The Cantor set before Cantor

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C &:= \bigcap_{n\in\mathbb{Z}^{\geq 0}} C_n
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4. Perfect

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$$S = S^{(1)}$$

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Should perfect imply continuum?

No, Cantor set counter example. Cantor introduced "connected."

Smith: On the Integration of discontinuous functions (1875)

Riemann...has given an important theorem which serves to determine whether a function $f(x)$ which is discontinuous, but not infinite, between the finite limits a and b, does or does not admit of integration...Some further discussion of this theorem would seem to be desirable, partly because, in one particular at least, Riemann's demonstration is wanting in formal accuracy, and partly because the theorem itself appears to have been misunderstood, and to have been made the basis of erroneous inferences.

Loose order

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> *A system of points is said to* **fill completely** *[be dense in] a given interval when, any segment of the interval being taken, however small, one point at least of the system lies on that segment ...*

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> *A system of points is said to* **fill completely** *[be dense in] a given interval when, any segment of the interval being taken, however small, one point at least of the system lies on that segment ...*
> *Next let us suppose that $f(x)$ in the interval $b - a$ has an infinite number of discontinuities surpassing a given quantity σ ... points are in* **loose order** *[nowhere dense] when they do not completely fill it or any part of it, however small.*

His construction: Generalized Cantor set

Divide the interval from 0 *to* 1 *into* m *equal parts, exempting the last segment from any further division; let us divide each of the remaining* $m - 1$ *segments by* m^2*, exempting the last segment of each segment; let us again divide each of the remaining* $(m-1)(m^2-1)$ *segments by* m^3*, exempting the last segment of each segment; and so on continually... The points of division* Q *exist in loose order over the whole interval... But a function having finite discontinuities at the points* Q *would be incapable of integration."*

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[He] asserts that, when a system of points is in loose order on a line, the line may be so divided as to make the sum of the segments containing the points less than any assignable line [and hence integrable] . . .
[his] demonstration is rigorous, if the number of points in the system is finite; but . . . the assertion that, when the system of points of discontinuity is in loose order the function is integrable, would seem . . . to be negatived by the result.

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Thank you very much!