Example 11 Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices.

(a) The statement $\forall \mathbf{A}\, \exists \mathbf{B}\ \mathbf{A} + \mathbf{B} = \mathbf{I}_n$ is read "for every $\mathbf{A}$ there is a $\mathbf{B}$ such that $\mathbf{A} + \mathbf{B} = \mathbf{I}_n$." For a given $\mathbf{A} = [\, a_{ij} \,]$, define $\mathbf{B} = [\, b_{ij} \,]$ as follows: $b_{ii} = 1 - a_{ii}$, $1 \le i \le n$ and $b_{ij} = -a_{ij}$, $i \ne j$, $1 \le i \le n$, $1 \le j \le n$. Then $\mathbf{A} + \mathbf{B} = \mathbf{I}_n$ and we have shown that $\forall \mathbf{A}\, \exists \mathbf{B}\ \mathbf{A} + \mathbf{B} = \mathbf{I}_n$ is a true statement.

(b) $\exists \mathbf{B}\, \forall \mathbf{A}\ \mathbf{A} + \mathbf{B} = \mathbf{I}_n$ is the statement "there is a $\mathbf{B}$ such that for all $\mathbf{A}$, $\mathbf{A} + \mathbf{B} = \mathbf{I}_n$." This statement is false; no single $\mathbf{B}$ has this property for all $\mathbf{A}$'s.

(c) $\exists \mathbf{B}\, \forall \mathbf{A}\ \mathbf{A} + \mathbf{B} = \mathbf{A}$ is true. What is the value for $\mathbf{B}$ that makes the statement true? ◆

Let p: $\forall x\ \mathrm{P}(x)$. The negation of p is false when p is true, and true when p is false. For p to be false there must be at least one value of x for which $\mathrm{P}(x)$ is false. Thus, p is false if $\exists x \sim\mathrm{P}(x)$ is true. On the other hand, if $\exists x \sim\mathrm{P}(x)$ is false, then for every x, $\sim\mathrm{P}(x)$ is false; that is, $\forall x\ \mathrm{P}(x)$ is true.

Example 12 (a) Let p: For all positive integers n, $n^2 + 41n + 41$ is a prime number. Then $\sim p$ is There is at least one positive integer n for which $n^2 + 41n + 41$ is not prime.

(b) Let q: There is some integer k for which $12 = 3k$. Then $\sim q$: For all integers k, $12 \ne 3k$. ◆

Example 13 Let p: The empty set is a subset of any set A. For p to be false, there must be an element of $\varnothing$ that is not in A, but this is impossible. Thus, p is true. ◆

2.1 Exercises

1. Which of the following are statements?
   (a) Is 2 a positive number?
   (b) $x^2 + x + 1 = 0$
   (c) Study logic.
   (d) There will be snow in January.
   (e) If stock prices fall, then I will lose money.
2. Give the negation of each of the following statements.
   (a) $2 + 7 \le 11$
   (b) 2 is an even integer and 8 is an odd integer.
3. Give the negation of each of the following statements.
   (a) It will rain tomorrow or it will snow tomorrow.
   (b) If you drive, then I will walk.
4. In each of the following, form the conjunction and the disjunction of p and q.
   (a) p: $3 + 1 < 5$ $\quad q$: $7 = 3 \times 6$
   (b) p: I am rich. $\quad q$: I am happy.
5. In each of the following, form the conjunction and the disjunction of p and q.
   (a) p: I will drive my car. $\quad q$: I will be late.
   (b) p: NUM > 10 $\quad q$: NUM $\le$ 15
6. Determine the truth or falsity of each of the following statements.
   (a) $2 < 3$ and 3 is a positive integer.
   (b) $2 \ge 3$ and 3 is a positive integer.
   (c) $2 < 3$ and 3 is not a positive integer.
   (d) $2 \ge 3$ and 3 is not a positive integer.
7. Determine the truth or falsity of each of the following statements.
   (a) $2 < 3$ or 3 is a positive integer.
   (b) $2 \ge 3$ or 3 is a positive integer.
   (c) $2 < 3$ or 3 is not a positive integer.
   (d) $2 \ge 3$ or 3 is not a positive integer.

In Exercises 8 and 9, find the truth value of each proposition if p and r are true and q is false.

8. (a) $\sim p \wedge \sim q$ $\quad$ (b) $(\sim p \vee q) \wedge r$
   (c) $p \vee q \vee r$ $\quad$ (d) $\sim(p \vee q) \wedge r$
9. (a) $\sim p \wedge (q \vee r)$ $\quad$ (b) $p \wedge (\sim(q \vee \sim r))$
   (c) $(r \wedge \sim q) \vee (p \vee r)$ $\quad$ (d) $(q \wedge r) \wedge (p \vee \sim r)$
10. Which of the following statements is the negation of the statement "2 is even and -3 is negative"?
    (a) 2 is even and -3 is not negative.
    (b) 2 is odd and -3 is not negative.
    (c) 2 is even or -3 is not negative.
    (d) 2 is odd or -3 is not negative.

11. Which of the following statements is the negation of the statement "2 is even or -3 is negative"?

(a) 2 is even or -3 is not negative.

(b) 2 is odd or -3 is not negative.

(c) 2 is even and -3 is not negative.

(d) 2 is odd and -3 is not negative.

In Exercises 12 and 13, use p: *Today is Monday;* q: *The grass is wet; and* r: *The dish ran away with the spoon.*

12. Write each of the following in terms of p, q, r, and logical connectives.

(a) Today is Monday and the dish did not run away with the spoon.

(b) Either the grass is wet or today is Monday.

(c) Today is not Monday and the grass is dry.

(d) The dish ran away with the spoon, but the grass is wet.

13. Write an English sentence that corresponds to each of the following.

(a) $\sim r \wedge q$ **(b)** $\sim q \vee r$

(c) $\sim(p \vee q)$ **(d)** $p \vee \sim r$

In Exercises 14 through 19, use $\mathrm{P}(x)$: *x is even;* $\mathrm{Q}(x)$: *x is a prime number;* $\mathrm{R}(x, y)$: *$x + y$ is even. The variables x and y represent integers.*

14. Write an English sentence corresponding to each of the following.

(a) $\forall x\ \mathrm{P}(x)$ **(b)** $\exists x\ \mathrm{Q}(x)$

15. Write an English sentence corresponding to each of the following.

(a) $\forall x\, \exists y\ \mathrm{R}(x, y)$ **(b)** $\exists x\, \forall y\ \mathrm{R}(x, y)$

16. Write an English sentence corresponding to each of the following.

(a) $\forall x\ (\sim\mathrm{Q}(x))$ **(b)** $\exists y\ (\sim\mathrm{P}(y))$

17. Write an English sentence corresponding to each of the following.

(a) $\sim(\exists x\ \mathrm{P}(x))$ **(b)** $\sim(\forall x\ \mathrm{Q}(x))$

18. Write each of the following in terms of $\mathrm{P}(x)$, $\mathrm{Q}(x)$, $\mathrm{R}(x, y)$, logical connectives, and quantifiers.

(a) Every integer is an odd integer.

(b) The sum of any two integers is an even number.

(c) There are no even prime numbers.

(d) Every integer is even or a prime.

19. Determine the truth value of each statement given in Exercises 14 through 18.

20. If $\mathrm{P}(x)$: $x^2 < 12$, then

(a) P(4) is the statement _____.

(b) P(1.5) is the statement _____.

21. If $\mathrm{Q}(n)$: $n + 3 = 6$, then

(a) Q(5) is the statement _____.

(b) $\mathrm{Q}(m)$ is the statement _____.

22. If $\mathrm{P}(y)$: $1 + 2 + \cdots + y = 0$, then

(a) P(1) is the statement _____.

(b) P(5) is the statement _____.

(c) $\mathrm{P}(k)$ is the statement _____.

23. If $\mathrm{Q}(m)$: $m \le 3^m$, then

(a) Q(0) is the statement _____.

(b) Q(2) is the statement _____.

(c) $\mathrm{Q}(k)$ is the statement _____.

24. Give a symbolic statement of the commutative property for addition of real numbers using appropriate quantifiers.

25. Give a symbolic statement of De Morgan's laws for sets using appropriate quantifiers.

26. Give a symbolic statement of the multiplicative inverse property for real numbers using appropriate quantifiers.

In Exercises 27 through 30, make a truth table for the statement.

27. $(\sim p \wedge q) \vee p$ **28.** $(p \vee q) \vee \sim q$

29. $(p \vee q) \wedge r$ **30.** $(\sim p \vee q) \wedge \sim r$

For Exercises 31 through 33, define $p \downarrow q$ to be a true statement if neither p nor q is true.

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

31. Make a truth table for $(p \downarrow q) \downarrow r$.

32. Make a truth table for $(p \downarrow q) \wedge (p \downarrow r)$.

33. Make a truth table for $(p \downarrow q) \downarrow (p \downarrow r)$.

For Exercises 34 through 36, define $p \vartriangle q$ to be true if either p or q, but not both, is true. Make a truth table for the statement.

34. **(a)** $p \vartriangle q$ **(b)** $p \vartriangle \sim p$

35. $(p \wedge q) \vartriangle p$

36. $(p \vartriangle q) \vartriangle (q \vartriangle r)$

In Exercises 37 through 40, revision of the given programming block is needed. Replace the guard $\mathrm{P}(x)$ with $\sim\mathrm{P}(x)$.

37. IF ($x \neq max$ and $y > 4$) THEN take action

38. WHILE (key = "open" or $t <$ limit) take action

39. WHILE (item $\neq$ sought and index < 101) take action

40. IF (cell > 0 or found) THEN take action

Proof

(a) was proved in Example 7 and (b) was proved in Example 4. Notice that (b) says a conditional statement is equivalent to its contrapositive.

(d) gives an alternate version for the negation of a conditional statement. This could be proved using truth tables, but it can also be proved by using previously proven facts. Since $(p \Rightarrow q) \equiv ((\sim p) \vee q)$, the negation of $p \Rightarrow q$ must be equivalent to $\sim((\sim p) \vee q)$. By De Morgan's laws, $\sim((\sim p) \vee q) \equiv (\sim(\sim p)) \wedge (\sim q)$ or $p \wedge (\sim q)$. Thus, $\sim(p \Rightarrow q) \equiv (p \wedge \sim q)$.

The remaining parts of Theorem 2 are left as exercises. ■

Theorem 3 states two results from Section 2.1, and several other properties for the universal and existential quantifiers.

THEOREM 3
(a) $\sim(\forall x\ \mathrm{P}(x)) \equiv \exists x \sim\mathrm{P}(x)$
(b) $\sim(\exists x\ \mathrm{P}(x)) \equiv \forall x\ (\sim\mathrm{P}(x))$
(c) $\exists x\ (\mathrm{P}(x) \Rightarrow \mathrm{Q}(x)) \equiv \forall x\ \mathrm{P}(x) \Rightarrow \exists x\ \mathrm{Q}(x)$
(d) $\exists x\ (\mathrm{P}(x) \vee \mathrm{Q}(x)) \equiv \exists x\ \mathrm{P}(x) \vee \exists x\ \mathrm{Q}(x)$
(e) $\forall x\ (\mathrm{P}(x) \wedge \mathrm{Q}(x)) \equiv \forall x\ \mathrm{P}(x) \wedge \forall x\ \mathrm{Q}(x)$
(f) $((\forall x\ \mathrm{P}(x)) \vee (\forall x\ \mathrm{Q}(x))) \Rightarrow \forall x\ (\mathrm{P}(x) \vee \mathrm{Q}(x))$ is a tautology.
(g) $\exists x\ (\mathrm{P}(x) \wedge \mathrm{Q}(x)) \Rightarrow \exists x\ \mathrm{P}(x) \wedge \exists x\ \mathrm{Q}(x)$ is a tautology. ■

The following theorem gives several important tautologies that are implications. These are used extensively in proving results in mathematics and computer science and we will illustrate them in Section 2.3.

THEOREM 4 Each of the following is a tautology.

(a) $(p \wedge q) \Rightarrow p$
(b) $(p \wedge q) \Rightarrow q$
(c) $p \Rightarrow (p \vee q)$
(d) $q \Rightarrow (p \vee q)$
(e) $\sim p \Rightarrow (p \Rightarrow q)$
(f) $\sim(p \Rightarrow q) \Rightarrow p$
(g) $(p \wedge (p \Rightarrow q)) \Rightarrow q$
(h) $(\sim p \wedge (p \vee q)) \Rightarrow q$
(i) $(\sim q \wedge (p \Rightarrow q)) \Rightarrow \sim p$
(j) $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$ ■

2.2 Exercises

In Exercises 1 and 2, use the following: p: I am awake; q: I work hard; r: I dream of home.

1. Write each of the following statements in terms of p, q, r, and logical connectives.
   (a) I am awake implies that I work hard.
   (b) I dream of home only if I am awake.
   (c) Working hard is sufficient for me to be awake.
   (d) Being awake is necessary for me not to dream of home.
2. Write each of the following statements in terms of p, q, r, and logical connectives.
   (a) I am not awake if and only if I dream of home.
   (b) If I dream of home, then I am awake and I work hard.
   (c) I do not work hard only if I am awake and I do not dream of home.
   (d) Not being awake and dreaming of home is sufficient for me to work hard.
3. State the converse of each of the following implications.
   (a) If $2 + 2 = 4$, then I am not the Queen of England.
   (b) If I am not President of the United Sates, then I will walk to work.
   (c) If I am late, then I did not take the train to work.
   (d) If I have time and I am not too tired, then I will go to the store.
   (e) If I have enough money, then I will buy a car and I will buy a house.
4. State the contrapositive of each implication in Exercise 3.

5. Determine the truth value for each of the following statements.
   (a) If 2 is even, then New York has a large population.
   (b) If 2 is even, then New York has a small population.
   (c) If 2 is odd, then New York has a large population.
   (d) If 2 is odd, then New York has a small population.

In Exercises 6 and 7, let p, q, and r be the following statements: p: I will study discrete structure; q: I will go to a movie; r: I am in a good mood.

6. Write the following statements in terms of p, q, r, and logical connectives.
   (a) If I am not in a good mood, then I will go to a movie.
   (b) I will not go to a movie and I will study discrete structures.
   (c) I will go to a movie only if I will not study discrete structures.
   (d) If I will not study discrete structures, then I am not in a good mood.
7. Write English sentences corresponding to the following statements.
   (a) $((\sim p) \wedge q) \Rightarrow r$ (b) $r \Rightarrow (p \vee q)$
   (c) $(\sim r) \Rightarrow ((\sim q \vee p)$ (d) $(q \wedge (\sim p)) \Leftrightarrow r$

In Exercises 8 and 9, let p, q, r, and s be the following statements: p: $4 > 1$; q: $4 < 5$; r: $3 \leq 3$; s: $2 > 2$.

8. Write the following statements in terms of p, q, r, and logical connectives.
   (a) Either $4 > 1$ or $4 < 5$.
   (b) If $3 \leq 3$, then $2 > 2$.
   (c) It is not the case that $2 > 2$ or $4 > 1$.
9. Write English sentences corresponding to the following statements.
   (a) $(p \wedge s) \Rightarrow q$ (b) $\sim(r \wedge q)$ (c) $(\sim r) \Rightarrow p$

In Exercises 10 through 12, construct truth tables to determine whether the given statement is a tautology, a contingency, or an absurdity.

10. (a) $p \wedge \sim p$ (b) $q \vee (\sim q \wedge p)$
11. (a) $p \Rightarrow (q \Rightarrow p)$ (b) $q \Rightarrow (q \Rightarrow p)$
12. (a) $(q \wedge p) \vee (q \wedge \sim p)$
    (b) $(p \wedge q) \Rightarrow p$ (c) $p \Rightarrow (q \wedge p)$
13. If $p \Rightarrow q$ is false, can you determine the truth value of $(\sim(p \wedge q)) \Rightarrow q$? Explain your answer.
14. If $p \Rightarrow q$ is false, can you determine the truth value of $(\sim p) \vee (p \Leftrightarrow q)$? Explain your answer.
15. If $p \Rightarrow q$ is true, can you determine the truth value of $(p \wedge q) \Rightarrow \sim q$? Explain your answer.
16. If $p \Rightarrow q$ is true, can you determine the truth value of $\sim(p \Rightarrow q) \wedge \sim p$? Explain your answer.
17. Fill the grid so that each row, column, and marked 2×2 square contains the letters M, A, T, H, with no repeats.

(a)

A			
	M		
T			M
		H	

(b)

T			
		A	
	M		
			H

18. Fill the grid so that each row, column, and marked 2×3 block contains 1, 2, 3, 4, 5, 6, with no repeats.

(a)

4			3		
	2	3			
			1	6	5
6	1	5			
			6	5	
		4			1

(b)

1		6		2	3
	2		1		
5		1		6	
			3		
4		5			2
	6			4	

19. Fill the grid so that each row, column, and marked 3×3 block contains 1, 2, 3, 4, 5, 6, 7, 8, 9, with no repeats.

	6		9		3	1		
1			2	8	7			6
	3	4	6		5		8	
6	2	8	1		4	9		5
				3			7	
5					9	4		1
		6						3
	1		7			5		
4		9	3			2	1	

In summary, if a statement claims that a property holds for all objects of a certain type, then to prove it, we must use steps that are valid for all objects of that type and that do not make references to any particular object. To disprove such a statement, we need only show one counterexample, that is, one particular object or set of objects for which the claim fails.

2.3 Exercises

In Exercises 1 through 11, state whether the argument given is valid or not. If it is valid, identify the tautology or tautologies on which it is based.

1. If I drive to work, then I will arrive tired.
I am not tired when I arrive at work.
∴ I do not drive to work.

2. If I drive to work, then I will arrive tired.
I arrive at work tired.
∴ I drive to work.

3. If I drive to work, then I will arrive tired.
I do not drive to work.
∴ I will not arrive tired.

4. If I drive to work, then I will arrive tired.
I drive to work.
∴ I will arrive tired.

5. I will become famous or I will not become a writer.
I will become a writer.
∴ I will become famous.

6. I will become famous or I will be a writer.
I will not be a writer.
∴ I will become famous.

7. If I try hard and I have talent, then I will become a musician.
If I become a musician, then I will be happy.
∴ If I will not be happy, then I did not try hard or I do not have talent.

8. If I graduate this semester, then I will have passed the physics course.
If I do not study physics for 10 hours a week, then I will not pass physics.
If I study physics for 10 hours a week, then I cannot play volleyball.
∴ If I play volleyball, I will not graduate this semester.

9. If my plumbing plans do not meet the construction code, then I cannot build my house.
If I hire a licensed contractor, then my plumbing plans will meet the construction code.
I hire a licensed contractor.
∴ I can build my house.

10. **(a)**
$$\begin{array}{rl} & p \vee q \\ & \underline{\sim q} \\ \therefore & p \end{array}$$
(b)
$$\begin{array}{rl} & p \Rightarrow q \\ & \underline{\sim p} \\ \therefore & \sim q \end{array}$$

11. Write each argument in Exercise 10 as a single compound statement.

12. **(a)**
$$\begin{array}{rl} & (p \Rightarrow q) \wedge (q \Rightarrow r) \\ & \underline{(\sim q) \wedge r} \\ \therefore & p \end{array}$$
(b)
$$\begin{array}{rl} & \sim(p \Rightarrow q) \\ & \underline{p} \\ \therefore & \sim q \end{array}$$

13. Write each argument in Exercise 12 as a single compound statement.

14. Prove that the sum of two even numbers is even.

15. Prove that the sum of two odd numbers is even.

16. Prove that the structure (even integers, +, *) is closed with respect to *.

17. Prove that the structure (odd integers, +, *) is closed with respect to *.

18. Prove that n^2 is even if and only if n is even.

19. Prove that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

20. Let A and B be subsets of a universal set U. Prove that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

21. Show that

(a) $A \subseteq B$ is a necessary and sufficient condition for $A \cup B = B$.

(b) $A \subseteq B$ is a necessary and sufficient condition for $A \cap B = A$.

22. Show that k is odd is a necessary and sufficient condition for k^3 to be odd.

23. Prove or disprove: $n^2 + 41n + 41$ is a prime number for every integer n.

24. Prove or disprove: The sum of any five consecutive integers is divisible by 5.

25. Prove or disprove that $3 \mid (n^3 - n)$ for every positive integer n.

26. Prove or disprove: $1 + 2^n > 3^n$, for all $n \in Z^+$.

27. Determine if the following is a valid argument. Explain your conclusion.

Prove: $\forall x\ x^3 > x^2$.

Proof: $\forall x\ x^2 > 0$ so $\forall x\ x^2(x-1) > 0(x-1)$ and $\forall x\ x^3 - x^2 > 0$. Hence $\forall x\ x^3 > x^2$.

28. Determine if the following is a valid argument. Explain your conclusion.

Prove: If **A** and **B** are matrices such that $\mathbf{AB} = \mathbf{0}$, then either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

Proof: There are two cases to consider: $\mathbf{A} = \mathbf{0}$ or $\mathbf{A} \neq \mathbf{0}$. If $\mathbf{A} = \mathbf{0}$, then we are done. If $\mathbf{A} \neq \mathbf{0}$, then $\mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}\mathbf{0}$ and $(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$.

29. Determine if the following is a valid argument. Explain your conclusion.

Let m and n be two relatively prime integers. Prove that if mn is a cube, then m and n are each cubes.

Proof: We first note that in the factorization of any cube into prime factors, each prime must have an exponent that is a multiple of 3. Write m and n each as a product of primes; $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $n = q_1^{b_1} q_2^{b_2} \cdots q_j^{b_j}$. Suppose m is not a cube. Then at least one a_i is not a multiple of 3. Since each prime factor of mn must have an exponent that is a multiple of 3, n must have a factor $p_i^{b_i}$ such that $b_i \neq 0$ and $a_i + b_i$ is a multiple of 3. But this means that m and n share a factor, p_i. This contradicts the fact that m and n are relatively prime.

30. Determine if the following is a valid argument. Explain your conclusion.

Prove: If x is an irrational number, then $1 - x$ is also an irrational number.

Proof: Suppose $1 - x$ is rational. Then we can write $1 - x$ as $\frac{a}{b}$, with $a, b \in Z$. Now we have $1 - \frac{a}{b} = x$ and $x = \frac{b-a}{b}$, a rational number. This is a contradiction. Hence, if x is irrational, so is $1 - x$.

31. Prove that the sum of two prime numbers, each larger than 2, is not a prime number.

32. Prove that if two lines are each perpendicular to a third line in the plane, then the two lines are parallel.

33. Prove that if x is a rational number and y is an irrational number, then $x + y$ is an irrational number.

34. Prove that if $2y$ is an irrational number, then y is an irrational number.

2.4 Mathematical Induction

Here we discuss another proof technique. Suppose the statement to be proved can be put in the form $\forall n \geq n_0\ P(n)$, where n_0 is some fixed integer. That is, suppose we wish to show that $P(n)$ is true for all integers $n \geq n_0$. The following result shows how this can be done. Suppose that (a) $P(n_0)$ is true and (b) If $P(k)$ is true for some $k \geq n_0$, then $P(k+1)$ must also be true. Then $P(n)$ is true for all $n \geq n_0$. This result is called the **principle of mathematical induction**. Thus to prove the truth of a statement $\forall n \geq n_0\ P(n)$, using the principle of mathematical induction, we must begin by proving directly that the first proposition $P(n_0)$ is true. This is called the **basis step** of the induction and is generally very easy.

Then we must prove that $P(k) \Rightarrow P(k+1)$ is a tautology for any choice of $k \geq n_0$. Since the only case where an implication is false is if the antecedent is true and the consequent is false, this step is usually done by showing that if $P(k)$ were true, then $P(k+1)$ would also have to be true. Note that this is not the same as assuming that $P(k)$ is true for some value of k. This step is called the **induction step**, and some work will usually be required to show that the implication is always true.

Example 1 Show, by mathematical induction, that for all $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution

Let $P(n)$ be the predicate $1+2+3+\cdots+n = \frac{n(n+1)}{2}$. In this example, $n_0 = 1$.

Basis Step

We must first show that $P(1)$ is true. $P(1)$ is the statement

$$1 = \frac{1(1+1)}{2},$$

which is clearly true.

2.5 Exercises

1. Prove or disprove that the cube of an even number is even.
2. Prove or disprove that the cube of an odd number is odd.
3. (a) Prove or disprove that the sum of three consecutive integers is divisible by 3.
   (b) Prove or disprove that the sum of four consecutive integers is divisible by 4.
4. Prove that the product of a nonzero rational number and an irrational number is a(n) ______ number.
5. Prove that the quotient of a nonzero rational number and an irrational number is a(n) ______ number.
6. State and prove the extension of $\overline{A \cup B} = \overline{A} \cap \overline{B}$ for three sets.
7. State and prove the extension of $\overline{A \cap B} = \overline{A} \cup \overline{B}$ for three sets.
8. Let **A**, **B**, **C**, and **D** be $n \times n$ Boolean matrices. State and prove the extension of $\mathbf{A} \vee (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{A} \vee \mathbf{B}) \wedge (\mathbf{A} \vee \mathbf{C})$ to four matrices.
9. Modify the conjecture and proof in Example 4 to give the general case for $m + 1$ Boolean matrices.
10. Let p and q be propositions. State and prove an extension of $\sim(p \vee q) \equiv \sim p \wedge \sim q$ to the case of
    (a) 3 propositions (b) n propositions.
11. Let p and q be propositions. State and prove an extension of $\sim(p \wedge q) \equiv \sim p \vee \sim q$ to the case of
    (a) 3 propositions (b) n propositions.
12. Prove that the commutative property holds for the mathematical structure (3×3 diagonal matrices, matrix multiplication).
13. Prove that the commutative property holds for the mathematical structure ($r \times r$ diagonal matrices, matrix multiplication).
14. State and prove a conjecture about the sum of the first n positive odd integers.

For Exercises 15 through 17, use the sequence 3, 9, 15, 21, 27, 33, ...

15. Give both a recursive and an explicit formula for this sequence.
16. Experiment with the sums of the terms of the sequence to produce a reasonable conjecture.
17. Prove the conjecture from Exercise 16.
18. State and prove a reasonable conjecture about the sum of the first n terms of the sequence 6, 10, 14, 18, 22, ...

For Exercises 19 through 22, use the recursively defined sequence $g_1 = 1$, $g_2 = 3$, $g_n = g_{n-1} + g_{n-2}$.

19. Write the first ten terms of the sequence.
20. Experiment with the sums of terms in the sequence that occupy even-numbered positions to produce a reasonable conjecture.
21. Experiment with the sums of terms of the sequence to produce a reasonable conjecture about the sum of the first n terms of the sequence.
22. Experiment with the sums of terms in the sequence that occupy odd-numbered positions to produce a reasonable conjecture.
23. Prove the conjecture from Exercise 20.
24. Prove the conjecture from Exercise 21.
25. Prove the conjecture from Exercise 22.
26. Produce and prove a conjecture about the sum of the powers of 3 beginning with 3^0.
27. Build on the work done in Exercise 3 to produce a conjecture about when the sum of k consecutive integers is divisible by k.
28. Prove the conjecture of Exercise 27.

2.6 Logic and Problem Solving

In previous sections, we investigated the use of logic to prove mathematical theorems and to verify the correctness of computational algorithms. However, logic is also valuable in less formal settings. Logic is used every day to decide between alternatives and investigate consequences. It is, in many situations, essentially the same as precise and careful thinking, particularly in cases where the details of the situation are complex. One of the most important uses of logic is to develop correct and efficient algorithms to solve problems. Part of this process may be to express the solution in terms of a computation, even if the problem does not at first seem computational. In Section 2.5, we noted that one common way to create mathematical conjectures is to apply old ideas in new settings. This approach is also a powerful problem-solving technique. To demonstrate these two ideas, we present a variety of seemingly unrelated problems that can be solved by a single method.

A set A is called **finite** if it has n distinct elements, where $n \in \mathbb{N}$. In this case, n is called the **cardinality** of A and is denoted by $|A|$. Thus, the sets of Examples 1, 2, 4, 5, and 6 are finite. A set that is not finite is called **infinite**. The sets introduced in Example 3 (except $\varnothing$) are infinite sets.

If A is a set, then the set of all subsets of A is called the **power set** of A and is denoted by $P(A)$. (Be sure to distinguish between $\mathrm{P}(A)$, a statement about A, and $P(A)$, the power set of A.)

Example 11 Let $A = \{1, 2, 3\}$. Then $P(A)$ consists of the following subsets of A: $\{\ \}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$ (or A). In a later section, we will count the number of subsets that a set can have. ◆

1.1 Exercises

1. Let $A = \{1, 2, 4, a, b, c\}$. Identify each of the following as true or false.

(a) $2 \in A$ (b) $3 \in A$ (c) $c \notin A$
(d) $\varnothing \in A$ (e) $\{\ \} \notin A$ (f) $A \in A$

2. Let $A = \{x \mid x$ is a real number and $x < 6\}$. Identify each of the following as true or false.

(a) $3 \in A$ (b) $6 \in A$ (c) $5 \notin A$
(d) $8 \notin A$ (e) $-8 \in A$ (f) $3.4 \notin A$

3. In each part, give the set of letters in each word by listing the elements of the set.

(a) AARDVARK (b) BOOK
(c) MISSISSIPPI

4. Give the set by listing its elements.

(a) The set of all positive integers that are less than ten.
(b) $\{x \mid x \in \mathbb{Z}$ and $x^2 < 12\}$

5. Let $A = \{1, \{2, 3\}, 4\}$. Identify each of the following as true or false.

(a) $3 \in A$ (b) $\{1, 4\} \subseteq A$ (c) $\{2, 3\} \subseteq A$
(d) $\{2, 3\} \in A$ (e) $\{4\} \in A$ (f) $\{1, 2, 3\} \subseteq A$

In Exercises 6 through 9, write the set in the form $\{x \mid P(x)\}$, where $P(x)$ is a property that describes the elements of the set.

6. $\{2, 4, 6, 8, 10\}$ **7.** $\{$a, e, i, o, u$\}$
8. $\{1, 8, 27, 64, 125\}$ **9.** $\{-2, -1, 0, 1, 2\}$

10. Let $A = \{1, 2, 3, 4, 5\}$. Which of the following sets are equal to A?

(a) $\{4, 1, 2, 3, 5\}$ (b) $\{2, 3, 4\}$ (c) $\{1, 2, 3, 4, 5, 6\}$
(d) $\{x \mid x$ is an integer and $x^2 \le 25\}$
(e) $\{x \mid x$ is a positive integer and $x \le 5\}$
(f) $\{x \mid x$ is a positive rational number and $x \le 5\}$

11. Which of the following sets are the empty set?

(a) $\{x \mid x$ is a real number and $x^2 - 1 = 0\}$
(b) $\{x \mid x$ is a real number and $x^2 + 1 = 0\}$
(c) $\{x \mid x$ is a real number and $x^2 = -9\}$
(d) $\{x \mid x$ is a real number and $x = 2x + 1\}$
(e) $\{x \mid x$ is a real number and $x = x + 1\}$

12. List all the subsets of $\{a, b\}$.

13. List all the subsets of {JAVA, PASCAL, C++}.

14. List all the subsets of $\{\ \}$.

15. Let $A = \{1, 2, 5, 8, 11\}$. Identify each of the following as true or false.

(a) $\{5, 1\} \subseteq A$ (b) $\{8, 1\} \in A$
(c) $\{1, 8, 2, 11, 5\} \not\subseteq A$ (d) $\varnothing \subseteq A$
(e) $\{1, 6\} \not\subseteq A$ (f) $\{2\} \subseteq A$
(g) $\{3\} \notin A$ (h) $A \subseteq \{11, 2, 5, 1, 8, 4\}$

16. Let $A = \{x \mid x$ is an integer and $x^2 < 16\}$. Identify each of the following as true or false.

(a) $\{0, 1, 2, 3\} \subseteq A$ (b) $\{-3, -2, -1\} \subseteq A$
(c) $\{\ \} \subseteq A$
(d) $\{x \mid x$ is an integer and $|x| < 4\} \subseteq A$
(e) $A \subseteq \{-3, -2, -1, 0, 1, 2, 3\}$

17. Let $A = \{1\}$, $B = \{1, a, 2, b, c\}$, $C = \{b, c\}$, $D = \{a, b\}$, and $E = \{1, a, 2, b, c, d\}$. For each part, replace the symbol $\square$ with either $\subseteq$ or $\not\subseteq$ to give a true statement.

(a) $A \square B$ (b) $\varnothing \square A$ (c) $B \square C$
(d) $C \square E$ (e) $D \square C$ (f) $B \square E$

In Exercises 18 through 20, find the set of smallest cardinality that contains the given sets as subsets.

18. $\{a, b, c\}$, $\{a, d, e, f\}$, $\{b, c, e, g\}$

19. $\{1, 2\}$, $\{1, 3\}$, $\varnothing$

20. $\{2, 4, 6, \ldots, 20\}$, $\{3, 6, 9, \ldots, 21\}$

21. Is it possible to have two different (appropriate) universal sets for a collection of sets? Would having different universal sets create any problems? Explain.

22. Use the Venn diagram in Figure 1.3 to identify each of the following as true or false.

(a) $A \subseteq B$ (b) $B \subseteq A$ (c) $C \subseteq B$
(d) $x \in B$ (e) $x \in A$ (f) $y \in B$

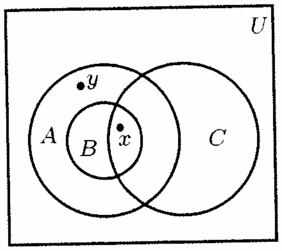

Figure 1.3

23. Use the Venn diagram in Figure 1.4 to identify each of the following as true or false.
(a) $B \subseteq A$ (b) $A \subseteq C$ (c) $C \subseteq B$
(d) $w \in A$ (e) $t \in A$ (f) $w \in B$

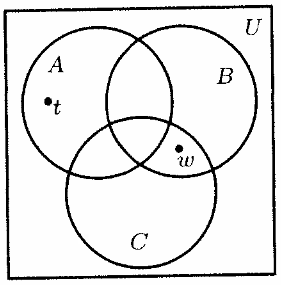

Figure 1.4

24. (a) Complete the following statement. A generic Venn diagram for a single set has ____ regions. Describe them in words.
(b) Complete the following statement. A generic Venn diagram for two sets has ____ regions. Describe them in words.

25. Complete the following statement. A generic Venn diagram for three sets has ____ regions. Describe them in words.

26. (a) If $A = \{3, 7\}$, find $P(A)$.
(b) What is $|A|$? (c) What is $|P(A)|$?

27. If $P(B) = \{\{\ \}, \{m\}, \{n\}, \{m, n\}\}$, then find B.

28. (a) If $A = \{3, 7, 2\}$, find $P(A)$.
(b) What is $|A|$? (c) What is $|P(A)|$?

29. If $P(B) = \{\{a\}, \{\ \}, \{c\}, \{b, c\}, \{a, b\}, \dots\}$ and $|P(B)| = 8$, then $B =$ ____.

In Exercises 30 through 32, draw a Venn diagram that represents these relationships.

30. $A \subseteq B$, $A \subseteq C$, $B \not\subseteq C$, and $C \not\subseteq B$

31. $x \in A$, $x \in B$, $x \notin C$, $y \in B$, $y \in C$, and $y \notin A$

32. $A \subseteq B$, $x \notin A$, $x \in B$, $A \not\subseteq C$, $y \in B$, $y \in C$

33. Describe all the subset relationships that hold for the sets given in Example 3.

34. Show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

35. The statement about sets in Exercise 34 can be restated as "Any subset of ____ is also a subset of any set that contains ____."

36. Suppose we know that set A has n subsets, $S_1, S_2, \dots, S_n$. If set B consists of the elements of A and one more element so $|B| = |A| + 1$, show that B must have $2n$ subsets.

37. Compare the results of Exercises 12, 13, 26, and 28 and complete the following: Any set with two elements has ____ subsets. Any set with three elements has ____ subsets.

1.2 Operations on Sets

In this section we will discuss several operations that will combine given sets to yield new sets. These operations, which are analogous to the familiar operations on the real numbers, will play a key role in the many applications and ideas that follow.

If A and B are sets, we define their **union** as the set consisting of all elements that belong to A or B and denote it by $A \cup B$. Thus

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Observe that $x \in A \cup B$ if $x \in A$ or $x \in B$ or x belongs to both A and B.

Example 1 Let $A = \{a, b, c, e, f\}$ and $B = \{b, d, r, s\}$. Find $A \cup B$.

Solution

Since $A \cup B$ consists of all the elements that belong to either A or B, $A \cup B = \{a, b, c, d, e, f, r, s\}$. ◆

We can illustrate the union of two sets with a Venn diagram as follows. If A and B are the sets in Figure 1.5(a), then $A \cup B$ is the set represented by the shaded region in Figure 1.5(b).

1.2 Exercises

In Exercises 1 through 4, let $U = \{a, b, c, d, e, f, g, h, k\}$, $A = \{a, b, c, g\}$, $B = \{d, e, f, g\}$, $C = \{a, c, f\}$, *and* $D = \{f, h, k\}$.

1. Compute

(a) $A \cup B$ **(b)** $B \cup C$ **(c)** $A \cap C$

(d) $B \cap D$ **(e)** $(A \cup B) - C$ **(f)** $A - B$

(g) $\overline{A}$ **(h)** $A \oplus B$ **(i)** $A \oplus C$

(j) $(A \cap B) - C$

2. Compute

(a) $A \cup D$ **(b)** $B \cup D$ **(c)** $C \cap D$

(d) $A \cap D$ **(e)** $(A \cup B) - (C \cup B)$

(f) $B - C$ **(g)** $\overline{B}$ **(h)** $C - B$

(i) $C \oplus D$ **(j)** $(A \cap B) - (B \cap D)$

3. Compute

(a) $A \cup B \cup C$ **(b)** $A \cap B \cap C$

(c) $A \cap (B \cup C)$ **(d)** $(A \cup B) \cap C$

(e) $\overline{A \cup B}$ **(f)** $\overline{A \cap B}$

4. Compute

(a) $A \cup \varnothing$ **(b)** $A \cup U$ **(c)** $B \cup B$

(d) $C \cap \{\,\}$ **(e)** $\overline{C \cup D}$ **(f)** $\overline{C \cap D}$

In Exercises 5 through 8, let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 4, 6, 8\}$, $B = \{2, 4, 5, 9\}$, $C = \{x \mid x$ *is a positive integer and* $x^2 \leq 16\}$, *and* $D = \{7, 8\}$.

5. Compute

(a) $A \cup B$ **(b)** $A \cup C$ **(c)** $A \cup D$

(d) $B \cup C$ **(e)** $A \cap C$ **(f)** $A \cap D$

(g) $B \cap C$ **(h)** $C \cap D$

6. Compute

(a) $A - B$ **(b)** $B - A$ **(c)** $C - D$

(d) $\overline{C}$ **(e)** $\overline{A}$ **(f)** $A \oplus B$

(g) $C \oplus D$ **(h)** $B \oplus C$

7. Compute

(a) $A \cup B \cup C$ **(b)** $A \cap B \cap C$

(c) $A \cap (B \cup C)$ **(d)** $(A \cup B) \cap D$

(e) $\overline{A \cup B}$ **(f)** $\overline{A \cap B}$

8. Compute

(a) $B \cup C \cup D$ **(b)** $B \cap C \cap D$

(c) $A \cup A$ **(d)** $A \cap \overline{A}$

(e) $A \cup \overline{A}$ **(f)** $A \cap (\overline{C} \cup D)$

In Exercises 9 and 10, let $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, c, f, g\}$, $B = \{a, e\}$, *and* $C = \{b, h\}$.

9. Compute

(a) $\overline{A}$ **(b)** $\overline{B}$ **(c)** $\overline{A \cup B}$

(d) $\overline{A \cap B}$ **(e)** $\overline{U}$ **(f)** $A - B$

10. Compute

(a) $\overline{A} \cap \overline{B}$ **(b)** $\overline{B} \cup \overline{C}$ **(c)** $\overline{A \cup A}$

(d) $\overline{C \cap C}$ **(e)** $A \oplus B$ **(f)** $B \oplus C$

11. Let U be the set of real numbers, $A = \{x \mid x$ is a solution of $x^2 - 1 = 0\}$, and $B = \{-1, 4\}$. Compute

(a) $\overline{A}$ **(b)** $\overline{B}$ **(c)** $\overline{A \cup B}$ **(d)** $\overline{A \cap B}$

In Exercises 12 and 13, refer to Figure 1.14.

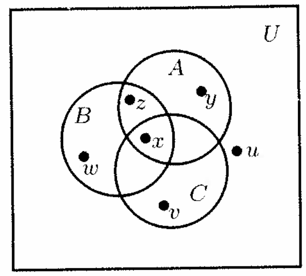

Figure 1.14

12. Identify the following as true or false.

(a) $y \in A \cap B$ **(b)** $x \in B \cup C$

(c) $w \in B \cap C$ **(d)** $u \notin C$

13. Identify the following as true or false.

(a) $x \in A \cap B \cap C$ **(b)** $y \in A \cup B \cup C$

(c) $z \in A \cap C$ **(d)** $v \in B \cap C$

14. Describe the shaded region shown in Figure 1.15 using unions and intersections of the sets A, B, and C. (Several descriptions are possible.)

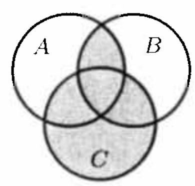

Figure 1.15

15. Let A, B, and C be finite sets with $|A| = 6$, $|B| = 8$, $|C| = 6$, $|A \cup B \cup C| = 11$, $|A \cap B| = 3$, $|A \cap C| = 2$, and $|B \cap C| = 5$. Find $|A \cap B \cap C|$.

In Exercises 16 through 18, verify Theorem 2 for the given sets.

16. **(a)** $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 6, 8\}$

(b) $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9\}$

17. **(a)** $A = \{a, b, c, d, e, f\}$, $B = \{a, c, f, g, h, i, r\}$

(b) $A = \{a, b, c, d, e\}$, $B = \{f, g, r, s, t, u\}$

18. **(a)** $A = \{x \mid x$ is a positive integer $< 8\}$, $B = \{x \mid x$ is an integer such that $2 \leq x \leq 5\}$

(b) $A = \{x \mid x$ is a positive integer and $x^2 \leq 16\}$, $B = \{x \mid x$ is a negative integer and $x^2 \leq 25\}$

19. If A and B are disjoint sets such that $|A \cup B| = |A|$, what must be true about B?

20. Write Property 14 of Theorem 1 in ordinary English.

21. Write Property 15 of Theorem 1 in ordinary English.

In Exercises 22 through 24, verify Theorem 3 for the given sets.

22. $A = \{a, b, c, d, e\}$, $B = \{d, e, f, g, h, i, k\}$, $C = \{a, c, d, e, k, r, s, t\}$

23. $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 7, 8, 9\}$, $C = \{1, 2, 4, 7, 10, 12\}$

24. $A = \{x \mid x \text{ is a positive integer} < 8\}$, $B = \{x \mid x \text{ is an integer such that } 2 \leq x \leq 4\}$, $C = \{x \mid x \text{ is an integer such that } x^2 < 16\}$

25. In a survey of 260 college students, the following data were obtained:

64 had taken a mathematics course,

94 had taken a computer science course,

58 had taken a business course,

28 had taken both a mathematics and a business course,

26 had taken both a mathematics and a computer science course,

22 had taken both a computer science and a business course, and

14 had taken all three types of courses.

(a) How many students were surveyed who had taken none of the three types of courses?

(b) Of the students surveyed, how many had taken only a computer science course?

26. A survey of 500 television watchers produced the following information: 285 watch football games, 195 watch hockey games, 115 watch basketball games, 45 watch football and basketball games, 70 watch football and hockey games, 50 watch hockey and basketball games, and 50 do not watch any of the three kinds of games.

(a) How many people in the survey watch all three kinds of games?

(b) How many people watch exactly one of the sports?

27. The Journalism 101 class recently took a survey to determine where the city's people obtained their news. Unfortunately, some of the reports were damaged. What we know is that 88 people said they obtained their news from television, 73 from the local paper, and 46 from a news magazine. Thirty-four people reported that they obtained news from television and the local paper, 16 said they obtained their news from television and a news magazine, and 12 obtained theirs from the local paper and a news magazine. A total of five people reported that they used all three media. If 166 people were surveyed, how many use none of the three media to obtain their news? How many obtain their news from a news magazine exclusively?

28. The college catering service must decide if the mix of food that is supplied for receptions is appropriate. Of 100 people questioned, 37 say they eat fruits, 33 say they eat vegetables, 9 say they eat cheese and fruits, 12 eat cheese and vegetables, 10 eat fruits and vegetables, 12 eat only cheese, and 3 report they eat all three offerings. How many people surveyed eat cheese? How many do not eat any of the offerings?

29. In a psychology experiment, the subjects under study were classified according to body type and gender as follows:

	ENDO-MORPH	ECTO-MORPH	MESO-MORPH
Male	72	54	36
Female	62	64	38

(a) How many male subjects were there?

(b) How many subjects were ectomorphs?

(c) How many subjects were either female or endomorphs?

(d) How many subjects were not male mesomorphs?

(e) How many subjects were either male, ectomorph, or mesomorph?

30. The following table displays information about the sophomore, junior, and senior classes at Old U.

Class	Major Declared (D)	Major Undeclared (U)
Sophomore (S)	143	289
Junior (J)	245	158
Senior (R)	392	36

For each of the following tell how many students are in the set and describe those students in words.

(a) $D \cap J$ **(b)** $\overline{U \cup R}$ **(c)** $(D \cup S) \cap \overline{R}$

31. Create a Venn diagram that displays the information in the table in Exercise 30.

32. Complete the following proof that $A \subseteq A \cup B$. Suppose $x \in A$. Then $x \in A \cup B$, because ______. Thus by the definition of subset $A \subseteq A \cup B$.

In Exercises 33 through 38, classify each statement as true, false, or not possible to identify as true or false.

33. Choose $x \in A \cap B$.

(a) $x \in A$ **(b)** $x \in B$ **(c)** $x \notin A$ **(d)** $x \notin B$

34. Choose $y \in A \cup B$.

(a) $y \in A$ **(b)** $y \in B$ **(c)** $y \notin A$

(d) $y \notin B$ **(e)** $y \in A \cap B$ **(f)** $y \notin A \cap B$

35. Choose $z \in A \cup (B \cap C)$.

(a) $z \in A$ **(b)** $z \in B$ **(c)** $z \in C$

(d) $z \in B \cap C$ **(e)** $z \notin A$ **(f)** $z \notin C$

36. Choose $w \in D \cap (E \cup F)$.

(a) $w \in D$ **(b)** $w \in E$ **(c)** $w \in F$
(d) $w \notin D$ **(e)** $w \in F \cup E$
(f) $w \in (D \cap E) \cup (D \cap F)$

37. Choose $t \in \overline{D \cap E}$.

(a) $t \in D$ **(b)** $t \in E$ **(c)** $t \notin D$
(d) $t \notin E$ **(e)** $t \in D \cup E$

38. Choose $x \in \overline{A} \cup (B \cap C)$.

(a) $x \in A$ **(b)** $x \in B$ **(c)** $x \in C$
(d) $x \in A \cup B$ **(e)** $x \in (\overline{A} \cup B) \cap (\overline{A} \cup C)$

39. Complete the following proof that $A \cap B \subseteq A$. Suppose $x \in A \cap B$. Then x belongs to ______. Thus $A \cap B \subseteq A$.

40. **(a)** Draw a Venn diagram to represent the situation $C \subseteq A$ and $C \subseteq B$.

(b) To prove $C \subseteq A \cup B$, we should choose an element from which set?

(c) Prove that if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cup B$.

41. **(a)** Draw a Venn diagram to represent the situation $A \subseteq C$ and $B \subseteq C$.

(b) To prove $A \cup B \subseteq C$, we should choose an element from which set?

(c) Prove that if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

42. Prove that $A - (A - B) \subseteq B$.

43. Suppose that $A \oplus B = A \oplus C$. Does this guarantee that $B = C$? Justify your conclusion.

44. Prove that $A - B = A \cap \overline{B}$.

45. If $A \cup B = A \cup C$, must $B = C$? Explain.

46. If $A \cap B = A \cap C$, must $B = C$? Explain.

47. Prove that if $A \subseteq B$ and $C \subseteq D$, then $A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$.

48. When is $A - B = B - A$? Explain.

49. Explain the last term in the sum in Theorem 3. Why is $|A \cap B \cap C|$ added and $|B \cap C|$ subtracted?

50. Write the four-set version of Theorem 3; that is, $|A \cup B \cup C \cup D| = \cdots$.

51. Describe in words the n-set version of Theorem 3.

1.3 Sequences

Some of the most important sets arise in connection with sequences. A **sequence** is simply a list of objects arranged in a definite order; a first element, second element, third element, and so on. The list may stop after n steps, $n \in \mathbb{N}$, or it may go on forever. In the first case we say that the sequence is **finite**, and in the second case we say that it is **infinite**. The elements may all be different, or some may be repeated.

Example 1 The sequence 1, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1 is a finite sequence with repeated items. The digit zero, for example, occurs as the second, third, fifth, seventh, and eighth elements of the sequence. ◆

Example 2 The list 3, 8, 13, 18, 23, ... is an infinite sequence. The three dots in the expression mean "and so on," that is, continue the pattern established by the first few elements. ◆

Example 3 Another infinite sequence is 1, 4, 9, 16, 25, ..., the list of the squares of all positive integers. ◆

It may happen that how a sequence is to continue is not clear from the first few terms. Also, it may be useful to have a compact notation to describe a sequence. Two kinds of formulas are commonly used to describe sequences. In Example 2, a natural description of the sequence is that successive terms are produced by adding 5 to the previous term. If we use a subscript to indicate a term's position in the sequence, we can describe the sequence in Example 2 as $a_1 = 3$, $a_n = a_{n-1} + 5$, $2 \leq n$. A formula, like this one, that refers to previous terms to define the next term is called **recursive**. Every recursive formula must include a starting place.

On the other hand, in Example 3 it is easy to describe a term using only its position number. In the nth position is the square of n; $b_n = n^2$, $1 \leq n$. This type of formula is called **explicit**, because it tells us exactly what value any particular term has.