general. This is always the problem when one is trying to formulate a new mathematical concept, to decide how general its definition should be. The definition finally settled on may seem a bit abstract, but as you work through the various ways of constructing topological spaces, you will get a better feeling for what the concept means.

Definition. A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a ***topological space***.

Properly speaking, a topological space is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X, but we often omit specific mention of $\mathcal{T}$ if no confusion will arise.

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an ***open set*** of X if U belongs to the collection $\mathcal{T}$. Using this terminology, one can say that a topological space is a set X together with a collection of subsets of X, called *open sets*, such that $\varnothing$ and X are both open, and such that arbitrary unions and finite intersections of open sets are open.

EXAMPLE 1. Let X be a three-element set, $X = \{a, b, c\}$. There are many possible topologies on X, some of which are indicated schematically in Figure 12.1. The diagram in the upper right-hand corner indicates the topology in which the open sets are X, $\varnothing$, $\{a, b\}$, $\{b\}$, and $\{b, c\}$. The topology in the upper left-hand corner contains only X and $\varnothing$, while the topology in the lower right-hand corner contains every subset of X. You can get other topologies on X by permuting a, b, and c.

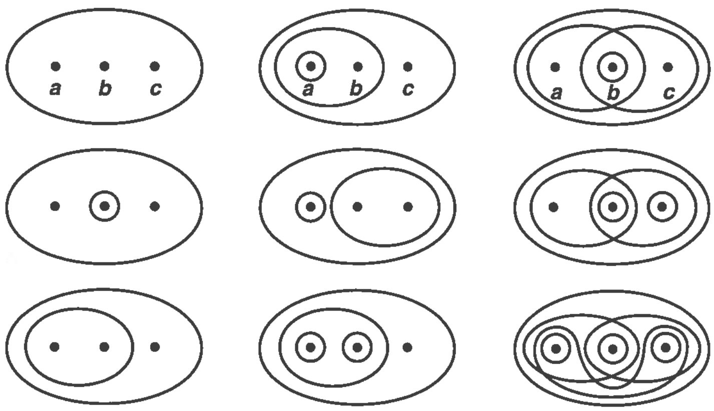

Figure 12.1

From this example, you can see that even a three-element set has many different topologies. But not every collection of subsets of X is a topology on X. Neither of the collections indicated in Figure 12.2 is a topology, for instance.