Solving a System of Linear Equations Using Ancient Chinese Methods

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Solving a system of linear equations is a skill essential to the modern mathematician, and one that is generally the first topic of conversation in an introductory linear algebra classroom. The standard modern algorithm is often titled Gaussian elimination, leading students to believe that it was invented by Carl Friedrich Gauss (1777-1855). However, the algorithm has a much longer history.

The ancient Chinese textbook *The Nine Chapters on the Mathematical Art* [Shen et al., 1999], compiled by the first century BCE, has a chapter entitled *Fangcheng* which is translated as Rectangular Arrays. The chapter introdes the *Fangcheng* Rule, which is a general method for solving a system of linear equations using rectangular arrays. Although the Chinese viewed the problems in a completely different context than the modern notions of equations with variables, their technique is equivalent to Gaussian elimination. The purpose of this lesson is to use the *Fangcheng* Rule of the ancient Chinese to introduce Gaussian elimination.

1 Historical Background

The history of Gaussian elimination is a long one, with many contributions from notable mathematicians through the centuries [Grcar, 2011a]. Systems of linear equations were not commonly part of ancient mathematics. There are a few isolated problems preserved, with no general solution method. The one exception to this rule is the mathematics of ancient China in the text *The Nine Chapters on the Mathematical Art*, originally compiled by the first century BCE. In Western mathematics, symbolic algebra began to develop in the late Renaissance, and systems of linear equations appeared as exercises in algebra textbooks in the sixteenth and seventeenth centuries. This 'schoolbook elimination' used the power of symbolic algebra to solve a system of equations (linear or possibly nonlinear) using a series of substitutions to eliminate one variable at a time. Contributors to these methods included textbooks by Isaac Newton [1720], Michel Rolle[1690], the banker Nathaniel Hammond [1742] and Thomas Simpson [1755] [Grcar, 2011a].

A very compelling need for an efficient method of solving simultaneous linear equations arose in the early nineteenth century with the development of the method of least squares. The two men recognized for independently developing the least squares method are Legendre (1805) and Gauss (1809) [Grcar, 2011a, p. 178]. The method of least squares was developed to address the problem of making accurate predictions from collected data, and involved solving a system of linear equations.

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Gauss developed the method of least squares when calculating the orbit of the dwarf planet Ceres in 1794 or 1795. The method was not published until 1809, after Legendre had published his version in 1805. Only years later was it called Gaussian elimination in honor of Gauss [Grcar, 2011a, p. 209].

Solving systems of linear equations using matrix algebra first appeared in the literature in the late 1930's [Grcar, 2011a, p. 200]. Von Neumann and Goldstone [1947] were some of the first authors to connect the elimination method of Gauss with matrix algebra developed in the previous 10 years [Grcar, 2011a, p. 205]. It is remarkable that the ancient Chinese found the power of using a rectangular array to solve systems of linear equations nearly 2000 years earlier!

The *Nine Chapters on the Mathematical Art*, hereafter referred to as the *Nine Chapters* for brevity, dominated the early history of Chinese mathematics [Shen et al., 1999, p. 1]. It played a central role in Chinese mathematics equivalent to that of Euclid's *Elements* in Western mathematics. It remains the fundamental source of traditional Chinese mathematics. The *Nine Chapters* is an anonymous text, compiled across generations of mathematicians. It is believed that the original text was compiled by the first century BC, but it is difficult to date precisely. Western mathematical ideas were not introduced into China until the first Chinese translation of Euclid's *Elements* by Xu Guangqi (1562-1633) and Matteo Ricci (1552-1610) appeared in 1606 [Shen et al., 1999, p. 21].

The *Nine Chapters* is a series of 246 Problems and their Solutions organized into nine chapters by topic. The topics indicate that the text was meant for addressing the practical needs of government, commerce and engineering. The problems and solutions do not generally include an explanation of why a particular solution method worked. Unlike the Greek emphasis on proofs, the Chinese emphasized algorithms for solving problems. This does not mean that they did not know why an algorithm worked, it only shows that the most important goal was to show students how to perform the calculations correctly.

The chapters of the book demonstrate that an extensive body of mathematical knowledge was known to the ancient Chinese:

1. Rectangular Fields: This chapter is concerned with land measurement and gives the formulas for finding the areas of fields of several shapes.

2. Millet and Rice: Chapters 2 and 3 contain a variety of problems from agriculture, manufacturing and commerce.

3. Distribution by Proportion

4. Short Width: The problems in this chapter involve changing the dimensions of a field while maintaining the same area and includes algorithms for finding square roots and working with circles.

5. Construction Consultations: This chapter includes formulas for volumes of various solids.

6. Fair Levies: The problems in this chapter come from taxes and distribution of labor.

7. Excess and Deficit: The rule of double false position for solving linear equations is used to solve a variety of problems in this chapter. [Double false position refers to a method of solving a linear equation using trial and error by using a series of prescribed steps to obtain the correct solution from information reported on incorrect guesses, ans is still a viable method today.]

8. Rectangular Arrays: The *Fangcheng* Rule is introduced to solve systems of linear equations.

9. Right-angled Triangles: This chapter includes the *Gougu* Rule, known to Western mathematicians as the Pythagorean Theorem.

The noted Chinese mathematician Liu Hui, who flourished in the third century CE, published an annotated version in the year 263 [Shen et al., 1999, p. 3] with detailed explanations of many of the solution methods. We know very little of his life beyond his notes accompanying the text of the *Nine Chapters*. Fortunately, in his introduction he gives his motivation for annotating the text. Read the portion of Liu Hui's introduction printed below and discover his reasons for publishing his comments [Shen et al., 1999, pp. 52-58].

> I read the *Nine Chapters* as a boy, and studied it in full detail when I was older. [I] observed the division between the dual natures of Yin and Yang [the positive and negative aspects] which sum up the fundamentals of mathematics. Thorough investigation shows the truth therein, which allows me to collect my ideas and take the liberty of commenting on it. Things are known to belong to various classifications. Just as the branches of a tree are to its trunk, so are a multitude of things to an archetype. Therefore I have tried to explain the whole theory as concisely as possible, with spatial forms shown in diagrams, so that the reader should have a reasonably good all-around understanding of it.

The original text of the *Nine Chapters* presents the *Fangcheng* Rule with no extra explanation. Without first-hand knowledge of how the procedure was done, it is difficult for modern readers to translate the instructions. Liu Hui's commentary will prove invaluable as we attempt to decipher the ancient Chinese instructions for solving rectangular arrays. This lesson will examine both the original text and the corresponding commentary together to discover how they applied the *Fangcheng* Rule to systems of linear equations.

2 Counting Rod Arithmetic

Modern mathematicians have calculators and computers available to perform tedious arithmetic. Before the advent of calculating machines, all computation had to be carried out by hand. Complex computations required methods of keeping track of the numbers. Ancient China developed a very efficient system of computation by physically manipulating counting rods. Counting rods and rod arithmetic were used in China from 500 BCE until approximately 1500 CE when counting rods were gradually replaced with the abacus [Shen et al., 1999, pp. 11-17].

China developed a base-ten place value system for numerals. Counting rods were used to represent the digits 1–9 and the arrangement of the rods on a counting board indicated the place value. Counting rods were small bamboo sticks, approximately 2.5 mm in diameter and 15 cm long. The rods were laid out either upright or horizontally, as in Figure 1. One horizontal rod set atop a number of vertical rods, or a vertical rod on top of some horizontal rods each represent five units in the digits 6, 7, 8 and 9. Numbers were formed by alternating upright numerals for units, hundreds, etc., with horizontal numerals for tens, thousands, etc. Places with zeros were left blank since there was no symbol for zero in the counting rod system. The alternating horizontal and vertical numerals helped

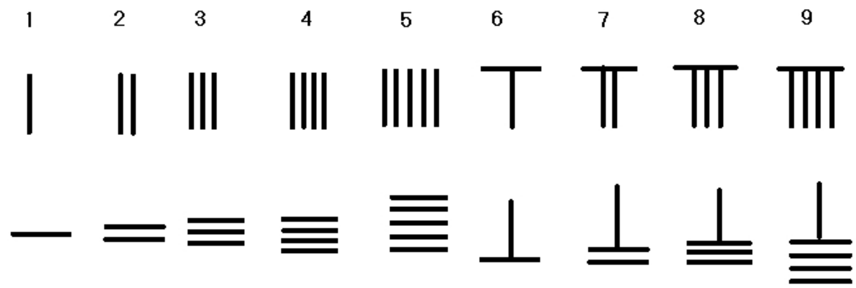

Figure 1: Vertical and Horizontal Counting Rod Numerals

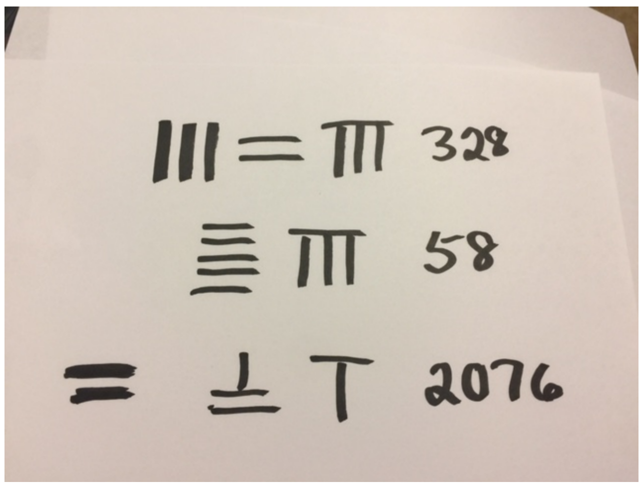

Figure 2: Examples of Rod Numbers

distinguish the places in the base ten numeral. The alternating orientation of the counting rods also served as a point of demarcation when one of the digits in the number was zero and the place in the written numeral was left blank.

Figure 2 illustrates the usefulness of the alternating orientation of the counting rods in the representations of the numbers 328, 58 and 2076. Notice that the alternating directions of the rods for numerals of successive powers of ten separates the 3 in the hundreds place and the 2 in the tens place, easily distinguishing 328 from 58. Also notice that the space in the counting rod representation of 2076 is marked by the fact that the numerals on either side of the space are horizontal. Numbers with more than one zero in a row, like 2003 or 400005, would need some context to help the reader interpret the space between the nonzero digits since the alternating horizontal and vertical rods would not obviously mark the missing digit. Do you see why zeros are so useful in our modern numerals?

Counting rod arithmetic was performed by manipulating the counting rods on a counting board. Unfortunately, we have no visual record of counting boards or how counting rod arithmetic was performed. Addition was carried out by the usual rules of combining the rods, replacing ten rods with one in the place to the left, and replacing five rods with one when recording digits greater than five. Subtraction was performed by taking away rods and borrowing one rod from one place and replacing it with ten rods in the next place to the right or replacing one rod with five to have enough rods to take away the required amount.

A multiplication problem was set up by placing rods representing the multiplicand in the top row of the counting board, and the rods representing the multiplier in the bottom row. As the procedure is performed, the answer is worked out in the middle row. The process of manipulating the rods kept no record of the intermediate steps, unlike our modern algorithm for multiplication. An example of how multiplication was performed will help us appreciate the value of counting rod arithmetic, so 48×67 will be illustrated.

Arrange the numbers on the counting board with each digit in a column representing a power of ten. All multiplication diagrams that follow will label the place values in each column in the header row, with the actual multiplication problem below. The multiplicand is laid in the top row and the multiplier is placed in the bottom row with its ones digit in the same column as the tens digit in the multiplicand.

1000	100	10	1
		4	**8**
	6	**7**	

(1)

Why place the 67 one column to the left of where we expect it to be? The number is not 670! The multiplication begins by performing 67×4, but the 4 is really 40 and we are going to shift the multiplier to the left one column so that the answer to 67×40 is actually in the correct columns. Begin by multiplying the 4 by the 6 and placing the answer in the middle row with the last digit in the same column as the 6.

1000	100	10	1
		4	**8**
2	4		
	6	**7**	

(2)

Then multiply the 4 by the 7 and place its last digit in the column with the 7 and add and carry where necessary.

1000	100	10	1
		4	**8**
2	$4 + 2 = 6$	8	
	6	**7**	

(3)

Next, remove the 4 from the multiplicand in the top row and slide the multiplier in the bottom row over one place to the right. The multiplier is slid to the right because the next step is to multiply 67×8, and shifting corrects the place values of the 67.

1000	100	10	1
			8
2	6	8	
		6	**7**

(4)

Then multiply the bottom 6 by the 8 in the top row and with the ones digit in the column with the 6 in the tens place in the multiplier.

1000	100	10	1
			8
2	$6+4$	$8+8$	
		6	**7**

(5)

Add the numbers in the middle row with carrying when needed.

1000	100	10	1
			8
3	1	6	
		6	**7**

(6)

Finally multiply the 8 by the 7 in the bottom row and add that answer to the numbers in the middle row.

1000	100	10	1
			8
3	1	$6+5$	6
		6	**7**

(7)

The last step is to add the numbers in the middle row and delete the numbers in the top and bottom row, showing only the answer.

1000	100	10	1
3	2	1	6

(8)

Task 1 The procedure for multiplication is actually easier to perform with the rods than it is to explain in words. Try multiplying 38×82 using counting rods. Use toothpicks or other small rods as counting rods and lay out a grid on your work space. Follow the steps for multiplication outline in the example above. How does this procedure compare to the algorithm you learned for multiplication? Where do you see a modern equivalent to shifting the multiplier to the left in the middle of our modern algorithm?

3 Forward Elimination

Chapter 8 of the *Nine Chapters* presents 18 problems and their solutions, all solved using the *Fangcheng* Rule for linear systems. All except one problem have the same number of unknowns and equations; problems range from solving a set of two equations in two unknowns to solving a set of five equations in five unknowns. Problem 13 is the exception: it has five equations and six unknown quantities. Problem 13 was solved by fixing the value of one of the variables and solving the resulting system of five equations in five unknowns. All of these systems arose from practical examples, and all had a unique solution. The *Fangcheng* Rule was used to find the single set of values for the unknowns which satisfy all equations simultaneously. Our introduction to Gaussian Elimination will focus on systems with unique solutions. Later lessons in a linear algebra course will illustrate how elimination is also used in a more general system.

The *Fangcheng* Rule gives step-by-step instructions for solving problems on a counting board which are equivalent to systems of linear equations. The physical manipulation of the counting rods and the visual organization of the array makes following the arithmetic rules straightforward. The Rule can be broken into two separate steps. The first step is often called **forward elimination** in modern mathematics, since it eliminates unknowns from equations from the top down, transforming them into systems which are substantially simpler to solve. Let's use a simple modern example to understand how useful elimination becomes. Suppose we want to solve the following system of equations.

$$\begin{aligned} x + 2y &= 7 \\ -x + 4y &= 11 \end{aligned}$$

If we add the two equations together and replace the second equation by the sum, we get the following equivalent system. Equivalent systems have the same solution set, so solving the second system gives the solution to the original system.

$$\begin{aligned} x + 2y &= 7 \\ 6y &= 18 \end{aligned}$$

$\boxed{\textbf{Task 2}}$ Do you see why the second system is simpler? What's its solution?

Consider the following system of three equations in three unknowns after forward elimination:

$$\begin{aligned} x - 4y + 2z &= 10 \\ 3y - z &= 5 \\ 5z &= 15 \end{aligned}$$

A straightforward solution procedure is to start from the bottom equation up, substituting values for the variables once they are known. This is what makes this system "simple" compared to one in which all variables are in all equations. Notice also that the shape of the system on the left-hand side of the equals sign resembles a triangle. This is called *upper triangular form.*

The ancient Chinese did not use equations and variables, only numbers organized into an array. However, the operations prescribed by the forward elimination steps in *Fangcheng* Rule transformed the array to one in which the nonzero entries formed a similar pattern. Since the Chinese did not use a mathematical symbol to represent zero at the time of the *Nine Chapters*, eliminating unknowns left blocks in the array blank and literally transformed the array into a triangle shape.

Once in the triangular form, the second half of the procedure solves the triangular array. We will refer to this as the **substitution** step, since one of the techniques is to substitute the value found for one unknown into the rest of the equations, reducing the number of unknowns. The ancient Chinese elimination step is identical in form to modern procedures, yet the steps for solving a triangular array using the *Fangcheng* Rule are different from modern techniques. In order to see the differences, forward elimination and substitution will be examined separately.

3.1 The Array on the Counting Board

Chapter 8 of the *Nine Chapters* starts by posing a problem. The English translation [Shen et al., 1999, p. 399] is presented below.

Problem 1: Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much paddy does one bundle of each grain yield? Answer: Top grade paddy yields $9\frac{1}{4}$ *dou* [per bundle]; medium grade paddy $4\frac{1}{4}$ *dou*; [and] low grade paddy $2\frac{3}{4}$ *dou*.

Problem 1 contains measures that are not familiar to modern readers, yet the basic structure of the problem looks exactly like many of the word problems in a modern linear algebra textbook. Paddy is rice. The term *dou* is an ancient unit of volume which translates to approximately 2 liters in modern units [Shen et al., 1999, p. 10]. Furthermore, the term *shi* in the *Fangcheng* Rule printed below means grain or fruit. It refers to the seeds of rice as it comes off the plant and before it is husked.

Task 3 The process for manually harvesting rice has not changed a great deal from the time of the *Nine Chapters* until modern times. The basic process is first to cut the rice stalks about 8 inches from the ground, then to gather the individual plants into bundles for transport. The bundles are then taken to be threshed, which separates the seeds of rice from the rest of the plant. For each grade of rice, the volume of rice seed obtained from one bundle of rice stalks is the quantity the problem asks us to find. Using modern language and modern measures for volume, restate Problem 1 in your own words.

Task 4 Write Problem 1 as a system of linear equations. For this system of equations, create a modern word problem unrelated to rice harvesting. For example, one could substitute different items on a lunch menu for the different grades of paddy, and prices of these items instead of yield of grain. Be creative!

The *Fangcheng* Rule for solving these types of problems appears immediately after Problem 1. The general rule is given by walking through the specific solution to Problem 1. This is the only rule given, and readers are supposed to follow the pattern on the counting board. The *Fangcheng* Rule starts by explaining how to organize the numbers on a counting board.

The Array (*Fangcheng*) Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. Similarly for the middle and left column.

In this lesson will will construct the array in the same way that the ancient Chinese did, as a grid with rows and columns. We will also use modern numerals throughout the rest of the lesson, but keep in mind that the Chinese used counting rod numerals and physically performed the arithmetic operations on the rods. Students are encouraged to do each of Tasks 5–9 on a separate sheet of paper, leaving room for solving the arrays in Tasks later in the lesson.

Task 5 Set up Problem 1 from Chapter 8 of the *Nine Chapters* using the instructions from the first two sentences of the Array Rule.

The pattern for Problem 1 is supposed to be followed when solving the other problems in the chapter. The problems become both more difficult to set up on the counting board and more difficult to solve.

Problem 3: Now there are 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 4 bundles of low grade paddy. Yield is each less than 1 *dou*. The top grade plus medium, the medium grade plus low [and] the low grade plus top, in each case adding one bundle, then the yield is one *dou*. Tell: What is the yield of 1 bundle of top, medium [and] low grade paddy?

Liu Hui's commentary helps us understand the question, as the wording was probably difficult for readers in his time, and definitely is not clear to modern readers.

Commentary on Problem 3 by Liu: Lay down 2 bundles of top grade paddy at the top of the right column, 3 bundles of medium grade paddy at the middle of the middle column, [and] 4 bundles of low grade paddy at the bottom of the left column; the 1 bundle to be added and the one *dou* yields in the appropriate places. Items that are added or borrowed in each column can follow this example.

Task 6 Liu tells us that the right column should be: 2 bundles of top grade paddy and one bundle of medium grade paddy yields one *dou* of grain. Use the *Fangcheng* Rule and Liu's commentary to set up the array for Problem 3. Remember that the Chinese counting rod arithmetic had no symbol for zero, so leave those entries blank!

The next problem should sound familiar from similar problems in high school algebra classes.

Problem 7: Now there are 5 cattle [and] 2 sheep costing 10 *liang* of silver. 2 cattle [and] 5 sheep costs 8 *liang* of silver. Tell: what is the cost of a cow and a sheep, respectively?

The *liang* is an ancient Chinese unit of weight which corresponds to approximately 16 grams in modern units [Shen et al., 1999, pp. 9-11].

Task 7 Set up Problem 7 from the *Nine Chapters*.

Problem 8 in the *Nine Chapters* requires the use of negative numbers in the formation of the array.

Problem 8: Now sell 2 cattle [and] 5 sheep, to buy 13 pigs. Surplus 1000 cash. Sell 3 cattle [and] 3 pigs to buy 9 sheep. [There is] exactly enough cash. Sell 6 sheep, [and] 8 pigs. Then buy 5 cattle. [There is] 600 coins deficit. Tell: what is the price of a cow, a sheep and a pig, respectively?

Task 8 Set up Problem 8 from the *Nine Chapters*. Consider selling as positive and buying as negative. Use negative numbers to represent a deficit. Read carefully, the numbers are not in the same order in each sentence!

Task 9 Set up the following Modern Problem using the instructions from the *Fangcheng* Rule: Three apples, 1 loaf of bread and 1 quart of milk cost \$7. One apple, 3 loaves of bread and 1 quart of milk cost \$10. Two apples, 4 loaves of bread and 3 quarts of milk cost \$18. Find the cost of each individual item.

The statements of the above problems from the *Nine Chapters* and the modern problem are reproduced in Appendix A for easy reference. These problems will be used throughout this lesson to practice the *Fangcheng* Rule and modern solution techniques.

3.2 The Sign Rule

Problem 8 clearly needs negative numbers to lay out the problem on a counting board. Even earlier, negative numbers are needed for Problem 3. The Sign Rule for arithmetic with negative numbers is included in the *Nine Chapters* following Problem 3 to explain how to handle addition and subtraction with negative numbers [Shen et al., 1999, pg. 404]. Negative and positive numbers were distinguished by differently colored counting rods, generally red for positive and black for negative.

The Sign *Zhengfu* Rule

Like signs subtract. Opposite signs add. Positive without extra, make negative; negative without extra make positive.

Opposite signs subtract; same signs add; positive without extra, make positive; negative without extra, make negative.

Task 10 The Sign Rule is actually two rules. The first rule (first sentence) explains how to perform $a - b$ for two numbers a and b depending on their signs. *Positive without extra make negative* in the first sentence refers to $0 - b$ when b is positive. The second sentence explains how to calculate $a + b$ depending on the signs of each. Explain the rules in your own words.

The following is a portion of Liu Hui's commentary on The Sign Rule. Note that the counting rods in different colors allowed the Chinese to represent excess as positive and deficit as negative in the normal course of solving practical problems.

Liu's Commentary on the Sign Rule: Now there are two opposite kinds of counting rods for gains and losses, let them be called positive and negative [respectively]. Red counting rods are positive, black counting rods are negative. Alternately distinguish [positive as] upright and [negative as] slanting. The rule for rectangular arrays [comprises] operations on the red and black entries from left to right. However, whether to add or subtract varies, so red and black [counting rods] are used to cancel one another. The operations of subtract or addition depend on the two types of entries in each column.

Task 11 The Sign Rule explains how to add or subtract positive and negative numbers, yet it makes no explicit mention of the sign of the answer. Why not? (Hint: Liu's Commentary gives us the clue that the rule was meant for rod arithmetic.)

3.3 Elimination by the *Fangcheng* Rule

Read the text of the *Fangcheng* Rule and try to follow it as you work the next Task.

Task 12 Try following the *Fangcheng* Rule with your array for Problem 1 before reading further. At what step were the instructions unclear?

The Array (*Fangcheng*) Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. Similarly for the middle and left column. Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge. Again multiply the next [and] follow the pivoting. Then use the remainder of the medium grade paddy in the middle column to multiply the left column and pivot. The remainder of the low grade paddy in the left column is the divisor, the entry below is the dividend. The quotient is the yield of low grade paddy. To solve for the medium grade paddy, use the divisor [of the left column] to multiply the *shi* in the middle column then subtract the value of the low grade paddy. To solve for the top grade paddy also take the divisor to multiply the *shi* of the right column then subtract the values of the low grade and the medium grade paddy. Divide by the number of bundles of top grade paddy. This is the number of bundles of top grade paddy. The constants are divided by the divisors. Each gives the *dou* of yield [of one bundle].

Did you follow the whole rule? Probably not! Don't be discouraged. The fact that Liu Hui gave detailed commentary to the *Fangcheng* Rule in his edition from 263 CE indicates that other ancient Chinese mathematicians needed help as well! The original students would have likely read the rule in conjunction with a visual demonstration of the procedure on a counting board. Since we no longer have any record of the visual part of the lesson, it will take a little more effort to translate the verbal description.

In this section we will use Liu Hui's commentary to help us understand how to perform the forward elimination steps in the *Fangcheng* Rule. We will use modern numerals to aid in understanding, but the layout will correspond to the ancient Chinese format in columns. Our goal is to reduce the array to triangular form. Read Liu's introduction to the *Fangcheng* Rule.

Liu's Commentary: The character *cheng* means comparing quantities. Given several different kinds of item, display [the number for] each as a number in an array with the sums (*shi*) [at the bottom]. Consider [the entries in] each column as rates, 2 items corresponds to a quantity twice, 3 items corresponds to a quantity 3 times, so the number of items is equal to the corresponding [number]. They are laid out in columns [from right to left], [and] therefore called a rectangular array (*fangcheng*). [Entries in each] column are distinct from one another and [these entries] are based on practical examples.

Task 13 What is the significance of Liu's statement that "Entries in each column are distinct from one another"? Why is it important that the numbers are based on practical examples?

We will now work through the *Fangcheng* Rule one step at a time, using Liu's commentary to help us understand the procedure.

Rule: [Let Problem 1 serve as example.] Lay down in the right column 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. Similarly for the middle and left column.

Liu's Commentary: This is the general rule [for arrays]. It is difficult to comprehend in mere words, so we simply use paddy to clarify. Lay down the middle and left column like the right column.

Task 14 Why is the *Fangcheng* Rule given using the numbers in a specific example instead of as a general procedure?

Traditional Chinese was read from top to bottom and then from right to left. Therefore, each combination of paddy and its yield is arranged in a column, with the numbers in order from top to bottom, starting from the right side of the counting board. The number of bundles of top grade paddy was in the first row in each column, medium grade in the second row, low grade paddy in the third row and the yield in the bottom row. The array for Problem 1 is the following:

1	2	3
2	3	2
3	1	1
26	34	39

(9)

The first step of the solution procedure is the following:

Rule: Use [the number of bundles of] top grade paddy in the right column to multiply the middle column then merge.

Liu's Commentary: The meaning of this rule is: subtract the column with smallest [top entry] repeatedly from the columns with larger [top entries], then the top entry must vanish. With the top entry gone, the column has one item absent. However, if the rates in one column are subtracted [from another column], this does not affect the proportions of the remainders. Eliminating the top entry means omitting one item from the sum (*shi*). In this way, subtract adjacent columns from one another. Determine whether [the sum is] positive or negative. Then one can obtain the answer. First take top grade paddy in the right column to multiply the middle column. This means homogenizing and uniformizing. To homogenize and uniformize means top grade paddy in the middle column also multiplies the right column. For the sake of simplicity, one omits saying homogenize and uniformize. From the point of view of homogenizing and uniformizing this reasoning is sound.

We are instructed to first multiply the middle column by 3, the number for top grade paddy in the right column. To multiply a column by a number means to multiply each entry in that column by the given number. Subtracting one row from another means to subtract the corresponding entries in every row of the given column. Modern mathematicians would record a zero if that was the result of subtraction. However, the ancient Chinese did not use a symbol for zero in counting board arithmetic, so we will follow the traditional procedure and leave the space blank.

Task 15 Use the instructions in this piece of the *Fangcheng* Rule to eliminate the number for the top grade paddy in the middle column of the array.

Liu next explains how to continue the process of elimination.

Liu's Commentary: Again eliminate the first entry in the left column. Then use the remainder of the medium grade paddy in the middle column to multiply the left column and pivot. Again, use the two adjacent columns to eliminate the medium grade paddy.

Task 16 Follow the same procedure Liu outlines.

1. Eliminate the 1 in the top row of the left column.
2. Follow the elimination procedure for the middle and left column with the medium grade paddy.

At this point your array should resemble a triangle. This is called *lower triangular form*.

The array for Problem 1 should now be in lower triangular form. Practice this procedure with the other Problems from the *Nine Chapters* and the Modern Problem by completing the following Tasks.

Task 17 Reduce the array you created in Task 6 for Problem 3 from the *Nine Chapters* to lower triangular form using the elimination procedure the *Fangcheng* Rule. Note that this problem involves using negative numbers in the elimination process.

Task 18 Reduce the array you created in Task 7 for Problem 7 from the *Nine Chapters* to lower triangular form using the elimination procedure the *Fangcheng* Rule.

Task 19 Reduce the array for the Modern Problem created in Task 9 to lower triangular form using the *Fangcheng* Rule.

The *Fangcheng* Rule, together with the Sign Rule, were used to solve problems that involved excess and deficit in the form of both positive and negative numbers. However, problems arose from practical examples, so the final solutions were always positive. In upper triangular form, the column containing only one unknown and a yield should be positive. Consider again Problem 8 in the *Nine Chapters*.

Problem 8: Now sell 2 cattle [and] 5 sheep, to buy 13 pigs. Surplus 1000 coins. Sell 3 cattle [and] 3 pigs to buy 9 sheep. [There is] exactly enough cash. Sell 6 sheep, [and] 8 pigs. Then buy 5 cattle. [There is] 600 coins deficit. Tell: what is the price of a cow, a sheep and a pig, respectively?

Answer: Cattle price 1200; sheep price 500; pig price 300.

Method: Use the Array Rule rule: lay down 2 cattle, 5 sheep, positive; 13 pigs, negative; surplus coins positive; next 3 cattle, positive; 9 sheep, negative; 3 pigs, positive; next 5 cattle, negative; 6 sheep, positive; 8 pigs, positive. Deficit coins, negative. Calculate using the Sign Rule.

Task 20 Perform elimination using the *Fangcheng* Rule on the array created for Problem 8 to reduce it to lower triangular form. Follow the steps prescribed below. This is the order suggested by Liu in his commentary to Problem 8.

1. Multiply the middle column by 2.
2. Replace the middle column with the middle column minus three times the right column.
3. Multiply the left column by 2 and then add 5 times the right column to eliminate the cattle number in the left column.
4. Use the sheep number in the left column to eliminate the sheep number in the middle column.

Is the resulting array in triangular form? How can it be transformed to triangular form? Why did Liu eliminate the sheep number in the middle column instead of eliminating the sheep number in the left column as the last step?

3.4 Elimination in Modern Mathematics

Modern solution methods for systems of linear equations take full advantage of the power of symbolic algebra. The unknown quantities in the statement of the problem are assigned variable representations and the stated conditions are translated into equations. Let x_1 be the yield of one bundle of top grade paddy, x_2 the yield of one bundle of medium grade paddy and x_3 be the yield of one bundle of low grade paddy. Then the equations for Problem 1 of the *Nine Chapters* have the following form.

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 39 \\ 2x_1 + 3x_2 + x_3 &= 34 \\ x_1 + 2x_2 + 3x_3 &= 26 \end{aligned}$$

The equations may be manipulated as equations, but the essential arithmetic in the problem comes down to operations with the coefficients of the variables and the yield numbers. Matrix notation is introduced to organize the numbers without needing all of the extra notation. We create an *augmented matrix* from the system of equations. An augmented matrix has one row for each equation and one column for each variable and the last column for the sum on the right hand side of the equation. The number in row i and column j of the matrix is the coefficient of variable x_j in equation i in the original system. The number in the last column of each row is the sum of the combination of variables in the equation. The augmented matrix for the system of equations for Problem 1 becomes the following:

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{pmatrix}$$

Compare the augmented matrix for Problem 1 with the layout on the Chinese counting board. Recall that traditional Chinese was read from top to bottom and then right to left, and modern English is read from left to right and then top to bottom. So the top right of the Chinese array would be the first number read, and the top left of the modern array is the first number read. Rotate the Chinese array counterclockwise to move the top grade paddy number from the Chinese first position to the modern first position and the result is the augmented matrix!

Task 21 Use modern notation to express Problem 3 in the *Nine Chapters* first as a system of linear equations, then as the equivalent augmented matrix.

The ancient Chinese elimination steps involved multiplying a row by a constant or subtracting one row from another row. Modern elimination does essentially the same procedures. However, modern mathematics demands a careful description of the operations that may be performed on an augmented matrix which do not change the solution set to the system of equations. The allowable operations for modern elimination are known as *elementary row operations*. There are three typse of elementary row operations which may be performed on an augmented matrix.

- Type 1: Replace a row by a nonzero multiple of that row.
- Type 2: Replace a row by the result of adding that row to a multiple of anther row.
- Type 3: Interchange two rows.

A Type 1 row operation on Row i multiplies every entry in Row i by the same nonzero constant c and will be denoted $R(i) \leftarrow c \times R(i)$, where the arrow $\leftarrow$ denotes replacement. A Type 2 row operation, replacing Row i with Row i plus a constant k times Row j, will be denoted $R(i) \leftarrow R(i) + k \times R(j)$. Interchanging two rows i and j will be denoted $R(i) \leftrightarrow R(j)$. Elementary row operations are performed one at a time.

Task 22 Elementary row operations do not change the solution set to a system of linear equations. Justify this statement for each type of row operation.

1. Type 1: If a row is multiplied by a nonzero constant, prove that the solution set does not change.
2. Type 2: Show that (x, y, z) is simultaneously a solution to the equations represented by $R(i)$ and $R(j)$ if and only if it is simultaneously a solution to the equations represented by $R(i)$ and $R(j) + k \times R(i)$ for any nonzero constant k and any natural numbers i, j.

3. Type 3: What does switching two rows in the augmented matrix correspond to relative to its associated system of linear equations? Does switching two rows alter the solution set? Explain.

Task 23 Elementary row operations are reversible. Reversing a Type 3 row operation is obvious, one simply switches the two equations back to the original configuration. How are Type 1 and Type 2 row operations reversed?

The purpose of elementary row operations is to eliminate nonzero entries in some equations in order to make the system easier to solve. The first goal is to transform the augmented matrix to one which is in upper triangular form.

The modern linear algebra term that encompasses upper triangular form of a matrix is *echelon form*. An *echelon* is a military formation of troops or equipment arranged in parallel rows with the end of each row projecting further than the one before. A matrix is in echelon form if the first nonzero entry in each row is to the right of the first nonzero entry in the row above it, and rows of all zeros are at the bottom of the matrix. Echelon form is more general than upper triangular form because matrices with any number of rows or columns can be brought to echelon form via elementary row operations, whereas not all matrices can be brought to a triangular form this way. The following matrix is in echelon form (Check this!).

$$\begin{pmatrix} 3 & 2 & 1 & 3 & 12 \\ 0 & 4 & 6 & 24 & 3 \\ 0 & 0 & 0 & 2 & 13 \end{pmatrix}$$

Task 24 Which of the following matrices is in echelon form? Justify your answers.

$$A = \begin{pmatrix} 3 & 5 & -2 & 4 \\ 0 & 0 & 53 & 5 \\ 0 & 1 & 12 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 2 & 4 & 5 \\ 0 & 0 & 1 & 5 & 7 \\ 0 & 0 & 0 & 2 & 13 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 0 & 5 & 7 \\ 0 & 5 & 1 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Row operations may be performed in any order, yet typically one starts in the top left corner of the matrix and works on eliminating nonzero entries below the diagonal in each column, in order from left to right and top to bottom. The systematic approach avoids the chance that one row operation will undo the progress made by the previous row operations.

The procedure will be demonstrated with the augmented matrix for Problem 1. A *pivot position* is defined as the position of the first nonzero entry in a row of the echelon form of the matrix. The number in that position is called the pivot in that row (or column) of the matrix and will be used as a multiplier to perform row operations of Type 1 or Type 2 that leave that position unchanged and eliminate nonzero entries in the pivot column and rows below the pivot. Begin with a nonzero number in the first row and left column. If this position contains a zero, switch two rows to put a nonzero number in that position. For Problem 1, the first pivot is the 3 which is highlighted in the matrix:

$$\begin{pmatrix} \boxed{3} & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{pmatrix}$$

At this point, our algorithm will follow the inspiration of the ancient Chinese because their method does not result in fractions in the echelon form of the matrix. Since the other numbers in the first column are not divisible by the pivot number 3, replace Row 2 by 3 times Row 2, $R(2) \leftarrow 3R(2)$. Then multiply Row 3 by 3 as well, $R(3) \leftarrow 3R(3)$.

$$\longrightarrow \begin{pmatrix} \boxed{3} & 2 & 1 & 39 \\ 6 & 9 & 3 & 102 \\ 3 & 6 & 9 & 78 \end{pmatrix}$$

Now perform row operations of Type 2 using the first row as the row to be added or subtracted to the other rows. First, replace Row 2 with the operation $R(2) \leftarrow R(2) - 2 \times R(1)$. Then, replace Row 3, $R(3) \leftarrow R(3) - R(1)$.

$$\longrightarrow \begin{pmatrix} \boxed{3} & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 3 & 6 & 9 & 78 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{3} & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 4 & 8 & 39 \end{pmatrix}$$

The next pivot is the 5 in the second row and second column. To eliminate the 4 below it, multiply the third row by 5, $R(3) \leftarrow 5 \times R(3)$. Then replace Row 3, $R(3) \leftarrow R(3) - 4 \times R(2)$.

$$\longrightarrow \begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & \boxed{5} & 1 & 24 \\ 0 & 4 & 8 & 39 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & \boxed{5} & 1 & 24 \\ 0 & 20 & 40 & 195 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{pmatrix}$$

The matrix is now in echelon form. This lesson will use the term *upper triangular form* instead of echelon form to emphasize its correspondence with lower triangular form in the Chinese array.

Task 25 Compare the steps just presented with the *Fangcheng* Rule steps on columns instead of rows that you performed in Tasks 15 and 16.

Task 26 Other orders of row operations are often suggested in modern linear algebra texts. Often the first step suggested is to switch rows if necessary to place a 1 in the first row and first column. Start with the matrix for Problem 1 and switch Row 3 and Row 1 in order to put a 1 in the upper left corner. Then pivot about the 1 by subtracting multiples of Row 1 from Row 2 and Row 3 to eliminate the nonzero entries below the 1 in the first column. What happens? Would you suggest this technique over the one outlined above? Does the order of the row operations change the final solution? How do you know?

The pattern for reducing the matrix from Problem 1 suggests the following steps for transforming a matrix into upper triangular form with row operations.

1. The first pivot position is the upper left corner. If this entry is a zero, perform a Type 3 row operation to switch Row 1 with another row which has a nonzero first entry. This entry is the first pivot p_1.

2. For each row with a nonzero entry in the first column below the first pivot, multiply that row by p_1.

3. For $j \geq 2$, if the first entry $a_{j,1}$ in Row j is nonzero, replace Row j with the operation $R(j) \leftarrow R(j) - a_{j,1} \times R(1)$. If the first entry $a_{j,1}$ is zero, do not change that row. Note that the notation $a_{j,k}$ refers to the number in row j and column k of the augmented matrix.

4. The second pivot position is in the second row and second column. If this position contains a zero, switch Row 2 with a row below it to put a nonzero number in the pivot position.

5. The second pivot p_2 is now the entry in the second row and second column. Multiply each row below Row 2 by p_2 if its entry in the second column is not zero.

6. For $j \geq 3$, if the entry $a_{j,2}$ in the second column of Row j is nonzero, replace Row j with the operation $R(j) \leftarrow R(j) - a_{j,2} \times R(2)$.

7. If there are more than 3 rows, continue by finding the next pivot in the next column from the left and use the pivot to eliminate nonzero entries below it in the same pattern.

This procedure assumes that the matrix has a pivot in every column corresponding to a variable, which is the case when the system has a unique solution. Further practice with elementary row operations will illustrate how variations in the orders of the steps are often done to make the arithmetic easier. The reader is encouraged to practice with this procedure.

Task 27 Create an augmented matrix for Problem 3 in the *Nine Chapters* and use the general procedure to reduce the matrix to upper triangular.

Task 28 Create an augmented matrix for Problem 7 in the *Nine Chapters* and use the general procedure to reduce the matrix to upper triangular.

Task 29 Create an augmented matrix for Problem 8 in the *Nine Chapters* and use the general procedure to reduce the matrix to upper triangular.

Task 30 Create an augmented matrix for the Modern Problem of Task 6 and use the general procedure to reduce the matrix to upper triangular.

4 Substitution: Solving the Triangular Array

Following the *Fangcheng* Rule to transform an array into lower triangular form or using modern elimination to reduce an augmented matrix to upper triangular form is only the first half of the process of solving the system of linear equations. Substitution, the second half of the solution procedure, may be performed in several different ways. We will first examine the instructions in the *Fangcheng* Rule. Next, we will use Liu Hui's commentary on the *Fangcheng* Rule to explain how the resulting formulas arise. Finally, we will discuss back substitution. Our goal is to compare the arithmetic in the *Fangcheng* Rule and in modern back substitution to discover each method's advantages and disadvantages.

4.1 Solving a Triangular Array by the *Fangcheng* Rule

The array for Problem 1 is in lower triangular form after the elimination steps:

		3
	5	2
36	1	1
99	24	39

(10)

The original *Fangcheng* Rule instructions explain how to find the values for the yields of one bundle of each grade of paddy.

The remainder of the low grade paddy in the left column is the divisor, the entry below is the dividend. The quotient is the yield of the low grade paddy. To solve for the medium grade paddy, usee the divisor [of the left column] to multiply the *shi* in the middle column then subtract the value of the low grade paddy. To solve for the top grade paddy, also take the divisor to multiply the *shi* of the right column then subtract the values of the low grade and the medium grade paddy. Divide by the number of bundles of top grade paddy. This is the number of bundles of top grade paddy. The constants are divided by the divisors. Each gives the *dou* of yield [in one bundle].

The instructions in the *Fangcheng* Rule begin easily enough. It is fairly straightforward to solve for the yield of low grade paddy. However, the instructions for medium grade and top grade paddy may be confusing to modern readers unfamiliar with the manipulations on a counting board.

Task 31 Although finding the low grade paddy yield is a simple division problem, the numbers in the left column will be needed for the rest of the procedure, so that division problem is delayed until the last step. Record the arithmetic described in the following steps of the solution process by writing out the arithmetic operations as well as the answers. This will help you compare this procedure with other methods.:

STEP 1 Multiply the number of bundles of low grade paddy in the left column by the yield in the middle column and subtract the number of bundles of low grade paddy in the middle column times the yield in the left column, then divide that answer by the number of bundles of medium grade paddy in the middle column. The result is the number of *dou* in the *shi* of the medium grade paddy.

STEP 2 For the top grade paddy, multiply the number bundles of low grade paddy times the yield in the *shi* of the right column and then subtract the number of bundles of medium grade paddy in the right column times the *dou* in the *shi* of medium grade paddy from Step 1 and subtract the number of *dou* in the *shi* of low grade paddy. Divide that result by the number of bundles of top grade paddy in the right column. The result is the number of *dou* in the *shi* of top grade paddy.

STEP 3 The final step is to divide the number of *dou* in the *shi* for each grade of paddy by the number of bundles of low grade paddy. The results are the final answers.

Did you obtain 153 as your answer for STEP 1 and 333 for your answer to STEP 2? The answer for STEP 3 should be the solution given by the *Nine Chapters*. Where do the number answers for STEP 1 and STEP 2 come from?

The instructions in the original *Fangcheng* Rule can be confusing to the modern reader. In [Hart, 2011, p. 95] the instructions are translated into solution steps for a general system of three equations and three unknowns that is in lower triangular form. Consider the following array. Note that the subscript notation reflects the fact that traditional Chinese was read from top to bottom and then right to left, so the first number counts the columns, starting from the right, and the second number counts the rows, starting from the top.

$$\begin{array}{|c|c|c|} \hline & & a_{11} \\ \hline & a_{22} & a_{12} \\ \hline a_{33} & a_{23} & a_{13} \\ \hline b_3 & b_2 & b_1 \\ \hline \end{array} \tag{11}$$

The steps for the general array with three equations and three unknowns are as follows.

STEP 1 Calculate

$$B_2 = (a_{33}b_2 - a_{23}b_3) \div a_{22}.$$

STEP 2 Using the result from the first step, calculate

$$B_1 = (a_{33}b_1 - a_{12}B_2 - a_{13}b_3) \div a_{11}.$$

STEP 3 Divide by the number in the third row of the left column, a_{33}, to solve for the unknowns.

$$x_1 = B_1 \div a_{33}.$$
$$x_2 = B_2 \div a_{33}.$$
$$x_3 = b_3 \div a_{33}.$$

Task 32 Solve the lower triangular array for Problem 3 in the *Nine Chapters* using the *Fangcheng* Rule procedure described in this section.

Task 33 Solve the lower triangular array for Problem 7 in the *Nine Chapters* using the *Fangcheng* Rule procedure.

Task 34 Solve the lower triangular array for Problem 8 in the *Nine Chapters* using the *Fangcheng* Rule procedure.

Task 35 Solve the lower triangular array for the Modern Problem using the *Fangcheng* Rule procedure.

4.2 Liu Hui's Commentary Illuminates the *Fangcheng* Rule Procedure

This part of the *Fangcheng* Rule is not obvious! Liu Hui admits that the procedure given is complicated, and suggests a variation that involves continuing the eliination steps to eliminate all but one variable in each column [Shen et al., 1999, p. 400].

Liu's Commentary: After eliminating the top grade and medium grade paddy, the remaining [bottom entry] is the yield of not just one bundle of low grade paddy. To reduce the yield on all the bundles one should take the number of bundles of paddy as divisor. Display this. Take the number of bundles of low grade paddy and multiply [the entries in] the second column. Merge, eliminate the entries of the low grade paddy. The *shi* is then divided by the number of bundles to give the number of *dou* of medium grade paddy. The calculations involved are complicated and inefficient, hence an alternative rule is introduced for simplification. However, if the old rule has to be used, this is a variation.

Liu Hui offers the reader a variation on the instruction in the *Fangcheng* Rule which is still complicated, but performs the arithmetic in a similar order to the original rule. The process is to continue to perform elimination steps so that each column contains only one entry for a different grade of paddy. The procedure is as follows:

Task 36 Follow the instructions by multiplying the middle column by the low grade paddy number in the left column. Then subtract the left column from the middle column and divide the middle column by 5. Did you notice that this Task performs STEP 1 of the *Fangcheng* Rule procedure for solving the triangular array? You should obtain the following array:

		3
	36	2
36		1
99	153	39

(12)

Task 37 Finish the solution process on the above array by performing the following steps.

1. Multiply the right column by 36.
2. Subtract twice the middle column from the right column.
3. Subtract the left column from the right column. Then divide the resulting right column by 3.
4. Divide each column by 36, giving the final answer.

Task 38 Follow the pattern explained by Liu Hui as in the previous Tasks to solve the system represented by the triangular array for Problem 3 in the *Nine Chapters* obtained in Task 17.

4.3 Back Substitution

Liu Hui suggests an alternate way to interpret the *Fangcheng* Rule instructions for solving the system corresponding to a lower triangular array. The following is the beginning of his instructions.

Liu: These are values for the medium [and] low grade paddy. Calculate the yield for one bundle of low grade paddy first. Substitute in the middle column to find [the yield of] the medium paddy.

Liu's comments suggest what is arguably the most intuitive way to solve the system corresponding to the triangular array. If the yield of one bundle of low grade paddy is known, then the middle column involving only low grade and medium grade paddy has only one unknown, which can be found. Then, the right column has only the top grade paddy as unknown, and its value may be found. The modern method that corresponds to this idea called is back substitution. This section demonstrates the process of back substitution on an upper triangular augmented matrix.

Consider the augmented matrix which corresponds to Problem 1 in the *Nine Chapters*.

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{pmatrix}$$

The last row, when translated back into equation form, reads $36x_3 = 99$. This gives $x_3 = \frac{99}{36} = 2\frac{3}{4}$.

The second row of the matrix is equivalent to the equation $5x_2 + x_3 = 24$. The value of x_3 can be substituted into this equation. Solving the equation for x_2 results in the following:

$$x_2 = \frac{24 - 2\frac{3}{4}}{5} = 4\frac{1}{4}$$

Then substitute the values of x_3 and x_2 into the equation corresponding to the first row, $3x_1 + 2x_2 + x_3 = 39$, to solve for x_1:

$$x_1 = \frac{39 - 2 \times 4\frac{1}{4} - 2\frac{3}{4}}{3} = 9\frac{1}{4}$$

The forward elimination algorithm transforms the array from the top line down, and then the back substitution algorithm works backwards from the bottom up – hence 'back' substitution. The procedure is straightforward to generalize: translate the upper triangular array into equations and solve them one at a time, starting from the bottom. Substitute the known values into subsequent equations as the process proceeds.

Task 39 Solve the upper triangular matrix for Problem 3 from the *Nine Chapters* using back substitution.

Task 40 Solve the upper triangular matrix for Problem 8 from the *Nine Chapters* using back substitution.

Task 41 Solve the upper triangular matrix from the Modern Problem using back substitution.

4.4 Comparing *Fangcheng* Rule and Back Substitution Arithmetic

The systems of linear equations we have examined in this lesson all have a single unique solution. The method used to obtain this solution does not change the final answer. However, the amount and complexity of the arithmetic that is involved in finding the solution changes with the choice of method for substitution. Furthermore, the precise order for performing the elimination or substitution operations may change the complexity of the arithmetic. For example, an arbitrary elementary row operation could be performed on an augmented matrix, followed immediately by the reverse operation. This involves extra arithmetic with no benefit. Roger Hart [Hart, 2011, pp. 99-110] points out that the *Fangcheng* Rule's counterintuitive method of solving the triangular array usually had the advantage of introducing fractions only in the last step of the substitution process.

Task 42 Compare the arithmetic for the *Fangcheng* Rule solution to Problem 1 of the *Nine Chapters* with the modern technique of forward elimination followed by back substitution. Which method would you prefer to practice if you had to manually solve similar problems on a regular basis? Why?

5 Conclusions

The *Fangcheng* Rule in Chapter 8 of the *Nine Chapters* was conceived about 2000 years ago, yet it feels very modern!

The *Fangcheng* Rule demonstrated the important consideration of designing a technique that keeps the arithmetic as simple as possible. We saw that the substitution step prescribed by the original rule was counterintuitive, yet had the advantage of delaying the introduction of fractions until the last step.

Liu Hui puts the importance of wisely choosing how to apply the *Fangcheng* Rule to make the arithmetic as simple as possible in his comments on Problem 18 [Shen et al., 1999, p. 426]. Problem 18 in the *Nine Chapters* has five variables and five equations. Clearly, the *Fangcheng* Rule requires many steps. As he introduces his new variation on the rule, Liu explains why it is important to think before blindly starting to calculate. We will not delve into the *New Array Rule* that Liu proposed because its complexity will not be enlightening. However, the following excerpt is instructive for students of mathematics at any level:

> Liu: . . . Some clumsy students of exact science use this Rule mechanically, or arrange a lot of counting rods on a carpet. They are meticulous but vulnerable, without thinking it irrational to behave so, but rather [saying] "the more the better". All the algorithms, even though each applies to special subjects individually, are mutually connected by basic principles. For certain problems, this [rule] succeeds; however, they are really quite fallible, and cannot be regarded as simple. Then there is a special solution. The skillful butcher always leaves plenty of room for the play of his knife [when slicing] through the muscles in dissecting an ox, so that the blade remains sharp even after a lengthy period. Mathematical rules are just like the blade, and simplicity conforms to the butcher's way. So if one takes care with the blade, one can solve problems both promptly and with few mistakes. . . .

Task 43 Does Liu Hui give us good advice? Have you experienced problems for which this advice would have helped make solving easier?

Although systems of linear equations may be solved using a computer in modern times, the question of streamlining the arithmetic remains an important consideration. The Chinese were amazingly ahead of their time!

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Appendix A: The Problems

Text from the *Nine Chapters* [Shen et al., 1999] for the problems in the Tasks and the Modern Problem in the Tasks are copied here for easy reference.

Problem 1: Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much paddy does one bundle of each grain yield? Answer: Top grade paddy yields $9\frac{1}{4}$ *dou* [per bundle]; medium grade paddy $4\frac{1}{4}$ *dou*; [and] low grade paddy $2\frac{3}{4}$ *dou*.

Problem 3: Now there are 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 4 bundles of low grade paddy. Yield is each less than 1 *dou*. The top grade plus medium, the medium grade plus low [and] the low grade plus top, in each case adding one bundle, then the yield is one *dou*. Tell: What is the yield of 1 bundle of top, medium [and] low grade paddy?

Problem 7: Now there are 5 cattle [and] 2 sheep costing 10 *liang* of silver. 2 cattle [and] 5 sheep costs 8 *liang* of silver. Tell: what is the cost of a cow and a sheep, respectively?

Problem 8: Now sell 2 cattle [and] 5 sheep, to buy 13 pigs. Surplus 1000 cash. Sell 3 cattle [and] 3 pigs to buy 9 sheep. [There is] exactly enough cash. Sell 6 sheep, [and] 8 pigs. Then buy 5 cattle. [There is] 600 coins deficit. Tell: what is the price of a cow, a sheep and a pig, respectively?

Modern Problem: Three apples, 1 loaf of bread and 1 quart of milk cost \$7. One apple, 3 loaves of bread and 1 quart of milk cost \$10. Two apples, 4 loaves of bread and 3 quarts of milk cost \$18. Find the cost of each individual item.

Appendix B: Results of Forward Elimination

This lesson used Problems 1, 3, 7 and 8 in the *Nine Chapters* as well as a Modern Problem to practice elimination and substitution. Later Tasks ask students to solve the Chinese array or matrix starting from the result of an earlier exercises. The answers to the Tasks concerned with setting up the array and performing forward elimination on the array are printed here for each of these Problems. Note that the upper or lower triangular form printed here is the result of following the algorithms in this lesson in the order given. Upper (lower) triangular form is not unique, other forms are possible.

Problem 1 in the *Nine Chapters*

The Chinese array for Problem 1 is as follows:

1	2	3
2	3	2
3	1	1
26	34	39

The Chinese array in lower triangular form:

		3
	5	2
36	1	1
99	24	39

The augmented matrix for Problem 1 is as follows:

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{pmatrix}$$

The augmented matrix for Problem 1 in upper triangular form:

$$\begin{pmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{pmatrix}$$

Problem 3 in the *Nine Chapters*

The Chinese array for Problem 3 is as follows:

1		2
	3	1
4	1	
1	1	1

The Chinese array in lower triangular form:

		2
	3	1
25	1	
4	1	1

The augmented matrix for Problem 3 is as follows:

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 4 & 1 \end{pmatrix}$$

The augmented matrix for Problem 3 in upper triangular form:

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 25 & 4 \end{pmatrix}$$

Problem 7 in the *Nine Chapters*

The Chinese array for Problem 7 is as follows:

2	5
5	2
8	10

The Chinese array in lower triangular form:

	5
21	2
20	10

The augmented matrix for Problem 7 is as follows:

$$\begin{pmatrix} 5 & 2 & 10 \\ 2 & 5 & 8 \end{pmatrix}$$

The augmented matrix for Problem 7 in upper triangular form:

$$\begin{pmatrix} 5 & 2 & 10 \\ 0 & 21 & 20 \end{pmatrix}$$

Problem 8 in the *Nine Chapters*

The Chinese array for Problem 8 is as follows:

-5	3	2
6	-9	5
8	3	-13
-600		1000

The Chinese array in (not quite) lower triangular form:

		2
37		5
-49	48	-13
3800	14400	1000

The augmented matrix for Problem 8 is as follows:

$$\begin{pmatrix} 2 & 5 & -13 & 1000 \\ 3 & -9 & 3 & 0 \\ -5 & 6 & 8 & -600 \end{pmatrix}$$

The augmented matrix for Problem 8 in upper triangular form:

$$\begin{pmatrix} 2 & 5 & -13 & 1000 \\ 0 & 37 & -49 & 3800 \\ 0 & 0 & 48 & 14400 \end{pmatrix}$$

The Modern Problem

The Chinese array for the Modern Problem is as follows:

2	1	3
4	3	1
3	1	1
18	10	7

The Chinese array in lower triangular form:

		3
	8	1
36	2	1
90	23	7

The augmented matrix for the Modern Problem is as follows:

$$\begin{pmatrix} 3 & 1 & 1 & 7 \\ 1 & 3 & 1 & 10 \\ 2 & 4 & 3 & 18 \end{pmatrix}$$

The augmented matrix for the Modern Problem in upper triangular form:

$$\begin{pmatrix} 3 & 1 & 1 & 7 \\ 0 & 8 & 2 & 23 \\ 0 & 0 & 36 & 90 \end{pmatrix}$$

The solution to the Modern Problem One apple costs \$0.75, one loaf of bread costs \$2.25 and on quart of milk costs \$2.50.

Instructor Notes

This lesson introduces Gaussian elimination on the augmented matrix for small systems of linear equations. All systems in this lesson have a unique solution. The instructor inn a typical linear algebra class can use this lesson in place of the introductory textbook section on reducing an augmented matrix to echelon form and solving by back substitution. The *Nine Chapters* does not address the issue of systems without a unique solution, therefore this lesson does not go into any detail on the techniques for handling more complicated systems. The instructor would follow this lesson with the more general instructions for elimination from a standard textbook. There are no specific prerequisites for undergraduate students.

It is estimated that it will take 2-3 weeks to implement the full PSP with a linear algebra class. A mixture of student individual preparation and in-class activities is optimal. The following lesson plan assumes a 75 minute class period. Instructors teaching a 50 minute class should allow 3 class periods for each two days listed.

- Day 1: Introduce the PSP and ask students to read through Section 3.3 and work Tasks 1-9.
- Day 2: Ask students to report on their reading and take questions. Check understanding of rod arithemtic and on setting up an array Chinese style. Then, break the students into small working groups for in-class work. Classroom work beginning with Task 12 on applying the *Fangcheng* Rule and choosing selected Tasks through Task 19. Every group should work through Problem 1, then different groups could practice on a different Problem in the *Nine Chapters*. Assign Tasks 10 and 11 on the Sign Rule as homework, reminding the students that it was written for calculating with rod arithmetic.
- Day 3: Take questions on the assignment on the Sign Rule. Review the *Fangcheng* Rule procedure and introduce modern elementary row operations. Work in groups on the procedure. Pay particular attention to Tasks 25 and 26 on comparing modern methods with those of the ancient Chinese and choosing the order in which the elementary row operations are performed. Assign Tasks 27-30 as homework.
- Day 4: Review forward elimination by asking for volunteers to share their solutions to Tasks 27-30. Then introduce substitution. Have the class work through Sections 4.1 and 4.2 in groups. Instructors may wish to do some of this as a whole class demonstration, then let student groups try selected Tasks from Section 4.1 and 4.2. Each group may try both versions of the Chinese procedures on one Problem from the *Nine Chapters*.
- Day 5: Finish the lesson by solving arrays by modern back substitution. Then compare the arithmetic. Assign Task 42 as homework.

The essential ideas of elimination can be presented in a linear algebra class by spending 3-4 class instruction hours on Section 3 alone. Courses in algebra for pre-service teachers should allow time to examine rod arithmetic in more detail, and emphasize the Sign Rule.

The Tasks in this lesson are intended as illustrations to help the reader practice the techniques as they read. Instructors are encouraged to require students to keep a notebook with all their work (showing each step in each algorithm) so that they can compare the complexity of the arithmetic using multiple approaches. The same four problems from the *Nine Chapters* and the same modern problem appear throughout the sections on elimination and substitution using both the ancient Chinese and the modern techniques. The purpose is to analyze the differences in the algorithms with essentially the same arithmetic operations in different orders. Appendix B provides the arrays and the reduced arrays after forward elimination for each of the Problems in the *Nine Chapters*. The purpose of including these hints are to let students check their work at the intermediate stage and to start later problems in the substitution section with the correct array.

The section on Counting Rod Arithmetic is important to understanding the use of rectangular arrays in Chinese algebra. An example of multiplication is shown to illustrate the very modern techniques used to multiply multi-digit numbers. Practicing the Chinese algorithm with Task 1 is useful even if done with modern numerals, as it forces students out of familiar patterns of arithmetic. Instructors who teach pre-service teachers might want to spend time on this section and actually create counting rods and learn to add, subtract and multiply Chinese style. A comparison with the standard algorithms with base ten blocks would be enlightening.

Task 11 on the Sign Rule is designed to have students understand that the rule was written for working with counting rods. Simply following the rule with modern numbers does not fully explain how the rule works. For example, if $a = 3$ and $b = 7$ then $a - b = -4$ as we know. The Sign Rule says that like signs subtract, so we would have three red (positive) rods and try to take away 7 red rods. Of course, we take away three and then we cannot take any more away. However, the rule says "Positive without extra, make negative", meaning the four remaining positive rods that we need to subtract are made negative (black) and thus the answer is -4. The physical operations with the rods dictates the Sign Rule. This is a particularly meaningful exercise for pre-service teachers, since it presents an unfamiliar twist on rules for arithmetic with signed numbers.

Task 13 asks students to think about why Liu states that "Entries in each column are distinct from one another". The point is that the columns are linearly independent. The fact that the problems come from practical examples implies that they have solutions.

The lesson emphasizes the *Fangcheng* Rule's order of operations that delays the introduction of fractions until the last step. Notice that the problems did not in general have simple integer answers.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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