MONSTERS IN THE MATHEMATICS CLASSROOM:

Learning analysis through the works of Gaston Darboux

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ABSTRACT

The drama of the rise of rigor in nineteenth century mathematical analysis has now been widely rehearsed. Notable within this saga is the appearance of functions with features so unexpected (e.g., everywhere continuous but nowhere differentiable) that contemporary critics described them as "bizarre", "pathological," or even "monsters." Among the "monster-makers," one of the most influential was Gaston Darboux (1842-1917). This paper reviews Darboux's mathematical and "backstage" contributions to the development of nineteenth century analysis, including some of his own favorite pet monsters, and explores the important role played by these "pathological functions" in the historical re-shaping of analysis. We then consider how these functions can be used in to help students develop a more robust understanding of modern analysis. We examine in particular how Darboux's proof of the result now known as "Darboux's Theorem" (i.e., all derivatives have the intermediate value property) in his 1875 *Mémoire sur les fonctions discontinues* can be used in today's analysis classroom.

1 Introduction

In 1904, Henri Poincaré wrote:

> Logic sometimes creates monsters. For half a century we have seen a host of bizarre functions which appear to be created so as to resemble as little as possible those honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. ….. ….. In former times when one invented a new function it was for a practical purpose; today one invents them expressly to show the defects in the reasoning of our fathers, *and nothing more can be drawn from them*. (Poincaré, 1902, p. 263, emphasis added)

This paper explores the question of whether there *is* something more that today's student of analysis can learn from the strange functions that Poincaré so roundly condemned. We begin in the next section by examining the motivations of one of the foremost "monster makers" of the nineteenth century, Gaston Darboux (1843-1917). In Section 3, we then briefly describe an instructional approach for bringing Darboux's thinking to the classroom through the use of excerpts from his original works in the form of a guided reading module for students. In our closing section, we return to the historical drama and examine Darboux's overall influence on the re-shaping of analysis that began in the late nineteenth century.

2 The Historical Players

In this section, we provide some background information on Darboux, introduce some of the other major players with whom he interacted, and examine his personal motivations for bringing "pathological" functions into the study of analysis, The larger historical drama in which Darboux's story is set – namely, the story of the rise of rigor in analysis in the nineteenth century - has been well rehearsed; see for example (Chorlay, 2016), (Lützen, 2003) and (Hochkirken, 2003). For details concerning Darboux's part in this drama, we also draw extensively on (Gispert, 1983), (Gispert, 1987) and (Gispert, 1990).

2.1 Gaston Darboux: Student, Teacher and Editor *par excellence*

Born on 14 August 1842, Darboux attended Lycée first in Nimes, and later in Montpellier. In 1861, he was admitted to both the École Polytechnique and the École Normale Supérieur; he chose to attend the École Normale. While a student there, he published his first paper on orthogonal surfaces; his 1866 doctoral thesis (under Michel Chasles) was on this same topic. From 1866-1867, Darboux taught at the Collège de France before spending five years at the Lycée Louis le Grand (1867-1872) and another four at the École Normale Supèrieur (1872-1881). He then moved to the Sorbonne where he taught for the remainder of his life.[1] While at the Sorbonne, Darboux demonstrated his excellence as both a teacher and an organizer. For the last 17 years of his life (1900-1917), his talents as an organizer were also put to use in his capacity as the Secrétaire Perpétuel de l'Académie des Sciences.

Darboux also excelled as an organizer and promoter of mathematical research in France. Of particular relevance to the story being told in this paper was his role as a founding editor of the *Bulletin des Sciences*, sometimes referred to as "Darboux's *Bulletin*" in recognition of his role as its co-founder in 1870.[2] The *Bulletin* published lists of titles of research papers from journals from outside of France, as well as summaries of the contents of the more important works and, when possible, complete translations of those papers. In this way, the *Bulletin* sought to provide the French mathematical community with access to cutting-edge mathematical research being conducted elsewhere that, for a variety of issues related to financial and infrastructure, was difficult to obtain inside France at the time. Darboux was especially concerned that, without proper exposure to the new research methodologies and standards then evolving outside of France, the training of future generations of French mathematicians as researchers would be compromised. Echoes of these concerns are heard in an early letter to the co-founder of the *Bulletin,* in which Darboux asserted:

> … we need to mend our [system of] higher education. I think you agree with me that the Germans get the better of us there, as elsewhere. If this continues, I believe the Italians will surpass us before too long. So let us try, with our *Bulletin*, to wake the holy fire and the French understanding that there are many things in the world that they do not suspect, and that even if we are still the Grrrand nation, no one abroad perceives this. [As quoted in (Gispert 1987, p. 160)]

Before looking further at the correspondence between these two men, we briefly introduce the recipient of Darboux's letter: Jules Houël (1823-1886).

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[1] Darboux's first academic position at the Sorbonne was as the assistant to Liouville who was then Chair of Rational Mechanics (1873-1878). From 1878-1880, he served as the assistant to Chasles in Chair of Higher Geometry, and moved into that position himself upon Chasles' death. From 1889-1903, Darboux also served as the Dean of the Faculty of Science at the Sorbonne.

[2] Recall that this was the same year that saw the start of the Franco-Prussian War, in which the French were defeated by a coalition of German states under the leadership of Prussia. Darboux resided in Paris throughout the period of this brief war (July 19, 1870–May 10, 1871).

2.2 Jules Houël: Translater *par excellence*

Senior to Darboux by twenty years, Houël completed Lycée at Caen, and then studied at the Collêge Rollin. Like Darboux, he received his initial mathematical training at the Ècole Normale Supérieur (entering in 1843) and his doctorate (in celestial mechanics) from the Sorbonne (in 1855). He then returned to his home town of Thaon for four years, pursuing mathematical research on his own despite an offer from Urbain Le Verrier for a post at the Paris Observatory. In 1859, Houël accepted the Chair of pure mathematics in Bordeaux, and remained in that position for the remainder of his life.

Prior to joining Darboux as co-founder of the *Bulletin*¸ Houël had already gained a reputation for excellence as a translator. An early proponent of non-Euclidean geometry – he expressed doubts about the parallel postulate as early as 1863, even before learning about the work of Lobachevski and Bolyai – Houël produced French translations of key papers by both these men, as well as other important works in non-Euclidean geometry by Beltrami, Helmholtz, and Riemann. After it was founded in 1870, Houël contributed numerous French translations to the *Bulletin*. Notable among these was his translation of Riemann's 1853 *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe*. First published in German in 1868, it was not until Houël's translation appeared in the *Bulletin* in 1873 that the contents of this important work, including Riemann's treatment of the integral, became generally known in France. That same year marked the beginning of an exchange between the *Bulletin*'s founding co-editors in which several "monster functions" made their debut as Darboux sought to convince Houël of the need for increased rigor in the latter's approach to analysis.

2.3 Monsters in the Darboux-Houël Correspondence

The impetus for the ten-year debate[3] concerning rigor in analysis in the Darboux-Houël correspondence was Houël's request for feedback on preliminary drafts of his intended textbook on differential calculus, eventually published as *Cours de Calcul infinitésimal* in 1878. Throughout this debate, Darboux offered various counterexamples in a (vain) attempt to convince Houël of the need for greater care in certain of his (Houël's) proofs. One such example [4]was the function given in modern notation by:

$$y = f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

As noted by Darboux, it is easy to see that $\lim\limits_{x \to 0} \frac{y}{x} = 0$, so that $f'(0) = 0$. Thus, f is a differentiable function with derivative function:

$$y' = f'(x) = \begin{cases} \cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases},$$

Since the derivative f' becomes indeterminate near $x = 0$, the function f thus provides an example of a differentiable function for which the derivative itself is not continuous. Faced

[3] See (Gispert 1983) for an in-depth analysis of this debate.

[4] See (Gispert, 1983, pp. 55-56) for Darboux's actual discussion of this example in his letters to Houël.

with Darboux's inevitable conclusion that derivatives are not necessarily continuous, Houël essentially responded by saying: *They are for all the functions that I consider!*

Throughout the course of the debate, one can hear Houël's increasing exacerbation with Darboux's examples in his description of them as "drôlatiques" (humorous), "bizarres" (bizarre), "dereglés" (disorderly), "saugrenues" (absurd), and "gênantes" (obstructive). Darboux too became increasingly vexed by Houël's apparent inability to understand the underlying purpose of these examples. The function $y = x^2 \sin(1/x)$, for instance, was put forward by Darboux in an attempt to explain to Houël that the proofs he proposed to include in his calculus textbook often relied on the assumption of *uniform* convergence, whereas only simple convergence had been assumed. Elsewhere in the correspondence, Darboux very explicitly explained this concern as follows:

> In the expression
>
> $$\frac{\mathrm{f(x+h)-f(x)}}{\mathrm{h}} - \mathrm{f'(x)} < \epsilon$$
>
> ϵ is a function of two variables x and h, where h approaches zero with x remaining fixed ….. but if x and h vary as in your proof, then each new subdivision of the interval $\mathrm{x_1 - x_1}$ introduces a new quantity ϵ. [As quoted in (Gispert, 1983, p. 54)]

Darboux's real message to Houël – a message which he tried repeatedly to convey by means of counterexamples and other explanations – might thus be paraphrased as follows:

> Your *proofs* often (implicitly) assume some assumption [e.g., uniform differentiability]. That's fine … but then you need to (explicitly) verify this condition *before* applying your theorems, and you never do! Even if you did bother to do this, you would be better off changing your proofs altogether – why not use the Mean Value Theorem as your foundation? That would be less complicated than verifying, for example, uniform differentiability each time you invoke a theorem for which you gave a proof that depends on it.

Houël's responses over the years as illustrated by the following quotations from their correspondence[5] seem to reveal frustration not only with Darboux's insistence, but with his own inability to even fully understand what Darboux is trying to accomplish with his monsters.

- I completely reject the simultaneous variation. I consider only successive variations.
- Get rid of your preoccupation with this simultaneous variation that does not belong here.
- …. and then you raise fatal worries about the points I thought the best established …

[5] These four quotes are taken from (Gispert, 1983 p. 56), and appeared in letters written by Houël on the following dates in 1875 respectively: January 9, January 24, January 31, February 15.

- I understand what Cauchy said, or when I do not understand, I can see what I am lacking. But your doctrine simply amazes me, and your abyss[6] all the same makes me insane.

2.4 Monsters in Darboux's Published Works in Analysis

Many of the monsters presented in Darboux's private letters to Houël remained hidden away from public sight until the publication study of that correspondence by Helene Gispert between 1983 and 1990. But other of his monster creations appeared in Darboux's three published works in analysis. In this section, we consider the contents of only the most influential[7] of the three, his 1875 publication *Mémoire sur les fonctions discontinues*.

Darboux's strong grasp of recent developments in analysis in the hands of his German contemporaries is well documented throughout his *Mémoire*. His citations included works by Hankel, Gilbert, Klein and Thomae, as well as Riemann's 1853 *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe*. Darboux greatly admired Riemanns' concept of the integral, as the latter described it in a brief (5-6 page) discussion in his 1853 paper. In fact, a primary goal of Darboux's *Mémoire* was to provide a rigorous reformulation of the Riemann integral.[8] Key components of this reformulation included:

- the first clear distinction between the concepts *supremum/infinum* and *maximum/minimum*
- the first definitions of upper and lower integrals;
- the rigorous establishment of necessary and sufficient conditions for integrability;
- the separation of discontinuous functions into two classes: integrable and non-integrable;
- rigorous proofs of the properties of integrable functions, including the following:
    - Every continuous function is integrable.
    - $F(x) = \int_a^x f(y)dy$ is continuous in x.
    - If f is continuous at x_0,
      then $F(x) = \int_a^x f(y)dy$ is differentiable at x_0 with $F'(x_0) = f(x_0)$.

With a rigorous definition of the Riemann integral in hand, Darboux next offered a collection of new functions of his own creation which included in particular a specimen of each of the following "monsters":

- A continuous, no-where differentiable function[9]
  (via a sum of uniformly convergent series of continuous functions):

$$\sum \frac{\sin[1.2.3 \dots (n+1)x]}{1.2.3 \dots n}$$

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[6] Houël is responding here to Darboux's earlier attempts to explain the abyss that exists between simple convergence and uniform convergence.

[7] Darboux's other two analysis publications were (Darboux, 1872) and (Darboux, 1879).

[8] For a discussion of both Riemann's and Darboux's treatment of the integral, see (Hochkirchen, 2003).

[9] Darboux's example of such a monster was, of course, not the only one put forward around this time.

- A continuous function that is neither increasing nor decreasing on any interval

$$\sum a_n \sin(nx\pi)^{2/3}, \text{ where } \sum a_n \text{ is absolutely convergent}$$

- A discontinuous function that satisfies the Intermediate Value Property

$$F(x) = \int_0^x f(y)dy = \sum \frac{a_n}{n} \phi(\sin nx\pi),$$

$$\text{where } \phi(y) = y^2 \sin\left(\frac{1}{y}\right) \text{ and } \sum a_n \text{ is absolutely convergent.}$$

Darboux's proof that this last example possesses Intermediate Value Property followed from a theorem that now bears his name – i.e., every derivative has the Intermediate Value Property. We include his proof of this theorem in the next section, in which we consider what role Darboux's monsters might play in today's analysis classroom.

3 Monsters in the Classroom?

In this section, we turn to the questions of why and how to bring Darboux's monster functions to the classroom. We accept as a given that these functions *do* belong in an analysis course. In fact, most modern undergraduate analysis textbooks already include example such as Darboux's $f(x) = x^a \sin(1/x)$. But missing from these modern treatments is a consideration of the historical context in which these examples were first considered. Why *were* these examples developed in the first place? What mathematical intuitions were refined and in what ways by studying them? Were they even accepted as legitimate examples of functions and, if not, why not? Because most students enter an analysis course with a general understanding of the calculus (and the concept of continuity in particular) that differs little from the views of mathematicians like Houël, exposing them to the historical context by sharing Darboux's motivations for considering such functions can be valuable in a number of ways.

One important role that the history of these functions can play in today's classroom is to help students to develop the more rigorous and critical view of the basic ideas of calculus that an introductory analysis course seeks to achieve. A companion goal is to help them to develop an understanding of the language, techniques and theorems of elementary analysis that developed when mathematicians adopted such a critical perspective in the nineteenth century. To achieve these two goals, it is especially important to focus students' attention on the underlying logical relationship of fundamental notions of analysis such a continuity and convergence, where both these concepts play pivotal roles in Darboux's monster functions.

How best to motivate this change of perspective on students' part and help them to develop an understanding of the mathematics that grew out of it? The answer proposed in this paper is to have students read the actual works of the mathematicians involved. Possible reactions to this proposal include a concern that the reading original sources is too difficult a task for students, and also one that is too far removed from the mathematical goals of the course; an analysis course is, after all, not a course in the history of mathematics. In reply to these legitimate pedagogical concerns, we propose placing the works in question within a "Primary Source Project" (PSP), following a guided reading approach developed with support from the US National Science Foundation.[10]

The aim of this particular approach is to provide students with sufficient guidance to allow them to successfully read an original source, while still allowing them the excitement of directly engaging with the thinking of its author. This is accomplished by interweaving three essential elements within the PSP:

- excerpts from the relevant original source(s);
- secondary commentary that discuss the historical context and mathematical significance of these excerpts; and
- a series of tasks that prompt students to develop their own understanding of the underlying concepts and theory.

For example, by reading selected excerpts from the writings of the nineteenth century mathematicians who led the initiative to raise the level of rigor in the field of analysis – as well as those who resisted or misunderstood this initiative – students' own understanding of and ability to work at the expected level of rigor can be refined. Grappling with the examples presented in Darboux's work through completion of the accompanying tasks also supports students' ability to develop mathematical ideas on their own, as well as their ability to formally communicate those ideas through reading and writing.

Throughout a PSP, the emphasis remains on the *mathematical* concepts and techniques involved, with historical and biographical information on the source's author playing only a supporting role. In this particular case, Darboux's monster functions and Houël's reactions to them thus take center stage. To illustrate how this can be accomplished, we close this section with a brief description of a portion of this particular PSP which is based on the proof of the theorem that now bears Darboux's name – i.e., every derivative has the Intermediate Value Property – from his *Mémoire sur les fonctions discontinues* (1875). (The complete proof in the original French is reproduced below.)

This particular excerpt from Darboux serves as the culmination of the PSP, following students' introduction to several of his monster functions. Students are first prompted to read the proof (in English translation) with those examples in mind. In a series of PSP tasks, students are then asked to provide either proofs or counterexamples from Darboux's menagerie of monster functions as a means to further explore the interplay of continuity, the Intermediate Value Property and anti-differentiability. For example, is the converse of this theorem true; that is, can a function satisfy the Intermediate Value Property without being a derivative? Can a continuous function fail to be anti-differentiable? Can a derivative function fail to be continuous? As a final task, students are asked to critique Darboux's proof from the perspective of modern standards of rigor – how would Darboux's proof fare if he were a student in the course they are completing? – and to provide their

[10] Examples of this approach can be found in a compendium of projects developed and tested scientists for the teaching of topics in discrete mathematics since 2008 by an interdisciplinary team (including the present author) of mathematicians and computer. Elsewhere, members of our team have written about the general pedagogical design goals of our work and analyzed specific projects in the collection relative to those goals. See, for example, (Barnett, 2014), (Barnett, Pengelly & Lodder, 2013) and (Barnett et al, 2011). A new collaboration of faculty from seven US institutions of higher education was recently awarded a five-year collaborative grant entitled *TRansforming Instruction in Undergraduate Mathematic via Primary Historical Sources (TRIUMPHS)* from the US National Science Foundation to continue and expand this work; more information about TRIUMPHS is available at http://webpages.ursinus.edu/nscoville/TRIUMPHS.html.

own version of the same proof, filling in any missing details and/or adapting the language employed to their own personal proof style.

> Soit, en effet, $F(x)$ une fonction dont la dérivée existe pour toute valeur de x, mais soit discontinue. Supposons que, pour $x = x_0$, $x = x_1$, la dérivée prenne les valuers
>
> $$F'(x_0) = A \ , \qquad F'(x_1) = B$$
>
> Je dis que, si x varie de x_0, á x_1, $f'(x)$ passe au moins une fois par toutes les valeurs intermèdaires entre A et B. Soit, en effet, M une des ces valuers,
>
> $$A > M > B,$$
>
> et formons la function
>
> $$F(x) \ - \ Mx.$$
>
> Cette fonction continue aura, pour $x = x_0$, une dérivée $A - M$ positive et, pour $x = x_1$, une dérivée $B - M$ négative.
>
> Elle commencera donc parêtre croissant quand x variera de x_0, á x_1, puis elle finira parêtre dècroissant pour $x = x_1$. Donc elle aura un maximum qu'elle atteindra pour une certaine valeur
>
> $$x_0 + \theta(x_1 - x_0),$$
>
> et pour lequel sa dérivée sera nulle; on aura donc
>
> $$f'(x_0 + \theta(x_1 - x_0)) - M = 0.$$
>
> Ainsi tout nombre M intermèdaire entre A et B est une valeur de la dérivée.
>
> (Darboux, 1875, pp.109-110)

4 Epilogue: Darboux's influence on mathematics, in and out of France

In this section, we return to the historical drama and consider the morals that may be drawn from it by a modern spectator.

Thus far in the tale, we have encountered several important features of Darboux's work in analysis, in both his published papers and his private correspondence with Houël. These include the identification of important conceptual distinctions such as simple versus uniform convergence, supremum/infinum versus maximum/minimum, continuity versus the Intermediate Value Property. Note that each of these topics is typically first encountered by undergraduates in an introductory analysis course. Darboux's skillful use of well-crafted counterexamples to explore the boundaries of function classes thus has a natural place in such a course. As was the case in the nineteenth century, the ways in which monster functions are able to "exemplify the general" can also assist students in making the transition to the more abstract level of thinking required at this level.[11]

Another (as yet unmentioned) feature of Darboux's work that students first seriously encounter in an introductory analysis course is the study of the *pointwise* behaviour of functions. The appearance of this new focus, which was characteristic of the approach taken by the leading nineteenth century analysts (e.g., Riemann, Dirichlet, Darboux), marked a new phase in the study of the calculus. In contrast, the focus throughout the early history of the calculus was on the *global* behaviour of functions, shifting to a focus on *local* behaviour (i.e., across an interval) only with the work of Cauchy.[12] Here again monsters can be helpful by bringing this new focus clearly into view: encountering a function with a positive derivative at a particular point that is not increasing across any interval containing that point underscores well the sharp contrast between these two viewpoints.

Yet another role played by Darboux's monsters was that of a "dissection tool" for uncovering implicit assumptions in a proof. Darboux's expertise with this tool was made especially clear in his letters to Houël (even though Houël himself managed to miss the point). The carefully crafted definitions and proofs that distinguish his published works in analysis offers further evidence of Darboux's mastery of this technique. His rigorous reformulation of the Riemann integral – yet another topic first encountered in an undergraduate analysis course – provides an especially lovely example of this.

In short, Darboux's work in analysis serves as an excellent representation of the changes that occurred in analysis in the latter half of the nineteenth century. His 1875 *Memoire* in particular drew notice even at the time as an outstanding embodiment of the new spirit of analysis, at least outside of France. The Italian analyst Dini cited it alongside works by various German analysts in his own highly commended *Fondamenti per la teoria delle funzio ni di variabili reali* of 1878, but named no other French mathematician. The reason for this was simple: no other French mathematician at the time understood the new direction and standards of analysis sufficiently well to merit international notice.

In fact, Darboux's view of analysis received such a chilly reception from his French colleagues that he eventually gave up analysis as a field of research altogether. The comments from Houël noted in Section 2.3 of this paper illustrate one extreme of the reactions to the new trends in analysis. The extent to which Houël rejected Darboux's view is evidenced by the fact that Houël declined Darboux's advice to base the proofs in his (Houël's) 1878 *Cours de Calcul infinitesimal* on the Mean Value Theorem. By this time, the divergence of their viewpoints on these issues was extremely clear, and Darboux in turn declined Houël's requests to consult on his infinitesimal calculus text.[13]

Even those who appeared to understand and share Darboux's goals appear not to have been directly influenced by Darboux's work. For instance, in the first edition of his *Cours d'analyse* of 1882, Jordan did not employ the Mean Value Theorem in the foundational role

[11] The expression "exemplify the general" is borrowed from Renaud Chorlay; see pp. 7 - 8 of version of (Chorlay 2016), posted on-line at www.sphere.univ-paris-ziderot.fr/IMG/pdf/Chorlay_chapitre_mis_en_forme_V2.pdf

[12] Interestingly, Cauchy also studied the behaviour of the function $f(x) = x^3 \sin(1/x)$ in a neighborhood of *0*, but made no mention of its differentiability properties.

[13] Differences of opinion had also begun to appear in terms of their editorial work as well, although Houël continued in this capacity until his death in 1883.

recommended to Houël by Darboux, yet in the second edition of 1887 he did do so. This change of heart was not a consequence of Darboux's views on this matter, however. Rather, Jordan followed the advice of Peano with whom he corresponded on issues in analysis between 1882 and 1884. As part of Peano's critique of the first edition of Jordan's text within that correspondence, the counterexample $f(x) = x^2 \sin(1/x)$ made yet another appearance. More than just a coincidence, this incident provides yet another indication of the role played by such monsters in helping heedful practitioners of the day understand the need for new standards of rigor in their proofs.

But while their creators viewed the existence of these monsters as a warning of the need for increased vigilance with respect to rigor, many French mathematicians continued to view them instead as evil omens to be avoided.[14] According to Lebesgue, Hermite once lamented to Stieltjes in a letter purportedly written in 1893

> I turn with horror and revulsion from this lamentable plague of functions that can have no derivative whatsoever. (Lebesgue, 1922, p.13)

As for how he himself was treated by his colleagues, Lebesgue further reported that:

> I became the man of the functions without derivatives …. whenever I tried to take part in a mathematical discussion there would always be an analyst who would say, "This won't interest you; we are discussing functions having derivatives." (Lebesgue, 1922, p.13-14)

Yet once set loose by Darboux and others, the monsters themselves could not be ignored. One important reason for this was described by Poincaré (who perhaps lamented their existence more than anyone) as part of the commentary with which this paper opened, quoted her in its entirety:

> Logic sometimes creates monsters. For half a century we have seen a host of bizarre functions which appear to be created so as to resemble as little as possible those honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. *What's more, from the logical point of view, these strange functions are the most common, those that we came across without looking for them now appear to be but a particular case. They are left with but a small corner.* In former times when one invented a new function it was for a practical purpose; today one invents them expressly to show the defects in the reasoning of our fathers, and nothing more can be drawn from them. (Poincaré, 1902, p. 263, emphasis added)

In other words, the monsters are everywhere!

In this paper, we have argued that bringing them into the classroom can provide today's students with a convincing motive for adopting a critical approach to the study of analysis, tools for constructing proofs that meet modern standards of rigor, important cognitive support for their efforts to develop an understanding of introductory analysis concepts, and

[14] The Latin root of the word "monster" is "monere", meaning "to warn". Interestingly, the English word "money" comes from this same Latin root.

more. As Darboux’s student Emile Borel pointed out, there is at least one other reason to become familiar with the monsters:

> Until now, no one could draw a clearly line between *straightforward* and *bizarre* functions; when studying the first you can never be certain you will not come across the others; thus they need to be known, if only to be able to rule them out. (Borel, 1972, p. 120)

Borel’s own contributions to the delineation of the boundaries of the small corner in which these straightforward functions reside, together with the works of Lebesgue and Baire, bear witness to the ultimate fulfilment of Darboux’s hope for the rejuvenation of French analysis.

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