Bolzano on Continuity and the Intermediate Value Theorem

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The foundations of calculus were not yet on firm ground in the early 1800s. Mathematicians such as Joseph-Louis Lagrange (1736-1813) made efforts to put limits and derivatives on a firmer logical foundation, but were not entirely successful.

Bernard Bolzano (1781-1848) was one of the great success stories of the foundations of analysis. He was a theologian with interests in mathematics and a contemporary of Gauss and Cauchy, but was not well known in mathematical circles. Despite his mathematical isolation in Prague, Bolzano was able to read works by Lagrange and others, and published work of his own.

This project investigates results from his important pamphlet *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*[1] [Bolzano, 1817]. We will read excerpts from this paper, as translated in [Russ, 2004], with very minor changes for readability by the project author. In particular, we examine two major theorems from this work. The first of these is the main theorem in Bolzano's Section 12, where he stated and proved a property of bounded sets. This inspired Weierstrass decades later to prove a version of the theorem nowadays called the Bolzano-Weierstrass Theorem.[2] The second major theorem, which appears in Section 15 of Bolzano's pamphlet, concerns continuous functions, and some version of this result is found in nearly every introductory calculus text. Naturally Bolzano's concept of continuity is vital for understanding both of these theorems, so we will first study it.

1 Bolzano's Definition of Continuity

Bolzano was very interested in logic, and he was dissatisfied with many contemporary attempts to prove theorems using methods he found inappropriate. In particular, Bolzano was interested in rigorously proving fundamental results that had often been considered obvious by other mathematicians. Here are excerpts from Bolzano's preface, as translated in [Russ, 2004], with very minor changes for readability by the PSP author. As you read, remember that when Bolzano wrote his pamphlet, there were not yet precise and universally agreed upon definitions of limit or continuity.

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[1]The title of Bolzano's pamphlet translates into English as *A Purely Analytic Proof of the Theorem that between two values which give results of opposite sign there lies at least one real root of the equation.*

[2]According to Kline, Bolzano's proof method "was used by Weierstrass in the 1860s, with due credit to Bolzano, to prove what is now called the Bolzano-Weierstrass theorem"[Kline, 1972, p. 953].

There are two propositions in the theory of equations for which, up until recently, it could still be said that a perfectly correct proof was unknown. One is the proposition: between every two values of the unknown quantity which give results of opposite sign there must always lie at least one real root of the equation.

...

We do find very distinguished mathematicians concerned with this proposition and various kinds of proof for it have already been attempted. Anyone wishing to be convinced of this need only compare the various treatments of this proposition given, for example, by Kästner, Clairaut, LaCroix, ... as well as by several others.

However, a more careful examination very soon shows that none of these kinds of proof can be regarded as satisfactory.

I. The most common kind of proof depends on a truth borrowed from geometry, namely: that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely), must necessarily intersect the $x-$axis[3] somewhere at a point lying between those ordinates. There is certainly nothing to be said against the correctness, nor against the obviousness of this geometrical proposition. But it is also equally clear that it is an unacceptable breach of good method to try to derive truths of pure (or general) mathematics (i.e. arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part of it, namely geometry.

Task 1 Rewrite Bolzano's Preface proposition ("between every two values ... at least one real root of the equation") in your own words with modern terminology. Sketch a diagram illustrating the proposition. What theorem from a first-year calculus course does this remind you of?

Task 2 Do you agree with Bolzano's philosophical criticism of using geometry to try to prove his Preface proposition ("between every two values ... at least one real root of the equation")? Explain why or why not. Start by restating Bolzano's criticism in your own words.

Later in his preface, Bolzano asserted a "correct" definition of continuity and gave an interesting example as a footnote. As you read his definition, think about whether you agree with it, and how you could rewrite it with modern language.

[3] Bolzano actually used the German word 'Abscissenlinie' here. Russ points out in [Russ, 2004] that 'Abscissenlinie' suggests a geometric measuring line in contrast to the more abstract 'x-axis'.

According to a correct definition, the expression that a function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits[1] means only that, if x is any such value the difference $f(x+\omega)-f(x)$ can be made smaller than any given quantity, provided ω can be taken as small as we please.

[Bolzano footnote] 1. There are functions which are continuously variable for all values of their argument, e.g., $\alpha+\beta x$. However, there are also others which vary according to the law of continuity only for values of their argument inside or outside certain limits. Thus $x+\sqrt{(1-x)(2-x)}$ varies continuously only for all values of x which are $<+1$ or $>+2$, but not for the values which lie between $+1$ and $+2$.

Task 3 For a function $f:\mathbb{R}\to\mathbb{R}$, rephrase Bolzano's "correct definition" of continuity at x using modern ϵ-δ terminology and appropriate quantifiers.

Task 4 Use this definition to give a modern ϵ-δ proof of the continuity of $f(x)=3x+47$ at $x=2$.

Task 5 Consider the function Bolzano discussed in his footnote.

(a) Sketch a graph of this function on the interval $[0,3]$.

(b) Based on the Preface proposition that Bolzano was discussing, why is this an interesting example?

(c) How could you adjust the function to make it better fit the issues surrounding the Preface proposition?

Task 6 Adjust your continuity definition in Task 3 to include the notion of domain, so it applies to functions defined on an interval I within $\mathbb{R}$. Do you think this footnote function should be continuous at $x=1$ and at $x=2$? Give an intuitive justification.

Task 7 Suppose a function h is continuous for all x in $[0,4]$ and $h(3)=6$. Show that there is a $\delta>0$ for which $h(x)\geq 5$ for all $x\in(3-\delta,3+\delta)$.

Task 8 Define $g(x)=3-5x^2$ with domain $I=[4,7]$. Show that g is continuous at an arbitrary $\alpha\in I$ using your continuity definition.

Bonus For Task 8, change the domain of g to be $\mathbb{R}$. Show that g is continuous at an arbitrary $\alpha\in\mathbb{R}$. You may need to adjust your proof from Task 8.

Task 9 We define a function to be continuous on an interval if it is continuous at each point in the interval. Suppose that functions f and g are both continuous on an interval I. Prove that $f-47g$ is also continuous on I, using your continuity definition.

For the next two tasks, the following properties of the sine and cosine functions will be useful:

$$\sin a - \sin b = 2\sin\left(\left(a-b\right)/2\right)\cos\left(\left(a+b\right)/2\right), \quad |\sin a| \le |a|$$
$$\cos a - \cos b = 2\sin\left(\left(b-a\right)/2\right)\sin\left(\left(a+b\right)/2\right), \quad |\sin a| \le 1, \quad |\cos a| \le 1 \quad \text{for } a, b \in \mathbb{R}$$

Task 10 Prove that $\sin x$ is continuous on $\mathbb{R}$.

Task 11 Prove that $\cos x$ is continuous on $\mathbb{R}$.

Task 12 Define $S(x) = x\sin(1/x)$ for $x \neq 0$. Find a value for $S(0)$ so that S will be continuous at $x = 0$. Prove your assertion.

In Section 3 of this project, we will return to Bolzano's proposition about equation roots that you examined in Task 1, and work through his proof of a related result. This material will involve continuous functions, but we will set continuity aside for now to study another important theorem from Bolzano's pamphlet.

2 Bolzano's Bounded Set Theorem

In this section we will leave continuity and study an important theorem from Bolzano about what we would today call bounded sets. The theory of sets had not been developed during Bolzano's era, so he did not use the same set theoretic language we might expect in a modern discussion of his ideas. As you read the next excerpt (taken from sections 11 and 12 of Bolzano's pamphlet), think about how you could translate his ideas into set terminology.

§11

Preamble. In investigations of applied mathematics it is often the case that we learn that a definite property M applies to all values of a [nonnegative[4]] variable quantity x which are smaller than a certain u without at the same time learning that this property M does not apply to values which are greater than u. In such cases there can still perhaps be some u' that is $> u$ for which, in the same way as it holds for u, all values of x lower than u' possess property M. Indeed this property M may even belong to all values of x without exception. But if this alone is known, that M does not belong to all x in general, then by combining these two conditions we will now be justified in concluding: there is a certain quantity U which is the greatest of those for which it is true that all smaller values of x possess property M. This is proved in the following theorem.

§12

Theorem. If a property M does not apply to all values of a [nonnegative] variable quantity x but does apply to all values smaller than a certain u, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x possess the property M.

[4]Bolzano intended to discuss only $x \ge 0$ in this note and his Section 12 theorem statement. The term "nonegative" has been included in this project for clarity.

Let's look at some examples of this concept that Bolzano was discussing.

Task 13 Let M be the property "$x^2 < 3$" applied to the set $\{x \in \mathbb{R} : x \geq 0\}$.

(a) Find rational numbers u, u' for this example (these values are not unique). What is the value of U for this example?

(b) Let S_M be the set of ω values for which all ω' values satisfying $0 \leq \omega' < \omega$ possess property M. Sketch S_M on a ω number line. Are the theorem hypotheses met for this property M?

(c) Does U possess property M?

Task 14 Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x$, and let $\alpha \in \mathbb{R}$ be arbitrary. Let M be the property "$f(\alpha + \omega) \leq f(\alpha) + 2$" applied to the set $\{\omega \in \mathbb{R} : \omega \geq 0\}$.

(a) Find rational numbers u, u' for this example. Are these values unique? What is the value of U for this example?

(b) Let S_M be the set of ω values for which all ω' values satisfying $0 \leq \omega' < \omega$ possess property M. Sketch S_M on a ω number line. Are the theorem hypotheses met for this property M?

Task 15 Rewrite Bolzano's theorem from his Section 12 using modern terminology and set notation.

We will refer to the theorem you stated in Task 15 as *Bolzano's Bounded Set Theorem*.

In Section 12 of his pamphlet, Bolzano went on to give a proof of his Bounded Set Theorem based on a Cauchy sequence-like convergence assumption for infinite series. While the proof is correct given that assumption, we will omit it for this project.[5] Instead, we next look at Section 15 of [Bolzano, 1817], to see how he used both his definition of continuity and his Bounded Set Theorem to prove his main result on the solution of certain equations involving continuous functions.

3 An Application of Bolzano's Bounded Set Theorem

We are now ready to work through Bolzano's main result, given in the excerpt below.[6] He broke his proof into three parts, and we will pause after each part to do tasks that will help unpack his proof and rephrase it with modern language.[7]

[5] You can explore the details of that proof in the related PSP *Investigations into Bolzano's Bounded Set Theorem*, available at `https://digitalcommons.ursinus.edu/triumphs_analysis/14`.

[6] This excerpt and all others that appear in this project are taken from section 15 of Bolzano's pamphlet.

[7] Throughout his theorem statement and proof below, Bolzano wrote fx where we would write $f(x)$, and similarly deleted argument parentheses for other functions and variables. The project author has inserted these parentheses to reduce distractions for the modern reader.

§15

Theorem. If two functions of x, $f(x)$ and $\phi(x)$, vary according to the law of continuity either for all values of x or for all those lying between α and β, and furthermore if $f(\alpha) < \phi(\alpha)$ and $f(\beta) > \phi(\beta)$, then there is always a certain value of x between α and β for which $f(x) = \phi(x)$.

Proof.

I. 1. Firstly assume that α and β are both positive and that (because it does not matter) β is the greater of the two, so $\beta = \alpha + i$, where i denotes a positive quantity. Now because $f(\alpha) < \phi(\alpha)$, if ω denotes a positive quantity which can become as small as we please, then also $f(\alpha + \omega) < \phi(\alpha + w)$. For because $f(x)$ and $\phi(x)$ are to vary continuously for all x lying between α and β, and $\alpha + \omega$ lies between α and β whenever we take $\omega < i$, then it must be possible to make $f(\alpha + \omega) - f(\alpha)$ and $\phi(\alpha + \omega) - \phi(\alpha)$ as small as we please if ω is taken small enough. Hence if Ω and Ω' denote quantities which can be made as small as we please, $f(\alpha + \omega) - f(\alpha) = \Omega$ and $\phi(\ alpha + \omega) - \phi(\alpha) = \Omega'$. Hence,

$$\phi(\alpha + \omega) - f(\alpha + \omega) = \phi(\alpha) - f(\alpha) + \Omega' - \Omega.$$

However, $\phi(\alpha) - f(\alpha)$ equals, by assumption, some positive quantity of constant value A. Therefore

$$\phi(\alpha + \omega) - f(\alpha + \omega) = A + \Omega' - \Omega\,,$$

which remains positive if Ω and Ω' are allowed to become small enough, i.e., if ω is given a very small value, and even more so for all smaller values of ω. Therefore it can be asserted that for all values of ω smaller than a certain value the two functions $f(\alpha+\omega)$ and $\phi(\alpha+w)$ stand in the relationship of smaller quantity to greater quantity. Let us denote this property of the variable quantity ω by M. Then we can say that all ω that are smaller than a certain one possess the property M. But nevertheless it is clear that this property M does not apply to all values of ω, namely not to the value $\omega = i$, because $f(\alpha + i) = f(\beta)$ which, by assumption, is not less than, but greater than $\phi(\alpha + \omega) = \phi(\beta)$. As a consequence of the theorem of §12 there must therefore be a certain quantity U which is the greatest of those of which it can be asserted that all ω which are less than U have the property M.[8]

Task 16 Sketch a diagram with graphs of f and ϕ that illustrates the theorem statement and label α, β and A. For an arbitrary ω possessing property M, label Ω' and Ω. Also draw an ω number line and label key values i, U, and the set of values ω possessing Property M.

Task 17 Bolzano stated that ω, Ω and Ω' can be made "as small as we please". Explain the dependencies between these quantities. Use ϵ-δ terminology to clarify what is going on.

[8] This is the theorem that appears in the excerpt on the bottom of page 4 of this project.

Task 18 Rewrite Bolzano's claim in the first two sentences of I. 1. using modern terminology and call this Lemma 1.

Task 19 Convert Bolzano's argument in I.1. into a proof of Lemma 1 with your modern definition of continuity.

Task 20 Rewrite with symbols Bolzano's definition of Property M in the context of Part I.1 of his proof. Then rephrase his statement that "all ω that are smaller than a certain one possess the property M" using set notation, and name this set S_M.

Task 21 As an example, consider the functions $f(x) = 4 + (x-2)(x-4)(x-6)$ and $\phi(x) = 4$ with $\alpha = 1$ and $\beta = 7$. Informally find the set S_M and the value of U for this example.

Task 22 We can summarize the results of Part I.1. of the proof by stating some facts about U. First, Bolzano showed that such a quantity U exists. What additional fact about U did Bolzano prove?

Now proceed to Bolzano's Part I.2. of his proof.

2. This U must lie between 0 and i. For firstly it cannot be equal to i because this would mean that $f(\alpha+\omega) < \phi(\alpha+w)$, whenever $\omega<i$, and however near it came to the value of i. But in exactly the same way that we have just proved that the assumption $f(\alpha) < \phi(\alpha)$ has the consequence $f(\alpha+\omega) < \phi(\alpha+w)$, provided ω is taken small enough, so we can also prove that the assumption $f(\alpha+i) > \phi(\alpha+i)$ leads to the consequence $f(\alpha+i-\omega) > \phi(\alpha+i-\omega)$, provided ω is taken small enough. It is therefore not true that the two functions $f(x)$ and $\phi(x)$ stand in the relationship of smaller quantity to greater quantity for all values of x which are $< \alpha + i$. Secondly, still less can it be that $\mathsf{U} > i$ because otherwise i would also be one of the values of ω which are $< \mathsf{U}$, and hence also $f(\alpha+i) < \phi(\alpha+i)$ which directly contradicts the assumption of the theorem. Therefore, since it is positive, U certainly lies between 0 and i and consequently $\alpha + \mathsf{U}$ lies between α and β.

Task 23 Rewrite Bolzano's claim: "the assumption $f(\alpha+i) > \phi(\alpha+i)$ leads to the consequence $f(\alpha+i-\omega) > \phi(\alpha+i-\omega)$, provided ω is taken small enough" using modern terminology and call this Lemma 2.

Task 24 We can summarize this part of Bolzano's plan as the claims "$0 < \mathsf{U} < i$" and "$\alpha < \alpha + \mathsf{U} < \beta$" followed by his proof of the claim for $\mathsf{U} < i$. Rewrite his proof using your own words and modern terms, referencing the set S_M and the Section 12 theorem given on page 4 of this project.

Now read Part I.3. of Bolzano's proof.

3. It may now be asked what relation holds between $f(x)$ and $\phi(x)$ for the value $x = \alpha + \mathsf{U}$? First of all, it cannot be that $f(\alpha + \mathsf{U}) < \phi(\alpha + \mathsf{U})$, for this would also give $f(\alpha + \mathsf{U} + \omega) < \phi(\alpha + \mathsf{U} + \omega)$, if ω were taken small enough, and consequently $\alpha + \mathsf{U}$ would not be the greatest value of which it can be asserted that all x below it have the property M. Secondly. just as little can it be that $f(\alpha + \mathsf{U}) > \phi(\alpha + \mathsf{U})$, because this would also give $f(\alpha + \mathsf{U} - \omega) > \phi(\alpha + \mathsf{U} - \omega)$ if ω were taken small enough and therefore, contrary to the assumption, the property M would not be true of all x less than $\alpha + \mathsf{U}$. Nothing else therefore remains but that $f(\alpha + \mathsf{U}) = \phi(\alpha + \mathsf{U})$, and so it is proved that there is a value of x lying between α and β, namely $\alpha + \mathsf{U}$, for which $f(x) = \phi(x)$.

Task 25 Adjust your Lemmas 1 & 2 to give modern justifications of the first two claims in this excerpt.

Task 26 What property of the real numbers justifies the statement "Nothing else therefore remains but that $f(\alpha + \mathsf{U}) = \phi(\alpha + \mathsf{U})$"?

Task 27 At the beginning of the proof in I.1., Bolzano made the assumption "that α and β are both positive". Can you find a place in the proof where he used this assumption?

Bolzano continued in Section 15 to address the cases α and β are both negative, one is zero, and of opposite sign. We will omit these proofs, as they are not terribly enlightening.

Now that we have completed our journey with Bolzano through his proof, let's return to his preface proposition that you examined in Task 1.

Task 28 Consider Bolzano's proposition from his preface that: "between every two values of the unknown quantity which give results of opposite sign there must always lie at least one real root of the equation".

(a) In Task 1, you restated this proposition using modern terminology. Look back at your answer to that task with Bolzano's Theorem in mind. Do the hypotheses of his Theorem apply to your restatement? If not, modify your restatement of the proposition as needed so that they do.

(b) Use Bolzano's theorem to prove your restated version of Bolzano's Preface proposition.

Task 29 Use Bolzano's theorem to prove the following result from a standard introductory Calculus text, which is typically referred to as the "Intermediate Value Theorem."

> Consider an interval $I = [a, b]$ in the real numbers $\mathbb{R}$ and a continuous function $f : I \to \mathbb{R}$. If $f(a) < L < f(b)$ then there is a $c \in (a, b)$ such that $f(c) = L$.

4 Conclusion

As the final task of the last section illustrates, Bolzano's Theorem is a generalized version of the Intermediate Value Theorem that is featured in a standard first-year Calculus textbook. Today, various versions of the Intermediate Value Theorem serve as powerful tools in numerical analysis and other areas of mathematics. The Bounded Set Theorem that Bolzano used to prove his version of the Intermediate Value Theorem was a highly original idea, and is closely linked to what are nowadays called the least upper bound and greatest lower bound existence properties of the real numbers. If you have studied these completeness properties, you might enjoy the following task.

Task 30 Let S be a nonempty subset of $\mathbb{R}$ such that $s > 47$ for all $s \in S$. Use Bolzano's Bounded Set Theorem to prove that S has a greatest lower bound.

Caution: S is a subset of the interval $(47, \infty)$, but don't assume $S = (47, \infty)$.

It is interesting to note that A. L. Cauchy (1789-1857), a key player in building the theory of calculus, also proved a version of Bolzano's preface proposition, probably a few years later than Bolzano.[9] Although Cauchy used a very different method of proof,[10] his approach also depended on the completeness property of the real numbers. Given the strong influence of the nineteenth century approach to reshaping calculus that was adopted by Bolzano, Cauchy and their contemporaries in the nineteenth century, it is no coincidence then that some version of the completeness property continues to lie at the foundation of the proofs of the Intermediate Value Theorem that are found in today's Analysis textbooks.

[9] Bolzano's proof was published in 1817, but wasn't widely known when it first appeared. Cauchy's proof appeared in a note on the numerical solution of equations that was included at the end of his textbook *Cours d'Analyse (Course on Analysis)*, published in 1821. [Lützen, 2003, p. 175]

[10] To learn how Cauchy's approach to the Intermediate Value Theorem compares to that of Bolzano (without actually reading Cauchy himself!), see the overviews of Cauchy's proof found in [Lützen, 2003, pp. 175–176] and [Grabiner, 2005, pp.69–74].

References

B. Bolzano. *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege (A Purely Analytic Proof of the Theorem that between two values which give results of opposite sign there lies at least one real root of the equation)*. Göttlieb Haase, Prague, 1817. English translations of the complete pamphlet appear in [Russ, 1980] and [Russ, 2004, pp. 251–278]; see references below.

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Notes to Instructors

PSP Content: Topics and Goals

This PSP is designed to introduce continuity and the Intermediate Value Theorem (IVT) for a course in Real Analysis. Specifically, its content goals are to:

1. Develop a modern continuity definition with quantifiers based on Bolzano's definition.

2. Develop facility with the modern continuity definition by applying it to various functions.

3. Analyze Bolzano's Bounded Set Property and rewrite it in modern terminology.

4. Modernize Bolzano's proof of his Intermediate Value Theorem.

5. Apply Bolzano's Intermediate Value Theorem to obtain two other formulations of the Intermediate Value Theorem.

Student Prerequisites

The PSP assumes that students have done a rigorous study of quantifiers and limits of real-valued functions. The project also assumes that students have seen the least upper bound property for bounded sets of real numbers, but the project could be used to introduce this concept, with instructor supplements.

PSP Design, and Task Commentary

This PSP is designed to take one or two weeks of classroom time, with some reading and tasks done outside class.

The first section contains excerpts from Bolzano introducing his version of the IVT and his definition of continuity. Most of the tasks in this section focus on developing the definition of continuity for a function defined on an interval, the appropriate setting for a discussion of the IVT. Getting a correct definition of continuity in Task 3 is crucial before going much further; a class discussion of Task 3 and the next problem can be helpful after students work on them for awhile or in groups.

Bolzano's choice of $x+\sqrt{(1-x)\,(2-x)}$ in his footnote is mildly perplexing, as it does not change signs in its domain $[1,2]$. Indeed, the first set of students using the project were rather critical of Bolzano's footnote function. They inspired Task 5. In Bolzano's defense, he discussed the function $x+\sqrt{(x-2)\,(x+1)}$, which lacks a root between -1 and 2, earlier in his very lengthy preface.

Task 7 foreshadows a crucial result in the next section, namely writing a modern ϵ-δ proof of Bolzano's "because $f\alpha < \phi\,(\alpha)$, if ω denotes a positive quantity which can become as small as we please, then also $f(\alpha+\omega) < \phi(\alpha+\omega)$". This is difficult for some students, and they may need a hint/reminder that THEY get to choose ϵ if f is known to be continuous.

The final group of Section 1 tasks, 8–12, are standard problems to sharpen skills in working with continuity. Instructors may sample the set for classroom examples or homework problems. However, they are not needed for the flow of Bolzano's discussion.

Section 2 is written with the assumption that students have seen the least upper bound property for bounded sets of real numbers. Bolzano's theorem basically asserts this property for a special class of bounded sets, but in a form students (and the PSP author!) have not seen before. It is a bit tricky to unravel and put into modern set notation. The first two tasks should help ease this process for students. Task 14 should help them with Bolzano's section 15, where the theorem is applied.

The fourth section focuses on Bolzano's proof of his IVT. Part I.1 of the IVT proof in his Section 15 is crucial and contains some subtleties. It is worth taking time to make sure students understand this part of the proof. In addition the symbols Ω', Ω may cause confusion for some students. Bolzano was not completely clear on whether he was looking for the U value for set $S_{\mathsf{M}} = \{\omega : f(\alpha+\omega') < \phi(\alpha+\omega') \text{ for all } \omega' \in [0,\omega)\}$ or for $\{\omega : f(\alpha+\omega) < \phi(\alpha+\omega)\ \}$. Task 21 illustrates the difference. An opportunity for discussion and careful reading!

In his original paper, Bolzano began the proof of the Section 15 theorem with the following:

> We must remember that in this theorem the values of the functions $f(x)$ and $\phi(x)$ are to be compared to one another simply in their absolute values, i.e., without regard to signs or as though they were quantities incapable of being of opposite signs. But the signs of α and β are important.

Bolzano then split his proof into several cases I-I-V, beginning with case $0 < \alpha < \beta$. Here is S.B. Russ' explanation (page 183 in [Russ, 1980]):

> "Bolzano always uses inequality signs to apply only to the magnitude of quantities and not to their position on a number line. This was common practice at the time as there was still no standard symbol for modulus. Thus in Bolzano's usage $x > -1$ means the range we should now describe as $x < -1$. For example, in Section 2 he stated that the general term of a geometric progression … And in Section 15.IV he described the range of values of x between α and β when α is negative and β positive as 'all values of x which if negative are $< \alpha$, and if positive are $< \beta$'."

These comments by Bolzano have been excluded from the project because they seem likely to cause considerable confusion with little payoff to most students of analysis. Instructors could bring up these issues in a class discussion of Task 27.

The statement and proof of Lemma 2 in the task set following Part I.2 of the IVT proof in Section 15 of Bolzano's pamphlet may seem a bit repetitive. However, it should clarify things for some students and serve as proof writing practice, especially if Lemma 1 is done in class.

LaTeXcode of this entire PSP is available from the author by request to facilitate preparation of 'in-class task sheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Suggestions for Classroom Implementation

This is roughly a one or two week project under the following methodology (basically David Pengelley's "A, B, C" method described on his website https://www.math.nmsu.edu/~davidp/):

1. Students do some advanced reading and light preparatory tasks before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the tasks or your grading load will get out of hand! Some instructor have students write questions or summaries based on the reading.

2. Class time is largely dedicated to students working in groups on the project—reading the material and working tasks. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a "participation" grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the tasks they missed. This is usually a good incentive not to miss class!

3. Some tasks are assigned for students to do and write up outside of class. Careful grading of these tasks is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project. If students have already studied continuity in a rigorous fashion, then the first section should move very quickly and many tasks can safely be skipped.

Sample Implementation Schedule (based on a 50-minute class period)

Students read through the introductory material and the beginning of Section 1, and do Tasks 1–3 before the first class. After discussing their results at the beginning of Class 1, students work on and discuss Tasks 4 and 7. Homework practice with the definition of continuity could be some subset of Tasks 5, 6, 8–12. However, none of these are essential for the following material.

As preparation for Class 2, students read the first Bolzano excerpt in Section 2 and do Task 13. After discussing their results at the beginning of Class 2, students work on and discuss Tasks 14 and 15. Students then begin Section 3 by reading the first part of Bolzano's IVT proof and doing Task 16, which is essentially drawing diagrams for the proof.

As preparation for Class 3, students do Tasks 17, 18. After discussing their results at the beginning of Class 3, students work on and discuss Tasks 19–22.

As preparation for Class 4, students read Bolzano's Part I.2 proof excerpt and do Task 23. After discussing their results at the beginning of Class 4, students work on and discuss Tasks 24–30. Some of these tasks can be given as homework, as time permits.

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students.

Connections to other Primary Source Projects

Other projects for real analysis written by the author of this PSP (Dave Ruch) are listed below. "Mini-PSPs," designed to be completed in 1–2 class periods, are designated with an asterisk (*).

- *Investigations into Bolzano's Bounded Set Theorem*
  https://digitalcommons.ursinus.edu/triumphs_analysis/14/
- *An Introduction to a Rigorous Definition of Derivative*
  https://digitalcommons.ursinus.edu/triumphs_analysis/7
- *The Mean Value Theorem*
  https://digitalcommons.ursinus.edu/triumphs_analysis/5/
- *The Definite Integrals of Cauchy and Riemann*
  https://digitalcommons.ursinus.edu/triumphs_analysis/11/
- *Investigations Into d'Alembert's Definition of Limit** (sequences)
  https://digitalcommons.ursinus.edu/triumphs_analysis/13/
- *Euler's Rediscovery of e**
  https://digitalcommons.ursinus.edu/triumphs_analysis/3/
- *Abel and Cauchy on a Rigorous Approach to Infinite Series*
  https://digitalcommons.ursinus.edu/triumphs_analysis/4/

Additional PSPs that are suitable for use in introductory real analysis courses include the following; the PSP author name for each is listed parenthetically.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Barnett)
  https://digitalcommons.ursinus.edu/triumphs_analysis/1/
- *Topology from Analysis** (Nick Scoville)
  Also suitable for use in a course on topology.
  https://digitalcommons.ursinus.edu/triumphs_topology/1/
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real Analysis* (Janet Barnett)
  https://digitalcommons.ursinus.edu/triumphs_analysis/10/
- *The Cantor set before Cantor** (Nick Scoville)
  Also suitable for use in a course on topology.
  https://digitalcommons.ursinus.edu/triumphs_topology/2/
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Barnett)
  https://digitalcommons.ursinus.edu/triumphs_analysis/2/

Recommendations for Further Reading

The translations by Russ in [Russ, 1980] and [Russ, 2004] also include interesting background on Bolzano as well as commentary on some of the subtleties of Bolzano's work. The articles in [Jahnke, 2003] give some perspective on other works in analysis during Bolzano's era.

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