Gaussian Guesswork:
Elliptic Integrals and Integration by Substitution

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Just prior to his 19$^{\text{th}}$ birthday, the mathematical genius Carl Friedrich Gauss (1777–1855) began a "mathematical diary" in which he recorded his mathematical discoveries for nearly 20 years. Among these discoveries is the existence of a beautiful relationship between three particular numbers:

- the ratio of the circumference of a circle to its diameter, or π;
- a specific value of a certain (elliptic[1]) integral, which Gauss denoted[2] by $\varpi = 2\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}}$; and
- a number called "the arithmetic-geometric mean" of 1 and $\sqrt{2}$, which he denoted as $\mu(1, \sqrt{2})$.

Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice that historian Adrian Rice calls "Gaussian Guesswork" in his *Math Horizons* article subtitled "why 1.19814023473559220744... is such a beautiful number" [Rice, November 2009].

This mini-project is one of a set of four mini-projects, based on excerpts from Gauss' mathematical diary [Gauss, 2005] and related manuscripts, that looks at the power of Gaussian guesswork via the story of his discovery of this beautiful relationship. In this mini-project, we focus especially on how Gauss' discovery led him to the creation of a powerful new technique for evaluating certain types of integrals, such as the one that defines the number ϖ.

1 Why study the integral $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$?

If you have studied the family of curves known as lemniscates, then you may recall that the integral $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ gives us the arc length of one-fourth of the lemniscate of equation $(x^2+y^2)^2 = x^2 - y^2$. Interest in the lemniscate and its arc length dates back to the earliest days of calculus, when the two Bernoulli brothers (Jacob and Johannes) studied them in the late 1690's. It was Jacob Bernoulli who named the curve *lemniscus*, a Latin word for a type of a decorative ribbon shaped somewhat like the current symbol ∞ for infinity. (See Figure 1.) From his notebooks, we know that Gauss himself studied this integral and

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[1] An integral of the form $\int_0^x \frac{1}{\sqrt{1-t^n}}dt$ is called an *elliptic* integral for $n=3$ and $n=4$; for $n>5$, it is called *hyperelliptic*. This terminology is historically related to the occurrence of this form in connection to the arc length of ellipses and other curves that naturally arise in astronomy and physics.

[2] The symbol 'ϖ' that Gauss used to denote this specific value is called "varpi;" it is a variant of the Greek letter π.

the lemniscate through the works of Euler, which he began to read by the age of 20. Gauss made note of these studies in his 51^{st} diary entry,[3] where he wrote:

I have begun to examine thoroughly the ~~elastic~~[4] lemniscatic curve depending on $\int (1-x^4)^{-1/2}dx$.

January 8, 1797

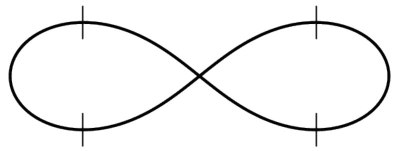

Figure 1: Sketch of the lemniscate from Gauss's notebook, from [Gauss, 1876]

Naturally, Gauss was interested in evaluating the integral for the arc length of the lemniscate. His first idea for doing this is one that perhaps has also occurred to you, based on your experience with trigonometric substitution:[5]

$$\ldots\ldots xx = \begin{matrix}\sin\\ \cos\end{matrix}\ \theta \qquad \int \frac{1}{\sqrt{\sin\theta}}d\theta \ = \ 2\int \frac{1}{\sqrt{1-x^4}}dx$$

.

January 7, 1797

Gauss' notation is a bit cryptic, as he seemed to be describing two different substitutions that would lead him to the result indicated. In the following task, you will verify just one of these two substitutions.

Task 1 In this task, we consider the integral $\int \frac{1}{\sqrt{1-x^4}}dx$.

(a) Consider the substitution $x^2 = \sin\theta$. Explain why this is a natural substitution to try here.

(b) Now verify that the substitution from part (a) gives the integral result that Gauss noted in his diary entry of January 7, 1797. What strategies might we use in order to evaluate the resulting $d\theta$ integral? What prevents these strategies from going through?

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[3] A German version of Gauss' Diary appears in [Gauss, 2005]. In this project, we use the English translation by Jeremy Grey that appears in [Dunnington, 2004, pp. 469-484].

[4] Originally, Gauss wrote 'elastic' here, only to cross it out at some later unknown date when he instead wrote in 'lemniscatic.' See the author's primary source project *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve* for information about the elastic curve and it relationship to the lemniscate, as well as their role in the early history of caclulus.

[5] The symbols used by Gauss in this and other diary entries has been changed slightly in order to maintain notational consistency throughout the project, which uses excerpts from several different works by Gauss.

Interestingly, there is a slight variation on the substitution from Task 1 which allowed Gauss to eventually succeed in evaluating the lemniscate arc length integral. In the next task, we will use this other substitution to transform this integral into a new form — a form that may not seem promising at first glance. In the subsequent sections of this project, we will then work through Gauss' treatment of the transformed integral to see how he was eventually able to evaluate it.

Task 2 **(a)** Let $x = \sin\theta$. Show $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2\theta}}$.

(b) Now re-write $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2\theta}}$ as an integral of the form $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}}$.
State the values of m and n clearly.

2 The Arithmetic-Geometric Mean

In the next section of this project, we will see how Gauss finally managed to evaluate integrals of the form $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}}$. In this section, we first take a look at the mathematical tool that would eventually allow him to tackle this tricky integral. This tool — a type of average called the *arithmetic-geometric mean* — was discovered by Gauss independently of his work on the lemniscate, but also quite early in his mathematical studies. Although his *Nachlass* contains extensive notes about the arithmetic-geometric mean and its properties, it was mentioned only once in his published works, in an important astronomical paper on the gravitational attraction of planets [Gauss, 1818]. Here is what Gauss had to say about the arithmetic-geometric mean in that paper[6]:

Let m, n be two positive quantities, and set[7]

$$m_1 = \frac{1}{2}(m+n), \quad n_1 = \sqrt{mn}$$

so that m_1, n_1 represent the arithmetic mean and the geometric mean, respectively, of m and n. The geometric mean will always be taken to be positive.

Similarly set

$$\begin{aligned} m_2 &= \tfrac{1}{2}(m_1+n_1), \qquad & n_2 &= \sqrt{m_1 n_1} \\ m_3 &= \tfrac{1}{2}(m_2+n_2), \qquad & n_3 &= \sqrt{m_2 n_2} \end{aligned}$$

and so on, by which manner [are obtained] the sequences m, m_1, m_2, m_3, etc., and n, n_1, n_2, n_3, etc., converging rapidly to a *common limit*, which we denote μ, and call simply the *arithmetic-geometric mean* between m and n.

[6] The excerpts from *Determinatio Attratctionis, quam in punctum quodvis positionis datae exerceret planeta, si eius massa per totam orbitam ratione temoris, quo singulae partes descibuntur, uniformiter esset dispertita* used in this project are taken from pages 352–353 of [Gauss, 1818]. The Latin translations of these excerpts were done by George W. Heine III, Math and Maps (`gheine@mathnmaps.com`).

[7] Gauss himself used prime notation (i.e., m', m'', m''') to denote the terms of the sequence. In this project, we instead use indexed notation (i.e., m_1, m_2, m_3) in keeping with current notational conventions. To fully adapt Gauss' notation to that used today, we could also write $m_0 = m$ and $n_0 = n$.

Notice that the value of the arithmetic-geometric mean μ depends on the starting values that are used for m and n; we will use the function notation $\mu(m, n)$ to remind us of this fact.[8] Although Gauss did not give any examples of how to compute arithmetic-geometric mean in his 1818 astronomy paper, here is one of the examples[9] that he worked out much earlier in one of his unpublished *Nachlass* papers, [Gauss, 1799].

Example 1: $m = 1, \quad n = 0.2$

$$
\begin{aligned}
m &= 1.00000\,00000\,00000\,00000\,0 & n &= 0.20000\,00000\,00000\,00000\,0 \\
m_1 &= 0.60000\,00000\,00000\,00000\,0 & n_1 &= 0.44721\,35954\,99957\,93928\,2 \\
m_2 &= 0.52360\,67977\,49978\,99964\,1 & n_2 &= 0.51800\,40128\,22268\,36005\,0 \\
m_3 &= 0.52080\,54052\,86123\,95414\,3 & n_3 &= 0.52080\,78709\,39876\,24344\,0 \\
m_4 &= 0.52080\,16381\,06187 & n_4 &= 0.52080\,16381\,06187
\end{aligned}
$$

Here m_5, n_5 differ in the 23^{rd} decimal place.

Task 3 This task examines examples from Gauss' 1799 paper on the arithmetic-geometric mean.

(a) Verify that the values given by Gauss in Example 1 above are correct. Are you able to use your calculator to obtain the same degree of accuracy (21 decimal places!) that Gauss obtained by hand calculations?

(b) In Example 1 above, notice how quickly the two sequences converge to the same limiting value, thereby allowing us to assert that $\mu(1, 0.2) \approx 0.52080\,16381\,06187$. Compile some additional numerical evidence concerning Gauss' claim that 'the sequences m, m_1, m_2, m_3, etc., and n, n_1, n_2, n_3, etc. converge rapidly to a common limit' by computing the first few terms of the sequences $(m_k), (n_k)$ in the following two examples from Gauss' 1799 paper. For each, compute a sufficient number of terms to approximate the value of the arithmetic-geometric mean $\mu(m, n)$ to at least 10 decimal places.

(i) $m =, n = 0.8$. (ii) $m = \sqrt{2}, n = 1$

Based on these three examples, how convincing do you find Gauss' assertion concerning the rapidity of the convergence of the two sequences to the value of $\mu(m, n)$?

In the next section, we will take a look at how Gauss put the arithmetic-geometric mean to work in evaluating elliptic integrals.[10]

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[8] Gauss also used this type of function notation in his unpublished writing about the arithmetic-geometric mean.

[9] We have changed Gauss' notation in this example slightly, in order to be consistent with the notation he used in [Gauss, 1799].

[10] Additional details of Gauss' work on the arithmetic-geometric mean can be found in the author's primary source project *Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean*, available at `http://webpages.ursinus.edu/nscoville/TRIUMPHS.html`.

3 Gauss' evaluation of the integral $\int \frac{d\theta}{\sqrt{m^2\cos^2\theta+n^2\sin^2\theta}}$

We continue now with our reading of Gauss' paper on the gravitational attraction of planets [Gauss, 1818]. Just after his definition of μ as the arithmetic-geometric mean of the numbers m and n, Gauss asserted the following:[11]

Now we shall demonstrate, $\frac{1}{\mu}$ to be the value of the integral

$$\int \frac{d\theta}{2\pi\sqrt{mm\cos^2\theta + nn\sin^2\theta}}$$

from $\theta = 0$ extended to $\theta = 2\pi$.

Let's pause before reading Gauss' proof of this theorem to take a look at how it can be applied, and especially how it relates to the lemniscate's arc length.

Task 4 Recall from Example 1 of the previous section that Gauss approximated the arithmetic-geometric mean μ for the values of $m = 1$ and $n = 0.2$, finding that $\mu(1, 0.2) \approx 0.52080\,16381\,06187$.

(a) Write down the integral corresponding to these values of m and n that Gauss' theorem tells us how to evaluate.

(b) Use Gauss' approximated value for $\mu(1, 0.2)$ in this theorem to give an approximation for the integral from part (a).

Task 5 Recall from Task 3 that $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2\theta + 2\sin^2\theta}}$, where this latter integral has the form $\int \frac{d\theta}{\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}}$ with $m = 1$ and $n = \sqrt{2}$. In Task 5 (b-ii), you also approximated the value of the arithmetic-geometric mean $\mu(1, \sqrt{2})$ by computing the first few terms of the associated arithmetic mean sequence (m_k) and the associated geometric mean sequence (n_k).

Use your approximate value for $\mu(1, \sqrt{2})$ to approximate the arc length of the full lemniscate curve.

Now that we have an idea of the power of Gauss' theorem for evalulating $\int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{mm\cos^2\theta + nn\sin^2\theta}}$, let's tackle his proof! **Read through this proof in the following excerpt at least twice**, making note of any questions or comments you may have. The tasks that follow this excerpt will then guide us through the full details.

[11] In this and subsequent excerpts from Gauss' paper, we have stated all limits of integration in radians; Gauss himself used degrees for this purpose.

Now we shall demonstrate, $\frac{1}{\mu}$ to be the value of the integral B

$$\int \frac{d\theta}{2\pi\sqrt{mm\cos^2\theta + nn\sin^2\theta}}$$

from $\theta = 0$ extended to $\theta = 2\pi$.

PROOF. We suppose the variable θ is expressed by another variable θ_1, so that

$$\sin\theta = \frac{2m\sin\theta_1}{(m+n)\cos^2\theta_1 + 2m\sin^2\theta_1}$$

B

[where] it is easily observed that while θ_1 is increased from 0 to $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π, θ also (although by different intervals) increases from 0 to $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π. The expansion duly performed, it is found to be that

$$\frac{d\theta}{\sqrt{mm\cos^2\theta + nn\sin^2\theta}} = \frac{d\theta_1}{\sqrt{m_1m_1\cos^2\theta_1 + n_1n_1\sin^2\theta_1}}$$

and indeed the values of the integrals

$$\int \frac{d\theta}{2\pi\sqrt{mm\cos^2\theta + nn\sin^2\theta}}, \quad \int \frac{d\theta_1}{2\pi\sqrt{m_1m_1\cos^2\theta_1 + n_1n_1\sin^2\theta_1}}$$

if each of the variables is extended continuously from the value 0 to the value 2π, equal to each other. And if this [process] is permitted to continue further, clearly these values also are equal to the integral value

$$\int \frac{d\theta}{2\pi\sqrt{\mu\mu\cos^2\theta + \mu\mu\sin^2\theta}}$$

from $\theta = 0$ to $\theta = 2\pi$, which evidently becomes $= \frac{1}{\mu}$.

Gauss' idea of setting up the substitution $\sin\theta = \dfrac{2m\sin\theta_1}{(m+n)\cos^2\theta_1 + 2m\sin^2\theta_1}$ certainly gives us some insight into why he is considered a mathematical genius! Before we look at the full details of this particular substitution, let's pause to look at Gauss' overall argument.

The critical idea here is that Gauss' (clever!) substitution produces a sequence of definite integrals that all have the same (constant!) value:

$$\int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}} = \int_0^{2\pi} \frac{d\theta_1}{2\pi\sqrt{m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1}} = \int_0^{2\pi} \frac{d\theta_1}{2\pi\sqrt{m_2^2\cos^2\theta_2 + n_2^2\sin^2\theta_2}} = \ldots$$

Notice also how the arithmetic mean sequence (m_k) and geometric mean sequence (n_k) associated with the starting values of m and n come into this integral sequence. As we know from Section 2, the sequences (m_k) and (n_k) both converge to the arithmetic-geometric mean $\mu(m,n)$. Connecting this fact to the constant sequence of definite integrals above gives us the following surprising, but quite powerful, equality:

$$\int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}} = \int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{\mu^2\cos^2\theta + \mu^2\sin^2\theta}}.$$

In the next task, you will give the details that show exactly why this "limit integral" is especially nice. The final task in this section will then guide us through the rather complicated algebraic details of the substitution that Gauss simply told us to 'duly perform'.

Task 6 Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{\mu^2\cos^2\theta + \mu^2\sin^2\theta}}$.

Then explain how this relates back to Gauss' claim that $\displaystyle\int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}} = \frac{1}{\mu}$.

Task 7 In this task, we examine the details of Gauss' substitution, which we restate here for ease of reference:

$$\boxed{\textbf{\textit{GIVEN:}} \qquad \sin\theta = \frac{2m\sin\theta_1}{(m+n)\cos^2\theta_1 + 2m\sin^2\theta_1}}$$

We will do this through a series of steps[12] to see how the "expansion duly performed" gives the following:

$$\boxed{\textbf{\textit{GOAL:}} \qquad \frac{d\theta}{\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}} = \frac{d\theta_1}{\sqrt{m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1}}}$$

The key to setting up the new integrand in θ_1 will be to find two different (but equal) expressions in θ_1 for the differential $d(\sin\theta)$.We first do this using the (original) constants m, n from the $d\theta$ integral throughout; the end result of this work will come from Differential Equality #2 in part (e) below. Part (f) will then look at how to bring the (new) constants m_1, n_1 into the $d\theta_1$ integrand.

In each part of this task, you should complete the details that appear in bold font.

(a) We start by proving a fact that will be useful in re-writing the given substitution (in the first box above) so that only one trigonometric function ($\sin\theta_1$) appears on its right-hand side; this will make certain steps (e.g., the derivative of the right hand side of the given substitution) easier to carry out. (Eventually, we will need to bring the cosine back into the equation.)

Use the Pythagorean identity $\cos^2\theta_1 = 1 - \sin^2\theta_1$ **to show that the following holds:**

$$\boxed{\textbf{Part (a) End Result} \qquad (m+n)\cos^2\theta_1 + 2m\sin^2\theta_1 = (m+n) + (m-n)\sin^2\theta_1}$$

[12] If you enjoy algebraic challenges, you could first try your hand at carrying the substitution through on your own first. But even if you decide not to take the full plunge, get ready for a bit of a work out!

(b) Using the end result of part (a), we can now re-write the given substitution as follows:

$$\boxed{\textbf{\textit{GIVEN (re-written):}} \qquad \sin\theta = \frac{2m\sin\theta_1}{(m+n)+(m-n)\sin^2\theta_1}}$$

Use the quotient rule to differentiate the right-hand side of this expression, then simplify in order to show that the following holds:

$$\boxed{\textbf{\textit{Differential Equality \#1:}} \qquad \cos\theta\, d\theta = \frac{2m\cos\theta_1[(m+n)-(m-n)\sin^2\theta_1]}{[(m+n)+(m-n)\sin^2\theta_1]^2}\, d\theta_1}$$

(c) Our next step will be to re-write $\cos\theta$ in terms of the variable θ_1.
(Eventually, we will substitute this into the left-hand side of Differential Equality #1).

(i) We start by again using the Pythagorean Identity $\cos^2\theta = 1-\sin^2\theta$ and the re-written form of the given substitution from part (b) as follows. *Be sure you see why each step holds!*

$$\begin{aligned}
\cos^2\theta \quad = \quad 1 - \sin^2\theta \quad &= \quad 1 - \left[\frac{2m\sin\theta_1}{(m+n)+(m-n)\sin^2\theta_1}\right]^2 \\
&= \frac{\left[(m+n)+(m-n)\sin^2\theta_1\right]^2 - 4m^2\sin^2\theta_1}{\left[(m+n)+(m-n)\sin^2\theta_1\right]^2} \\
&= \frac{\overbrace{(m+n)^2 + 2(m+n)(m-n)\sin^2\theta_1 + (m-n)^2\sin^4\theta_1 - 4m^2\sin^2\theta_1}^{\text{Set this equal to } V.}}{\left[(m+n)+(m-n)\sin^2\theta_1\right]^2}
\end{aligned}$$

Simplify the numerator V of the right-hand side of this last expression to show that the following holds:

$$V = (m+n)^2 - 2(m^2+n^2)\sin^2\theta_1 + (m-n)^2\sin^4\theta_1 \qquad (\diamondsuit)$$

Next show that the expression for V given in ($\diamondsuit$) can be re-written to get:[13]

$$V = \cos^2\theta_1\left[(m+n)^2 - (m-n)^2\sin^2\theta_1\right] \qquad (\heartsuit)$$

Finally, substitute the expression given for V in ($\heartsuit$) back into the numerator of the expression we obtained above for $\cos^2\theta_1$ in order to get the following.

$$\boxed{\textbf{Part (c-i) End Result} \qquad \cos^2\theta = \frac{\cos^2\theta_1\left[(m+n)^2 - (m-n)^2\sin^2\theta_1\right]}{\left[(m+n)+(m-n)\sin^2\theta_1\right]^2}}$$

(ii) **Now use the result of part (i) to write an expression for $\cos\theta$.**
(There will be a square root in this expression!)

[13] A sophisticated way to do this is to use the equality $2(m^2+n^2) = (m+n)^2 + (m-n)^2$ to rewrite the middle term of ($\diamondsuit$), then factor ($\diamondsuit$) by grouping. Another option is to start with the right-hand side of ($\heartsuit$) and work backwards to get ($\diamondsuit$). Either way, the Pythagorean Identity will be useful.

(d) Before we go back to the differential expression, it will be helpful to also re-write the expression $\sqrt{m^2\cos^2\theta+n^2\sin^2\theta}$ in terms of $\sin\theta_1$. *(Do you remember why we are interested in the expression $\sqrt{m^2\cos^2\theta+n^2\sin^2\theta}$? If not, look back at Task 6!)*

Dropping the square root for now, let's start by substituting the expressions for $\sin\theta$ and $\cos^2\theta$ from parts (b) and (c) respectively into this expression, simplifying a bit, and using the Pythagorean Identity once more. *Again, be sure you see why each step holds!*

$$\begin{aligned}
m^2\cos^2\theta+n^2\sin^2\theta &= m^2\left[\frac{\cos^2\theta_1\left[(m+n)^2-(m-n)^2\sin^2\theta_1\right]}{\left[(m+n)+(m-n)\sin^2\theta_1\right]^2}\right]+n^2\left[\frac{2m\sin\theta_1}{(m+n)+(m-n)\sin^2\theta_1}\right]^2 \\
&= \frac{m^2\cos^2\theta_1\left[(m+n)^2-(m-n)^2\sin^2\theta_1\right]+4m^2n^2\sin^2\theta_1}{\left[(m+n)+(m-n)\sin^2\theta_1\right]^2} \\
&= m^2\frac{\overbrace{(1-\sin^2\theta_1)\left[(m+n)^2-(m-n)^2\sin^2\theta_1\right]+4n^2\sin^2\theta_1}^{\text{Set this equal to } W}}{\left[(m+n)+(m-n)\sin^2\theta_1\right]^2}
\end{aligned}$$

Expand, then simplify, the expression W; then show that the following holds:

$$W=\left[(m+n)-(m-n)\sin^2\theta_1\right]^2$$

Conclude that the following holds:

$$m^2\cos^2\theta+n^2\sin^2\theta=\left[m\frac{(m+n)-(m-n)\sin^2\theta_1}{(m+n)+(m-n)\sin^2\theta_1}\right]^2,$$

or equivalently,

$$\boxed{\textbf{Part (d) End Result}\qquad \sqrt{m^2\cos^2\theta+n^2\sin^2\theta}=m\frac{(m+n)-(m-n)\sin^2\theta_1}{(m+n)+(m-n)\sin^2\theta_1}},$$

(e) We're now ready to go back to the differential equality that we found in part (b):

$$\boxed{\textbf{\textit{Differential Equality \#1:}}\qquad \cos\theta\,d\theta=\frac{2m\cos\theta_1[(m+n)-(m-n)\sin^2\theta_1]}{[(m+n)+(m-n)\sin^2\theta_1]^2}\,d\theta_1}$$

Substituting the expression for $\cos\theta$ from part (c-ii) into Differential Equality #1, we get:

Differential Equality #2:

$$\frac{\cos\theta_1\sqrt{(m+n)^2-(m-n)^2\sin^2\theta_1}}{(m+n)+(m-n)\sin^2\theta_1}\,d\theta=\frac{2m\cos\theta_1[(m+n)-(m-n)\sin^2\theta_1]}{[(m+n)+(m-n)\sin^2\theta_1]^2}\,d\theta_1$$

Use the end result of part (d) and some simplification to show that we can re-write Differential Equality #2 as follows:

$$\boxed{\textbf{Part (e) End Result}\qquad \sqrt{(m+n)^2-(m-n)^2\sin^2\theta_1}\,d\theta=2\sqrt{m^2\cos^2\theta+n^2\sin^2\theta}\,d\theta_1}$$

(f) Rearranging the result of part (e) gives us the following, which is very close to what we want:

$$\boxed{\textbf{\textit{Differential Equality \#3:}} \qquad \frac{d\theta}{\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}} = \frac{2d\theta_1}{\sqrt{(m+n)^2 - (m-n)^2\sin^2\theta_1}}}$$

All that remains is to see how the expression involving the constants m, n on the right-hand side relates to one involving the constants m_1, n_1. The key to doing this is to recall that $m_1 = \frac{m+n}{2}$ and $n_1 = \sqrt{mn}$. **Use these facts to first show the following:**

$$\boxed{\textbf{Part (f) Result \#1} \qquad 4(m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1) = (m+n)^2 - (m-n)^2\sin^2\theta_1}$$

Hints: The Pythagorean Identity will be useful once more!

Then use this Part (f) Result #1 to show that the right-hand side of Differential Equation #3 can be re-written as follows:

$$\boxed{\textbf{Part (f) Result \#2} \qquad \frac{2d\theta_1}{\sqrt{(m+n)^2 - (m-n)^2\sin^2\theta_1}} = \frac{d\theta_1}{\sqrt{m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1}}}$$

(g) Substituting Result #2 of Part (f) into Differential Equation #3 leaves us with just one last piece to explain about Gauss' final conclusion that

$$\int_0^{2\pi} \frac{d\theta}{2\pi\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}} = \int_0^{2\pi} \frac{d\theta_1}{2\pi\sqrt{m_1^2\cos^2\theta_1 + n_1^2\sin^2\theta_1}}.$$

Complete the proof by explaining why the limits of integration do not change under Gauss' substitution. *(Gauss commented on these limits in the first sentence of his proof.)*

4 Why *is* 1.19814023473559220744... such a beautiful number?

In the introduction to this project, we mentioned that historian Adrian Rice's article "Gaussian Guesswork" was cleverly subtitled "why 1.19814023473559220744... is such a beautiful number" [Rice, November 2009]. Since then, we've met this very same number in Task 5, where we found that 1.19814023473559220744... is a good approximation for the arithmetic-geometric mean $\mu(1, \sqrt{2})$. Task 5 also highlights one of the reasons why the specific arithmetic-geometric mean $\mu(1, \sqrt{2})$ was of special interest to Gauss; namely

$$\frac{1}{\mu(1,\sqrt{2})} = \int_0^{\pi/2} \frac{d\theta}{2\pi\sqrt{\cos^2\theta + 2\sin^2\theta}}$$

Now recall that $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2\theta + 2\sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$, where $\displaystyle\varpi = 2\int_0^1 \frac{1}{\sqrt{1-t^4}}dt$. Rearranging things as needed *(do this!)*, this lets us conclude (along with Gauss) that $\mu(1, \sqrt{2}) = \dfrac{\varpi}{\pi}$ ...a rather surprising connection between three apparently unrelated numbers!!

Here is Gauss' recording of his initial *discovery* of this relationship:

We have established that the arithmetic-geometric mean between 1 and $\sqrt{2}$ is π/ϖ to 11 places; the proof of this fact will certainly open up a new field of analysis.

May 30, 1799

Interestingly, his initial discovery was largely based on numerical guesswork! In this short project, we have only been able to hint at aspects of that guesswork. Gauss' work with the arithmetic-geometric mean illustrates this somewhat: he first calculated the value of $\mu(1, \sqrt{2}) = 1.19814023473559220744\ldots$ as one of his examples in [Gauss, 1799], well before he saw its connection to the elliptic integral $\int_0^1 \frac{1}{\sqrt{1-t^4}} dt$. Gauss also used power series techniques on this integral in order to numerically estimate the value of ϖ, finding that $\varpi = 2.62205755429211981046\ldots$. And, of course, he was familiar with the numerical value of $\pi = 3.14159265358979323846\ldots$. Although it is straightforward to numerically check that the relationship $\mu(1, \sqrt{2}) = \dfrac{\varpi}{\pi}$ is reasonable once these three estimates are put in front of us in this way — you should do this! — we will likely never know exactly what stroke of genius led Gauss to think about their possible connection in the first place.

Following his discovery, it took Gauss nearly another full year to prove that his guesswork about the numerical relationship $\mu(1, \sqrt{2}) = \dfrac{\varpi}{\pi}$ was correct. But, as he predicted, this discovery went well beyond just this one numerical relationship. We have seen, for instance, how Gauss' theorem allows us to numerically estimate *any* integral of the form $\int_a^b \frac{d\theta}{2\pi\sqrt{mm\cos^2\theta + nn\sin^2\theta}}$ by simply making use of the very rapid convergence of the arithmetic-geometric mean. The 'new field of analysis' that opened up in connection with this proof led him well beyond the study of elliptic functions of a single one real-valued variable, and into the realm of functions of several complex-valued variables. Today, a special class of such functions known as the 'theta functions' provide a powerful tool that is used in a wide range of applications throughout mathematics — providing yet one more piece of evidence of Gauss' extraordinary ability as a mathematician and a guesswork genius!

The arithmetic-geometric mean itself is an integral quantity. Proved.

December 23, 1799

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Notes to Instructors

This mini-Primary Source Project (mini-PSP) is one of a set of four mini-PSPs designed to consolidate student proficiency of the following traditional topics from a first-year Calculus course:[14]

- Gaussian Guesswork: Arc Length and the Numerical Approximation of Integrals
- Gaussian Guesswork: Elliptic Integrals and Integration by Substitution
- Gaussian Guesswork: Polar coordinates, Arc Length and the Lemniscate Curve
- Gaussian Guesswork: Sequences and the Arithmetic-Geometric Mean

Each of the four mini-PSPs can be used either alone or in conjunction with any of the other three. All four are based on excerpts from Gauss's mathematical diary [Gauss, 2005] and related primary texts that will introduce students to the power of numerical experimentation via the story of his discovery of a relationship between three particular numbers: the ratio of the circumference of a circle to its diameter (π), a specific value (ϖ) of the elliptic integral $u = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$; and the Arithmetic-Geometric Mean of 1 and $\sqrt{2}$. Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice referred to as "Gaussian Guesswork" by historian Adrian Rice in his *Math Horizons* article subtitled "Why 1.19814023473559220744 ... is such a beautiful number" [Rice, November 2009].

The primary content goal of this particular mini-PSP is to consolidate students' understanding of integration by substitution. Some familiarity with the technique of trigonometric substitution is also assumed, but only for the completion of Task 1. Basic knowledge of trigonometric identities and derivatives (for the sine and cosine function only) is essential, however, as is an understanding of the principles of algebraic manipulation. Although infinite sequences also appear in Section 2 of this project, they do so in a fairly straightforward way that requires little more than the ability to numerically compute arithmetic and geometric means. The project could thus easily be used in the course prior to the formal study of infinite sequences.

The core content of the project appears in Task 7, in which students work through the details of a sophisticated substitution used by Gauss to evaluate integrals of the form $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}}$. Although this particular substitution is not itself part of the standard Calculus II curriculum, working through its details provides an excellent opportunity for Calculus II students to apply and consolidate core concepts and techniques, and to witness their interplay within the context of some amazingly beautiful, and important, mathematics!

Classroom implementation of this and other mini-PSPs in the collection may be accomplished through individually assigned work, small group work and/or whole class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project.

To reap the full mathematical benefits offered by the PSP approach, students should be required to read assigned sections in advance of in-class work, and to work through primary source excerpts together in small

[14] As of August 2018, the first of these four mini-PSPs is not yet completed.

groups in class. The author's method of ensuring that advance reading takes place is to require student completion of "Reading Guides" (or "Entrance Tickets"); see pages 15–17 below for two sample guides based on this particular mini-PSP. Reading Guides typically include "Classroom Preparation" exercises (drawn from the PSP Tasks) for students to complete prior to arriving in class; they may also include "Discussion Questions" that ask students only to read a given task and jot down some notes in preparation for class work. On occasion, tasks are also assigned as follow-up to a prior class discussion. In addition to supporting students' advance preparation efforts, these guides provide helpful feedback to the instructor about individual and whole class understanding of the material. The author's students receive credit for completion of each Reading Guide (with no penalty for errors in solutions).

To complete this particular mini-PSP in its entirety, the following specific implementation schedule is recommended:

- **Advance Preparation Work for Day 1** (to be completed before class): Read pages 1 – 4, completing Tasks 1, 2 and 3 for class discussion along the way, per the sample Reading Guide on pages 15–17.
- **Class Work for Day 1** (based on a 75-minute class period)
  - Brief whole group or small group comparison of answers to Tasks 1, 2 and 3.
  - Small group work on Tasks 4 – 7 (supplemented by whole class discussion as deemed appropriate by the instructor). Note that Task 7 forms the core of the material in this mini-PSP, but will require careful attention to algebraic details on the part of the students. As needed (or desired), continued work on Task 7 could be assigned as follow-up to the work completed in class. **A in-class worksheet for Task 7 is included at the end of these Notes to Instructors.**
- **Homework (optional)**: A complete formal write-up of student work on some or all of Tasks 1 – 6 could be assigned, to be due at a later date (e.g., one week after completion of the in-class work).
- **Advance Preparation Work for Day 2** (to be completed before class): Read the concluding section (pages 10 – 11). As needed (or desired), continued "advance work" on Task 7 could also be assigned in preparation for small group or whole class discussion on Day 2.
- **Class Work for Day 2** (based on a 75-minute class period)
  The amount of time required to complete the following on Day 2 will naturally vary depending on the instructor's goals and student backgrounds, but is not expected to require an entire class period.
  - As appropriate, continued small group work on Task 7; if advance preparation work was assigned on this task, this could consist simply of a comparison of individual student answers.
  - Concluding whole group discussion of the mathematical ideas in the project.
- **Homework (optional)**: A complete formal write-up of student work on Task 7 could be assigned, to be due at a later date (e.g., one week after completion of the in-class work).

LaTeX Code

LaTeX code of the entire PSP is available from the author by request to facilitate preparation of reading guides or 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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For more information about the NSF-funded project TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources (TRIUMPHS), visit:

http://webpages.ursinus.edu/nscoville/TRIUMPHS.html

SAMPLE READING GUIDE: Advance Preparation Work for Day 1

Background Information: The goal of the advance reading and tasks assigned in this 3-page reading guide is to lay the ground work for in-class work on the tasks in Section 3 of this project, by having students first consider possible techniques for approaching the integral in question (Tasks 1 – 2) and introducting them to the basic idea of the arithmetic-geometric mean (Task 3) which is central to the substitution used by Gauss in the proof that forms the core of Section 3.

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Reading Assignment - *Gaussian Guesswork: Elliptic Integrals and Integration by Substitution* - pp. 1–4

1. Read the introduction on page 1.
   Any questions or comments?

2. In Section 1, read pages 1 – 2, stopping at Task 1.
   Any questions or comments?

3. **Class Prep** Complete **Task 1** from page 2 here:

   Task 1 In this task, we consider the integral $\int \frac{1}{\sqrt{1-x^4}}dx$.

   (a) Consider the substitution $x^2 = \sin\theta$. Explain why this is a natural substitution to try here.

   (b) Now verify that the substitution from part (a) gives the integral result that Gauss noted in his diary entry of January 7, 1797. What strategies might we use in order to evaluate the resulting $d\theta$ integral? What prevents these strategies from going through?

This Reading Guide continues on the next page.

4. Read the top of page 3.
   Any questions or comments?

5. **Class Prep** Complete **Task 2** from page 3 here:

   Task 2

   (a) Let $x = \sin\theta$. Show $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2\theta}}$.

   (b) Now re-write $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2\theta}}$ as an integral of the form $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{m^2\cos^2\theta + n^2\sin^2\theta}}$.
   State the values of m and n clearly.

6. In Section 2, read pages 3 – 4, stopping at Task 3.
   Any questions or comments?

This Reading Guide continues on the next page.

7. **Class Prep** Complete **Task 3** from page 4 here:

Task 3 This task examines examples from Gauss' 1799 paper on the arithmetic-geometric mean.

(a) Verify that the values given by Gauss in Example 1 (in the excerpt on page 4) are correct. Are you able to use your calculator to obtain the same degree of accuracy (21 decimal places!) that Gauss obtained by hand calculations?

(b) In Example 1 above, notice how quickly the two sequences converge to the same limiting value, thereby allowing us to assert that $\mu(1, 0.2) = 0.52080\,16381\,06187\ldots$. Compile some additional numerical evidence concerning Gauss' claim that 'the sequences $m, m_1, m_2, m_3,$ etc., and $n, n_1, n_2, n_3,$ etc. converge rapidly to a common limit' by computing the first few terms of the sequences $(m_k), (n_k)$ in the following two examples from Gauss' 1799 paper. For each, compute a sufficient number of terms to approximate the value of the arithmetic-geometric mean $\mu(m, n)$ to at least 10 decimal places.

(i) $m = 1$, $n = 0.8$.

(ii) $m = \sqrt{2}$, $n = 1$

Based on these three examples, how convincing do you find Gauss' assertion concerning the rapidity of the convergence of the two sequences to the value of $\mu(m, n)$?