An Introduction to a Rigorous Definition of Derivative

April 14, 2016

1 Introduction

The concept of a derivative evolved over a great deal of time, originally driven by problems in physics and geometry. Mathematicians found methods that worked, but justifications were not always very convincing by modern standards. A co-inventors of Calculus, Isaac Newton (1642-1727), is generally regarded as one of the most influential scientists in human history. Newton used the term *fluxion* for what we now call a derivative. He developed most of his methods around 1665, but did not publish them immediately. Later Newton wrote articles and books about his fluxion methods, which influenced many mathematicians across Europe.

Newton and fluxions. The modern definition of a function had not yet been created when Newton developed his fluxion theory. The context for Newton's methods of fluxions is a particle tracing out a curve in the $x-y$ plane. The x and y coordinates of the moving particle are fluents or flowing quantities. The horizontal and vertical velocities are the fluxions of x and y, respectively, associated with the flux of time. In the excerpt below from [N], Newton is considering the curve $y = x^n$ and wants to find the fluxion of y.

Let the quantity x flow uniformly, and let it be proposed to find the fluxion of x^n.

In the same time that the quantity x, by flowing, becomes $x+o$, the quantity x^n will become $(x+o)^n$, that is, ... $x^n + nox^{n-1} + \frac{n^2-n}{2}oox^{n-2} + \&c.$ And the augments o and $nox^{n-1} + \frac{n^2-n}{2}oox^{n-2} + \&c.$ are to one another as 1 and $nx^{n-1} + \frac{n^2-n}{2}ox^{n-2} + \&c.$

Now let these augments vanish, and their ultimate ratio will be 1 to nx^{n-1}.

Exercise 1 *Write out the algebraic details of Newton's fluxion method for $n = 3$ using modern algebraic notation.*

Exercise 2 *Convert Newton's argument that $(x^n)' = nx^{n-1}$ to one with modern limit notation for the case where n is a positive integer. You may use modern limit laws.*

It is important to remember that when Newton developed his fluxion method, there was no theory of mathematical limits. Critics of Newton's fluxion method were not happy with having augment o be a seemingly nonzero value at the beginning of the method, and then have the augment o "vanish" at the end of the argument. Indeed, the philosopher and theologian George Berkeley (1685-1753) wrote a 1734 paper [B] attacking Newton's methods. Berkeley rejected:

> "...the fallacious way of proceeding to a certain Point on the Supposition of an Increment, and then at once shifting your Supposition to that of no Increment ... Since if this second Supposition had been made before the common Division by o, all had vanished at once, and you must have got nothing by your Supposition. Whereas by this Artifice of first dividing, and then changing your Supposition, you retain 1 and nx^{n-1}. But, notwithstanding all this address to cover it, the fallacy is still the same. And what are these Fluxions? The Velocities of evanescent Increments? And what are these same evanescent Increments? They are neither finite Quantities nor Quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?"

Some historians argue that Berkeley was attacking the Calculus itself, while others argue that he was instead claiming that Calculus was no more logically rigorous than theology. Certainly, many a Calculus I student has had the same feeling that Berkeley had about the limiting process! As we shall see in Section 2, by the early 1800's the mathematical community was close to putting limits and derivatives on a firmer mathematical foundation.

L'Hôpital on the differential of a Product. While Newton was working on his fluxion methods, G. Leibniz (1646-1716) was independently developing calculus in Germany during the same time period. Many of Leibniz's ideas appeared in the 1696 book *Analyse des infinitment petits* [LH] by G. L'Hôpital (1661-1701). A key idea for Leibniz was the *differential.* Here is an excerpt from L'Hôpital's book.

Definition II. The infinitely small part whereby a variable quantity is continually increased or decreased, is called the differential of that quantity.

...

Proposition II. To find the differentials of the product of several quantities multiplied.

The differential of xy is $y\,dx + x\,dy$: for y becomes $y + dy$, when x becomes $x + dx$; and therefore xy becomes $xy + y\,dx + x\,dy + dx\,dy$. Which is the product of $x + dx$ into $y + dy$, and the differential thereof will be $ydx + x\,dy + dx\,dy$, that is, $y\,dx + x\,dy$: because $dx\,dy$ is a quantity infinitely small, in respect of the other terms $y\,dx$ and $x\,dy$: For if, for example, you divide $y\,dx$ and $dx\,dy$ by dx, we shall have the quotients y and dx, the latter of which is infinitely less than the former.

Exercise 3 *Rewrite this argument in your own words with modern algebraic notation.*

Exercise 4 *What do you think of L'Hôpital's argument that he could eventually ignore $dx\,dy$ because it is "a quantity infinitely small, in respect of the other terms $y\,dx$ and $x\,dy$"?*

After reading Cauchy in Section 2, you will have an opportunity to write and prove a modern version of L'Hôpital's Proposition II.

A number of mathematicians were stung by Berkeley's criticism's and attempted to define the derivative in a more logically satisfying manner. One of the leaders of this movement was J. L. Lagrange (1736-1813). While Lagrange was not entirely successful in his efforts, his ideas were influential, particularly with A. L. Cauchy (1789-1857), whom he knew in Paris. Cauchy developed a theory of limits, the derivative and the definite integral in his 1823 textbook *Calcul Infinitésimal* [C], which we will read from in Section 2 of this project.

2 Cauchy's Definition of Derivative

As a prelude to reading about Cauchy's definition of derivative, it is worthwhile to read an excerpt on limits from his *Calcul Infinitésimal*.

When the vales successively attributed to the same variable approach a fixed value indefinitely, in such a manner as to end up differing from it by as little as we wish, this last value is called the *limit* of all the others.

...

We obviously have, for very small numerical values of α,

$$\frac{\sin\alpha}{\sin\alpha} > \frac{\sin\alpha}{\alpha} > \frac{\sin\alpha}{\tan\alpha}.$$

By consequence, the ratio $\frac{\sin\alpha}{\alpha}$, always contained between the two quantities $\frac{\sin\alpha}{\sin\alpha} = 1$, and $\frac{\sin\alpha}{\tan\alpha} = \cos\alpha$, the first of which serves to limit the second, will itself have unity for a limit.

Exercise 5 *Comment on Cauchy's definition of limit and his proof that* $\lim_{\alpha\to 0}\sin\alpha/\alpha = 1$. *What adjustments, if any, are needed to conform to the modern definition of limit?*

Here is Cauchy on the derivative. In this passage from [C], when Cauchy uses the term "between" two values, he means to include the two values.

When the function $y = f(x)$ remains continuous between two given limits of the variable x, and that we assign to this variable a value contained between the two limits in question, an infinitely small increment attributed to the variable produces an infinitely small increment of the function itself. By consequence, if we then set $\Delta x = i$, the two terms of the *ratio of differences*

$$\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i} \tag{1}$$

will be infinitely small quantities. But, while these two terms indefinitely and simultaneously will approach the limit of zero, the ratio itself may be able to converge toward another limit, either positive or negative. This limit, when it exists, has a determined value for each particular value of x; but, it varies along with x. Thus, for example, if we take $f(x) = x^m$, m designating an integer number, the ratio between the infinitely small differences will be

$$\frac{(x+i)^m - x^m}{i} = mx^{m-1} + \frac{m(m-1)}{1 \cdot 2} x^{m-2} i + \cdots + i^{m-1},$$

and it will have for a limit the quantity mx^{m-1}, that is to say, a new function of the variable x. It will be the same in general, only the form of the new function, which will serve as the limit of the ratio $\frac{f(x+i)-f(x)}{i}$, will depend on the form of the proposed function $y = f(x)$. To indicate this dependence, we give to the new function the name of *derivative function*, and we represent it, with the help of an accent mark, by the notation

$$y' \quad \text{or} \quad f'(x).$$

Let's introduce some modern notation and terminology for Cauchy's definition. First assume like Cauchy that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. If f' exists at some x, we say f is **differentiable** at x. If f is differentiable at each x in an interval I, we say f is differentiable on I.

Exercise 6 *Rewrite Cauchy's definition of $f'(x)$ as a limit using $\epsilon - \delta$ quantifiers and modern terminology. Your definition should allow for the possibility that f' exists at endpoints a, b.*

Exercise 7 *Cauchy assumes f is continuous on $[a,b]$. This suggests that continuity is a necessary condition for the existence of f'. Prove this using limit properties and algebraic fact*

$$f(x+i) - f(x) = \frac{f(x+i) - f(x)}{i} \cdot i$$

for $i \neq 0$.

Exercise 8 *Consider Cauchy's proof that $(x^n)' = nx^{n-1}$.*

(a) Compare and contrast with Newton's argument.

(b) Modernize Cauchy's argument using properties of limits.

Exercise 9 *Let $f(x) = 5x^2$ and let $c \in \mathbb{R}$. Use your modern definition of f' with an $\epsilon - \delta$ argument to show $f'(x) = 10x$*

Exercise 10 *Suppose that $f, g : [a,b] \to \mathbb{R}$ and both f and g are differentiable on I. Use L'Hôpital's proof ideas from his Proposition II to give a modern proof that $f \cdot g$ is differentiable on I.*

After the excerpt above, Cauchy gives some derivative examples, including:

$$\text{For } y = \sin x, \quad \frac{\Delta y}{\Delta x} = \frac{\sin \frac{1}{2}i}{\frac{1}{2}i} \cos\left(x + \frac{1}{2}i\right), \quad y' = \cos x \tag{2}$$

Exercise 11 *(a) Use a trig identity to justify Cauchy's expression for* $\frac{\Delta y}{\Delta x}$ *in (2)*

(b) Use Cauchy's expression for $\frac{\Delta y}{\Delta x}$*, limit properties and the continuity of the cosine function to show that* $y' = \cos x$

Cauchy defines the derivative $f'(x)$ when the limit $\lim_{i \to 0} \frac{f(x+i) - f(x)}{i}$ exists. Implicitly this means $f'(c)$ may not exist for some c values.

Exercise 12 *Show that for* $f(x) = |x|$,

(a) $f'(0)$ *does not exist.*

(b) $f'(x)$ *is* 1 *for* $x > 0$*, and is* -1 *for* $x < 0$

In a discussion about proving the Mean Value Theorem, the mathematician Giuseppe Peano (1858–1932) references the behavior of the function $P(x) = x^2 \sin(1/x)$ near $x = 0$.

Exercise 13 *Consider Peano's function* P*.*

(a) Find a value for $P(0)$ *so that* P *is continuous at* 0*. Prove your assertion.*

(b) Find a value for $P'(0)$ *so that* P *is differentiable at* 0*. Prove your assertion with the definition of derivative.*

Exercise 14 *Consider Peano's function* P*.*

(a) Use standard calculus rules to find $P'(x)$ *for* $x \neq 0$*.*

(b) Is P' *continuous at* $x = 0$*? Prove your assertion.*

3 Conclusion

Mathematicians figured out how to define and apply the derivative well before they put the concept on a firm mathematical foundation. After giving his definition of f', Cauchy went on to prove theorems with it, including the all important Mean Value Theorem. This result is fundamental in proving many results from an introductory Calculus course.

References

[B] Berkeley, G. 1734. *The Analyst*, London.

[C] Cauchy, A.L. 1823. *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal*. Paris: De Bure.

[LH] L'Hôpital, G. 1696. *Analyse des infiniment petits pour l'intelligence des lignes courbes.* Paris : Montalant.

[N] Newton, I. 1704, *Tractatus de quadratura curvarum*. London.