Abel and Cauchy on a Rigorous Approach to Infinite Series

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1 Introduction

Infinite series were of fundamental importance in the development of calculus by Newton, Euler and other mathematicians during the late 1600's and 1700's. Questions of rigor and convergence were of secondary importance in these times, but things began to change in the early 1800's. When the brilliant young mathematician Niels Abel moved to Paris in 1826 at age 24, he was aware of many paradoxes with infinite series and wanted big changes. Indeed, in a letter to his friend Holmboe he wrote that "I shall devote all my efforts to bring light into the immense obscurity that today reigns in Analysis. It so lacks any plan or system, that one is really astonished that there are so many people who devote themselves to it – and, still worse, it is absolutely devoid of rigor."

Abel was born and raised in Norway, far from the centers of mathematical activity in his time. His work was largely unrecognized during his lifetime through a series of misfortunes. Nevertheless he managed to get to Paris and attend lectures by mathematical stars such as Adrien-Marie Legendre (1752-1833) and Augustin Louis Cauchy (1789-1857). Abel was particularly taken by Cauchy and his efforts to introduce rigor into analysis, writing to Holmboe that Cauchy "is the only man who knows how mathematics should be treated. What he does is excellent".

In this project, we will read excerpts from 1820's work by Abel and Cauchy as we rigorously develop infinite series and examine some of the tough infinite series problems of their day.

Of course, the study of infinite series go back to antiquity. The Greek mathematician Archimedes used infinite series to help calculate the area under the arc of a parabola, and geometric series such as

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$$

were well known and used extensively in the development of calculus. The notion of divergent series was not clearly understood and somewhat controversial in the times leading up to Abel and Cauchy. For a simple example, here are two groupings and "sums" of the series $1-1+1-1+1-1+\cdots$

$$\begin{aligned}(1-1)+(1-1)+(1-1)+\cdots &= 0+0+0+\cdots=0\\ 1+(-1+1)+(-1+1)+(-1+1)+\cdots &= 1+0+0+0+\cdots=1\end{aligned}$$

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which suggests that $1 = 0$ (*oops!*). Some mathematicians in the 1700's suggested that the sum of this series should split the difference and be $1/2$, and others argued that the series did not converge and had no sum. We next read a short excerpt from Abel in another 1826 letter home to Holmboe from Paris. Abel references a much more sophisticated series example (equation (1) below) that Euler had discussed in 1750. This series is quite important historically, as Joseph Fourier used it in his development of Fourier series and his model of heat transfer during the early 1800's.

Divergent series are in their entirety an invention of the devil and it is a disgrace to base the slightest demonstration on them. You can take out whatever you want when you use them, and they are what has produced so many failures and paradoxes. ... The following example shows how one can err. One can rigorously demonstrate that

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots \tag{1}$$

for all values of x smaller than π. It seems that consequently the same formula must be true for $x = \pi$; but this will give

$$\frac{\pi}{2} = \sin\pi - \frac{1}{2}\sin 2\pi + \frac{1}{3}\sin 3\pi - \text{etc.} \quad = 0.$$

On can find innumerable examples of this kind.

In general the theory of infinite series, up to the present, is very poorly established. One performs every kind of operation on infinite series, as if they were finite, but is it permissible? Never at all. Where has it been demonstrated that one can obtain the derivative of an infinite series by taking the derivative of each term? It is easy to cite examples where this is not right ... By taking derivatives [of (1)], one has

$$\frac{1}{2} = \cos x - \cos 2x + \cos 3x - \text{ etc.} \tag{2}$$

A completely false result, because this series is divergent.

Exercise 1 *Find a few values of x less than π which, substituted into (2), produce strange results and support Abel's contention that the series in (2) is divergent.*

In Abel's day, there was no standard terminology for "absolute value", and mathematicians were not always clear whether they meant the absolute value of a number or the number itself.

Exercise 2 *In the excerpt above, Abel claims that the series in equation (1) is valid "for all values of x smaller than π". Do you think he meant x itself or $|x|$?*

Exercise 3 *Let's try to visualize equation (1). Use a Computer Algebra System (CAS) to graph $y = x/2$ and $y = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots - \frac{1}{10}\sin 10x$ together for $-2\pi \le x \le 2\pi$. What do you observe at $x = \pm\pi$? If you were to plot even more terms from the infinite series, the "wiggly" parts of the sine curve would grow even closer to straight line for $|x| < \pi$. What do you think of Abel's comments about this series and its derivative?*

We won't try to tackle during this project all the issues Abel raises with this example. However, we can see why mathematicians of his time were struggling with infinite series at the same time that they were amazed by their power!

Another series that bothered Abel comes from the binomial theorem. Newton had discovered that the standard *finite* binomial expansion $(1+x)^m$ for positive integer m could be be generalized to an *infinite* series for *non-integer* values of m (equation (3) below), and he was able to use this series to produce a number of new results. While Newton thought this series converged only for $|x| < 1$, no one had produced a convergence proof that fully convinced Abel, and he set out to do so in an 1826 paper. Here is an excerpt from the introduction to Abel's paper.

Investigations on the series:

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \ldots \text{ etc.} \tag{3}$$

1.

If one subjects to a more precise examination the reasoning that one generally uses when dealing with infinite series, one will find that, taken as a whole, it is not very satisfactory, and that consequently the number of theorems concerning infinite series that may be considered rigorously based is very limited. On normally applies the operations of analysis to infinite series as if the series were finite. This does not seem to me permissible without special proof. ...

One of the most remarkable series in algebraic analysis is (3). When m is a positive whole number, one knows that the sum of this series, which in this case is finite, may be expressed as $(1+x)^m$. When m is not a whole number, the series becomes infinite, and it will be convergent or divergent, according to different values that one gives to m and x. In this case one writes in the same way

$$(1+x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \ldots \text{ etc.;} \tag{4}$$

... One assumes that numerical equality will always hold when the series is convergent; but this is what until now has not yet been proved. No one has even examined all the cases where the series is convergent ...

The aim of this memoir is to try to fill a gap with the complete solution of the following problem:

"Find the sum of the series (3) for all real or imaginary values of x and m for which the series is convergent."

2.

We are first going to establish some necessary theorems on series. The excellent work of Cauchy "Cours d'analyse de l'école polytechnique", which must be read by every analyst who loves rigor in mathematical research, will serve as our guide.

Exercise 4 *To get a sense of the binomial series equality for whole numbers m, verify (4) with $m = 3$.*

Exercise 5 *To get a sense of Abel's concerns about the binomial series with non-integer m, use a CAS to graph $y = \sqrt{1+x}$ and the first five terms of series (3) together for $-1.2 \leq x \leq 1.2$. Newton claimed this series converges for $|x| < 1$. Does your plot suggest this is correct? Does the plot suggest the series converges at $x = \pm 1$? What about for $|x| > 1$?*

We will now take Abel's advice and read Cauchy on infinite series in the next part of this project. Then we will return to Section 2 of Abel's paper, where he develops some new infinite series results and tackles a controversial theorem of Cauchy. This work is important in its own right, independent of the binomial theorem, and will serve as the primary focus of our project.

2 Cauchy on Infinite Series

Augustin Louis Cauchy was a renowned figure in 1826 Paris. After graduating in 1810 from the École Polytechnique in Paris, he published much impressive mathematics and became a professor at this same institution. Cauchy loved pure mathematics and was convinced of the need for a rigorous approach to analysis. He wrote his 1821 *Cours d'Analyse* as a text for his teaching, and he constructed it with his philosophy of rigor. Abel had read this text before coming to Paris, and was inspired to use its methods and spirit in his own research. One radical aspect of Cauchy's book was his study of convergence of series without necessarily finding the sum of the series, which was quite a departure from the eighteenth century tradition of focusing on series sums with little attention to convergence issues.

We now start reading Chapter 6 on infinite series of Cauchy's *Cours d'Analyse.*

6.1 General Considerations on series.

We call a series an indefinite sequence of quantities,

$$u_0, u_1, u_2, u_3, \ldots,$$

which follow from one to another according to a determined law. These quantities themselves are the various terms of the series under consideration. Let

$$s_n = u_0 + u_1 + u_2 + \ldots + u_{n-1} \tag{5}$$

be the sum of the first n terms, where n denotes any integer number. If, for ever increasing values of n, the sum s_n indefinitely approaches a certain limit s, the series is said to be convergent, and the limit in question is called the sum of the series. On the contrary, if the sum s_n does not approach any fixed limit as n increases indefinitely, the series is divergent, and does not have a sum. In either case, the term which corresponds to the index n, that is u_n, is what we call the general term. For the series to be completely determined, it is enough that we give this general term as a function of the index n.

One of the simplest series is the geometric progression

$$1, x, x^2, x^3, \ldots,$$

which has x^n for its general term, that is to say the nth power of the quantity x. If we form the sum of the first n terms of this series, then we find

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1}{1-x} - \frac{x^n}{1-x} \tag{6}$$

As the values of n increase, the numerical value of the fraction $\frac{x^n}{1-x}$ converges towards the limit zero, or increases beyond all limits, according to whether we suppose that the numerical value of x is less than or greater than 1. Under the first hypothesis, we ought to conclude that the progression

$$1, x, x^2, x^3, \ldots,$$

is a convergent series which has $\frac{1}{1-x}$ as its sum, whereas, under the second hypothesis, the same progression is a divergent series which does not have a sum.

Cauchy's terminology and notation is close to what we use today. His definition that a divergent sum does not have a sum was not universally accepted in his day, but is now standard. The s_n expression defined in (5) above is nowadays called the *nth partial sum.* Notice that we can form a sequence of partial sums (s_n), and convergence of the **series** $u_0, u_1 u_2, u_3, \ldots$ is equivalent to convergence of the **sequence** (s_n). Most modern texts formally define the infinite series generated by the u_k terms to be the sequence (s_n) of partial sums.

Exercise 6 *Rewrite Cauchy's definition of series convergence and the sum s in terms of the sequence of partial sums (s_n), using modern notation.*

Exercise 7 *Verify the algebra in (6). This is often called the **finite** geometric series formula. For what x values is this formula valid?*

Exercise 8 *When Cauchy discusses the convergence of the geometric series, note his language "whether we suppose that the numerical value of x is less than or greater than 1." Explain why the geometric series diverges for $x = -2$. In modern terminology, what do you think Cauchy means by the "numerical value of x"? Cauchy frequently uses the term "numerical value" with this meaning.*

Notice that Cauchy does not use sigma summation notation $\sum_{k=0}^{\infty} u_k$ in this 1821 work, nor does Abel use it in his 1826 paper. The sigma summation did not come into common use until later in the 1800's. It is now conventional to denote both the infinite series and its sum using the symbols $\sum u_i$ or $\sum_{i=0}^{\infty} u_i$.

Exercise 9 *Rewrite Cauchy's proof for the geometric series when $|x| < 1$ using modern notation and results from a modern treatment of sequences.*

Exercise 10 *Suppose a series $\sum a_i$ converges, and $c \in \mathbb{R}$. Prove the series $\sum (ca_i)$ converges with sum $c \sum a_i$. If $\sum a_i$ diverges, what can you say about $\sum (ca_i)$?*

Exercise 11 *Suppose series $\sum a_i$ and $\sum b_i$ converge to A and B, respectively. Prove the series $\sum (4a_i + 7b_i)$ converges. Write its sum in terms of A and B. Generalize the results of this exercise.*

Exercise 12 *Use the results above to determine convergence and sum, or divergence, of the following series.*

(a) $6 - \frac{2}{3} + \frac{2}{27} - \frac{2}{243} + \cdots$

(b) $\sum_{k=2}^{\infty} \frac{5^{k-1}}{4^{k+1}}$

Let's return to Cauchy. As you read this next excerpt, pay careful attention to how Cauchy used the terms "necessary" and "sufficient" in his claims.

(Section 6.1 continued)

Following the principles established above, in order that the series

$$u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots \tag{7}$$

be convergent, it is necessary and it suffices that increasing values of n make the sum

$$s_n = u_0 + u_1 + u_2 + \ldots + u_{n-1}$$

converge indefinitely towards a fixed limit s. In other words, it is necessary and it suffices that, for infinitely large values of the number n, the sums

$$s_n, s_{n+1}, s_{n+2}, \ldots$$

differ from the limit s, and consequently from one another, by infinitely small quantities. Moreover, the successive differences between the first sum s_n and each of the following sums are determined, respectively, by the equations

$$\begin{aligned} s_{n+1} - s_n &= u_n \\ s_{n+2} - s_n &= u_n + u_{n+1} \\ s_{n+3} - s_n &= u_n + u_{n+1} + u_{n+2} \end{aligned}$$

....

Hence, in order for series (7) to be convergent, it is first of all necessary that the general term u_n decrease indefinitely as n increases. But this condition does not suffice, and it is also necessary that, for increasing values of n, the different sums,

$$\begin{aligned} &u_n + u_{n+1} \\ &u_n + u_{n+1} + u_{n+2} \end{aligned}$$

..........................,

that is to say, the sums of as many of the quantities

$$u_n, u_{n+1} u_{n+2}, \ldots,$$

as we may wish, beginning with the first one, eventually constantly assume numerical values less than any assignable limit. Conversely, whenever these various conditions are fulfilled, the convergence of the series is guaranteed.

Exercise 13 *List all the "necessary" claims in this excerpt, expressing each as an implication. Then list all the "sufficient" claims, expressing each as an implication.*

Exercise 14 *Carefully reread Cauchy's sentence beginning with "In other words ..." and notice that he is making two separate equivalence claims, from a modern viewpoint. Rewrite each equivalence claim in Cauchy's sentence with modern $\epsilon - N$ terminology.*

Cauchy's statements that "in order for series (7) to be convergent, it is first of all necessary that the general term u_n decrease indefinitely as n increases. But this condition does not suffice" are worth a clarification, a proof and an example.

Exercise 15 *First, clarify what Cauchy means by "the general term u_n decrease indefinitely as n increases". Second, write the claim "it is first of all necessary that the general term u_n decrease indefinitely as n increases" as a theorem, and give a modern proof using Cauchy's equation $s_{n+1} - s_n = u_n$ and modern sequence limit laws.*

Exercise 16 *Write down the contrapositive of your theorem implication from Exercise 15. Does this result remind you of an infinite series "test" from your Introductory Calculus course?*

Exercise 17 *Apply your result in Exercise 15 to the following series, where possible.*

(a) $1 - 1 + 1 - 1 + 1 - 1 + \cdots$

(b) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

(c) $\frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \cdots$

Exercise 18 *Consider the statement "the sums of as many of the quantities*

$$u_n, u_{n+1}, u_{n+2}, \ldots,$$

as we may wish, beginning with the first one, eventually constantly assume numerical values less than any assignable limit." Using finite sums, convert this statement into modern $\epsilon - N$ terminology. What is this saying about the sequence (s_n), in modern terminology?

Exercise 19 *Consider the statement "Conversely, whenever these various conditions are fulfilled, the convergence of the series is guaranteed." What modern theorem about sequences of real numbers justifies this statement?*

It is interesting that Cauchy, and many of his contemporaries, thought this last necessary and sufficient condition for convergence of a series was obvious and did not need a proof. As we shall see, Cauchy and Abel use this criterion, nowadays named after Cauchy, to prove some convergence results.

Exercise 20 *Rewrite this new "Cauchy criterion" for series convergence in modern $\epsilon - N$ terminology.*

Exercise 21 *Suppose an infinite series $\sum u_n$ is convergent and $\epsilon_0 > 0$, and define Q_m by*

$$Q_m = \sup\left\{\left|\sum_{k=m}^{m+n} u_k\right| : n \in \mathbb{N}\right\}.$$

Prove there exists $N \in \mathbb{N}$ such that $Q_m < \epsilon_0$ for any $m \geq N$.

This result will come in handy when we read Abel.

Hint: Use a property of the convergent sequence of partial sums (s_n) of $\sum u_n$.

Exercise 22 *Suppose series $\sum |x_k|$ converges. Use the Cauchy criterion to prove that $\sum x_k$ must converge.*

Now that we have carefully analyzed some fundamental results by Cauchy, let's return to his discussion, where he considered two important examples.

(Section 6.1 continued)

Let us take, for example, the geometric progression

$$1, x, x^2, x^3, ... \tag{8}$$

If the numerical value of x is greater than 1, that of the general term x^n increases indefinitely with n, and this remark alone suffices to establish the divergence of the series. The series is still divergent if we let $x = \pm 1$, because the numerical value of the general term x^n, which is 1, does not decrease indefinitely for increasing values of n. However, if the numerical value of x is less than 1, then the sums of any number of terms of the series, beginning with x^n, namely:

$$\begin{aligned} & x^n, \\ x^n + x^{n+1} &= x^n \frac{1-x^2}{1-x}, \\ x^n + x^{n+1} + x^{n+2} &= x^n \frac{1-x^3}{1-x}, \\ & \ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

are all contained between the limits

$$x^n \quad \text{and} \quad \frac{x^n}{1-x},$$

each of which becomes infinitely small for infinitely large values of n. Consequently, the series is convergent, as we already knew.

As a second example, let us take the numerical series

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots \tag{9}$$

The general term of this series, namely $\frac{1}{n+1}$, decreases indefinitely as n increases. Nevertheless, the series is not convergent, because the sum of the terms from $\frac{1}{n+1}$ up to $\frac{1}{2n}$ inclusive, namely

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}$$

is always greater than the product

$$n\frac{1}{2n}=\frac{1}{2}$$

whatever the value of n. As a consequence, this sum does not decrease indefinitely with increasing values of n, as would be the case if the series were convergent.

Exercise 23 *In this part of Section 6.1, Cauchy gave a proof that the geometric series is convergent for $|x|<1$, using the new Cauchy criterion for series convergence that you put into modern form in Exercise 20. Notice that he is a bit cavalier for the negative x case when stating that terms are "all contained between the limits...". Write a careful modern version of his proof using the modern form of the Cauchy criterion for series convergence.*

Now we turn to Cauchy's second example, where he argues that the series (9) diverges.

Exercise 24 *Write a "Cauchy criterion" for series divergence in modern $\epsilon - N$ terminology.*
Hint: Negate your definition from Exercise 20.

Exercise 25 *Justify Cauchy's claim after (9) that*

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}>\frac{1}{2}.$$

Exercise 26 *Rewrite Cauchy's proof that series (9) diverges with modern terminology and quantifiers, using the new Cauchy criterion.*
Hint: Consider $s_{2n}-s_n$.

Let's go back to Cauchy for another important example, which he analyzes with another useful technique.

(Cauchy Section 6.1 continued)
Let us further consider the numerical series

$$1,\frac{1}{1},\frac{1}{1\cdot 2},\frac{1}{1\cdot 2\cdot 3},....,\frac{1}{1\cdot 2\cdot 3\ldots n}\cdots \tag{10}$$

The terms of this series with index greater than n, namely

$$\frac{1}{1\cdot 2\cdot 3\ldots n},\frac{1}{1\cdot 2\cdot 3\ldots n\,(n+1)},\frac{1}{1\cdot 2\cdot 3\ldots n\,(n+1)\,(n+2)},\cdots,$$

are, respectively, less than the corresponding terms of the geometric progression

$$\frac{1}{1\cdot 2\cdot 3\ldots n}, \frac{1}{1\cdot 2\cdot 3\ldots n}\frac{1}{n}, \frac{1}{1\cdot 2\cdot 3\ldots n}\frac{1}{n^2}, \ldots ,$$

As a consequence, the sum of however many of the initial terms as we may wish is always less than the sum of the corresponding terms of the geometric progression, which is a convergent series, and so *a fortiori*, it is less than the sum of this series, which is to say

$$\frac{1}{1\cdot 2\cdot 3\ldots n}\frac{1}{1-\frac{1}{n}} = \frac{1}{1\cdot 2\cdot 3\ldots (n-1)}\frac{1}{n-1}\,. \tag{11}$$

Because this last sum decreases indefinitely as n increases, it follows that series (10) is itself convergent. It is conventional to denote the sum of this series by the letter e. By adding together the first n terms, we obtain an approximate value of the number e,

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\ldots.+\frac{1}{1\cdot 2\cdot 3\ldots (n-1)}\,.$$

According to what we have just said, the error made will be smaller than the product of the nth term by $\frac{1}{n-1}$. Therefore, for example, if we let $n = 11$, we find as the approximate value of e

$$e = 2.7182818..., \tag{12}$$

and the error made in this case is less than the product of the fraction $\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}$ by $\frac{1}{10}$, that is $\frac{1}{36{,}288{,}000}$, so that it does not affect the seventh decimal place.

The number e, determined as we have just said, is often used in the summation of series and in the infinitesimal Calculus. Logarithms taken in the system with this number as its base are called Napierian, for Napier, the inventor of logarithms, or hyperbolic, because they measure the various parts of the area between the equilateral hyperbola and its asymptotes.[1]

In general, we denote the sum of a convergent series by the sum of the first terms, followed by an ellipsis. Thus, when the series

$$u_0, u_1 u_2, u_3, \ldots,$$

is convergent, the sum of this series is denoted

$$u_0 + u_1 + u_2 + u_3 + \ldots$$

By virtue of this convention, the value of the number e is determined by the equation

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\ldots, \tag{13}$$

and, if one considers the geometric progression

$$1, x, x^2, x^3, ...,$$

we have, for numerical values of x less than 1,

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x}\,. \tag{14}$$

[1] Cauchy meant the area under the curve $y = 1/x$, using standard terminology of his time.

Euler derived this series expression (10) for e by 1743 from his definition $e = \lim (1+1/n)^n$ using an infinitesimal argument with the binomial theorem. Cauchy proved convergence of the e series by comparing each series term to a *larger* geometric series value, which he could sum precisely. You will generalize this method in Exercise 29 below.

Exercise 27 *Fill the algebraic details of Cauchy's argument between (10) and (11) that, for arbitrary $n, m \in \mathbb{N}$, the difference $s_{n+m} - s_n$ for the e series is less than $\frac{1}{1 \cdot 2 \cdot 3 \ldots (n-1)} \frac{1}{n-1}$.*

Exercise 28 *Use Cauchy's approach with the Cauchy criterion to give a modern $\epsilon - N$ proof that the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ converges.*

Exercise 29 *Use Cauchy's comparison ideas for the e series to fill in the blanks below (with $\sum a_i$ or $\sum b_i$) and create a valid theorem.*

Theorem 30 *Suppose (a_i) and (b_i) are sequences and there exists a $K \in \mathbb{N}$ for which $0 \le a_i \le b_i$ whenever $i \ge K$. If _ _ _ _ _ converges, then _ _ _ _ _ converges.*

Exercise 31 *Prove Theorem 30.*

Hint: Use partial sum sequences and the Monotone Convergence Theorem.

Exercise 32 *Use a contrapositive to state and prove a "divergence" version of Theorem 30.*

Exercise 33 *Use your results to determine convergence or divergence of the following series.*

(a) $\dfrac{4/7}{1} + \dfrac{5/7}{2} + \dfrac{6/7}{3} + \dfrac{1}{4} + \dfrac{8/7}{5} + \cdots$

(b) $\sum_{n=2}^{\infty} \dfrac{(3n+1)\, 4^{2n-1}}{7^{n+5}}$

(c) $\dfrac{1}{6} + \dfrac{1/2}{18} + \dfrac{1/3}{54} + \dfrac{1/4}{162} + \cdots$

In the next excerpt, we will see how Cauchy tried to extend his ideas on series of real numbers to series of a function x. His argument has problems from a modern point of view, so read it carefully.

Denoting the sum of the convergent series

$$u_0, u_1 u_2, u_3, \ldots,$$

by s and the sum of the first n terms by s_n, we have

$$\begin{aligned} s &= u_0 + u_1 + u_2 + \ldots + u_{n-1} + u_n + u_{n+1} + \ldots \\ &= s_n + u_n + u_{n+1} + \ldots, \end{aligned}$$

and, as a consequence,

$$s - s_n = u_n + u_{n+1} + \ldots$$

From this last equation, it follows that the quantities

$$u_n, u_{n+1}, u_{n+2}, \ldots$$

form a new convergent series, the sum of which is equal to $s - s_n$. If we represent this sum by r_n, we have

$$s = s_n + r_n\,,$$

and r_n is called the remainder of series (7) beginning from the nth term.

Suppose the terms of series (7) involve some variable x. If the series is convergent and its various terms are continuous functions of x in a neighborhood of some particular value of this variable, then

$$s_n, r_n \text{ and } s$$

are also three functions of the variable x, the first of which is obviously continuous with respect to x in a neighborhood of the particular value in question. Given this, let us consider the increments in these three functions when we increase x by an infinitely small quantity α. For all possible values of n, the increment in s_n is an infinitely small quantity. The increment of r_n, as well as r_n itself, becomes infinitely small for very large values of n. Consequently, the increment in the function s must be infinitely small. From this remark, we immediately deduce the following proposition:

Theorem I — *When the various terms of series (7) are functions of the same variable x, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighborhood of this particular value.*

By virtue of this theorem, the sum of series (8) must be a continuous function of the variable x between the limits $x = -1$ and $x = 1$, as we may verify by considering the values of s given by the equation

$$s = \frac{1}{1 - x}\,.$$

For simplicity we will take Cauchy's meaning of the term "neighborhood" about a value x to be a small open interval $(x - \delta, x + \delta)$ centered at the value x.

Exercise 34 *In his proof outline for Theorem I, Cauchy states that s_n is "obviously continuous with respect to x in a neighborhood of the particular value in question." In what sense is this statement correct from a modern viewpoint?*

Exercise 35 *Critique Cauchy's proof outline from a modern point of view. How might you try to adjust it for a modern proof?*

Cauchy's claim in this theorem seems pretty reasonable: if we add up some continuous functions that converge to a limit function at a point x, it seems plausible that the limit function is also continuous at x. Unfortunately, this is not always the case at all x values[2]. Indeed, Abel noticed this issue. Here is a footnote from Abel's 1826 paper where he addresses this problem:

In the work by M. Cauchy one will find the following theorem: "When the various terms of series $u_0 + u_1 + u_2 + \cdots$ are functions of the same variable x, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighborhood of this particular value."

But it seems to me that this theorem admits of exceptions. For example the series

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots \tag{15}$$

is discontinuous for any value $(2m+1)\pi$, where m is a whole number.

Exercise 36 *Go back to the project introduction and reread Abel's discussion of this series. As Abel states in the introduction, this series (15) can be shown to converge to $x/2$ for any value of x with $|x| < \pi$, although the method, Fourier series convergence, is beyond the scope of this project. Carefully explain why Cauchy's theorem is not valid for the series (15) at $x = \pi$.*

Note that Abel politely mentions this example as an "exception" to Cauchy's theorem. A blunter interpretation would be that Abel had found a counterexample to this theorem, rendering it invalid without stronger hypotheses. He did not identify the problem in Cauchy's proof, but he did prove a correct variation of this theorem with significantly stronger hypotheses. We will examine Abel's theorem in the next section of this project. Other major mathematicians worked hard during the mid-1800's to prove other corrected variations on Cauchy's Theorem I. This indicates the subtlety of Cauchy's error and the difficulty involved in fixing it!

3 Abel's 1826 Paper

Abel was aware of the difficulties with Cauchy's theorem on a series of continuous functions. Abel did not identify the problem in Cauchy's proof, but he was able to prove an important theorem on the convergence of power series in his 1826 paper. Along the way, he proved other important results. These ideas are the focus of this section of the project. We now turn to Abel's paper, with his first two theorems. As you read them, see if they remind you of convergence tests from your Introductory Calculus course.

[2]Some historians argue that Cauchy meant the convergence was *uniform*, which makes his theorem valid. For more on this debate, see [Jahnke] and [Bot].

Theorem I. If one denotes a series of positive quantities by $\rho_0, \rho_1, \rho_2 \ldots$, and the quotient $\frac{\rho_{m+1}}{\rho_m}$, for ever increasing values of m, approaches a limit α greater than 1, then the series

$$c_0\rho_0 + c_1\rho_1 + c_2\rho_2 + \cdots + c_m\rho_m + \cdots ,$$

where c_m is a quantity which, for ever increasing values of m, does not approach zero, will be necessarily divergent.

Theorem II. If in a series of positive quantities $\rho_0 + \rho_1 + \rho_2 + \cdots + \rho_m + \cdots$ the quotient $\frac{\rho_{m+1}}{\rho_m}$, for ever increasing values of m, approaches a limit α smaller than 1, then the series

$$c_0\rho_0 + c_1\rho_1 + c_2\rho_2 + \cdots + c_m\rho_m + \cdots , \tag{16}$$

where c_0, c_1, c_2 etc. are quantities that are never greater than one, will be necessarily convergent.[3]

Exercise 37 *Interpret Theorems I and II in the special case where $c_m = 1$ for all m. What name did we give these results in an Introductory Calculus course? Cauchy actually gave these results in the special case where $c_m = 1$ in his book; Abel generalized them for his needs in Theorem IV later in his 1826 paper.*

Abel did not give a proof of his first theorem, but he did supply a proof of his Theorem II, given in the next excerpt from his paper. Read it carefully, because it needs some minor adjustments for a modern level of rigor.

Indeed, by assumption, one may always take m large enough that $\rho_{m+1} < \alpha\rho_m$, $\rho_{m+2} < \alpha\rho_{m+1}$, $\cdots$ $\rho_{m+n} < \alpha\rho_{m+n-1}$. It follows from there that $\rho_{m+k} < \alpha^k\rho_m$, and consequently

$$\rho_m + \rho_{m+1} + \cdots + \rho_{m+n} < \rho_m\left(1 + \alpha + \alpha^2 + \cdots + \alpha^n\right) < \frac{\rho_m}{1-\alpha}, \tag{17}$$

therefore, for all the more reason,

$$c_m\rho_m + c_{m+1}\rho_{m+1} + \cdots + c_{m+n}\rho_{m+n} < \frac{\rho_m}{1-\alpha}.$$

Now, since $\rho_{m+k} < \alpha^k\rho_m$ and $\alpha < 1$, it is clear that ρ_{m+1} and consequently the sum

$$c_m\rho_m + c_{m+1}\rho_{m+1} + \cdots + c_{m+n}\rho_{m+n}$$

will approach zero.

The above series [(16)] is therefore convergent.

[3] We have used c where Abel wrote ε in these theorems, in order to reduce confusion in modern ε arguments.

Exercise 38 *In his proof of Theorem II, Abel states that "one may always take* m *large enough that* $\rho_{m+1} < \alpha\rho_m$*". Use the example* $\rho_n = \dfrac{n}{2^n}$ *to show that this is not quite true.*

Exercise 39 *You will fix Abel's proof in this exercise.*

(a) Show that if $\lim \dfrac{\rho_{m+1}}{\rho_m} < 1$ *then we can find (i) a number* β *such that* $\lim \dfrac{\rho_{m+1}}{\rho_m} < \beta < 1$*, and (ii)* $N \in \mathbb{N}$ *for which* $m \geq N$ *implies that* $\rho_{m+1} < \beta\rho_m$*.*

(b) Use part (a) and Abel's ideas to show that the sequence (ρ_m) *converges to* 0*.*

(c) Use part (a) and Abel's ideas to prove a statement analogous to (17).

Exercise 40 *To understand Abel's Theorem II statement completely, we need to remember to interpret the statement "where* c_0 *etc. are quantities that are never greater than one" carefully. To see this, set* $\rho_m = 1/2^m$ *and* $c_m = -2^m$ *and show that the series* $\sum \rho_m c_m$ *diverges.*

Exercise 41 *Based on the exercise above, let's interpret Abel's Theorem II hypotheses about the* c_k *as "the quantities* $|c_k|$ *are never greater than one". With this adjustment,*

(a) Write a modern version of Abel's Theorem II.

(b) Use Abel's proof method and your results from Exercise 39 to give a modern $\epsilon - N$ *proof of your modern version of Theorem II.*

Abel did not give a proof of his first theorem, perhaps thinking it obvious. See if you can verify his claim in the next exercise.

Exercise 42 *Give a modern proof of Theorem I. Here are some suggestions:*

(a) First explain why the sequence (ρ_k) *diverges with* $\lim \rho_k = \infty$*.*

Hint: Think about Abel's argument relating ρ_{m+k} *and* ρ_m *in his Theorem II proof.*

(b) Translate the hypothesis about the sequence (c_k) *to a statement about a subsequence* $(|c_{n_k}|)$ *of* $(|c_k|)$*.*

(c) Draw a conclusion based on the limiting behavior of a subsequence of $(|c_k\rho_k|)$ *corresponding to* $(|c_{n_k}|)$*.*

Exercise 43 *In the 1700's, Euler derived a power series for* $\ln(x+1)$ *:*

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{18}$$

Use Abel's theorems and standard sequence theorems to prove this series converges at a given real number x *when* $|x| < 1$ *and diverges when* $|x| > 1$*. Be sure to clearly identify the* c_k, ρ_k *and* α *values.*

Exercise 44 *Generalize Abel's Theorem II to a theorem with hypothesis "(c_k) bounded" in place of "$|c_k|$ are never greater than one". Prove your claim.*

Exercise 45 *Apply these theorems from Exercises 41, 44 and standard sequence theorems to determine convergence or divergence of the series below. Be sure to clearly identify the c_k, ρ_k and α values and make sure the theorem hypotheses are met.*

(a) $\sum \frac{3k-1}{2^k(k+1)}$

(b) $\sum \frac{k+1+(-1)^k}{k+1} \frac{3^{k+1}}{2^{k-1}}$

(c) $\sum y^k/k!$ *for fixed, arbitrary* $y \in \mathbb{R}$.

Exercise 46 *Recall from the project introduction that Abel was interested in proving convergence of the generalized binomial series (3)*

$$1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \dots$$

for various m and x values. For $m \in \mathbb{R}$ but not necessarily an integer, use Abel's theorems to show that series (3) converges when $|x| < 1$ and diverges when $|x| > 1$.

We next return to Abel for his third theorem, which he needs as a tool for proving his major power series result in Theorem IV.

(Abel Section 2 continued)

Theorem III. On denoting by $t_0, t_1, t_2, \dots t_m, \dots$ a series of any quantities whatever, if $p_m = t_0 + t_1 + t_2 + \dots t_m$ is always less than a determined quantity[4] B, one will have

$$r = c_0 t_0 + c_1 t_1 + c_2 t_2 + \dots + c_m t_m < B c_0$$

where $c_0, c_1, c_2 \dots$ denote positive decreasing quantities.

Indeed, one has

$$t_0 = p_0, \quad t_1 = p_1 - p_0, \quad t_2 = p_2 - p_1, \quad \text{etc.}$$

therefore

$$r = c_0 p_0 + c_1 (p_1 - p_0) + c_2 (p_2 - p_1) + \dots + c_m (p_m - p_{m-1}) \tag{19}$$

or rather

$$r = p_0 (c_0 - c_1) + p_1 (c_1 - c_2) + \dots + p_{m-1} (c_{m-1} - c_m) + p_m c_m \tag{20}$$

But $c_0 - c_1, c_1 - c_2, \cdots$ are positive, so the quantity r will clearly be less than Bc_0.

[4] We have used B where Abel wrote δ in these theorems, in order to reduce confusion in modern ϵ-δ proofs.

Let's examine this result and Abel's proof from a modern viewpoint.

Exercise 47 *Rewrite the theorem statement with appropriate quantifiers, and clarify the phrase "decreasing quantities".*

Exercise 48 *Justify the algebraic rearrangement of terms in r, between (19) and (20).*

Exercise 49 *Justify Abel's claim in his Theorem III proof that "quantity r will clearly be less than Bc_0".*

Exercise 50 *It will be helpful to have a version of this theorem with stronger conclusion $|r| < Bc_0$ instead of $r < Bc_0$. This stronger conclusion naturally requires a stronger hypothesis on the partial sums p_m. State and prove an "absolute value" version of Theorem III with the stronger hypothesis $|p_k| < B$ for all k and conclusion $|r| < Bc_0$. Abel's beautiful rearrangement of the terms in r will still be crucial for your proof!*

Exercise 51 *Consider the partial sums s_m of series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \tag{21}$$

(a) Use your "absolute value" version of Theorem III from Exercise 50 to prove that $|s_m| < 1.01$ for all m.

(b) For this same series (21), find m so that for any n we have $|s_{m+n} - s_m| < 0.001$, and prove your claim using your "absolute value" version of Theorem III from Exercise 50.

Hint: Choose t_0, c_0, B to align with the mth term of series (21).

The series (21) is an example of an alternating series, which you may recall from your Introductory Calculus course. Here is a useful theorem for guaranteeing convergence of a certain class of alternating series.

Theorem 52 *If (d_k) is a decreasing sequence of positive numbers with $\lim(d_k) = 0$, then the alternating series $\sum (-1)^k d_k$ is convergent.*

Exercise 53 *Prove Theorem 52 using the Cauchy criterion, your "absolute value" version of Theorem III, and the ideas used in Exercise 51 (b).*

Exercise 54 *Use Theorem 52 to prove the following series converge, or explain why the theorem cannot be applied to the particular series.*

(a) $\sum_{k=0}^{\infty} (-1)^k \dfrac{1}{k+1}$

(b) $\dfrac{2}{1} - \dfrac{3}{2} + \dfrac{4}{3} - \dfrac{5}{4} + \cdots$

(c) $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \dfrac{1}{\sqrt{5}} - \cdots$

Like Cauchy, Abel wanted to apply his results on numerical series to variable series, and to address the issue of continuity. We investigate his efforts in the remainder of the project.

3.1 Abel's power series theorem

After developing his first three theorems on numerical series, Abel considered the situation where a function f was defined by an infinite series in terms of a variable α. In the theorem below, he states an important result for power series.

Definition. A function $f(x)$ will be said to be a continuous function of x between the limits $x = a$ and $x = b$, if for any value of x contained between these limits, the quantity $f(x - \beta)$, for ever decreasing values of β, approaches the limit $f(x)$.

***Theorem* IV.** If the series

$$f(\alpha) = v_0 + v_1\alpha + v_2\alpha^2 + \cdots + v_m\alpha^m + \ldots$$

converges for a certain value d[5] of α, it will also converge for every value smaller than d and, for this kind of series, for ever decreasing values of β, the function $f(\alpha - \beta)$ will approach the limit $f(\alpha)$, assuming that α is equal to or less than d.

We begin by examining the statement of Theorem IV. First notice that Abel is making two claims. First, he claims that the infinite series $f(\alpha)$ will "converge for every value smaller than d". From our previous readings, we suspect that this is not to be taken literally. Let's take his meaning on α to be: for every α value, $0 \leq \alpha < d$ where d is positive.

Abel's second claim is that "the quantity $f(\alpha - \beta)$, for ever decreasing values of β, approaches the limit $f(\alpha)$" for $0 \leq \alpha \leq d$.

Exercise 55 *Abel's definition of continuity is essentially the same as Cauchy's. To make this definition consistent with our modern definition, what do you think he means by "for ever decreasing values of β"? Rewrite his continuity claim with modern terminology. Be careful with the special case $\alpha = d$.*

Exercise 56 *Explain how Abel's Theorem IV is a variant of Cauchy's Theorem I, page 12. Be sure to compare and contrast both the hypotheses and the conclusions.*

Exercise 57 *As a first application of this theorem, consider the power series (18), which we showed converges for $x = 1$ in Exercise 54. For what other x values does Abel's Theorem IV guarantee convergence of this power series?*

We next read Abel's proof of his Theorem 4. As you read his proof, think about how you can adjust it for a modern proof. In particular, notice how he uses the symbol ω.

[5] We are using d where Abel wrote δ, in order to avoid confusion with ϵ-δ proofs.

For brevity, in this memoir we will understand by ω a quantity which may be smaller than any given quantity, however small.[6]

***Theorem* IV.** If the series

$$f(\alpha) = v_0 + v_1\alpha + v_2\alpha^2 + \cdots + v_m\alpha^m + \ldots$$

converges for a certain value d[7] of α, it will also converge for every value smaller than d and, for this kind of series, for ever decreasing values of β, the function $f(\alpha-\beta)$ will approach the limit $f(\alpha)$, assuming that α is equal to or less than d.

Suppose

$$\begin{aligned} v_0 + v_1\alpha + v_2\alpha^2 + \cdots + v_{m-1}\alpha^{m-1} &= \varphi(\alpha), \\ v_m\alpha^m + v_{m+1}\alpha^{m+1} + v_{m+2}\alpha^{m+2} + \text{ etc. } \cdots &= \psi(\alpha), \end{aligned}$$

so

$$\psi(\alpha) = \left(\frac{\alpha}{d}\right)^m v_m d^m + \left(\frac{\alpha}{d}\right)^{m+1} v_{m+1} d^{m+1} + \text{etc.},$$

therefore, from Theorem III, $\psi(\alpha) < \left(\frac{\alpha}{d}\right)^m p$ where p denotes the greatest of the quantities $v_m d^m, v_m d^m + v_{m+1}d^{m+1}, v_m d^m + v_{m+1}d^{m+1} + v_{m+2}d^{m+2}$ etc. Therefore for every value of α, equal to or less than d, one may take m large enough that one will have

$$\psi(\alpha) = \omega.$$

Now $f(\alpha) = \varphi(\alpha) + \psi(\alpha)$, so $f(\alpha) - f(\alpha-\beta) = \varphi(\alpha) - \varphi(\alpha-\beta) + \omega$. Further, $\varphi(\alpha)$ is a polynomial is α, so one may take β small enough that

$$\varphi(\alpha) - \varphi(\alpha-\beta) = \omega;$$

so also one has in the same way

$$f(\alpha) - f(\alpha-\beta) = \omega,$$

which it was required to prove.

Exercise 58 *What part of this proof is needed for Abel's first claim, that the infinite series $f(\alpha)$ will converge for $0 \le \alpha < d$? On what variables does ω depend for this part of his proof? How can we translate Abel's phrase "a quantity which may be smaller than any given quantity, however small" for a modern proof?*

Exercise 59 *Abel lets p denote the greatest of an infinite number of quantities. From a modern point of view, how would you criticize this?*

[6] Abel put this remark into a footnote.

[7] We are using d where Abel wrote δ, in order to avoid confusion with ϵ-δ proofs.

3.2 Modernizing Abel's proof that $f(\alpha)$ converges for $\alpha < d$

Notice that Abel uses Theorem III in his Theorem IV proof with an *infinite* sum, the remainder term $\psi(\alpha)$, but the r in Theorem III involves a *finite* sum. Also observe that for a modern proof, we can't use his infinite series "tail" $\psi(\alpha)$ until we know it converges. Moreover, we don't have a candidate for this series sum, so a modern proof will need to use the Cauchy criterion. For these reasons, let's introduce the notation

$$\begin{aligned}\varphi_m(\alpha) &= v_0 + v_1\alpha + v_2\alpha^2 + \cdots + v_{m-1}\alpha^{m-1} \\ \psi_{m,n}(\alpha) &= v_m\alpha^m + v_{m+1}\alpha^{m+1} + v_{m+2}\alpha^{m+2} + \cdots + v_{m+n-1}\alpha^{m+n-1}.\end{aligned}$$

Exercise 60 *Show that for arbitrary α,m,n we have $\psi_{m,n}(\alpha) = \varphi_{m+n}(\alpha) - \varphi_m(\alpha)$.*

We need to adjust Abel's p definition for our modern proof with the Cauchy criterion as follows:

$$P_m = \sup\left\{\left|v_m d^m + v_{m+1}d^{m+1} + v_{m+2}d^{m+2} + \cdots + v_{m+n-1}d^{m+n-1}\right| : n \in \mathbb{N}\right\}$$

Observe that P_m depends only on d, m and the coefficients v_k.

Exercise 61 *Let $\epsilon > 0$. Prove there exists $N \in \mathbb{N}$ such that $P_m < \epsilon/3$ for all $m \geq N$, using Exercise 21 and the Theorem IV hypotheses.*

Exercise 62 *Let $m, n \in \mathbb{N}$ and $0 \leq \alpha < d$. Use Abel's ideas from his Theorem IV proof and your "absolute value" version of Theorem III from Exercise 50 to show that*

$$\left|\psi_{m,n}(\alpha)\right| \leq \left(\frac{\alpha}{d}\right)^m P_m.$$

Be sure to make clear which factors in $\psi_{m,n}(\alpha)$ correspond to which c_k and t_k values in Theorem III.

Exercise 63 *Let α be fixed with $0 \leq \alpha < d$. Use the exercise results above and the Cauchy criterion to give a modern $\epsilon - N$ proof that the sequence of partial sums $\{\varphi_m(\alpha)\}$ converges. We will call the sum $f(\alpha)$, in keeping with Abel's name for this convergent infinite series.*

Exercise 63 gives us a modern proof that the infinite series $f(\alpha)$ in Theorem IV will converge for $0 \leq \alpha \leq d$. Now we tackle the second part of Abel's proof regarding the continuity of f.

3.3 Modernizing Abel's proof that $f(\alpha)$ is continuous for $\alpha \leq d$

To prove continuity with modern terminology, given $\alpha \leq d$ and $\epsilon > 0$ we need to find a $\delta > 0$ so that $|\beta| < \delta$ implies that $|f(\alpha - \beta) - f(\alpha)| < \epsilon$. Observe that Abel uses the symbol ω four times. Recalling his statement "we will understand by ω a quantity which may be smaller than any given quantity, however small", we can see that he might not mean for these four ω's to be literally identical.

Exercise 64 *On what variables does ω depend for the last three times he uses it?*

Now that we know the infinite series $f(\alpha)$ converges from the first part of the proof, we can safely use Abel's infinite term remainder $\psi(\alpha)$, but must take care to remember it depends on m as well as α. When this is particularly important, we can use $\psi_m(\alpha)$ in place of $\psi(\alpha)$ for emphasis.

Exercise 65 *Let α be fixed with $0 \leq \alpha < d$. Explain why $\psi_m(\alpha) = \lim_{n\to\infty} \psi_{mn}(\alpha)$.*

The next exercise will help us modernize Abel's claim that "$f(\alpha) - f(\alpha - \beta) = \varphi(\alpha) - \varphi(\alpha - \beta) + \omega$."

Exercise 66 *Let m be arbitrary. Show that*

$$|f(\alpha) - f(\alpha - \beta)| \leq |\varphi_m(\alpha) - \varphi_m(\alpha - \beta)| + |\psi_m(\alpha)| + |\psi_m(\alpha - \beta)| .$$

For what β values is this valid?

We need to convert Abel's ω statements into appropriate $\epsilon - \delta$ statements. We need an ϵ bound on $|\psi_m(\alpha)|$ and another bound on $|\varphi(\alpha) - \varphi(\alpha - \beta)|$. Less obviously, we need a bound on $|\psi_m(\alpha - \beta)|$, which Abel absorbs into one of his ω's in the claim $f(\alpha) - f(\alpha - \beta) = \varphi(\alpha) - \varphi(\alpha - \beta) + \omega$.

It turns out that the trickiest of these three bounds is for $|\psi_m(\alpha - \beta)|$, because we need the bound to work *for all* β with $|\beta| < \delta$, not just a single β. To get this bound, look at your definitions of $\psi_{m,n}(\alpha)$ and P_m just before Exercise 61. Notice P_m does not depend on n or α! In fact observe that

$$\left|\psi_{m,n}(\alpha)\right| \leq \left(\frac{\alpha}{d}\right)^m P_m \tag{22}$$

for all n and for any α, $0 \leq \alpha \leq d$.

Exercise 67 *For a given $\epsilon > 0$ and $0 \leq \alpha \leq d$, find $N \in \mathbb{N}$ so that $m \geq N$ and $0 \leq \alpha - \beta \leq d$ imply that*

$$|\psi_m(\alpha - \beta)| < \frac{\epsilon}{3}$$

The bound (22) and Exercise 61 should be helpful!

Exercise 68 *Using a modern $\epsilon - \delta$ argument with your results from the past few exercises, rewrite Abel's proof that the infinite series $f(\alpha)$ is continuous at each α for $0 \leq \alpha \leq d$.*

Hint: Remember that $\varphi_N(\alpha)$ is a polynomial, hence continuous.

Exercise 69 *Define a function f by*

$$f(x) = x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \frac{x^5}{\sqrt{5}} - \cdots$$

Use Abel's theorems to find x values at which f is continuous.

Hint: Recall Exercise 54.

4 Conclusion

We have examined Cauchy's flawed Theorem I and Abel's response to it using the series

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots \tag{23}$$

on a neighborhood of $x = \pi$. Part of the difficulty for Cauchy was that mathematicians had not yet worked out the issues for an infinite series involving a variable x converging at a *single* x value, versus converging *uniformly* for a *set* of x values. Here is a modern definition of *uniform* convergence for a power series on a set A.

Definition 70 *Suppose $\sum_{k=0}^{\infty} v_k x^k$ is a power series, and define partial sum $\varphi_m(x) = \sum_{k=0}^{m-1} v_k x^k$ for each $m \in \mathbb{N}$. We say the series converges* ***uniformly*** *to $f(x)$ on set $A \subset \mathbb{R}$ if for each $\epsilon > 0$ there is a natural number $K(\epsilon)$ such that if $n \geq K(\epsilon)$ then*

$$|f(x) - \varphi_n(x)| < \epsilon \quad \textit{for all } x \in A.$$

Note in particular that $K(\epsilon)$ is independent of the points x in A.

It turns out that Cauchy's Theorem I is valid if we insist on *uniform* convergence of his series functions $u_n(x)$ on a set A. Part of the success of Abel's Theorem IV proof is that it actually shows uniform convergence on the set $[0, d]$, even if Abel didn't explicitly say so in 1826. Indeed, the terminology for uniform convergence did not exist at the time.

Exercise 71 *Carefully review your modern proof of the power series convergence on $[0, d]$ for Abel's Theorem IV, and explain why the convergence is uniform.*

Exercise 72 *Prove that the geometric series converges uniformly to $\frac{1}{1-x}$ on the set $A = \left[-\frac{1}{3}, \frac{1}{3}\right]$.*

Hint: Use Cauchy's expression (6) for the finite geometric series.

Often a series will converge "pointwise" at each x in a set, but will fail to converge uniformly.

Exercise 73 *In this exercise you will show that the geometric series does* ***not*** *converge uniformly to $\frac{1}{1-x}$ on $(-1, 1)$.*

(a) Carefully write out the negation of the uniform convergence definition for the geometric series on $(-1, 1)$.

(b) Prove that the convergence of the geometric series to $\frac{1}{1-x}$ on $(-1, 1)$ is not uniform.

Hint: Consider the sequence $\left((1/2)^{1/n}\right)$, which converges to 1.

As you might guess, the trigonometric series (23) does not converge uniformly to $x/2$ on set $(-\pi, \pi)$. Moreover, it turns out that conditions for continuity and convergence of power series are quite a bit different than conditions for continuity and convergence of a series of trigonometric functions. These challenges would keep mathematicians busy in the decades after Cauchy's *Cours d'analyse* and Abel's 1826 paper.

Regarding the binomial series, Abel went on in his 1826 paper to prove a number of rigorous convergence results for complex numbers x, which is outside the scope of this project.

Sadly, Abel would only have a few more years to work on mathematics, for he contracted tuberculosis while on his Paris visit, and died in 1829 at the age of 27. Nevertheless, he did an amazing amount of first class mathematics in his short lifetime and has been much celebrated for it.

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5 Instructor Notes

This project is designed for a course in Real Analysis. The project starts with Abel's concerns about infinite series in general and the amazing Fourier series $x/2 = \sum (-1)^k \sin(kx)/k$ to give the students a quick hook into the topic of infinite series and immerse them into the problems of the day. Then the project moves through the first section on infinite series in Cauchy's *Cours d'Analyse*, which reads a lot like most current analysis text introductions to infinite series. However, he doesn't quite phrase things in terms of $\epsilon - N$ language, so there are plenty of challenging details for students to iron out. Cauchy finishes the section with a famous "near miss" Theorem I on the continuity of an infinite series of continuous functions. We return to Abel, who has gives a counterexample to this theorem.

In the next section of the project, the project works through several of Abel's results, including a slightly generalized ratio test, and then a clever rearrangement theorem which is often referred to as Abel's Lemma in modern books. This theorem is used, via some exercises, to prove the Alternating Series Test from Introductory Calculus. Abel's results are then used to prove his Theorem IV on power series convergence, now often referred to as one of Abel's theorems on infinite series, which is a partial patch to Cauchy's problematic Theorem I.

From this point, it would be natural for a analysis class to move into a discussion of power series and intervals of convergence, and the various convergence tests not covered in this project, such as the root test. The project also motivates the need to pin down the concept of *uniform* convergence - the fundamental source of problems with Cauchy's Theorem I. This topic is discussed briefly in the project conclusion.

Project Content Goals

1. Develop a modern convergence definition with quantifiers for infinite series based on Cauchy's definition.

2. Analyze convergence for geometric and harmonic series using Cauchy's arguments.

3. Develop and apply modern versions of the "Divergence Test", the "Comparison Test" and a Cauchy condition for convergence based on Cauchy's work.

4. Develop and apply modern versions of the "Ratio Test" and the "Alternating Series Test" based on Abel's work.

5. Develop modern proofs of Abel's theorem on convergence and continuity of power series based on Abel's proofs.

Preparation of Students

Students have done a rigorous study of sequences and limits and continuity for real-valued functions. In particular, students should know the Cauchy criterion for sequences. This equivalence between Cauchy and convergent sequences in $\mathbb{R}$ is treated as obvious by Abel and Cauchy in this project's excerpts from their writing, and the project doesn't dwell too much on this.

Preparation for the Instructor

This is roughly a three or four week project under the following methodology (basically David Pengelley's "A, B, C" method described on his website):

1. Students do some advanced reading and light preparatory exercises before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the exercises or your grading load will get out of hand! Some instructor have students write questions or summaries based on the reading.

2. Class time is largely dedicated to students working in groups on the project - reading the material and working exercises. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a "participation" grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the exercises they missed. This is usually a good incentive not to miss class!

3. Some exercises are assigned for students to do and write up outside of class. Careful grading of these exercises is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project.

Section 1 Introduction

The discussion and examples here are largely to provoke thought in the students and motivate the need for a systematic approach to series. Euler's series (1), stimulated by the vibrating string controversy, is a lovely gateway into Fourier series. A complete investigation of this topic deserves plenty of time and is not the focus of this project. This series exposes a problem with Cauchy's Theorem I that we meet at the end of Section 2.

Section 2 Cauchy

Exercise 8 is important for giving students some historical perspective. There was no universal terminology or notation for the magnitude/absolute value of a real number in the 1820's, and both Abel and Cauchy need to be read with this in mind. There are a number of exercises sprinkled throughout the section establishing basic series properties that are not necessary for the flow of Cauchy's arguments, but which are useful elsewhere and provide good homework practice.

Section 3 Abel

Abel does not need Theorems I and II to prove his Theorems III, IV, so their proofs can be treated lightly if desired. On the other hand, as suggested in Exercise 37, they provide a "backdoor" approach to the important Ratio Test. While Cauchy lists this theorem in Section 6.2 of his *Cours d'Analyse*, he does not give its proof there.

As noted in the discussion before inequality (22), the quantity p does not depend on n or α, and in some sense this inequality captures the uniformity of convergence. Indeed, elements of Abel's proof of Theorem IV are very similar to the modern proof that a sequence of continuous functions, converging uniformly to a function f, result in a continuous limit function f. Abel tried to stretch his proof to the case where the v_m are continuous functions in a following Theorem V, but his proof fails due to the uniform convergence issue and lack of conditions on the v_m functions.

The rest of Abel's proof for the Binomial Theorem involves copious algebraic manipulation of complex numbers, and is not addressed in this project.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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