

Investigations Into d'Alembert's Definition of Limit

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1 Introduction

The modern definition of a limit evolved over many decades. One of the earliest attempts at a precise definition is credited to Jean-Baptiste le Rond d'Alembert (1717 – 1783), a French mathematician, philosopher and physicist. Among his many accomplishments, he argued in a 1754 article of the *Encyclopédie* that the theory of limits should be put on a firm foundation.

2 D'Alembert's Limit Definition

By 1754 mathematical techniques were quite advanced. d'Alembert won a 1747 prize for his work in partial differential equations, but became embroiled in arguments with Euler and others over methodology and foundational issues. Perhaps these squabbles led to his interest in clearing up the foundations of limits and convergence.

Here is d'Alembert's limit definition from the *Encyclopédie* [1]:



Limit. One says that one quantity is the limit of another quantity, when the second can approach nearer to the first than any given quantity, however small; nevertheless, without the quantity which is approaching ever being able to surpass the quantity which it approaches; so that the difference between a quantity and its limit is absolutely inassignable.

For example, consider two polygons, one inscribed and the other circumscribed about a circle, it is evident that one can increase the sides however one wishes; and in this case, each polygon will more and more closely approach the circumference of the circle, the contour of the inscribed polygon increasing, and that of the circumscribed polygon decreasing; but the perimeter or the contour of the first will never surpass the length of the circumference, and that of the second will never be smaller than the same circumference; the circumference of the circle is thus the limit of the increase of the first polygon, and the decrease of the second.

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[Claim] 1st. If two values are the limit of the same quantity, these two quantities are equal to one another.

[Claim] 2nd. Let $A \times B$ be the product of two quantities A , B . We suppose that C is the limit of the quantity A , and D the limit of the quantity B ; I say that $C \times D$, the product of the limits, will necessarily be the limit of $A \times B$, the product of the two quantities A , B .

....

Properly speaking, the limit never coincides, or is never equal to the quantity of which it is the limit; but it is approached more and more, and can differ by as little as one wants. The circle, for example, is the limit of the inscribed and circumscribed polygons; because it never merges with them, though they can approach it ad infinitum.



Note that this definition is lacking in precise, modern mathematical notation. Also observe that the polygon/circle example is for the limit of a *sequence*. Here is a standard first year calculus book definition of limit for a sequence:

First Year Calculus Definition. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

Let's examine some examples.

Exercise 1 Use modern subscript notation for an appropriate sequence to rewrite d'Alembert's inscribed polygon \rightarrow circle limit example. Assume for simplicity that the inscribed polygons are regular with n sides centered at the circle's center. These polygons have perimeter formula

$$\text{perimeter} = 2n \cdot \text{radius} \cdot \sin(\pi/n).$$

As a bonus, derive the perimeter formula, and use Calculus to confirm this limit.

Exercise 2 Consider d'Alembert's polygon \rightarrow circle limit example and his definition. For "given quantity" 0.1 and a circle of radius 1, how many sides for the inscribed polygon are needed to guarantee the "second can approach nearer to the first than" given quantity 0.1? Technology will be helpful! How many sides are needed for given quantity 0.01?

Exercise 3 Consider the sequence $\{a_n\}$ with $a_n = \frac{n}{2n+1}$. Find its limit by any means. For "given quantity" 0.01, suppose we want a_n and its limit to "differ by as little as" 0.01. What is "sufficiently large" for n to guarantee that a_n and its limit differ by 0.01 or less?

Now let us generalize this example a bit, replacing "given quantity" 0.01 by a generic value c .

Exercise 4 For sequence $\left\{ \frac{n}{2n+1} \right\}$, let c be an arbitrary small positive number. Suppose we want a_n and its limit to differ by less than c . In terms of c , what is "sufficiently large" for n ?

Exercise 5 Look closely at d'Alembert's phrase "Properly speaking, the limit never coincides, or is never equal to the quantity of which it is the limit" and notice that it does not appear in the First Year Calculus definition. Can you think of a simple convergent sequence that violates this requirement of d'Alembert's limit definition?

Exercise 6 Consider d'Alembert's phrase "without the quantity which is approaching ever being able to surpass the quantity which it approaches" and notice that it does not appear in the First Year Calculus definition. Find a simple convergent sequence that violates this requirement of d'Alembert's limit definition.

Exercise 7 Use modern notation to help rewrite d'Alembert's limit definition for sequences using quantifiers "for all", and "there exists". The First Year Calculus Definition and a graph of the sequence $\{a_n\}$ should be helpful! You should introduce a variable c to bound the distance between the quantities, and another variable M to measure n being "sufficiently large".

As we have seen, d'Alembert's limit definition doesn't fully apply to some types of sequences studied by today's mathematicians. The First Year Calculus definition avoids them, but is too vague for actually constructing complex proofs.

Exercise 8 Use quantifiers "for all", and "there exists" to rewrite the First Year Calculus limit definition for sequences.

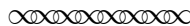
Exercise 9 Use your definition from Exercise 8 to prove that sequence $\left\{ \frac{n}{2n+1} \right\}$ converges.

3 Limit Properties

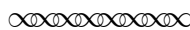
D'Alembert makes a couple assertions about limit properties. One quality of a good modern definition is that it should be useful in constructing proofs of a concept's properties. Let's investigate d'Alembert's first uniqueness claim and his proof.

Exercise 10 Write d'Alembert's Claim 1st for sequences in modern notation.

Here is his proof of uniqueness from [2].



Given Z and X , limits of the same quantity Y , I say that $X = Z$, because if there is some difference between them, such as V , it would be $X = Z \pm V$. By hypothesis, the quantity Y can approach X as closely as we desire. That is to say that the difference between Y and X can be as small as wished. Therefore, since Z differs from X by the quantity V , it follows that Y cannot approach Z any closer than the quantity V , and consequently, that Z is not the limit of Y , which is contrary to the hypothesis.



Exercise 11 Rewrite this uniqueness proof using your modern definition from Exercise 8.

The proof of d'Alembert's Claim 2nd is harder and d'Alembert does not give one in his article.

Exercise 12 Write d'Alembert's Claim 2nd for sequences in modern notation.

The next exercise investigates a proof for a special case of the second claim on the product of sequences to give you some appreciation of the challenges. It may give you ideas for writing a general proof!

Exercise 13 Suppose you know a sequence $\{a_n\}$ is within 0.01 of its limit $C = 5$ if n is larger than some integer $N_1 = 47$. Also suppose you know a sequence $\{b_n\}$ is within 0.01 of its limit $D = 3$ if n is larger than some integer $N_2 = 92$. Can you determine how far you must go with sequence $\{a_nb_n\}$ to get close to the product of limits $CD = 15$? How little difference between a_nb_n and CD can you guarantee if you go out far enough?

References

- [1] Alembert, Jean le Rond d' and La Chapelle, Jean-Baptiste de, Limite. In *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, vol. 9. Paris, 1765. Translated by Jeff Suzuki. The Encyclopedia of Diderot & d'Alembert Collaborative Translation Project. Ann Arbor: Michigan Publishing, University of Michigan Library, 2003.
<http://quod.lib.umich.edu/d/did/did2222.0000.123/-limit?rgn=main;view=fulltext;q1=Mathematics>.
- [2] Alembert, Jean le Rond d' and La Chapelle, Jean-Baptiste de, Calcul différentiel. In *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, vol. 9. Paris, 1765. Translated by Gregory Bringman. The Encyclopedia of Diderot & d'Alembert Collaborative Translation Project. Ann Arbor: Michigan Publishing, University of Michigan Library, 2012.
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Instructor Notes

This project is designed to be used to introduce the definition of limit for sequences in an Introductory Analysis Course.

Project Content Goals

1. Develop a modern limit definition with quantifiers for sequences based on d'Alembert's definition and an Introductory Calculus text definition.
2. Analyze subtleties of the limit definitions: whether sequence terms can “surpass” or coincide with the limit.
3. Develop facility with the modern limit definition by using it to prove a given sequence converges.
4. Students also analyze the uniqueness property of limits and begin exploring the limit of a *product* of convergent sequences. Instructors could use this exploration as a sequeway into a general proof of the limit of a *product* of convergent sequences.

Preparation of Students

The project is written for a course in Real Analysis with the assumption that students have become somewhat comfortable with quantifiers.

Preparation for the Instructor

This is roughly a one week project under the following methodology (basically David Pengelley's “A, B, C” method described on his website):

1. Students do some advanced reading and light preparatory exercises before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the exercises or your grading load will get out of hand! Some instructors have students write questions or summaries based on the reading.
2. Class time is largely dedicated to students working in groups on the project - reading the material and working exercises. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a “participation” grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the exercises they missed. This is usually a good incentive not to miss class!
3. Some exercises are assigned for students to do and write up outside of class. Careful grading of these exercises is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project.

Section 2 Comments

1. Instructors may want to use Exercise 4 to discuss the “obvious” Archimedean property.
2. Exercise 7 may be difficult for students. Encouraging students to draw a plot and labels for c and M should help. Leading questions to help them realize that the modern definition needs to start with “for all $c > 0$ ” may also be helpful.

Section 3 Comments

1. Note that d’Alembert’s uniqueness proof, using contradiction, is different than most traditional Analysis book proofs. Students may need help with the inequalities for Exercise 11.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of ‘in-class task sheets’ based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Acknowledgments

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