Discovery through Figurate Numbers

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Teaching with Historical Sources

We present a few excerpt from historical sources covering the figurate numbers. This material could be used as a separate module for general education in mathematics, for a beginning course in discrete mathematics, for courses in algebra or computer science. The sources:

- Nicomachus of Gerasa (first century CE), *Introduction to Arithmetic*.
- Pierre de Fermat (1601–1665), letters to Marin Mersenne and Gilles Persone de Roberval.
- Blaise Pascal (1623–1662), *Treatise on the Arithmetical Triangle*.

Introduction to Arithmetic

Figure: Nicomachus of Gerasa.

Nicomachus begins with the linear numbers:

The number 1, α

The number 2, $\alpha \quad \alpha$

The number 3, $\alpha \quad \alpha \quad \alpha$

The number 4, $\alpha \quad \alpha \quad \alpha \quad \alpha$

The number 5, $\alpha \quad \alpha \quad \alpha \quad \alpha \quad \alpha$

These are the counting numbers. Nicomachus uses no algebraic symbols in his treatise, and only refers to numbers by their values.

Exercise: If, instead, we let L_n denote the nth linear number for a counting number n, then (fill in the blank)

$$L_n = \boxed{}.$$

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Exercise: If, instead, we let L_n denote the nth linear number for a counting number n, then (fill in the blank)

$$L_n = \boxed{}.$$

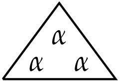
Solution: $L_n = \boxed{n}$.

The Triangular Numbers

Nicomachus: *Now a triangular number is one which shapes into triangular form the equilateral placement of its parts in a plane. 3, 6, 10, 15, 21, 28, and so on, are examples of it . . . . I take the first term and have the triangular number which is 1,* 

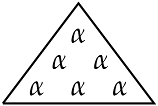
Using algebra of today, let's set $T_1 = 1$ for the first triangular number.

Two units, side by side, are set beneath one unit, and the number three is made a triangle:



Today: Set $T_2 = 3$ for the second triangular number.

Nicomachus: *Then the following number, 3, is added, . . . , and joined to the former [triangle]; it gives 6:*



This is a recursive description of how the third triangular number can be constructed from the second.

Exercise: Fill in the blank

$$T_3 = T_2 + \boxed{\quad}.$$

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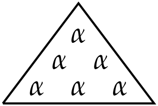

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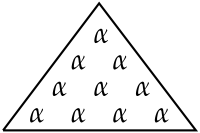
Exercise: Fill in the blank

$$T_3 = T_2 + \boxed{}.$$

Solution: $T_3 = T_2 + \boxed{3}$ or $T_3 = T_2 + \boxed{L_3}$.

Nicomachus: *The number that naturally follows, 4, added in and set down below the former [triangle], . . . , gives the one in order next after the aforesaid, 10, and takes a triangular form:*



Recursively, $T_4 = T_3 + 4$ or $T_4 = T_3 + L_4$.
Exercise: Find a recursive formula for T_n, the nth triangular number.

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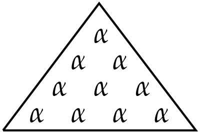

Recursively, $T_4 = T_3 + 4$ or $T_4 = T_3 + L_4$.
Exercise: Find a recursive formula for T_n, the nth triangular number.
Solution: Either $T_n = T_{n-1} + n$ or $T_n = T_{n-1} + L_n$.

Another Method for Computing the Triangular Numbers

Nicomachus: *The triangular number is produced from the natural series of number set forth in a line, and by the continued addition of successive terms, one by one, from the beginning ...* .

This represents an iterative construction of the triangular numbers.

Exercise: Find an algebraic formula that reflects the above statement for computing T_n.

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Exercise: Find an algebraic formula that reflects the above statement for computing T_n.

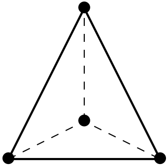
Solution: $T_n = 1 + 2 + 3 + \ldots + n$ or $T_n = L_1 + L_2 + L_3 + \ldots + L_n$.

The Pyramidal Numbers

Nicomachus: *The pyramids with a triangular base, in their proper order, are these: 1, 4, 10, 20, 35, 56, 84, and so on; and their origin is the piling up of the triangular numbers one upon the other, ...*

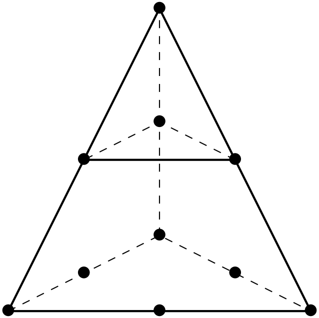
For the first pyramidal number, set $P_1 = 1$.

The Second Pyramid



Set $P_2 = 4$.

The Third Pyramid



Exercise: What is the total number of dots in the above pyramid? Call this P_3. So, $P_3 = \boxed{}$.

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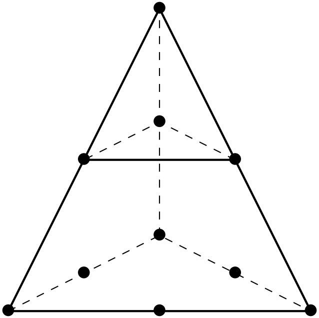

Exercise: What is the total number of dots in the above pyramid? Call this P_3. So, $P_3 = \boxed{}$. Solution: $P_3 = 10$.

Student Exercise

Compute the first eight pyramidal numbers,

$$P_1,\ P_2,\ P_3,\ \dots,\ P_8$$

and devise a strategy for the their computation.

Exercise for conference participants: Articulate two strategies for computing P_n.

Student Exercise

Compute the first eight pyramidal numbers,

$$P_1,\ P_2,\ P_3,\ \dots,P_8$$

and devise a strategy for the their computation.
Exercise for conference participants: Articulate two strategies for computing P_n.
Solution: The iterative formula:

$$P_n = T_1 + T_2 + T_3 + \ldots + T_n.$$

The recursive formula:

$$P_n = P_{n-1} + T_n.$$

The Triangulo-Triangular Numbers

Fermat writes of certain four-dimensional numbers, called the triangulo-triangular numbers. Let Q_n denote the nth triangulo-triangular number.

Exercise: How should the four-dimensional figurate numbers be constructed from the three-dimensional figurate numbers in analogy to how the pyramidal numbers are constructed from the triangular numbers?

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Exercise: How should the four-dimensional figurate numbers be constructed from the three-dimensional figurate numbers in analogy to how the pyramidal numbers are constructed from the triangular numbers?

The iterative formula for Q_n:

$$Q_n = P_1 + P_2 + P_3 + \ldots + P_n.$$

The recursive formula for Q_n:

$$Q_n = Q_{n-1} + P_n.$$

There are figurate numbers in each dimension after this.

A Closed Formula for the Triangular Numbers

Figure: Pierre de Fermat.

Fermat writes: "Ultimum latus in latus proxime majus facit duplum trianglui." Let's decipher this in a discovery exercise. What would result if the area of a triangle were divided by the length of its base? In the case of the triangular numbers, what would result if T_n were divided by L_{n+1}? Note that T_n and L_{n+1} are the two numbers used in the recursive sum for T_{n+1}.

Exercise: Use the values of the triangular numbers in the table

n	1	2	3	4	5	6	7	8
T_n	1	3	6	10	15	21	28	36

to compute

$$\frac{T_1}{2} = \boxed{}, \quad \frac{T_2}{3} = \boxed{}, \quad \frac{T_3}{4} = \boxed{}, \quad \frac{T_4}{5} = \boxed{}, \quad \frac{T_5}{6} = \boxed{}.$$

To clear fractions, compute

$$\frac{2T_1}{2} = \boxed{}, \quad \frac{2T_2}{3} = \boxed{}, \quad \frac{2T_3}{4} = \boxed{}, \quad \frac{2T_4}{5} = \boxed{}, \quad \frac{2T_5}{6} = \boxed{}.$$

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Solution: $\frac{2T_1}{2} = \boxed{1}, \quad \frac{2T_2}{3} = \boxed{2}, \quad \frac{2T_3}{4} = \boxed{3}, \quad \frac{2T_4}{5} = \boxed{4}, \quad \frac{2T_5}{6} = \boxed{5}.$

Exercise: Guess a pattern for

$$\frac{2T_n}{n+1} = \boxed{}.$$

Write this in the form $n(n+1) = \boxed{}$, and state this equation in words.

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Fermat's statement "Ultimum latus in latus proxime majus facit duplum trianglui" has been translated as "The last side [n] multiplied by the next larger [$n+1$] makes twice the triangle [$2T_n$]."

Solution: $n(n+1) = 2T_n$.

A Closed Formula for the Pyramidal Numbers

Fermat: "Ultimum latus in triangulum lateris proxime majoris facit triplum pyramidis."

Exercise: Use the values of the pyramidal numbers in the table

n	1	2	3	4	5	6	7
T_n	1	3	6	10	15	21	28
P_n	1	4	10	20	35	56	84

to compute

$\frac{P_1}{T_2} = \boxed{}, \quad \frac{P_2}{T_3} = \boxed{}, \quad \frac{P_3}{T_4} = \boxed{}, \quad \frac{P_4}{T_5} = \boxed{}, \quad \frac{P_5}{T_6} = \boxed{}.$

To clear fractions, compute

$\frac{3P_1}{T_2} = \boxed{}, \quad \frac{3P_2}{T_3} = \boxed{}, \quad \frac{3P_3}{T_4} = \boxed{}, \quad \frac{3P_4}{T_5} = \boxed{}, \quad \frac{3P_5}{T_6} = \boxed{}.$

A Closed Formula for the Pyramidal Numbers

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Solution:

$$\frac{3P_1}{T_2} = \boxed{1}, \quad \frac{3P_2}{T_3} = \boxed{2}, \quad \frac{3P_3}{T_4} = \boxed{3}, \quad \frac{3P_4}{T_5} = \boxed{4}, \quad \frac{3P_5}{T_6} = \boxed{5}.$$

Exercise: Guess a formula for

$$\frac{3P_n}{T_{n+1}} = \boxed{}.$$

Write this formula as $nT_{n+1} = \boxed{}$.

State the above equation in words.

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Fermat's statement about the pyramidal numbers has been translated as "The last side [n] multiplied by the triangle of the next larger [T_{n+1}] makes three times the pyramid [$3P_n$]."

Solution: $nT_{n+1} = 3P_n$.

Formulas for the Higher-Dimensional Figurate Numbers?

Fermat writes: "The last side multiplied by the pyramid of the next greater makes four times the triangulo-triangle," which in our notation becomes

$$nP_{n+1} = 4Q_n.$$

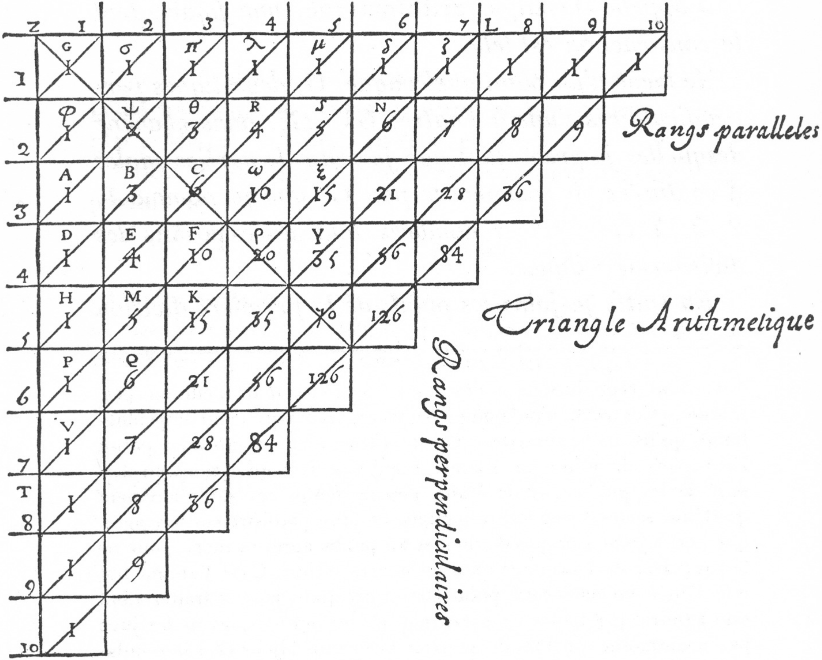
Of the higher-dimensional figurate numbers, Fermat writes "And so on by the same progression in infinitum."

How would you express such a progression verbally or algebraically?

Pascal's Treatise on the Arithmetical Triangle

Figure: Blaise Pascal.

Pascal arranges the figurate numbers across all dimensions into one table.


Rangs paralleles
Triangle Arithmetique
Rangs perpendiculaires

Pascal's Triangle

The first row contains "numbers of the first order," which are the zero-dimensional figurate numbers.

The second row contains "numbers of the second order," which are the linear numbers.

The third row contains "numbers of the third order," which are the triangular numbers.

The fourth row contains "numbers of the fourth order," which are the pyramidal numbers, etc.

Exercise: Is there an all-inclusive recursion relation for the figurate numbers across all dimensions?

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Exercise: Is there an all-inclusive recursion relation for the figurate numbers across all dimensions?

The construction principle for Pascal's table is: "The number of each cell is equal to the sum of the numbers of the perpendicular and parallel cells immediately preceding."

Pascal uses single letters, Greek or Roman, to stand for an entry in his table without index or subscript notation to denote the row (order) or column (root) number.

Let $E_{k,\,n}$ denote the entry in row k (order) and column n (root) in Pascal's table.

Exercise: Express the all-inclusive recursion relation for the figurate numbers (Pascal's construction principle) in $E_{k,\,n}$ notation.

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Exercise: Express the all-inclusive recursion relation for the figurate numbers (Pascal's construction principle) in $E_{k,\,n}$ notation.

Solution:

$$E_{k,\,n} = E_{k-1,\,n} + E_{k,\,n-1}.$$

Z	1	2	3	4	5	6	7	L 8	9	10
1	G 1	σ 1	π 1	λ 1	μ 1	δ 1	ζ 1	1	1	1
2	φ 1	ψ 2	θ 3	R 4	S 5	N 6	7	8	9	
3	A 1	B 3	C 6	ω 10	ξ 15	21	28	36		
4	D 1	E 4	F 10	ρ 20	Y 35	56	84			
5	H 1	M 5	K 15	35	70	126				
6	P 1	Q 6	21	56	126					
7	V 1	7	28	84						
T 8	1	8	36							
9	1	9								
10	1									

Write down the values of $E_{10,\,1}$, $E_{9,\,2}$, $E_{8,\,3}$, $E_{7,\,4}$, $E_{6,\,5}$, $E_{5,\,6}$, $E_{4,\,7}$, $E_{3,\,8}$, $E_{2,\,9}$, $E_{10,1}$.

A Discovery Exercise

Find a pattern in the following ratios of adjacent entries in the tenth base of Pascal's table:

$$\frac{E_{9,2}}{E_{10,1}} = \boxed{}, \quad \frac{E_{8,3}}{E_{9,2}} = \boxed{}, \quad \frac{E_{7,4}}{E_{8,3}} = \boxed{}, \quad \frac{E_{6,5}}{E_{7,4}} = \boxed{}, \quad \frac{E_{5,6}}{E_{6,5}} = \boxed{}$$

$$\frac{E_{4,7}}{E_{5,6}} = \boxed{}, \quad \frac{E_{3,8}}{E_{4,7}} = \boxed{}, \quad \frac{E_{2,9}}{E_{3,8}} = \boxed{}, \quad \frac{E_{1,10}}{E_{2,9}} = \boxed{}.$$

Solution:

$$\frac{E_{9,\,2}}{E_{10,\,1}} = \frac{9}{1}, \quad \frac{E_{8,\,3}}{E_{9,\,2}} = \frac{8}{2}, \quad \frac{E_{7,\,4}}{E_{8,\,3}} = \frac{7}{3}, \quad \frac{E_{6,\,5}}{E_{7,\,4}} = \frac{6}{4}, \quad \frac{E_{5,\,6}}{E_{6,\,5}} = \frac{5}{5}$$

$$\frac{E_{4,\,7}}{E_{5,\,6}} = \frac{4}{6}, \quad \frac{E_{3,\,8}}{E_{4,\,7}} = \frac{3}{7}, \quad \frac{E_{2,\,9}}{E_{3,\,8}} = \frac{2}{8}, \quad \frac{E_{1,\,10}}{E_{2,\,9}} = \frac{1}{9}.$$

Exercise: Write this pattern in $E_{n,\,k}$ notation.

Solution:

$$\frac{E_{9,\,2}}{E_{10,\,1}} = \frac{9}{1}, \quad \frac{E_{8,\,3}}{E_{9,\,2}} = \frac{8}{2}, \quad \frac{E_{7,\,4}}{E_{8,\,3}} = \frac{7}{3}, \quad \frac{E_{6,\,5}}{E_{7,\,4}} = \frac{6}{4}, \quad \frac{E_{5,\,6}}{E_{6,\,5}} = \frac{5}{5}$$

$$\frac{E_{4,\,7}}{E_{5,\,6}} = \frac{4}{6}, \quad \frac{E_{3,\,8}}{E_{4,\,7}} = \frac{3}{7}, \quad \frac{E_{2,\,9}}{E_{3,\,8}} = \frac{2}{8}, \quad \frac{E_{1,\,10}}{E_{2,\,9}} = \frac{1}{9}.$$

Exercise: Write this pattern in $E_{n,\,k}$ notation.

Solution: Either

(1)

$$\frac{E_{k-1,\,n+1}}{E_{k,\,n}} = \frac{k-1}{n}$$

(2)

$$\frac{E_{k,\,n}}{E_{k+1,\,n-1}} = \frac{k}{n-1}.$$

Note that (1)

$$\frac{E_{k-1,\,n+1}}{E_{k,\,n}} = \frac{k-1}{n}$$

becomes $n \cdot E_{k-1,\,n+1} = (k-1)E_{k,\,n}$, which is the generalization of Fermat's patterns.

Exercise: Write (2) $\frac{E_{k,\,n}}{E_{k+1,\,n-1}} = \frac{k}{n-1}$ in words using Pascal's order and root terminology.

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Exercise: Write (2) $\frac{E_{k,\,n}}{E_{k+1,\,n-1}} = \frac{k}{n-1}$ in words using Pascal's order and root terminology.

Pascal: "A number [$E_{k,\,n}$, $k =$ order, $n =$root] of whatever order [k] when multiplied by the preceding root [$n-1$] is equal [=] to the exponent of its order [k] multiplied by the preceding number [$E_{*,\,n-1}$] of the following order [$* = k+1$]." Thus,

$$(n-1)E_{k,\,n} = k \cdot E_{k+1,\,n-1}.$$

Homework: By iterating $\frac{E_{k-1,\,n+1}}{E_{k,\,n}} = \frac{k-1}{n}$, find a factorial formula formula for $E_{k,\,n}$.

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Solution:

$$E_{k,\,n} = \frac{k(k+1)(k+2)\cdots(n+k-2)}{(n-1)(n-2)(n-3)\cdots(1)}\,,$$

$$E_{k,\,n} = \frac{(n+k-2)!}{(n-1)!\,(k-1)!}\,.$$

Our Web Site

"Transforming Instruction in Undergraduate Mathematics via Primary Historical Sources," `https://blogs.ursinus.edu/triumphs/`

Also see:

- "A General Education Course from Primary Historical Sources," *The Newsletter of the Consortium for Mathematics and Its Applications*, 113, ISSN 0889-5392 (2017), pp. 5–11.
- "Teaching Discrete Mathematics Entirely from Primary Historical Sources," joint with J. Barnett, G. Bezhanishvili, D. Pengelley, *Primus*, Vol. 26, 7 (2016), pp. 657–675.

Advantages of Teaching with Historical Sources

- No specialized background is required to understand the source. The material on the figurate numbers can be understood with a knowledge of the counting numbers, addition, and division.
- Pattern recognition came effortlessly to students.
- Unlike many algebra classes, students had little difficulty transitioning from actual numbers to variables such as n, T_n, P_n, etc.
- Historical sources can be used to motivate content in most any mathematics course, such as geometry, calculus, or analysis.

Thank You!

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