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PRIMARY HISTORICAL SOURCES IN THE CLASSROOM: GRAPH THEORY AND SPANNING TREES

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We examine the pedagogical benefits of teaching from the historical curricular module "Networks and Spanning Trees," based on the original works of Arthur Cayley, Heinz Prüfer and Otakar Borůvka. Cayley identifies a compelling pattern in the enumeration of (labeled) trees, although his counting argument is incomplete. Prüfer provides an alternate proof of "Cayley's formula" by counting all railway networks connecting n towns that contain the least number of segments. Borůvka develops one of the first algorithms for finding a minimal spanning tree by considering how best to connect n towns to an electrical network.

INTRODUCTION

We examine the pedagogical benefits afforded by teaching from primary source documents by studying the details of one classroom module: "Networks and Spanning Trees" (Lodder, 2013). This is only one of several curricular modules covering topics in discrete mathematics and computer science, available online (Barnett 2013). "Networks and Spanning Trees" highlights the work of three scholars, Arthur Cayley (1821—1895), Heinz Prüfer (1896—1934) and Otakar Borůvka (1899—1995) on the use and enumeration of (labeled) trees as well as one of the first algorithms for finding a *minimal spanning tree,* all written before the subject of modern graph had been developed. These historical sources provide context to the subject matter, with the authors stating a compelling problem whose solution involves key concepts or constructions that have become abstract definitions or theorems in present-day textbooks. A verbal description of the problem, without specialized vocabulary, offers a more inviting and understandable entry into the subject. The sources provide motivation for study, once the original problem has been stated and its significance is understood. When arranged over time, the sources provide direction to the subject matter not apparent when reading the final axiomatized version in a textbook. Also, studying from primary sources leads to an interdisciplinary approach to learning, since the sources were almost always written before the modern division of scholarship into collegiate departments. Many concepts in the module "Networks and Spanning Trees" are germane to both discrete mathematics and computer science. In fact, algorithms for finding a minimal spanning tree have been a topic of research in computer science, while combinatorial arguments for counting trees are primarily taught in mathematics courses.

In terms of a theoretical framework, the classroom module "Networks and Spanning Trees" uses history-as-tool (Jankvist, 2009) to learn the inner issues of graph theory and some of its applications. To elaborate, the primary goal of introducing historical sources in the classroom is to learn mathematics. The history of the subject is used as a tool to help in the understanding of mathematics. The inner issues refer to the lemmas, theorems, procedures or reasoning processes within the subject. In this module we read that Prüfer is motivated by finding all ways of connecting *n* towns to a railway network using the least number of railway segments as possible. Borůvka is motivated by a different problem, namely of all possible networks connecting *n* towns to an electrical grid, which network uses the least amount of electrical cable. Neither author uses any specialized vocabulary in the statement or solution to these problems, not even the word tree, which displays an advantage of learning from primary sources articulated by Jahnke (2000), namely an ease of understanding the motivational problem. Students and instructors may feel a bit of cognitive dissonance, or *dépaysement* (Barbin, 1997) when reading a historical source and find no modern theorems. Primary source documents are not written like textbooks, and cognitive dissonance arises when the reader encounters the unexpected, particularly in what has become the formalized subject of mathematics. In the language of Sfard (2000, p. 161) Prüfer and Borůvka are using object-level rules to find solutions to their respective problems, while a meta-discursive discussion is needed to formulate these results in terms of modern theorems or algorithms in graph theory, a discussion worth pursuing in the classroom. Cayley uses the rules of algebra (associativity, commutativity, distributivity) when manipulating polynomials to represent trees, and these are the rules of the "objects" (object-level rules) in his treatment of counting trees. Prüfer counts trees with what today would be called Cartesian products of sets, which requires a different set of rules for the enumeration of their elements. To formulate the modern definition of a tree, however, we must go beyond counting arguments, and formalize the fundamental properties of the objects we wish to count. This requires reasoning beyond the object-level rules, and the beginning of a meta-discursive discussion. Throughout the paper, we mention which definitions or theorems have evolved from observations in the primary source documents. In fact, Kjeldsen and Blomhøj (2012) suggest that reading original sources may be essential for raising students' awareness of the meta-discursive rules that govern the current mathematical paradigm.

DESIGN FEATURES OF THE MODULE

The module "Networks and Spanning Trees" was originally written to explore and explain the ideas behind the modern definition of a tree, appearing today in many textbooks on graph theory or discrete mathematics. There a tree is defined as a connected graph containing no cycles, which serves as the starting point for many lemmas and theorems about trees. While this definition has intuitive appeal after a study of the subject is complete, the definition remains opaque and in fact arbitrary for novices. The modern definition of a tree, stated by Oswald Veblen (1922), is an outgrowth of the study of the connectivity of a topological space and not an exploration of combinatorial problems that can be solved using the structure of a tree. In fact, explaining the mathematics behind any opaque definition or procedure could be the staring point of a historical curricular module. The pedagogical idea is to replace the memorization of technical definitions with the study of more engaging and compelling mathematical problems whose solutions involve the constructions appearing in modern definitions.

Once knowledge of the historical background of the topic is acquired, often from a few key primary sources, authorship of the module can begin. The modes of reasoning and standards of rigor from historical sources are often very different from those of today. Care should be taken to avoid an anachronistic or Whigish (Fried, 2001) view of history by evaluating sources in terms of the modern mathematical paradigm of the subject. For example, although Cayley uses the term "tree," he offers no mathematical definition of this term. Its use is intuitive and he arrives at a striking pattern for the enumeration of certain types of trees based on simple counting arguments that involve no specialized algorithms. Students are more often able to participate in the reasoning process when the cognitive demand (Schoenfeld, 2014) is eased via the less formal description of a problem from historical sources. Careful study then reveals the need for more rigorous reasoning, often developed in later sources by scholars or mathematicians confronted with the same situation. Thus, we see Prüfer offer a rigorous proof of "Cayley's formula." Additionally we see how concepts and definitions evolve over time. Although Prüfer presents no formal definition of a tree, nor does he even use the word "tree," he seeks to count all railway networks between n-many towns that are connected and contain the least number of railway segments. This reflects the characterization of a tree as a connected, minimally-connected graph, which is logically equivalent to a connected graph containing no cycles (the textbook definition of a tree). By studying how modern textbook concepts evolved from solving problems of the past, students (and instructors) are able to resolve the cognitive dissonance encountered when first reading an original source.

A historical module should have a focal point, a main result with significance outside mathematical formalism. For "Networks and Spanning Trees," the ultimate goal is to understand Borůvka's algorithm for finding a minimal spanning tree, which is a tree of shortest total edge length that connects n-many towns (or points). The origin of this problem is to connect n towns to an electrical network using the least amount of cable, which Borůvka solved in 1926. Once the significance of this result is understood, we see how the concept of tree defines the domain of study for the minimal spanning tree algorithm, and we see how Cayley's and Prüfer's work enumerates the elements in the domain. Borůvka's work can then be understood as an algorithm for finding a tree of minimal total edge length over this domain. In this way we witness how the historical pieces fit together to form a coherent whole. Let's now outline the specific mathematical content of this module.

CAYLEY'S "THEOREM ON TREES"

Although not motivated by a problem as broad in scope as those stated by Prüfer or Borüvka, Cayley (1857) does introduce the term tree, without definition, to describe the logical branching when iterating the fundamental process of (partial) differentiation. In a later publication Cayley (1889) counts trees in which each knot (his term for vertex or node) carries a particular label or letter. Two trees are counted as the same if and only if the same pairs of vertices are directly connected by an edge. Cayley associates to each tree a certain polynomial constructed from the vertex labels (letters). He then adds all polynomials for a fixed number of vertices and arrives at a striking pattern, which he claims (without proper justification) continues for all values of n, where n is the number of vertices. The object-level rules (Sfard, 2000, p. 161) for working with polynomials are those of algebra, namely associativity and commutativity of addition and multiplication, and distributivity of multiplication across addition. To understand Cayley's use of polynomials, first consider trees with three fixed vertices, labeled α , β, γ in Figure 1:

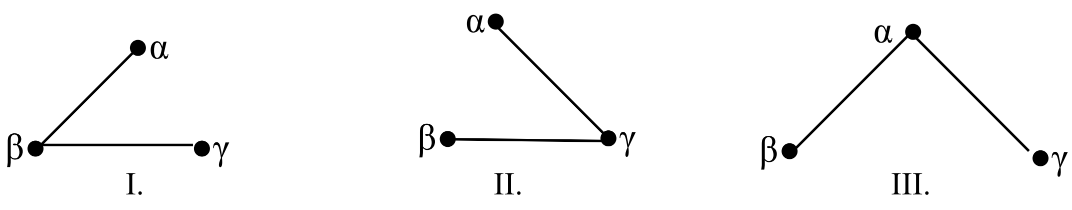


Figure 1.

In tree I above, the vertices α and γ are not directly connected by an edge, while in tree II, α and β are not directly connected, and in tree III, β and γ are not directly connected. Should these three trees be counted as distinct? Cayley does so in his 1889 paper, and introduces a method of counting trees based on assigning polynomials to trees. Let's construct polynomials for the above trees by multiplying all pairs of vertices in the given tree that are directly connected by an edge, and then follow the rules of algebra. For tree I, α is directly connected to β and β is directly connected to γ. Thus, the Cayley polynomial is given by $(\alpha\beta)(\beta\gamma) = \beta(\alpha\beta\gamma)$, which uses the associativity and commutativity of multiplication. For tree II, α is directly connected to γ and γ is directly connected to β. In this case, the Cayley polynomial is $(\alpha\gamma)(\gamma\beta) = \gamma(\alpha\beta\gamma)$. For tree III, the Cayley polynomial is $(\beta\alpha)(\alpha\gamma) = \alpha(\alpha\beta\gamma)$. Adding all polynomials for labeled trees on three vertices, we have $(\alpha+\beta+\gamma)(\alpha\beta\gamma)$.

To follow Cayley's argument (Cayley, 1889), the reader is asked to consider the number of possible trees on four vertices $\alpha, \beta, \gamma, \delta$. First, arrange the vertices in a fixed configuration, such as a diamond in Figure 2:

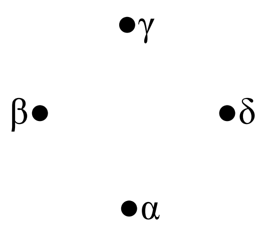


Figure 2.

Then begin to connect the vertices via edges to form trees. Of course, one vertex, for example α, could be connected to each of the other vertices, β, γ, δ, forming the polynomial $(\alpha\beta)(\alpha\gamma)(\alpha\delta)= \alpha^2(\alpha\beta\gamma\delta)$. The list of all Cayley polynomials that can be constructed could be either a homework problem or an in-class activity. For reference, the complete list is:

$(\alpha\beta)(\alpha\beta\gamma\delta)$,	$(\alpha\gamma)(\alpha\beta\gamma\delta)$,	$(\alpha\delta)(\alpha\beta\gamma\delta)$,	$\alpha^2(\alpha\beta\gamma\delta)$,
$(\beta\alpha)(\alpha\beta\gamma\delta)$,	$(\beta\gamma)(\alpha\beta\gamma\delta)$,	$(\beta\delta)(\alpha\beta\gamma\delta)$,	$\beta^2(\alpha\beta\gamma\delta)$,
$(\gamma\alpha)(\alpha\beta\gamma\delta)$,	$(\gamma\beta)(\alpha\beta\gamma\delta)$,	$(\gamma\delta)(\alpha\beta\gamma\delta)$,	$\gamma^2(\alpha\beta\gamma\delta)$,
$(\delta\alpha)(\alpha\beta\gamma\delta)$,	$(\delta\beta)(\alpha\beta\gamma\delta)$,	$(\delta\gamma)(\alpha\beta\gamma\delta)$,	$\delta^2(\alpha\beta\gamma\delta)$.

Although this list is a bit lengthy, adding all of the above terms, we have:

$$(\alpha + \beta + \gamma + \delta)^2(\alpha\beta\gamma\delta),$$

which hints at a simple pattern for counting labeled trees on *n* vertices. A discovery exercise could be to articulate what this pattern is. Note that the commutativity of polynomials, such as $\alpha\beta = \beta\alpha$, loses information about how the tree is constructed, a point the instructor may wish to explore with the class. Might there be a better symbolic device other than polynomials that encodes the construction of a tree? That question will be answered in the next section. Cayley (1889) discusses the number of trees on six vertices $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ in detail, arriving at the expression $(\alpha+\beta+\gamma+\delta+\varepsilon+\zeta)^4(\alpha\beta\gamma\delta\varepsilon\zeta)$. After the six-vertex example, Cayley (1889) writes "It will be at once seen that the proof for this particular case is applicable for any value whatever of n," although the inverse correspondence between polynomials and trees is not mentioned. To illustrate the difficulties with the inverse correspondence, ask students to find all trees with polynomial $\alpha^2\beta^2(\alpha\beta\gamma\delta\varepsilon\zeta)$ in the six-vertex example. Nonetheless, a compelling pattern in the number of labeled trees with *n* vertices has been identified, corresponding to the number of terms in an expansion of the form $(\alpha+\beta+\gamma+ \ldots +\omega)^{n-2}$, where there are *n*-many letters in the list $\alpha, \beta, \gamma, \ldots, \omega$. This suggests that there are n^{n-2} labeled trees on *n* vertices.

PRÜFER'S ENUMERATION OF TREES

Heinz Prüfer begins his paper (Prüfer 1918) with the geometric problem of counting all railway networks connecting *n*-many towns so that: (1.) the least number of railway segments is used; and (2.) a person can travel from each town to any other town by some sequence of connected segments. The ideas expressed here, that the least number of railway segments is used, yet travel remains possible between any two towns, are recognized today as properties that characterize such a railway network as a tree. Since the town names (labels) are fixed, the modern concept of a labeled tree is an excellent model for this problem. Prüfer also states several properties about such a network that have become modern theorems in graph theory. For example, the statement that every network connecting *n* towns has exactly *n-1* many single segments has become the theorem that every tree on *n* vertices has *n-1* edges. Also, the statement that every network has an endpoint has become the theorem that every tree has a leaf (a vertex with only one edge connected to it). Prüfer assigns to each tree with *n* vertices a "symbol" consisting of *n-2* numbers (or characters) taken from the labels of the vertices. Moreover, he establishes that each tree corresponds to only one symbol, and each symbol corresponds to only one tree. Thus, the problem of counting trees is reduced to the problem of counting sequences of length *n-2* taken from a set of *n* numbers (or characters), where the characters may be repeated. Two symbols are considered the same if and only if all corresponding entries are the same. The resulting number of symbols is n^{n-2} for $n > 1$.

Prüfer (1918) writes:

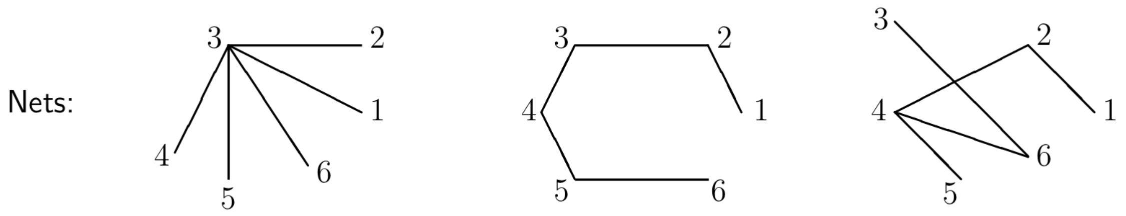
> Consider a country with n towns. These towns must be connected by a railway network of n-1 single segments (the smallest possible number) in such a way that one can travel from each town to every other town. There are n^{n-2} different railway networks of this kind.
> By a single segment is meant a stretch of railway that connects only two towns. The theorem can be proved by assigning to each railway network, in a unique way, a symbol $\{a_1, a_2, \ldots, a_{n-2}\}$, whose n - 2 elements can be selected independently from any of the numbers 1, 2, ..., n. There are n^{n-2} such symbols, and this fact, together with the one-to-one correspondence between networks and symbols, will complete the proof.

Today, a Prüfer symbol would be written with parentheses as delimiters, i.e., $(a_1, a_2, \ldots, a_{n-2})$, and considered as an element of the Cartesian product $V^{n-2} = V \times V \times \ldots \times V$, where *V* is the set of vertex labels (or letters). The object-level rules (Sfard, 2000, p. 161) of counting Prüfer symbols are the rules of enumerating the elements of Cartesian products, where elements are not subject to the commutativity of their components. To understand the construction of a Prüfer symbol, we quote from the original paper (Prüfer, 1918):

> In the case n = 2, the empty symbol corresponds to the only possible network, consisting of just one single segment that connects both towns. If n > 2, we denote the towns by the numbers 1, 2, ..., n and specify them in a fixed sequence. The towns at which only one segment terminates we call the endpoints.

> … In order to define the symbol belonging to a given net for n > 2, we proceed as follows. Let b_1 be the first town which is an endpoint of the net, and a_1 the town which is directly joined to b_1. Then a_1 is the first element of the symbol. We now strike out the town b_1 and the segment b_1 a_1. There remains a net containing n - 2 segments that connects n - 1 towns in such a way that one can travel from each town to any other.

The above characterizes the new graph, after deleting vertex b_1 and edge b_1a_1, as a tree. The process may be iterated, or a recursive construction can be formulated to yield a Prüfer symbol. Prüfer offers several examples of how to construct symbols from nets (trees) in his paper, given below.



Symbols: (3, 3, 3, 3) (2, 3, 4, 5) (2, 4, 6, 4)

The ambiguity raised in the last section over which trees on six vertices have Cayley polynomial $\alpha^2\beta^2(\alpha\beta\gamma\delta\varepsilon\zeta)$ can now be solved. First, use vertex labels $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ ordered as $\alpha< \beta< \gamma< \delta< \varepsilon< \zeta$. Realizing that the Prüfer symbols for such trees are carried by the factor $\alpha^2\beta^2$, we see that threre are six possible Prüfer symbols corresponding to $\alpha^2\beta^2$, namely:

$(\alpha, \alpha, \beta, \beta)$, $(\alpha, \beta, \alpha, \beta)$, $(\beta, \alpha, \alpha, \beta)$, $(\alpha, \beta, \beta, \alpha)$, $(\beta, \alpha, \beta, \alpha)$, $(\beta, \beta, \alpha, \alpha)$.

Then apply Prüfer's algorithm to produce trees from symbols, given in his original paper. Recall that Prüfer uses braces, { … }, to delimit his symbols. He writes (Prüfer 1918):

> Conversely, if we are given a particular symbol $\{a_1, a_2, \ldots, a_{n-2}\}$, other than the empty symbol, then we write down the numbers 1, 2, …, n, and find the first number that does not appear in the symbol. Let this be b_1. Then we connect the towns b_1 and a_1 by a segment. We now strike out the first element of the symbol and the number b_1.
> If $\{a_2, a_3, \ldots, a_{n-2}\}$ is also not the empty symbol, then we find b_2, the first of the n-1 remaining numbers that does not appear in the symbol. Connect the towns b_2 and a_2. Then strike out the number b_2 and the element a_2 in the symbol.
> In this way we eventually obtain the empty symbol. When that happens, we join the last two towns not yet crossed out.

Let's study how Prüfer's mostly verbal description of the tree corresponding to a symbol can be applied to $(\alpha, \alpha, \beta, \beta)$, using the modern notation for a symbol. Since Prüfer speaks of "the first town" or of "the first number," we see that the town names or vertex labels are ordered. For the vertex labels $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, simply use the alphabetical ordering. Now, set

$$(a_1, a_2, a_3, a_4) = (\alpha, \alpha, \beta, \beta).$$

Thus, $a_1 = \alpha$, $a_2 = \alpha$, $a_3 = \beta$, $a_4 = \beta$. The letters that appear in the symbol are just α, β. The first letter that does not appear in the symbol is γ. Set $b_1 = \gamma$. Since $b_1 = \gamma$ and $a_1 = \alpha$ are connected by a segment, the first edge in the construction of the tree has form (Figure 3):

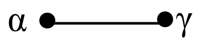


Figure 3.

The updated symbol is now $(a_2, a_3, a_4) = (\alpha, \beta, \beta)$. The updated list of vertex labels, after striking out γ, is: $\alpha, \beta, \delta, \varepsilon, \zeta$. The first symbol in this list that does not occur in (a_2, a_3, a_4) is δ. Thus, $b_2 = \delta$, which is connected to $a_2 = \alpha$ by a segment. Including the second edge in the tree, we have (Figure 4):

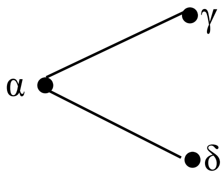


Figure 4.

The updated symbol is now $(a_3, a_4) = (\beta, \beta)$. The updated list of vertex labels, after crossing out δ, is: $\alpha, \beta, \varepsilon, \zeta$. The first element in the above list that does not occur in (a_3, a_4) is α. Thus, $b_3 = \alpha$, and b_3 is connected to $a_3 = \beta$ by a segment. The tree now has from (Figure 5):

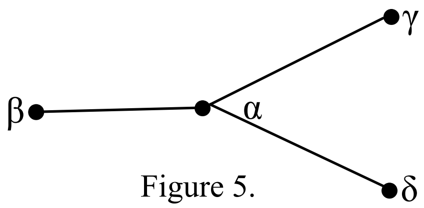


Figure 5.

The updated symbol is now just $a_4 = \beta$. After crossing out α, the updated list of vertex labels is: $\beta, \varepsilon, \zeta$. The first element in the this list that does not contain a_4 is ε. Thus, $b_4 = \varepsilon$, and b_4 is connected to $a_4 = \beta$ by a segment. The tree now has form (Figure 6):

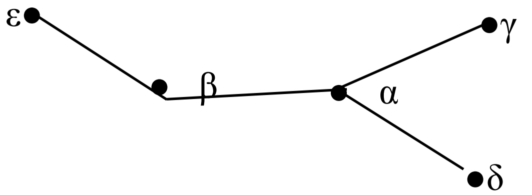


Figure 6.

The updated symbol is now empty. The updated list of vertex labels, after crossing out ε, is just β, ζ. Prüfer writes: "In this way we eventually obtain the empty symbol. When that happens, we join the last two towns not yet crossed out." Thus, the final step in the construction of the tree is to join β and ζ with a segment (Figure 7):

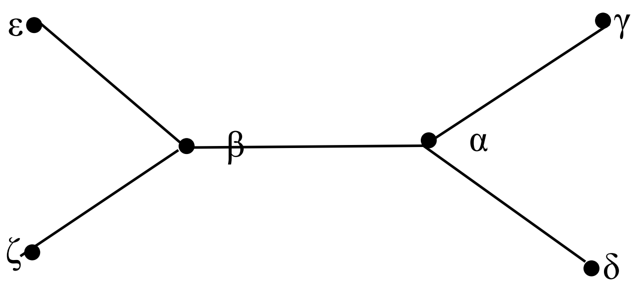


Figure 7.

Figure 7 above is a tree corresponding to the Prüfer symbol $(\alpha, \alpha, \beta, \beta)$. The exact positioning of the edges is not given by the algorithm to construct a tree and may depend on the location of the vertices, provided that information is given at the outset of the problem. An interesting exercise for students or instructors is to verify that the symbol for the above tree is actually $(\alpha, \alpha, \beta, \beta)$. Also, trees corresponding to the symbols $(\alpha, \beta, \alpha, \beta)$, $(\beta, \alpha, \alpha, \beta)$, $(\alpha, \beta, \beta, \alpha)$, $(\beta, \alpha, \beta, \alpha)$, $(\beta, \beta, \alpha, \alpha)$ could now be assigned as homework problems. Further exercises in the student module (Lodder, 2013) develop a precise algorithmic formulation of the tree correspoding to a given symbol, which can be gleaned from Prüfer's original paper, excerpted above. The one-to-one correspondence between symbols and trees is further developed in the exercises of the student module, while a study of trees having the same Prüfer symbol leads to the idea of a graph isomorphism.

BORŮVKA'S SOLUTION TO A MINIMIZATION PROBLEM

Perhaps more important than counting trees are the applications that this structure have found in modern day mathematics and computer science. Well before graph theory was a subject in the present-day curriculum, Otakar Borůvka (1926a, 1926b) published the solution to an applied problem of immediate benefit for constructing an electrical power network in the Southern Moravia Region, now part of the Czech Republic. He describes his own involvement in this project as (Graham, Hell, 1985):

> My studies at polytechnical schools made me feel very close to engineering sciences and made me fully appreciate technical and other applications of mathematics. Soon after the end of World War I, at the beginnings of the 1920s, the Electrical Power Company of Western Moravia, Brno, was engaged in rural electrification of Southern Moravia. In the framework of my friendly relations with some of their employees, I was asked to solve, from a mathematical standpoint, the question of the most economical construction of an electric power network. I succeeded in finding a construction … which I published in 1926 … .

He phrased the problem as follows (Borůvka 1926b):

> There are n points in the plane (in space) whose mutual distances are all different. We wish to join them by a net such that: (1.) Any two points are joined either directly or by means of some other points; and (2.) The total length of the net would be the shortest possible.

How does this problem differ from that posed by Prüfer? Prüfer wishes to find a network that requires the least number of single segments, while Borůvka wishes to find a network of shortest possible total length. Both authors require that all towns in their respective applications be connected to the network (railway or electrical). Are these identical problems? No, since Prüfer never considers the length of a railway segment connecting two towns. Are these problems related? Yes, since a network of shortest total length is recognized today as a tree. Thus, of all possible n^{n-2} labeled trees on n points (towns), which tree or trees have the shortest possible total length? Borůvka offers a solution to this problem that is rather algorithmic in nature, and has become the basis for finding what today is called a *minimum spanning tree*. With the advent of the electronic programmable computer in the late 1940s and early 1950s, algorithms for finding minimal spanning trees became a topic of research in computer science with both Joseph Kruskal (1956) and Robert Prim (1957) publishing their own methods for finding such as tree. Some thirty years before this Borůvka (1926b) had published "A contribution to the solution of a problem on the economical construction of power networks," which outlines how to find a network of shortest total edge length in a very visible and compelling example. He uses no modern terminology in his 1926 papers, not even the word "tree."

Borůvka proposes a simple algorithm to find such a net of minimum total length, based on the guiding principle "I shall join each of the given points with the point nearest to it" (Borůvka, 1926b). Of course, given points $v_1, v_2, v_3, \ldots$ in the plane, if the closest point to v_1 is v_2, then it is not necessarily the case that the closest point to v_2 is v_1. Also, if the only connections made are those resulting from connecting a vertex to its nearest neighbor, then a connected graph would not necessarily result, but would consist of several connected components. If this is the case, Borůvka uses the term "polygonal stroke" to refer to a connected component. He then devises an ingenious method to iterate this algorithm by connecting each polygonal stroke to its nearest polygonal stroke. The distance between two polygonal strokes G_0 and G_1 is given by $min\ d(v_i, v_j)$, where v_i ranges over the vertices of G_0, v_j ranges over the vertices of G_1, and $d(v_i, v_j)$ denotes the distance between v_i and v_j. Of course, after connecting a polygonal stroke to its nearest polygonal stroke, a connected graph may still not necessarily result, but the algorithm can be iterated until a connected graph does result, beginning with a finite number of vertices initially. Borůvka vividly illustrates his algorithm with the following example (Borůvka, 1926b). Given the 40 towns (points) in Figure 1, find the network of least total length that connects them.

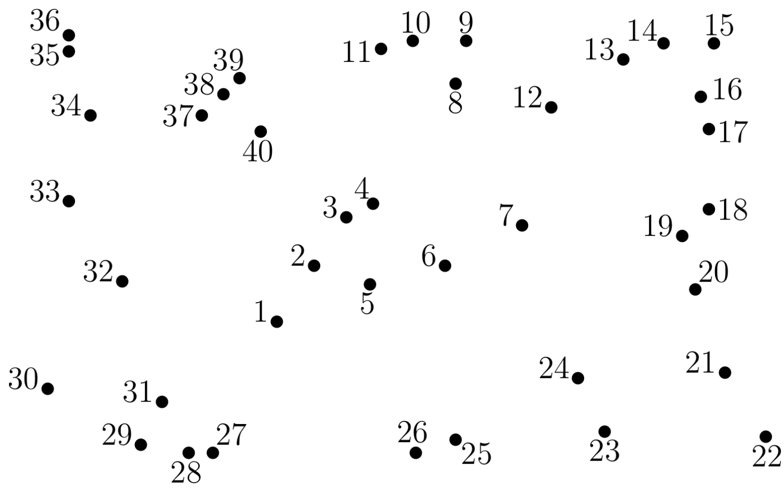


Fig. 1.

First, “join each of the given points with the point nearest to it,” resulting in “a sequence of polygonal strokes” in Figure 2.

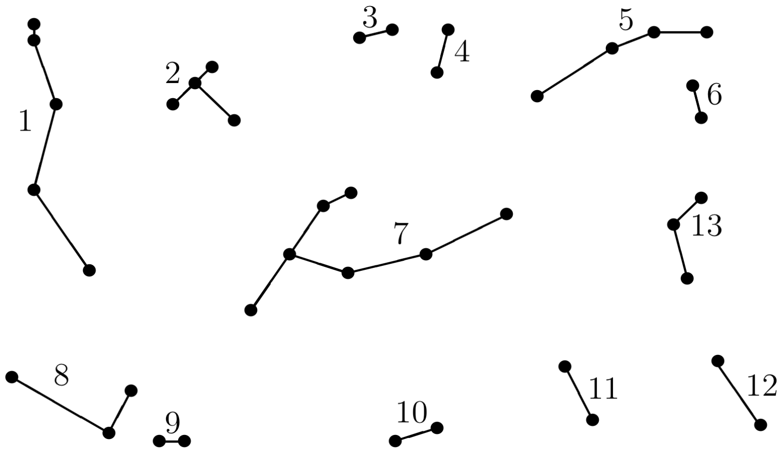


Fig. 2.

Then “join each of these strokes with the nearest stroke in the shortest possible way,” resulting in Figure 3.

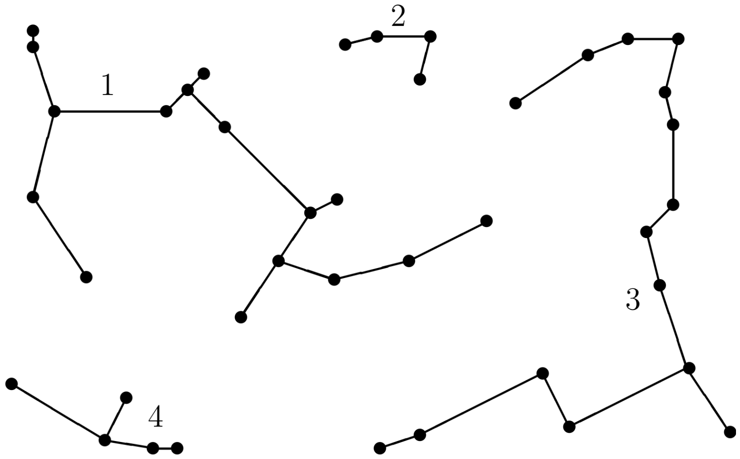


Fig. 3.

Borůvka concludes (Borůvka, 1926b):

> I shall join each of these strokes in the shortest way with the nearest stroke. Thus stroke 1 with stoke 3, stroke 2 with stroke 3 (stroke 3 with stroke 1), stroke 4 with stroke 1. I shall finally obtain a single polygonal stroke (Fig. 4) which solves the given problem.

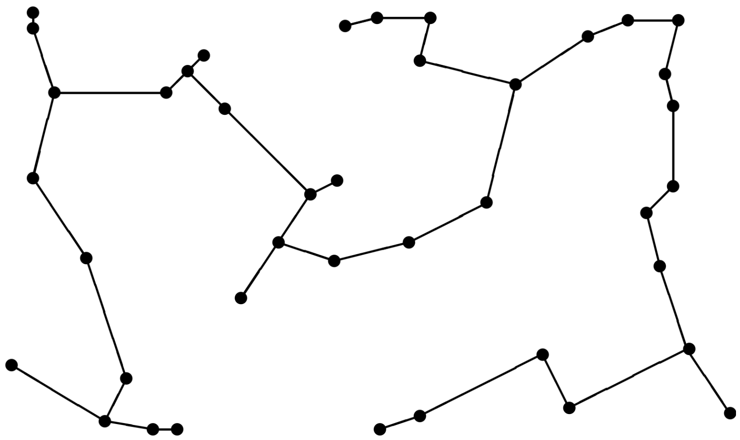


Fig. 4.

It remains to be checked that Borůvka's algorithm produces a minimal spanning tree. Borůvka proves so in his first publication (Borůvka, 1926a), which is a rather algebraic formulation of his algorithm using matrix notation. The visual appeal of the above example (Borůvka, 1926b) remains a compelling illustration of this procedure. Under the assumption that all edge lengths are distinct, there is only one (a unique) solution to the minimal spanning tree problem for a given set of vertices. For further exercises on a precise formulation of Borůvka's algorithm as well as an exploration of why a minimal spanning tree is produced, see the student module "Networks and Spanning Trees," (Lodder, 2013).

IMPLEMENTATION OF THE MODULE

The module “Networks and Spanning Trees” has been tested in both a mathematics course on combinatorics and a computer science course on algorithm design. In the combinatorics course, student work on constructing trees with Cayley polynomial $\alpha^2\beta^2(\alpha\beta\gamma\delta\varepsilon\zeta)$ was graded with leniency, since the goal of this exercise is to raise doubt in the students' minds whether Cayley has correctly counted tree. The complete solution to this problem must wait until the conclusion of the Prüfer section. While working through the Prüfer section, most students were able to construct the correct symbol from a given labeled tree. However, students needed guidence when working in the reservse direction, when constructing a labeled tree from a Prüfer symbol. During class, instructors may wish to work through the example of this given above. Also, Prüfer uses an induction argument on *n,* the number of vertices of a tree, to prove that there is a one-to-one correspondence between labeled trees and his symbols, which the instructor may wish to explore with the class.

For use in computer-science courses, the need for an algorithm to find a minimal spanning tree can be vividly demonstrated from Borůvka's example of 40 towns. We know that there are $40^{38} \approx 7.55 \times 10^{60}$ labeled trees on 40 vertices, and finding the minimal spanning tree by checking the least value over this entire domain, even electronically, is virtually impossible. A more systematic method is necessary. The efficiency of Borůvka's algorithm can be explored by comparing the running time to find a minimal spanning tree with other algorithms, such as those proposed by Kruskal (1956) or Prim (1957). Today, Borůvka's algorithm is known as a “greedy” algorithm, since at each step, a vertex is connected to the vertex closest to it (in some iteration of the algorithm), and this is characterized as a “greedy” choice. In fact an entire subject, combinatorial optimization, (Lawler, 1976) has arisen to discuss these algorithms.

At the conclusion of these courses using historical curricular modules, students were asked to complete a questionnaire about their attitudes towards leaning mathematics. Students were asked to offer, in free response, what are the benefits of learning from historical sources, and separately what are the drawbacks of learnig from historical sources. Stated drawbacks include “the language may be difficult to read,” and “math may not be state-of-the-art.” Although anecdotal, stated benefits are encourgaging, and include:

“You get answers to questions like 'where did all of this come from?' ”

“It helps me understand the reason why things were put together like they are.”

“I like to see where everything comes from and how it works, especially when I am able to make sense of it.”

“You learn the concept.”

“It makes me care about learning.”

CONCLUDING REMARKS

Pedagogical advantages of teaching from historical curricular modules include:

(1.) The study of engaging problems rather than the memorization of technical definitions. Modern textbooks on graph theory begin with the formal definition of a graph and the definition of a tree, followed by lemmas and theorems about these structures and proofs of the results. By contrast, in a historical module we see from the outset what problems humankind confronted and the thought processes developed to find a solution. Cognitive dissonance or *dépaysement* (Barbin, 1997) may occur when reading a verbal description of these problems without a knowledge of the formal definitions of a graph or tree. A reader may not view the verbal description as mathematics.

(2.) The description of problems without the use of specialized vocabulary. Both Prüfer and Borůvka offer verbal descriptions of problems they wish to solve without reference to any technical terms. This represents an advantage of learning from original sources articulated by Jahnke (2000). No specialized use of vocabulary is required at the outset.

(3.) An ease of the cognitive demand in understanding the reasoning process. Modern graph-theory textbooks provide a proof of "Cayley's formula" for the number of labeled trees on *n* vertices, although such proofs are often difficult for students to follow, if students read these at all. The first attempts to justify a result are often intuitive and do not require the assimilation of a body of technical results, easing the cognitive demand (Schoenfeld, 2014) for understanding. Thus, we see Cayley's attempt to count trees with an algebraic construct already familiar to him, and familiar to most students at this point, namely polynomials, subject only to the rules of algebra (associativity, commutativity, distributivity). Once it becomes clear how a monomial is constructed from a tree, then the object-level rules (Sfard, 2000, p. 161) of the algebra of polynomials become the object-level rules of representing trees.

(4.) An appreciation of rigor by observing how the subject has evolved over time. By studying the first attempts to verify a result, inconsistencies, omissions or unanticipated subtleties often arise. When students are afforded the opportunity to find or understand these gaps in reasoning, they appreciate the efforts of other scholars to offer a more rigorous justification of the same result. Thus, we see Prüfer provide a proof of "Cayley's formula" by using a one-to-one correspondence between labeled trees and Prüfer symbols. A Prüfer symbol for a labeled tree on *n* vertices can be interpreted as an element of the Cartesian product, $V^{n-2} = V \times V \times \ldots \times V$, where V is the set of vertex labels. Once the construction of a Prüfer symbol is understood, then the object-level rules of Cartesian products (as sets) become the object-level rules of counting trees. When moving from Cayley to Prüfer, we see the development of rigor by replacing one set of object-level rules (polynomials as objects) with a more subtle set of object-level rules (Cartesian products as objects). Of course, elements of a Cartesian product are not subject to commutativity of their entries.

(5.) An understanding of the origin of modern mathematical definitions and procedures. Modern mathematics attempts to codify the key properties of a structure or procedure in the form of a definition or an axiom. After a study of trees and their applications is complete, we see why a tree might be defined as a "connected, minimally-connected graph," and why this might be reformulated as a "connected graph with no cycles," since a cycle always contains more edges than necessary in order for the graph to be "connected and minimally-connected." The formulation of a modern definition requires a knowledge of the various historical works (discussants) about the subject, and a meta-discursive (Sfard, 2000) synthesis of these works into one definition. This offers an example of raising students' awareness of the meta-discursive rules that govern the current mathematical paradigm (Kjeldsen and Blomhøj, 2012).

(6.) Context, motivation and direction for the subject. As a summary statement, the three historical sources in this curricular module provide context, motivation and direction for a course on graphs and trees.

Instructors seeking more information on this teaching module should consult Lodder (2014).

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