Investigating Difference Equations

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1 Introduction

Early in the eighteenth century a number of mathematicians were interested in the solving what we now call *difference equations,* or recurrence relations. Abraham De Moivre (1667-1754) is often given credit for the first systematic method for solving a general linear difference equation with constant coefficients. He did this by creating and using a general theory that he named recurring series. A classic example that pops up throughout mathematical history is the sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots \tag{1}$$

that is obtained by starting with the terms $1, 1$ and thereafter computing each term by summing the two preceding terms. In modern terms the difference equation governing this sequence is $a_k = a_{k-1} + a_{k-2}$, and we are interested in solving it; that is, finding a formula for a_k in terms of k. This is quite useful if we want to find a distant term with large k using the defining recursion $a_k = a_{k-1} + a_{k-2}$. As De Moivre himself put it, "if that place be very distant from the beginning of the series, the continuation of those terms may prove laborious". For our example, we can quickly find the next term $8 + 13 = 21$, but what about the thousandth term?

Here is a simpler example: solve the difference equation $a_k = a_{k-1} + 3$ beginning with $a_0 = 2$. After working out that $a_1 = 2 + 3 = 5$, $a_2 = 8$, $a_3 = 11$ and so on, you can probably see the pattern and write down the solution $a_k = 2 + 3k$. The example (1) above is more challenging, because the given rule to find a term a_k depends on *two* preceding terms, and a formula for a_k in terms of k is not so obvious.

Exercise 1 *Write a program in the language of your choice that will find the nth term of* $a_k = a_{k-1} + a_{k-2}$ *given starting terms* $a_0 = 1$, $a_1 = 1$ *and user input for* n*. Check your program with* $n = 7$ *to verify that* $a_7 = 21$ *and then use your program to find* a_{47}*.*

As we shall see in the first half of this project, De Moivre worked out a clever method for solving many types of linear difference equations where the given rule to find a term a_k depends on several preceding terms. He did this over a period of time between 1710 and 1718, and published fundamental components of his method in his 1718 book *Doctrine of Chances* and another 1718 manuscript that he gave to the Royal Society but did not make public for several years because he was hiding some key ideas from a rival.

Another big contribution in this area came in a 1728 paper by Daniel Bernoulli (1700-1782). In particular, Bernoulli came up with a way to solve another class of linear difference equations that De Moivre did not address. We will read about Bernoulli's contributions in the second part of this project. Even more types of linear difference equations were left unsolved by these pioneers, and were tackled later by giants such as Euler, Laplace and Lagrange.

2 De Moivre Recurrence Method

De Moivre developed the notion of a *recurring series*, which is governed by what he termed the *scale* of the relation which we will investigate momentarily. From these ideas he made the following bold claim.

> "In a recurring Series, any Term may be obtained whose place is assigned. It is very plain, from what we have said, that after having taken so many Terms of the Series as there is in the Scale of Relation, the Series may be protracted till it reach the place assigned; however if that place be very distant from the beginning of the Series, the continuation of those Terms may prove laborious, especially if there be many parts in the Scale. But there being frequent Cases wherein that inconveniency may be avoided, it will be proper to shew by what Rule this may be known; and then to shew how we are to proceed."

Here is De Moivre's definition of recurring series:

If a series is so constituted that, no matter how many terms it contains, each successive term always has the same ratios as the previous ones, I call this kind of series a recurring series; here is a series of this type:

$$\begin{array}{ccccccccccc} \mathsf{A} & & \mathsf{B} & & \mathsf{C} & & \mathsf{D} & & \mathsf{E} & & \mathsf{F} \\ 1 & + & 2x & + & 3xx & + & 10x^3 & + & 34x^4 & + & 97x^5 \end{array} \text{ etc., in which}$$

$D = 3Cx - 2Bxx + 5Ax^3$
$E = 3Dx - 2Cxx + 5Bx^3$
$F = 3Ex - 2Dxx + 5Cx^3$
Etc.

However, the quantities $3x - 2xx + 5x^3$ or even $3 - 2 + 5$ taken together and connected with their own individual signs I call the index or scale of the relation.

Many authors in De Moivre's day, such as Newton and Euler, wrote xx instead of x^2. In modern notation we could write De Moivre's example series above as $\sum_{n=0}^{\infty} a_n x^n$ with $a_0 = 1, a_1 = 2, a_2 = 3$ and so on. We will use his terms "index or scale" interchangeably.

Exercise 2 *Find a_6 and a_7 in De Moivre's example series using the scale (index) of the relation.*

Exercise 3 *What is the corresponding difference equation for De Moivre's example series in modern notation?*

Exercise 4 *Write out the first seven terms of the recurring series for the introductory sequence (1). What is the scale for the relation?*

Exercise 5 *Write out the first seven terms of the recurring series with scale $2x + 3xx$ for the sequence*

$$-1, 3, 3, 15, 39, \cdots \tag{2}$$

What is the corresponding difference equation in modern notation?

De Moivre explains how to sum up these infinite series into a nice neat formula as follows (from the 1718 *Doctrine of Chances*).

LEMMA II.

If in any series, the terms A, B, C, D, E, F etc. be continually decreasing, and be so related to one another that each of them may have to the same number of preceding terms a given relation, always expressible by the same index; I say, that the sum of all the terms of that series ad infinitum may always be obtained.

First, let the relation of each term to the two preceding ones be expressed in this manner, viz. Let C be = m Bx + n Axx; and let D likewise be = m Cx + n Bxx, and so on. The sum of that infinite series be equal to $\dfrac{\mathsf{A + B - mxA}}{\mathsf{1 - mx - nxx}}$. [1]

Thus, if it be proposed to find the sum of the following series, viz.

$$\begin{array}{ccccccccccccccc} \mathsf{A} & & \mathsf{B} & & \mathsf{C} & & \mathsf{D} & & \mathsf{E} & & \mathsf{F} & & \mathsf{G} & & \\ 1 & + & 3\mathsf{x} & + & 5\mathsf{xx} & + & 7\mathsf{x}^3 & + & 9\mathsf{x}^4 & + & 11\mathsf{x}^5 & + & 13\mathsf{x}^6 & & \text{etc.} \end{array}$$

whose terms are related to one another in this manner, viz. C = 2x B - 1 xx, D = 2x C - 1 xx B etc. Let m and n be respectively equal to 2 and -1, and these numerical quantities being substituted, in the room of the literal ones, in the general theorem, the sum of the terms of the foregoing series will be found equal to $\dfrac{\mathsf{1 + 3x - 2x}}{\mathsf{1 - 2x + xx}}$, or to $\dfrac{\mathsf{1 + x}}{\mathsf{(1 - x)^2}}$. [2]

Exercise 6 *Use De Moivre's Lemma II to find the sum of the series in Exercise 4.*

[1] De Moivre actually uses "$-n$" where we have "$+n$" throughout Lemma II and its Demonstration below. This notational change was made for clarity at the urging of the first students working through this Primary Source Project.

[2] De Moivre actually wrote $\overline{1-x}|^2$ instead of $(1-x)^2$ in the denominator. Parentheses notation was not universally used in 1718.

Exercise 7 *Use De Moivre's Lemma II to find the sum of the recurring series generated from the sequence (2).*

Exercise 8 *Find the first four terms of the recurring series for the following sequence, and then use De Moivre's Lemma II to find the sum of the series.*

$$a_0 = 1, a_1 = -3, \quad a_{k+2} = -3a_{k+1} + 10a_k \tag{3}$$

Exercise 9 *Write the Lemma II difference equation using modern notation.*

Exercise 10 *Rewrite the sum formula* $\dfrac{A + B - mxA}{1 - mx - nxx}$ *by substituting modern notation* a_0 *and* a_1 *appropriately.*

Exercise 11 *(Optional) What do you think De Moivre meant by "continually decreasing" in Lemma II? Why do you think he mentioned it? Can you relate this requirement to results you have seen about series in your other math courses?*

Here is De Moivre's proof of Lemma II:

DEMONSTRATION

Let the following scheme be written down, viz.

$$\begin{aligned} A &= A \\ B &= B \\ C &= mBx + nAxx \\ D &= mCx + nBxx \\ E &= mDx + nCxx \\ F &= mEx + nDxx \\ &\quad \text{etc.} \end{aligned}$$

This being done, if the sum of the terms A, B, C, D, E, F etc. ad infinitum, composing the first column, be supposed equal to S, then the sum of the terms of the other two columns will be found thus: by hypothesis, $A + B + C + D + E$ etc. $= S$, or $B + C + D + E$ etc. $= S - A$; and multiplying both sides of this equation by mx, it will follow that

$$mBx + mCx + mDx + mEx \text{ etc. } = mxS - mxA.$$

Again, adding $A + B$ to both sides, we shall have the sum of the terms of the second column, viz. $A + B + mBx + mCx + mDx$ etc. equal to $A + B + mxS - mxA$. The sum of the terms of the third column will be found by bare inspection to be $nxxS$. But the sum of the terms contained in the first column, is equal to the other two sums contained in the other two columns. Wherefore the following equation will be had, viz. $S = A + B + mxS - mxA + nxxS$; from whence it follows that the value of S, or the sum of all the terms $A + B + C + D + E$ etc. will be equal to $\dfrac{A + B - mxA}{1 - mx - nxx}$.

Exercise 12 *Work out the details for the last step in the proof for obtaining the final claim "the sum of all the terms $A+B+C+D+E$ etc. will be equal to $\dfrac{A+B-mxA}{1-mx-nxx}$."*

De Moivre next writes down the analogous result for summing the recurring series with the three term scale $mx+nxx+px^3$, claiming the sum of all the terms $A+B+C+D+E$ etc. will be equal to

$$\frac{A+B+C-mxA-mxB-nxxA}{1-mx-nxx-px^3} \tag{4}$$

but he doesn't supply the proof.

Exercise 13 *Prove this 3-scale summation result. Hint: Mimic De Moivre's proof method for the 2-term scale version.*

Exercise 14 *Explain how to extend De Moivre's summing method to a recurring series with a 4-term scale $mx+nxx+px^3+qx^4$.*

Exercise 15 *Prove your 4-scale summing method is correct.*

De Moivre also produces the sum for the 5-scale and then states that "The law of the continuation of these theorems being manifest, they may be all easily comprehended under one general rule." This is a pretty common approach at the time of De Moivre: prove a result for small orders two, three and four, spot the general pattern, and then write down a general rule.

Exercise 16 *(Optional) De Moivre doesn't explicitly state this general Rule. See if you can use modern notation to do so for a scale of length L with difference equation $a_{k+L}=\sum_{j=0}^{L-1} c_j a_{k+j}$*

Exercise 17 *Use De Moivre's proof technique to find the geometric series sum $a+ax+ax^2+ax^3+\cdots$*

The theory of recurring series that De Moivre published in the *Doctrine of Chances* is certainly some clever mathematics, but it is not obvious how it can be used to solve difference equations. That is, how "any Term may be obtained whose place is assigned" using recurring series. In fact, De Moivre deliberately omitted the rest of his method from publication because he was involved in a heated rivalry with another mathematician named Pierre Rémond de Montmort (1678-1719). However, De Moivre wanted to ensure his claim to his guarded methods, so he very publicly gave a copy of his manuscript to Sir Isaac Newton (1643-1727) and the Royal Society in 1718, where it stayed in a vault until Montmort died in 1720. After that, De Moivre published his results for the world to see. Let's take a look! Here is the first result from this 1722 paper.

Proposition I.

Let there be any fraction , as $\dfrac{1}{1-ex+fx^2-gx^3}$ etc. whose numerator is any given quantity, and denominator a multinomial anyhow composed of the given quantities $1, e, f, g$ etc. and the indeterminant x; I say that the fraction above will be be reducible to more simple fractions.

Case 1. Let the proposed fraction be $\dfrac{1}{1-ex+fx^2}$; making $x^2-ex+f=0$, let r, p be the roots; put $A=\dfrac{r}{r-p}$ and $B=\dfrac{p}{p-r}$ and the proposed fraction will be equal to the sum $\dfrac{A}{1-rx}+\dfrac{B}{1-px}$

Case 2. Let the proposed fraction be $\dfrac{1}{1-ex+fx^2-gx^3}$; suppose $x^3-ex^2+fx-g=0$, let r, p, q be the roots of this equation; put $A=\dfrac{r^2}{(r-p)(r-q)}$ and $B=\dfrac{p^2}{(p-r)(p-q)}$, $C=\dfrac{q^2}{(q-r)(q-p)}$ and the proposed fraction will be equal to the sum $\dfrac{A}{1-rx}+\dfrac{B}{1-px}+\dfrac{C}{1-qx}$

Let's take a closer look at this result.

Exercise 18 *Use De Moivre's formulas to reduce* $\dfrac{1}{1+3x-10x^2}$ *to "more simple fractions."*

Exercise 19 *De Moivre's Proposition I should remind you of partial fractions in Calculus, where you split up rational functions. Do you recall formulas like his for* A, B, C *or are these new to you? Use the partial fractions method to reduce* $\dfrac{1}{1+3x-10x^2}$ *to "more simple fractions."*

Exercise 20 *What happens when you try De Moivre's method of reducing fractions for the sum in his Lemma II example* $\dfrac{1}{1-2x+x^2}$*?*

Exercise 21 *Based on what you found in Exercise 20, clarify De Moivre's claim for Case 1 of Proposition I, and then prove that De Moivre's Case 1 formula for a quadratic denominator is valid with your clarification.*

De Moivre goes on to give explicit formulas for two more cases, when the denominator degree is four and five. He also makes some very brief comments on repeated and complex roots without proof or examples.

After showing how to "reduce" fractions, De Moivre tells how to write them as infinite series in his Proposition III. As in Proposition I, r, p are the roots of $x^2-ex+f=0$.

Proposition III.

If unity be divided by a trinomial,however composed of the given quantities $1, e, f, g$ etc. and the indeterminant x; I say that every term of this series resulting from this division will be assignable.

Let the trinomial be $1-ex+fx^2$; suppose $x^2-ex+f=0$, and let $n+1$ be the place of the desired term; that is, let n express the interval between the first term and the term sought. Make $A=\dfrac{r}{r-p}$ and $B=\dfrac{p}{p-r}$ and the desired term will be, viz. $(Ar^n+Bp^n)\cdot x^n$

Exercise 22 *According to De Moivre's Proposition III, what is the formula for the general term* $(Ar^n + Bp^n) \cdot x^n$ *of the infinite series resulting from* $\dfrac{1}{1+3x-10x^2}$*? Use it to write out the first four terms of the infinite series.*

Exercise 23 *Use De Moivre's Proposition I and Proposition III to find the formula for the general term* $(Ar^n + Bp^n) \cdot x^n$ *of the infinite series resulting from* $\dfrac{1}{1-x-x^2}$*. You should find the roots involve radicals, so simplifying the numerical values of* $(Ar^n + Bp^n)$ *exactly by hand is tedious for large* n*. For now, just use a calculator or a computer algebra system to find the first four terms.*

De Moivre doesn't produce a proof of this Proposition III in his 1722 paper, but his later writings suggest that he used the sum of geometric series

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots = \sum_{n=0}^{\infty} y^n \tag{5}$$

to find these formulas. Geometric series were certainly well known during the time of De Moivre.

Exercise 24 *Use the geometric series expansion (5) together with De Moivre's Proposition I to prove his Proposition III.*

De Moivre also give an analogous result for the cubic polynomial case, and states that "the same law will hold good in any multinomial whatsoever." Rather than pursue this path of higher degree polynomials, let's see if we can use what we have to finish our investigation of solving difference equations for 2-scale relations.

Let's reflect now on the tools De Moivre has given us for solving a 2-scale linear difference equation with sequence $a_0, a_1, \ldots$. First, in 1718 De Moivre explained how to write the corresponding recurring series as a rational function. Second, in 1722 he explained how a rational function can be expanded as an infinite series and gives a formula for the nth degree term. These two infinite series must be equal, so their x^n coefficients must be equal for each n. This gives us an explicit formula for a_n.

Let's apply this to the sequence (3) and solve the difference equation $a_{k+2} = -3a_{k+1} + 10a_k$.

Exercise 25 *Use your results from Exercises 8 and 22 to find two infinite series equal to* $\dfrac{1}{1+3x-10x^2}$*. Since they are equal, their* x^n *coefficients must be equal for each* n*. What is this nth coefficient formula? Explain why this give you a formula for the nth term* a_n *in the sequence (3). Equivalently, this gives you a solution to the difference equation* $a_{n+2} = -3a_{n+1} + 10a_n$ *with initial values* $a_0 = 1$*,* $a_1 = -3$*! Verify your solution for* a_n*,* $n = 0, 1, 2, 3$ *using sequence (3).*

Exercise 26 *Use the ideas above to solve the difference equation* $a_{n+2} = a_{n+1} + a_n$ *with initial values* $a_0 = a_1 = 1$*. The results from exercises 6 and 23 will be helpful. Verify your solution for* a_n*,*

$n = 0, 1, 2, 3$ *using sequence (1). 6 we know* $\dfrac{1}{1-x-x^2} = \sum_{n=0}^{\infty} a_n x^n = 1 + 1x + 2xx + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \cdots$ *so we by matching coefficients of* a_n *we have*

$$a_n = \frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1-\sqrt{5}}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Plug in $n = 0, 1, 2, 3$ *to verify.*

De Moivre's method needs adjustments for some 2-scale difference equations. In Exercise 7, you showed that the recurring series for sequence (2) sums to $\dfrac{5x-1}{1-2x-3x^2}$. Unfortunately, in this 1722 paper De Moivre didn't directly address this situation where the numerator is not simply one ("unity"), leaving some work for the reader. Let's try it now.

Exercise 27 *Use Partial Fractions to rewrite* $\dfrac{5x-1}{1-2x-3x^2}$ *into a sum* $\dfrac{N_1}{1-3x} + \dfrac{N_2}{1+x}$*. Then use geometric series to rewrite this sum as the sum of two series. Finally, combine these series into a single infinite series.*

Exercise 28 *Now you have two different ways of writing* $\dfrac{5x-1}{1-2x-3x^2}$ *as an infinite series, using Exercises 7 and 27. Since the two series are equal, their* x^n *coefficients must be equal for each* n*. Use this fact to find a solution to the sequence (2). Check your work by finding the tenth term* a_9 *in the sequence using your solution, and also using the original definition of the sequence* $a_{k+2} = 2a_{k+1} + 3a_k$*. Of course your values should be the same!*

Exercise 29 *Use the ideas above to solve the difference equation* $a_{n+2} = 5a_{n+1} - 6a_n$ *with initial values* $a_0 = 12$, $a_1 = 29$.

Exercise 30 *(Putting it all together) Using your work thus far, write out an algorithm using De Moivre's ideas to solve a 2-scale difference equation* $a_{k+2} = m \cdot a_{k+1} - n \cdot a_k$ *where* a_0 *and* a_1 *are given. Be sure to clarify the situations where the algorithm does and doesn't apply.*

De Moivre's method works beautifully for longer scales, although some of the formulas are more complex, and there are more cases to consider.

Exercise 31 *(Optional) Write out a general approach to solving 3-scale difference equations using De Moivre's results, especially Lemma II and formula (4), as well as Case 2 of Proposition I. Test out your approach by solving the difference equation*

$$a_{n+3} = -a_{n+2} + 4a_{n+1} + 4a_n, \quad a_0 = 1, a_1 = -1, a_2 = 5$$

Hint: $\pm 2, -1$ *are roots of a key polynomial for this example.*

Can you see some cases where this method might run into difficulties?

3 D. Bernoulli Recurrence Method

After De Moivre's pioneering work, Daniel Bernoulli wrote an important 1728 article on this topic [1]. Here is an excerpt from his introduction. Note his use of the term "series" where we would say "sequence".

1. It was nearly five years ago, while I was in Venice, that I had communicated certain hasty observations about series to a Venetian Nobleman, which, shortly afterward, that Nobleman arranged to have printed, but in my name, together with other geometrical theorems, under the title *Geometrical Exercises*. There I had made mention of series in which each term is equal to the sum of the two preceding terms, not knowing that those had been treated, first by Kepler, afterwards by Mr. Cassini, and finally had been explored with great success and indeed in a much more general form by the Celebrated Geometers, Messrs. Montmort, Moivre, Goldbach, my Cousin Nicolas Bernoulli, and others. Indeed, those most learned men had accomplished something that at that time I did not think possible, namely, to find a general term for all series of that kind, of which I had produced an example contrary to the view expressed by the most erudite Mr. Goldbach, *that all series that follow a definite law of progression can be reduced to some formula or general term*. Mr. Nicolas Bernoulli first informed me of this in a letter of 21 Nov. 1724, sent to me in Venice, and afterwards so did Mr. Goldbach himself, both adding formulas of their own for the general term of the series in question. Having grasped these things, I had soon penetrated the whole mystery of the problem by myself, and indeed to the extent that it seemed that nothing further could be desired in the matter; nor indeed would I have thought any further about those series, had not my aforementioned Cousin recommended a new examination of them in his most recent letter to me of 22 August 1728.

In the next three sections of this paper, Bernoulli discusses a different type of sequence outside the scope of this project. In Section 5, he takes up our problem of finding the general term of a sequence generated by a difference equation. Here is a key lemma from his Section 5, followed by his Sections 6 and 7.

Lemma. Whatever may be the coefficients $m, n, p, ...q$, it will always be possible to exhibit a series of numbers in continued proportion, such that, again denoting by the letters $A, B, C, D, ...E$ contiguous terms in reverse order taken from the series, the number of which is indicated by N, we have

$$A = mB + nC + pD + \ldots + qE. \tag{6}$$

Demonstration. Indeed, if we let the general term in the required geometric progression be r^x, and if this is taken to be equal to A, then
$B = r^{x-1}$, $C = r^{x-2}$, $D = r^{x-3}$ $\ldots$ $E = r^{x-N+1}$,

and from this, by setting up an equation according to the law of the proposition and dividing by r^{x-N+1}, we will have

$$r^{N-1} = m.r^{N-2} + n.r^{N-3} + p.r^{N-4} + \ldots + q, \tag{7}$$

by means of which equation, which I call the *primary* equation, the value of r will be found, and since the equation has $N-1$ roots, just that many geometric progressions will be obtained satisfying the requirement, of which any can be multiplied by a constant number.

6. If now the roots of the preceding equation (7) are $P, Q, R.....S$, it is clear without any difficulty, that all possible series satisfying the condition of the preceding lemma are comprehended under this general term

$$\beta.P^x + \gamma.Q^x + \delta.R^x \ldots + \epsilon.S^x, \tag{8}$$

and whenever these same series begin with as many arbitrary terms as there are units in $N-1$, that is, as many as there are roots $P, Q, R, ...S$, the coefficients $\beta, \gamma, \delta, \ldots \epsilon$ will serve to determine those arbitrary terms, whence therefore the universal method for finding the general terms of all our [recurring] series becomes clear.

7. It will be preferable indeed to illustrate the rule we have set forth here by a particular example, rather than to explain it further in words. Suppose that the general term of the following series is to be found.

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \quad \text{etc.} \tag{9}$$

in which any term is the sum of the two preceding, and which begins with two arbitrary terms $1, 1$. The *primary* equation (6) of Section 5 will be

$$rr = r + 1,$$

whose roots are

$$r = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad r = \frac{1-\sqrt{5}}{2},$$

to be indicated by the letters P and Q, whence the general term of all series, whose terms are everywhere equal to the sum of the two preceding, becomes

$$\beta\left(\frac{1+\sqrt{5}}{2}\right)^x + \gamma\left(\frac{1-\sqrt{5}}{2}\right)^x,$$

which will serve as a particular example of the method we have presented, if, taking successively $x = 0$, $x = 1$ the resulting quantities are taken equal to nothing and to unity (which are the terms whose exponents are 0 and 1). Therefore

$$\beta + \gamma = 0, \quad \text{and} \quad \beta\left(\frac{1+\sqrt{5}}{2}\right) + \gamma\left(\frac{1-\sqrt{5}}{2}\right) = 1\,; \tag{10}$$

or

$$\beta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \gamma = -\frac{1}{\sqrt{5}},$$

so that finally the general term of the given series would be

$$\left[\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1-\sqrt{5}}{2}\right)^x\right] : \sqrt{5}. \tag{11}$$

There is quite a bit to digest in this material! The next set of exercises should help you think through what Bernoulli is saying.

Exercise 32 *Write out difference equation (6) in modern notation, with $A = a_{x+N}$ using x as our indexing variable like Bernoulli.*

Let's use the familiar example of Section 7 to make sure you fully understand the notation in Bernoulli's Sections 5 and 6.

Exercise 33 *Set $A = r^x, B = r^{x-1}, C = r^{x-2}$ and write out equation (6) for sequence (9). Then show how Bernoulli derived the primary equation $rr = r + 1$.*

Exercise 34 *Clarify the algebraic steps Bernoulli used to get the equations in (10) and then $\beta = \frac{1}{\sqrt{5}}, \gamma = -\frac{1}{\sqrt{5}}$ from the information above (10).*

Exercise 35 *Note that when Bernoulli uses $x = 0$, he is in effect creating a term a_0 that he doesn't list in (9), and he has chosen $a_1 = 1, a_2 = 1$. Give the modern sequence notation for the sequence (9) and the corresponding difference equation. Verify that letting $a_0 = 0$ is consistent with the difference equation.*

Exercise 36 *Explain why Bernoulli's solution (11) looks different than the solution you found in Exercise (26).*

Exercise 37 *In his Section 7 example Bernoulli mentions "arbitrary terms". Suppose that instead of $a_1 = 1, a_2 = 1$, this example began with $a_1 = 0, a_2 = 3$. Find the next two terms, and also solve for the general term a_n using Bernoulli's method.*

Exercise 38 *Now use Bernoulli's ideas to reexamine the sequence (2) discussed in Exercise 5 from the Bernoulli perspective, with "arbitrary terms" $a_0 = -1, a_1 = 3$. Apply Bernoulli's Lemma to this sequence: find and solve the primary equation.*

Exercise 39 *Now use Bernoulli's method from his Section 6 to find the general term a_n for the sequence (2).*

Exercise 40 *Let's revisit Bernoulli's proof of his Lemma in Section 5. He makes the claim about solutions r^x that "of which any can be multiplied by a constant number", but he doesn't really back this up. Clarify what he meant here, and why it is true.*

Exercise 41 *At the beginning of Section 6, Bernoulli states that "it is clear without any difficulty, that all possible series satisfying the condition of the preceding lemma are comprehended under this general term", expression (8), but he doesn't provide a proof of his claim. Give a proof in the following case with just two roots considered: Assume P and Q are distinct roots of (7), and show that $a_x = \beta.P^x + \gamma.Q^x$ satisfies the corresponding difference equation (6) for all $x = 3, \ldots$*

Exercise 42 *Later in Section 6, Bernoulli states that "the coefficients $\beta, \gamma, \delta, \ldots \epsilon$ will serve to determine those arbitrary terms, whence therefore the universal method for finding the general terms of all our [recurring] series becomes clear". It is not obvious that we can always find the coefficients from the arbitrary terms. See if you can prove this when there are just two distinct roots P and Q. That is, find formulas for coefficients β, γ in terms of P, Q and "arbitrary terms" a_0 and a_1. Where do you use the fact that $P \neq Q$?*

Exercise 43 *Recall that Bernoulli was well aware of De Moivre's work in this area. Compare and contrast the methods of these mathematicians for solving difference equations where the primary equation has two distinct roots. Start by discussing De Moivre's scale or index and Bernoulli's "primary equation".*

Bernoulli now goes on to tackle the problem of roots of multiplicity greater than one, an issue not directly addressed by De Moivre.

8. It is plain from this that one could hardly exhibit the general term, unless there were as many roots of the primary equation as there were terms that concur to form the sequence. What then if in that equation two or more roots were the same? But a remedy for this difficulty will be brought about, if the root in the general term is understood to be multiplied by

$$b + cx + dxx \ldots + ex^{m-1},$$

where m indicates how many times the root is contained in the equation, and this is to be observed for each of the roots. E.g., let the general term of this series

$$0, 0, 0, 0, 1, 0, 15, -10, 165, -228, \quad \text{etc.} \tag{12}$$

be sought for, beginning with five arbitrary terms, and the law of which requires that everywhere

$$A = 0\,B + 15C - 10D - 60E + 72F.$$

Here the *primary* equation of the lemma gives

$$r^5 - 15r^3 + 10rr + 60r - 72 = 0,$$

of which the five roots are

$$r = 2, \quad r = 2, \quad r = 2, \quad r = -3, \quad \text{and } r = -3.$$

I say therefore that the general term of the proposed series will be

$$(b + cx + dxx) \cdot 2^x + (e + fx) \cdot (-3)^x \tag{13}$$

and that the values of the letters $b, c, d, e,$ and f can be obtained by five comparisons of the general term just constructed with the corresponding terms of the series, so that finally the general term will be

$$\frac{(1026 - 1035x + 225xx) \cdot 2^x + (224 - 80x) \cdot (-3)^x}{90000}.$$

Exercise 44 *Write the difference equation for the sequence (12) using modern a_n notation, and calculate a_{11} both from the recursion and the general term formula given by Bernoulli. Remember that Bernoulli uses $x = 1$ for the first term. For example, $a_7 = 15$ for his formula.*

Exercise 45 *With the aid of a computer algebra system, verify the algebra needed to find the general term*

$$a_x = \frac{(1026 - 1035x + 225xx) \cdot 2^x + (224 - 80x) \cdot (-3)^x}{90000}$$

Bernoulli did this by hand - give him a salute!

Note that Bernoulli's new method in Section 8 allows us to do more than what De Moivre presented.

Exercise 46 *Use Bernoulli's method to find the general term for the sequence $1, 3, 5, 7, 9, 11, \ldots$ discussed in De Moivre's Lemma II (page 3).*

Bernoulli doesn't attempt to justify his "general term" formula (13) beyond this example. In the next exercise, you will prove it is valid in the special case where the primary equation is quadratic and has just one repeated root.

Exercise 47 *Suppose the primary equation for a sequence $\{a_n\}$ is quadratic and has just one repeated root r, so according to Bernoulli the solution is $a_n = (b + cn) \cdot r^n$. Prove this claim.*

Exercise 48 *For the special case in Exercise 47 above, verify that you can always find the coefficients b, c from the arbitrary terms. That is, find formulas for coefficients b and c in terms of "arbitrary terms" a_0 and a_1.*

Unfortunately, Bernoulli doesn't discuss how he came up with his results for repeated roots. Nevertheless we can speculate about his thinking. This even brings De Moivre back into the conversation as well!

Bernoulli was well aware of Newton's celebrated infinite binomial series expansion, which can be stated for our purposes as

$$\frac{1}{(1-rx)^m} = \sum_{k=0}^{\infty} \binom{k+m-1}{k} r^k x^k \tag{14}$$

for positive integer r, where we use binomial coefficient notation $\binom{k+m-1}{k} = \dfrac{(k+m-1)(k+m-2)\cdots(k+1)}{(m-1)(m-2)\cdots(2)(1)}$. If Bernoulli used this fact along with De Moivre's Lemma II method for summing infinite series, he could find the general term in a recurring series with repeated roots in the primary equation. Let's explore this with an example.

Consider the recurring series

$$1 + 6x + 24x^2 + 80x^3 + \cdots \tag{15}$$

where the coefficients satisfy the difference equation $a_{n+3} = 6a_{n+2} - 12a_{n+1} + 8a_n$ with initial values $a_0 = 1, a_1 = 6, a_2 = 24$.

Exercise 49 *Use De Moivre's Lemma II equation (4) to sum up this series (15).*

Exercise 50 *Use Newton's binomial series expansion (14) to expand the sum of the series you found in the previous exercise. Hint: the denominator is a perfect cube.*

Exercise 51 *Now combine your results to find the coefficient x^n in the recurring series* $1 + 6x + 24x^2 + 80x^3 + \cdots$

Once Bernoulli had this general form when his primary equation has just repeated roots, he might have integrated it with his "undetermined coefficients" approach to create (13).

Exercise 52 *Use Bernoulli's method with (13) to directly solve the difference equation $a_{n+3} = 6a_{n+2} - 12a_{n+1} + 8a_n$ with initial values $a_0 = 1, a_1 = 6, a_2 = 24$.*

Bernoulli did not investigate the case where his primary equation has complex roots. Euler tackled this problem in Chapter XIII of his 1740 masterpiece *Introduction to Analysis of the Infinite* [4]. With some regret we chose not to include Euler's contribution in this project to keep the time frame manageable. For the inquisitive reader, Euler's treatment is well written, if rather technical.

References

[1] Bernoulli, D. 1728. Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* Bd. III: 85-100. Translation by Stacy Langton, personal communication.

[2] De Moivre, A. 1718. *The Doctrine of Chances, or, a Method of Calculating the Probability of Events in Play*. London: W. Pearson.

[3] De Moivre, A. 1722. De fractionibus algebrais radicalitate immunibus ad fractiones simpliciores reducendis, de que summandis terminis quarundam serierum aequali intervallo a se distantibus. *Philisophical Transactions* 32: 162-178.

[4] Euler, L. 1988. *Introduction to Analysis of the Infinite, Book I.* New York: Springer-Verlag.