.323	262.994	263.338
Vla	17.5	0.5	195.99772	196.2192	196.2192	G3	197.658	196.553	195.847	196.176
Cel	17.5	0.5	130.81279	130.81279	130.81279	C3	129.709	131.428	130.08	132.07
Vl1	18.0	0.5	293.66477	294.3288	294.3288	D4	295.783	293.734	294.706	296.415
Vl2	18.0	0.5	246.94165	248.3399	245.274	B3	247.509	246.218	246.607	247.339
Vla	18.0	0.5	195.99772	196.2192	196.2192	G3	195.601	196.246	195.908	197.432
Cel	18.0	0.5	146.83239	147.1644	147.1644	D3	146.678	147.205	147.399	146.764
Vl1	19.0	0.5	293.66477	294.3288	294.3288	D4	293.357	294.604	294.318	296.198
Vl2	19.0	0.5	246.94165	248.3399	245.274	B3	247.778	245.573	246.32	247.25
Vla	19.0	0.5	195.99772	196.2192	196.2192	G3	196.659	196	196.011	197.689
Cel	19.0	0.5	146.83239	147.1644	147.1644	D3	145.61	147.87	146.121	146.714
Vl1	19.5	0.5	293.66477	294.3288	294.3288	D4	294.565	296.273	294.134	293.538
Vl2	19.5	0.5	220.00000	220.7466	220.7466	A3	220.137	220.961	221.032	221.421
Vla	19.5	0.5	184.99721	186.2549*	183.9555	F#3	186.038	183.755	184.717	185.774
Cel	19.5	0.5	146.83239	147.1644	147.1644	D3	145.79	147.87	146.202	147.169
Vl1	20.0	0.5	329.62756	331.1199	327.03200	E4	331.259	331.103	331.41	329.195
Vl2	20.0	0.5	246.94165	248.3400	245.2740	B3	247.928	246.331	249.123	248.16
Vla	20.0	0.5	195.99772	196.2192	196.2192	G3	195.501	196.041	196.839	195.316
Cel	20.0	0.5	73.41619	73.5822	73.5822	D2	73.5249	72.4475	73.6761	73.9574
Vl1	20.5	0.5	369.99442	372.5099	367.9110	F#4	367.345	369.189	369.721	368.729
Vl2	20.5	0.5	261.62556	261.62556	261.62556	C4	261.304	261.821	261.931	263.872
Vla	20.5	0.5	220.00000	220.7466	220.7466	A3	220.227	219.875	221.534	219.81
Cel	20.5	0.5	73.41619	73.5822	73.5822	D2	73.5239	72.9051	73.8142	73.556
Vl1	21.0	1.5	440.00000	441.4932	441.4932	A4	443.869	444.665	441.461	441.874
Vl2	21.0	1.5	369.99442	372.5099	367.9110	F#4	376.908	368.79	370.57	368.709
Vla	21.0	1.5	261.62556	261.62556	261.62556	C4	261.673	261.924	261.798	261.311
Cel	21.0	1.5	73.41619	73.5822	73.5822	D2	73.2424	73.2962	73.6219	73.7943
Vl1	22.5	0.5	391.99544	392.4384	392.4384	G4	385.143	391.181	393.146	387.448
Vl2	22.5	0.5	391.99544	392.4384	392.4384	G4	385.143	391.181	393.146	387.448
Vla	22.5	0.5	246.94165	248.3400	245.2740	B3	249.216	246.566	246.75	248.17
Cel	22.5	0.5	97.99886	98.1096	98.1096	G2	97.6688	99.1384	97.7485	98.8288

Table 84. Normalized frequency data converted into distance, in cents, from middle C (sorted by note)

Instrument	Start	Dur	E.T.	Pyth	Just	Note	Berg	Quator	Salomon	Vienna
Cel	0.00	0.50	-2399.99	-2399.99	-2399.99	C2	-2393.48	-2407.59	-2403.57	-2399.33
Cel	1.00	0.50	-2399.99	-2399.99	-2399.99	C2	-2383.53	-2403.90	-2403.38	-2399.20
Cel	1.50	0.50	-2399.99	-2399.99	-2399.99	C2	-2397.96	-2410.53	-2413.56	-2417.74
Cel	2.00	0.50	-2399.99	-2399.99	-2399.99	C2	-2399.28	-2403.57	-2402.24	-2397.23
Cel	2.50	0.50	-2399.99	-2399.99	-2399.99	C2	-2379.98	-2403.87	-2398.05	-2424.52
Cel	6.00	0.50	-2199.99	-2196.08	-2217.59	D2I	-2201.43	-2206.17	-2185.34	-2201.99
Cel	7.00	0.50	-2199.99	-2196.08	-2217.59	D2I	-2197.41	-2204.59	-2196.88	-2217.56
Cel	7.50	0.50	-2199.99	-2196.08	-2217.59	D2I	-2200.35	-2195.64	-2210.96	-2215.45
Cel	8.00	0.50	-2199.99	-2196.08	-2217.59	D2I	-2198.75	-2190.66	-2193.61	-2198.68
Cel	8.50	0.50	-2199.99	-2196.08	-2217.59	D2I	-2196.26	-2192.89	-2194.48	-2210.14
Cel	20.00	0.50	-2199.99	-2196.08	-2196.08	D2II	-2197.43	-2222.99	-2193.88	-2187.28
Cel	20.50	0.50	-2199.99	-2196.08	-2196.08	D2II	-2197.46	-2212.09	-2190.63	-2196.70
Cel	21.00	1.50	-2199.99	-2196.08	-2196.08	D2II	-2204.10	-2202.83	-2195.15	-2191.10
Cel	3.00	1.50	-1999.99	-1992.17	-2013.68	E2	-1998.52	-2015.57	-2015.81	-2029.43
Cel	9.00	1.50	-1899.99	-1901.95	-1901.95	F2	-1874.66	-1886.30	-1893.62	-1912.44
Cel	4.50	0.50	-1699.99	-1698.04	-1698.04	G2	-1712.02	-1692.92	-1722.51	-1700.86
Cel	14.00	0.50	-1699.99	-1698.04	-1698.04	G2	-1708.63	-1697.18	-1700.98	-1682.18
Cel	17.00	0.50	-1699.99	-1698.04	-1698.04	G2	-1710.02	-1694.76	-1701.15	-1680.62
Cel	22.50	0.50	-1699.99	-1698.04	-1698.04	G2	-1705.84	-1679.98	-1704.42	-1685.40
Cel	10.50	0.50	-1500.00	-1494.13	-1515.63	A2I	-1501.03	-1498.41	-1505.16	-1507.19
Cel	13.50	0.50	-1299.99	-1290.26	-1311.73	B2	-1294.35	-1288.67	-1304.15	-1301.72
Cel	16.50	0.50	-1299.99	-1290.26	-1311.73	B2	-1301.27	-1308.61	-1300.41	-1294.38
Cel	13.00	0.50	-1199.99	-1199.99	-1199.99	C3	-1165.15	-1216.73	-1196.01	-1182.53
Cel	14.50	0.50	-1199.99	-1199.99	-1199.99	C3	-1211.34	-1175.77	-1198.31	-1210.88
Cel	16.00	0.50	-1199.99	-1199.99	-1199.99	C3	-1239.90	-1199.71	-1195.88	-1212.58
Cel	17.50	0.50	-1199.99	-1199.99	-1199.99	C3	-1214.66	-1191.87	-1209.72	-1183.44
Cel	12.00	0.50	-1000.00	-996.09	-1017.59	D3I	-1013.52	-1007.57	-997.42	-1015.44
Cel	15.00	0.50	-1000.00	-996.09	-1017.59	D3I	-995.57	-1000.44	-997.42	-1007.93
Cel	18.00	0.50	-1000.00	-996.09	-996.09	D3II	-1001.82	-995.61	-993.33	-1000.80
Cel	19.00	0.50	-1000.00	-996.09	-996.09	D3II	-1014.47	-987.80	-1008.40	-1001.39
Cel	19.50	0.50	-1000.00	-996.09	-996.09	D3II	-1012.33	-987.80	-1007.44	-996.03
Vla	15.00	0.50	-700.00	-701.95	-701.95	F3	-683.89	-684.85	-699.94	-686.37

Instrument	Start	Dur	E.T.	Pyth	Just	Note	Berg	Quator	Salomon	Vienna
Vla	19.50	0.50	-600.00	-588.26	-609.77	F#3	-590.28	-611.66	-602.62	-592.74
Vla	0.00	0.50	-499.99	-498.04	-498.04	G3	-491.61	-507.95	-499.88	-500.87
Vla	1.00	0.50	-499.99	-498.04	-498.04	G3	-481.73	-502.96	-501.51	-501.48
Vla	1.50	0.50	-499.99	-498.04	-498.04	G3	-481.73	-504.23	-505.40	-496.67
Vla	2.00	0.50	-499.99	-498.04	-498.04	G3	-505.97	-504.95	-491.90	-497.28
Vla	2.50	0.50	-499.99	-498.04	-498.04	G3	-495.56	-490.42	-499.61	-494.84
Vla	3.00	2.00	-499.99	-498.04	-498.04	G3	-505.18	-504.41	-493.84	-496.50
Vla	4.50	0.50	-499.99	-498.04	-498.04	G3	-499.18	-488.17	-515.95	-507.87
Vla	13.50	0.50	-499.99	-498.04	-498.04	G3	-503.24	-504.68	-491.23	-487.64
Vla	14.00	0.50	-499.99	-498.04	-498.04	G3	-505.44	-501.42	-488.71	-494.41
Vla	14.50	0.50	-499.99	-498.04	-498.04	G3	-512.27	-498.44	-492.31	-508.31
Vla	16.50	0.50	-499.99	-498.04	-498.04	G3	-502.71	-492.22	-494.65	-484.86
Vla	17.00	0.50	-499.99	-498.04	-498.04	G3	-505.62	-493.84	-502.87	-495.46
Vla	17.50	0.50	-499.99	-498.04	-498.04	G3	-485.39	-495.10	-501.33	-498.42
Vla	18.00	0.50	-499.99	-498.04	-498.04	G3	-503.50	-497.80	-500.79	-487.37
Vla	19.00	0.50	-499.99	-498.04	-498.04	G3	-494.16	-499.97	-499.88	-485.12
Vla	20.00	0.50	-499.99	-498.04	-498.04	G3	-504.39	-499.61	-492.58	-506.03
Vla	6.00	0.50	-300.00	-294.13	-315.64	A3I	-300.17	-300.49	-283.50	-305.62
Vla	7.00	0.50	-300.00	-294.13	-315.64	A3I	-288.50	-292.93	-282.78	-306.18
Vla	7.50	0.50	-300.00	-294.13	-315.64	A3I	-287.80	-291.33	-293.41	-294.19
Vla	8.00	0.50	-300.00	-294.13	-315.64	A3I	-288.27	-293.50	-292.05	-291.32
Vla	8.50	0.50	-300.00	-294.13	-315.64	A3I	-287.72	-292.93	-307.02	-293.03
Vla	9.00	2.00	-300.00	-294.13	-315.64	A3I	-290.77	-302.83	-296.54	-292.17
Vla	10.50	0.50	-300.00	-294.13	-315.64	A3I	-310.18	-301.14	-294.53	-293.57
Vla	12.00	0.50	-300.00	-294.13	-315.64	A3I	-288.90	-302.83	-299.68	-294.42
Vla	13.00	0.50	-300.00	-294.13	-315.64	A3I	-284.76	-293.81	-294.70	-289.69
Vl2	15.00	0.50	-300.00	-294.13	-315.64	A3I	-298.13	-310.67	-300.16	-296.59
Vla	16.00	0.50	-300.00	-294.13	-315.64	A3I	-295.38	-270.93	-288.13	-306.18
Vl2	19.50	0.50	-300.00	-294.13	-294.13	A3II	-298.92	-292.45	-291.89	-288.85
Vla	20.50	0.50	-300.00	-294.13	-294.13	A3II	-298.21	-300.98	-287.97	-301.49
Vl2	4.50	0.50	-99.99	-90.22	-111.73	B3	-95.05	-112.65	-100.69	-121.12
Vl2	14.00	0.50	-99.99	-90.22	-111.73	B3	-92.12	-93.45	-92.54	-70.36
Vl2	17.00	0.50	-99.99	-90.22	-111.73	B3	-102.95	-93.52	-84.06	-91.68
Vl2	18.00	0.50	-99.99	-90.22	-111.73	B3	-96.02	-105.08	-102.34	-97.21
Vl2	19.00	0.50	-99.99	-90.22	-111.73	B3	-94.14	-109.62	-104.36	-97.83
Vl2	20.00	0.50	-99.99	-90.22	-111.73	B3	-93.09	-104.28	-84.77	-91.47
Vla	22.50	0.50	-99.99	-90.22	-111.73	B3	-84.12	-102.63	-101.34	-91.40
Vl2	0.00	0.50	0.00	0.00	0.00	C4	-2.12	-0.80	-0.34	-0.24
Vl2	1.00	0.50	0.00	0.00	0.00	C4	4.86	-7.73	0.52	4.19
Vl2	1.50	0.50	0.00	0.00	0.00	C4	-7.82	-1.00	-1.02	1.38
Vl2	2.00	0.50	0.00	0.00	0.00	C4	3.67	0.61	-1.22	1.45
Vl2	2.50	0.50	0.00	0.00	0.00	C4	-2.39	0.88	-1.02	0.99
Vl2	3.00	1.50	0.00	0.00	0.00	C4	-3.78	0.14	0.73	-3.52
Vl2	14.50	0.50	0.00	0.00	0.00	C4	-2.59	12.56	9.77	0.01
Vl2	17.50	0.50	0.00	0.00	0.00	C4	-2.12	4.60	9.03	11.29
Vl2	20.50	0.50	0.00	0.00	0.00	C4	-2.12	1.29	2.01	14.80
Vla	21.00	1.50	0.00	0.00	0.00	C4	0.31	1.97	1.14	-2.08
Vl2	10.50	0.50	100.00	90.22	70.67	C#4	100.45	93.88	98.85	100.18
Vl2	6.00	0.50	200.00	203.91	182.40	D4I	212.50	194.66	188.61	190.71
Vl2	7.00	0.50	200.00	203.91	182.40	D4I	214.30	194.60	193.58	206.81
Vl2	7.50	0.50	200.00	203.91	182.40	D4I	204.65	198.11	210.15	207.45
Vl2	8.00	0.50	200.00	203.91	182.40	D4I	213.54	197.26	195.58	191.94
Vl2	8.50	0.50	200.00	203.91	182.40	D4I	197.94	205.76	196.49	199.25
Vl2	9.00	1.50	200.00	203.91	182.40	D4I	199.42	202.39	201.73	196.68
Vl2	12.00	0.50	200.00	203.91	182.40	D4I	197.24	189.99	189.76	206.93
Vl2	13.00	0.50	200.00	203.91	182.40	D4I	205.00	197.74	196.96	189.77
Vl2	16.00	0.50	200.00	203.91	182.40	D4I	212.61	198.35	206.79	212.03
Vl1	4.50	0.50	200.00	203.91	203.91	D4II	209.51	213.81	205.10	206.81
Vl2	13.50	0.50	200.00	203.91	203.91	D4II	206.93	181.53	196.24	212.84
Vl2	16.50	0.50	200.00	203.91	203.91	D4II	212.26	197.62	212.84	208.09
Vl1	18.00	0.50	200.00	203.91	203.91	D4II	212.44	200.40	206.12	216.13
Vl1	19.00	0.50	200.00	203.91	203.91	D4II	198.18	205.52	203.84	214.87
Vl1	19.50	0.50	200.00	203.91	203.91	D4II	205.29	215.30	202.76	199.25
Vl1	0.00	0.50	400.00	407.82	386.31	E4	399.97	388.73	396.57	385.46
Vl1	1.00	0.50	400.00	407.82	386.31	E4	400.60	387.37	403.82	390.01
Vl1	1.50	0.50	400.00	407.82	386.31	E4	391.87	387.16	396.19	393.87
Vl1	2.00	0.50	400.00	407.82	386.31	E4	399.49	387.37	401.40	390.95
Vl1	2.50	0.50	400.00	407.82	386.31	E4	399.44	389.43	398.07	392.78
Vl1	10.50	0.50	400.00	407.82	386.31	E4	391.29	407.19	411.52	399.07

Instrument	Start	Dur	E.T.	Pyth	Just	Note	Berg	Quator	Salomon	Vienna
Vl1	14.50	0.50	400.00	407.82	386.31	E4	402.90	403.65	396.67	412.63
Vl1	17.50	0.50	400.00	407.82	386.31	E4	406.40	403.65	403.12	406.34
Vl1	20.00	0.50	400.00	407.82	386.31	E4	408.54	407.73	409.33	397.72
Vl1	6.00	0.50	499.99	498.04	498.04	F4	497.66	505.45	496.10	505.29
Vl1	7.00	0.50	499.99	498.04	498.04	F4	497.17	507.02	504.77	502.64
Vl1	7.50	0.50	499.99	498.04	498.04	F4	499.00	512.57	513.29	507.87
Vl1	8.00	0.50	499.99	498.04	498.04	F4	496.57	513.38	512.89	509.44
Vl1	8.50	0.50	499.99	498.04	498.04	F4	499.79	517.45	503.65	511.38
Vl1	12.00	0.50	499.99	498.04	498.04	F4	496.72	502.26	512.99	497.25
Vl1	13.00	0.50	499.99	498.04	498.04	F4	503.99	502.21	509.21	501.66
Vl1	13.50	0.50	499.99	498.04	498.04	F4	497.27	499.87	497.37	503.77
Vl1	14.00	0.50	499.99	498.04	498.04	F4	498.11	504.69	501.88	508.65
Vl1	15.00	0.50	499.99	498.04	498.04	F4	496.53	499.78	510.07	504.65
Vl1	16.00	0.50	499.99	498.04	498.04	F4	496.97	499.52	504.72	504.55
Vl1	16.50	0.50	499.99	498.04	498.04	F4	497.27	504.69	501.93	504.45
Vl1	17.00	0.50	499.99	498.04	498.04	F4	495.98	503.43	503.25	509.09
Vl1	20.50	0.50	599.99	611.73	590.22	F#4	587.55	596.22	598.72	594.06
Vl2	21.00	1.50	599.99	611.73	590.22	F#4	632.05	594.35	602.69	593.97
Vl1	3.00	1.50	699.99	701.95	701.95	G4	695.61	693.13	704.58	685.37
Vl1	22.50	0.50	699.99	701.95	701.95	G4	669.46	696.39	705.07	679.79
Vl2	22.50	0.50	699.99	701.95	701.95	G4	669.46	696.39	705.07	679.79
Vl1	9.00	1.50	899.99	905.86	884.36	A4I	907.58	902.28	902.80	909.13
Vl1	21.00	1.50	899.99	905.86	905.86	A4II	915.15	918.25	905.73	907.35

Table 85. Major triad normalization tables (Berg Quartet)

Instrument	Start	Dur	Note	ET - Berg	Adj ET-Berg	Pyth - Berg	Adj Pyth - Berg	Just - Berg	Adj Just - Berg
ROOT POSITION									
C Major									
Vl1	0	0.5	E4	0.03	3.215	7.85	8.5925	-13.66	-7.54
Vl2	0	0.5	C4	2.12	5.305	2.12	2.8625	2.12	8.24
Vla	0	0.5	G3	-8.38	-5.195	-6.43	-5.6875	-6.43	-0.31
Cel	0	0.5	C2	-6.51	-3.325	-6.51	-5.7675	-6.51	-0.39
				-3.185	AVE:	-0.7425	AVE:	-6.12	
Vl1	1	0.5	E4	-0.6	9.445	7.22	14.8225	-14.29	-1.31
Vl2	1	0.5	C4	-4.86	5.185	-4.86	2.7425	-4.86	8.12
Vla	1	0.5	G3	-18.26	-8.215	-16.31	-8.7075	-16.31	-3.33
Cel	1	0.5	C2	-16.46	-6.415	-16.46	-8.8575	-16.46	-3.48
				-10.045	AVE:	-7.6025	AVE:	-12.98	
Vl1	1.5	0.5	E4	8.13	9.215	15.95	14.5925	-5.56	-1.54
Vl2	1.5	0.5	C4	7.82	8.905	7.82	6.4625	7.82	11.84
Vla	1.5	0.5	G3	-18.26	-17.175	-16.31	-17.6675	-16.31	-12.29
Cel	1.5	0.5	C2	-2.03	-0.945	-2.03	-3.3875	-2.03	1.99
				-1.085	AVE:	1.3575	AVE:	-4.02	
Vl1	2	0.5	E4	0.51	-0.0175	8.33	5.36	-13.18	-10.7725
Vl2	2	0.5	C4	-3.67	-4.1975	-3.67	-6.64	-3.67	-1.2625
Vla	2	0.5	G3	5.98	5.4525	7.93	4.96	7.93	10.3375
Cel	2	0.5	C2	-0.71	-1.2375	-0.71	-3.68	-0.71	1.6975
				0.5275	AVE:	2.97	AVE:	-2.4075	
Vl1	2.5	0.5	E4	0.56	5.9325	8.38	11.31	-13.13	-4.8225
Vl2	2.5	0.5	C4	2.39	7.7625	2.39	5.32	2.39	10.6975
Vla	2.5	0.5	G3	-4.43	0.9425	-2.48	0.45	-2.48	5.8275
Cel	2.5	0.5	C2	-20.01	-14.6375	-20.01	-17.08	-20.01	-11.7025
				-5.3725	AVE:	-2.93	AVE:	-8.3075	
Vl1	14.5	0.5	E4	-2.9	-8.73	4.92	-3.3525	-16.59	-19.485
Vl2	14.5	0.5	C4	2.59	-3.24	2.59	-5.6825	2.59	-0.305
Vla	14.5	0.5	G3	12.28	6.45	14.23	5.9575	14.23	11.335
Cel	14.5	0.5	C3	11.35	5.52	11.35	3.0775	11.35	8.455
				5.83	AVE:	8.2725	AVE:	2.895	

Instrument	Start	Dur	Note	ET - Berg	Adj ET-Berg	Pyth - Berg	Adj Pyth - Berg	Just - Berg	Adj Just - Berg
Vl1	17.5	0.5	E4	-6.4	-5.3475	1.42	0.03	-20.09	-16.1025
Vl2	17.5	0.5	C4	2.12	3.1725	2.12	0.73	2.12	6.1075
Vla	17.5	0.5	G3	-14.6	-13.5475	-12.65	-14.04	-12.65	-8.6625
Cel	17.5	0.5	C3	14.67	15.7225	14.67	13.28	14.67	18.6575
				-1.0525	AVE:	1.39	AVE:	-3.9875	
G Major									
Vl1	4.5	0.5	D4II	-9.51	-8.7025	-5.6	-9.1875	-5.6	-3.81
Vl2	4.5	0.5	B3	-4.94	-4.1325	4.83	1.2425	-16.68	-14.89
Vla	4.5	0.5	G3	-0.81	-0.0025	1.14	-2.4475	1.14	2.93
Cel	4.5	0.5	G2	12.03	12.8375	13.98	10.3925	13.98	15.77
				-0.8075	AVE:	3.5875	AVE:	-1.79	
Vl1	22.5	0.5	G4	30.53	17.77	32.49	15.82	32.49	21.1975
Vl2	22.5	0.5	G4	30.53	17.77	32.49	15.82	32.49	21.1975
Vla	22.5	0.5	B3	-15.87	-28.63	-6.1	-22.77	-27.61	-38.9025
Cel	22.5	0.5	G2	5.85	-6.91	7.8	-8.87	7.8	-3.4925
				12.76	AVE:	16.67	AVE:	11.2925	
A Major									
Vl1	10.5	0.5	E4	8.71	3.8425	16.53	9.2175	-4.98	8.725
Vl2	10.5	0.5	C#4	-0.45	-5.3175	-10.23	-17.5425	-29.78	-16.075
Vla	10.5	0.5	A3I	10.18	5.3125	16.05	8.7375	-5.46	8.245
Cel	10.5	0.5	A2I	1.03	-3.8375	6.9	-0.4125	-14.6	-0.895
				4.8675	AVE:	7.3125	AVE:	-13.705	
D Major									
Vl1	19.5	0.5	D4II	-5.29	-4.35	-1.38	-6.7975	-1.38	-1.42
Vl2	19.5	0.5	A3II	-1.08	-0.14	4.79	-0.6275	4.79	4.75
Vla	19.5	0.5	F#3	-9.72	-8.78	2.02	-3.3975	-19.49	-19.53
Cel	19.5	0.5	D3II	12.33	13.27	16.24	10.8225	16.24	16.2
				-0.94	AVE:	5.4175	AVE:	0.04	
FIRST INVERSION									
C Major									
Vl1	3	1.5	G4	4.38	1.41	6.34	0.4375	6.34	5.815
Vl2	3	1.5	C4	3.78	0.81	3.78	-2.1225	3.78	3.255
Vla	3	2	G3	5.19	2.22	7.14	1.2375	7.14	6.615
Cel	3	1.5	E2	-1.47	-4.44	6.35	0.4475	-15.16	-15.685
				2.97	AVE:	5.9025	AVE:	0.525	
SECOND INVERSION									
G Major									
Vl1	18	0.5	D4II	-12.44	-9.67	-8.53	-10.645	-8.53	-5.2675
Vl2	18	0.5	B3	-3.97	-1.2	5.8	3.685	-15.71	-12.4475
Vla	18	0.5	G3	3.51	6.28	5.46	3.345	5.46	8.7225
Cel	18	0.5	D3II	1.82	4.59	5.73	3.615	5.73	8.9925
				-2.77	AVE:	2.115	AVE:	-3.2625	
Vl1	19	0.5	D4II	1.82	0.6675	5.73	-0.3075	5.73	5.07
Vl2	19	0.5	B3	-5.85	-7.0025	3.92	-2.1175	-17.59	-18.25
Vla	19	0.5	G3	-5.83	-6.9825	-3.88	-9.9175	-3.88	-4.54
Cel	19	0.5	D3II	14.47	13.3175	18.38	12.3425	18.38	17.72
				1.1525	AVE:	6.0375	AVE:	0.66	

Table 86. Minor triad normalization tables (Berg Quartet)

Instrument	Start	Dur	Note	ET - Berg	Adj ET- Berg	Pyth - Berg	Adj Pyth - Berg	Just - Berg	Adj Just - Berg
ROOT POSITION									
D Minor									
Vl1	6	0.5	F4	2.33	4.47	0.38	-0.415	0.38	15.7175
Vl2	6	0.5	D4I	-12.5	-10.36	-8.59	-9.385	-30.1	-14.7625
Vla	6	0.5	A3I	0.17	2.31	6.04	5.245	-15.47	-0.1325
Cel	6	0.5	D2I	1.44	3.58	5.35	4.555	-16.16	-0.8225
				-2.14	AVE:	0.795	AVE:	-15.3375	
Vl1	7	0.5	F4	2.82	9.21	0.87	4.325	0.87	20.4575
Vl2	7	0.5	D4I	-14.3	-7.91	-10.39	-6.935	-31.9	-12.3125
Vla	7	0.5	A3I	-11.5	-5.11	-5.63	-2.175	-27.14	-7.5525
Cel	7	0.5	D2I	-2.58	3.81	1.33	4.785	-20.18	-0.5925
				-6.39	AVE:	-3.455	AVE:	-19.5875	
Vl1	7.5	0.5	F4	0.99	4.865	-0.96	-0.02	-0.96	16.1125
Vl2	7.5	0.5	D4I	-4.65	-0.775	-0.74	0.2	-22.25	-5.1775
Vla	7.5	0.5	A3I	-12.2	-8.325	-6.33	-5.39	-27.84	-10.7675
Cel	7.5	0.5	D2I	0.36	4.235	4.27	5.21	-17.24	-0.1675
				-3.875	AVE:	-0.94	AVE:	-17.0725	
Vl1	8	0.5	F4	3.42	9.1925	1.47	4.3075	1.47	20.44
Vl2	8	0.5	D4I	-13.54	-7.7675	-9.63	-6.7925	-31.14	-12.17
Vla	8	0.5	A3I	-11.73	-5.9575	-5.86	-3.0225	-27.37	-8.4
Cel	8	0.5	D2I	-1.24	4.5325	2.67	5.5075	-18.84	0.13
				-5.7725	AVE:	-2.8375	AVE:	-18.97	
Vl1	8.5	0.5	F4	0.2	3.6375	-1.75	-1.2475	-1.75	14.885
Vl2	8.5	0.5	D4I	2.06	5.4975	5.97	6.4725	-15.54	1.095
Vla	8.5	0.5	A3I	-12.28	-8.8425	-6.41	-5.9075	-27.92	-11.285
Cel	8.5	0.5	D2I	-3.73	-0.2925	0.18	0.6825	-21.33	-4.695
				-3.4375	AVE:	-0.5025	AVE:	-16.635	
Vl1	12	0.5	F4	3.27	1.1575	1.32	-3.7275	1.32	12.4025
Vl2	12	0.5	D4I	2.76	0.6475	6.67	1.6225	-14.84	-3.7575
Vla	12	0.5	A3I	-11.1	-13.2125	-5.23	-10.2775	-26.74	-15.6575
Cel	12	0.5	D3I	13.52	11.4075	17.43	12.3825	-4.07	7.0125
				2.1125	AVE:	5.0475	AVE:	-11.0825	
Vl1	15	0.5	F4	3.46	8.1975	1.51	4.7775	1.51	15.53
Vl2	15	0.5	A3I	-1.87	2.8675	4	7.2675	-17.51	-3.49
Vla	15	0.5	F3	-16.11	-11.3725	-18.06	-14.7925	-18.06	-4.04
Cel	15	0.5	D3I	-4.43	0.3075	-0.52	2.7475	-22.02	-8
				-4.7375	AVE:	-3.2675	AVE:	-14.02	
FIRST INVERSION									
D Minor									
Vl1	9	1.5	A4I	-7.59	2.8025	-1.72	5.25	-23.22	-0.12
Vl2	9	1.5	D4I	0.58	10.9725	4.49	11.46	-17.02	6.08
Vla	9	2	A3I	-9.23	1.1625	-3.36	3.61	-24.87	-1.77
Cel	9	1.5	F2	-25.33	-14.9375	-27.29	-20.32	-27.29	-4.19
				-10.3925	AVE:	-6.97	AVE:	-23.1	

Table 87. Dominant seventh normalization tables (Berg Quartet)

Instrument	Start	Dur	Note	ET - Berg	Adj ET-Berg	Pyth - Berg	Adj Pyth - Berg	Just - Berg	Adj Just - Berg
ROOT POSITION									
G Dominant Seventh									
Vl1	14	0.5	F4	1.88	-0.145	-0.07	-5.025	-0.07	0.3525
Vl2	14	0.5	B3	-7.87	-9.895	1.9	-3.055	-19.61	-19.1875
Vla	14	0.5	G3	5.45	3.425	7.4	2.445	7.4	7.8225
Cel	14	0.5	G2	8.64	6.615	10.59	5.635	10.59	11.0125
				2.025	AVE:	4.955	AVE:	-0.4225	
Vl1	17	0.5	F4	4.01	-1.6475	2.06	-6.5275	2.06	-1.15
Vl2	17	0.5	B3	2.96	-2.6975	12.73	4.1425	-8.78	-11.99
Vla	17	0.5	G3	5.63	-0.0275	7.58	-1.0075	7.58	4.37
Cel	17	0.5	G2	10.03	4.3725	11.98	3.3925	11.98	8.77
				5.6575	AVE:	8.5875	AVE:	3.21	
D Dominant Seventh									
Vl1	20.5	0.5	F#4	12.44	9.88	24.18	16.24	2.67	0.1075
Vl2	20.5	0.5	C4	2.12	-0.44	2.12	-5.82	2.12	-0.4425
Vla	20.5	0.5	A3II	-1.79	-4.35	4.08	-3.86	4.08	1.5175
Cel	20.5	0.5	D2II	-2.53	-5.09	1.38	-6.56	1.38	-1.1825
				2.56	AVE:	7.94	AVE:	2.5625	
Vl1	21	1.5	A4II	-15.16	-4.305	-9.29	-3.815	-9.29	1.5625
Vl2	21	1.5	F#4	-32.06	-21.205	-20.32	-14.845	-41.83	-30.9775
Vla	21	1.5	C4	-0.31	10.545	-0.31	5.165	-0.31	10.5425
Cel	21	1.5	D2II	4.11	14.965	8.02	13.495	8.02	18.8725
				-10.855	AVE:	-5.475	AVE:	-10.8525	
FIRST INVERSION									
G Dominant Seventh									
Vl1	13.5	0.5	F4	2.72	4.37	0.77	-0.99	0.77	4.3775
Vl2	13.5	0.5	D4II	-6.93	-5.28	-3.02	-4.78	-3.02	0.5875
Vla	13.5	0.5	G3	3.25	4.9	5.2	3.44	5.2	8.8075
Cel	13.5	0.5	B2	-5.64	-3.99	4.09	2.33	-17.38	-13.7725
				-1.65	AVE:	1.76	AVE:	-3.6075	
Vl1	16.5	0.5	F4	2.72	4.105	0.77	-1.255	0.77	4.1125
Vl2	16.5	0.5	D4II	-12.26	-10.875	-8.35	-10.375	-8.35	-5.0075
Vla	16.5	0.5	G3	2.72	4.105	4.67	2.645	4.67	8.0125
Cel	16.5	0.5	B2	1.28	2.665	11.01	8.985	-10.46	-7.1175
				-1.385	AVE:	2.025	AVE:	-3.3425	

Table 88. Minor seventh normalization tables (Berg Quartet)

Instrument	Start	Dur	Note	ET - Berg	Adj ET-Berg	Pyth - Berg	Adj Pyth – Berg	Just - Berg	Adj Just - Berg
THIRD INVERSION									
D Minor Seventh									
Vl1	13	0.5	F4	-4	10.77	-5.95	6.8625	-5.95	17.6175
Vl2	13	0.5	D4I	-5	9.77	-1.09	11.7225	-22.6	0.9675
Vla	13	0.5	A3I	-15.24	-0.47	-9.37	3.4425	-30.88	-7.3125
Cel	13	0.5	C3	-34.84	-20.07	-34.84	-22.0275	-34.84	-11.2725
				-14.77	AVE:	-12.8125	AVE:	-23.5675	
Vl1	16	0.5	F4	3.02	-3.405	1.07	-7.3125	1.07	3.4425
Vl2	16	0.5	D4I	-12.61	-19.035	-8.7	-17.0825	-30.21	-27.8375
Vla	16	0.5	A3I	-4.62	-11.045	1.25	-7.1325	-20.26	-17.8875
Cel	16	0.5	C3	39.91	33.485	39.91	31.5275	39.91	42.2825
				6.425	AVE:	8.3825	AVE:	-2.3725	

Instrument	Start	Dur	Note	ET - Berg	Adj ET-Berg	Pyth - Berg	Adj Pyth – Berg	Just - Berg	Adj Just - Berg
THIRD INVERSION									
E Minor Seventh									
Vl1	20	0.5	E4	-8.54	-5.14	-0.72	-3.1825	-22.23	-13.9375
Vl2	20	0.5	B3	-6.9	-3.5	2.87	0.4075	-18.64	-10.3475
Vla	20	0.5	G3	4.4	7.8	6.35	3.8875	6.35	14.6425
Cel	20	0.5	D2II	-2.56	0.84	1.35	-1.1125	1.35	9.6425
				-3.4	AVE:	2.4625	AVE:	-8.2925	

Table 89. Major triad normalization tables (Quator Quartet)

Instrument	Start	Dur	Note	ET - Quator	Adj ET-Quator	Pyth - Quator	Adj Pyth - Quator	Just - Quator	Adj Just - Quator
ROOT POSITION									
C Major									
Vl1	0	0.5	E4	11.27	4.3625	19.09	9.74	-2.42	-6.3925
Vl2	0	0.5	C4	0.8	-6.1075	0.8	-8.55	0.8	-3.1725
Vla	0	0.5	G3	7.96	1.0525	9.91	0.56	9.91	5.9375
Cel	0	0.5	C2	7.6	0.6925	7.6	-1.75	7.6	3.6275
				6.9075	AVE:	9.35	AVE:	3.9725	
Vl1	1	0.5	E4	12.63	5.82	20.45	11.1975	-1.06	-4.935
Vl2	1	0.5	C4	7.73	0.92	7.73	-1.5225	7.73	3.855
Vla	1	0.5	G3	2.97	-3.84	4.92	-4.3325	4.92	1.045
Cel	1	0.5	C2	3.91	-2.9	3.91	-5.3425	3.91	0.035
				6.81	AVE:	9.2525	AVE:	3.875	
Vl1	1.5	0.5	E4	12.84	5.685	20.66	11.0625	-0.85	-5.07
Vl2	1.5	0.5	C4	1	-6.155	1	-8.5975	1	-3.22
Vla	1.5	0.5	G3	4.24	-2.915	6.19	-3.4075	6.19	1.97
Cel	1.5	0.5	C2	10.54	3.385	10.54	0.9425	10.54	6.32
				7.155	AVE:	9.5975	AVE:	4.22	
Vl1	2	0.5	E4	12.63	7.49	20.45	12.8675	-1.06	-3.265
Vl2	2	0.5	C4	-0.61	-5.75	-0.61	-8.1925	-0.61	-2.815
Vla	2	0.5	G3	4.96	-0.18	6.91	-0.6725	6.91	4.705
Cel	2	0.5	C2	3.58	-1.56	3.58	-4.0025	3.58	1.375
				5.14	AVE:	7.5825	AVE:	2.205	
Vl1	2.5	0.5	E4	10.57	9.57	18.39	14.9475	-3.12	-1.185
Vl2	2.5	0.5	C4	-0.88	-1.88	-0.88	-4.3225	-0.88	1.055
Vla	2.5	0.5	G3	-9.57	-10.57	-7.62	-11.0625	-7.62	-5.685
Cel	2.5	0.5	C2	3.88	2.88	3.88	0.4375	3.88	5.815
				1	AVE:	3.4425	AVE:	-1.935	
Vl1	14.5	0.5	E4	-3.65	6.845	4.17	12.2225	-17.34	-3.91
Vl2	14.5	0.5	C4	-12.56	-2.065	-12.56	-4.5075	-12.56	0.87
Vla	14.5	0.5	G3	-1.55	8.945	0.4	8.4525	0.4	13.83
Cel	14.5	0.5	C3	-24.22	-13.725	-24.22	-16.1675	-24.22	-10.79
				-10.495	AVE:	-8.0525	AVE:	-13.43	
Vl1	17.5	0.5	E4	-3.65	1.665	4.17	7.0425	-17.34	-9.09
Vl2	17.5	0.5	C4	-4.6	0.715	-4.6	-1.7275	-4.6	3.65
Vla	17.5	0.5	G3	-4.89	0.425	-2.94	-0.0675	-2.94	5.31
Cel	17.5	0.5	C3	-8.12	-2.805	-8.12	-5.2475	-8.12	0.13
				-5.315	AVE:	-2.8725	AVE:	-8.25	
G Major									
Vl1	4.5	0.5	D4II	-13.81	-8.8	-9.9	-9.285	-9.9	-3.9075
Vl2	4.5	0.5	B3	12.66	17.67	22.43	23.045	0.92	6.9125
Vla	4.5	0.5	G3	-11.82	-6.81	-9.87	-9.255	-9.87	-3.8775
Cel	4.5	0.5	G2	-7.07	-2.06	-5.12	-4.505	-5.12	0.8725
				-5.01	AVE:	-0.615	AVE:	-5.9925	

Instrument	Start	Dur	Note	ET - Quator	Adj ET-Quator	Pyth - Quator	Adj Pyth - Quator	Just - Quator	Adj Just - Quator
Vl1	22.5	0.5	G4	3.6	6.1425	5.56	4.1925	5.56	9.57
Vl2	22.5	0.5	G4	3.6	6.1425	5.56	4.1925	5.56	9.57
Vla	22.5	0.5	B3	2.64	5.1825	12.41	11.0425	-9.1	-5.09
Cel	22.5	0.5	G2	-20.01	-17.4675	-18.06	-19.4275	-18.06	-14.05
				-2.5425	AVE:	1.3675	AVE:	-4.01	
A Major									
Vl1	10.5	0.5	E4	-7.19	-6.81	0.63	-1.435	-20.88	-1.9275
Vl2	10.5	0.5	C#4	6.12	6.5	-3.66	-5.725	-23.21	-4.2575
Vla	10.5	0.5	A3I	1.14	1.52	7.01	4.945	-14.5	4.4525
Cel	10.5	0.5	A2I	-1.59	-1.21	4.28	2.215	-17.22	1.7325
				-0.38	AVE:	2.065	AVE:	-18.9525	
D Major									
Vl1	19.5	0.5	D4II	-15.3	-9.4525	-11.39	-11.9	-11.39	-6.5225
Vl2	19.5	0.5	A3II	-7.55	-1.7025	-1.68	-2.19	-1.68	3.1875
Vla	19.5	0.5	F#3	11.66	17.5075	23.4	22.89	1.89	6.7575
Cel	19.5	0.5	D3II	-12.2	-6.3525	-8.29	-8.8	-8.29	-3.4225
				-5.8475	AVE:	0.51	AVE:	-4.8675	
FIRST INVERSION									
C Major									
Vl1	3	1.5	G4	6.86	0.18	8.82	-0.7925	8.82	4.585
Vl2	3	1.5	C4	-0.14	-6.82	-0.14	-9.7525	-0.14	-4.375
Vla	3	2	G3	4.42	-2.26	6.37	-3.2425	6.37	2.135
Cel	3	1.5	E2	15.58	8.9	23.4	13.7875	1.89	-2.345
				6.68	AVE:	9.6125	AVE:	4.235	
SECOND INVERSION									
G Major									
Vl1	18	0.5	D4II	-0.4	0.0725	3.51	-0.9025	3.51	4.475
Vl2	18	0.5	B3	5.09	5.5625	14.86	10.4475	-6.65	-5.685
Vla	18	0.5	G3	-2.19	-1.7175	-0.24	-4.6525	-0.24	0.725
Cel	18	0.5	D3II	-4.39	-3.9175	-0.48	-4.8925	-0.48	0.485
				-0.4725	AVE:	4.4125	AVE:	-0.965	
Vl1	19	0.5	D4II	-5.52	-3.4925	-1.61	-4.4675	-1.61	0.91
Vl2	19	0.5	B3	9.63	11.6575	19.4	16.5425	-2.11	0.41
Vla	19	0.5	G3	-0.02	2.0075	1.93	-0.9275	1.93	4.45
Cel	19	0.5	D3II	-12.2	-10.1725	-8.29	-11.1475	-8.29	-5.77
				-2.0275	AVE:	2.8575	AVE:	-2.52	

Table 90. Minor triad normalization tables (Quator Quartet)

Instrument	Start	Dur	Note	ET - Quator	Adj ET-Quator	Ptyh - Quator	Adj Pyth - Quator	Just - Quator	Adj Just - Quator
ROOT POSITION									
D Minor									
Vl1	6	0.5	F4	-5.46	-7.0975	-7.41	-11.9825	-7.41	4.15
Vl2	6	0.5	D4I	5.34	3.7025	9.25	4.6775	-12.26	-0.7
Vla	6	0.5	A3I	0.49	-1.1475	6.36	1.7875	-15.15	-3.59
Cel	6	0.5	D2I	6.18	4.5425	10.09	5.5175	-11.42	0.14
				1.6375	AVE:	4.5725	AVE:	-11.56	
Vl1	7	0.5	F4	-7.03	-6.005	-8.98	-10.89	-8.98	5.2425
Vl2	7	0.5	D4I	5.4	6.425	9.31	7.4	-12.2	2.0225
Vla	7	0.5	A3I	-7.07	-6.045	-1.2	-3.11	-22.71	-8.4875
Cel	7	0.5	D2I	4.6	5.625	8.51	6.6	-13	1.2225
				-1.025	AVE:	1.91	AVE:	-14.2225	

Instrument	Start	Dur	Note	ET - Quator	Adj ET- Quator	Ptyh - Quator	Adj Pyth - Quator	Just - Quator	Adj Just - Quator
Vl1	7.5	0.5	F4	-12.58	-6.6525	-14.53	-11.5375	-14.53	4.595
Vl2	7.5	0.5	D4I	1.89	7.8175	5.8	8.7925	-15.71	3.415
Vla	7.5	0.5	A3I	-8.67	-2.7425	-2.8	0.1925	-24.31	-5.185
Cel	7.5	0.5	D2I	-4.35	1.5775	-0.44	2.5525	-21.95	-2.825
				-5.9275	AVE:	-2.9925	AVE:	-19.125	
Vl1	8	0.5	F4	-13.39	-6.77	-15.34	-11.655	-15.34	4.4775
Vl2	8	0.5	D4I	2.74	9.36	6.65	10.335	-14.86	4.9575
Vla	8	0.5	A3I	-6.5	0.12	-0.63	3.055	-22.14	-2.3225
Cel	8	0.5	D2I	-9.33	-2.71	-5.42	-1.735	-26.93	-7.1125
				-6.62	AVE:	-3.685	AVE:	-19.8175	
Vl1	8.5	0.5	F4	-17.46	-8.1125	-19.41	-12.9975	-19.41	3.135
Vl2	8.5	0.5	D4I	-5.76	3.5875	-1.85	4.5625	-23.36	-0.815
Vla	8.5	0.5	A3I	-7.07	2.2775	-1.2	5.2125	-22.71	-0.165
Cel	8.5	0.5	D2I	-7.1	2.2475	-3.19	3.2225	-24.7	-2.155
				-9.3475	AVE:	-6.4125	AVE:	-22.545	
Vl1	12	0.5	F4	-2.27	-6.805	-4.22	-11.69	-4.22	4.44
Vl2	12	0.5	D4I	10.01	5.475	13.92	6.45	-7.59	1.07
Vla	12	0.5	A3I	2.83	-1.705	8.7	1.23	-12.81	-4.15
Cel	12	0.5	D3I	7.57	3.035	11.48	4.01	-10.02	-1.36
				4.535	AVE:	7.47	AVE:	-8.66	
Vl1	15	0.5	F4	0.21	1.1675	-1.74	-2.2525	-1.74	8.5
Vl2	15	0.5	A3I	10.67	11.6275	16.54	16.0275	-4.97	5.27
Vla	15	0.5	F3	-15.15	-14.1925	-17.1	-17.6125	-17.1	-6.86
Cel	15	0.5	D3I	0.44	1.3975	4.35	3.8375	-17.15	-6.91
				-0.9575	AVE:	0.5125	AVE:	-10.24	
FIRST INVERSION									
D Minor									
Vl1	9	1.5	A4I	-2.29	1.595	3.58	4.0425	-17.92	-1.3275
Vl2	9	1.5	D4I	-2.39	1.495	1.52	1.9825	-19.99	-3.3975
Vla	9	2	A3I	2.83	6.715	8.7	9.1625	-12.81	3.7825
Cel	9	1.5	F2	-13.69	-9.805	-15.65	-15.1875	-15.65	0.9425
				-3.885	AVE:	-0.4625	AVE:	-16.5925	

Table 91. Dominant seventh normalization tables (Quator Quartet)

Instrument	Start	Dur	Note	ET - Quator	Adj ET- Quator	Ptyh - Quator	Adj Pyth - Quator	Just - Quator	Adj Just - Quator
ROOT POSITION									
Vl1	14	0.5	F4	-4.7	-1.545	-6.65	-6.425	-6.65	-1.0475
Vl2	14	0.5	B3	-6.54	-3.385	3.23	3.455	-18.28	-12.6775
Vla	14	0.5	G3	1.43	4.585	3.38	3.605	3.38	8.9825
Cel	14	0.5	G2	-2.81	0.345	-0.86	-0.635	-0.86	4.7425
				-3.155	AVE:	-0.225	AVE:	-5.6025	
Vl1	17	0.5	F4	-3.44	1.8825	-5.39	-2.9975	-5.39	2.38
Vl2	17	0.5	B3	-6.47	-1.1475	3.3	5.6925	-18.21	-10.44
Vla	17	0.5	G3	-6.15	-0.8275	-4.2	-1.8075	-4.2	3.57
Cel	17	0.5	G2	-5.23	0.0925	-3.28	-0.8875	-3.28	4.49
				-5.3225	AVE:	-2.3925	AVE:	-7.77	
D Dominant Seventh									
Vl1	20.5	0.5	F#4	3.77	-0.12	15.51	6.24	-6	-9.8925
Vl2	20.5	0.5	C4	-1.29	-5.18	-1.29	-10.56	-1.29	-5.1825
Vla	20.5	0.5	A3II	0.98	-2.91	6.85	-2.42	6.85	2.9575
Cel	20.5	0.5	D2II	12.1	8.21	16.01	6.74	16.01	12.1175
				3.89	AVE:	9.27	AVE:	3.8925	

Instrument	Start	Dur	Note	ET - Quator	Adj ET-Quator	Ptyh - Quator	Adj Pyth - Quator	Just - Quator	Adj Just - Quator
Vl1	21	1.5	A4II	-18.26	-15.3225	-12.39	-14.8325	-12.39	-9.455
Vl2	21	1.5	F#4	5.64	8.5775	17.38	14.9375	-4.13	-1.195
Vla	21	1.5	C4	-1.97	0.9675	-1.97	-4.4125	-1.97	0.965
Cel	21	1.5	D2II	2.84	5.7775	6.75	4.3075	6.75	9.685
				-2.9375	AVE:	2.4425	AVE:	-2.935	
FIRST INVERSION									
G Dominant Seventh									
Vl1	13.5	0.5	F4	0.12	-2.87	-1.83	-8.23	-1.83	-2.8625
Vl2	13.5	0.5	D4II	18.47	15.48	22.38	15.98	22.38	21.3475
Vla	13.5	0.5	G3	4.69	1.7	6.64	0.24	6.64	5.6075
Cel	13.5	0.5	B2	-11.32	-14.31	-1.59	-7.99	-23.06	-24.0925
				2.99	AVE:	6.4	AVE:	1.0325	
Vl1	16.5	0.5	F4	-4.7	-4.3325	-6.65	-9.6925	-6.65	-4.325
Vl2	16.5	0.5	D4II	2.38	2.7475	6.29	3.2475	6.29	8.615
Vla	16.5	0.5	G3	-7.77	-7.4025	-5.82	-8.8625	-5.82	-3.495
Cel	16.5	0.5	B2	8.62	8.9875	18.35	15.3075	-3.12	-0.795
				-0.3675	AVE:	3.0425	AVE:	-2.325	

Table 92. Minor seventh normalization tables (Quator Quartet)

Instrument	Start	Dur	Note	ET - Quator	Adj ET-Quator	Ptyh - Quator	Adj Pyth - Quator	Just - Quator	Adj Just - Quator
THIRD INVERSION									
D Minor Seventh									
Vl1	13	0.5	F4	-2.22	-4.8675	-4.17	-8.775	-4.17	1.98
Vl2	13	0.5	D4I	2.26	-0.3875	6.17	1.565	-15.34	-9.19
Vla	13	0.5	A3I	-6.19	-8.8375	-0.32	-4.925	-21.83	-15.68
Cel	13	0.5	C3	16.74	14.0925	16.74	12.135	16.74	22.89
				2.6475	AVE:	4.605	AVE:	-6.15	
Vl1	16	0.5	F4	0.47	7.2775	-1.48	3.37	-1.48	14.125
Vl2	16	0.5	D4I	1.65	8.4575	5.56	10.41	-15.95	-0.345
Vla	16	0.5	A3I	-29.07	-22.2625	-23.2	-18.35	-44.71	-29.105
Cel	16	0.5	C3	-0.28	6.5275	-0.28	4.57	-0.28	15.325
				-6.8075	AVE:	-4.85	AVE:	-15.605	
E Minor Seventh									
Vl1	20	0.5	E4	-7.73	-12.525	0.09	-10.5675	-21.42	-21.3225
Vl2	20	0.5	B3	4.29	-0.505	14.06	3.4025	-7.45	-7.3525
Vla	20	0.5	G3	-0.38	-5.175	1.57	-9.0875	1.57	1.6675
Cel	20	0.5	D2II	23	18.205	26.91	16.2525	26.91	27.0075
				4.795	AVE:	10.6575	AVE:	-0.0975	

Table 93. Major triad normalization tables (Salomon Quartet)

Instrument	Start	Dur	Note	ET - Salomon	Adj ET-Salomon	Ptyh - Salomon	Adj Pyth - Salomon	Just - Salomon	Adj Just - Salomon
ROOT POSITION									
C Major									
Vl1	0	0.5	E4	3.43	1.62	11.25	6.9975	-10.26	-9.135
Vl2	0	0.5	C4	0.34	-1.47	0.34	-3.9125	0.34	1.465
Vla	0	0.5	G3	-0.11	-1.92	1.84	-2.4125	1.84	2.965
Cel	0	0.5	C2	3.58	1.77	3.58	-0.6725	3.58	4.705
				1.81	AVE:	4.2525	AVE:	-1.125	

Instrument	Start	Dur	Note	ET - Salomon	Adj ET- Salomon	Ptyh - Salomon	Adj Pyth - Salomon	Just - Salomon	Adj Just - Salomon
Vl1	1	0.5	E4	-3.82	-3.9625	4	1.415	-17.51	-14.7175
Vl2	1	0.5	C4	-0.52	-0.6625	-0.52	-3.105	-0.52	2.2725
Vla	1	0.5	G3	1.52	1.3775	3.47	0.885	3.47	6.2625
Cel	1	0.5	C2	3.39	3.2475	3.39	0.805	3.39	6.1825
				0.1425	AVE:	2.585	AVE:	-2.7925	
Vl1	1.5	0.5	E4	3.81	-2.1425	11.63	3.235	-9.88	-12.8975
Vl2	1.5	0.5	C4	1.02	-4.9325	1.02	-7.375	1.02	-1.9975
Vla	1.5	0.5	G3	5.41	-0.5425	7.36	-1.035	7.36	4.3425
Cel	1.5	0.5	C2	13.57	7.6175	13.57	5.175	13.57	10.5525
				5.9525	AVE:	8.395	AVE:	3.0175	
Vl1	2	0.5	E4	-1.4	0.105	6.42	5.4825	-15.09	-10.65
Vl2	2	0.5	C4	1.22	2.725	1.22	0.2825	1.22	5.66
Vla	2	0.5	G3	-8.09	-6.585	-6.14	-7.0775	-6.14	-1.7
Cel	2	0.5	C2	2.25	3.755	2.25	1.3125	2.25	6.69
				-1.505	AVE:	0.9375	AVE:	-4.44	
Vl1	2.5	0.5	E4	1.93	1.7725	9.75	7.15	-11.76	-8.9825
Vl2	2.5	0.5	C4	1.02	0.8625	1.02	-1.58	1.02	3.7975
Vla	2.5	0.5	G3	-0.38	-0.5375	1.57	-1.03	1.57	4.3475
Cel	2.5	0.5	C2	-1.94	-2.0975	-1.94	-4.54	-1.94	0.8375
				0.1575	AVE:	2.6	AVE:	-2.7775	
Vl1	14.5	0.5	E4	3.33	7.28	11.15	12.6575	-10.36	-3.475
Vl2	14.5	0.5	C4	-9.77	-5.82	-9.77	-8.2625	-9.77	-2.885
Vla	14.5	0.5	G3	-7.68	-3.73	-5.73	-4.2225	-5.73	1.155
Cel	14.5	0.5	C3	-1.68	2.27	-1.68	-0.1725	-1.68	5.205
				-3.95	AVE:	-1.5075	AVE:	-6.885	
Vl1	17.5	0.5	E4	-3.12	-2.85	4.7	2.5275	-16.81	-13.605
Vl2	17.5	0.5	C4	-9.03	-8.76	-9.03	-11.2025	-9.03	-5.825
Vla	17.5	0.5	G3	1.34	1.61	3.29	1.1175	3.29	6.495
Cel	17.5	0.5	C3	9.73	10	9.73	7.5575	9.73	12.935
				-0.27	AVE:	2.1725	AVE:	-3.205	
G Major									
Vl1	4.5	0.5	D4II	-5.1	-13.62	-1.19	-14.105	-1.19	-8.7275
Vl2	4.5	0.5	B3	0.7	-7.82	10.47	-2.445	-11.04	-18.5775
Vla	4.5	0.5	G3	15.96	7.44	17.91	4.995	17.91	10.3725
Cel	4.5	0.5	G2	22.52	14	24.47	11.555	24.47	16.9325
				8.52	AVE:	12.915	AVE:	7.5375	
Vl1	22.5	0.5	G4	-5.08	-3.985	-3.12	-5.935	-3.12	-0.5575
Vl2	22.5	0.5	G4	-5.08	-3.985	-3.12	-5.935	-3.12	-0.5575
Vla	22.5	0.5	B3	1.35	2.445	11.12	8.305	-10.39	-7.8275
Cel	22.5	0.5	G2	4.43	5.525	6.38	3.565	6.38	8.9425
				-1.095	AVE:	2.815	AVE:	-2.5625	
A Major									
Vl1	10.5	0.5	E4	-11.52	-8.85	-3.7	-3.475	-25.21	-3.9675
Vl2	10.5	0.5	C#4	1.15	3.82	-8.63	-8.405	-28.18	-6.9375
Vla	10.5	0.5	A3I	-5.47	-2.8	0.4	0.625	-21.11	0.1325
Cel	10.5	0.5	A2I	5.16	7.83	11.03	11.255	-10.47	10.7725
				-2.67	AVE:	-0.225	AVE:	-21.2425	
D Major									
Vl1	19.5	0.5	D4II	-2.76	-2.5575	1.15	-5.005	1.15	0.3725
Vl2	19.5	0.5	A3II	-8.11	-7.9075	-2.24	-8.395	-2.24	-3.0175
Vla	19.5	0.5	F#3	2.62	2.8225	14.36	8.205	-7.15	-7.9275
Cel	19.5	0.5	D3II	7.44	7.6425	11.35	5.195	11.35	10.5725
				-0.2025	AVE:	6.155	AVE:	0.7775	

Instrument	Start	Dur	Note	ET - Salomon	Adj ET- Salomon	Ptyh - Salomon	Adj Pyth - Salomon	Just - Salomon	Adj Just - Salomon
FIRST INVERSION									
C Major									
Vl1	3	1.5	G4	-4.59	-5.6775	-2.63	-6.65	-2.63	-1.2725
Vl2	3	1.5	C4	-0.73	-1.8175	-0.73	-4.75	-0.73	0.6275
Vla	3	2	G3	-6.15	-7.2375	-4.2	-8.22	-4.2	-2.8425
Cel	3	1.5	E2	15.82	14.7325	23.64	19.62	2.13	3.4875
				1.0875	AVE:	4.02	AVE:	-1.3575	
SECOND INVERSION									
G Major									
Vl1	18	0.5	D4II	-6.12	-3.71	-2.21	-4.685	-2.21	0.6925
Vl2	18	0.5	B3	2.35	4.76	12.12	9.645	-9.39	-6.4875
Vla	18	0.5	G3	0.8	3.21	2.75	0.275	2.75	5.6525
Cel	18	0.5	D3II	-6.67	-4.26	-2.76	-5.235	-2.76	0.1425
				-2.41	AVE:	2.475	AVE:	-2.9025	
Vl1	19	0.5	D4II	-3.84	-6.045	0.07	-7.02	0.07	-1.6425
Vl2	19	0.5	B3	4.37	2.165	14.14	7.05	-7.37	-9.0825
Vla	19	0.5	G3	-0.11	-2.315	1.84	-5.25	1.84	0.1275
Cel	19	0.5	D3II	8.4	6.195	12.31	5.22	12.31	10.5975
				2.205	AVE:	7.09	AVE:	1.7125	

Table 94. Minor triad normalization tables (Salomon Quartet)

Instrument	Start	Dur	Note	ET - Salomon	Adj ET- Salomon	Ptyh - Salomon	Adj Pyth - Salomon	Just - Salomon	Adj Just - Salomon
ROOT POSITION									
D Minor									
Vl1	6	0.5	F4	3.89	7.8575	1.94	2.9725	1.94	19.105
Vl2	6	0.5	D4I	11.39	15.3575	15.3	16.3325	-6.21	10.955
Vla	6	0.5	A3I	-16.5	-12.5325	-10.63	-9.5975	-32.14	-14.975
Cel	6	0.5	D2I	-14.65	-10.6825	-10.74	-9.7075	-32.25	-15.085
				-3.9675	AVE:	-1.0325	AVE:	-17.165	
Vl1	7	0.5	F4	-4.78	-0.1075	-6.73	-4.9925	-6.73	11.14
Vl2	7	0.5	D4I	6.42	11.0925	10.33	12.0675	-11.18	6.69
Vla	7	0.5	A3I	-17.22	-12.5475	-11.35	-9.6125	-32.86	-14.99
Cel	7	0.5	D2I	-3.11	1.5625	0.8	2.5375	-20.71	-2.84
				-4.6725	AVE:	-1.7375	AVE:	-17.87	
Vl1	7.5	0.5	F4	-13.3	-8.5325	-15.25	-13.4175	-15.25	2.715
Vl2	7.5	0.5	D4I	-10.15	-5.3825	-6.24	-4.4075	-27.75	-9.785
Vla	7.5	0.5	A3I	-6.59	-1.8225	-0.72	1.1125	-22.23	-4.265
Cel	7.5	0.5	D2I	10.97	15.7375	14.88	16.7125	-6.63	11.335
				-4.7675	AVE:	-1.8325	AVE:	-17.965	
Vl1	8	0.5	F4	-12.9	-7.1975	-14.85	-12.0825	-14.85	4.05
Vl2	8	0.5	D4I	4.42	10.1225	8.33	11.0975	-13.18	5.72
Vla	8	0.5	A3I	-7.95	-2.2475	-2.08	0.6875	-23.59	-4.69
Cel	8	0.5	D2I	-6.38	-0.6775	-2.47	0.2975	-23.98	-5.08
				-5.7025	AVE:	-2.7675	AVE:	-18.9	
Vl1	8.5	0.5	F4	-3.66	-4	-5.61	-8.885	-5.61	7.2475
Vl2	8.5	0.5	D4I	3.51	3.17	7.42	4.145	-14.09	-1.2325
Vla	8.5	0.5	A3I	7.02	6.68	12.89	9.615	-8.62	4.2375
Cel	8.5	0.5	D2I	-5.51	-5.85	-1.6	-4.875	-23.11	-10.2525
				0.34	AVE:	3.275	AVE:	-12.8575	
Vl1	12	0.5	F4	-13	-11.585	-14.95	-16.47	-14.95	-0.34
Vl2	12	0.5	D4I	10.24	11.655	14.15	12.63	-7.36	7.25
Vla	12	0.5	A3I	-0.32	1.095	5.55	4.03	-15.96	-1.35
Cel	12	0.5	D3I	-2.58	-1.165	1.33	-0.19	-20.17	-5.56
				-1.415	AVE:	1.52	AVE:	-14.61	

Instrument	Start	Dur	Note	ET - Salomon	Adj ET- Salomon	Ptyh - Salomon	Adj Pyth - Salomon	Just - Salomon	Adj Just - Salomon
Vl1	15	0.5	F4	-10.08	-6.94	-12.03	-10.36	-12.03	0.3925
Vl2	15	0.5	A3I	0.16	3.3	6.03	7.7	-15.48	-3.0575
Vla	15	0.5	F3	-0.06	3.08	-2.01	-0.34	-2.01	10.4125
Cel	15	0.5	D3I	-2.58	0.56	1.33	3	-20.17	-7.7475
				-3.14	AVE:	-1.67	AVE:	-12.4225	
FIRST INVERSION									
D Minor									
Vl1	9	1.5	A4I	-2.81	0.7825	3.06	3.23	-18.44	-2.14
Vl2	9	1.5	D4I	-1.73	1.8625	2.18	2.35	-19.33	-3.03
Vla	9	2	A3I	-3.46	0.1325	2.41	2.58	-19.1	-2.8
Cel	9	1.5	F2	-6.37	-2.7775	-8.33	-8.16	-8.33	7.97
				-3.5925	AVE:	-0.17	AVE:	-16.3	

Table 95. Dominant seventh normalization tables (Salomon Quartet)

Instrument	Start	Dur	Note	ET - Salomon	Adj ET- Salomon	Ptyh - Salomon	Adj Pyth - Salomon	Just - Salomon	Adj Just - Salomon
ROOT POSITION									
G Dominant Seventh									
Vl1	14	0.5	F4	-1.89	3.0175	-3.84	-1.8625	-3.84	3.515
Vl2	14	0.5	B3	-7.45	-2.5425	2.32	4.2975	-19.19	-11.835
Vla	14	0.5	G3	-11.28	-6.3725	-9.33	-7.3525	-9.33	-1.975
Cel	14	0.5	G2	0.99	5.8975	2.94	4.9175	2.94	10.295
				-4.9075	AVE:	-1.9775	AVE:	-7.355	
Vl1	17	0.5	F4	-3.26	0.5275	-5.21	-4.3525	-5.21	1.025
Vl2	17	0.5	B3	-15.93	-12.1425	-6.16	-5.3025	-27.67	-21.435
Vla	17	0.5	G3	2.88	6.6675	4.83	5.6875	4.83	11.065
Cel	17	0.5	G2	1.16	4.9475	3.11	3.9675	3.11	9.345
				-3.7875	AVE:	-0.8575	AVE:	-6.235	
D Dominant Seventh									
Vl1	20.5	0.5	F#4	1.27	6.8025	13.01	13.1625	-8.5	-2.97
Vl2	20.5	0.5	C4	-2.01	3.5225	-2.01	-1.8575	-2.01	3.52
Vla	20.5	0.5	A3II	-12.03	-6.4975	-6.16	-6.0075	-6.16	-0.63
Cel	20.5	0.5	D2II	-9.36	-3.8275	-5.45	-5.2975	-5.45	0.08
				-5.5325	AVE:	-0.1525	AVE:	-5.53	
Vl1	21	1.5	A4II	-5.74	-2.135	0.13	-1.645	0.13	3.7325
Vl2	21	1.5	F#4	-2.7	0.905	9.04	7.265	-12.47	-8.8675
Vla	21	1.5	C4	-1.14	2.465	-1.14	-2.915	-1.14	2.4625
Cel	21	1.5	D2II	-4.84	-1.235	-0.93	-2.705	-0.93	2.6725
				-3.605	AVE:	1.775	AVE:	-3.6025	
FIRST INVERSION									
G Dominant Seventh									
Vl1	13.5	0.5	F4	2.62	2.175	0.67	-3.185	0.67	2.1825
Vl2	13.5	0.5	D4II	3.76	3.315	7.67	3.815	7.67	9.1825
Vla	13.5	0.5	G3	-8.76	-9.205	-6.81	-10.665	-6.81	-5.2975
Cel	13.5	0.5	B2	4.16	3.715	13.89	10.035	-7.58	-6.0675
				0.445	AVE:	3.855	AVE:	-1.5125	
Vl1	16.5	0.5	F4	-1.94	2.985	-3.89	-2.375	-3.89	2.9925
Vl2	16.5	0.5	D4II	-12.84	-7.915	-8.93	-7.415	-8.93	-2.0475
Vla	16.5	0.5	G3	-5.34	-0.415	-3.39	-1.875	-3.39	3.4925
Cel	16.5	0.5	B2	0.42	5.345	10.15	11.665	-11.32	-4.4375
				-4.925	AVE:	-1.515	AVE:	-6.8825	

Table 96. Minor seventh normalization tables (Salomon Quartet)

Instrument	Start	Dur	Note	ET - Salomon	Adj ET- Salomon	Ptyh - Salomon	Adj Pyth - Salomon	Just - Salomon	Adj Just - Salomon
THIRD INVERSION									
D Minor Seventh									
Vl1	13	0.5	F4	-9.22	-5.355	-11.17	-9.2625	-11.17	1.4925
Vl2	13	0.5	D4I	3.04	6.905	6.95	8.8575	-14.56	-1.8975
Vla	13	0.5	A3I	-5.3	-1.435	0.57	2.4775	-20.94	-8.2775
Cel	13	0.5	C3	-3.98	-0.115	-3.98	-2.0725	-3.98	8.6825
				-3.865	AVE:	-1.9075	AVE:	-12.6625	
Vl1	16	0.5	F4	-4.73	2.145	-6.68	-1.7625	-6.68	8.9925
Vl2	16	0.5	D4I	-6.79	0.085	-2.88	2.0375	-24.39	-8.7175
Vla	16	0.5	A3I	-11.87	-4.995	-6	-1.0825	-27.51	-11.8375
Cel	16	0.5	C3	-4.11	2.765	-4.11	0.8075	-4.11	11.5625
				-6.875	AVE:	-4.9175	AVE:	-15.6725	
E Minor Seventh									
Vl1	20	0.5	E4	-9.33	0.1875	-1.51	2.145	-23.02	-8.61
Vl2	20	0.5	B3	-15.22	-5.7025	-5.45	-1.795	-26.96	-12.55
Vla	20	0.5	G3	-7.41	2.1075	-5.46	-1.805	-5.46	8.95
Cel	20	0.5	D2II	-6.11	3.4075	-2.2	1.455	-2.2	12.21
				-9.5175	AVE:	-3.655	AVE:	-14.41	

Table 97. Major triad normalization tables (Vienna Quartet)

Instrument	Start	Dur	Note	ET - Vienna	Adj ET- Vienna	Ptyh - Vienna	Adj Pyth - Vienna	Just - Vienna	Adj Just - Vienna
ROOT POSITION									
C Major									
Vl1	0	0.5	E4	14.54	10.79	22.36	16.1675	0.85	0.035
Vl2	0	0.5	C4	0.24	-3.51	0.24	-5.9525	0.24	-0.575
Vla	0	0.5	G3	0.88	-2.87	2.83	-3.3625	2.83	2.015
Cel	0	0.5	C2	-0.66	-4.41	-0.66	-6.8525	-0.66	-1.475
				3.75	AVE:	6.1925	AVE:	0.815	
Vl1	1	0.5	E4	9.99	8.365	17.81	13.7425	-3.7	-2.39
Vl2	1	0.5	C4	-4.19	-5.815	-4.19	-8.2575	-4.19	-2.88
Vla	1	0.5	G3	1.49	-0.135	3.44	-0.6275	3.44	4.75
Cel	1	0.5	C2	-0.79	-2.415	-0.79	-4.8575	-0.79	0.52
				1.625	AVE:	4.0675	AVE:	-1.31	
Vl1	1.5	0.5	E4	6.13	1.335	13.95	6.7125	-7.56	-9.42
Vl2	1.5	0.5	C4	-1.38	-6.175	-1.38	-8.6175	-1.38	-3.24
Vla	1.5	0.5	G3	-3.32	-8.115	-1.37	-8.6075	-1.37	-3.23
Cel	1.5	0.5	C2	17.75	12.955	17.75	10.5125	17.75	15.89
				4.795	AVE:	7.2375	AVE:	1.86	
Vl1	2	0.5	E4	9.05	8.5175	16.87	13.895	-4.64	-2.2375
Vl2	2	0.5	C4	-1.45	-1.9825	-1.45	-4.425	-1.45	0.9525
Vla	2	0.5	G3	-2.71	-3.2425	-0.76	-3.735	-0.76	1.6425
Cel	2	0.5	C2	-2.76	-3.2925	-2.76	-5.735	-2.76	-0.3575
				0.5325	AVE:	2.975	AVE:	-2.4025	
Vl1	2.5	0.5	E4	7.22	0.8175	15.04	6.195	-6.47	-9.9375
Vl2	2.5	0.5	C4	-0.99	-7.3925	-0.99	-9.835	-0.99	-4.4575
Vla	2.5	0.5	G3	-5.15	-11.5525	-3.2	-12.045	-3.2	-6.6675
Cel	2.5	0.5	C2	24.53	18.1275	24.53	15.685	24.53	21.0625
				6.4025	AVE:	8.845	AVE:	3.4675	
Vl1	14.5	0.5	E4	-12.63	-14.2725	-4.81	-8.895	-26.32	-25.0275
Vl2	14.5	0.5	C4	-0.01	-1.6525	-0.01	-4.095	-0.01	1.2825
Vla	14.5	0.5	G3	8.32	6.6775	10.27	6.185	10.27	11.5625
Cel	14.5	0.5	C3	10.89	9.2475	10.89	6.805	10.89	12.1825
				1.6425	AVE:	4.085	AVE:	-1.2925	

Instrument	Start	Dur	Note	ET - Vienna	Adj ET- Vienna	Ptyh - Vienna	Adj Pyth - Vienna	Just - Vienna	Adj Just - Vienna
Vl1	17.5	0.5	E4	-6.34	2.5975	1.48	7.975	-20.03	-8.1575
Vl2	17.5	0.5	C4	-11.29	-2.3525	-11.29	-4.795	-11.29	0.5825
Vla	17.5	0.5	G3	-1.57	7.3675	0.38	6.875	0.38	12.2525
Cel	17.5	0.5	C3	-16.55	-7.6125	-16.55	-10.055	-16.55	-4.6775
				-8.9375	AVE:	-6.495	AVE:	-11.8725	
G Major									
Vl1	4.5	0.5	D4II	-6.81	-12.5775	-2.9	-13.0625	-2.9	-7.685
Vl2	4.5	0.5	B3	21.13	15.3625	30.9	20.7375	9.39	4.605
Vla	4.5	0.5	G3	7.88	2.1125	9.83	-0.3325	9.83	5.045
Cel	4.5	0.5	G2	0.87	-4.8975	2.82	-7.3425	2.82	-1.965
				5.7675	AVE:	10.1625	AVE:	4.785	
Vl1	22.5	0.5	G4	20.2	15.895	22.16	13.945	22.16	19.3225
Vl2	22.5	0.5	G4	20.2	15.895	22.16	13.945	22.16	19.3225
Vla	22.5	0.5	B3	-8.59	-12.895	1.18	-7.035	-20.33	-23.1675
Cel	22.5	0.5	G2	-14.59	-18.895	-12.64	-20.855	-12.64	-15.4775
				4.305	AVE:	8.215	AVE:	2.8375	
A Major									
Vl1	10.5	0.5	E4	0.93	0.5525	8.75	5.9275	-12.76	5.435
Vl2	10.5	0.5	C#4	-0.18	-0.5575	-9.96	-12.7825	-29.51	-11.315
Vla	10.5	0.5	A3I	-6.43	-6.8075	-0.56	-3.3825	-22.07	-3.875
Cel	10.5	0.5	A2I	7.19	6.8125	13.06	10.2375	-8.44	9.755
				0.3775	AVE:	2.8225	AVE:	-18.195	
D Major									
Vl1	19.5	0.5	D4II	0.75	6.1575	4.66	3.71	4.66	9.0875
Vl2	19.5	0.5	A3II	-11.15	-5.7425	-5.28	-6.23	-5.28	-0.8525
Vla	19.5	0.5	F#3	-7.26	-1.8525	4.48	3.53	-17.03	-12.6025
Cel	19.5	0.5	D3II	-3.97	1.4375	-0.06	-1.01	-0.06	4.3675
				-5.4075	AVE:	0.95	AVE:	-4.4275	
FIRST INVERSION									
C Major									
Vl1	3	1.5	G4	14.62	3.5975	16.58	2.625	16.58	8.0025
Vl2	3	1.5	C4	3.52	-7.5025	3.52	-10.435	3.52	-5.0575
Vla	3	2	G3	-3.49	-14.5125	-1.54	-15.495	-1.54	-10.1175
Cel	3	1.5	E2	29.44	18.4175	37.26	23.305	15.75	7.1725
				11.0225	AVE:	13.955	AVE:	8.5775	
SECOND INVERSION									
G Major									
Vl1	18	0.5	D4II	-16.13	-8.4475	-12.22	-9.4225	-12.22	-4.045
Vl2	18	0.5	B3	-2.78	4.9025	6.99	9.7875	-14.52	-6.345
Vla	18	0.5	G3	-12.62	-4.9375	-10.67	-7.8725	-10.67	-2.495
Cel	18	0.5	D3II	0.8	8.4825	4.71	7.5075	4.71	12.885
				-7.6825	AVE:	-2.7975	AVE:	-8.175	
Vl1	19	0.5	D4II	-14.87	-7.2425	-10.96	-8.2175	-10.96	-2.84
Vl2	19	0.5	B3	-2.16	5.4675	7.61	10.3525	-13.9	-5.78
Vla	19	0.5	G3	-14.87	-7.2425	-12.92	-10.1775	-12.92	-4.8
Cel	19	0.5	D3II	1.39	9.0175	5.3	8.0425	5.3	13.42
				-7.6275	AVE:	-2.7425	AVE:	-8.12	

Table 98. Minor triad normalization tables (Vienna Quartet)

Instrument	Start	Dur	Note	ET - Vienna	Adj ET- Vienna	Ptyh - Vienna	Adj Pyth - Vienna	Just - Vienna	Adj Just - Vienna
ROOT POSITION									
D Minor									
Vl1	6	0.5	F4	-5.3	-8.2025	-7.25	-13.0875	-7.25	3.045
Vl2	6	0.5	D4I	9.29	6.3875	13.2	7.3625	-8.31	1.985
Vla	6	0.5	A3I	5.62	2.7175	11.49	5.6525	-10.02	0.275
Cel	6	0.5	D2I	2	-0.9025	5.91	0.0725	-15.6	-5.305
				2.9025	AVE:	5.8375	AVE:	-10.295	

Instrument	Start	Dur	Note	ET - Vienna	Adj ET- Vienna	Ptyh - Vienna	Adj Pyth - Vienna	Just - Vienna	Adj Just - Vienna
Vl1	7	0.5	F4	-2.65	-6.2225	-4.6	-11.1075	-4.6	5.025
Vl2	7	0.5	D4I	-6.81	-10.3825	-2.9	-9.4075	-24.41	-14.785
Vla	7	0.5	A3I	6.18	2.6075	12.05	5.5425	-9.46	0.165
Cel	7	0.5	D2I	17.57	13.9975	21.48	14.9725	-0.03	9.595
				3.5725	AVE:	6.5075	AVE:	-9.625	
Vl1	7.5	0.5	F4	-7.88	-6.46	-9.83	-11.345	-9.83	4.7875
Vl2	7.5	0.5	D4I	-7.45	-6.03	-3.54	-5.055	-25.05	-10.4325
Vla	7.5	0.5	A3I	-5.81	-4.39	0.06	-1.455	-21.45	-6.8325
Cel	7.5	0.5	D2I	15.46	16.88	19.37	17.855	-2.14	12.4775
				-1.42	AVE:	1.515	AVE:	-14.6175	
Vl1	8	0.5	F4	-9.45	-6.605	-11.4	-11.49	-11.4	4.6425
Vl2	8	0.5	D4I	8.06	10.905	11.97	11.88	-9.54	6.5025
Vla	8	0.5	A3I	-8.68	-5.835	-2.81	-2.9	-24.32	-8.2775
Cel	8	0.5	D2I	-1.31	1.535	2.6	2.51	-18.91	-2.8675
				-2.845	AVE:	0.09	AVE:	-16.0425	
Vl1	8.5	0.5	F4	-11.39	-9.525	-13.34	-14.41	-13.34	1.7225
Vl2	8.5	0.5	D4I	0.75	2.615	4.66	3.59	-16.85	-1.7875
Vla	8.5	0.5	A3I	-6.97	-5.105	-1.1	-2.17	-22.61	-7.5475
Cel	8.5	0.5	D2I	10.15	12.015	14.06	12.99	-7.45	7.6125
				-1.865	AVE:	1.07	AVE:	-15.0625	
Vl1	12	0.5	F4	2.74	1.3225	0.79	-3.5625	0.79	12.5675
Vl2	12	0.5	D4I	-6.93	-8.3475	-3.02	-7.3725	-24.53	-12.7525
Vla	12	0.5	A3I	-5.58	-6.9975	0.29	-4.0625	-21.22	-9.4425
Cel	12	0.5	D3I	15.44	14.0225	19.35	14.9975	-2.15	9.6275
				1.4175	AVE:	4.3525	AVE:	-11.7775	
Vl1	15	0.5	F4	-4.66	-1.2175	-6.61	-4.6375	-6.61	6.115
Vl2	15	0.5	A3I	-3.41	0.0325	2.46	4.4325	-19.05	-6.325
Vla	15	0.5	F3	-13.63	-10.1875	-15.58	-13.6075	-15.58	-2.855
Cel	15	0.5	D3I	7.93	11.3725	11.84	13.8125	-9.66	3.065
				-3.4425	AVE:	-1.9725	AVE:	-12.725	
FIRST INVERSION									
D Minor									
Vl1	9	1.5	A4I	-9.14	-8.84	-3.27	-6.3925	-24.77	-11.7625
Vl2	9	1.5	D4I	3.32	3.62	7.23	4.1075	-14.28	-1.2725
Vla	9	2	A3I	-7.83	-7.53	-1.96	-5.0825	-23.47	-10.4625
Cel	9	1.5	F2	12.45	12.75	10.49	7.3675	10.49	23.4975
				-0.3	AVE:	3.1225	AVE:	-13.0075	

Table 99. Dominant seventh normalization tables (Vienna Quartet)

Instrument	Start	Dur	Note	ET - Vienna	Adj ET- Vienna	Ptyh - Vienna	Adj Pyth - Vienna	Just - Vienna	Adj Just - Vienna
ROOT POSITION									
G Dominant Seventh									
Vl1	14	0.5	F4	-8.66	6.76	-10.61	1.88	-10.61	7.2575
Vl2	14	0.5	B3	-29.63	-14.21	-19.86	-7.37	-41.37	-23.5025
Vla	14	0.5	G3	-5.58	9.84	-3.63	8.86	-3.63	14.2375
Cel	14	0.5	G2	-17.81	-2.39	-15.86	-3.37	-15.86	2.0075
				-15.42	AVE:	-12.49	AVE:	-17.8675	
Vl1	17	0.5	F4	-9.1	1.2275	-11.05	-3.6525	-11.05	1.725
Vl2	17	0.5	B3	-8.31	2.0175	1.46	8.8575	-20.05	-7.275
Vla	17	0.5	G3	-4.53	5.7975	-2.58	4.8175	-2.58	10.195
Cel	17	0.5	G2	-19.37	-9.0425	-17.42	-10.0225	-17.42	-4.645
				-10.3275	AVE:	-7.3975	AVE:	-12.775	

Instrument	Start	Dur	Note	ET - Vienna	Adj ET-Vienna	Ptyh - Vienna	Adj Pyth - Vienna	Just - Vienna	Adj Just - Vienna
D Dominant Seventh									
Vl1	20.5	0.5	F#4	5.93	8.5975	17.67	14.9575	-3.84	-1.175
Vl2	20.5	0.5	C4	-14.8	-12.1325	-14.8	-17.5125	-14.8	-12.135
Vla	20.5	0.5	A3II	1.49	4.1575	7.36	4.6475	7.36	10.025
Cel	20.5	0.5	D2II	-3.29	-0.6225	0.62	-2.0925	0.62	3.285
				-2.6675	AVE:	2.7125	AVE:	-2.665	
Vl1	21	1.5	A4II	-7.36	-5.3225	-1.49	-4.8325	-1.49	0.545
Vl2	21	1.5	F#4	6.02	8.0575	17.76	14.4175	-3.75	-1.715
Vla	21	1.5	C4	2.08	4.1175	2.08	-1.2625	2.08	4.115
Cel	21	1.5	D2II	-8.89	-6.8525	-4.98	-8.3225	-4.98	-2.945
				-2.0375	AVE:	3.3425	AVE:	-2.035	
FIRST INVERSION									
G Dominant Seventh									
Vl1	13.5	0.5	F4	-3.78	3.03	-5.73	-2.33	-5.73	3.0375
Vl2	13.5	0.5	D4II	-12.84	-6.03	-8.93	-5.53	-8.93	-0.1625
Vla	13.5	0.5	G3	-12.35	-5.54	-10.4	-7	-10.4	-1.6325
Cel	13.5	0.5	B2	1.73	8.54	11.46	14.86	-10.01	-1.2425
				-6.81	AVE:	-3.4	AVE:	-8.7675	
Vl1	16.5	0.5	F4	-4.46	3.8625	-6.41	-1.4975	-6.41	3.87
Vl2	16.5	0.5	D4II	-8.09	0.2325	-4.18	0.7325	-4.18	6.1
Vla	16.5	0.5	G3	-15.13	-6.8075	-13.18	-8.2675	-13.18	-2.9
Cel	16.5	0.5	B2	-5.61	2.7125	4.12	9.0325	-17.35	-7.07
				-8.3225	AVE:	-4.9125	AVE:	-10.28	

Table 100. Minor seventh normalization tables (Vienna Quartet)

Instrument	Start	Dur	Note	ET - Vienna	Adj ET-Vienna	Ptyh - Vienna	Adj Pyth - Vienna	Just - Vienna	Adj Just - Vienna
THIRD INVERSION									
D Minor Seventh									
Vl1	13	0.5	F4	-1.67	3.1325	-3.62	-0.775	-3.62	9.98
Vl2	13	0.5	D4I	10.23	15.0325	14.14	16.985	-7.37	6.23
Vla	13	0.5	A3I	-10.31	-5.5075	-4.44	-1.595	-25.95	-12.35
Cel	13	0.5	C3	-17.46	-12.6575	-17.46	-14.615	-17.46	-3.86
				-4.8025	AVE:	-2.845	AVE:	-13.6	
Vl1	16	0.5	F4	-4.56	-5.105	-6.51	-9.0125	-6.51	1.7425
Vl2	16	0.5	D4I	-12.03	-12.575	-8.12	-10.6225	-29.63	-21.3775
Vla	16	0.5	A3I	6.18	5.635	12.05	9.5475	-9.46	-1.2075
Cel	16	0.5	C3	12.59	12.045	12.59	10.0875	12.59	20.8425
				0.545	AVE:	2.5025	AVE:	-8.2525	
E Minor Seventh									
Vl1	20	0.5	E4	2.28	5.5075	10.1	7.465	-11.41	-3.29
Vl2	20	0.5	B3	-8.52	-5.2925	1.25	-1.385	-20.26	-12.14
Vla	20	0.5	G3	6.04	9.2675	7.99	5.355	7.99	16.11
Cel	20	0.5	D2II	-12.71	-9.4825	-8.8	-11.435	-8.8	-0.68
				-3.2275	AVE:	2.635	AVE:	-8.12	

Appendix II: Listing of All Chorales

Table 101 provides detailed information about each chorale used in this study, including Riemenschneider and Kalmus listings. Also given are the original keys, the keys actually used, and comments.

9.3 Key to Comments

K = Key analysed different than key given

M = Mode analysed different than mode given

Table 101. Listing of all chorales used in Bach Chorale Tuning Database

Music ID	Riemenschneider	Kalmus	Key Given	Key Analysed	Comments
000106B_.MID		378	F maj	F maj	
000206B_.MID	262	7	D min	G min	K
000306B_.MID	156, 308	8	A maj	A maj	
000408B_.MID	184	41	E min	E min	
000507B_.MID	304	28	G min	G min	
000606B_.MID	72	79	G min	G min	
000707B_.MID		44	B min	E min	K
000907B_.MID	290	87	E maj	E maj	
001007B_.MID	358	122	G min	G min	
001106B_.MID	343	82	D maj	D maj	
001207B_.MID		340	B♭ maj	B♭ maj	
001306B_.MID	103	295	B♭ maj	B♭ maj	
001405B_.MID	182	330	G min	G min	
001606B_.MID	99	125	A min	A min	
001707B_.MID	7	271	A maj	A maj	
001805B_.MID	100	73	G min	C min	K
001907B_.MID	298	99	C maj	C maj	
002007B_.MID	26	276	F maj	F maj	
002406BS.MID	337		F maj	F maj	
002506B_.MID	254, 282		C maj	C maj	
002606B_.MID	48	11	A min	A min	
002706B_.MID	150	350	B♭ maj	B♭ maj	
002806B_.MID	88, 23	124	A min	A min	
002908B_.MID	116	272	D maj	D maj	
003006B_.MID	76	103	A maj	A maj	
003109B_.MID		357	C maj	C maj	
003206B_.MID	29	102	G maj	G maj	
003306B_.MID	13	16	C maj	C maj	
003604B2.MID	86, 195, 305	377	D maj	D maj	
003608B2.MID	28	264	B min	B min	
003706B_.MID	341	178	A maj	A maj	
003806B_.MID	10	31	A min	A min	
003907B_.MID	67	104	G maj	G maj	
004003B_.MID	321	379	G min	G min	
004006B_.MID	142	305	D min	D min	
004008B_.MID	8	105	C min	F min	K
004106B_.MID	11	204	C maj	C maj	
004207B_.MID	91, 259	322	F# min	F# min	
004311B_.MID	102	81	G maj	G maj	

Music ID	Riemenschneider	Kalmus	Key Given	Key Analysed	Comments
004407B_.MID	355	296	B♭ maj	B♭ maj	
004507B_.MID	85	278	E maj	E maj	
004606BS.MID			D min	G min	K
004705B_.MID	94	333	G min	G min	
004803B_.MID	279	4	B♭ maj	B♭ maj	
004807B_.MID	266	144	G min	G min	
005206B_.MID		212	F maj	F maj	
005505B_.MID	95	362	B♭ maj	B♭ maj	
005605B_.MID	87	72	G min	C min	K
005708B_.MID	90	231	B♭ maj	B♭ maj	
005903B_.MID		220	G maj	G maj	
006005B_.MID	216	91	A maj	A maj	
006206B_.MID	170	265	A min	A min	
006402B_.MID	160	108	A min	G maj	K, M
006404B_.MID	255	280	D maj	D maj	
006408B_.MID	138	200	E min	E min	
006502B_.MID	12	302	C maj	A min	M
006507B_.MID	41	346	A min	A min	
006606B_.MID		37	F# min	F# min	
006707B_.MID	42	68	A maj	A maj	
006906B_.MID	333	97	D maj	D maj	
006906BA.MID	293		G maj	G maj	
007007B_.MID		98	G maj	G maj	
007011B_.MID	348	243	C maj	C maj	
007205B_.MID		344	A min	A min	
007305B_.MID	191	328	A min	A min	
007408B_.MID	370	223	A min	A min	
007706B_.MID	253	6	D min	G min	K
007807B_.MID	297	188	G min	G min	
007906B_.MID		267	G maj	G maj	
008008B_.MID	273	76	D maj	D maj	
008107B_.MID	324	197	E min	E min	
008305B_.MID	325	250	A min	D min	K
008405B_.MID	112	373	B min	B min	
008506B_.MID	122	216	G min	C min	K
008606B_.MID	4	86	E maj	E maj	
008707B_.MID	96	201	D min	D min	
008807B_.MID	104	368	B min	B min	
008906B_.MID	281	26	G min	G min	
009005B_.MID	267	319	D min	D min	
009106B_.MID	51	109	G maj	G maj	
009209B_.MID		347	B min	B min	
009307B_.MID		369	C min	C min	
009408B_.MID	291	281	D maj	D maj	
009507B_.MID		356	G maj	G maj	
009606B_.MID	303	128	D min	F maj	M
009709B_.MID		297	B♭ maj	B♭ maj	
009906B_.MID		341	G maj	G maj	
010107B_.MID	292	318	D min	D min	
010207B_.MID	110	320	G min	C min	K
010306B_.MID	120, 349	348	B min	B min	
010406B_.MID	326	13	A maj	A maj	
010406BG.MID	125		G maj	G maj	
010806B_.MID	45	224	B min	B min	
011007B_.MID	55	380	B min	B min	
011106B_.MID		345	A min	A min	

Music ID	Riemenschneider	Kalmus	Key Given	Key Analysed	Comments
011308B_.MID	294	142	B min	B min	
011407B_.MID	301	386	G min	G min	
011506B_.MID	38	312	G maj	G maj	
011606B_.MID		69	A maj	A maj	
011704B_.MID	248, 354	90	G maj	G maj	
011909B_.MID		134	C maj	C maj	
012006B_.MID		135	B min	B min	
012008BA.MID		230	D maj	D maj	
012206B_.MID	53, 178	57	G min	G min	
012306B_.MID	194	229	B min	B min	
012406B_.MID		246	E maj	E maj	
012506B_.MID		251	E min	E min	
012606B_.MID	215	321	G min	G min	
012705B_.MID	284	147	F maj	F maj	
012805B_.MID		279	G maj	G maj	
013006B_.MID		131	C maj	C maj	
013306B_.MID	60	181	D maj	D maj	
013506B_.MID		156	A min	A min	
013606B_.MID	331	27	B min	B min	
013705B_.MID		230	C maj	C maj	
013906B_.MID		238	E maj	E maj	
014007B_.MID	179	329	E♭ maj	E♭ maj	
014403B_.MID	65	338	G maj	G maj	
014406B_.MID	265	343	B min	B min	
014500B_.MID	338	209	D maj	D maj	
014505B_.MID	17	84	B min	F# min	K
014608B_.MID		360	F maj	F maj	
014806B_.MID	25	29	F# min	F# min	
014907B_.MID		155	C maj	C maj	
015105B_.MID	54	235	G maj	G maj	
015301B_.MID	3	5	E min	A min	K
015305B_.MID	21	160	A min	A min	
015309B_.MID	217	9	C maj	C maj	
015403B_.MID	233	365	A maj	A maj	
015408B_.MID	152	244	D maj	D maj	
015505B_.MID	335	88	F maj	F maj	
015606B_.MID	317	150	C maj	C maj	
015705B_.MID		245	D maj	D maj	
015804B_.MID	261	40	E min	E min	
015905B_.MID	61	194	E♭ maj	E♭ maj	
016106B_.MID	270	161	A min	A min	
016206B_.MID		18	A min	A min	
016406B_.MID	101	127	B♭ maj	B♭ maj	
016506B_.MID		266	G maj	G maj	
016606B_.MID	204	372	B♭ maj	G min	M
016806B_.MID	92	143	B min	B min	
016907B_.MID	97	256	A maj	A maj	
017206B_.MID	323	376	F maj	F maj	
017405B_.MID	58	153	D maj	D maj	
017606B_.MID	119	45	G min	F min	K
017705B_.MID	71	183	G min	F min	K
017807B_.MID		384	A min	A min	
017906B_.MID	339	371	A min	A min	
018007B_.MID	22	304	F maj	F maj	
018305B_.MID	123	126	A min	A min	
018400BX.MID	14		G maj	G maj	

Music ID	Riemenschneider	Kalmus	Key Given	Key Analysed	Comments
018405B_.MID		283	D maj	D maj	
018506B_.MID		184	F# min	F# min	
018707B_.MID	109	308	G min	G min	
018806B_.MID		25	A min	A min	
019406B_.MID	256	100	B♭ maj	B♭ maj	
019412B_.MID	93, 257		B♭ maj	B♭ maj	
019506B_.MID		236	G maj	G maj	
019705B_.MID	84	255	A maj	A maj	
019710B_.MID	62	370	B min	B min	
022604B_.MID	69	221	B♭ maj	B♭ maj	
022701B_.MID	263	196	E min	E min	
022703B_.MID		198	E min	E min	
022707B_.MID	283	199	E min	E min	
022709B_.MID			A min	A min	
022906B_.MID		222	G min	G min	
024401BB.MID		247	E maj	E maj	
024403B_.MID	78	166	B min	B min	
024410B_.MID	117	294	A♭ maj	A♭ maj	
024415B_.MID	98	163	D maj	D maj	
024425B_.MID	115	342	B min	B min	
024432B_.MID	118	213	B♭ maj	B♭ maj	
024437B_.MID	50	292	F maj	F maj	
024440B_.MID	121	361	F# min	A maj	M
024444B_.MID	80	159	D maj	D maj	
024446B_.MID	105	167	B min	B min	
024454B_.MID	74	162	D min	F maj	M
024462B_.MID	89	164	B min	B min	
024503B_.MID	59	168	B♭ maj	G min	M
024505B_.MID		317	D min	D min	
024511B_.MID	63	293	A maj	A maj	
024514B_.MID	83	192	F# min	A maj	M
024515B_.MID	81	49	A min	A min	
024517B_.MID	111	169	A min	A min	
024522B_.MID	310	239	E maj	E maj	
024526B_.MID	108	315	E♭ maj	E♭ maj	
024528B_.MID	106	193	A maj	A maj	
024537B_.MID	113	50	C min	B♭ min	K
024540B_.MID	107	154	E♭ maj	E♭ maj	
024805B_.MID	345	165	A min	A min	
024809BS.MID	46		D maj	D maj	
024812B_.MID	9, 361	80	G maj	G maj	
024817B_.MID		323	C maj	C maj	
024823B_.MID	344		G maj	G maj	
024823BS.MID			G maj	G maj	
024833B_.MID	139	335	G maj	G maj	
024835B_.MID	360	381	F# min	F# min	
024842BS.MID	368		F maj	F maj	
024846B_.MID	77	214	A maj	A maj	
024853B_.MID	35	114	A maj	A maj	
024859B_.MID	362	263	G maj	G maj	
025000B_.MID	347	339	G maj	G maj	
025100B_.MID	329	89	G maj	G maj	
025200B_.MID	330	258	G maj	G maj	
025300B_.MID	177	1	A maj	A maj	
025400B_.MID	186	2	A min	D min	K
025500B_.MID	40	3	C maj	C maj	

Music ID	Riemenschneider	Kalmus	Key Given	Key Analysed	Comments
025600B_.MID	31	385	A min	A min	
025700B_.MID	285	388	A min	A min	
025800B_.MID	336	383	B min	B min	
025900B_.MID	39	10	E min	E min	
026000B_.MID	249	12	G maj	G maj	
026200B_.MID	153	17	D maj	D maj	
026300B_.MID	128	19	G maj	G maj	
026400B_.MID	159	20	G maj	G maj	
026500B_.MID	180	21	A min	D min	K
026600B_.MID	208	22	E min	E min	
026700B_.MID	5	23	G maj	G maj	
026800B_.MID	124	24	G maj	G maj	
026900B_.MID	1	30	G maj	G maj	
027000B_.MID	286	157	B min	B min	
027100B_.MID	367	158	D maj	B min	M
027200B_.MID	340	32	D min	D min	
027300B_.MID	230	33	G min	G min	
027400B_.MID	245	34	B♭ maj	G min	M
027500B_.MID	210	35	C maj	D min	K, M
027600B_.MID	197	36	C maj	D min	K, M
027700B_.MID	15	38	A min	D min	K
027800B_.MID	371	39	E min	E min	
028000B_.MID	66	43	A min	D min	K
028100B_.MID	6	46	F maj	F maj	
028200B_.MID	316	47	G maj	G maj	
028500B_.MID	196	52	C min	C min	
028600B_.MID	228	53	A min	A min	
028700B_.MID	311	54	F maj	F maj	
028800B_.MID	162	55	A min	D min	K
028900B_.MID	314	56	E min	E min	
029000B_.MID	224	58	F maj	F maj	
029100B_.MID	75	59	D min	D min	
029200B_.MID	239	60	C maj	C maj	
029400B_.MID	158	62	G maj	G maj	
029500B_.MID	207	63	D min	D min	
029600B_.MID	231	64	C maj	G maj	K
029700B_.MID	232	65	A min	D min	K
029800B_.MID	127	66	C maj	C maj	
029900B_.MID	209	67	B♭ maj	B♭ maj	
030000B_.MID	167	70	E min	E min	
030100B_.MID	134	71	D min	D min	
030200B_.MID	20	74	D maj	D maj	
030300B_.MID	250	75	D maj	D maj	
030400B_.MID	280	77	D maj	D maj	
030500B_.MID	34	78	A min	A min	
030600B_.MID	176	85	F maj	F maj	
030700B_.MID	260	262	B♭ maj	B♭ maj	
030800B_.MID	27	92	B♭ maj	B♭ maj	
030900B_.MID	166	93	D min	G min	K
031000B_.MID	238	94	E min	E min	
031100B_.MID	16	95	B min	B min	
031200B_.MID	352	96	A min	A min	
031300B_.MID	163	106	G min	G min	
031500B_.MID	271	111	E min	E min	
031600B_.MID	225	112	G min	G min	
031700B_.MID	135	113	D maj	D maj	

Music ID	Riemenschneider	Kalmus	Key Given	Key Analysed	Comments
031800B_.MID	18	115	G maj	G maj	
031900B_.MID	181	116	E min	E min	
032000B_.MID	234	117	F maj	F maj	
032100B_.MID	192	118	B♭ maj	B♭ maj	
032200B_.MID	70	119	C maj	C maj	
032500B_.MID	235, 319	123	F maj	F maj	
032600B_.MID	164	129	B♭ maj	B♭ maj	
032700B_.MID	334	132	D maj	D maj	
032900B_.MID	212	136	E♭ maj	E♭ maj	
033000B_.MID	33	137	A min	A min	
033100B_.MID	287	138	A min	A min	
033200B_.MID	136	139	G maj	G maj	
033300B_.MID	226	140	G min	G min	
033400B_.MID	73	141	G min	G min	
033500B_.MID	295, 236	145	E min	E min	
033600B_.MID	189	146	A maj	A maj	
033700B_.MID	190	148	A min	A min	
033800B_.MID	221	149	A min	A min	
033900B_.MID	318, 144	151	A maj	A maj	
034000B_.MID	277	152	C maj	C maj	
034100B_.MID	168	170	D min	G min	K
034200B_.MID	79	171	A min	A min	
034300B_.MID	302, 199	172	D min	G min	K
034500B_.MID	251	174	G min	G min	
034600B_.MID	223	175	C maj	C maj	
034700B_.MID	2	176	A maj	A maj	
034800B_.MID	272	177	B♭ maj	B♭ maj	
034900B_.MID	188	179	F maj	F maj	
035000B_.MID	229	180	G min	G min	
035100B_.MID	19	182	D min	G min	K
035200B_.MID	37	185	A min	A min	
035300B_.MID	269	186	G min	G min	
035400B_.MID	369	187	F min	B♭ min	K, M
035500B_.MID	169	189	A maj	A maj	
035600B_.MID	243	190	G min	G min	
035700B_.MID	244	191	G min	C min	K
035800B_.MID	356	195	D min	D min	
035900B_.MID	365	363	A maj	A maj	
036000B_.MID	350	364	B♭ maj	B♭ maj	
036100B_.MID	264	202	B♭ maj	B♭ maj	
036200B_.MID	252	203	B♭ maj	B♭ maj	
036300B_.MID	30	206	E min	E min	
036500B_.MID	175	208	C maj	C maj	
036600B_.MID	161	210	D min	D min	
036700B_.MID	140	211	C maj	C maj	
036800B_.MID	143	215	F maj	F maj	
036900B_.MID	129	217	E min	E min	
037000B_.MID	187	218	C maj	C maj	
037200B_.MID	218	226	D min	G min	K
037300B_.MID	131, 328	228	G maj	G maj	
037400B_.MID	227	232	D min	G min	K
037500B_.MID	276	233	G maj	G maj	
037600B_.MID	342	234	A maj	A maj	
037700B_.MID	44	237	D maj	D maj	
037800B_.MID	258	240	G maj	G maj	
037900B_.MID	151	241	G maj	G maj	

Music ID	Riemenschneider	Kalmus	Key Given	Key Analysed	Comments
038000B_.MID	299	242	E♭ maj	E♭ maj	
038100B_.MID	346	248	E min	E min	
038200B_.MID	49	249	A min	D min	K
038300B_.MID	214	252	A min	A min	
038400B_.MID	149	253	E♭ maj	C min	M
038500B_.MID	36	254	A maj	A maj	
038700B_.MID	185	260	C maj	D min	K, M
038900B_.MID	268	269	C maj	C maj	
039000B_.MID	296	270	C maj	C maj	
039100B_.MID	222	273	G maj	G maj	
039200B_.MID	289	298	B♭ maj	B♭ maj	
039300B_.MID	275	289	A maj	A maj	
039400B_.MID	366	290	A maj	A maj	
039500B_.MID	363	291	A maj	A maj	
039600B_.MID	240	274	A min	A min	
039700B_.MID	274	275	F maj	F maj	
039900B_.MID	315	282	E min	G maj	M
040000B_.MID	173	284	E♭ maj	E♭ maj	
040100B_.MID	165	285	F maj	F maj	
040200B_.MID	201, 306	286	E♭ maj	E♭ maj	
040300B_.MID	203	287	D min	G min	K
040400B_.MID	57	288	A min	A min	
040500B_.MID	213	299	D min	D min	
040600B_.MID	219	300	D min	D min	
040700B_.MID	202	301	D maj	D maj	
040800B_.MID	171	303	D min	G min	K
040900B_.MID	141	306	A maj	A maj	
041000B_.MID	172	307	G min	G min	
041100B_.MID	246	309	G maj	G maj	
041200B_.MID	206	310	D min	G min	K
041300B_.MID	220	311	D min	D min	
041400B_.MID	148	313	G maj	G maj	
041500B_.MID	24	314	D maj	D maj	
041600B_.MID	47	316	D min	D min	
041700B_.MID	364	324	B min	B min	
041800B_.MID	332	325	A min	A min	
041900B_.MID	114	326	A min	A min	
042000B_.MID	145	331	A min	A min	
042100B_.MID	300	332	A min	A min	
042200B_.MID	357	334	C maj	C maj	
042300B_.MID	237	336	D min	G min	K
042400B_.MID	193	337	A min	A min	
042500B_.MID	241	349	A min	D min	K
042600B_.MID	211	351	C maj	C maj	
042700B_.MID	147	352	E♭ maj	E♭ maj	
042800B_.MID	322	353	G maj	G maj	
042900B_.MID	52	354	A maj	A maj	
043000B_.MID	351	355	A maj	A maj	
043100B_.MID	68	358	F maj	F maj	
043200B_.MID	247	359	G maj	G maj	
043300B_.MID	137	366	G maj	G maj	
043400B_.MID	146	367	A min	A min	
043500B_.MID	242	374	E min	E min	
043600B_.MID	278	375	E maj	E maj	
043700B_.MID	133	382	A min	D min	K
043800B_.MID	157	389	F maj	F maj	

Appendix III: Chord Indices for the Bach Chorale Tuning Database

Table 102 includes every chord and its associated index used in generating the Bach Chorale Tuning Database of Chapter 4, beginning on p. 65. Each line begins with the ‘#‘ character, followed by the index value. Immediately following the index value is the number of times the chord represented by the index appears in the entire collection. The remaining characters, representing the notes present in the chord, originated as MIDI values, and therefore might be enharmonic equivalents for those actually used. The decision of whether to adopt a sharp or flat for a given note was an intuitive one. There are no doubt instances when F# was used instead of G♭, as both are accounted for by the same MIDI value.

Table 102. Indices generated for Bach Chorale Tuning Database

```
#1 1 C# C# D E F
#2 2 C# C# D F A
#3 1 C# C# E
#4 1 C# C# E F
#5 4 C# C# E F A
#6 2 C# C# E F# A
#7 10 C# C# E G
#8 24 C# C# E G B♭
#9 73 C# C# E G A
#10 2 C#/D♭ C#/D♭ F/E#
#11 10 C#/D♭ C#/D♭ F/E# G#/A♭
#12 1 D♭ B♭ D♭ E♭ F
#13 10 D♭ B♭ D♭ F
#14 1 C# B C# E F#
#15 2 C# B C# E G
#16 2 D♭ C D♭ E♭ G
#17 1 D♭ C D♭ F
#18 2 D♭ F G B♭ D♭
#19 1 D♭ F A♭ C D♭
#20 1 C# F# G A C#
#21 1 C# F# A B C#
#22 1 D♭ G B♭ D♭
#23 7 D♭ G B♭ D♭ E♭
#24 2 C# G A C#
#25 2 D♭ A♭ C D♭ E♭
#26 1 C# A C# D E
#27 40 C# A C# E
#28 3 A# C# E G A#
#29 1 A#/B♭ A#/B♭
#30 5 B♭ B♭ C D
#31 4 B♭ B♭ C D E♭
#32 14 B♭ B♭ C D F
#33 8 B♭ B♭ C E♭
#34 4 B♭ B♭ C E♭ F
#35 3 B♭ B♭ C E
#36 1 A# A# C# D E
#37 1 A# A# C# E
#38 8 B♭ B♭ D♭ F
#39 26 B♭ B♭ D
#40 11 B♭ B♭ D E♭
#41 6 B♭ B♭ D E♭ F
#42 1 B♭ B♭ D E
#43 663 B♭ B♭ D F
#44 13 B♭ B♭ E♭
#45 50 B♭ D E♭ G B♭
#46 18 B♭ D E G B♭
#47 21 B♭ D F G B♭
#48 1 B♭ D F G A♭ B♭
#49 150 B♭ D F A♭ B♭
#50 18 B♭ D F A B♭
#51 1 B♭ D F# B♭
#52 2 B♭ D F# A B♭
#53 75 B♭ E♭ F B♭
#54 3 B♭ E♭ F G B♭
#55 4 B♭ E♭ F A♭ B♭
#56 90 B♭ E♭ G B♭
#57 1 B♭ E♭ G A♭ B♭
#58 3 B♭ E♭ G A B♭
#59 13 B♭ E♭ A♭ B♭
#60 15 A#/B♭ E G A#/B♭
#61 70 B♭ E G B♭ C
#62 1 A#/B♭ E G A A#/B♭
#63 6 B♭ F B♭
#64 20 B♭ F B♭ C
#65 4 B♭ F G B♭
#66 5 B♭ F G B♭ C
#67 2 B♭ F G B♭ C D
#68 5 B♭ F G B♭ D♭
#69 6 B♭ F A♭ B♭
#70 28 B♭ F A♭ B♭ C
```

#71 4 A#/B♭ E#/F G#/A♭ A#/B♭ C#/D♭
#72 5 B♭ F A B♭ C
#73 1 A#/B♭ F#/G♭ A A#/B♭ C
#74 3 A#/B♭ G A#/B♭
#75 4 B♭ G B♭ C
#76 22 B♭ G B♭ C D
#77 3 B♭ G B♭ C D E♭
#78 1 B♭ G B♭ C D E
#79 159 B♭ G B♭ C E♭
#80 5 A#/B♭ G A#/B♭ C#/D♭
#81 2 B♭ G B♭ D♭ E♭
#82 174 B♭ G B♭ D
#83 1 B♭ G A♭ B♭ C
#84 1 B♭ G A B♭
#85 1 B♭ G A B♭ C#/D♭
#86 3 B♭ G A B♭ D
#87 5 B♭ A♭ B♭ C D
#88 10 B♭ A♭ B♭ C E♭
#89 38 B♭ A♭ B♭ D
#90 12 B♭ A♭ B♭ D E♭
#91 1 B♭ A B♭
#92 4 B♭ A B♭ C D
#93 2 B♭ A B♭ C E♭
#94 15 B♭ A B♭ D
#95 3 B B C D
#96 1 B B C D E♭
#97 5 B B C D E
#98 5 B B C D E G
#99 12 B B C D F
#100 1 B B C D F#
#101 20 B B C E
#102 2 B B C E F
#103 5 B B C F
#104 1 B B C# D E
#105 1 B B C# E G
#106 5 B B D
#107 1 B B D D#/E♭ F
#108 3 B B D E
#109 1 B B D E F G
#110 60 B B D E G
#111 86 B B D F
#112 373 B B D F G
#113 11 B B D F#/G♭
#114 1 B B D#/E♭
#115 6 B B D# F#
#116 2 B B E
#117 1 B D D#/E♭ F G#/A♭ B
#118 1 B D F G A#/B♭ B
#119 4 B D F G A B
#120 60 B D F G#/A♭ B
#121 1 B/C♭ D#/E♭ F G B/C♭
#122 2 B/C♭ D#/E♭ E#/F G#/A♭ B/C♭
#123 1 B/C♭ D#/E♭ G B/C♭
#124 1 B/C♭ D#/E♭ G G#/A♭ B
#125 1 B E F# B
#126 36 B E G B
#127 176 B E G B C
#128 1 B E G A B
#129 2 B E G A B C
#130 3 B E G# B
#131 2 B E G# B C
#132 6 B E A B
#133 7 B F G B
#134 2 B F G B C
#135 5 B F G#/A♭ B
#136 2 B F G#/A♭ B C
#137 2 B F A B
#138 42 B F A B C
#139 3 B F# A B C
#140 8 B F# A B D
#141 4 B G B
#142 3 B G B C
#143 29 B G B C D
#144 4 B G B C D#/E♭
#145 863 B G B D
#146 13 B G B D D#/E♭
#147 3 B G G#/A♭ B D
#148 1 B G A B C
#149 2 B G A B C# D E
#150 9 B G A B D
#151 1 B G#/A♭ B C
#152 36 B G#/A♭ B D
#153 6 B G#/A♭ B D E
#154 2 B G#/A♭ B D#/E♭
#155 1 B A B C
#156 7 B A B C D
#157 8 B A B C D F
#158 21 B A B C E
#159 1 B A B C#/D♭ E
#160 37 B A B D
#161 4 B A B D E
#162 57 B A B D F
#163 8 B A B D#
#164 4 C B♭ C D E♭
#165 1 C B♭ C D E♭ F
#166 1 C B♭ C D E
#167 16 C B♭ C D F
#168 20 C B♭ C E♭
#169 14 C B♭ C E♭ F
#170 33 C B♭ C E
#171 1 C B♭ C E F
#172 5 C B C D E
#173 3 C B C D E F G
#174 6 C B C D E G
#175 22 C B C D F
#176 3 C B C D F G
#177 4 C B C D#/E♭
#178 2 C B C D# F#
#179 8 C B C E
#180 4 C B C F
#181 6 C C
#182 2 C C C#/D♭ E G
#183 4 C C D
#184 14 C C D E♭
#185 4 C C D E♭ F
#186 117 C C D E♭ G
#187 6 C C D E
#188 3 C C D E F
#189 10 C C D E F G
#190 143 C C D E G

#191 62 C C D F
#192 126 C C D F G
#193 44 C C D F A♭
#194 7 C C D F#
#195 1 C C D F# G
#196 90 C C E♭
#197 28 C C E♭ F
#198 10 C C E♭ F G
#199 2 C C E♭ F G A♭
#200 19 C C E♭ F A♭
#201 7 C C D#/E♭ F#/G♭
#202 1694 C C E♭ G
#203 100 C C E
#204 19 C C E F
#205 17 C C E F G
#206 5 C C E F#
#207 4 C C E F# G
#208 2670 C C E G
#209 17 C C F
#210 9 C D#/E♭ F#/G♭ A C
#211 13 C E F G A C
#212 1 C E F A♭ C
#213 43 C E F A C
#214 2 C E F# A B C
#215 8 C E F# A C
#216 70 C E G B♭ C
#217 86 C E G B C
#218 1 C E G A♭ C
#219 1 C E G A B C
#220 129 C E G A C
#221 2 C E G#/A♭ B C
#222 5 C E G#/A♭ C
#223 5 C F B♭ C
#224 2 C F G B♭ C
#225 5 C F G B C
#226 148 C F G C
#227 3 C F G A♭ C
#228 2 C F G A C
#229 2 C F A♭ B♭ C
#230 2 C F A♭ B C
#231 41 C F A♭ C
#232 1 C F A♭ C D♭
#233 1 C F A B C
#234 76 C F A C
#235 1 C G♭ B♭ C D
#236 1 C F# G C
#237 1 C F#/G♭ G#/A♭ C D
#238 1 C F#/G♭ A A#/B♭ C
#239 22 C F#/G♭ A C
#240 77 C F# A C D
#241 25 C G B♭ C
#242 1 C G B♭ C D♭
#243 30 C G B♭ C D
#244 1 C G B♭ C D E
#245 78 C G B♭ C E♭
#246 5 C G B C
#247 39 C G B C D
#248 12 C G B C E♭
#249 46 C G C
#250 119 C G C D
#251 2 C G A♭ C
#252 8 C G A♭ C E♭
#253 2 C G A B♭ C
#254 1 C G A B♭ C E♭
#255 13 C G A C
#256 8 C G A C D
#257 1 C G A C D E
#258 47 C G A C E♭
#259 3 C A♭ B♭ C E♭
#260 1 C A♭ B C D
#261 1 C A♭ C
#262 1 C A♭ C D
#263 4 C A♭ C D E♭
#264 1 C A♭ C D E
#265 144 C A♭ C E♭
#266 1 C A B♭ C D
#267 8 C A B C E
#268 4 C A C
#269 2 C A C D
#270 13 C A C D E
#271 4 C A C D E F
#272 53 C A C D F
#273 120 C A C D#/E♭
#274 25 C A C E♭ F
#275 164 C A C E
#276 5 D C# D F A
#277 2 D C# D G
#278 1 D B♭ C D E♭
#279 1 D B♭ C D E
#280 3 D B♭ C D F
#281 2 D B♭ D♭ D E
#282 2 D B♭ D
#283 7 D B♭ D E♭ F
#284 2 D B♭ D E F
#285 289 D B♭ D F
#286 2 D B C D E
#287 6 D B C D F
#288 2 D B C D F G
#289 1 D B C D F#
#290 4 D B D
#291 5 D B D E♭
#292 5 D B D E♭ F
#293 2 D B D E
#294 6 D B D E F
#295 1 D B D E F G
#296 34 D B D E G
#297 462 D B D F
#298 101 D B D F G
#299 35 D B D F#
#300 5 D C D
#301 4 D C D E♭
#302 7 D C D E♭ F
#303 76 D C D E♭ G
#304 1 D C D E
#305 16 D C D E F
#306 3 D C D E F G
#307 2 D C D E F#
#308 92 D C D E G
#309 179 D C D F
#310 131 D C D F G
#311 21 D C D F A♭
#312 67 D C D F#

#313 7 D C D F# G
#314 4 D D
#315 6 D D E♭ F G
#316 9 D D E♭ F A
#317 16 D D E♭ G
#318 48 D D E♭ G B♭
#319 4 D D E♭ G A
#320 1 D D E♭ A♭
#321 2 D D E F
#322 5 D D E F G
#323 1 D D E F G#/A♭
#324 41 D D E F A
#325 6 D D E F# A
#326 11 D D E G
#327 11 D D E G B♭
#328 12 D D E G A
#329 1 D D E G A B
#330 1 D D E G#
#331 23 D D E A
#332 33 D D F
#333 39 D D F G
#334 53 D D F G B♭
#335 4 D D F G A
#336 1 D D F G A B
#337 42 D D F G#/A♭
#338 69 D D F A♭ B♭
#339 40 D D F G#/A♭ B
#340 493 D D F A
#341 14 D D F#
#342 14 D D F#/G♭ A#/B♭
#343 11 D D F# G
#344 1 D D F#/G♭ G#/A♭
#345 378 D D F# A
#346 2 D D F# A B♭
#347 15 D D G
#348 1 D D G G#/A♭
#349 89 D D G A
#350 2 D D G#/A♭
#351 1 D F G A C D
#352 6 D F# G B D
#353 1 D F# G A B D
#354 1 D F# G A C D
#355 1 D F#/G♭ G#/A♭ C D
#356 11 D F# A B D
#357 203 D F# A C D
#358 3 D G B♭ C D
#359 48 D G B♭ D
#360 5 D G B C D
#361 94 D G B D
#362 9 D G B D E♭
#363 68 D G C D
#364 1 D G A B♭ D
#365 4 D G A B D
#366 27 D G A C D
#367 3 D A♭ B♭ D
#368 4 D G#/A♭ B D
#369 3 D G#/A♭ B D E
#370 13 D A♭ C D
#371 15 D A♭ C D E♭
#372 2 D A B♭ D
#373 1 D A B D
#374 4 D A B D E
#375 30 D A B D F
#376 43 D A C D
#377 3 D A C D E♭
#378 34 D A C D E
#379 1 D A C D E F
#380 87 D A C D F
#381 2 D A C# D E
#382 10 D A D
#383 3 E♭ D♭ E♭ G
#384 3 E♭ B♭ C E♭
#385 3 E♭ B♭ C E♭ F
#386 1 E♭ B♭ D♭ E♭ F
#387 1 E♭ B♭ D E♭
#388 14 E♭ B♭ D E♭ F
#389 16 E♭ B♭ D#/E♭
#390 13 D#/E♭ B D D#/E♭ F
#391 1 D# B D# E F#
#392 16 D# B D# F#
#393 2 E♭ C D E♭
#394 21 E♭ C D E♭ G
#395 14 D#/E♭ C D#/E♭
#396 5 E♭ C E♭ F
#397 10 E♭ C E♭ F G
#398 24 E♭ C E♭ F A♭
#399 4 E♭ C E♭ F#/G♭
#400 541 E♭ C E♭ G
#401 3 E♭ D E♭ F
#402 8 E♭ D E♭ F G
#403 2 E♭ D E♭ F G B♭
#404 1 E♭ D E♭ F A♭
#405 35 E♭ D E♭ G
#406 47 E♭ D E♭ G B♭
#407 1 E♭ D E♭ G A♭
#408 1 E♭ D E♭ G A
#409 5 D#/E♭ D#/E♭
#410 1 D# D# E F# A C
#411 1 D# D# E G B
#412 2 D# D# E A
#413 3 D#/E♭ D#/E♭ F
#414 39 E♭ E♭ F B♭
#415 8 E♭ E♭ F G
#416 55 E♭ E♭ F G B♭
#417 5 E♭ E♭ F G B
#418 1 E♭ E♭ F G A♭
#419 1 E♭ E♭ F G A
#420 1 E♭ E♭ F G A C
#421 14 E♭ E♭ F A♭
#422 8 E♭ E♭ F A♭ B♭
#423 1 E♭ E♭ F A♭ B
#424 2 E♭ E♭ F A
#425 1 E♭ E♭ F A B♭
#426 1 E♭ E♭ F# G B
#427 1 D#/E♭ D#/E♭ F#/G♭ G A C
#428 2 D#/E♭ D#/E♭ F#/G♭ A
#429 13 D# D# F# A B
#430 14 D#/E♭ D#/E♭ F#/G♭ A C
#431 60 E♭ E♭ G
#432 808 E♭ E♭ G B♭
#433 10 E♭ E♭ G B
#434 1 E♭ E♭ G A♭ B♭

#435 7 E♭ E♭ G A
#436 1 E♭ E♭ G A B♭
#437 2 D#/E♭ D#/E♭ G#/A♭
#438 26 E♭ E♭ A♭ B♭
#439 33 E♭ G B♭ C E♭
#440 22 E♭ G B♭ D♭ E♭
#441 1 E♭ G B C E♭
#442 9 E♭ G B D E♭
#443 10 E♭ G A♭ C E♭
#444 30 E♭ G A C E♭
#445 1 E♭ A♭ B♭ C E♭
#446 1 E♭ A♭ B♭ D E♭
#447 8 E♭ A♭ C E♭
#448 2 E♭ A B♭ E♭
#449 1 D#/E♭ A B D#/E♭
#450 15 D#/E♭ A C D#/E♭
#451 50 E♭ A C E♭ F
#452 1 E C# D E G
#453 1 E C# E
#454 4 E C# E F A
#455 49 E C# E G
#456 24 E C# E G A#/B♭
#457 1 E C#/D♭ E G G#/A♭
#458 7 E C# E G A
#459 1 E B♭ C D E
#460 1 E B♭ C E
#461 1 E B♭ C E F
#462 2 E A#/B♭ C#/D♭ E
#463 4 E B♭ D E
#464 3 E B♭ D E F
#465 1 E B C D E
#466 1 E B C D E G
#467 7 E B D E
#468 31 E B D E F
#469 3 E B D E F G
#470 7 E B D E F#
#471 91 E B D E G
#472 1 E B E
#473 3 E B E F
#474 1 E C D E
#475 1 E C D E F#
#476 34 E C D E G
#477 10 E C E
#478 7 E C E F
#479 17 E C E F G
#480 3 E C E F#
#481 1 E C E F# G
#482 968 E C E G
#483 1 E D E F
#484 13 E D E F G
#485 1 E D E F G B♭
#486 17 E D E F A
#487 1 E D E F# G
#488 3 E D E F# A
#489 50 E D E G
#490 12 E D E G B♭
#491 8 E D E G A
#492 13 E D E G#
#493 9 E D E A
#494 1 E E
#495 1 E E A#/B♭
#496 2 E E F G
#497 2 E E F G B♭
#498 2 E E F G B♭ C
#499 19 E E F G B
#500 3 E E F G A
#501 3 E E F G A C
#502 1 E E F G#/A♭ B
#503 6 E E F A
#504 1 E E F A B
#505 62 E E F A C
#506 2 E E F# B
#507 5 E E F# G B
#508 5 E E F# A
#509 2 E E F# A B
#510 4 E E F# A C
#511 27 E E G
#512 23 E E G A#/B♭
#513 98 E E G B♭ C
#514 288 E E G B
#515 28 E E G B C
#516 2 E E G A♭ C
#517 3 E E G A
#518 1 E E G A B♭
#519 1 E E G A B
#520 44 E E G A C
#521 5 E E G#
#522 113 E E G# B
#523 3 E E G# B C
#524 8 E E G#/A♭ C
#525 11 E E A
#526 2 E E A B♭
#527 34 E E A B
#528 42 E G# B D E
#529 12 E A C D E
#530 1 E A C D# E
#531 53 E A C E
#532 5 E A C# E
#533 9 F C# E F G
#534 2 F C#/D♭ E F A
#535 1 F C#/D♭ F G A
#536 3 F D♭ F A♭
#537 1 F B♭ C D F
#538 4 F B♭ C E♭ F
#539 3 F B♭ D♭ F
#540 1 F B♭ D E♭ F
#541 24 F B♭ D F
#542 2 F B C D F
#543 1 F B C E F
#544 6 F B C F
#545 57 F B D F
#546 316 F B D F G
#547 1 F B E F
#548 1 F C D E♭ F G
#549 3 F C D E♭ F A♭
#550 1 F C D E F G
#551 27 F C D F
#552 32 F C D F G
#553 233 F C D F A♭
#554 44 F C E♭ F
#555 106 F C E♭ F G
#556 1 F C E♭ F G A♭

#557 99 F C E♭ F A♭
#558 10 F C E F
#559 134 F C E F G
#560 14 F C F
#561 4 F D E♭ F G
#562 15 F D E♭ F A♭
#563 1 F D E F G
#564 1 F D E F G A
#565 5 F D E F A
#566 5 F D F
#567 21 F D F G
#568 41 F D F G B♭
#569 6 F D F G A♭
#570 2 F D F G A♭ B♭
#571 28 F D F G A
#572 7 F D F G A B
#573 228 F D F A♭
#574 21 F D F A♭ B♭
#575 22 F D F G#/A♭ B
#576 263 F D F A
#577 6 F D F A B♭
#578 15 F E♭ F B♭
#579 3 F E♭ F G
#580 50 F E♭ F G B♭
#581 1 F E♭ F G B
#582 4 F E♭ F G A♭
#583 2 F E♭ F G A♭ B♭
#584 3 F E♭ F G A
#585 1 F E♭ F G A C
#586 29 F E♭ F A♭
#587 24 F E♭ F A♭ B♭
#588 21 F E♭ F A
#589 1 F E F G
#590 4 F E F G B
#591 1 F E F G A
#592 9 F E F G A C
#593 4 F E F G#/A♭ C
#594 15 F E F A
#595 1 F E F A B
#596 159 F E F A C
#597 6 F F
#598 8 F F B♭
#599 34 F F B♭ C
#600 1 F F F#/G♭ A C
#601 2 F F G
#602 15 F F G B♭
#603 22 F F G B♭ C
#604 13 F F G B♭ D♭
#605 13 F F G B
#606 3 F F G B C
#607 77 F F G C
#608 2 F F G A♭ B♭
#609 44 F F G A♭ C
#610 5 F F G A
#611 1 F F G A B♭
#612 68 F F G A C
#613 1 F F G A C D
#614 12 F F G#/A♭
#615 4 F F A♭ B♭
#616 1 F F A♭ B♭ C
#617 20 F F G#/A♭ B
#618 1 F F A♭ B C
#619 309 F F A♭ C
#620 1 F F G#/A♭ A C
#621 37 F F A
#622 7 F F A B♭
#623 2 F F A B♭ C
#624 11 F F A B
#625 7 F F A B C
#626 734 F F A C
#627 1 F A B C D F
#628 78 F A B D F
#629 13 F A C D E F
#630 245 F A C D F
#631 46 F A C E♭ F
#632 1 F# B C D# F#
#633 1 F# B C E F#
#634 1 F# B D E F#
#635 1 F#/G♭ B/C♭ D E#/F F#/G♭
#636 7 F# B D F#
#637 1 F#/G♭ C D D#/E♭ F#/G♭
#638 5 F# C D F#
#639 2 F# C D F# G
#640 4 F#/G♭ C D#/E♭ F#/G♭
#641 4 G♭ C E♭ G♭ A♭
#642 2 F# C E F#
#643 6 F# C E F# G
#644 2 F# D E F# A
#645 1 F#/G♭ D F#/G♭ A#/B♭
#646 4 F# D F# G A
#647 1 F# D F# G#/A♭
#648 134 F# D F# A
#649 1 F# D F# A B♭
#650 1 F#/G♭ D#/E♭ F#/G♭ A#/B♭
#651 1 F# D# F# G B
#652 6 F#/G♭ D#/E♭ F#/G♭ A
#653 2 F#/G♭ D#/E♭ F#/G♭ A B
#654 80 F#/G♭ D#/E♭ F#/G♭ A C
#655 1 F# E F# B
#656 3 F# E F# G B
#657 2 F# E F# G A
#658 1 F# E F# G A C
#659 8 F# E F# A
#660 1 F# E F# A B
#661 35 F# E F# A C
#662 2 F#/G♭ F F#/G♭ A C
#663 15 F# F# G B D
#664 1 F# F# G A B
#665 1 F# F# G A B D
#666 2 F# F# G A C
#667 2 F#/G♭ F#/G♭ A A#/B♭
#668 10 F# F# A B D
#669 21 F#/G♭ F#/G♭ A C
#670 199 F# F# A C D
#671 1 F# F# A C#
#672 1 F# A C D E F#
#673 1 G C#/D♭ D#/E♭ G
#674 8 G C#/D♭ E G A#/B♭
#675 8 G C# E G A
#676 2 G D♭ F G
#677 3 G B C D E G
#678 1 G B D E F G

#679 48 G B D E G
#680 811 G B D F G
#681 11 G C D E♭ G
#682 1 G C D E F G
#683 1 G C D E F G A
#684 27 G C D E G
#685 51 G C D F G
#686 2 G C D F G A♭
#687 1 G C D F# G
#688 7 G C E♭ F G
#689 181 G C E♭ G
#690 12 G C E F G
#691 1 G C E F# G
#692 188 G C E G
#693 1 G D E♭ F G
#694 8 G D E♭ G
#695 14 G D E♭ G B♭
#696 3 G D E♭ G A♭
#697 1 G D E F G A
#698 7 G D E G
#699 27 G D E G B♭
#700 7 G D E G A
#701 42 G D F G
#702 61 G D F G B♭
#703 26 G D F G A♭
#704 2 G D F G A♭ B♭
#705 28 G D F G A
#706 2 G D F G A B
#707 1 G D F# G
#708 1 G D F# G B♭
#709 1 G D F# G A
#710 47 G D G
#711 52 G D G A
#712 1 G E♭ F G
#713 10 G E♭ F G B♭
#714 1 G E♭ F G B♭ C
#715 3 G E♭ F G B
#716 4 G E♭ F G A♭
#717 1 G E♭ F G A
#718 2 G E♭ F# G B♭
#719 8 G E♭ G
#720 382 G E♭ G B♭
#721 53 G D#/E♭ G B
#722 1 G E♭ G A♭
#723 2 G E♭ G A♭ B♭
#724 1 G E♭ G A♭ B
#725 2 G E♭ G A
#726 6 G E F G B
#727 2 G E F G A C
#728 1 G E F# G B
#729 2 G E F# G A
#730 5 G E G
#731 46 G E G A#/B♭
#732 15 G E G B♭ C
#733 143 G E G B
#734 38 G E G B C
#735 1 G E G A♭ C
#736 6 G E G A
#737 11 G E G A B♭
#738 9 G E G A B
#739 7 G E G A B C
#740 141 G E G A C
#741 5 G F G
#742 28 G F G B♭
#743 17 G F G B♭ C
#744 5 G F G B♭ D♭
#745 159 G F G B
#746 43 G F G B C
#747 53 G F G C
#748 5 G F G A♭ B♭
#749 3 G F G A♭ B
#750 40 G F G A♭ C
#751 5 G F G A
#752 2 G F G A B♭
#753 3 G F G A B
#754 2 G F G A B C
#755 43 G F G A C
#756 5 G F G A C D
#757 6 G F# G B D
#758 1 G F# G A
#759 1 G F# G A B C D
#760 6 G F# G A C
#761 6 G F# G A C D
#762 6 G G
#763 29 G G A#/B♭
#764 13 G G B♭ C
#765 1 G G B♭ C D
#766 1 G G B♭ C D E
#767 39 G G B♭ C E♭
#768 1 G G B♭ C#/D♭
#769 4 G G B♭ D♭ E♭
#770 352 G G B♭ D
#771 120 G G B
#772 44 G G B C
#773 13 G G B C D
#774 7 G G B C E♭
#775 2655 G G B D
#776 16 G G B D E♭
#777 55 G G C
#778 440 G G C D
#779 1 G G A♭ B♭
#780 3 G G A♭ B♭ C
#781 6 G G A♭ B♭ D
#782 4 G G A♭ B D
#783 1 G G A♭ B D E♭
#784 18 G G A♭ C
#785 12 G G A♭ C D
#786 48 G G A♭ C E♭
#787 12 G G A
#788 2 G G A B♭
#789 1 G G A B♭ C
#790 16 G G A B♭ D
#791 10 G G A B C
#792 46 G G A B D
#793 41 G G A C
#794 24 G G A C D
#795 1 G G A C D E
#796 13 G G A C E♭
#797 1 G G A C#
#798 1 G#/A♭ C#/D♭ D#/E♭ G#/A♭
#799 1 G#/A♭ C#/D♭ E#/F G#/A♭
#800 108 A♭ C D F A♭

#801 130 A♭ C E♭ F A♭
#802 6 A♭ C E♭ G♭ A♭
#803 1 A♭ D E♭ F A♭
#804 2 A♭ D E♭ G A♭
#805 5 A♭ D E♭ A♭
#806 3 G# D E F G# B
#807 39 G#/A♭ D F G#/A♭
#808 75 A♭ D F A♭ B♭
#809 45 G#/A♭ D F G#/A♭ B
#810 5 A♭ D G A♭
#811 1 A♭ E♭ F G A♭
#812 7 A♭ E♭ F A♭
#813 5 A♭ E♭ F A♭ B♭
#814 4 A♭ E♭ F A♭ B
#815 2 G#/A♭ D#/E♭ F#/G♭ G#/A♭ A#/B♭
#816 1 A♭ E♭ G A♭
#817 38 A♭ E♭ G A♭ B♭
#818 2 A♭ E♭ G A♭ B
#819 16 A♭ E♭ A♭ B♭
#820 2 G#/A♭ E F G#/A♭ B
#821 1 G#/A♭ E F G#/A♭ C
#822 2 A♭ E G A♭ B♭
#823 4 A♭ E G A♭ C
#824 1 A♭ E A♭ B♭ C
#825 103 G# E G# B
#826 4 G# E G# B C
#827 3 G#/A♭ E G#/A♭ C
#828 23 A♭ F G A♭ C
#829 5 G#/A♭ F G#/A♭
#830 2 A♭ F A♭ B♭
#831 4 A♭ F A♭ B♭ C
#832 2 A♭ F A♭ B♭ C D
#833 8 G#/A♭ F G#/A♭ B
#834 4 G#/A♭ F G#/A♭ B C
#835 384 A♭ F A♭ C
#836 2 A♭ G♭ A♭ C
#837 2 A♭ G♭ A♭ C D
#838 1 A♭ G A♭ B♭
#839 11 A♭ G A♭ B♭ C
#840 2 A♭ G A♭ B♭ C E♭
#841 4 A♭ G A♭ B♭ D
#842 2 A♭ G A♭ B C
#843 8 A♭ G A♭ B D
#844 52 A♭ G A♭ C
#845 1 A♭ G A♭ C D♭ E♭
#846 1 A♭ G A♭ C D
#847 68 A♭ G A♭ C E♭
#848 1 G#/A♭ G#/A♭
#849 1 A♭ A♭ B♭ C
#850 1 A♭ A♭ B♭ C D
#851 28 A♭ A♭ B♭ C E♭
#852 2 A♭ A♭ B♭ D
#853 3 A♭ A♭ B C E♭
#854 14 G#/A♭ G#/A♭ B D
#855 64 G# G# B D E
#856 3 G#/A♭ G#/A♭ B D#/E♭
#857 13 G#/A♭ G#/A♭ C
#858 11 G#/A♭ G#/A♭ C D
#859 7 G#/A♭ G#/A♭ C D D#/E♭
#860 352 A♭ A♭ C E♭
#861 1 G# G# A C E♭
#862 1 A C# D F A
#863 3 A C# E F A
#864 41 A C# E G A
#865 2 A C# F G A
#866 7 A C# F A
#867 1 A C D E F G A
#868 1 A D E♭ F A
#869 4 A D E F A
#870 2 A D E G A
#871 1 A D E G A B
#872 1 A D E G# A
#873 34 A D E A
#874 3 A D F G A
#875 1 A D F G A B
#876 2 A D F G# A
#877 44 A D F A
#878 14 A D F A B♭
#879 8 A D F# A
#880 21 A D G A
#881 1 A E♭ F G A C
#882 7 A E♭ F A
#883 1 A E♭ F A B♭
#884 20 A D#/E♭ F#/G♭ A C
#885 3 A E♭ G A
#886 1 A E♭ G A B♭
#887 1 A D#/E♭ A
#888 1 A E F G A C C#
#889 59 A E F A C
#890 3 A E F# A
#891 8 A E F# A C
#892 22 A E G A
#893 3 A E G A B♭
#894 1 A E G A B♭ C
#895 20 A E G A B
#896 2 A E G A B C
#897 97 A E G A C
#898 1 A E G A C C#
#899 1 A E G# A B
#900 1 A E G# A B C
#901 6 A E A
#902 14 A E A B
#903 1 A F G A B♭
#904 3 A F G A B
#905 14 A F G A C
#906 4 A F A
#907 1 A F A B♭
#908 5 A F A B♭ C
#909 6 A F A B
#910 7 A F A B C
#911 588 A F A C
#912 2 A F# G A C
#913 2 A F# G A C D
#914 2 A F# A B C
#915 159 A F# A C
#916 26 A F# A C D
#917 2 A G A B♭ C
#918 1 A G A B♭ C#
#919 14 A G A B♭ D
#920 11 A G A B C
#921 3 A G A B C D

#922 22 A G A B D
#923 106 A G A C
#924 56 A G A C D
#925 43 A G A C E♭
#926 5 A G A C#
#927 3 A G A C# D
#928 3 A G# A B
#929 1 A G# A C D#/E♭
#930 1 A A
#931 1 A A B♭ C D F
#932 2 A A B♭ C E♭
#933 4 A A B♭ C E
#934 1 A A B♭ D E
#935 2 A A B C
#936 3 A A B C D
#937 1 A A B C D E F
#938 2 A A B C D F
#939 60 A A B C E
#940 5 A A B D
#941 15 A A B D E
#942 26 A A B D F
#943 40 A A C
#944 1 A A C C#/D♭ E
#945 16 A A C D
#946 6 A A C D E
#947 1 A A C D E F#
#948 69 A A C D F
#949 16 A A C E♭
#950 101 A A C E♭ F
#951 836 A A C E
#952 1 A A C#
#953 1 A A C# D
#954 100 A A C# E
#955 8 A A D

Appendix IV: Top 100 Chords by Tonal Function

These tables list by tonal function the 100 most frequently occurring chords discussed in Chapter 4, and found in Table 38 on p. 111. Many chords, such as $ii^{6/4}$, are omitted from the tonal function count because their respective rates of occurrence do not warrant inclusion.

Tonic Major Triad

Chord Type	Index	Occurrences
I	208	2670
I^{6}	482	968
$I^{6/4}$	692	188
		3826

Tonic Minor Triad

Chord Type	Index	Occurrences
i	202	1694
i^{6}	400	541
$i^{6/4}$	689	181
		2416

Tonic Major Seventh

Chord Type	Index	Occurrences
I^{7}	217	86
$I^{4/2}$	127	176
		262

Tonic Minor Seventh

Chord Type	Index	Occurrences
i^{7}	245	78
$i^{4/2}$	79	159
		237

Supertonic Minor Triad

Chord Type	Index	Occurrences
ii	340	493
ii^{6}	576	263
		756

Supertonic Diminished Triad

Chord Type	Index	Occurrences
ii^{o6}	573	228

$ii^{\phi 7}$, $vii^{\phi 7}$/III

Chord Type	Index	Occurrences
$ii^{\phi 6/5}$ $vii^{\phi 6/5}$/III	553	233
$ii^{\phi 4/3}$ $vii^{\phi 4/3}$/III	800	108
		341

Supertonic Minor Seventh

Chord Type	Index	Occurrences
ii^{7}	380	87
$ii^{6/5}$	630	245
$ii^{4/3}$	948	69
		401

Mediant Major Triad

Chord Type	Index	Occurrences
III	432	808
III^{6}	720	382
$III^{6/4}$	56	90
		1280

Mediant Minor Triad

Chord Type	Index	Occurrences
iii	514	288
iii^{6}	733	143
		431

Subdominant Major Triad

Chord Type	Index	Occurrences
IV	626	734
IV^{6}	911	588
$IV^{6/4}$	234	76
		1398

Subdominant Minor Triad

Chord Type	Index	Occurrences
iv	619	309
iv^{6}	835	384
		693

Subdominant Major Seventh

Chord Type	Index	Occurrences
IV^{7}	596	159
$IV^{4/2}$	505	62
		221

Subdominant Minor Seventh

Chord Type	Index	Occurrences
iv^{7}	557	99
$iv^{6/5}$	801	130
		229

Dominant Major Triad

Chord Type	Index	Occurrences
V	775	2655
V^{6}	145	863
$V^{6/4}$	361	94
		3612

Dominant Seventh

Chord Type	Index	Occurrences
V^{7}	680	811
V^{7} (-5)	745	159
$V^{6/5}$	112	373
$V^{4/2}$	546	316
$V^{4/3}$	298	101
		1760

Submediant Major Triad

Chord Type	Index	Occurrences
VI	860	352
VI^{6}	265	144
		496

Submediant Minor Triad

Chord Type	Index	Occurrences
vi	951	836
vi^{6}	275	164
		1000

Submediant Minor Seventh

Chord Type	Index	Occurrences
vi^{7}	897	97
vi^{7} (-5), vii^{6} (susp. 2)/V	923	106
$vi^{4/2}$	740	141
$vi^{6/5}$	220	129
		473

Leading Tone Dim Triad

Chord Type	Index	Occurrences
vii^{o}	111	86
vii^{o6}	297	462
		548

$vii^{\phi7}$, $ii^{\phi7}$/vi

Chord Type	Index	Occurrences
$vii^{\phi4/3}$, $ii^{\phi4/3}$/vi	628	78

iii/III

Chord Type	Index	Occurrences
iii/III	770	352
iii^{6}/III	82	174
		526

V (susp. 4)

Chord Type	Index	Occurrences
V(susp. 4), $V^{6/5}$(-5, susp. 4)/V	778	440
V(susp. 4, 4 bass), $V^{4/2}$(-5, susp. 4)/V	250	119
$V^{6/4}$ (susp. 4), V^{7}(-5, susp. 4)/V	363	68
		627

V/III

Chord Type	Index	Occurrences
V/III	43	663
V^{6}/III	285	289
		952

V/ii

Chord Type	Index	Occurrences
V/ii	954	100

V/V

Chord Type	Index	Occurrences
V/V	345	378
V^{6}/V	648	134
		512

V/vi

Chord Type	Index	Occurrences
V/vi	522	113
V^{6}/vi	825	103
		216

V^{7}/III

Chord Type	Index	Occurrences
V^{7}/III	49	150
$V^{4/2}$/III	808	75
$V^{6/5}$/III	338	69
		294

V^{7}/IV, V^{7}/iv

Chord Type	Index	Occurrences
V^{7}/IV, V^{7}/iv	216	70
$V^{4/2}$/IV, $V^{4/2}$/iv	61	70
$V^{6/5}$/IV, $V^{6/5}$/iv	513	98
		238

V^7/V

Chord Type	Index	Occurrences
V^7/V	357	203
$V^7(-5)/V$	312	67
$V^{6/5}/V$	670	199
$V^{4/2}/V$	240	77
		546

$V^7/V/III$

Chord Type	Index	Occurrences
$V^{6/5}/V/III$	950	101

vii^o/V

Chord Type	Index	Occurrences
vii^{o6}/V	915	159

V^7 (susp. 4)

Chord Type	Index	Occurrences
$V^{4/3}$ (susp. 4)	310	131
$V^{6/5}$ (susp. 4)	192	126
		257

I (+2)

Chord Type	Index	Occurrences
I (+2)	190	143
I (+2, 2 bass)	308	92
		235

i (+2)

Chord Type	Index	Occurrences
i (+2)	186	117
i (+2, 2 bass)	303	76
		193

I (+4)

Chord Type	Index	Occurrences
I (+4, 4 bass)	559	134

i (+4)

Chord Type	Index	Occurrences
i (+4, 4 bass)	555	106

I (susp. 4)

Chord Type	Index	Occurrences
I (susp. 4), $V^{6/5}$(-5, susp. 4)	226	148

vii^o (susp. 2)

Chord Type	Index	Occurrences
vii^{o6} (susp. 2)	309	179
vii^o (susp. 2)	191	62
		241

Other

Chord Type	Index	Occurrences
iii^7	471	91
V(susp. 4)/V, $V^{6/5}$(-5, susp. 4)/V/V	349	89
I^6(susp. 2), $V^{4/2}$(-5, susp. 4)	607	77
V(susp. 4)/III, $V^{6/5}$(-5)/V/III	53	75
$V^{6/5}/ii$, $V^{6/5}/ii^o$	9	73
IV(+2)	612	68
VI^7	847	68
$V^{6/5}/vi$	855	64
iii^7/III	702	61
		666

Ambiguous

Chord Type	Index	Occurrences
AMB	771	120
AMB	203	100
AMB	196	90
AMB	654	80
AMB	273	120
		510

Appendix V: Heuristic Analysis of Bach Chorales

9.4 Overview

This appendix is provided to demonstrate that entire pieces of music can be tuned according to the precise heuristics of 5-limit just intonation given in section 5.11, starting on p. 145. Detailed tunings are given for every vertical sonority contained in the following five chorales:

1) “Meine Seele erhebet den Herr” (Riemenschneider #130)

2) “Danket dem Herren, den er ist sehr freunclich” (Riemenschneider #228)

3) “Christus, der ist mein Leben” (Riemenschneider #6)

4) “Nun, sich der Tag geendet hat” (Riemenschneider #240)

5) “Ach Gott, wie manches Herzeleid”(Riemenschneider #156)

9.5 Explanation of Tables

The following tables were initially created using intermediate output from the CONVERT program used to convert Csound scores into collections of discrete chords. Each line beginning with the letter ‘i’ is a Csound event representing a chord “snapshot” in which no voices enters, exits or changes. Data from these tables are directly converted back into a Csound score, which is then synthesized to aurally confirm accuracy. The format for each event is as follows:

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7
Instrument number	Starting time	Duration	Note and octave	Numerator of relative frequency	Denominator of relative frequency	Heuristic (See “Heuristics for Tuning Chords” on p. 145) and comments.

Chorale File Number: 032400b_.sco
Riemenschneider #130
"Meine Seele erhebet den Herr"
Key: E Minor
1/1 = E1 (E1 = 82.407 cps)

Measure 1 Beat 1 Chord Type: i						
i1	0	2	B3	6	1	2.c
i2	0	2	E3	4	1	2.a
i3	0	2	G2	12	5	2.b
i4	0	2	E2	2	1	2.a
Measure 1 Beat 3 Chord Type: V^{6}/III						
i1	2	2	D4	36	5	15.a
i2	2	2	D3	18	5	15.a
i3	2	2	A2	27	10	15.c
i4	2	2	F#2	9	4	15.b
Measure 2 Beat 1 Chord Type: III						
i1	4	1	B3	6	1	8.b
i2	4	1	D3	18	5	8.c
i3	4	1	B2	3	1	8.b
i4	4	1	G2	12	5	8.a
Measure 2 Beat 2 Chord Type: $V^{4/3}$						
i1	5	1	B3	6	1	16.a
i2	5	1	D#3	15	4	16.b
i3	5	1	A2	8	3	16.d
i4	5	1	F#2	9	4	16.c
Measure 2 Beat 3 Chord Type: i						
i1	6	1	B3	6	1	2.c
i2	6	1	E3	4	1	2.a
i3	6	1	G2	12	5	2.b
i4	6	1	E2	2	1	2.a
Measure 2 Beat 4 Chord Type: V^{6}						
i1	7	1	B3	6	1	15.a
i2	7	1	F#3	9	2	15.c
i3	7	1	F#2	9	4	15.c
I4	7	1	D#2	15	8	15.b
Measure 3 Beat 1 Chord Type: VI^{6}						
i1	8	1	C4	32	5	17.a
i2	8	1	E3	4	1	17.b
i3	8	1	G2	12	5	17.c
i4	8	1	E2	2	1	17.b
Measure 3 Beat 2 Chord Type: $V^{6/5}$/III						
i1	9	1	C4	32	5	16.d
i2	9	1	D3	18	5	16.a
i3	9	1	A2	27	10	16.c
i4	9	1	F#2	9	4	16.b
Measure 3 Beat 3 Chord Type: III (+2)						
i1	10	1	B3	6	1	8.b
i2	10	1	D3	18	5	8.c
i3	10	1	A2	27	10	23(D->A)
i4	10	1	G2	12	5	8.a
Measure 3 Beat 4 Chord Type: i^{7}						
i1	11	1	B3	6	1	4.c
i2	11	1	D3	18	5	4.d
i3	11	1	G2	12	5	4.b
i4	11	1	E2	2	1	4.a
Measure 4 Beat 1 Chord Type: $iv^{6/5}$						
i1	12	1	A3	16	3	14.a
i2	12	1	E3	4	1	14.c
i3	12	1	G2	12	5	14.d
i4	12	1	C2	8	5	14.b

Measure 4 Beat 2 Chord Type: iv^{7} (-3)						
i1	13	1	A3	16	3	14.a
i2	13	1	E3	4	1	14.c
i3	13	1	G2	12	5	14.d
i4	13	1	A1	4	3	14.a
Measure 4 Beat 3 Chord Type: V/III (microtonal change in soprano)						
i1	14	2	A3	27	5	15.c
i2	14	2	D3	18	5	15.a
i3	14	2	F#2	9	4	15.b
i4	14	2	D2	9	5	15.a
Measure 5 Beat 1 Chord Type: III						
i1	16	4	G3	24	5	8.a
i2	16	4	D3	18	5	8.c
i3	16	4	B2	3	1	8.b
i4	16	4	G1	6	5	8.a
Measure 6 Beat 1 Chord Type: III						
i1	20	2	B3	6	1	8.b
i2	20	2	D3	18	5	8.c
i3	20	2	G2	12	5	8.a
i4	20	2	G2	12	5	8.a
Measure 6 Beat 3 Chord Type: III^{6}						
i1	22	2	D4	36	5	8.c
i2	22	2	D3	18	5	8.c
i3	22	2	G2	12	5	8.a
i4	22	2	B1	3	2	8.b
Measure 7 Beat 1 Chord Type: V/III						
i1	24	8	A3	27	5	15.c
i2	24	8	D3	18	5	15.a
i3	24	8	F#2	9	4	15.b
i4	24	8	D2	9	5	15.a
Measure 9 Beat 1 Chord Type: V/III						
i1	32	8	A3	27	5	15.c
i2	32	8	D3	18	5	15.a
i3	32	8	F#2	9	4	15.b
i4	32	8	D2	9	5	15.a
Measure 11 Beat 1 Chord Type: iv^{6}						
i1	40	2	E3	4	1	13.c
i2	40	2	E3	4	1	13.c
i3	40	2	A2	8	3	13.a
i4	40	2	C2	8	5	13.b
Measure 12 Beat 1 Chord Type: i$^{6/4}$						
i1	42	1	G3	24	5	2.b
i2	42	1	E3	4	1	2.a
i3	42	1	B2	3	1	2.c
i4	42	1	B1	3	2	2.c
Measure 12 Beat 2 Chord Type: i$^{6/4}$ (+4, 4 in bass)						
i1	43	1	G3	24	5	2.b
i2	43	1	E3	4	1	2.a
i3	43	1	B2	3	1	2.c
i4	43	1	A1	4	3	22(E->A)
Measure 12 Beat 3 Chord Type: V						
i1	44	1	F#3	9	2	15.c
i2	44	1	D#3	15	4	15.b
i3	44	1	B2	3	1	15.a
i4	44	1	B1	3	2	15.a

Measure 12 Beat 4 Chord Type: V^7						
i1	45	1	F#3	9	2	16.c
i2	45	1	D#3	15	4	16.b
i3	45	1	A2	8	3	16.d
i4	45	1	B1	3	2	16.a
Measure 13 Beat 1 Chord Type: i						
i1	46	4	E3	4	1	2.a
i2	46	4	B2	3	1	2.c
i3	46	4	G2	12	5	2.b
i4	46	4	E1	1	1	2.a

Chorale File Number: 028600b_.sco
Riemenschneider #228
"Danket dem Herren, den er ist sehr freunclich"
Key: A Minor
2/1 = A1 (A1 = 110 cps)

Measure 1 Beat 4 Chord Type: i						
i1	3	1	A3	8	1	2.a
i2	3	1	E3	6	1	2.c
i3	3	1	C3	24	5	2.b
i4	3	1	A1	2	1	2.a
Measure 2 Beat 1 Chord Type: i						
i1	4	1	C4	48	5	2.b
i2	4	1	A3	8	1	2.a
i3	4	1	E3	6	1	2.c
i4	4	1	A2	4	1	2.a
Measure 2 Beat 2 Chord Type: V^6 (susp. 4, 4 in bass)						
i1	5	0.5	B3	9	1	15.c
i2	5	0.5	B3	9	1	15.c
i3	5	0.5	E3	6	1	15.a
i4	5	0.5	A2	4	1	22(E->A)
Measure 2 Beat 2.5 Chord Type: V^6						
i1	5.5	0.5	B3	9	1	15.c
i2	5.5	0.5	B3	9	1	15.c
i3	5.5	0.5	E3	6	1	15.a
i4	5.5	0.5	G#2	15	4	15.b
Measure 2 Beat 3 Chord Type: i (+2)						
i1	6	0.5	C4	48	5	2.b
i2	6	0.5	B3	9	1	22(E->B)
i3	6	0.5	E3	6	1	2.c
i4	6	0.5	A2	4	1	2.a
Measure 2 Beat 3.5 Chord Type: i						
i1	6.5	0.5	C4	48	5	2.b
i2	6.5	0.5	A3	8	1	2.a
i3	6.5	0.5	E3	6	1	2.c
i4	6.5	0.5	A1	2	1	2.a
Measure 2 Beat 4 Chord Type: $vii^{\phi7}$/III (CONTRADICTION – Rule 21; See Half-Diminished viiϕ7 Chord, p. 144)						
i1	7	0.5	D4			
i2	7	0.5	A3			
i3	7	0.5	D3			
i4	7	0.5	B1			
Measure 2 Beat 4.5 Chord Type: V/III						
i1	7.5	0.5	D4	54	5	15.c
i2	7.5	0.5	G3	36	5	15.a
i3	7.5	0.5	D3	27	5	15.c
i4	7.5	0.5	B1	9	4	15.b

Measure 3 Beat 1 Chord Type: III (+2)						
i1	8	0.5	E4	12	1	8.b
i2	8	0.5	G3	36	5	8.c
i3	8	0.5	D3	27	5	23(G->D)
i4	8	0.5	C2	12	5	8.a
Measure 3 Beat 1.5 Chord Type: III (+2, 2 in bass)						
i1	8.5	0.25	E4	12	1	8.b
i2	8.5	0.25	G3	36	5	8.c
i3	8.5	0.25	C3	24	5	8.a
i4	8.5	0.25	D2	27	10	23(G->D)
Measure 3 Beat 1.75 Chord Type: iii$^{4/2}$/III						
i1	8.75	0.25	E4	12	1	10.a
i2	8.75	0.25	G3	36	5	10.b
i3	8.75	0.25	B2	9	2	10.c
i4	8.75	0.25	D2	27	10	10.d
Measure 3 Beat 2 Chord Type: III^{6}						
i1	9	0.5	E4	12	1	8.b
i2	9	0.5	G3	36	5	8.c
i3	9	0.5	C3	24	5	8.a
i4	9	0.5	E2	3	1	8.b
Measure 3 Beat 2.5 Chord Type: III^{6} (+4, 4 in bass)						
i1	9.5	0.5	E4	12	1	8.b
i2	9.5	0.5	G3	36	5	8.c
i3	9.5	0.5	C3	24	5	8.a
i4	9.5	0.5	F2	16	5	22(C->F)
Measure 3 Beat 3 Chord Type: V/III						
i1	10	0.5	D4	54	5	15.c
i2	10	0.5	G3	36	5	15.a
i3	10	0.5	B2	9	2	15.b
i4	10	0.5	G2	18	5	15.a
Measure 3 Beat 3.5 Chord Type: V^{7}/III						
i1	10.5	0.5	D4	54	5	16.c
i2	10.5	0.5	F3	32	5	16.d
i3	10.5	0.5	B2	9	2	16.b
i4	10.5	0.5	G2	18	5	16.a
Measure 3 Beat 4 Chord Type: i (+2)						
i1	11	0.5	C4	48	5	2.b
i2	11	0.5	E3	6	1	2.c
i3	11	0.5	B2	9	2	23(E->B)
i4	11	0.5	A2	4	1	2.a
Measure 3 Beat 4.5 Chord Type: i						
i1	11.5	0.5	C4	48	5	2.b
i2	11.5	0.5	E3	6	1	2.c
i3	11.5	0.5	A2	4	1	2.a
i4	11.5	0.5	A2	4	1	2.a
Measure 4 Beat 1 Chord Type: V (susp. 4)						
i1	12	1	B3	9	1	15.c
i2	12	1	E3	6	1	15.a
i3	12	1	A2	4	1	22(E->A) 23(E->A)
i4	12	1	E2	3	1	15.a
Measure 4 Beat 2 Chord Type: V						
i1	13	0.5	B3	9	1	15.c
i2	13	0.5	E3	6	1	15.a
i3	13	0.5	G#2	15	4	15.b
i4	13	0.5	E2	3	1	15.a
Measure 4 Beat 2.5 Chord Type: V (-3 +4)						
i1	13.5	0.5	B3	9	1	15.c
i2	13.5	0.5	E3	6	1	15.a
i3	13.5	0.5	F#2	27	8	23(B->F#)
i4	13.5	0.5	E2	3	1	15.a

Measure 4 Beat 3 Chord Type: V						
i1	14	1	B3	9	1	15.c
i2	14	1	E3	6	1	15.a
i3	14	1	G#2	15	4	15.b
i4	14	1	E1	3	2	15.a
Measure 4 Beat 4 Chord Type: V						
i1	15	0.5	B3	9	1	15.c
i2	15	0.5	E3	6	1	15.a
i3	15	0.5	G#2	15	4	15.b
i4	15	0.5	E2	3	1	15.a
Measure 4 Beat 4.5 Chord Type: $V^{4/2}$						
i1	15.5	0.5	B3	9	1	16.c
i2	15.5	0.5	E3	6	1	16.a
i3	15.5	0.5	G#2	15	4	16.b
i4	15.5	0.5	D2	8	3	16.d
Measure 5 Beat 1 Chord Type: i^6						
i1	16	1	E4	12	1	2.c
i2	16	1	E3	6	1	2.c
i3	16	1	A2	4	1	2.a
i4	16	1	C2	12	5	2.b
Measure 5 Beat 2 Chord Type: $V^{4/2}$/V/III (mutable tone A in A minor chorale!)						
i1	17	1	D4	54	5	16.a
i2	17	1	F#3	27	4	16.b
i3	17	1	A2	81	20	16.c
i4	17	1	C2	12	5	16.d
Measure 5 Beat 3 Chord Type: vii° (Mutable repeated tone D in melody)						
i1	18	1	D4	32	3	20.c
i2	18	1	G#3	15	4	20.a
i3	18	1	B2	9	2	20.b
i4	18	1	B1	9	4	20.b
Measure 5 Beat 4 Chord Type: i (-5)						
i1	19	0.5	C4	48	5	2.b
i2	19	0.5	A3	8	1	2.a
i3	19	0.5	C3	24	5	2.b
i4	19	0.5	A1	2	1	2.a
Measure 5 Beat 4.5 Chord Type: $iv^{4/3}$ (-3)						
i1	19.5	0.5	C4	48	5	14.d
i2	19.5	0.5	A3	8	1	14.c
i3	19.5	0.5	D3	16	3	14.a
i4	19.5	0.5	A1	2	1	14.c
Measure 6 Beat 1 Chord Type: V (susp. 4)						
i1	20	0.5	B3	9	1	15.c
i2	20	0.5	A3	8	1	23(A->E)
i3	20	0.5	E3	6	1	15.a
i4	20	0.5	E2	3	1	15.a
Measure 6 Beat 1.5 Chord Type: V						
i1	20.5	0.5	B3	9	1	15.c
i2	20.5	0.5	G#3	15	4	15.b
i3	20.5	0.5	E3	6	1	15.a
i4	20.5	0.5	E2	3	1	15.a
Measure 6 Beat 2 Chord Type: i						
i1	21	0.5	C4	48	5	2.b
i2	21	0.5	A3	8	1	2.a
i3	21	0.5	E3	6	1	2.c
i4	21	0.5	A2	4	1	2.a
Measure 6 Beat 2.5 Chord Type: $i^{4/2}$						
i1	21.5	0.5	C4	48	5	4.b
i2	21.5	0.5	A3	8	1	4.a
i3	21.5	0.5	E3	6	1	4.c
i4	21.5	0.5	G2	18	5	4.d

Measure 6 Beat 3 Chord Type: iv^6						
i1	22	0.5	D4	32	3	13.a
i2	22	0.5	A3	8	1	13.c
i3	22	0.5	D3	16	3	13.a
i4	22	0.5	F2	16	5	13.b
Measure 6 Beat 3.5 Chord Type: vii^{o7} (-3) (fully dim. 7 is CONTRADICTION. See "A Contradiction with the Doubly Diminished Seventh Chord" on p. 141. NOTE that a computer program might have identified this chord as a D diminished triad, when it is really a G# diminished seventh minus the third.)						
i1	22.5	0.5	D4			
i2	22.5	0.5	G#3			
i3	22.5	0.5	D3			
i4	22.5	0.5	F2			
Measure 6 Beat 4 Chord Type: $i^{6/4}$						
i1	23	1	C4	48	5	2.b
i2	23	1	A3	8	1	2.a
i3	23	1	E3	6	1	2.c
i4	23	1	E2	3	1	2.c
Measure 7 Beat 1 Chord Type: half-dim. $ii^{\phi 6/5}$						
i1	24	1	B3	9	1	7.a
i2	24	1	A3	8	1	7.d
i3	24	1	F3	32	5	7.c
i4	24	1	D2	8	3	7.b
Measure 7 Beat 2 Chord Type: V (susp. 4)						
i1	25	0.5	B3	9	1	15.c
i2	25	0.5	A3	8	1	23(E->A)
i3	25	0.5	E3	6	1	15.a
i4	25	0.5	E2	3	1	15.a
Measure 7 Beat 2.5 Chord Type: V						
i1	25.5	0.25	B3	9	1	15.c
i2	25.5	0.25	G#3	15	4	15.b
i3	25.5	0.25	E3	6	1	15.a
i4	25.5	0.25	E2	3	1	15.a
Measure 7 Beat 2.75 Chord Type: V^7						
i1	25.75	0.25	B3	9	1	16.c
i2	25.75	0.25	G#3	15	4	16.b
i3	25.75	0.25	D3	16	3	16.d
i4	25.75	0.25	E2	3	1	16.a
Measure 7 Beat 3 Chord Type: I						
i1	26	2	A3	8	1	1.a
i2	26	2	E3	6	1	1.c
i3	26	2	C#3	5	1	1.b
i4	26	2	A1	2	1	1.a

Chorale File Number: 028100b_.sco
Riemenschneider #6
"Christus, der ist mein Leben"
Key: F Major
2/1 = F1 (F1 = 87.307 cps)

Measure 1 Beat 4 Chord Type: I						
i1	3	1	F3	8	1	1.a
i2	3	1	C3	6	1	1.c
i3	3	1	A2	5	1	1.b
i4	3	1	F1	2	1	1.a
Measure 2 Beat 1 Chord Type: I						
i1	4	1	A3	10	1	1.b
i2	4	1	F3	8	1	1.a
i3	4	1	C3	6	1	1.c
i4	4	1	F2	4	1	1.a
Measure 2 Beat 2 Chord Type: V^6						
i1	5	1	G3	9	1	15.c
i2	5	1	G3	9	1	15.c
i3	5	1	C3	6	1	15.a
i4	5	1	E2	15	4	15.b
Measure 2 Beat 3 Chord Type: $V^{4/2}$/IV						
i1	6	1	A3	10	1	16.b
i2	6	1	F3	8	1	16.a
i3	6	1	C3	6	1	16.c
i4	6	1	D#2	32	9	16.d
Measure 2 Beat 4 Chord Type: IV^6						
i1	7	1	A#3	32	3	11.a
i2	7	1	F3	8	1	11.c
i3	7	1	D3	20	3	11.b
i4	7	1	D2	10	3	11.b
Measure 3 Beat 1 Chord Type: V						
i1	8	0.5	C4	12	1	15.a
i2	8	0.5	E3	15	2	15.b
i3	8	0.5	G2	9	2	15.c
i4	8	0.5	C2	3	1	15.a
Measure 3 Beat 1.5 Chord Type: V (-3 +2 +4)						
i1	8.5	0.5	C4	12	1	15.c
i2	8.5	0.5	F3	8	1	15.a
i3	8.5	0.5	G2	9	2	23(C->G)
i4	8.5	0.5	D2 (mutable)	27	8	23(G->D)
Measure 3 Beat 2 Chord Type: V^6						
i1	9	0.5	C4	12	1	15.a
i2	9	0.5	G3	9	1	15.c
i3	9	0.5	C3	6	1	15.a
i4	9	0.5	E2	15	4	15.b
Measure 3 Beat 2.5 Chord Type: V (-5)						
i1	9.5	0.5	C4	12	1	15.a
i2	9.5	0.5	E3	15	2	15.b
i3	9.5	0.5	C3	6	1	15.a
i4	9.5	0.5	C2	3	1	15.a
Measure 3 Beat 3 Chord Type: I						
i1	10	1	A3	10	1	1.b
i2	10	1	F3	8	1	1.a
i3	10	1	C3	6	1	1.c
i4	10	1	F2	4	1	1.a
Measure 3 Beat 4 Chord Type: IV						
i1	11	1	D4	40	3	11.b
i2	11	1	F3	8	1	11.c
i3	11	1	A#2	16	3	11.a
i4	11	1	A#1	8	3	11.a

Measure 4 Beat 1 Chord Type: I^{6}						
i1	12	1	C4	12	1	1.c
i2	12	1	F3	8	1	1.a
i3	12	1	C3	6	1	1.c
i4	12	1	A1	5	2	1.b
Measure 4 Beat 2 Chord Type: ii^{7}						
i1	13	0.5	A#3	32	3	6.b
i2	13	0.5	F3	8	1	6.d
i3	13	0.5	D3	20	3	6.c
i4	13	0.5	G1 (mutable)	20	9	6.a
Measure 4 Beat 2.5 Chord Type: vii^{o6}						
i1	13.5	0.5	A#3	32	3	20.c
i2	13.5	0.5	E3	15	2	20.a
i3	13.5	0.5	A#2	16	3	20.c
i4	13.5	0.5	G1 (mutable held tone)	9	4	20.b
Measure 4 Beat 3 Chord Type: I						
i1	14	0.5	A3	10	1	1.b
i2	14	0.5	F3	8	1	1.a
i3	14	0.5	C3	6	1	1.c
i4	14	0.5	F1	2	1	1.a
Measure 4 Beat 3.5 Chord Type: I^{6}						
i1	14.5	0.5	A3	10	1	1.b
i2	14.5	0.5	F3	8	1	1.a
i3	14.5	0.5	C3	6	1	1.c
i4	14.5	0.5	A1	5	2	1.b
Measure 4 Beat 4 Chord Type: V (susp. 4)						
i1	15	0.5	G3	9	1	15.c
i2	15	0.5	F3	8	1	23(C->F)
i3	15	0.5	C3	6	1	15.a
i4	15	0.5	C2	3	1	15.a
Measure 4 Beat 4.5 Chord Type: V						
i1	15.5	0.5	G3	9	1	15.c
i2	15.5	0.5	E3	15	2	15.b
i3	15.5	0.5	C3	6	1	15.a
i4	15.5	0.5	C2	3	1	15.a
Measure 5 Beat 1 Chord Type: I						
i1	16	2	A3	10	1	1.b
i2	16	2	F3	8	1	1.a
i3	16	2	C3	6	1	1.c
i4	16	2	F1	2	1	1.a
Measure 5 Beat 4 Chord Type: I^{6}						
i1	19	1	C4	12	1	1.c
i2	19	1	A3	10	1	1.b
i3	19	1	F3	8	1	1.a
i4	19	1	A2	5	1	1.b
Measure 6 Beat 1 Chord Type: $vii^{ϕ7}$/V (Contradiction; See “Half-Diminished viiϕ7 Chord”, p. 144)						
i1	20	0.5	D4			
i2	20	0.5	A3			
i3	20	0.5	F3			
i4	20	0.5	B1			
Measure 6 Beat 1.5 Chord Type: $V^{6/5}$/V						
i1	20.5	0.5	D4	27	2	16.c
i2	20.5	0.5	G3	9	1	16.a
i3	20.5	0.5	F3	8	1	16.d
i4	20.5	0.5	B1	45	16	16.b
Measure 6 Beat 2 Chord Type: V						
i1	21	0.5	E4	15	1	15.b
i2	21	0.5	G3	9	1	15.c
i3	21	0.5	E3	15	2	15.b
i4	21	0.5	C2	3	1	15.a

Measure 6 Beat 2.5 Chord Type: V (-5)						
i1	21.5	0.25	E4	15	1	15.b
i2	21.5	0.25	C4	12	1	15.a
i3	21.5	0.25	E3	15	2	15.b
i4	21.5	0.25	C2	3	1	15.a
Measure 6 Beat 2.75 Chord Type: V^7 (-5)						
i1	21.75	0.25	E4	15	1	16.b
i2	21.75	0.25	A#3	32	3	16.d
i3	21.75	0.25	E3	15	2	16.b
i4	21.75	0.25	C2	3	1	16.a
Measure 6 Beat 3 Chord Type: vi (+2)						
i1	22	0.5	F4	16	1	18.b
i2	22	0.5	A3	10	1	18.c
i3	22	0.5	E3	15	2	23(A->E)
i4	22	0.5	D2	10	3	18.a
Measure 6 Beat 3.5 Chord Type: vi^6						
i1	22.5	0.5	F4	16	1	18.b
i2	22.5	0.5	A3	10	1	18.c
i3	22.5	0.5	D3	20	3	18.a
i4	22.5	0.5	F2	4	1	18.b
Measure 6 Beat 4 Chord Type: iii						
i1	23	0.5	E4	15	1	9.c
i2	23	0.5	A3	10	1	9.a
i3	23	0.5	C3	6	1	9.b
i4	23	0.5	A2	5	1	9.a
Measure 6 Beat 4.5 Chord Type: $iii^{4/2}$						
i1	23.5	0.5	E4	15	1	10.c
i2	23.5	0.5	A3	10	1	10.a
i3	23.5	0.5	C3	6	1	10.b
i4	23.5	0.5	G2	9	2	10.d
Measure 7 Beat 1 Chord Type: $vi^{6/5}$						
i1	24	0.5	D4	40	3	19.a
i2	24	0.5	A3	10	1	19.c
i3	24	0.5	C3	6	1	19.d
i4	24	0.5	F2	4	1	19.b
Measure 7 Beat 1.5 Chord Type: vi^7 (-3)						
i1	24.5	0.5	D4	40	3	19.a
i2	24.5	0.5	A3	10	1	19.c
i3	24.5	0.5	C3	6	1	19.d
i4	24.5	0.5	D2	10	3	19.a
Measure 7 Beat 2 Chord Type: V/V						
i1	25	1	D4 (Mutable held tone)	27	2	15.c
i2	25	1	G3	9	1	15.a
i3	25	1	B2	45	8	15.b
i4	25	1	G2	9	2	15.a
Measure 7 Beat 3 Chord Type: V						
i1	26	1	C4	12	1	15.a
i2	26	1	G3	9	1	15.c
i3	26	1	E3	15	2	15.b
i4	26	1	C2	3	1	15.a
Measure 7 Beat 4 Chord Type: I						
i1	27	1	A3	10	1	1.b
i2	27	1	F3	8	1	1.a
i3	27	1	C3	6	1	1.c
i4	27	1	F1	2	1	1.a
Measure 8 Beat 1 Chord Type: ii^7 (-5)						
i1	28	0.5	A#3	32	3	6.b
i2	28	0.5	F3	8	1	6.d
i3	28	0.5	A#2	16	3	6.b
i4	28	0.5	G1	20	9	6.a

Measure 8 Beat 1.5 Chord Type: vii^{o6}						
i1	28.5	0.5	A#3	32	3	20.c
i2	28.5	0.5	E3	15	2	20.a
i3	28.5	0.5	A#2	16	3	20.c
i4	28.5	0.5	G1 (Mutable held tone)	9	4	20.b
Measure 8 Beat 2 Chord Type: I^6						
i1	29	1	A3	10	1	1.b
i2	29	1	F3	8	1	1.a
i3	29	1	C3	6	1	1.c
i4	29	1	A1	5	2	1.b
Measure 8 Beat 3 Chord Type: $ii^{\phi 6/5}$						
i1	30	1	G3	9	1	7.a
i2	30	1	F3	8	1	7.d
i3	30	1	C#3	32	5	7.c
i4	30	1	A#1	8	3	7.b
Measure 8 Beat 4 Chord Type: V						
i1	31	0.75	G3	9	1	15.c
i2	31	0.75	E3	15	2	15.b
i3	31	0.75	C3	6	1	15.a
i4	31	0.75	C2	3	1	15.a
Measure 8 Beat 4.75 Chord Type: V^7						
i1	31.75	0.25	G3	9	1	16.c
i2	31.75	0.25	E3	15	2	16.b
i3	31.75	0.25	A#2	16	3	16.d
i4	31.75	0.25	C2	3	1	16.a
Measure 9 Beat 1 Chord Type: I						
i1	32	3	F3	8	1	1.a
i2	32	3	C3	6	1	1.c
i3	32	3	A2	5	1	1.b
i4	32	3	F1	2	1	1.a

Chorale File Number: 039600b_.sco
Riemenschneider #240
“Nun, sich der Tag geendet hat”
Key: A Minor
2/1 = A1 (A1 = 110 cps)

Measure 1 Beat 4 Chord Type: i						
i1	3	0.5	E4	12	1	2.c
i2	3	0.5	A3	8	1	2.a
i3	3	0.5	C3	24	5	2.b
i4	3	0.5	A1	2	1	2.a
Measure Beat 4.5 Chord Type: i^6 (+2, 2 in bass)						
i1	3.5	0.5	E4	12	1	2.c
i2	3.5	0.5	A3	8	1	2.a
i3	3.5	0.5	C3	24	5	2.b
i4	3.5	0.5	B1	9	4	23(E->B)
Measure 2 Beat 1 Chord Type: i^6						
i1	4	0.5	A3	8	1	2.a
i2	4	0.5	E3	6	1	2.c
i3	4	0.5	C3	24	5	2.b
i4	4	0.5	C2	12	5	2.b
Measure 2 Beat 1.5 Chord Type: i^6 (+2, 2 in bass)						
i1	4.5	0.5	A3	8	1	2.a
i2	4.5	0.5	E3	6	1	2.c
i3	4.5	0.5	C3	24	5	2.b
i4	4.5	0.5	B1	9	4	23(E->B)

Measure 2 Beat 2 Chord Type: V^6 (susp. 4)						
i1	5	0.5	B3	9	1	15.c
i2	5	0.5	E3	6	1	15.a
i3	5	0.5	B2	9	2	15.c
i4	5	0.5	A1	2	1	22(E->A)
Measure 2 Beat 2.5 Chord Type: V^6						
i1	5.5	0.5	B3	9	1	15.c
i2	5.5	0.5	E3	6	1	15.a
i3	5.5	0.5	B2	9	2	15.c
i4	5.5	0.5	G#1	15	8	15.b
Measure 2 Beat 3 Chord Type: i (+2)						
i1	6	0.5	C4	48	5	2.b
i2	6	0.5	E3	6	1	2.c
i3	6	0.5	B2	9	2	23(E->B)
i4	6	0.5	A1	2	1	2.a
Measure 2 Beat 3.5 Chord Type: VI^6						
i1	6.5	0.5	C4	48	5	17.c
i2	6.5	0.5	F3	32	5	17.a
i3	6.5	0.5	A2	4	1	17.b
i4	6.5	0.5	A1	2	1	17.b
Measure 2 Beat 4 Chord Type: III^6						
i1	7	0.5	C4	48	5	8.a
i2	7	0.5	G3	36	5	8.c
i3	7	0.5	G2	18	5	8.c
i4	7	0.5	E1	3	2	8.b
Measure 2 Beat 4.5 Chord Type: III^6						
i1	7.5	0.5	C4	48	5	8.a
i2	7.5	0.5	G3	36	5	8.c
i3	7.5	0.5	C3	24	5	8.a
i4	7.5	0.5	E1	3	2	8.b
Measure 3 Beat 1 Chord Type: $iv^{6/5}$ ($ii^{6/5}$/III)						
i1	8	0.5	D4	32	3	14.a
i2	8	0.5	A3	8	1	14.c
i3	8	0.5	C3	24	5	14.d
i4	8	0.5	F1	8	5	14.b
Measure 3 Beat 1.5 Chord Type: $ii^{\phi7}$						
i1	8.5	0.25	D4	32	3	7.b
i2	8.5	0.25	A3	8	1	7.d
i3	8.5	0.25	B2	9	2	7.a
i4	8.5	0.25	F1	8	5	7.c
Measure 3 Beat 1.75 Chord Type: iv^6 (ii^6/III)						
i1	8.75	0.25	D4	32	3	13.a
i2	8.75	0.25	A3	8	1	13.c
i3	8.75	0.25	A2	4	1	13.c
i4	8.75	0.25	F1	8	5	13.b
Measure 3 Beat 2 Chord Type: V/III						
i1	9	1	D4	54	5	15.c
i2	9	1	G3	18	5	15.a
i3	9	1	B2	9	2	15.b
i4	9	1	G1	9	5	15.a
Measure 3 Beat 3 Chord Type: III						
i1	10	1	E4	12	1	8.b
i2	10	1	G3	36	5	8.c
i3	10	1	C3	24	5	8.a
i4	10	1	C1	6	5	8.a
Measure 3 Beat 4 Chord Type: III						
i1	11	1	E4	12	1	8.b
i2	11	1	G3	36	5	8.c
i3	11	1	C3	24	5	8.a
i4	11	1	C2	12	5	8.a

Measure 4 Beat 1 Chord Type: $V^{6/5}$/V/III (Mutable A in A minor chorale!)						
i1	12	1	D4	54	5	16.a
i2	12	1	A3	81	10	16.c
i3	12	1	C3	24	5	16.d
i4	12	1	F#1	27	16	16.b
Measure 4 Beat 2 Chord Type: V^{6}						
i1	13	1	E4	12	1	15.a
i2	13	1	E3	6	1	15.a
i3	13	1	B2	9	2	15.c
i4	13	1	G#1	15	8	15.b
Measure 4 Beat 3 Chord Type: i (+2)						
i1	14	0.5	C4	48	5	2.b
i2	14	0.5	E3	6	1	2.c
i3	14	0.5	B2	9	2	23(E->B)
i4	14	0.5	A1	2	1	2.a
Measure 4 Beat 3.5 Chord Type: i (susp. 2, 2 in bass)						
i1	14.5	0.5	C4	48	5	2.b
i2	14.5	0.5	E3	6	1	2.c
i3	14.5	0.5	B2	9	2	23(E->B)
i4	14.5	0.5	B1	9	4	23(E->B)
Measure 4 Beat 4 Chord Type: i^{6}						
i1	15	0.5	C4	48	5	2.b
i2	15	0.5	E3	6	1	2.c
i3	15	0.5	A2	4	1	2.a
i4	15	0.5	C2	12	5	2.b
Measure 4 Beat 4.5 Chord Type: i (+4, 4 in bass)						
i1	15.5	0.5	C4	48	5	2.b
i2	15.5	0.5	E3	6	1	2.c
i3	15.5	0.5	A2	4	1	2.a
i4	15.5	0.5	D2	8	3	22(A->D)
Measure 5 Beat 1 Chord Type: V (susp. 4)						
i1	16	1	B3	9	1	15.c
i2	16	1	E3	6	1	15.a
i3	16	1	A2	4	1	22(E->A) 23(E->A)
i4	16	1	E2	3	1	15.a
Measure 5 Beat 2 Chord Type: V						
i1	17	1	B3	9	1	15.c
i2	17	1	E3	6	1	15.a
i3	17	1	G#2	15	4	15.b
i4	17	1	E2	3	1	15.a
Measure 5 Beat 4 Chord Type: V^{6}						
i1	19	1	E4	12	1	15.a
i2	19	1	B3	9	1	15.c
i3	19	1	E3	6	1	15.a
i4	19	1	G#1	15	4	15.b
Measure 6 Beat 1 Chord Type: i (+2)						
i1	20	0.5	C4	48	5	2.b
i2	20	0.5	B3	9	1	22(E->B)
i3	20	0.5	E3	6	1	2.c
i4	20	0.5	A1	2	1	2.a
Measure 6 Beat 1.5 Chord Type: i						
i1	20.5	0.5	C4	48	5	2.b
i2	20.5	0.5	A3	8	1	2.a
i3	20.5	0.5	E3	6	1	2.c
i4	20.5	0.5	A1	2	1	2.a
Measure 6 Beat 2 Chord Type: V						
i1	21	0.5	B3	9	1	15.c
i2	21	0.5	G#3	15	2	15.b
i3	21	0.5	E3	6	1	15.a
i4	21	0.5	E2	3	1	15.a

Measure 6 Beat 2.5 Chord Type: $V^{4/2}$						
i1	21.5	0.5	B3	9	1	16.c
i2	21.5	0.5	G#3	15	2	16.b
i3	21.5	0.5	E3	6	1	16.a
i4	21.5	0.5	D2	8	3	16.d
Measure 6 Beat 3 Chord Type: i^6						
i1	22	0.5	C4	48	5	2.b
i2	22	0.5	A3	8	1	2.a
i3	22	0.5	E3	6	1	2.c
i4	22	0.5	C2	12	5	2.b
Measure 6 Beat 3.5 Chord Type: III						
i1	22.5	0.5	C4	48	5	8.a
i2	22.5	0.5	G3	36	5	8.c
i3	22.5	0.5	E3	6	1	8.b
i4	22.5	0.5	C2	12	5	8.a
Measure 6 Beat 4 Chord Type: iv						
i1	23	0.5	A3	8	1	13.c
i2	23	0.5	F3	32	5	13.b
i3	23	0.5	D3	16	3	13.a
i4	23	0.5	D2	8	3	13.a
Measure 6 Beat 4.5 Chord Type: $iv^{4/2}$						
i1	23.5	0.5	A3	8	1	14.c
i2	23.5	0.5	F3	32	5	14.b
i3	23.5	0.5	D3	16	3	14.a
i4	23.5	0.5	C2	12	5	14.d
Measure 7 Beat 1 Chord Type: fully diminished vii^{o7} (Contradiction: See "A Contradiction with the Doubly Diminished Seventh Chord" on p. 141.)						
i1	24	0.5	G#3			
i2	24	0.5	F3			
i3	24	0.5	D3			
i4	24	0.5	B1			
Measure 7 Beat 1.5 Chord Type: V						
i1	24.5	0.5	G#3	15	2	15.b
i2	24.5	0.5	E3	6	1	15.a
i3	24.5	0.5	B2	9	2	15.c
i4	24.5	0.5	E2	3	1	15.a
Measure 7 Beat 2 Chord Type: i^6						
i1	25	0.5	A3	8	1	2.a
i2	25	0.5	E3	6	1	2.c
i3	25	0.5	C3	24	5	2.b
i4	25	0.5	C2	12	5	2.b
Measure 7 Beat 2.5 Chord Type: vii^{o6}/V/III (Mutable A in A minor chorale!)						
i1	25.5	0.5	A3	81	10	20.b
i2	25.5	0.5	F#3	27	4	20.a
i3	25.5	0.5	C3	24	5	20.c
i4	25.5	0.5	A1	81	40	20.b
Measure 7 Beat 3 Chord Type: V						
i1	26	1	B3	9	1	15.c
i2	26	1	G#3	15	2	15.b
i3	26	1	B2	9	2	15.c
i4	26	1	E2	3	1	15.a
Measure 7 Beat 4 Chord Type: i^6						
i1	27	1	E4	12	1	2.c
i2	27	1	A3	8	1	2.a
i3	27	1	A2	4	1	2.a
i4	27	1	C2	12	5	2.b

Measure 8 Beat 1 Chord Type: ii^{ø7} (– 5)						
i1	28	0.5	D4	32	3	7.b
i2	28	0.5	A3	8	1	7.d
i3	28	0.5	D3	16	3	7.b
i4	28	0.5	B1	9	4	7.a
Measure 8 Beat 1.5 Chord Type: vii^{o6}						
i1	28.5	0.5	D4	32	3	20.c
i2	28.5	0.5	G#3	15	4	20.a
i3	28.5	0.5	D3	16	3	20.c
i4	28.5	0.5	B1	9	4	20.b
Measure 8 Beat 2 Chord Type: i						
i1	29	1	C4	48	5	2.b
i2	29	1	A3	8	1	2.a
i3	29	1	E3	6	1	2.c
i4	29	1	A1	2	1	2.a
Measure 8 Beat 3 Chord Type: ii$^{ø6/5}$						
i1	30	0.5	B3	9	1	7.a
i2	30	0.5	A3	8	1	7.d
i3	30	0.5	F3	32	5	7.c
i4	30	0.5	D2	8	3	7.b
Measure 8 Beat 3.5 Chord Type: vii^{ø7} (– 5)						
i1	30.5	0.5	B3	9	1	7.a
i2	30.5	0.5	A3	8	1	7.d
i3	30.5	0.5	D3	16	3	7.b
i4	30.5	0.5	D2	8	3	7.b
Measure 8 Beat 4 Chord Type: V						
i1	31	0.5	B3	9	1	15.c
i2	31	0.5	G#3	15	2	15.b
i3	31	0.5	B2	9	2	15.c
i4	31	0.5	E2	3	1	15.a
Measure 8 Beat 4.5 Chord Type: V						
i1	31.5	0.25	B3	9	1	15.c
i2	31.5	0.25	G#3	15	2	15.b
i3	31.5	0.25	E3	6	1	15.a
i4	31.5	0.25	E2	3	1	15.a
Measure 8 Beat 4.75 Chord Type: V^{7}						
i1	31.75	0.25	B3	9	1	16.c
i2	31.75	0.25	G#3	15	2	16.b
i3	31.75	0.25	D3	16	3	16.d
i4	31.75	0.25	E2	3	1	16.a
Measure 9 Beat 1 Chord Type: I						
i1	32	4	A3	8	1	1.a
i2	32	4	E3	6	1	1.c
i3	32	4	C#3	5	1	1.b
i4	32	4	A1	2	1	1.a

Chorale File Number: 000306b_.sco
Riemenschneider #156
"Ach Gott, wie manches Herzeleid"
Key: A Major
2/1 = A1 (A1 = 110 cps)

Measure 1 Chord Type: I (-5)						
i1	3	0.5	A3	8	1	1.a
i2	3	0.5	C#3	5	1	1.b
i3	3	0.5	A2	4	1	1.a
i4	3	0.5	A2	4	1	1.a
Measure 1 Beat 4.5 Chord Type: I (Embedded Dom Passing)						
i1	3.5	0.5	A3	8	1	1.a
i2	3.5	0.5	D3	16	3	22(A->D) 23(A->D)
i3	3.5	0.5	B2	9	2	27(D->B)
i4	3.5	0.5	A2	4	1	1.a
Measure 2 Beat 1 Chord Type: I						
i1	4	0.5	A3	8	1	1.a
i2	4	0.5	E3	6	1	1.c
i3	4	0.5	C#3	5	1	1.b
i4	4	0.5	A1	2	1	1.a
Measure 2 Beat 1.5 Chord Type: I (Embedded Dom Passing)						
i1	4.5	0.5	A3	8	1	1.a
i2	4.5	0.5	E3	6	1	1.c
i3	4.5	0.5	D3	16	3	22(A->D)
i4	4.5	0.5	B1	9	4	23(E->B)
Measure 2 Beat 2 Chord Type: I^{6}						
i1	5	1	A3	8	1	1.a
i2	5	1	A2	4	1	1.a
i3	5	1	E3	6	1	1.c
i4	5	1	C#2	5	2	1.b
Measure 2 Beat 3 Chord Type: IV						
i1	6	0.5	F#3	20	3	11.b
i2	6	0.5	D3	16	3	11.a
i3	6	0.5	A2	4	1	11.c
i4	6	0.5	D2	8	3	11.a
Measure 2 Beat 3.5 Chord Type: IV (Embedded Dom Passing)						
i1	6.5	0.5	G#3	15	2	24(E->G#)
i2	6.5	0.5	D3	16	3	11.a
i3	6.5	0.5	A2	4	1	11.c
i4	6.5	0.5	E2	3	1	23(A->E)
Measure 2 Beat 4 Chord Type: vi						
i1	7	0.5	A3	8	1	18.b
i2	7	0.5	C#3	5	1	18.c
i3	7	0.5	A2	4	1	18.b
i4	7	0.5	F#2	10	3	18.a
Measure 2 Beat 4.5 Chord Type: $I^{4/3}$						
i1	7.5	0.5	A3	8	1	3.a
i2	7.5	0.5	C#3	5	1	3.b
i3	7.5	0.5	G#2	15	4	3.d
i4	7.5	0.5	E2	3	1	3.c
Measure 3 Beat 1 Chord Type: V^{6}/V						
i1	8	0.5	B3	9	1	15.a
i2	8	0.5	B2	9	2	15.a
i3	8	0.5	F#2	27	8	15.c
i4	8	0.5	D#2	45	16	15.b
Measure 3 Beat 1.5 Chord Type: $V^{6/5}$/V						
i1	8.5	0.5	A3	8	1	16.d
i2	8.5	0.5	B2	9	2	16.a
i3	8.5	0.5	F#2	27	8	16.c
i4	8.5	0.5	D#2	45	16	16.b

Measure 3 Beat 2 Chord Type: V (-5)						
i1	9	0.5	G#3	15	2	15.b
i2	9	0.5	E3	6	1	15.a
i3	9	0.5	G#2	15	4	15.b
i4	9	0.5	E2	3	1	15.a
Measure 3 Beat 2.5 Chord Type: $I^{4/3}$ (-3)						
i1	9.5	0.5	G#3	15	2	3.d
i2	9.5	0.5	E3	6	1	3.c
i3	9.5	0.5	A2	4	1	3.a
i4	9.5	0.5	E2	3	1	3.c
Measure 3 Beat 3 Chord Type: V/V						
i1	10	1	F#3	27	4	15.c
i2	10	1	D#3	45	8	15.b
i3	10	1	B2	9	2	15.a
i4	10	1	B1	9	4	15.a
Measure 3 Beat 4 Chord Type: V/V						
i1	11	0.5	F#3	27	4	15.c
i2	11	0.5	D#3	45	8	15.b
i3	11	0.5	B2	9	2	15.a
i4	11	0.5	B2	9	4	15.a
Measure 3 Beat 4.5 Chord Type: $V^{4/2}$/V						
i1	11.5	0.5	F#3	27	4	16.c
i2	11.5	0.5	D#3	45	8	16.b
i3	11.5	0.5	B2	9	2	16.a
i4	11.5	0.5	A2	4	1	16.d
Measure 4 Beat 1 Chord Type: V^{6}						
i1	12	0.5	G#3	15	2	15.b
i2	12	0.5	E3	6	1	15.a
i3	12	0.5	B2	9	2	15.c
i4	12	0.5	G#2	15	4	15.b
Measure 4 Beat 1.5 Chord Type: $V^{6/4}$ (+4, 4 in bass)						
i1	12.5	0.5	G#3	15	2	15.b
i2	12.5	0.5	E3	6	1	15.a
i3	12.5	0.5	B2	9	2	15.c
i4	12.5	0.5	A2	4	1	22 (E->A)
Measure 4 Beat 2 Chord Type: vii^{o}/V (+4, 4 in bass)						
i1	13	0.5	A3	8	1	20.c
i2	13	0.5	F#3	27	4	20.b
i3	13	0.5	D#3	45	8	20.a
i4	13	0.5	G#2	15	4	22(D#->G#)
Measure 4 Beat 2.5 Chord Type: vii^{o6}/V						
i1	13.5	0.5	A3	8	1	20.c
i2	13.5	0.5	F#3	27	4	20.b
i3	13.5	0.5	D#3	45	8	20.a
i4	13.5	0.5	F#2	27	8	20.b
Measure 4 Beat 3 Chord Type: V (+2)						
i1	14	0.5	B3	9	1	15.c
i2	14	0.5	F#3	27	4	23(B->F#)
i3	14	0.5	G#2	15	4	15.b
i4	14	0.5	E2	3	1	15.a
Measure 4 Beat 3.5 Chord Type: $V^{4/3}$/V						
i1	14.5	0.5	B3	9	1	16.a
i2	14.5	0.5	D#3	45	8	16.b
i3	14.5	0.5	A2	4	1	16.d
i4	14.5	0.5	F#2	27	8	16.c
Measure 4 Beat 4 Chord Type: V^{6}						
i1	15	0.5	E3	6	1	15.a
i2	15	0.5	E3	6	1	15.a
i3	15	0.5	B2	9	2	15.c
i4	15	0.5	G#2	15	4	15.b

Measure 4 Beat 4.5 Chord Type: $vi^{6/5}$						
i1	15.5	0.5	F#3	20	3	19.a
i2	15.5	0.5	E3	6	1	19.d
i3	15.5	0.5	C#3	5	1	19.c
i4	15.5	0.5	A2	4	1	19.b
Measure 5 Beat 1 Chord Type: $V^{6/4}$						
i1	16	0.5	G#3	15	2	15.b
i2	16	0.5	E3	6	1	15.a
i3	16	0.5	B2	9	2	15.c
i4	16	0.5	B2	9	2	15.c
Measure 5 Beat 1.5 Chord Type: V^7/V (-3)						
i1	16.5	0.5	A3	8	1	16.d
i2	16.5	0.5	F#3	27	4	16.c
i3	16.5	0.5	B2	9	2	16.a
i4	16.5	0.5	B2	9	2	16.a
Measure 5 Beat 2 Chord Type: V/V						
i1	17	0.5	F#3	27	4	15.c
i2	17	0.5	D#3	45	8	15.b
i3	17	0.5	B2	9	2	15.a
i4	17	0.5	B1	9	4	15.a
Measure 5 Beat 2.5 Chord Type: V^7/V						
i1	17.5	0.5	F#3	27	4	16.c
i2	17.5	0.5	D#3	45	8	16.b
i3	17.5	0.5	A2	4	1	16.d
i4	17.5	0.5	B1	9	4	16.a
Measure 5 Beat 3 Chord Type: V						
i1	18	1	E3	6	1	15.a
i2	18	1	B2	9	2	15.c
i3	18	1	G#2	15	4	15.b
i4	18	1	E2	3	1	15.a
Measure 5 Beat 4 Chord Type: V						
i1	19	1	B3	9	1	15.c
i2	19	1	G#3	15	2	15.b
i3	19	1	E3	6	1	15.a
i4	19	1	E2	3	1	15.a
Measure 6 Beat 1 Chord Type: I						
i1	20	0.5	C#4	10	1	1.b
i2	20	0.5	A3	8	1	1.a
i3	20	0.5	E3	6	1	1.c
i4	20	0.5	A2	4	1	1.a
Measure 6 Beat 1.5 Chord Type: vi (+2, 2 in bass)						
i1	20.5	0.5	C#4	10	1	18.c
i2	20.5	0.5	A3	8	1	18.b
i3	20.5	0.5	F#3	20	3	18.a
i4	20.5	0.5	G#2	15	4	23(C#->G#)
Measure 6 Beat 2 Chord Type: V/vi (susp. 4, 4 in bass)						
i1	21	0.5	C#4	10	1	15.a
i2	21	0.5	C#3	5	1	15.a
i3	21	0.5	G#3	15	2	15.c
i4	21	0.5	F#2	10	3	22(C#->F#)
Measure 6 Beat 2.5 Chord Type: $V^{6/5}$/vi						
i1	21.5	0.5	C#4	10	1	15.a
i2	21.5	0.5	C#3	5	1	15.a
i3	21.5	0.5	G#3	15	2	15.c
i4	21.5	0.5	F2	25	8	15.b
Measure 6 Beat 3 Chord Type: vi						
i1	22	0.5	A3	8	1	18.b
i2	22	0.5	F#3	20	3	18.a
i3	22	0.5	C#3	5	1	18.c
i4	22	0.5	F#2	10	3	18.a

Measure 6 Beat 3.5 Chord Type: V^7						
i1	22.5	0.5	B3	9	1	16.c
i2	22.5	0.5	G#3	15	2	16.b
i3	22.5	0.5	D3	16	3	16.d
i4	22.5	0.5	E2	3	1	16.a
Measure 6 Beat 4 Chord Type: vii^o/ii (passing D)						
i1	23	0.5	C#4	10	1	20.b
i2	23	0.5	A#3	25	3	20.a
i3	23	0.5	E3	160	27	20.c
i4	23	0.5	D2			CONTR Rule 24 (A#->D) D=125:48 violates Rule 29 (C#->D) D=8:3
Measure 6 Beat 4.5 Chord Type: vii^{o6}/ii						
i1	23.5	0.5	C#4	10	1	20.b
i2	23.5	0.5	A#3	25	3	20.a
i3	23.5	0.5	E3	160	27	20.c
i4	23.5	0.5	C#2	5	2	20.b
Measure 7 Beat 1 Chord Type: ii						
i1	24	0.5	D4	32	3	5.b
i2	24	0.5	B3	80	9	5.a
i3	24	0.5	F#3	20	3	5.c
i4	24	0.5	B1	20	9	5.a
Measure 7 Beat 1.5 Chord Type: vii^{o6}						
i1	24.5	0.5	D4	32	3	20.c
i2	24.5	0.5	B3	9	1	20.b (Held Mutable Tone)
i3	24.5	0.5	G#3	15	2	20.a
i4	24.5	0.5	B1	9	4	20.b (Held mutable tone)
Measure 7 Beat 2 Chord Type: I^6						
i1	25	0.5	C#4	10	1	1.b
i2	25	0.5	E3	6	1	1.c
i3	25	0.5	A3	8	1	1.a
i4	25	0.5	C#2	5	2	1.b
Measure 7 Beat 2.5 Chord Type: IV^7						
i1	25.5	0.5	C#4	10	1	12.d
i2	25.5	0.5	F#3	20	3	12.b
i3	25.5	0.5	A3	8	1	12.c
i4	25.5	0.5	D2	8	3	12.a
Measure 7 Beat 3 Chord Type: V						
i1	26	1	B3	9	1	15.c
i2	26	1	G#3	15	2	15.b
i3	26	1	E3	6	1	15.a
i4	26	1	E2	3	1	15.a
Measure 7 Beat 4 Chord Type: vi						
i1	27	1	A3	8	1	18.b
i2	27	1	F#3	20	3	18.a
i3	27	1	C#3	5	1	18.c
i4	27	1	F#2	10	3	18.a
Measure 8 Beat 1 Chord Type: V^6						
i1	28	0.5	B3	9	1	15.c
i2	28	0.5	E3	6	1	15.a
i3	28	0.5	B2	9	2	15.c
i4	28	0.5	G#2	15	4	15.b

Measure 8 Beat 1.5 Chord Type: vii°						
i1	28.5	0.5	B3	9	1	20.b
i2	28.5	0.5	D3	16	3	20.c
i3	28.5	0.5	B2	9	2	20.b
i4	28.5	0.5	G#2	15	4	20.a
Measure 8 Beat 2 Chord Type: I						
i1	29	0.5	C#4	10	1	1.b
i2	29	0.5	C#3	5	1	1.b
i3	29	0.5	E2	3	1	1.c
i4	29	0.5	A2	4	1	1.a
Measure 8 Beat 2.5 Chord Type: vii$^{\phi7}$/V (Rule 21.a: CONTRADICTION. Please see "Half-Diminished vii$^{\phi7}$ Chord" on p. 144.)						
i1	29.5	0.5	C#4			
i2	29.5	0.5	D#3			
i3	29.5	0.5	F#2			
i4	29.5	0.5	A2			
Measure 8 Beat 3 Chord Type: V						
i1	30	0.5	B3	9	1	15.c
i2	30	0.5	E3	6	1	15.a
i3	30	0.5	G#2	15	4	15.b
i4	30	0.5	E2	3	1	15.a
Measure 8 Beat 3.5 Chord Type: V^{7}						
i1	30.5	0.5	B3	9	1	16.c
i2	30.5	0.5	D3	16	3	16.d
i3	30.5	0.5	G#2	15	4	16.b
i4	30.5	0.5	E2	3	1	16.a
Measure 8 Beat 4 Chord Type: vi						
i1	31	0.5	A3	8	1	18.b
i2	31	0.5	C#3	5	1	18.c
i3	31	0.5	A2	4	1	18.b
i4	31	0.5	F#2	10	3	18.a
Measure 8 Beat 4.5 Chord Type: I$^{6/4}$						
i1	31.5	0.5	A3	8	1	1.a
i2	31.5	0.5	C#3	5	1	1.b
i3	31.5	0.5	A2	4	1	1.a
i4	31.5	0.5	E2	3	1	1.c
Measure 9 Beat 1 Chord Type: V$^{6/5}$/V						
i1	32	0.5	A3	8	1	16.d
i2	32	0.5	F#3	27	4	16.c
i3	32	0.5	B2	9	2	16.a
i4	32	0.5	D#2	45	16	16.b
Measure 9 Beat 1.5 Chord Type: vii$^{\phi7}$/V (Rule 21.a: CONTRADICTION Please see "Half-Diminished vii$^{\phi7}$ Chord" on p. 144.)						
i1	32.5	0.5	A3			
i2	32.5	0.5	F#3			
i3	32.5	0.5	C#3			
i4	32.5	0.5	D#2			
Measure 9 Beat 2 Chord Type: V^{7} (-5)						
i1	33	1	G#3	15	2	16.b
i2	33	1	E3	6	1	16.a
i3	33	1	D3	16	3	16.d
i4	33	1	E2	3	1	16.a
Measure 9 Beat 3 Chord Type: I						
i1	34	2	A3	8	1	1.a
i2	34	2	E3	6	1	1.c
i3	34	2	C#3	5	1	1.b
i4	34	2	A1	2	1	1.a

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