Bolzano's Definition of Continuity, his Bounded Set Theorem, and an Application to Continuous Functions

David Ruch*

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1 Introduction

The foundations of calculus were not yet on firm ground in the early 1800s. Mathematicians such as Joseph-Louis Lagrange (1736-1813) made efforts to put limits and derivatives on a firmer logical foundation, but were not entirely successful.

Bernard Bolzano (1781-1848) was one of the great success stories of the foundations of analysis. He was a theologian with interests in mathematics and a contemporary of Gauss and Cauchy, but was not well known in mathematical circles. Despite his mathematical isolation in Prague, Bolzano was able to read works by Lagrange and others, and published work of his own.

This project investigates results from his important pamphlet *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* (Prague 1817) [Bz]. In particular, his proof of the main theorem in Section 12 on a property of bounded sets inspired Weierstrass decades later, and some version of his theorem in Section 15 is found in nearly every introductory calculus text.

Bolzano was very interested in logic, and he was dissatisfied with many contemporary attempts to prove theorems using methods he found inappropriate. In particular, Bolzano was interested in rigorously proving fundamental results that had often been considered obvious by other mathematicians. Here are excerpts from Bolzano's preface, as translated in [Russ], with minor changes. As you read, remember that when Bolzano wrote his pamphlet, there were not yet precise and universally agreed upon definitions of limit or continuity.

There are two propositions in the theory of equations for which, up until recently, it could still be said that a perfectly correct proof was unknown. One is the proposition: between every two values of the unknown quantity which give results of opposite sign there must always lie at least one real root of the equation.

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*Department of Mathematical and Computer Sciences, Metropolitan State University of Denver, Denver, CO; emailruch@msudenver.edu

We do find very distinguished mathematicians concerned with this proposition and various kinds of proof for it have already been attempted. Anyone wishing to be convinced of this need only compare the various treatments of this proposition given, for example, by Kästner ... as well as by several others.

However, a more careful examination very soon shows that none of these kinds of proof can be regarded as satisfactory. The most common kind of proof depends on a truth borrowed from geometry, namely: that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the $x-$axis somewhere at a point lying between those ordinates. There is certainly nothing to be said against the correctness, nor against the obviousness of this geometrical proposition. But it is equally clear that it is an unacceptable breach of good method to try to derive truths of pure (or general) mathematics (i.e. arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part of it, namely geometry.

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According to a correct definition, the expression that a function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits[1] means only that, if x is any such value the difference $f(x+\omega)-f(x)$ can be made smaller than any given quantity provided ω can be taken as small as we please.

[Bolzano footnote] 1. There are functions which vary continuously for all values of their root, e.g., $\alpha+\beta x$. But there are others which are continuous only for values of their root inside or outside certain limits. Thus $x+\sqrt{(1-x)(2-x)}$ is continuous only for values of $x<+1$ or $>+2$ but not for values between $+1$ and $+2$.

Exercise 1 *Do you agree with Bolzano's philosophical criticism of geometrical proof attempts of the Preface proposition "between every two values ... at least one real root of the equation"?*

Exercise 2 *Rewrite Bolzano's Preface proposition "between every two values ... at least one real root of the equation" in your own words with modern terminology. Sketch a diagram illustrating the proposition.*

Exercise 3 *For a function $f:\mathbb{R}\to\mathbb{R}$, rephrase Bolzano's "correct definition" of continuity at x using modern ϵ-δ terminology and appropriate quantifiers.*

Exercise 4 *Use this definition to give a modern ϵ-δ proof of the continuity of $f(x)=3x+47$ at $x=2$.*

Exercise 5 *Consider the function Bolzano discusses in his footnote. Based on the Preface proposition he is discussing, why is this an interesting example? How could you adjust the function to make it better fit the issues surrounding the Preface proposition?*

Exercise 6 *Adjust your continuity definition in Exercise 3 to include the notion of domain, so it applies to functions defined on an interval I within $\mathbb{R}$. Do you think this footnote function should be continuous at $x=1$ and at $x=2$? Give an intuitive justification.*

Exercise 7 *Suppose a function h is continuous for all x in $[0,4]$ and $h(3)=6$. Show that there is a $\delta>0$ for which $h(x)\geq 5$ for all $x\in(3-\delta,3+\delta)$.*

Exercise 8 *Define $g(x)=3-5x^2$ with domain $I=[4,7]$. Show that g is continuous at an arbitrary $\alpha\in I$ using your continuity definition.*

Bonus For Exercise 8, change the domain of g to be $\mathbb{R}$. Show that g is continuous at an arbitrary $\alpha\in\mathbb{R}$. You may need to adjust your proof from Exercise 8.

Exercise 9 *We define a function to be continuouus on an interval if it is continuous at each point in the interval. Suppose that functions f and g are both continuous on an interval I. Prove that $f-47g$ is also continuous on I, using your continuity definition.*

Use the following properties of the sine and cosine functions for the exercises below.

$$\begin{aligned}\sin a-\sin b &= 2\sin((a-b)/2)\cos((a+b)/2)\ , \quad |\sin a|\leq|a| \\ \cos a-\cos b &= 2\sin((b-a)/2)\sin((a+b)/2)\ , \quad |\sin a|\leq 1\,, \quad |\cos a|\leq 1 \quad \text{for } a,b\in\mathbb{R}\end{aligned}$$

Exercise 10 *Prove that $\sin x$ is continuous on $\mathbb{R}$.*

Exercise 11 *Prove that $\cos x$ is continuous on $\mathbb{R}$.*

Exercise 12 *Define $S(x)=x\sin(1/x)$ for $x\neq 0$. Find a value for $S(0)$ so that S will be continuous at $x=0$. Prove your assertion.*

2 Bolzano's Bounded Set Theorem and an Application

In Sections 1-10 of paper [Bz], Bolzano discusses infinite series and their convergence. He uses his results in his Section 12 to prove a very important theorem about certain bounded sets.

§11

Preliminary note. In investigations of applied mathematics it is often the case that we learn that a definite property M applies to all values of a [nonnegative[1]] variable quantity x which are smaller than a certain u without at the same time learning that this property M does not apply to values which are greater than u. In such cases there can still perhaps be some u' that is $>u$ for which in the same way as it holds for u, all values of x lower than u' possess property M. Indeed this property M may even belong to all values of x without exception. But if this alone is known, that M does not belong to all x in general then by combining these two conditions we will now be justified in concluding: there is a certain quantity U which is the greatest of those for which it is true that all smaller values of x possess property M. This is proved in the following theorem.

[1] Bolzano intends to discuss only $x\geq 0$ in this note and his Section 12 theorem statement. The term "nonegative" has been included in this project for clarity.

§12

Theorem. If a property M does not apply to all values of a [nonnegative] variable quantity x but does apply to all values smaller than a certain u, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x possess the property M.

Let's look at some examples of this concept Bolzano is discussing.

Exercise 13 *Let* M *be the property "$x^2 < 3$" applied to the set $\{x \in \mathbb{R} : x \geq 0\}$.*

(a) Find rational numbers u, u' for this example (these values are not unique). What is the value of U *for this example?*

(b) Let S_M be the set of ω values that possess property M *and for which all ω' values satisfying $0 \leq \omega' \leq \omega$ also possess property* M*. Sketch S_M on a ω number line. Are the theorem hypotheses met for this property* M*?*

(c) Does U *possess property* M*?*

Exercise 14 *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x$, and let $\alpha \in \mathbb{R}$ be arbitrary. Let* M *be the property "$f(\alpha + \omega) \leq f(\alpha) + 2$" applied to the set $\{\omega \in \mathbb{R} : \omega \geq 0\}$.*

(a) Find rational numbers u, u' for this example. Are these values unique? What is the value of U *for this example?*

(b) Let S_M be the set of ω values that possess property M *and for which all ω' values satisfying $0 \leq \omega' \leq \omega$ also possess property* M*. Sketch S_M on a ω number line. Are the theorem hypotheses met for this property* M*?*

(c) Does U *possess property* M*?*

Exercise 15 *Rewrite this theorem using modern terminology and set notation.*

Bolzano's proof of the theorem is correct, based on a modern definition of real numbers and Cauchy sequence-like convergence assumption for infinite series. However, the proof is long and difficult, so we will omit it for this project. This theorem is crucial for Bolzano's proof of his main result for solving equations, which he gives in Section 15.

§15

Theorem. If two functions of x, fx and ϕx vary according to the law of continuity either for all values of x or for all those lying between α and β, and furthermore if $f\alpha < \phi\alpha$ and $f\beta > \phi\beta$, then there is always a certain value of x between α and β for which $fx = \phi x$.

Proof.

I. 1. Firstly assume that α and β are both positive and that (because it does not matter) β is the greater of the two, so $\beta = \alpha + i$, where i denotes a positive quantity. Now because $f\alpha < \phi\alpha$, if ω denotes a positive quantity which can become as small as we please, then also $f(\alpha+\omega) < \phi(\alpha+w)$. For because fx and ϕx vary continuously for all x lying between α and β, and $\alpha+\omega$ lies between α and β whenever we take $\omega < i$, then it must be possible to make $f(\alpha+\omega) - f\alpha$ and $\phi(\alpha+\omega) - \phi\alpha$ as small as we please if ω is taken small enough. Hence if Ω and $\Omega^{'}$ denote quantities which can be made as small as we please, $f(\alpha+\omega) - f\alpha = \Omega$ and $\phi(\alpha+\omega) - \phi\alpha = \Omega^{'}$. Hence,

$$\phi(\alpha+\omega) - f\,(\alpha+\omega) = \phi\alpha - f\alpha + \Omega^{'} - \Omega.$$

However, $\phi\alpha - f\alpha$ equals, by assumption, some positive quantity of constant value A. Therefore

$$\phi(\alpha+\omega) - f\,(\alpha+\omega) = A + \Omega^{'} - \Omega\,,$$

which remains positive if Ω and $\Omega^{'}$ are allowed to become small enough, i.e., if ω is given a very small value, and even more so for all smaller values of ω. Therefore it can be asserted that for all values of ω smaller than a certain value the two functions $f(\alpha+\omega)$ and $\phi(\alpha+w)$ stand in the relationship of smaller quantity to greater quantity. Let us denote this property of the variable quantity ω by M. Then we can say that all ω that are smaller than a certain one possess the property M. But nevertheless it is clear that this property M does not apply to all values of ω, namely not to the value $\omega = i$, because $f\,(\alpha+i) = f\beta$ which, by assumption, is not less than, but greater than $\phi(\alpha+\omega) = \phi\beta$. As a consequence of the theorem of §12 there must therefore be a certain quantity U which is the greatest of those of which it can be asserted that all ω which are less than U have the property M.

Exercise 16 *Sketch a diagram with graphs of f and ϕ that illustrates the theorem statement and label α, β and A. For an arbitrary ω possessing property M, label $\Omega^{'}$ and Ω. Also draw an ω number line and label key values i, U, and the set of values ω possessing Property M.*

Exercise 17 *Bolzano states that ω, Ω and $\Omega^{'}$ can be made "as small as we please". Explain the dependencies between these quantities. Use ϵ-δ terminology to clarify what is going on.*

Exercise 18 *Rewrite Bolzano's claim in the first two sentences of I. 1. using modern terminology and call this Lemma 1.*

Exercise 19 *Convert Bolzano's argument in I.1. into a proof of Lemma 1 with your modern definition of continuity.*

Exercise 20 *Rewrite with symbols Bolzano's definition of Property M in the context of Section I.1. Then rephrase his statement that "all ω that are smaller than a certain one possess the property M" using set notation, and name this set S_M.*

Exercise 21 *As an example, consider the functions $f(x) = 4+(x-2)(x-4)(x-6)$ and $\phi(x) = 4$ with $\alpha = 1$ and $\beta = 7$. Informally find the set S_M and the value of U for this example.*

Exercise 22 *We can summarize the results of Section I.1. of the proof by stating a couple facts about U. First, that such a quantity exists. What else?*

Now proceed to Bolzano's Section I.2. of his proof.

2. This U must lie between 0 and i. For firstly it cannot be equal to i because this would mean that $f(\alpha+\omega) < \phi(\alpha+w)$, whenever $\omega<i$, and however near it came to the value of i. But in exactly the same way that we have just proved that the assumption $f\alpha < \phi\alpha$ has the consequence $f(\alpha+\omega) < \phi(\alpha+w)$, provided ω is taken small enough, so we can also prove that the assumption $f(\alpha+i) > \phi(\alpha+i)$ leads to the consequence $f(\alpha+i-\omega) > \phi(\alpha+i-\omega)$, provided ω is taken small enough. It is therefore not true that the two functions fx and ϕx stand in the relationship of smaller quantity to greater quantity for all values of x which are $< \alpha+i$. Secondly, still less can it be true that $\mathsf{U} > i$ because otherwise i would also be one of the values of ω which are $< \mathsf{U}$, and hence also $f(\alpha+i) < \phi(\alpha+i)$ which directly contradicts the assumption of the theorem. Therefore, since it is positive, U certainly lies between 0 and i and consequently $\alpha+\mathsf{U}$ lies between α and β.

Exercise 23 *Rewrite Bolzano's claim: "the assumption $f(\alpha+i) > \phi(\alpha+i)$ leads to the consequence $f(\alpha+i-\omega) > \phi(\alpha+i-\omega)$, provided ω is taken small enough" using modern terminology and call this Lemma 2.*

Exercise 24 *We can summarize this section as the claims "$0 < \mathsf{U} < i$" and "$\alpha < \alpha+\mathsf{U} < \beta$" followed by Bolzano's proof of the claim for $\mathsf{U} < i$. Rewrite his proof using your own words and modern terms, referencing the set S_M and the Section 12 theorem.*

Now read Section I.3. of Bolzano's proof.

3. It may now be asked, what relation holds between fx and ϕx for the value $x = \alpha+\mathsf{U}$? First of all, it cannot be that $f(\alpha+\mathsf{U}) < \phi(\alpha+\mathsf{U})$, for this would also give $f(\alpha+\mathsf{U}+\omega) < \phi(\alpha+\mathsf{U}+\omega)$, if ω were taken small enough, and consequently $\alpha+\mathsf{U}$ would not be the greatest value of which it can be asserted that all x below it have the property M. Secondly. just as little can it be that $f(\alpha+\mathsf{U}) > \phi(\alpha+\mathsf{U})$, because this would also give $f(\alpha+\mathsf{U}-\omega) > \phi(\alpha+\mathsf{U}-\omega)$ if ω were taken small enough and therefore, contrary to the assumption, the property M would not be true of all x less than $\alpha+\mathsf{U}$. Nothing else therefore remains but that $f(\alpha+\mathsf{U}) = \phi(\alpha+\mathsf{U})$, and so it is proved that there is a value of x lying between α and β, namely $\alpha+\mathsf{U}$, for which $fx = \phi x$.

Exercise 25 *Adjust your Lemmas 1 & 2 to give modern justifications of the first two claims in this section.*

Exercise 26 *What property of the real numbers justifies the statement "Nothing else therefore remains but that $f(\alpha + \mathsf{U}) = \phi(\alpha + \mathsf{U})$"?*

Exercise 27 *At the beginning of the proof in I.1., Bolzano makes the assumption "that α and β are both positive". Can you find a place in the proof where he uses this assumption?*

Bolzano continues in Section 15 to address the cases α and β are both negative, one is zero, and of opposite sign. We will omit these proofs, as they are not terribly enlightening.

Exercise 28 *Use Bolzano's theorem to state and prove a result, with modern terminology and methods, making precise the proposition "between every two values of the unknown quantity which give results of opposite sign there must always lie at least one real root of the equation" from Bolzano's preface.*

Exercise 29 *Use Bolzano's theorem to prove the following result from a standard introductory Calculus text:*

Consider an interval $I = [a, b]$ in the real numbers $\mathbb{R}$ and a continuous function $f : I \to \mathbb{R}$. If $f(a) < L < f(b)$ then there is a $c \in (a, b)$ such that $f(c) = L$.

References

[Bz] Bolzano, B., *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* (Prague 1817). Translated to English by Russ (see references below).

[Russ] Russ, S. B., *A Translation of Bolzano's Paper on the Intermediate Value Theorem*, Historia Mathematica 7, 1980, 156-185.

Instructor Notes

This project can be used to introduce continuity and the Intermediate Value Theorem (IVT) for a course in Real Analysis.

Project Content Goals

1. Develop a modern continuity definition with quantifiers based on Bolzano's definition.
2. Develop facility with the modern continuity definition by applying it to various functions.
3. Analyze Bolzano's version of the Least Upper Bound Property and rewrite it in modern terminology.
4. Modernize Bolzano's proof of his Intermediate Value Theorem.
5. Apply Bolzano's Intermediate Value Theorem to obtain two other formulations of the Intermediate Value Theorem.

Preparation of Students

The project is written for a course in Real Analysis with the assumption that students have seen the least upper bound property for bounded sets of real numbers, but the project could be used to introduce this concept, with instructor supplements. The project also assumes that students have done a rigorous study of quantifiers and limits of real-valued functions.

Preparation for the Instructor

This is roughly a one week project under the following methodology (basically David Pengelley's "A, B, C" method described on his website):

1. Students do some advanced reading and light preparatory exercises before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the exercises or your grading load will get out of hand! Some instructor have students write questions or summaries based on the reading.
2. Class time is largely dedicated to students working in groups on the project - reading the material and working exercises. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a "participation" grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the exercises they missed. This is usually a good incentive not to miss class!
3. Some exercises are assigned for students to do and write up outside of class. Careful grading of these exercises is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project. If students have already studied continuity in a rigorous fashion, then the first section should move very quickly and many exercises can safely be skipped.

Section 1 Comments

1. The first set of exercises develops the definition for a function defined on an interval, the appropriate setting for a discussion of the IVT. While many textbooks give a definition of continuity on a more general domain, we avoid this with the IVT as our main objective.

2. Getting a correct definition of continuity in Exercise 3 is crucial before going much further; a class discussion of Exercise 3 and the next problem can be helpful after students work on them for awhile or in groups.

3. Bolzano's choice of $x + \sqrt{(1-x)(2-x)}$ in his footnote is mildly perplexing, as it does not change signs in its domain $[1, 2]$. Indeed, the first set of students using the project were rather critical of Bolzano's footnote function. They inspired Exercise 5. In Bolzano's defense, he discusses the function $x + \sqrt{(x-2)(x+1)}$, which lacks a root between -1 and 2, earlier in his very lengthy preface.

4. Exercise 7 foreshadows a crucial result in the next section, namely writing a modern ϵ-δ proof of Bolzano's "because $f\alpha < \phi\alpha$, if ω denotes a positive quantity which can become as small as we please, then also $f(\alpha + \omega) < \phi(\alpha + \omega)$". This is difficult for some students, and they may need a hint/reminder that THEY get to choose ϵ if f is known to be continuous.

5. The final group of Section 1 exercises, 8 - 12, are standard problems to sharpen skills in working with continuity. Instructors may sample the set for classroom examples or homework problems. However, they are not needed for the flow of Bolzano's discussion.

Section 2 Comments

1. The project is written with the assumption that students have seen the least upper bound property for bounded sets of real numbers. Bolzano's theorem basically asserts this property for a special class of bounded sets, but in a form students (and the PSP author!) have not seen before. It is a bit tricky to unravel and put into modern set notation. The first two exercises should help ease this process for students. Exercise 14 should help them with Bolzano's next section (15), where the theorem is applied.

2. Section I.1 of the IVT proof in Section 15 is crucial and contains some subtleties. It is worth taking time to make sure students understand this part of the proof. In addition the symbols Ω', Ω may cause confusion for some students. Bolzano is not completely clear on whether he is looking for the U value for set $S_{\mathsf{M}} = \{\omega : f(\alpha + \omega') < \phi(\alpha + \omega') \text{ for all } \omega' \in [0, \omega)\}$ or for $\{\omega : f(\alpha + \omega) < \phi(\alpha + \omega)\}$. Exercise 21 illustrates the difference. An opportunity for discussion and careful reading!

3. In his original paper, Bolzano begins the proof of the Section 15 theorem with the following:

We must remember that in this theorem the values of the functions fx and ϕx are to be compared to one another simply in their absolute values, i.e., without regard to signs or as though they were quantities incapable of being of opposite signs. But the signs of α and β are important.

He then splits his proof into several cases I-IV, beginning with case $0 < \alpha < \beta$. Here is S.B. Russ' explanation (page 183 in [Russ]):

"Bolzano always uses inequality signs to apply only to the magnitude of quantities and not to their position on a number line. This was common practice at the time as there was still no standard symbol for modulus. Thus in Bolzano's usage $x > -1$ means the range we should now describe as $x < -1$. For example, in Section 2 he states that the general term of a geometric progression … And in Section 15.IV he describes the range of values of x between α and β when α is negative and β positive as 'all values of x which if negative are $< \alpha$, and if positive are $< \beta$'."

These comments by Bolzano have been excluded from the project because they seem likely to cause considerable confusion with little payoff to most students of analysis. Instructors could bring up these issues in a class discussion of Exercise 27.

4. The statement and proof of Lemma 2 in the exercise set following Section I.2 of the IVT proof in Section 15 may seem a bit repetitive. However, it should clarify things for some students and serve as proof writing practice, especially if Lemma 1 is done in class.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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