Solving Linear First-Order Differential Equations
Bernoulli's (Almost) Variation of Parameters Method

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1 Introduction

In 1926, British mathematician E. L. Ince (1891–1941) described the typical evolution of solution techniques from calculus (and differential equations and science in general).[1]

> The early history of the infinitesimal calculus abounds in instances of problems solved through the agency of what were virtually differential equations; it is even true to say that the problem of integration, which may be regarded as the solution of the simplest of all types of differential equations, was a practical problem even in the middle of the sixteenth century. Particular cases of the inverse problem of tangents, that is the problem of determining a curve whose tangents are subjected to a particular law, were successfully dealt with before the invention of the calculus.
>
> But the historical value of a science depends not upon the number of particular phenomena it can present but rather upon the power it has of coordinating diverse facts and subjecting them to one simple code. [Ince, 1926]

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[1] Ince himself is part of at least one such story within differential equations. He developed the so called *Ince Equation* (in about 1923),

$$(1 + a\cos(2t))y''(t) + (b\sin(2t))y'(t) + (\lambda + d\cos(2t))y(t) = 0,$$

which generalized at least two other well known equations from about 1868 and 1914, respectively. Letting $a = b = 0$ and $d = -2q$, we obtain *Mathieu's equation* (which model elliptical drumheads),

$$y''(t) + (\lambda - 2q\cos(2t))y(t) = 0,$$

and letting $a = 0, b = -4q$, and $d = 4q(\nu - 1)$, we obtain the *Whittaker-Hill equation* (with applications to lunar stability and quantum mechanics)

$$y''(t) - 4q(\sin(2t))y'(t) + (\lambda + 4q(\nu - 1)\cos(2t))y(t) = 0.$$

Ince's equation then is itself a special case of *generalized Ince equations* (studied in [Moussa, 2014])

$$(1 + \epsilon A(t))y''(t) + \epsilon B(t)y'(t) + (\lambda + \epsilon D(t))y(t) = 0.$$

This is exactly the evolution of solution methods for first-order linear ordinary differential equations. First, particular problems were solved with "one-off" methods that didn't have general applications beyond that specific problem. But then those results were combined and generalized until a unified theory developed.

Task 1 In the above passage, Ince made a connection between the solution of the simplest of all types of differential equations and the problem of determining a curve whose tangents are subjected to a particular law. Connect these two statements. If the differential equation is

$$\frac{dy}{dx} = f(x, y),$$

then what is the "curve," the "'tangents," and the "'particular law"?

Task 2 Recall that non-homogenous first-order linear ordinary differential equations have the following form

$$p(x)\frac{dy}{dx} + q(x)y = f(x), \tag{1}$$

or if made monic

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2}$$

Explain how to make Equation (1) monic like Equation (2). In particular why can we assume that $p(x)$ isn't identically zero? Write $P(x)$ and $Q(x)$ in terms of $p(x), q(x)$ and $f(x)$.

The theme of this project is a solution method due to Johann Bernoulli (1667–1748) which considered first-order linear differential equations as special cases of what are now called "Bernoulli[2] differential equations." Prior to Bernoulli's contribution, Gottfried Leibniz[3] (1646-1-716) published a paper in 1694 in which he solved first-order linear ordinary differential equations by intuiting a solution then checking that it worked. That method wasn't general or part of a larger theory. But as Ince noted, science aims to combine diverse techniques into coordinated theories.[4]

[2]Which one? Was it brothers Jacob or Johann, nephew Daniel, Nicolaus I or II, Johann II or III, Jacob II, ... conservatively there are at least 8 first-rate mathematicians in the Bernoulli Family.

[3]Leibniz was a German mathematician and philosopher who created (probably independently of Newton) the calculus along with the notation that we currently use for it.

[4]The story of Leibniz' method and that of the post-Bernoulli work of Leonhard Euler (1707–1783) on exact differential equations can be found in the two other projects of this "Solving First-Order Linear Differential Equations" series. Each of the three projects in the series can be completed individually or in any combination with the others. They are available at `https://digitalcommons.ursinus.edu/triumphs_differ/`.

2 Bernoulli's First Solution

Recall that a Bernoulli differential equation has the form

$$p(x)\frac{dy}{dx} + q(x)y = f(x)y^n$$

or if made monic

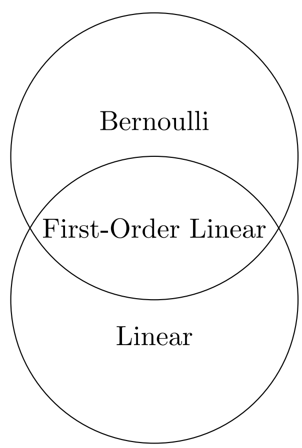
$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Task 3

(a) In general, is a Bernoulli differential equation linear? Why or why not?

(b) Why is it easy to solve a Bernoulli differential equation when $n = 1$?

(c) Verify that first-order *linear* differential equations are a special case of Bernoulli when $n = 0$. Thus if we had a method to solve all Bernoulli equations, we would have a method to solve first-order linear equations.



The history of the Bernoulli differential equation is interesting in its own right [Parker, 2013]. The short version is that in December of 1695, Jacob Bernoulli[5] (1654–1705) asked for solutions to these equations [Bernoulli, December 1695].[6]

> *Problem*: To somehow construct the equation $a\,dy = yp\,dx + by^n q\,dx$ (where a and b denote given quantities and constants, n any power of the letter y, p and q quantities of any sort given in terms of x) by means of quadratures, that is, to mutually separate in this [equation] the indeterminate letters x and y along with their differentials.

[5] Jacob Bernoulli was a Swiss mathematician and eldest son of Niklaus. He was the first of the Bernoulli scientific family tree, taught his Brother Johann at Basel, and is best known for his work in probability.

[6] All translations in this project by Danny Otero of Xavier University, 2020.

In March of 1696, Leibniz published a "solution" which contained no details about how to actually execute his process [Leibniz, 1696].

. . . The problem presented there concerning the differential equation $a\,dy = yp\,dx + by^n.q\,dx$ [is one that] I can solve. I reduce it to an equation whose form is $\ldots dv + \ldots v\,dz + \ldots dz = 0$, where by the dots one should understand quantities given somehow in terms of z. Moreover, I have reduced this equation generally to quadratures, by an argument that I communicated to friends a while ago, which I do not think necessary to detail here, being content to have worked it out, so that that most intelligent Author of the problem would be able to perceive that the method is (as I suspect) not dissimilar to his own. For I do not doubt that it has been made known to him. . . .

Task 4 What did Leibniz omit that would be necessary to actually carry out his "solution"? What would your professor do if you tried to turn this in as a proof?

In March of 1697, Johann Bernoulli[7] published some of Leibniz' missing details [Bernoulli, 1697]. Specifically, he revealed the change of variable that allowed Leibniz to convert the Bernoulli differential equation into one of the form $\ldots dv + \ldots v\,dz + \ldots dz = 0$:

The proposed equation is this one: $a\,dy = yp\,dx + by^n q\,dx$ (where a and b denote given quantities and constants, n any power of the letter y; p and q quantities of any sort given in terms of x); [the terms] should be mutually separated into the indeterminate letters x and y along with their differentials, so that somehow there may be constructed their quadratures; this then is what I do. To remove the power n, we let $y^n = v^{n:(1-n)}$, when the proposed [equation] transforms into the final resolved form $\frac{1}{1-n} a\,dv = vp\,dx + bq\,dx$, which corresponds to the formula that Leibniz delivered in [the *Acta*,] March 1696.[8]

[7] Johann Bernoulli was a very talented Swiss mathematician and third son of Niklaus. His unpleasant personality and desire for fame eventually ruined his relationships with both his brother Jacob, and his son, Daniel.

[8] When this paper was republished in Johann Bernoulli's *Opera Omnia* of collected works [Bernoulli and de la Sainte Trinité de la Compagnie de Jésus, 1742], it contained a typographical error; we have used the correct statement of Bernoulli's substitution as it appeared in the original publication of the paper in the *Acta*.

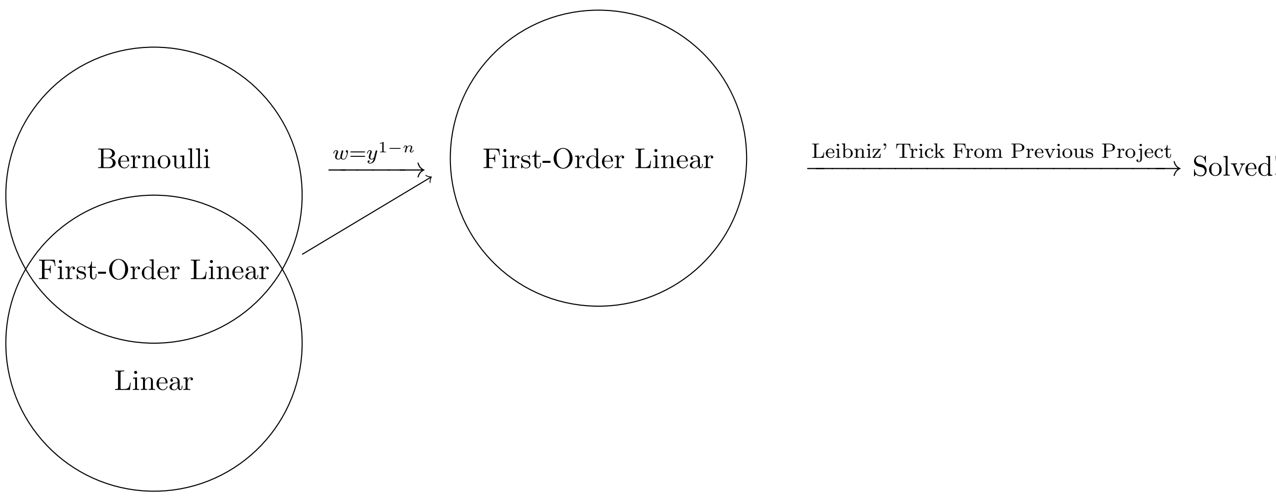
In Bernoulli's statement of the substitution, $y^n = v^{n:(1-n)}$, we would today write the exponent '$n:(1-n)$' as '$\frac{n}{1-n}$.' In other words, Bernoulli set

$$y^n = v^{\frac{n}{1-n}}.$$

It is then straightforward to see how to transform this into the more familiar form of the substitution found in today's textbooks:

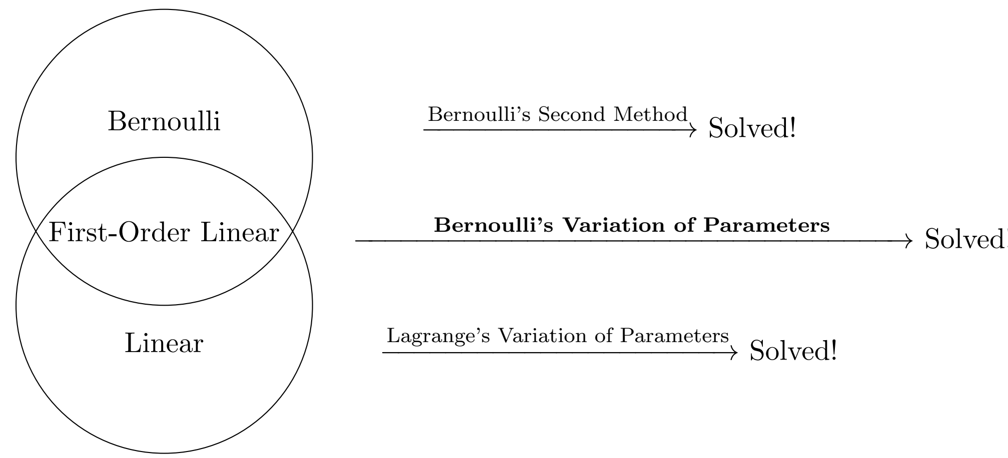
$$y^{1-n} = v.$$

This change of variables converts any Bernoulli equation into a first-order linear equation that could be solved using Leibniz' 1694 "Intuition and Check" method. Unfortunately, applying this to an already linear Bernoulli equation simply returns the exact same equation we started with. We may think of this as:



So, why do we care about Bernoulli differential equations if the canonical method of solving them still uses Leibniz' method? Well, because Johann gave a *second* method of solution immediately after clarifying Leibniz'! There are few things to know about this method.

- There isn't a name for this method and so we creatively call it "Bernoulli's Second Solution."
- This second solution is direct as it didn't require use of Leibniz' method at any point, but rather reduced the problem to two separable differential equations.
- Unlike Leibniz' change of variables, this second solution *can* be applied to non-homogenous first-order linear differential equations!
- When applied to arbitrary order non-homogenous linear differential equations, this method is called "variation of parameters."



3 Bernoulli's Second Solution

In March of 1697, Johann Bernoulli published *De conoidibus et spaeroidibus quaedam. Solutio analytica æquationis in Actis A. 1695, pag. 553 propositæ(A Fratre Jac. Bernoullio).*[9] Bernoulli's Second Solution is:

> The proposed equation is this one: $a\,dy = yp\,dx + by^n q\,dx$ (where a and b denote given quantities and constants, n any power of the letter y; p and q quantities of any sort given in terms of x)...
>
> But this difficulty has no power to distress me; for immediately I pursue my goal by setting $y = mz$, whence $dy = m\,dz + z\,dm$; by which substitution into the proposed equation yields $az\,dm + am\,dz = mzp\,dx + bm^n z^n q\,dx$. Now in order to reduce this equation from four terms to two, I set $am\,dz = mzp\,dx$, that is, $\frac{a\,dz}{z} = p\,dx$, whence, because we get z in terms of x not algebraically but rather transcendentally, let $z = \zeta$ (by ζ I understand a quantity given in terms of x and constants.) In fact, since this transmutation cancels $m\,dz$ and $mzp\,dx$ in the equation, what remains is $az\,dm = bm^n z^n q\,dx$, or, when I substitute this value for z, $a\zeta\,dm = bm^n \zeta^n q\,dx$, that is, $am^{-n}\,dm = b\zeta^{n-1} q\,dx$, hence, we evidently get likewise m in terms of x : $\frac{a}{-n+1} m^{-n+1} = b \int \zeta^{n-1} q\,dx$, so I assume that $m = X$ (a quantity similarly composed from x and constants), from which we obtain $y = (zm) = \zeta X =$ a quantity purely dependent on x and constants. Which is what was to be done.

Let us examine this passage more closely.

[9] Or, in English: *On certain conoids and spheroids. An analytic solution to the equation proposed in Acta A. 1695, p. 553 (by my brother, Jacob Bernoulli))*

Task 5 (a) What common rule explains how Bernoulli derived $dy = m\,dz + z\,dm$ from $y = mz$.

(b) Verify that substituting y and dy into the proposed equation gives what Bernoulli claimed.

(c) Bernoulli then required that z satisfy $amdz = mzp\,dx$. How is this differential equation related to the proposed one? If the proposed differential equation was linear, the relationship has a special name.

(d) Algebraically show how Bernoulli went from $am\,dz = mzp\,dx$ to $\frac{a\,dz}{z} = p\,dx$ and explain in your own words why this is always solvable. Bernoulli called that solution ζ.

(e) Explain in your own words why replacing z with ζ reduced $az\,dm + am\,dz = mzp\,dx + bm^n z^n q\,dx$ to $a\zeta\,dm = bm^n\zeta^n q\,dx$.

(f) Algebraically show how Bernoulli went from $a\zeta\,dm = bm^n\zeta^n q\,dx$ to $am^{-n}\,dm = b\zeta^{n-1}q\,dx$ and explain in your own words why this is always solvable. Bernoulli called that solution X.

Since we have found $z = \zeta$ and $m = X$ we know the solution y.

4 Examples

Task 6 We will work through Bernoulli's technique, using

$$x\frac{dy}{dx} + y = 3x^2 \tag{3}$$

as an example.

(a) Rewrite Equation 3 in the form that Johann used to begin his process.[10] What are the functions p, q and the constant n?

(b) Johann then wrote the solution as $y = mz$, where z would satisfy what equation? Solve that equation and call the solution ζ.

(c) Substituting $z = \zeta$ into the equation from part (a), we find that m must satisfy what equation? Solve that equation and call the solution X.

(d) Finally, write $y = zm = \zeta X$.

Task 7 Solve

$$\frac{dy}{dx} - y = xe^x$$

using the above method.

[10] Notice that Bernoulli's form does not require the proposed equation to be monic, but does require the only coefficient on $\frac{dy}{dx}$ be a constant. It is probably easiest to just make the proposed equation monic if it isn't already (so $a = 1$).

5 Conclusion

Writing a non-homogenous solution to a linear differential equation as $y = mz$ where z is a solution to the corresponding homogenous differential equation and mz solves the proposed equation is known as *variation of parameters.* It is credited to Joseph-Louis Lagrange[11] (1736–1813) in his 1775 "Recherche sur les suites récurrentes" [Lagrange, 1775]. And while that is certainly where we see its power on non-homogenous linear equations of higher order, Johann Bernoulli's Second Solution for solving Bernoulli differential equations reduces to this method for first order equation, and was used decades earlier.

It is likely that your ODE class utilizes an integrating factor μ to solve first-order linear differential equations

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad\qquad \mu = e^{\int P(x)dx}. \tag{4}$$

While μ was not derived by Bernoulli, it is interesting that his z function is our modern μ![12]

$$a\,dy = yp\,dx + by^n q\,dx \qquad\qquad \frac{a\,dz}{z} = p\,dx. \tag{5}$$

Task 8 Show that μ from Equation (4) is the same as z from Equation (5).

You should notice that Bernoulli's method of solution to a non-homogenous first order linear differential equation is actually a special case of a more general method. This is how science proceeds. Sporadic solutions are combined into a general theory and in this case the theory is variation of parameters.

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Ja. Bernoulli. Explicationes, annotationes et additiones ad ea quæ in Actis superiorum annorum de Curva Elastica, Isochrona Paracentrica, & Velaria, hin inde memorata, & partim controversa lenuntur; ubi de Linea mediarum directionum, aliisque novis [Explanations, notes and additions to that in the Acts of the preceding year about the Elastica, Paracentric Isochrone and Velara Curves, thence from this recounted, the controversial part read, where concerning the line of the middle direction]. *Acta Eruditorum*, pages 537–553, December 1695. Also in *Opera Omnia*, Volume V.1, pp. 639–662.

[11]Lagrange was a shy, submissive, yet extremely talented and driven Italian / French mathematician. He succeeded Euler in Berlin when Euler went to St. Petersburg and later became the first professor of analysis at the École Polytechnique in Paris. He is perhaps best known for placing mechanics on solid mathematical footing.

[12]It should be noted that this equivalency requires that the given differential equation is made monic, as that is the form that Leibniz starts from.

Jo. Bernoulli. De conoidibus et spaeroidibus quaedam. Solutio analytica æquationis in Actis A. 1695, pag. 553 propositæ(a Fratre Jac. Bernoullio) [On certain conoids and spheroids. An analytic solution to the equation proposed in Acta A., 1695, p. 553 (by my brother, jacob bernoulli)]. *Acta Eruditorum*, Mar.:113–118, 1697.

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Notes to Instructors

This set of notes accompanies the Primary Source Project "Solving Linear First-Order Differential Equations: Johann Bernoulli's (Almost) Variation of Parameters Method" written as part of the TRIUMPHS project. (See end of these Notes for details about TRIUMPHS).

PSP Content: Topics and Goals

This mini-Primary Source Project (mini-PSP) is one of a set of three mini-PSPs that share the name "Solving Linear First Order Differential Equations: Johann Bernoulli's (Almost) Variation of Parameters Method," designed to show three solutions to non-homogenous first-order linear differential equations, each from a different context. Recall that a non-homogenous first-order linear differential equation has the form

$$a(x)\frac{dy}{dx} + p(x)y = q(x).$$

- The mini-PSP subtitled "Gottfried Leibniz' "Intuition and Check" Method" explains how in 1694 Leibniz solved these equations using one-off method applicable only to this specific problem. Strictly speaking, Leibniz didn't solve the equation, but asserted a solution and then showed it worked. Part of his proposed solution will be familiar to the students: it is the standard integrating factor method we teach today.
- The mini-PSP subtitled "Johann Bernoulli's (Almost) Variation of Parameters Method" explains how in 1697, Bernoulli provided a method for solving Bernoulli differential equations that reduces to variation of parameters when applied to first-order linear equations. This was decades before Lagrange received credit for the technique. Again, part of Bernoulli's solution will be the standard integrating factor.
- The mini-PSP subtitled "Leonard Euler's Integrating Factor Method" explains how in 1763, Euler solved these equations as a special case of exact differential equations by finding an integrating factor. His integrating factor is the same as the one as today's students see. This mini-PSP is a bit longer than the others, and may require a bit more time or pre-preparation.

All three of these mini-PSPs are designed for use in an Ordinary Differential Equations course but can be used in three different ways. They work best after at least presenting the standard integrating method of solution found in modern textbooks.

- Since the *type* of equation (first-order linear) has been introduced, all three projects can be immediately done. This would require the instructor to "preview" techniques that will be introduced more fully later. While this is somewhat awkward, it does mimic how these techniques were actually developed.
- The "Gottfried Leibniz' "Intuition and Check" Method" project can be done immediately, but the the other projects done after the respective *method* of solution (variation of parameters, exactness) are first introduced. Showing how those techniques can solve first-order linear differential equations makes a great first example of each technique. This is typically the way that I utilize the project.
- With a bit of revision of the first section, each of these projects can stand on their own as they don't necessarily build on the others (though they do create a richer experience together). Additionally, students gain confidence as they proceed through the three projects.

Student Prerequisites

This mini-PSP requires some algebraic manipulation of differentials along with differentiation up to the product rule. It also needs knowledge of separable Differential Equations. Other techniques of integration needed are typically dictated by the examples used. Finally, the project benefits from the students being aware of the modern integrating factor method.

PSP Design, and Task Commentary

This PSP consists of five sections:

1. The first section contains a short introduction to what first-order linear differential equations are, along with a description of the way that mathematics often evolves. Mathematicians might first solve a specific problem using any tool at their disposal. They then attempt to see if they could find a class of problems (of which the initial one belongs) that can also be attacked using that technique. This closely mimics the evolution of how first-order linear differential equations were solved. Much of this section is the same for all three projects, so if the first project has been covered, this section can be skipped

2. The second section includes some historical discussion of the Bernoulli differential equation that is not contained in other projects. Specifically, the standard change of variables that we see in our textbooks is presented.

3. The next section is devoted to Bernoulli's Second Solution. A translation is provided along with a few tasks to explain his method. Bernoulli's method sets the bqy^n portion of the presented differential equation equal to zero. When the Bernoulli equation is linear ($n = 0$), this is the *homogenous* form of the presented equation, and the method is *variation of parameters*. Those terms are typically reserved for linear differential equations, and so we avoid using those terms when discussing Bernoulli's general method, though the similarities are obvious.

4. The next section consists of two first-order linear differential equations to be solved with Leibniz' method. The first is broken into steps, while the second requires the student to solve it on their own. These can be swapped with any examples you wish—in particular so that the integrations utilize techniques your students are comfortable with.

   One of the TRIUMPHS reviewers of this PSP correctedly observed that these examples could be chosen better. They are not historical when there are plenty of primary source examples available. They may utilize techniques not known at the time. And, neither of the examples in this section are Bernoulli equations. This is so that they mimic the examples from the other projects, but is awkward given the central theme Bernoulli equations play in this project. The author is very open to suggestions on improving these examples.

   As mentioned, it is possible to use this project for general Bernoulli differential equations, and not just for when they are linear. If so, then the following task can be assigned as part of this section as well.

Task 9 Consider the differential equation

$$\frac{dy}{dx} + y = xy^2.$$

(a) Solve this differential equation by using Leibniz' change of variables to convert it into a linear differential equation, and then solve the linear equation using the integration factor method, Leibniz' intuition method, or Bernoulli's variation of parameters.

(b) Solve this directly using Bernoulli's variation of parameters, without first converting to a linear equation.

5. The final section reiterates what we saw in the first section. There is a task to show how Bernoulli's Second Solution has the modern integrating factor method embedded in it, to connect this project to the first in the trilogy along with the integration factor method the students may know.

Suggestions for Classroom Implementation

Please see student requirements and implementation schedule for suggestions.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

The first section of this mini-PSP should be out of class homework. The instructor should present some guidance on the amount of Bernoulli differential equation history that the students should learn. The second and third sections are important and shouldn't be skipped. The Task 8 is useful to complete the integration of this PSP into the material the student sees in their textbook. Also, the Task 7 can be assigned as homework after class. The Task 9 that appears in these Notes can be included if Bernoulli differential equations are being covered. With these types of revisions, this is a doable activity in one 50-minute class period.

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students. This project is typically done in groups.[13] One reviewer warned that, "The groups often want to take a divide and conquer approach, which is just utterly useless for these documents, because the only person who is going to make any progress is the person who is working on Intro/Section 1. These are all designed to be read top to bottom in slow careful detail, and the later parts of the PSP rarely make any sense unless you've seen the earlier parts."

Connections to other Primary Source Projects

As mentioned above, this mini-PSP is part of a series of three, all which are intended for use in an Ordinary Differential Equations course.

- Solving Linear First-Order Differential Equations: Gottfried Leibniz' Intuition and Check Method.
- Solving Linear First-Order Differential Equations: Johann Bernoulli's (Almost) Variation of Parameters Method.

[13]...though with COVID, who knows?!?!

- Solving Linear First-Order Differential Equations: Leonard Euler's Integrating Factor Method.

Additionally, the author has written a fourth mini-PSP for use in an Ordinary Differential Equations course, based on works by Peano:

- Wronskians and Linear Independence: A Theorem Misunderstood by Many.

All of the above projects can be found at `https://digitalcommons.ursinus.edu/triumphs_differ/`.

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.