TRIUMPHS Student Projects: Detailed Descriptions
TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources

F 01. A Genetic Context for Understanding the Trigonometric Functions

In this project, we explore the genesis of the trigonometric functions: sine, cosine, tangent, cotangent, secant, and cosecant. The goal is to provide the typical student in a pre-calculus course some context for understanding these concepts that is generally missing from standard textbook developments. Trigonometry emerged in the ancient Greek world (and, it is suspected, independently in China and India as well) from the geometrical analyses needed to solve basic astronomical problems regarding the relative positions and motions of celestial objects. While the Greeks (Hipparchus, Ptolemy) recognized the usefulness of tabulating chords of central angles in a circle as aids to solving problems of spherical geometry, Hindu mathematicians, like Varahamahira (505–587), in his *Pancasiddhantika* [57], found it more expedient to tabulate half-chords, from whence the use of the sine and cosine became popular. We examine an excerpt from this work, wherein Varahamahira described a few of the standard modern relationships between sine and cosine in the course of creating a sine table. In the eleventh century, the Arabic scholar and expert on Hindu science Abu l-Rayhan Muhammad al-Biruni (973–1055) published *The Exhaustive Treatise on Shadows* (c. 1021) [50]. In this work, we see how Biruni presented geometrical methods for the use of sundials; the relations within right triangles made by the gnomon of a sundial and the shadow cast on its face lead to the study and tabulation of values of the tangent and cotangent, secant and cosecant. Biruni also worked out the relationships that these quantities have with the sines and cosines of the angles. However, the modern terminology for the standard trigonometric quantities was not established until the European Renaissance. Foremost in this development is the landmark *On Triangles* (1463) by Regiomontanus (Johannes Müller) [44]. Regiomontanus exposed trigonometry in a purely geometrical form and then applies the ideas to problems in circular and spherical geometry. We examine a few of the theorems that explore the trigonometric relations and which are used to solve triangle problems.

This project is intended for courses in pre-calculus, trigonometry, the history of mathematics, or as a capstone course for teachers. Author: Danny Otero.

F 02. Determining the Determinant

This project in linear algebra illustrates how the mathematicians of the eighteenth and nineteenth centuries dealt with solving systems of linear equations in many variables, a complicated problem that ultimately required attention to issues of the notation and representation of equations as well as careful development of the auxiliary notion of a "derangement" or "permutation." Colin Maclaurin (1698–1746) taught a course in algebra at the University of Edinburgh in 1730 whose lecture notes includeds formulas for solving systems of linear equations in 2 and 3 variables; an examination of these lecture notes [42] illustrate the forms of the modern determinant long before the notion was formally crystalized. In 1750, Gabriel Cramer (1704–1752) published his landmark *Introduction a l'Analyse des Lignes Courbes algébriques* (*Introduction to the Analysis of Algebraic Curves*) [22]. In an appendix to this work, Cramer tackles the solution of linear systems more systematically, providing a formula for the solution to such a system, today known as Cramer's Rule. More significantly, he pointed out the rules for formation of the determinantal expressions that appear in the formulas for the solution quantities, using the term "derangement" to refer to the complex permuting of variables and their coefficients that gives structure to these expressions. These ideas reach maturity in an 1812 memoir by Augustin-Louis Cauchy (1789–1857) entitled *M'emoire sur les functions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferement* (*Memoir on those functions which take only two values, equal but of opposite sign, as a result of transpositions performed* *on the variables which they contain*) [20]. In this work, Cauchy provided a full development of the determinant and its permutational properties in an essentially modern form. Cauchy used the term "determinant" (adopted from Gauss (1777–1855)) to refer to these expressions and even adopted an early form of matrix notation to express the formulas for solving a linear system.

This project is intended for courses in linear algebra. Author: Danny Otero.

F 03. Solving a System of Linear Equations Using Ancient Chinese Methods

Gaussian Elimination for solving systems of linear equations is one of the first topics in a standard linear algebra class. The algorithm is named in honor of Carl Friedrich Gauss (1777–1855), but the technique was not his invention. In fact, Chinese mathematicians were solving linear equations with a version of elimination as early as 100 CE. This project has the students study portions of Chapter 8 Rectangular Arrays in *The Nine Chapters on the Mathematical Art* [49] to learn the technique known to the Chinese by 100 CE. Students then read the commentary to Chapter 8 of *Nine Chapters* given in 263 by Chinese mathemtician Liu Hui (fl. 3rd century CE) and are asked how his commentary helps understanding. The method of the *Nine Chapters* is compared to the modern algorithm. The similarity between the ancient Chinese and the modern algorithm exemplifies the sophisticated level of ancient Chinese mathematics. The format of the *Nine Chapters* as a series of practical problems and solutions reinforces the concept that mathematics is connected to everyday life.

This project is appropriate for an introductory linear algebra class, and may be used in a more advanced class with appropriate choice of the more challenging exercises. Author: Mary Flagg.

F 04. Investigating Difference Equations

Abraham de Moivre (1667–1754) is generally given credit for the first systematic method for solving a general linear difference equation with constant coefficients. He did this by creating and using a general theory of recurrent series, the details of which appeared in his 1718 *Doctrine of Chances* and a second manuscript written that same year. While de Moivre's methods are accessible to students in a sophomore/junior discrete math course, they are not as clear or straightforward as the methods found in today's textbooks. Building on de Moivre's work, Daniel Bernoulli (1700–1782) published a 1728 paper, Observations about series produced by adding or subtracting their consecutive terms which are particularly useful for determining all the roots of algebraic equations, in which he laid out a simpler approach, along with illuminating examples and a superior exposition. The first part of the project develops de Moivre's approach with excerpts from original sources. The second part gives Bernoulli's 1728 methodology, no doubt more attractive to most students. Ideally, this project will help students understand and appreciate how mathematics is developed over time, in addition to learning how to solve a general linear difference equation with constant coefficients.

This project is intended for courses in discrete mathematics. Author: David Ruch.

F 05. Quantifying Certainty: The p-value

The history of statistics is closely linked to our ability to quantify uncertainty in predictions based on partial information. In modern statistics, this rather complex idea is crystallized in one concept: the p-value. Understanding p-values is famously difficult for students, and statistics professors often have trouble getting their students to understand the rather precise nuances involved in the definition. In this project, students work to build a robust understanding of p-values by working through some early texts on probability and certainty. These include the famous text *Statistical Methods for Research Workers* by Sir Ronald Fisher (1890–1962), as well as earlier attempts that came very close to the modern concept, such as Buffon's *Essai d'Arithmtique Morale* [51].

This project is intended for courses in statistics. Author: Dominic Klyve.

F 06. The Exigency of the Parallel Postulate

In this project, we examine the use of the parallel postulate for such basic constructions as the distance formula between two points and the angle sum of a triangle (in Euclidean space). Beginning with Book I of Euclid's (c. 300 BCE) *Elements* [28], we witness the necessity of the parallel postulate for constructing such basic figures as parallelograms, rectangles and squares. This is followed by Euclid's demonstration that parallelograms on the same base and between the same parallels have equal area, an observation essential for the proof of the Pythagorean Theorem. Given a right triangle, Euclid constructed squares on the three sides of the triangle, and showed that the area of the square on the hypotenuse is equal to the combined area of the squares on the other two sides. The proof is a geometric puzzle with the pieces found between parallel lines and on the same base. The project stresses the ancient Greek view of area, which greatly facilitates an understanding of the Pythagorean Theorem. This theorem is then essential for the modern distance formula between two points, often used in high school and college mathematics, engineering and science courses.

The project is designed for courses in geometry taken both by mathematics majors and secondary education majors. Author: Jerry Lodder.

F 07. The Failure of the Parallel Postulate

This project develops the non-Euclidean geometry pioneered by János Bolyai (1802–1860), Nikolai Lobachevsky (1792–1856) and Carl Friedrich Gauss (1777–1855). Beginning with Adrien-Marie Legendre's (1752–1833) failed proof of the parallel postulate [53], the project begins by questioning the validity of the Euclidean parallel postulate and the consequences of doing so. How would distance be measured without this axiom, how would "rectangles" be constructed, and what would the angle sum of a triangle be? The project continues with Lobachevsky's work [7], where he stated that in the uncertainty whether there is only one line though a given point parallel to a given line, he considered the possibility of multiple parallels, and continued to study the resulting geometry, limiting parallels, and proprieties of triangles in this new world. This is followed by a discussion of distance in hyperbolic geometry from the work of Bolyai [39] and Lobachevksy [7]. The project shows that all triangles in hyperbolic geometry have angle sum less than 180°, with zero being the sharp lower bound for such a sum, as anticipated by Gauss [40, p. 244]. The project continues with the unit disk model of hyperbolic geometry provided by Henri Poincaré (1854–1912) [61], and, following the work of Albert Einstein (1879–1955) [27], closes with the open question of whether the universe is best modeled by Euclidean or non-Euclidean geometry.

This project is designed for courses in geometry taken both by mathematics majors and secondary education majors. Author: Jerry Lodder.

F 08. Richard Dedekind and the Creation of an Ideal: Early Developments in Ring Theory

As with other structures in modern Abstract Algebra, the ring concept has deep historical roots in several nineteenth century mathematical developments, including the work of Richard Dedekind (1831–1916) on algebraic number theory. This project draws on Dedekind's 1877 text *Theory of Algebraic Numbers*[25] as a means to introduce students to the elementary theory of commutative rings and ideals. Characteristics of Dedekind's work that make it an excellent vehicle for students in a first course on abstract algebra include his emphasis on abstraction, his continual quest for generality and his careful methodology. The 1877 version of his ideal theory (the third of four versions he developed in all) is an especially good choice for students to read, due to the care Dedekind devoted therein to motivating why ideals are of interest to mathematicians by way of examples from number theory that are readily accessible to students at this level.

The project begins with Dedekind's discussion of several specific integral domains, including the example of $\mathcal{Z}[\sqrt{-5}]$ which fails to satisfy certain expected number theoretic properties (e.g., a prime divisor of a product should divide one of the factors of that product). Having thus set the stage for his eventual introduction of the concept of an *ideal*, the project next offers students the opportunity to explore the general algebraic structures of a ring, integral domain and fields. Following this short detour from the historical story — rings themselves were first singled out as a structure separate from ideals only in Emmy Noether's later work—the project returns to Dedekind's exploration of ideals and their basic properties. Starting only with his formal definition of an ideal, project tasks lead students to explore the basic concept of and elementary theorems about ideals (e.g., the difference between ideals and subrings, how properties of subrings and ideals may differ from the properties of the larger ring, properties of ideals in rings with unity). Subsequent project tasks based on excerpts from Dedekind's study of principal ideals and divisibility relationships between ideals conclude with his (very modern!) proofs that the least common multiple and the greatest common divisor of two ideals are also ideals. The project closes by returning to Dedekind's original motivation for developing a theory of ideals, and considers the sense in which ideals serve to recover the essential properties of divisibility — such as the fact that a prime divides a product of two rational integer factors only if it divides one of the factors — for rings like $\mathcal{Z}[\sqrt{-5}]$ that fail to satisfy these properties.

No prior familiarity with ring theory is assumed in the project. Although some familiarity with elementary group theory can be useful in certain portions of the project, it has also been successfully used with students who had not yet studied group theory. For those who have not yet studied group theory (or those who have forgotten it!), basic definitions and results about identities, inverses and subgroups are fully stated when they are first used within the project (with the minor exception of Lagrange's Theorem for Finite Groups which is needed in one project task). The only number theory concepts required should be familiar to students from their K-12 experiences; namely, the definitions (within $\mathcal{Z}$) of *prime*, *composite*, *factor*, *multiple*, *divisor*, *least common multiple*, and *greatest common divisor*.

This project is suitable for use in either a general abstract algebra courses at the introductory level, or as part of a junior or senior level courses in ring theory. Author: Janet Heine Barnett.

F 09. Primes, Divisibility, and Factoring

Questions about primality, divisibility, and the factorization of integers have been part of mathematics since at least the time of Euclid (c. 300 BCE). Today, they comprise a large part of an introductory class in number theory, and they are equally important in contemporary research. In this project, students investigate the development of the modern theory of these three topics by reading a remarkable 1732 paper by Leonhard Euler (1707–1783). This, Euler's first paper in number theory, contains a surprising number of new ideas in the theory of numbers. In a few short pages, he provided for the first time a factorization of $2^{2^5}+1$ (believed by Fermat to be prime), discussed the factorization of 2^n-1 and 2^n+1, and began to develop the ideas that would later lead to the first proof of what we now call Fermat's Little Theorem. In this work, Euler provided few proofs. By providing these, students develop an intimacy with the techniques of number theory, and simultaneously come to discover the importance of modern ideas and notation in the field.

This project is intended for courses in number theory. Author: Dominic Klyve.

F 10. The Pell Equation in Indian Mathematics

The Pell Equation is the Diophantine Equation

$$x^2 - Ny^2 = 1 \tag{1}$$

where N is a non-square, positive integer. The equation has infinitely many solutions in positive integers x and y, though finding a solution is not trivial.

In modern mathematics, the method of solving the Pell equation via continued fractions was developed by Lagrange (1736–1813). However, much earlier, Indian mathematicians made significant contributions to the study of the Pell equation and its solution. Brahmagupta (b. 598 CE) discovered that the Pell equation (1) can be solved if a solution to

$$x^2 - Ny^2 = k \tag{2}$$

where $k = -1, 2, -2, 4, -4$ is known. Later a method, a cyclic algorithm known in Sanskrit as *cakravāla*, to solve the Pell equation was developed by Jayadeva (fl. ninth century CE) and Bhāskara II (b. 1114 CE). While the project touches on the Pell equation in modern mathematics, the main focus is on its solution in Sanskrit mathematical texts. This approach will not only familiarize the students with the Pell equation and how it can be solved, but also expose them to significant mathematical work from a nonwestern culture.

This project is intended for a number theory course. Authors: Toke Knudsen and Keith Jones.

F 11. The Greatest Common Divisor: Algorithm and Proof

Finding the greatest common divisor of two integers is a foundational skill in mathematics, needed for tasks from simplifying fractions to cryptography. Yet, the best place to look for a simple algorithm for finding the greatest common divisor is not in a modern textbook, but in the writings of the ancient Chinese and the *Elements* [28] of Euclid (c. 300 BCE) in ancient Greece. In this project, students explore how the mutual subtraction algorithm evolved in ancient China, starting from a text dated c. 200 BCE, to the version of the algorithm in *The Nine Chapters on the Mathematical Art* [49], to the explanation of the *Nine Chapters* algorithm given by Liu Hui (fl. 3rd century BCE). They then explore the algorithm of Euclid and examine his careful proof. Parallel to the story of the development of the algorithm is a beautiful illustration of the history of proof. Proof in ancient China was not based on prepositional logic, but on demonstrating the correctness of an algorithm. Euclid was the pioneer of logical proof, yet his proof has flaws when examined in the light of modern rigor. Therefore, the project finishes by explicitly stating the properties of integers assumed in the proof of Euclid, and justifying the correctness of Euclid's iterative method using the power of inductive proof.

The project is suitable for a introduction to proof class, including junior level courses in algebra, discrete math or number theory. Author: Mary Flagg.

F 12. Determining Primality

One of the oldest problems in arithmetic is to determine whether a given number is prime. Thousands of years after it was first considered by Euclid (c. 300 BCE), Gauss (1777–1855) wrote "The problem of distinguishing prime numbers from composite numbers and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic." This project leads students through several mathematicians' attempts to solve this problem. Beginning with Euclid's proof of the infinitude of primes and his geometric approach to primes, we trace multiple approaches for finding or determining prime numbers. We shall examine approaches by Euler (1707–1783), Gauss, and Wilson (1741–1793).

This project could be suitable for use in a course (a) a number theory course, (b) a history of math class, (c) a capstone course for high school math teachers, or (d) a professional development unit for high school teachers. Author: Diana White.

F 13. Bolzano's Definition of Continuity, his Bounded Set Theorem, and an Application to Continuous Functions

The foundations of calculus were not yet on firm ground in early 1800's. Students read from 1817 paper [6] by Bernard Bolzano (1781–1848) in which he gave a definition of continuity and formulated a version of the least upper bound property of the real numbers. Students then read Bolzano's proof of the Intermediate Value Theorem.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 14. Rigorous Debates over Debatable Rigor: Monster Functions in Introductory Analysis

Although students in an introductory analysis course will have already encountered the majority of concepts studied in such a course during their earlier calculus experience, the study of analysis requires them to re-examine these concepts through a new set of powerful lenses. Among the new creatures revealed by these lenses are the family of functions defined by $f_\alpha(x) = x^\alpha \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f_\alpha(0) = 0$. In the late nineteenth century, Gaston Darboux (1842–1917) and Giuseppe Peano (1858–1932) each used members of this function family to critique the level of rigor in certain contemporaneous proofs. Reflecting on the introduction of such functions into analysis for this purpose, Henri Poincaré (1854–1912) lamented in [61]: "Logic sometimes begets monsters. The last half-century saw the emergence of a crowd of bizarre functions, which seem to strive to be as different as possible from those honest [honnêtes] functions that serve a purpose. No more continuity, or continuity without differentiability, etc. What's more, from the logical point of view, it is these strange functions which are the most general, [while] those which arise without being looked for appear only as a particular case. They are left with but a small corner. In the old days, when a new function was invented, it was for a practical purpose; nowadays, they are invented for the very purpose of finding fault in our father's reasoning, and nothing more will come out of it." Yet in [8], Émile Borel (1871–1956) proposed two reasons why these "refined subtleties with no practical use" should not be ignored: "[O]n the one hand, until now, no one could draw a clear line between straightforward and bizarre functions; when studying the first, you can never be certain you will not come across the others; thus they need to be known, if only to be able to rule them out. On the other hand, one cannot decide, from the outset, to ignore the wealth of works by outstanding mathematicians; these works have to be studied before they can be criticized."

In this project, students come to know these "monster" functions directly from the writings of the influential French mathematician Darboux and one of the mathematicians whose works he critiqued, Guillaume Houël (1823–1886). Project tasks based on the sources [23, 38] prompt students to refine their intuitions about continuity, differentiability and their relationship, and also includes an optional section that introduces them to the concept of uniform differentiability. The project closes with an examination of Darboux's proof of the theorem that now bears his name: every derivative has the intermediate value property. The project thus fosters students' ability to read and critique proofs in modern analysis, thereby enhancing their understanding of current standards of proof and rigor in mathematics more generally.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: Janet Heine Barnett.

F 15. An Introduction to Algebra and Geometry in the Complex Plane

In this project, students sudy the basic definitions, as well as geometric and algebraic properties, of complex numbers via Wessel's 1797 paper *On the Analytical Representation of Direction. An attempt Applied Chiefly to Solving Plane and Spherical Polygons* [72], the first to develop the geometry of complex numbers.

This project is suitable for a first course in complex variables, or a capstone course for high school math teachers. Authors: Diana White and Nick Scoville.

F 16. Nearness without Distance

Point-set topology is often described as "nearness without distance." Although this phrase is intended to convey some intuitive notion of the study of topology, the student is often left feeling underwhelmed after seeing this idea made precise in the definition of a topology. This project follows the development of topology, starting with a question in analysis, into a theory of nearness of points that took place over several decades. Motivated by a question of uniqueness of a Fourier expansion [13], Cantor (1845–1918) developed a theory of nearness based on the notion of limit points over several papers written over a decade, beginning in 1872 [14, 15, 16, 17, 18, 19]. Borel then took Cantor's ideas and began to apply them to a more general setting. Finally, Hausdorff (1868–1842) developed a coherent theory of topology in his famous 1914 book *Grundzüge der Mengenlehre* [41]. The purpose of this project is to introduce the student to the ways in which we can have nearness of points without a concept of distance by studying these contributions of Cantor, Borel, and Hausdorff.

This project is intended for courses in point–set topology or introductory topology. Author: Nick Scoville.

F 17. Connectedness–Its Evolution and Applications

The need to define the concept of "connected" is first seen in an 1883 work of Cantor (1845–1918) where he gives a rigourous definition of a continuum. After its inception by Cantor, definitions of connectedness were given by Jordan (1838–1922) and Schoenflies (1853–1928), among others, culminating with the current definition proposed by Lennes (1874–1951) in 1905. This led to connectedness being studied for its own sake by Knaster and Kuratowski. In this project, we trace the development of the concept of connectedness through the works of these authors [18, 47, 4, 54, 67], proving many fundamental properties of connectedness along the way.

This project is intended for courses in point–set topology or introductory topology. Author: Nick Scoville.

F 18. Construction of the Figurate Numbers

This project is accessible to a wide audience, requiring only arithmetic and elementary high school algebra as a prerequisite. The project opens by studying the triangular numbers, which enumerate the number of dots in regularly shaped triangles, forming the sequence 1, 3, 6, 10, 15, 21, etc. Student activities include sketching certain of these triangles, counting the dots, and studying how the nth triangular number, T_n, is constructed from the previous triangular number, T_{n-1}. Further exercises focus on tabulating the values of T_n, conjecturing an additive pattern based on the first differences $T_n - T_{n-1}$, and conjecturing a multiplicative pattern based on the quotients T_n/n. The triangular numbers are related to probability by enumerating the number of ways two objects can be chosen from n (given that order does not matter). Other sequences of two-dimensional numbers based on squares, regular pentagons, etc. are studied from the work of Nicomachus (c. 60–120 CE) [58].

The project continues with the development of the pyramidal numbers, P_n, which enumerate the number of dots in regularly shaped pyramids, forming the sequence 1, 4, 10, 20, 35, etc. Student activities again include sketching certain of these pyramids, tabulating the values of P_n, conjecturing an additive pattern based on the first differences $P_n - P_{n-1}$, and conjecturing a multiplicative pattern based on the quotients P_n/T_n. The pyramidal numbers are related to probability by counting the number of ways three objects can be chosen from n. Similar exercises are provided for the four-dimensional (triangulo-triangular) numbers and the five-dimensional (triangulo-pyramidal) numbers. The multiplicative patterns for these figurate numbers are compared to those stated by Pierre de Fermat (1601–1665), such as "The last number multiplied by the triangle of the next larger is three times the collateral pyramid" [55, p. 230f], which, when generalized, hint at a method for computing the n-dimensional figurate numbers similar to an integration formula.

This project is designed for a general education course in mathematics. Author: Jerry Lodder.

F 19. Pascal's Triangle and Mathematical Induction

In this project, students build on their knowledge of the figurate numbers gleaned in the previous project (F 18). The material centered around excerpts from Blasie Pascal's (1623–1662) "Treatise on the Arithmetical Triangle" [60], in which Pascal employs a simple organizational tool by arranging the figurate numbers into the columns of one table. The nth column contains the n-dimensional figurate numbers, beginning the process with $n = 0$. Pascal identifies a simple principle for the construction of the table, based on the additive patterns for the figurate numbers. He then notices many other patters in the table, which he calls consequences of this construction principle. To verify that the patterns continue no matter how far the table is constructed, Pascal states verbally what has become known as mathematical induction. Students read Pascal's actual formulation of this method, discuss its validity, and compare it to other types of reasoning used in the sciences and humanities today. Finally, students are asked to verify Pascal's twelfth consequence, where he identifies a pattern in the quotient of two figurate numbers in the same base of the triangle. This then leads to the modern formula for the combination numbers (binomial coefficients) in terms of factorials.

This project is designed for a general education course in mathematics. Author: Jerry Lodder.

F 20. Investigations Into d'Alembert's Definition of Limit

The modern definition of a limit evolved over many decades. One of the earliest attempts at a precise definition is credited to d'Alembert (1717–1783). Students read his limit definition and his propositions on uniqueness and the product of limits. Students formulate the modern definition of limit for sequences, and then explore modern proofs of d'Alembert's proposition on the limit of a product.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 21. An Introduction to a Rigorous Definition of Derivative

Cauchy (1789–1857) is generally credited with being among the first to define and use the derivative in a near-modern fashion. This project is designed to introduce the derivative with some historical background from Newton (1643–1727), Berkeley (1685–1783) and L'Hôpital (1661-1704). Students then read Cauchy's definition and examples from [21], and explore relevant examples and basic properties.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 22. Investigations Into Bolzano's Formulation of the Least Upper Bound Property

Bernard Bolzano (1781–1848) was among the first mathematicians to rigorously analyze the completeness property of the real numbers. This project investigates his formulation of the least upper bound property from his 1817 paper [6]. Students read his proof of a theorem on this property, a proof that inspired Karl Weierstrass (1815–1897) decades later in his proof of what is now known as the Bolzano-Weierstrass Theorem.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 23. The Mean Value Theorem

The Mean Value Theorem has come to be recognized as a fundamental result in a modern theory of the differential calculus. Iin this project, students read from the efforts of Cauchy (1789–1857) in [21] to rigorously prove this theorem for a function with continuous derivative. Later in the project, students explore a very different approach that was developed some forty years after Cauchy's proof, by mathematicians Serret and Bonnet [68].

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 24. Abel and Cauchy on a Rigorous Approach to Infinite Series.

Infinite series were of fundamental importance in the development of the calculus. Questions of rigor and convergence were of secondary importance early on, but things began to change in the early 1800s. When Niels Abel (1802–1829) moved to Paris in 1826, he was aware of certain paradoxes concerning infinite series and wanted big changes. In this project, students read from the 1821 *Cours d'Analyse* [11], in which Cauchy (1789–1857) carefully defined infinite series and proved some properties. Students then read from the paper [1], in which Able attempted to correct a flawed series convergence theorem from Cauchy's book.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 25. The Definite Integrals of Cauchy and Riemann

Rigorous attempts to define the definite integral began in earnest in the early 1800s. One of the pioneers in this development was Augustin-Louis Cauchy (1789–1857). In this project, students read from his 1823 study of the definite integral for continuous functions [21]. They then read from the 1854 paper [66], in which Bernard Riemann (1826–1846) developed a more general concept of the definite integral that could be applied to functions with infinite discontinuities.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 26. Gaussian Integers and Dedekind's Creation of an Ideal: A Number Theory Project

In the historical development of mathematics, the nineteenth century was a time of extraordinary change during which the discipline became more abstract, more formal and more rigorous than ever before. Within the subdiscipline of algebra, these tendencies led to a new focus on studying the underlying *structure* of various number (and number-like) systems related to the solution of various equations. The concept of a *group*, for example, was singled out by Évariste Galois (1811–1832) as an important algebraic structure related to the problem of finding all complex solutions of a general polynomial equation. Two other important algebraic structures — *ideals* and *rings* — emerged later in that century from the problem of finding all integer solutions of various equations in number theory. In their efforts to solve these equations, nineteenth century number theorists were led to introduce generalizations of the seemingly simple and quite ancient concept of an integer. This project examines the ideas from algebraic number theory that eventually led to the new algebraic concepts of an 'ideal' and a 'ring' via excerpts from the work of German mathematician Richard Dedekind (1831–1916).

A key feature of Dedekind's approach was the formulation of a new conceptual framework for studying problems that were previously treated algorithmically. Dedekind himself described his interest in solving problems through the introduction of new concepts as follows [26, p. 16]:

> The greatest and most fruitful progress in mathematics and other sciences is through the creation and introduction of new concepts; those to which we are impelled by the

frequent recurrence of compound phenomena which are only understood with great difficulty in the older view.

In this project, students encounter Dedekind's creative talents first hand through excerpts from his 1877 *Theory of Algebraic Integers* [25]. The project begins with Dedekind's description of the number theoretic properties of two specific integral domains: the set of rational integers $\mathbf{Z}$, and the set of Guassian integers $\mathbf{Z}[i]$. The basic properties of Gaussian integer divisibility are then introduced, and connections between Gaussian Primes and number theory results such as The Two Squares Theorem are explored. The project next delves deeper into the essential properties of rational primes in $\mathbf{Z}$ — namely, the Prime Divisibility Property and Unique Factorization — to see how these are mirrored by properties of the Guassian Primes in $\mathbb{Z}[i]$. Concluding sections of the project then draw on Dedekind's treatment of indecomposables in the integral domain $\mathbf{Z}[\sqrt{-5}]$, in which Prime Divisibility Property and Unique Factorization both break down, and briefly consider the mathematical after-effects of this 'break down' in Dedekind's creation of an *ideal*.

This project is intended for junior level courses in number theory, and assumes no prior knowledge of abstract algebra. Author: Janet Heine Barnett.

F 27. Otto Hölder's Formal Christening of the Quotient Group Concept

Today's undergraduate students are typically introduced to quotient groups only after meeting the concepts of equivalence, normal subgroups and cosets. Not surprisingly, the historical record reveals a different course of development. Although quotient groups implicitly appeared in work on algebraic solvability done by Galois (1811–1832) in the 1830s, that work itself pre-dated the development of an abstract group concept. Even the 1854 paper by Cayley (1821–1895) which marks the first appearance of a definition of an abstract group was premature, and went essentially ignored by mathematicians for decades. Permutation groups were extensively studied during that time, however, with implicit uses of quotient groups naturally arising as part of those studies. Camille Jordan (1838–1922), for example, used the idea of congruence of group elements modulo a subgroup to produce a quotient group structure [45, 46]. Thus, when Otto Hölder (1859–1937) gave what is now considered to be the first "modern" definition of quotient groups in 1889, he was able to treat the concept as neither new nor difficult [43].

This project examines Hölder's own treatment of the quotient group concept, leading up to a statement of the Fundamental Homomorphism Theorem. The evolution of the concept of abstract quotient groups within the context of earlier work done by Jordan and others who paved the way for Hölder is also treated in optional appendices to the project.

This project is intended is intended for introductory courses in abstract algebra or group theory. Author: Janet Heine Barnett.

F 28. Roots of Early Group Theory in the Works of Lagrange

This project studies works by one of the early precursors of abstract group, French mathematician J. L. Lagrange (1736–1813). An important figure in the development of group theory, Lagrange made the first real advance in the problem of solving polynomial equations by radicals since the work of Cardano (1501–1576) and his sixteeth-century contemporaries. In particular, Lagrange was the first to suggest a relation between permutations and the solution of equations of radicals that was later exploited by the mathematicians Abel (1802–1829) and Galois (1811–1832). Lagrange's description of his search for a general method of algebraically solving all polynomial equations is a model of mathematical research that make him a master well worth reading even today. In addition to the concept of a permutation, the project employs excerpts from Lagrange's work on roots of unity to develop concepts related to finite cyclic groups. Through their guided reading of excerpts from Lagrange, abstract algebra students encounter his original motivations while develop their own understanding of these important group-theoretic concepts.

This project is intended is intended for introductory courses in abstract algebra or group theory. Author: Janet Heine Barnett.

F 29. The Radius of Curvature According to Christiaan Huygens

Curvature is a topic in calculus and physics used today to describe motion (velocity and acceleration) of vector-valued functions. Many modern textbooks introduce curvature via a rather opaque definition, namely the magnitude of the rate of change of the unit tangent vector with respect to arc length. Such a definition offers little insight into what curvature was designed to capture, not to mention its rich historical origins. This project offers Christiaan Huygens's (1629–1695) highly original work on the radius of curvature and its use in the construction of an isochronous pendulum clock. A perfect time-keeper, if one could be constructed to operate at sea, would solve the longitude problem for naval navigation during the Age of Exploration.

Amazingly, Huygens identified the path of the isochrone as a cycloid, a curve that had been studied intensely and independently during the seventeenth century. To force a pendulum bob to swing along a cycloidal path, Huygens constrained the thread of the pendulum with metal or wooden plates. He dubbed the curve for the plates an evolute of the cycloid and described the evolutes of curves more general than cycloids. Given a curve and a point B on this curve, consider the circle that best matches the curve at B. Suppose that this circle has center A. Segment AB became known as the radius of curvature of the original curve at B, and the collection of all centers A as B varies over the curve form the evolute. Note that the radius of curvature AB is perpendicular to the original curve at B. For an object moving along this curve, AB helps in the identification of the perpendicular component of the force necessary to cause the object to traverse the curve. This is the key insight into the meaning of curvature.

This project is intended is intended for courses in multivariable or vector calculus. Author: Jerry Lodder.

F 30. A Proof and Application of Cotes's Theorem

The goal of this project is to develop and prove a theorem due to English mathematician Robert Cotes (1682–1716). Because no proof of Cotes's Theorem from the pen of Cotes himself is known today, the project instead follows the paper [72] by Caspar Wessel (1745-1818), a Danish surveyor by trade who made significant contributions to mathematics.

This project is suitable for a first course in complex variables, or a capstone course for high school math teachers. Authors: Diana White and Nick Scoville.

F 31. Cross Cultural Comparisons: The Art of Computing the Greatest Common Divisor

Finding the greatest common divisor between two or more numbers is fundamental to basic number theory. There are three algorithms taught to pre-service elementary teachers: finding the largest element in the intersection of the sets of factors of each number, using prime factorization and the Euclidean algorithm. This project has students investigate a fourth method found in *The Nine Chapters on the Mathematical Art* [49], an important text in the history of Chinese mathematics that dates from before 100 CE. This project asks students to read the translated original text instructions for finding the gcd of two numbers using repeated subtraction. Then students are asked to compare this method with the other modern methods taught. Students are led to discover that the Chinese method is equivalent to the Euclidean algorithm.

The project is well-suited to a basic algebra course for pre-service elementary and middle school teachers. Author: Mary Flagg.

F 32. A Look at Desargues' Theorem from Dual Perspectives

Girard Desargues (1591–1661) is often cited as one of the founders of Projective Geometry. Desargues was, at least in part, motivated by perspective drawing and other practical applications. However, this project focuses on Desargues' Theorem from a mathematical point of view. The theorem that today goes by his name is central to modern Projective Geometry. This porject, in fact, starts with a modern statement of Desargues' Theorem in order to more quickly appreciate the elegant beauty of the statement. Desargues' own proof of the theorem is, perhaps ironically, buried at the end of the treatise [9], which was written by his student Abraham Bosse (164–1676). The primary focus of this project is to understand Desargues' proof of the theorem from a classical perspective. To achieve this goal we read the proof given by Bosse, which requires a visit to other results of Desargues in his more famous work on conics [70], to classical results of Euclid (c. 300 BCE) from the *Elements* [28], and to a result of Menelaus (c. 100 CE) which we find both in Desargues' own colorful writings [70] and in those of Ptolemy (c. 100 CE) [71]. The project concludes with a view of Desargues' Theorem from a modern perspective. We also use the work of Jean Victor Poncelet (1788–1867) to reexamine Desargues' Theorem with the assumption that parallel lines meet at a point at infinity and with the principle of duality [62].

The development of the project is intended to both convey the geometrical content and help students learn to *do* math. It is meant to be accessible to students at the "Introduction to Proofs" level. Many of the exercises explicitly go through a read-understand-experiment-prove cycle. Some experience proving theorems in the spirit of Euclid would be helpful, but not absolutely necessary. A few optional exercises (whose answers could easily be found in a modern text) are left more open.

This project is designed for students in a Modern Geometry course or an Introduction to Proofs course. Author: Carl Lienert.

F 33. Completing the Square: From the Beginnings of Algebra

This project seeks to provide a deep understanding of the standard algebraic method of completing the square, the universal procedure for solving quadratic equations, through the reading of selections from *The Compendious Book on Calculation by Restoration and Reduction* [2, 64], written in the ninth century in Baghdad by Muḥammad ibn Mūsā al-Khwārizmī (c. 780–850 CE), better known today simply as al-Khwārizmī. At the same time, students become acquainted with a sense of how algebraic problem solving was successfully carried out in its earliest days even in the absence of symbolic notation, thereby conveying the importance of modern symbolic practices.

Future high school mathematics teachers who will be responsible for teaching algebra courses in their own classrooms will be well-served by working through this classroom module. It is also suitable for use in a general history of mathematics course as an introduction to the role of early Islamic era mathematics in the development of algebra as a major branch of mathematics, and is of value to instructors of higher algebra courses who are interested in conveying a sense of the early history of the theory of equations to their students. Author: Danny Otero.

F 34. Argand's Development of the Complex Plane

Complex numbers are a puzzling concept for today's student of mathematics. This is not entirely surprising, as complex numbers were not immediately embraced by mathematicians either. Complex numbers showed up somewhat sporadically in works such as those of Cardano (1501–1576), Tartaglia (1499–1557), Bombelli (1526–1572), and Wallis (1616–1703), but a systematic treatment of complex numbers was given in an essay titled *Imaginary Quantities: Their Geometrical Interpretation* [3], written by Swiss mathematician Jean-Robert Argand (1768–1822). This project studies the basic definitions, as well as geometric and algebraic properties, of complex numbers via Argand's essay.

This project is suitable for a first course in complex variables, or a capstone course for high school math teachers. Authors: Diana White and Nick Scoville.

Mini-Primary Source Project Descriptions

M 01. Babylonian numeration

Rather than being taught a different system of numeration, students in this project discover one for themselves. Students are given an accuracy recreation of a cuneiform tablet from Nippur with no initial introduction to Babylonian numerals. Unknown to the students, the table contains some simple mathematics – a list of the first 13 integers and their squares. Their challenge is threefold: to deduce how the numerals represent values, to work out the mathematics on the tablet, and to decide how to write the number "seventy two" using Babylonian numerals.
The Notes to Instructors for the project also suggests the small optional extension of asking students to compare the good and bad traits of several numeration systems.

This project is intended for "Math for the Liberal Arts" and Elementary Ed classes. Author: Dominic Klyve.

M 02. Regression to the Mean

Over a century ago, Francis Galton (1822–1911) noted the curious fact that tall parents usually have children shorter than they, and that short parents, in turn, have taller children. This observation was the beginning of what is now called "regression to the mean" – the phenomenon that extreme observations are generally followed by more average ones. In this project, students engage with Galton's original work on the subject [36], and build an understanding of the underlying causes for this sometimes non-intuitive phenomenon.

This project is intended for classes in Statistics, and would also be useful in a general education class on quantitative reasoning. Author: Dominic Klyve.

M 03. The Derivatives of the Sine and Cosine Functions

Working through the standard presentation of computing the derivative of $\sin(x)$ is a difficult task for a first-year mathematics student. Often, explaining "why" cosine is the derivative of sine is done via ad-hoc handwaving and pictures. Using an older definition of the derivative, Leonhard Euler (1707–1783) gave a very interesting and accessible presentation of finding the derivative of $\sin(x)$ in his *Institutiones Calculi Differentialis* [31]. The entire process can be mastered quite easily in a day's class, and leads to a deeper understanding of the nature of the derivative and of the sine function.

This project is intended for Calculus 1. Author: Dominic Klyve.

M 04. Beyond Riemann Sums

The purpose of this project is to introduce the method of integration developed by Fermat (1601–1665), in which he essentially used Riemann sums, but allowed the width of the rectangles to vary. Students work through Fermat's text [34], with the goal of better understanding the method of approximating areas with rectangles.

This project is intended for Calculus 1. Author: Dominic Klyve.

M 05. Fermat's Method for Finding Maxima and Minima

In his 1636 article "Method for the Study of Maxima and Minima" [33], Pierre de Fermat (1601–1665) proposed his method of *adequality* for optimization. In this work, he provided a rather cryptic sounding paragraph of instructions regarding how to find maxima and minima. Afterwards, he claimed that "It is impossible to give a more general method." Here, we trace through his instructions and see how it ends up being mostly equivalent to the standard modern textbook approach of taking a derivative and setting it equal to zero.

This project is intended for Calculus 1. Author: Kenneth Monks.

M 06. Euler's Calculation of the Sum of the Reciprocals of the Squares

This project introduces students to p-series via a proof of the divergence of the harmonic series in *Quaestiones super Geometriam Euclidis* [59], written by Nicole Oresme (c. 1325–1382) in approximately 1350. It continues with the proof via an infinite product formula for $\sin(s)/s$ that was given by Leonhard Euler (1707–1783) in his 1740 "De summis serierum reciprocarum" [29], showing that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

This project is intended for Calculus 2. Author: Kenneth Monks.

M 07. Braess' Paradox in City Planning: An Application of Multivariable Optimization

On December 5, 1990, The New York Times published an article titled *What if They Closed 42nd Street and Nobody Noticed?* Two of the early paragraphs in this article summarize what happened:

> "On Earth Day this year, New York City's Transportation Commissioner decided to close 42nd Street, which as every New Yorker knows is always congested. 'Many predicted it would be doomsday,' said the Commissioner, Lucius J. Riccio. 'You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem."

But to everyone's surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed."

This very counterintuitive phenomenon, in which the removal of an edge in a congested network actually results in improved flow, is known as Braess' Paradox. This paradox had actually been studied decades earlier not by rocket scientists, but by mathematicians. In the 1968 paper "On a paradox of traffic planning" [12], Dietrich Braess (1938–) described a framework for detecting this paradox in a network. In this project, we see how the examples he provided can be analyzed using standard optimization techniques from a multivariable calculus course.

This project is suitable for a course in multi-variable calculus, as well as a course in combinatorial optimization/network flows. Author: Kenneth Monks.

M 08: The Origin of the Prime Number Theorem

Near the end of the eighteenth century, Adrien-Marie Legendre (1752–1833) and Carl Friederich Gauss (1777–1855) seemingly independently began a study of the primes – more specifically, of what we now call their *density*. It would seem fairly clear to anyone who considered the matter that prime numbers are more rare among larger values than among smaller ones, but describing this difference mathematically seems not to have occurred to anyone earlier. Indeed, there's arguably no *a priori* reason to assume that there is a nice function that describes the density of primes at all. Yet both Gauss and Legendre managed to provide exactly that: a nice function for estimating the density of primes. Gauss claimed merely to have looked at the data and seen the pattern (His complete statement reads "I soon recognized that behind all of its fluctuations, this frequency is on the average inversely proportional to the logarithm.") Legendre gave even less indication of the origin of his estimate. In this project, students explore how they may have arrived at their conjectures, compare their similar (though not identical) estimates for the number of primes up to x, and examine some of the ideas related to different formulations of the Prime Number Theorem. Using a letter written by Gauss, they then examine the error in their respective estimates.

This project is intended for courses in number theory. Author: Dominic Klyve.

M 09–11. How to Calculate π

Most students have no idea how they might, even in theory, calculate π. Demonstrating ways that it can be calculated is fun, and provides a useful demonstration of how the mathematics they are learning can be applied. This series of mini-projects, any of which can be completed in one class period, leads students through different ways to calculate π. For a capstone or honors course, an instructor may choose to have students study all three methods, and then compare their efficiency. The sources on which the projects are based include [51, 56, 32].

The intended course for each mini-project is indicated below. Author of all mini-PSPs in this series: Dominic Klyve.

- M 09. **How to Calculate π: Machin's Inverse Tangents** In this mini-PSP, students rediscover the work of John Machin (1681–1751) and Leonhard Euler (1707–1783), who used a tangent identity to calculate π by hand to almost 100 digits.
- M 10. **How to Calculate π: Buffon's Needle** This project explores the clever experimental method for calculating π by throwing a needle on a floor on which several parallel lines have been drawn developed by Georges-Louis Leclerc, Comte de Buffon (1707–1788). It is available in two versions, as described below. Basic notions of geometric probability are introduced in both versions of the project.
  - M 10.1 **How to Calculate π - Buffon's Needle (Non-Calculus Version)** This version requires some basic trigonometry, but uses no calculus. It is suitable for use with students who have completed a course in pre-calculus or trigonometry.
  - M 10.2 **How to Calculate π - Buffon's Needle (Calculus Version)** This calculus-based version requires the ability to perform integration by parts. It is suitable for use in Calculus 2, capstone courses for secondary teachers and history of mathematics.
- M 11. **How to Calculate π: Euler (for Calculus 2)**

M 12–15. Gaussian Guesswork

Just prior to his nineteenth birthday, the mathematical genius Carl Friederich Gauss (1777–1855) began a "mathematical diary" in which he recorded his mathematical discoveries for nearly 20 years. Among these discoveries was the existence of a beautiful relationship relationship between three particular numbers: the ratio of the circumference of a circle to its diameter (π), a specific value (ϖ) of the elliptic integral $u = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$; and the Arithmetic-Geometric Mean of 1 and $\sqrt{2}$. Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice referred to as "Gaussian Guesswork" by historian Adrian Rice in his *Math Horizons* article "Gaussian Guesswork, or why 1.19814023473559220744 . . . is such a beautiful number" [65].

This set of four mini-projects, based on excerpts from Gauss' mathematical diary [37] and related texts, introduces students to the power of numerical experimentation via the story of his discovery of this beautiful relationship, while also serving to consolidate student proficiency of the following traditional topics from a Calculus 2 course:

- M 12: Arc Length and the Numerical Approximation of Integrals
- **M 13: Gaussian Guesswork: Elliptic Integrals and Integration by Substitution**
- **M 14: Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve**
- **M 15: Infinite Sequences and the Arithmetic-Geometric Mean**

Each of the four mini-PSPs can be used either alone or in conjunction with any of the other three.

This project is intended for Calculus 2. Author: Janet Heine Barnett.

M 16. The Logarithm of -1

Understanding the behavior of multiple-valued functions can be a difficult mental hurdle to overcome in the early study of complex analysis. Many eighteenth-century mathematicians also found this difficult. This one-day project looks at excerpts from letters (taken from [10]) in the correspondence between Euler (1707–1783) and Jean Le Rond d'Alembert in which they argued about the value of $\log(-1)$. This argument between Euler and d'Alembert not only set the stage for the rise of complex analysis, but helped to end a longstanding friendship.

This project is intended for complex variables classes. Author: Dominic Klyve.

M 17. Why be so Critical? Nineteenth Century Mathematical and the Origins of Analysis

The seventeenth century witnessed the development of calculus as the study of curves in the hands of Newton and Leibniz, with Euler (1707–1783) transforming the subject into the study of analytic functions in the eighteenth century. Soon thereafter, mathematicians began to express concerns about the relation of calculus (analysis) to geometry, as well as the status of calculus (analysis) more generally. The language, techniques and theorems that developed as the result of the critical perspective adopted in response to these concerns are precisely those which students encounter in an introductory analysis course — but without the context that motivated nineteenth-century mathematicians. This project employs excerpts from the texts [1, 5, 11, 24], written by Abel (1802–1829), Bolzano (1781–1848), Cauchy (1789–1857) and Dedekind (1831–1916) respectively, as a means to introduce students to that larger context in order to motivate and support development of the more rigorous and critical view required of students for success in an analysis course.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: Janet Heine Barnett.

M 18. Topology from Analysis: Making the Connection

Topology is often described as having no notion of distance, but a notion of nearness. How can such a thing be possible? Isn't this just a distinction without a difference? In this project, students discover the notion of "nearness without distance" by studying the work of Georg Cantor [13] on a problem involving Fourier series. In this work, they see that it is the relationship of points to each other, and not their distances per se, that is essential. In this way, students are led to see the roots of topology organically springing from analysis.

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 19. Connecting Connectedness

Connectedness has become a fundamental concept in modern topology. The concept seems clear enough-a space is connected if it is a "single piece." Yet the definition of connectedness we use today was not what was originally written down. In fact, today's definition of connectedness is a classic example of a definition that took decades to evolve. The first definition of this concept was given by Georg Cantor in an 1872 paper [13]. After investigating his definition, the project traces the evolution of the definition of connectedness through works of Jordan [48] and Schoenflies [67], culminating with the modern definition given by Lennes [54].

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 20. The Cantor Set before Cantor

A special construction used in both analysis and topology today is known as the Cantor set. Cantor used this set in a paper in the 1880s. Yet a variation of this set appeared as early as 1875, in the paper *On the Integration of Discontinuous Functions* [69] by the Irish mathematician Henry John Stephen Smith (1826–1883). Smith, who is best known for the Smith-normal form of a matrix, was a professor at Oxford who made great contributions in matrix theory and number theory. This project explores the concept of nowhere dense sets in general, and the Cantor set in particular, through his 1875 paper.

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 21. A Compact Introduction to a Generalized Extreme Value Theorem

In a short paper published just one year prior to his thesis, Maurice Frechet (1878–1973) gave a simple generalization of what we today call the Extreme Value Theorem: continuous real-valued funtions attain a maximum and a minimum on a closed bounded interval. Developing this generalization was a simple matter of coming up with "the right" definitions in order to make things work. In this mini-PSP, students work through Frechet's entire 1.5-page long paper [35] to give an extreme value theorem for a more general topological spaces: those which, to use Frechet's newly-coined term, are compact.

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 22. From Sets to Metric Spaces to Topological Spaces

One of the significant contributions that Hausdorff made in his 1914 book *Grundzüge der Mengenlehre* (*Fundamentals of Set Theory*) [41] was to clearly lay out for the reader the differences and similarities between sets, metric spaces, and topological spaces. It is easily seen how metric and topological spaces are built upon sets as a foundation, while also clearly seeing what is "added" to sets in order to obtain metric and topological spaces. In this project, we follow Hausdorff as he builds topology "from the ground up" with sets as his starting point.

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 23. The Closure Operation as the Foundation of Topology.

The axioms for a topology are well established- closure under unions of open sets, closure under finite intersections of open sets, and the entire space and empty set are open. However, in the early twentieth century, multiple systems were being proposed as equivalent options for a topology. Once such system was based on the closure property, and it was the subject of Polish mathematician K. Kuratowski's doctoral thesis. In this mini-project, students work through a proof that today's axioms for a topology are equivalent to Kuratowski's closure axioms by studying excerpts from both Kuratowski and Hausdorff.

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 24. Euler's Rediscovery of e.

The famous constant e appears periodically in the history of mathematics. In this mini-project, students read Euler (1707–1783) on e and logarithms from his 1748 book *Introductio in Analysin Infinitorum* [30], and use Euler's ideas to justify the modern definition: $e = \lim_{j\to\infty}(1 + 1/j)^j$.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

M 25. Henri Lebesgue and the Development of the Integral Concept

The primary goal of this project is to consolidate students' understanding of the Riemann integral, and its relative strengths and weaknesses. This is accomplished by contrasting the Riemann integral with the Lebesgue integral, as described by Lebesgue himself in a relatively non-technical 1926 paper [52]. A second mathematical goal of this project is to introduce the important concept of the Lebesgue integral, which is rarely discussed in an undergraduate course on analysis. Additionally, by offering an overview of the evolution of the integral concept, students are exposed to the ways in which mathematicians hone various tools of their trade (e.g., definitions, theorems).

In light of the project's goals, it is assumed that students have studied the rigorous definition of the Riemann integral as it is presented in an undergraduate textbook on analysis. Familiarity with the Dirichlet function is also useful for two project tasks. These tasks also refer to pointwise convergence of function sequences, but no prior familiarity with function sequences is required.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: Janet Heine Barnett.

M 26. Generating Pythagorean Triples via Gnomons

This mini-PSP is designed to provide students an opportunity to explore the number-theoretic concept of a Pythagorean triple. Using excerpts from Proclus' *Commentary on Euclid's Elements* [63], it focuses on developing an understanding of two now-standard formulas for such triples, commonly referred to as 'Plato's method' and 'Pythagoras' method' respectively. The project further explores how those formulas may be developed/proved via figurate number diagrams involving gnomons.

- **M 26.1 Generating Pythagorean Triples via Gnomons: The Methods of Pythagoras and of Plato via Gnomons**

  In this less open-ended version, students begin by completing tasks based on Proclus' verbal descriptions of the two methods, and are presented with the task of connecting the method in question to gnomons in a figurate number diagram only after assimilating its verbal formulation. *This version of the project may be more suitable for use in lower division mathematics courses for non-majors or prospective elementary teachers.*

- **M 26.2 Generating Pythagorean Triples via Gnomons: A Gnomonic Exploration**

  In this more open-ended version, students begin with the task of using gnomons in a figurate number diagram to first come up with procedures for generating new Pythagorean triples themselves, and are presented with Proclus' verbal description of each method only after completing the associated exploratory tasks. *This version of the project may be more suitable for use in upper division courses in number theory and discrete mathematics, or in capstone courses for prospective secondary teachers.*

Although more advanced students will naturally find the algebraic simplifications involved in certain tasks to be more straightforward, the only mathematical content pre-requisites are required in either version is some basic arithmetic and (high school level) algebraic skills. The major distinction between the two versions of this project is instead the degree of general mathematical maturity expected. Both versions include an open-ended "comparisons and conjectures" penultimate section that could be omitted (or expanded upon) depending on the instructor's goals for the course.

M 27. Seeing and Understanding Data

Modern data-driven decision-making includes the ubiquitous use of visualizations, mainly in the form of graphs or charts. This project explores the parallel development of thinking about data visually and the technological means for sharing data through pictures rather than words, tables, or lists. Students are provided the opportunity to consider both the data and the construction methods along with impact that broadening access to data has had on social concerns. Beginning with a tenth-century graph that was hand-drawn in a manuscript, students experience data displays printed with woodcuts and plates through those generated by digital typesetting and dynamic online or video-recorded presentations of data. Early uses of bar charts, pie charts, histograms, line charts, boxplots, and stem-and-leaf plots are compared with modern thoughts on graphical excellence.

This project is intended for courses in statistics, and is also well-suited to use in courses for general education and elementary education audiences that treat graphical displays of data. Authors: Beverly Wood and Charlotte Bolch.

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