1. Express each of the following statements as a conditional statement in "if-then" form or as a universally quantified statement.

   a) Every odd number is prime.

   b) The sum of the angles of a triangle is 180 degrees.

   c) Passing the test requires solving all of the problems.

   d) Lockers must be turned in by the last day of class.

   e) Haste makes waste.

   **Solution 1.** *a) If $a$ is odd, then $a$ is prime.*

   *b) If $T$ is a triangle, then the sum of the angles of $T$ is 180 degrees.*

   *c) If you pass the test, then you solved all the problems.*

   *d) If it is the last day of class, then lockers must be turned in.*

2. Prove that if $0 < a < b$, then $a^2 < ab < b^2$ and $0 < \sqrt{a} < \sqrt{b}$.

   *Proof.* Assume $0 < a < b$. Multiplying by $a$ yields $0 < a^2 < ab$. Again, multiplying the original inequality by $b$ yields $0 < ab < b^2$. Combining these two inequalities, we have $0 < a^2 < ab < b^2$, the desired result.

Now we wish to show that if $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$. We have

$$\begin{aligned}
0 &< a < b \\
a - b &< 0 \\
(\sqrt{a})^2 - (\sqrt{b})^2 &< 0 \\
(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) &< 0 \\
\sqrt{a} - \sqrt{b} &< 0 \\
\sqrt{a} &< \sqrt{b}.
\end{aligned}$$

□

3. Let $P(x)$ be the assertion "$x$ is odd", and let $Q(x)$ be the assertion "$x$ is twice an integer." Determine whether the following statements are true:

a) $(\forall x \in \mathbb{Z})[P(x) \implies Q(x)]$.

b) $(\forall x \in \mathbb{Z})[Q(x) \implies P(x)]$.

**Solution 2.** *a) We claim that $(\forall x \in \mathbb{Z})[P(x) \implies Q(x)]$ is false. Consider $x = 3$. Then $3 = 2(1) + 1$ so 3 is odd. If $3 = 2m$ for some integer $m$, then $m = \frac{3}{2}$ whence $m$ is not an integer. Thus 3 is not $2m$ for any integer $m$, and the assertion is false.*

*b) This is similar to a).*

4. Show that the following statement is false: "If $a$ and $b$ are integers, then there are integers $m, n$ such that $a = m + n$ and $b = m - n$." What can be added to the hypothesis of the statement to make it true?

**Solution 3.** *Let $a = 0$ and $b = 1$. If $a = m + n$ and $b = m - n$, then $0 = m + n$ and $1 = m - n$. Adding the two equations together yields $1 = 2m$ for some integer $m$, which is impossible. Thus there do not exist $m, n \in \mathbb{Z}$ for $a = 0, b = 1$.*

*The new hypothesis reads "If $a, b$ are integers such that $a - b$ is even, then there are integers $m, n$ such that $a = m + n$ and $b = m - n$.*

5. The statement below is not always true for $x, y \in \mathbb{R}$. Give an example where it is false, and add a hypothesis on $y$ that makes it a true statement.
   "If $x$ and $y$ are nonzero real numbers and $x > y$, then $(-1/x) > (-1/y)$."

   **Solution 4.** *Consider $x = 1$ and $y = -1$. Then $x > y$, but $(-1/1) \not> (-1/-1) = 1$.*

   *The new hypothesis reads: "If $x$ and $y$ are nonzero real numbers and $x > y$ such that $xy > 0$, then $(-1/x) > (-1/y)$."*

6. Prove that if $x$ and $y$ are distinct real numbers, then $(x+1)^2 = (y+1)^2$ if and only if $x + y = -2$. How does the conclusion change if we allow $x = y$?

   *Proof.* ($\Rightarrow$) Assume $(x+1)^2 = (y+1)^2$. We wish to show that $x + y = -2$. We have

$$\begin{aligned} (x+1)^2 &= (y+1)^2 \\ x^2 + 2x + 1 &= y^2 + 2y + 1 \\ x^2 - y^2 &= -2(x-y) \\ (x-y)(x+y) &= -2(x-y) \\ x + y &= -2. \end{aligned}$$

   Observe that we are justified in dividing by $x - y$ since $x \neq y$ by hypothesis. This proves the forward direction.

   ($\Leftarrow$) Now assume that $x + y = -2$. We show that $(x+1)^2 = (y+1)^2$. We have

$$\begin{aligned} x + y &= -2 \\ (x-y)(x+y) &= -2(x-y) \\ x^2 - y^2 &= -2(x-y) \\ x^2 + 2x + 1 &= y^2 + 2y + 1 \\ (x+1)^2 &= (y+1)^2 \end{aligned}$$

which proves the reverse direction.

If we allow $x = y$, then if $x + y = -2$, then $(x+1)^2 = (y+1)^2$, but the other direction does not follow, for if $x = y = 3$, then $(3+1)^2 = (3+1)^2$ but $3 + 3 \neq -2$. $\square$

7. Two opposite squares corner squares are deleted from an eight by eight checkerboard. Prove that the remaining squares cannot be covered exactly by dominoes (rectangles consisting of two adjacent squares). [Hint: Use proof by contradiction.]

   *Proof.* Assume, by way of contradiction, that such a covering is possible. Then we have the modified chess board with 62 squares covered using 31 dominoes. Observe that a domino covers precisely one black and one white square so that there are 31 black squares and 31 white squares covered on the modified chess board. However, the two squares that were removed must be the same color (say black) leaving 32 white squares and 30 black squares. But our covering of 62 squares necessarily covers 31 black squares and 31 white squares, a contradiction. Thus, no such covering is possible. $\square$

8. Prove that the statements $P \implies Q$ and $Q \implies R$ imply $P \implies R$. This property of implication is called *transitivity*.

   *Proof.* The following truth table proves the assertion:

| $P$ | $Q$ | $R$ | $P \Rightarrow Q$ | $Q \Rightarrow R$ | $P \Rightarrow R$ | $P \Rightarrow Q \wedge Q \Rightarrow R$ | $[P \Rightarrow Q \wedge Q \Rightarrow R] \Rightarrow [P \Rightarrow R]$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $T$ | $F$ | $T$ |

$\square$