Solving Linear First-Order Differential Equations
Leonard Euler's Integrating Factor Method

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1 Introduction

In 1926, British mathematician E. L. Ince (1891–1941) described the typical evolution of solution techniques from calculus (and differential equations and science in general).[1]

> The early history of the infinitesimal calculus abounds in instances of problems solved through the agency of what were virtually differential equations; it is even true to say that the problem of integration, which may be regarded as the solution of the simplest of all types of differential equations, was a practical problem even in the middle of the sixteenth century. Particular cases of the inverse problem of tangents, that is the problem of determining a curve whose tangents are subjected to a particular law, were successfully dealt with before the invention of the calculus.
>
> But the historical value of a science depends not upon the number of particular phenomena it can present but rather upon the power it has of coordinating diverse facts and subjecting them to one simple code. [Ince, 1926]

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[1]Ince himself is part of at least one such story within differential equations. He developed the so called *Ince Equation* (in about 1923),

$$(1 + a\cos(2t))y''(t) + (b\sin(2t))y'(t) + (\lambda + d\cos(2t))y(t) = 0,$$

which generalized at least two other well known equations from about 1868 and 1914, respectively. Letting $a = b = 0$ and $d = -2q$, we obtain *Mathieu's equation* (which model elliptical drumheads),

$$y''(t) + (\lambda - 2q\cos(2t))y(t) = 0,$$

and letting $a = 0, b = -4q$, and $d = 4q(\nu - 1)$, we obtain the *Whittaker-Hill equation* (with applications to lunar stability and quantum mechanics)

$$y''(t) - 4q(\sin(2t))y'(t) + (\lambda + 4q(\nu - 1)\cos(2t))y(t) = 0.$$

Ince's equation then is itself a special case of *generalized Ince equations* (studied in [Moussa, 2014])

$$(1 + \epsilon A(t))y''(t) + \epsilon B(t)y'(t) + (\lambda + \epsilon D(t))y(t) = 0.$$

This is exactly the evolution of solution methods for first-order linear ordinary differential equations. First, particular problems were solved with “one-off” methods that didn’t have general applications beyond that specific problem. But then those results were combined and generalized until a unified theory developed.

Task 1 In the above passage, Ince made a connection between the solution of the simplest of all types of differential equations and the problem of determining a curve whose tangents are subjected to a particular law. Connect these two statements. If the differential equation is

$$\frac{dy}{dx} = f(x, y),$$

then what is the “curve,” the “‘tangents,” and the “‘particular law”?

Task 2 Recall that non-homogenous first-order linear ordinary differential equations have the following form

$$p(x)\frac{dy}{dx} + q(x)y = f(x), \tag{1}$$

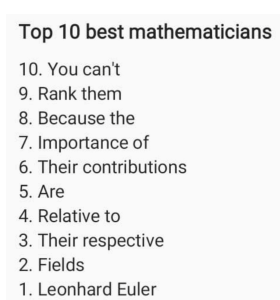
or if made monic

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2}$$

Explain how to make Equation (1) monic like Equation (2). In particular why can we assume that $p(x)$ isn’t identically zero? Write $P(x)$ and $Q(x)$ in terms of $p(x), q(x)$ and $f(x)$.

The theme of this project is a method due to Leonard Euler[2] (1707–1883) which considered a first-order linear differential equations as being “almost” exact. Prior to Euler’s contribution, Gottfried Leibniz[3] (1646–1716) published a paper in 1694 in which he solved first-order linear ordinary differential equations by intuiting a solution then checking that it worked. That method wasn’t general or part of a larger theory. But as Ince noted, science aims to combine diverse techniques into coordinated theories. Johann Bernoulli[4] (1667–1748) did exactly this when he considered first-order linear differential equations as special cases of what are now called “Bernoulli differential equations”

[2] I think this is all that needs said about the Swiss mathematician Euler is:



[3] Leibniz was a German mathematician and philosopher who created (probably independently of Newton) the calculus along with the notation that we currently use for it.

[4] Johann Bernoulli was a very talented Swiss mathematician and third son of Niklaus. His unpleasant personality and desire for fame eventually ruined his relationships with both his brother Jacob, and his son, Daniel.

and used a similar method to “variation of parameters” to solve them.[5] The work of Euler that we consider in this project took even greater strides in terms of presenting a unified theory.

2 Exactly What Background Do We Need?

Recall that a differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{3}$$

is *exact* if there exists an equation $f(x,y) = c$ with c a constant, such that the proposed differential equation is the total differential[6] of both sides of $f(x,y) = c$. In other words, there exists a function $f(x,y)$ such that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)\,dx + N(x,y)\,dy = 0.$$

Under reasonable assumptions, a necessary and sufficient condition to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, after which finding f is well understood; indeed, Euler solved this as Problem I in his paper [Euler, 1763]. This may look familiar if you have taken a multivariable calculus class that included conservative vector fields and their potentials. We start with the left side of Equation (3). I stress that understanding the following *process* is far more important than memorizing the formula.

- First notice that $\frac{\partial f}{\partial x} = M(x,y)$. *Integrating* both sides with respect to x then gives

  $$f(x,y) = \int_x \frac{\partial f}{\partial x} = \int_x M + \text{ “constant” } = \int_x M + g(y)$$

  because the variable y is a constant when integrating with respect to x.

- Now notice that $\frac{\partial f}{\partial y} = N(x,y)$. *Differentiating* the previous equation with respect to y, gives

  $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_x M + \frac{dg}{dy} = N.$$

- *Integrating* the previous with respect to y then gives

  $$f(x,y) = \int_y \frac{\partial}{\partial y}\int_x M + g(y) + c = \int_y N$$

  where c is a genuine constant. We thus get the solution:[7]

  $$f(x,y) = \int_x M + g(y) = \int_x M + \int_y N - \int_y \frac{\partial}{\partial y}\int_x M = c.$$

[5] The story of Leibniz’s and Bernoulli’s methods can be found in the two other projects of this “Solving First-Order Linear Differential Equations” series. Each of the three projects in the series can be completed individually or in any combination with the others. They are available at `https://digitalcommons.ursinus.edu/triumphs_differ/`.

[6] The *total differential* of a function $f(x,y)$ is $\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$ which immediately gives $dc = 0$.

[7] Remember when I stressed that understanding the *process* is far more important than memorizing the solution formula? I meant that.

This solution will be *implicit* and it may be impossible to solve for y to make it *explicit*. Thus, in order to check that your answer is correct you will need think back to implicit differentiation from Calc 1.

$\boxed{\textbf{Task 3}}$ Here is Euler's Example 1 in [Euler, 1763]. Solve this equation using the above method, then check it with implicit differentiation.

EXAMPLE 1

§12. To integrate this differential equation

$$2axy\,dx + axx\,dy - y^3\,dx - 3xyy\,dy = 0.$$

$\boxed{\textbf{Task 4}}$ If you need additional practice, you can try Euler's Example 2 using the above method.

EXAMPLE 2

§13. To integrate this differential equation:

$$\frac{y\,dy + x\,dx - 2y\,dx}{(y-x)^2} = 0.$$

Even if the differential equation is not exact, it may become exact if we multiply it by an *integrating factor* $L(x, y)$. This is the first Theorem in [Euler, 1763]

THEOREM

§16. If in the differential equation

$$M\,dx + N\,dy = 0$$

is not

$$\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$$

always a multiplicator is given, multiplied by which the formula $M\,dx + N\,dy$ become integrable.

In other words $(LM)dx + (LN)dy = 0$ becomes exact. Equivalently, by the exactness test,

$$\frac{\partial(LM)}{\partial y} = \frac{\partial(LN)}{\partial x},$$

which means after applying the product rule that L solves the following PDE. Again, I stress that it is far more important to understand the process than memorizing this formula[8].

$$L\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) + M\frac{\partial L}{\partial y} - N\frac{\partial L}{\partial x} = 0. \tag{4}$$

Task 5 Euler assumed in the above Theorem that the differential equation was not exact. If it were, then there is a very easy solution to Equation (4). What is that L? Keep in mind that if the differential equation were exact we'd just do the technique at the beginning of this section.

In general it is difficult to solve Equation (4), but if we can, we could make the initial differential equation exact and solve it using the above method. This general technique was first published by Alexis-Claude Clairaut[9] (1713–1765) in his paper "Recherches générales sur le calcul intégral"[10] [Clairaut, 1739]. As is so often the case, Euler independently discovered the technique [Euler, 1740], which was published in 1740 though written in 1734.

3 Euler's Method (Not That Euler's Method)

In 1763, Leonard Euler published the paper "De integration aequationum differentialium"[11] [Euler, 1763]. In typical Euler form, the publication consists of Problems and Solutions. We are interested in his "Problema 4."

Problem 4.

§34. Suppose the differential equation

$$P\,dx + Qy\,dx + R\,dy = 0$$

is proposed, where P, Q and R denote functions of x of any sort, and so that the other variable y has no more than one dimension; to find the factor which allows it to be integrated.

[8] Remember when I stressed that understanding the *process* is far more important than memorizing the solution formula? I meant that.

[9] Clairaut was an extremely talented French mathematician and astronomer. Unfortunately he never reached his potential as "he maintained an active social life" [Judson, 2000]. In 1734, he went to Basel where he studied with Johann Bernoulli and Euler wrote that Clairaut's work on the three body problem was "…the most important and profound discovery that has ever been made in mathematics" [O'Connor and Robertson, 1998].

[10] Or, in English, "General research on integral calculus"

[11] Or, in English, "On the Integration of Differential Equations." According to the Euler Archive (`https://scholarlycommons.pacific.edu/euler-works/269/`), Euler actually wrote this paper as early as 1755.

Solution.

Comparing this equation with the form $M\,dx + N\,dy = 0$ we get $M = P + Qy$ and $N = R$, whence

$$\left(\frac{dM}{dy}\right) = Q \qquad \text{and} \qquad \left(\frac{dN}{dx}\right) = \frac{dR}{dx}$$

Now let L stand for the required factor, and so $dL = pdx + qdy$, and whence it is necessary that it satisfy this equation:

$$\frac{Np - Mq}{L} = Q - \frac{dR}{dx} = \frac{Rp - (P + Qy)q}{L}. \tag{5}$$

Since now $Q - \frac{dR}{dx}$ is a function only of x we should also take for L a function only of x, so that $q = 0$ and $dL = p\,dx$; whence:

$$Q - \frac{dR}{dx} = \frac{Np}{L}, \qquad \text{or} \qquad Qdx - dR = \frac{RdL}{L},$$

and therefore, $\frac{dL}{L} = \frac{Qdx}{R} - \frac{dR}{R}$. Wherefore, integrating, we obtain $\log L = \int \frac{Qdx}{R} - \log R$, and assuming that e is the number whose hyperbolic logarithm is unity[12], it yields

$$L = \frac{1}{R} e^{\int \frac{Q\,dx}{R}}.$$

In addition with this factor, the integral equation becomes:

$$\int \frac{P\,dx}{R} e^{\int \frac{Q\,dx}{R}} + y e^{\int \frac{Q\,dx}{R}} = \text{Constant}^{13}$$

Let's examine this passage more closely.

Euler started with the differential equation

$$P\,dx + Qy\,dx + R\,dy = 0 \tag{6}$$

and then rewrote it in the starting form of every problem from this text

$$M\,dx + N\,dy = 0,^{14}$$

with the intent to apply the exactness test:

$$\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x}.$$

[12]Notice that the notation of e is so new that Euler felt the need to explain it (and used hyberbolic logarithm instead of natural logarithm). Using e for the constant $2.71828\ldots$ first appeared in a letter from Euler to Goldbach in 1731 [O'Connor and Robertson, 2001].

[13]Euler translations in Section 3 of this project by Danny Otero of Xavier University, 2020.

[14]Indeed the very first sentence in the text is, "§1 Here, I consider differential equations of first degree, which involve only two variables and which therefore can be represented in this general form $M\,dx + N\,dy = 0$ if M and N denote any arbitrary function of the two variables x and y."

Task 6 For Euler's example $P\,dx + Qy\,dx + R\,dy = 0$,

(a) Verify his answers to the following questions What is M? N? $\frac{\partial M}{\partial y}$? $\frac{\partial N}{\partial x}$

(b) Is this differential equation exact in general? Under what very restrictive condition will it be exact?

Euler then defined the "required factor"[15] L, and immediately wrote $dL \equiv p\,dx + q\,dy$. This simply meant that

$$p = \frac{\partial L}{\partial x} \qquad \text{and} \qquad q = \frac{\partial L}{\partial y}.$$

Task 7 Use Euler's definitions of p and q to show that Equation (5) is just a restatement of Equation (4) with the values of Task 6.

We had noted that solving Equation (4) is typically hard. But for first-order linear differential equations, Equation (5) is actually a separable ODE and can be solved. Euler arrived at this by noticing that L is only a function of x.

Task 8

(a) Explain in your own words why L must only be a function of x.

(b) Explain why this means that $q = 0$.

(c) Following Euler, derive the formula for the integrating factor.

(d) Using the method described above, find a solution to Equation (6).

By construction, multiplication by L creates an exact differential equation, at which point Euler used exact techniques and integrated both sides of Equation (6).

Task 9 (Hint: For what follows, it may be simpler to not use the explicit form of L but leave it as L and simply note that $\frac{dL}{dx} = \frac{L\left(Q - \frac{dR}{dx}\right)}{R}$ and $\frac{dL}{dy} = 0$, until the end of the problem.)

(a) Verify that

$$(LM)dx + (LN)dy = 0$$

is exact for the values of L, M and N from Problem 4.

(b) Show the solution you found in Task 8 is equivalent to Euler's.

[15] Today this is called an "integrating factor."

4 Examples

Task 10 We will now work through Euler's solution for the linear differential equation

$$x\frac{dy}{dx} + y = 3x^2. \tag{7}$$

as an example.

(a) Write Equation (7) in the form of $M\,dx + N\,dy = P\,dx + Qy\,dx + R\,dy = 0$.

(b) What is Equation (5) for this example?

(c) Solve the above equation to find an integrating factor L. What would that indicate about our original differential equation?

(d) Verify that $(LM)\,dx + (LN)\,dy = 0$ is exact.

(e) Solve this exact differential equation.

Task 11 Solve the following using the above method.

$$\frac{dy}{dx} - y = xe^x$$

5 Conclusion

We have come to the end of the story, and Ince predicted our path. At the beginning, Leibniz presented his results with no explanation and no generalization to other problems. But, soon after Bernoulli wrote a solution in a bit more generality with more explanation. Then came Euler, who presented clearly the theory of integrating factors with lots of explanation and examples along the way.

It is likely that your ODE class utilizes an integrating factor μ to solve first-order linear differential equations

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad\qquad \mu = e^{\int P(x)\,dx}. \tag{8}$$

Euler derived his integrating factor by noticing the PDE:

$$M(x,y)\,dx + N(x,y)\,dy = 0 \qquad\qquad L\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) + M\frac{\partial L}{\partial y} - N\frac{\partial L}{\partial x} = 0. \tag{9}$$

was a separable ODE for first-order linear differential equations. We conclude by showing that the L he defined is equivalent to today's μ.

Task 12 Show that μ from Equation (8) is the same as L from Equation (9).

References

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Notes to Instructors

This set of notes accompanies the Primary Source Project "Solving Linear First-Order Differential Equations: Leonard Euler's Integrating Factor Method" written as part of the TRIUMPHS project. (See end of notes for details about TRIUMPHS).

PSP Content: Topics and Goals

This mini-Primary Source Project (mini-PSP) is one of a set of three mini-PSPs that share the name "Solving Linear First Order Differential Equations," designed to show three solutions to non-homogenous first-order linear differential equations, each from a different context. Recall that a non-homogenous first-order linear differential equation has the form

$$a(x)\frac{dy}{dx} + p(x)y = q(x).$$

- The mini-PSP subtitled "Gottfried Leibniz' "Intuition and Check" Method" explains how in 1694 Leibniz solved these equations using one-off method applicable only to this specific problem. Strictly speaking, Leibniz didn't solve the equation, but asserted a solution and then showed it worked. Part of his proposed solution will be familiar to the students: it is the standard integrating factor method we teach today.
- The mini-PSP subtitled "Johann Bernoulli's (Almost) Variation of Parameters Method" explains how in 1697, Bernoulli provided a method for solving Bernoulli differential equations that reduces to variation of parameters when applied to first-order linear equations. This was decades before Lagrange received credit for the technique. Again, part of Bernoulli's solution will be the standard integrating factor.
- The mini-PSP subtitled "Leonard Euler's Integrating Factor Method" explains how in 1763, Euler solved these equations as a special case of exact differential equations by finding an integrating factor. His integrating factor is the same as the one as the students would have seen. This mini-PSP is a bit longer than the others, and may require a bit more time or pre-preparation.

All three of these mini-PSPs are designed for use in an Ordinary Differential Equations course but can be used in three different ways. They work best after at least presenting the standard integrating method of solution found in modern textbooks.

- Since the *type* of equation (first-order linear) has been introduced, all three projects can be immediately done. This would require the instructor to "preview" techniques that will be introduced more fully later. While this is somewhat awkward, it does mimic how these techniques were actually developed.
- The "Gottfried Leibniz' "Intuition and Check" Method" project can be done immediately, but the the other projects done after the respective *method* of solution (variation of parameters, exactness) are first introduced. Showing how those techniques can solve first-order linear differential equations makes a great first example of each technique. This is typically the way that I utilize the project.
- With a bit of revision of the first section, each of these projects can stand on their own as they don't necessarily build on the others (though they do create a richer experience together). Additionally, students gain confidence as they proceed through the three projects.

Student Prerequisites

This mini-PSP requires some algebraic manipulation of differentials along with knowledge of partial differentiation and integration. The differential equation defining the integrating factor is separable, so knowledge of separable techniques would be helpful. Other techniques of integration needed are dictated by the examples used, where instructors can modify the project itself in order to substitute any example they wish in place of those included in the tasks. Knowledge of exact differential equations (or conservative vector fields) will make this project go much faster, though are not necessary based on the background provided.

PSP Design, and Task Commentary

This PSP consists of five sections:

1. The first section contains a short introduction to what first-order linear differential equations are, along with a description of the way that mathematics often evolves. Mathematicians might first solve a specific problem using any tool at their disposal. They then attempt to see if they could find a class of problems (to which the initial one belongs) that can also be attacked using that technique. This closely mimics the evolution of how first-order linear differential equations were solved. Much of this section is the same for all three projects so if either of the first two has been covered, this section can be skipped.

2. The second section includes some background on exact differential equations. This section includes

   (a) Definition of exact differential equations.

   (b) The test for exactness.

   (c) How to solve exact differential equations.

   (d) Finally the PDE defining the integrating factor.

   It is my hope to soon produce a mini-PSP that uses primary sources to derive the equality of mixed partials, the test for exactness, gives the definitions of an exact differential equation and conserved vector field, and shows primary sources for the solution method that I've given. Much of this can be found in [Euler, 1763] and [Clairaut, 1739]; indeed references to specific sections and examples from [Euler, 1763] are provided in the present mini-PSP. When produced, the planned mini-PSP could be used to expand upon the cursory descriptions that are provided here in Section 2.

3. The third section is devoted to Euler's method of solution. A translation is provided along with tasks to explain his method. The main take-away is that for a first-order linear differential equation, the PDE defining the integrating factor is actually a separable ODE that can be solved and therefore the proposed equation can be made exact. There are a fair amount of numbered equations which are referred to later which can be confusing. Thus, I always encourage students to learn the process and not the formulas. I do that all the time but I think it is especially helpful here.

4. The fourth section consists of two first-order linear differential equation examples to be solved with Euler's method. The first is broken into steps that mimic the primary source, while the second requires the student to solve it on their own. These can be swapped with any examples

you wish—in particular, so that the integrals utilize techniques with which your students are comfortable.

One of the TRIUMPHS reviewers of this project correctly observed that these examples could be chosen better. They are not historical when there are plenty of primary source examples available. They may utilize techniques not known at the time. And, the first example given is already exact and so Euler's integrating factor is not necessary (though works). The author is very open to suggestions on improving these examples.

5. The final section concludes the three project series. For the third time, students are asked to show that the modern integrating factor μ can be found in each of the methods that are studied, but Euler's is the only one that derives it as what we know as an integrating factor. The section does briefly reference the historical episodes that are treated in the first two projects so if you've not assigned them, just a word of clarification will be useful.

Suggestions for Classroom Implementation

Please see student requirements and implementation schedule for suggestions.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

The first section of this project should be out of class homework and could be skipped entirely if either of the previous projects has been assigned. If you have already covered a "modern" method for exact differential equations, then Section 2 can be skipped. If you have not covered that, Section 2 is necessary. Section 3 and 4 are required, along with encouraging students to understand process rather than memorizing the formulas. The task in the final section is the "point" of the series as it shows that Euler derived the modern integrating factor. But, that can be assigned as homework as well.

This is a doable activity in one 50-minute class period, depending on students' background. though it may be best just to be safe and schedule one and half periods as it can be very rushed for students unfamiliar with exact differential equations or for which Section 2 doesn't make sense. To be safe completing this in 50 minutes, I would drop Task 9. If needed, I'll work more slowly through the first task in Section 4 and then leave the second example as homework.

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students. This project is typically done in groups.[16] One reviewer warned that, "The groups often want to take a divide and conquer approach, which is just utterly useless for these documents, because the only person who is going to make any progress is the person who is working on Intro/Section 1. These are all designed to be read top to bottom in slow careful detail, and the later parts of the PSP rarely make any sense unless you've seen the earlier parts."

Connections to other Primary Source Projects

As mentioned above, this mini-PSP is part of a series of three, all which are intended for use in an Ordinary Differential Equations course.

[16]...though with COVID, who knows?!?!

- Solving Linear First-Order Differential Equations: Gottfried Leibniz' Intuition and Check Method.
- Solving Linear First-Order Differential Equations: Johann Bernoulli's (Almost) Variation of Parameters Method.
- Solving Linear First-Order Differential Equations: Leonard Euler's Integrating Factor Method.

Additionally, the author has written a fourth mini-PSP for use in an Ordinary Differential Equations course, based on works by Peano:

- Wronskians and Linear Independence: A Theorem Misunderstood by Many.

All of the above projects can be found at `https://digitalcommons.ursinus.edu/triumphs_differ/`.

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.