NAME:____________________________

MATH 112 EXAM 3

December 8, 2010

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam.

- Print your name clearly in the space provided.
- You may use a calculator.

HONOR STATEMENT:

I have neither given nor received help on this exam, and all of the answers are my own.

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Signature

Question	Points	Score
1	10	
2	15	
3	10	
4	12	
5	12	
6	21	
7	20	
8	0	
Total:	100	

You may use

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots \text{ for} |x| < 1.$$

1. [10 points] Choose either (a) or (b).

(a) Determine whether or not $\sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}}$ converges.

$b_n = \frac{1}{n^2}$

$$\lim \frac{a_n}{b_n} = \lim \frac{\frac{1}{n^2-\sqrt{n}}}{1/n^2} = \lim \frac{n^2}{n^2-\sqrt{n}} = 1$$

So converges since $L > 0$ and b_n converges.

(b) Determine whether or not $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

$$\int_2^{\infty} \frac{1}{x \ln x^2}\, dx = \lim_R \int_2^R \frac{1}{x(\ln x)^2}\, dx = \lim_R \int_{\ln 2}^{\ln R} du$$

$$= -\lim_R \left(\frac{1}{\ln R} - \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$$ so converge.

2. [15 points] Use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ to find the exact value of $\sum_{n=2}^{\infty} \left(2^{n+1} + \frac{3^n}{n^2}\right) \frac{1}{3^n}$.

$$\sum_{n=2}^{\infty} \frac{2^{n+1}}{3^n} + \frac{1}{n^2} = 2\sum_{n=2}^{\infty} \frac{2^n}{3^n} + \sum_{n=2}^{\infty} \frac{1}{n^2}$$

$$= 2\left(\frac{4/3}{1-2/3}\right) + \frac{\pi^2}{6} - 1$$

$$= 2(4/3) + \frac{\pi^2}{6} - 1$$

$$= 5/3 + \frac{\pi^2}{6}$$

3. [10 points] Use the ratio test to determine whether or not $\sum_{n=1}^{\infty} n! e^{-n^2}$ converges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_n \frac{(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{n!} = \lim_n \frac{n+1}{e^{2n+1}} \longrightarrow 0$$

So converges

4. [12 points] Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}x^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{[(n+1)!]^3}{[3n+3]!} \cdot \frac{(3n)!}{(n!)^3}$$

$$= \left(\frac{(n+1)!}{n!}\right)^3 \frac{1}{(3n+3)(3n+2)(3n+1)}$$

$$= \frac{(n+1)^3}{27n^3 + \text{lower order terms}}$$

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{27}$$

$$\therefore \left|\frac{x}{27}\right| < 1$$

$$|x| < 27$$

5. [12 points] Find the first three terms of the Maclaurin series for $f(x) = e^x \cos(x)$.

$$\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)$$

$$\Rightarrow 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+x-\frac{x^3}{2!}+\frac{x^5}{4!}+\frac{x^2}{2!}-\frac{x^4}{2!2!}+\frac{x^6}{4!2!}+\frac{x^3}{3!}$$

$$\Rightarrow 1+x+\left(\frac{1}{3!}-\frac{1}{2!}\right)x^3$$

$$=1+x-\frac{1}{3}x^3$$

6. (a) [5 points] Find the Maclaurin series for $\frac{1}{1+x}$.

$$\frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

(b) [10 points] Find the Maclaurin series for $\ln(1+x)$.

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int 1 - x + x^2 - x^3 + x^4 + \cdots \; dx$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + C$$

Let $x = C$. Then $c + \ln(1+0) = C + 0 = 0$

(c) [6 points] Use your answer in (b) to show that

$$\ln\frac{3}{2} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots.$$

Let $x = 1/2$

$$\ln(3/2) = \ln(1 + 1/2) = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^{3}} - \frac{1}{4\cdot 2^4} + \cdots$$

7. [20 points] Label each statement as true or false (no ambiguous letters that look like both a T and an F please). Below, a_n and b_n are sequences.

(a) F If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

(b) T If $a_n \leq b_n$ for all n, then $\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} b_n$

(c) F A convergent sequence is monotonic.

(d) T The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges otherwise.

(e) T If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

(f) F The series $\sum_{n=2}^{\infty} (-1)^n \frac{\sqrt{n}}{\ln n}$ converges absolutely.

(g) T $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$

(h) F The ratio test can be used to determine whether $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

? (think true)

(i) If a_n and b_n are positive monotone-decreasing to 0, then $\sum_{n=1}^{\infty} a_n + b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

(j) F If $\sum_{n=1}^{\infty} a_n 6^n$ is convergent, then $\sum_{n=1}^{\infty} a_n(-6)^n$ is convergent.

8. (Extra Credit) Show that $e^{\pi i} + 1 = 0$ where $i = \sqrt{-1}$. [Hint: Use Maclaurin series]

$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$\vdots$

$$e^{ix} = 1 + \frac{ix}{1} - \frac{x^2}{2} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots$$

$$\cos x + i\sin x = 1 + ix - \frac{x^2}{2} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} -$$

$\therefore\ e^{ix} = \cos x + i\sin x$

Let $x = \pi$

$e^{\pi i} = -1$

$e^{\pi i} + 1 = 0$