

The Mean Value Theorem

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1 Introduction

The basic ideas and tools of differential calculus were developed well before 1800, but mathematicians of that time were still struggling to build a rigorous foundation for these ideas¹. Joseph-Louis Lagrange was one of the leaders of the movement to create a solid theory of the derivative. He tried to create his theory of the derivative around Taylor series expansions during the period 1790–1810, but he was not entirely successful. Augustin-Louis Cauchy, a pivotal character in building the theory of calculus, also built a theory of the derivative early in the 1800’s. He was more successful than Lagrange with his work, mostly published in his 1823 *Calcul Infinitésimal* [C]. Cauchy gave a good, near-modern definition of limits, defined the derivative as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

and then used it to prove a number of basic derivative properties, in many places building on efforts by Lagrange and others.

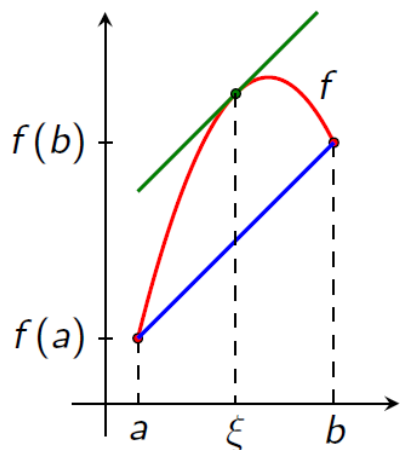
The Mean Value Theorem (MVT) has come to be recognized as a fundamental result in a modern theory of the differential calculus. As you may recall from introductory calculus courses, under suitable conditions for a function $f : [a, b] \rightarrow \mathbb{R}$, we can find a value ξ so that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(\xi)(b - a) \quad (1)$$

This result is quite plausible from a geometric argument, as the diagram below indicates. The MVT provides a crucial link between change in function values and the derivative at a point. While this result looks pretty clear, it is not so simple to prove analytically, without recourse to a proof by picture. In this project, we will read in Section 2 about Cauchy’s efforts to tackle this problem and his proposal for “suitable conditions” on f . Then in Section 3 we will explore a very different approach some forty years later by mathematicians Serret and Bonnet.

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¹See PSP “Why be so Critical” [Barn2] in the references for more on this.



Exercise 1 Use the diagram above to explain why (1) makes sense geometrically.

The next exercise should give you some initial appreciation of the “suitable conditions” issue for the MVT to be valid.

Exercise 2 Show that there is no such ξ value for (1) with $f(x) = 2|x - 3| + 1$, and $a = 0, b = 4$. Interpret this result in terms of the graph of f .

2 Cauchy’s Mean Value Theorem

As mentioned in the Introduction, Cauchy defined the derivative as a limit of the difference quotient:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Building on the work of predecessors such as Lagrange, he knew he needed to state and prove an acceptable version of the MVT in order to expand his rigorous theory of the calculus.

Before Cauchy proved his MVT, he first proposed a crucial inequality. Here is Cauchy, stating what is generally called his Mean Value Inequality theorem.



We now make known a relationship worthy of remark which exists between the derivative $f'(x)$ of any function $f(x)$, and the ratio of the finite differences

$$\frac{f(x+h) - f(x)}{h}.$$

If, in this ratio, we attribute to x a particular value x_0 , and if we make, in addition, $x_0 + h = X$, it will take the form

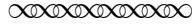
$$\frac{f(X) - f(x_0)}{X - x_0}.$$

This granted, we will establish the following without difficulty.

THEOREM. If, the function $f(x)$ being continuous between the limits $x = x_0, x = X$, we denote by A the smallest, and by B the largest values that the derivative function receives² in this interval, the ratio of the finite differences,

$$\frac{f(X) - f(x_0)}{X - x_0} \quad (2)$$

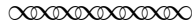
will necessarily be contained between A and B .



It turns out that Cauchy made some strong (perhaps hidden) assumptions in his proof of this theorem.

Exercise 3 Try applying Cauchy's Mean Value Inequality theorem to the function $f(x) = \sqrt{x}$ on the interval with endpoints $x_0 = 0, X = 4$. What are A, B for this example? Identify any ambiguities in Cauchy's theorem statement.

The first part of Cauchy's proof is given below. Read it carefully, thinking about the assumptions that he seems to be making.



Proof. Denote by δ, ϵ two very small numbers, the first being selected so that, for the numerical values³ of i less than δ , and for any value of x contained between the limits x_0, X , the ratio

$$\frac{f(x+i) - f(x)}{i}$$

always remains greater than $f'(x) - \epsilon$ and less than $f'(x) + \epsilon$. If, between the limits x_0, X , we interpose $n - 1$ new values of the variable x , namely,

$$x_1, x_2, \dots, x_{n-1}$$

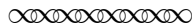
in a manner to divide the difference $X - x_0$ into elements

$$x_1 - x_0, \quad x_2 - x_1, \quad \dots, \quad X - x_{n-1},$$

which, all being of the same sign and having numerical values less than δ , the fractions

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad \dots, \quad \frac{f(X) - f(x_{n-1})}{X - x_{n-1}} \quad (3)$$

are found contained, the first between the limits $f'(x_0) - \epsilon, f'(x_0) + \epsilon$, the second between the limits $f'(x_1) - \epsilon, f'(x_1) + \epsilon, \dots$, will all be greater than the $A - \epsilon$ and less than the quantity $B + \epsilon$.



²Cauchy means A and B are "output" function values of the derivative.

³Cauchy uses the term "numerical value" to mean the absolute value, in modern terminology.

We should celebrate this proof: historians credit it as the first time the symbols ϵ and δ appear in a published analysis proof! However, there are some subtle problems with Cauchy's work from a modern viewpoint.

Exercise 4 *Based on Exercise 3 and your reading of the proof, what assumption(s) is Cauchy making about the derivative function f' in his theorem statement and proof?*

Let's decipher the proof. The very first sentence is especially important and subtle.

Exercise 5 *How do you think Cauchy would have justified the very first sentence of the proof? What does the value of δ seem to depend on?*

Exercise 6 *Rewrite the very first sentence of Cauchy's proof using modern quantifiers to clarify the dependencies between ϵ, δ, i and x .*

As we saw in Exercise 3, we need more hypothesis on f' than Cauchy gave in his theorem statement to make Cauchy's proof solid by modern standards. In 1884, 61 years after Cauchy's proof was published, the mathematician Giuseppe Peano (1858–1932) offered the following example [Peano], which demonstrates a problem in Cauchy's first sentence without the additional hypothesis implied by the first sentence of Cauchy's proof :

$$f(x) = x^2 \sin \frac{1}{x} \quad (4)$$

for $n \in \mathbb{N}$ sufficiently large⁴. Exercises 7, 8, 9 below will reveal Peano's concern. We first record the modern version of Cauchy's first sentence in his proof, which asserts a sort of *uniform differentiability* property for f on the interval. We will call it Property UD (we use "UD" to indicate "uniform differentiability"):

Property UD. For all $\epsilon > 0$, there exists $\delta > 0$ such that if $x, x + i \in [x_0, X]$ and $|i| < \delta$ then

$$\left| \frac{f(x+i) - f(x)}{i} - f'(x) \right| < \epsilon.$$

Exercise 7 *This exercise analyzes the behavior of Peano's function f at $x = 0$.*

- (a) *Find a value for $f(0)$ so that Peano's function f is continuous at 0. Prove your assertion.*
- (b) *Find a value for $f'(0)$ so that f is differentiable at 0. Prove your assertion with the definition of derivative.*
- (c) *Use standard calculus rules to find $f'(x)$ for $x \neq 0$.*
- (d) *Show that f' is not continuous at $x = 0$.*

⁴The mathematician Gaston Darboux used this function example in some 1875 correspondence, but it not known whether Peano knew of it or devised his example independently.

Exercise 8 We are now ready for Peano's key observation. Suppose $n \in \mathbb{N}$. Choose interval endpoints $x_0 = 0, X = 1$. Consider points

$$x_1 = \frac{1}{(2n+1)\pi}, \quad x_2 = \frac{1}{2n\pi}.$$

Show that for Peano's f defined in (4) and Exercise 7, and arbitrary ϵ with $0 < \epsilon < 1$, the statement

$$f'(x_1) - \epsilon \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq f'(x_1) + \epsilon$$

is false.

Exercise 9 Use the previous exercises to explain why Peano's example function does not have Property UD even though its derivative exists and is bounded on the interval $[x_0, X] = [0, 1]$.

Hint: For any ϵ with $0 < \epsilon < 1$ and any $\delta > 0$, first show you can choose n so that $|x_1 - x_2| < \delta$.

What about the problematic Property UD in Cauchy's proof? This uniform differentiability property for f turns out to be valid once we add the crucial hypothesis that f' is *uniformly continuous* on the interval. However, the proof of Property UD for a function with uniformly continuous derivative is surprisingly difficult without using the MVT, so we omit it.

The next key idea in Cauchy's proof was to divide the interval $[x_0, X]$ finely enough so that he could apply Property UD. After this, Cauchy used some elaborate algebra and bounding to show that

$$A \leq \frac{f(X) - f(x_0)}{X - x_0} \leq B,$$

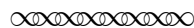
which completes the proof of his mean value inequality theorem - with the added hypothesis that f' is *uniformly continuous* on the interval. As we can see from this proof by Cauchy, mathematicians in his era had not fully grasped all the subtleties of the dependence of δ on x as well as ϵ in the definition of the derivative. More on the history of these issues can be found in [Grab].

Exercise 10 Write a corrected version of Cauchy's mean value inequality theorem with modern notation. Be sure to include both (i) the assumption that f' is uniformly continuous on the interval, and (ii) his conclusion

$$A \leq \frac{f(X) - f(x_0)}{X - x_0} \leq B.$$

Then add an explanation for why A and B exist as finite values under the hypothesis f' is uniformly continuous on the interval.

Immediately after Cauchy gave his proof of the mean value inequality theorem, he stated the following result.



Corollary. If the derivative function $f'(x)$ is itself continuous between the limits $x = x_0, x = X$, by passing from one limit to the other, this function will vary in a manner to always remain contained between the two values A and B , and to successively take all the intermediate values. Therefore, any

number⁵ between A and B will then be a value of $f'(x)$ corresponding to a value of x included between the limits x_0 and $X = x_0 + h$, or to what amounts to the same thing, to a value of x of the form

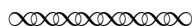
$$x_0 + \theta h = x_0 + \theta (X - x_0),$$

θ denoting a number less than unity. By applying this remark to expression (2), we will conclude that there exists between the limits 0 and 1, a value of θ that works to satisfy the equation

$$\frac{f(X) - f(x_0)}{X - x_0} = f'[x_0 + \theta (X - x_0)]$$

or what amounts to the same thing, the following

$$\frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0 + \theta h). \quad (5)$$



We can recognize this corollary as being a version of the Mean Value Theorem mentioned in the introduction! Recall that in his Mean Value Inequality theorem, Cauchy proved that

$$A \leq \frac{f(X) - f(x_0)}{X - x_0} \leq B. \quad (6)$$

Exercise 11 *What theorem is Cauchy applying to assert the existence of his θ value? In what interval must the value $c = x_0 + \theta h$ lie?*

Exercise 12 *Write Cauchy's corollary (his mean value theorem) with modern notation.*

While Cauchy's *proof* requires the uniform continuity of f' on the closed interval $[x_0, X]$, there are examples where the MVT holds true even without the continuity of f' on the closed interval $[x_0, X]$. In the next section, we will read about a very different approach to proving an improved version of the MVT, developed well after Cauchy, one which does not require the uniform continuity of f' .

Exercise 13 *Define $f : [0, 1] \rightarrow \mathbb{R}$ by $f(x) = \sqrt{1 - x^2}$.*

- (a) *Find a value θ for which equation (5) holds for f on $[0, 1]$*
- (b) *Explain why we can't apply Cauchy's version of the MVT to this example.*

The following exercises are not needed for the flow of the project, but will sharpen your skills in working with Mean Value theorems, and show the power of Cauchy's version of the MVT.

Exercise 14 *Suppose f is differentiable on $I = [a, b]$ with $f'(x) = 0$ on I . Use Cauchy's MVT to show that f is a constant function on I .*

⁵Cauchy actually uses the term “average quantity” in place of “number” for purposes of his proof.

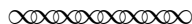
Sometimes properties of a function or functions are more easily shown by considering an “auxiliary function” defined in terms of the original function(s). We will see an example of this in the next section. The next exercise also demonstrates this technique.

Exercise 15 Suppose that f and g are differentiable on $I = [a, b]$ with $f'(x) = g'(x)$ on I . Show there is a constant C for which $f(x) = g(x) + C$ on I . Hint: Apply the prior exercise to “auxiliary function” $f - g$.

Exercise 16 Explain why the previous exercise justifies the “+C” you tacked onto antiderivatives in your introductory calculus courses.

3 Bonnet’s Version of the Mean Value Theorem.

Cauchy’s version of the MVT was the best available for another four decades after he proved it. A different version with weaker hypotheses and a very different proof appeared in Joseph Serret’s 1868 *Cours de calcul infinitésimal*. Serret (1819-1885) credited his colleague Pierre Ossian Bonnet (1819-1892) with the proof. Both men were accomplished French mathematicians. Here is the theorem and proof from Serret’s book. As you read this proof, try to identify the three separate “subclaims” that Serret proves to complete the poof.



THEOREM I. Suppose $f(x)$ is a function of x that is continuous for x values between given limits, and which, for these values, has a specific derivative $f'(x)$. If x_0 and X designate two values of x between these same limits, we have

$$\frac{f(X) - f(x_0)}{X - x_0} = f'(x_1),$$

x_1 being a value between x_0 and X .

Indeed, the ratio

$$\frac{f(X) - f(x_0)}{X - x_0}$$

has, by hypothesis, a finite value, and if one names this value A , one has

$$[f(X) - AX] - [f(x_0) - Ax_0] = 0. \quad (7)$$

Designating by $\phi(x)$ the function of x defined by formula

$$\phi(x) = [f(x) - Ax] - [f(x_0) - Ax_0], \quad (8)$$

one has, due to equality (8),

$$\phi(x_0) = 0, \quad \phi(X) = 0, \quad (9)$$

so that $\phi(x)$ vanishes at x_0 and X .

Suppose, to fix the ideas, $X > x_0$ and as x increases from x_0 to X ; the function $\phi(x)$ is initially zero. If we assume that it is not constantly zero, for the values of x between x_0 and X , it will have to

begin to increase through positive values or to decrease by taking negative values, either from $x = x_0$ or from a value of x between x_0 and X . If the values in question are positive, as $\phi(x)$ is continuous and vanishes at X , obviously there will be a value x_1 between x_0 and X for which

$$\phi(x_1)$$

is greater than or at least equal to nearby values

$$\phi(x_1 - h), \quad \phi(x_1 + h),$$

h being an amount as small as we please⁶. If the function $\phi(x)$, ceasing to be zero, takes negative values, the same reasoning proves that there is a value x_1 between x_0 and X for which

$$\phi(x_1)$$

is less than or at most equal to nearby values

$$\phi(x_1 - h), \quad \phi(x_1 + h).$$

So, in both cases, the value of x_1 will be such that the differences

$$\phi(x_1 - h) - \phi(x_1), \quad \phi(x_1 + h) - \phi(x_1),$$

will have the same sign, and, therefore, the ratios

$$\frac{\phi(x_1 - h) - \phi(x_1)}{-h}, \quad \frac{\phi(x_1 + h) - \phi(x_1)}{h}, \quad (10)$$

will have opposite signs.

Note that we do not exclude the hypothesis in which one of the previous ratios evaluates to zero, which requires the function $\phi(x)$ to have the same value for values of x in a finite interval. In particular, if the $\phi(x)$ function is always zero for x values between x_0 and X , the ratios (10) are both zero.

The ratios (10) tend to the same limit when h tends to zero, because we assume that the $f(x)$ function has a specific derivative, and the same thing occurs, consequently, with respect to $\phi(x)$. Anyway these ratios are of opposite signs, so the limit is zero. Thus we have

$$\lim \frac{\phi(x_1 + h) - \phi(x_1)}{h} = 0, \quad (11)$$

or, because of the equation (8),

$$\lim \left[\frac{f(x_1 + h) - f(x_1)}{h} - A \right] = 0,$$

that is to say

$$A = \lim \frac{f(x_1 + h) - f(x_1)}{h} = f'(x_1) \quad (12)$$

⁶Serret means h is positive throughout his proof.

Therefore we have

$$\frac{f(X) - f(x_0)}{X - x_0} = f'(x_1)$$

or

$$f(X) - f(x_0) = (X - x_0) f'(x_1) \quad (13)$$

as was promised.

We assumed $X > x_0$, but as the above formula does not change by permuting the letters x_0, X , it [(13)] is obviously independent of this assumption.

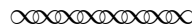
If we set

$$X = x_0 + h,$$

the quantity x_1 , between x_0 and $x_0 + h$, may be represented by $x_0 + \theta h$, θ being a quantity between 0 and 1; we can write

$$f(x_0 + h) - f(x_0) = hf'(x_0 + \theta h) \quad (14)$$

REMARK. The preceding demonstration is due to M. Ossian Bonnet. It should be noted that it in no way implies the continuity of the derivative $f'(x)$; it only requires that the derivative exists and has a specific value.



Exercise 17 Compare Serret's MVT statement with Cauchy's, Exercise 12. In what way is Serret's theorem superior to Cauchy's?

Hint: Peek at Exercise 13.

Exercise 18 To begin analyzing Serret's proof, first outline the proof's main features. Try to identify three distinct major steps.

Exercise 19 To get a sense of Serret's "auxiliary" function ϕ , consider $f(x) = \sqrt{1 - x^2}$ on $[0, 1]$ from Exercise 13. Sketch the graph of ϕ for this example, and find the value of x_1 . What are the properties of ϕ at x_1 ?

Now look carefully at the first part of Serret's proof, where he showed the existence of x_1 .

Exercise 20 Show that the equations in (7) and (9) are valid.

In his theorem statement, Serret stated that f is continuous "between" two given limits and that x_1 is "between" x_0 and X , but it is not clear whether he meant *strictly* between, or meant to include the endpoints. He was similarly vague about where f' exists. Let's try to clarify these issues as we dissect the proof.

Exercise 21 Using modern terminology and facts about continuous functions, how would you justify the existence of x_1 for ϕ ? Does f need to be continuous on $[x_0, X]$ or just on (x_0, X) ? Explain.

In his existence proof for x_1 , Serret mentioned three cases for ϕ , one being the case where "the values in question are positive".

Exercise 22 What are the other two cases for ϕ ? Explain why we claim the existence of an x_1 in the open interval (x_0, X) for all three cases.

Let's record our findings as a lemma, for which you should be able to provide a modern proof.

Lemma 23 Suppose $g : [a, b] \rightarrow \mathbb{R}$ is continuous on closed interval $[a, b]$. If $g(a) = g(b)$, then g has a local extremum at an interior point $c \in (a, b)$.

Exercise 24 Use Serret's ideas and the discussion above to give a modern proof of Lemma 23 with a case argument.

Now that we have the first part of Serret's proof fully clarified, let's turn to the second part, where he proved the limit (11) for ϕ at this point x_1 . We recognize this limit as the derivative of ϕ at x_1 . Recall that Serret assumed the existence of f' at x values "between given limits", which is a bit vague.

Exercise 25 What derivative laws justify Serret's claim that ϕ has a derivative at x_1 ? Explain why we only need to require the hypothesis that f' exists on the open interval (x_0, X) .

Exercise 26 Use Serret's ideas to give a modern proof of the following Theorem 27:

Theorem 27 Suppose $g : [a, b] \rightarrow \mathbb{R}$. If g has a local extremum at an interior point $c \in (a, b)$ and $g'(c)$ exists, then $g'(c) = 0$.

The third part of Serret's proof is perhaps the easiest, since he just had to repackage his statements about ϕ into a conclusion about f , which he restated in a few ways. Observe that (14) looks a lot like Cauchy's (5). This plus Serret's final remark that his own proof "only requires that the derivative exists" suggest that Serret was aware of Cauchy's result and its deficiency.

Exercise 28 For what θ values is (14) valid? Explain.

Exercise 29 State a modern version of the MVT based on Serret's work, adjusting his theorem statement as needed.

Exercise 30 Show that your modern MVT statement applies to the example $f(x) = \sqrt{1-x^2}$ on $[0, 1]$ discussed in Exercise 13.

Exercise 31 Consider the function $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ studied in Exercise 7. Show that the hypotheses of the modern MVT are met for f on $[0, 1]$ while the hypotheses for Cauchy's version are not met for f on $[0, 1]$.

Theorem 27 was considered pretty obvious, using geometrical arguments, to mathematicians of the 18th century. This result should be familiar from introductory calculus, where it is used to solve max/min problems. This result is often credited to French mathematician Pierre Fermat (1601-1665), who gave a version of the result for the special case of quadratic functions in terms of tangent lines in [F], before Newton and Leibniz even developed calculus! The modern version is often called Fermat's theorem, but be aware that Fermat has several famous theorems named after him.

We can combine Lemma 23 and Theorem 27 to get a proof of the following theorem.

Theorem 32 (Rolle) Suppose $g : [a, b] \rightarrow \mathbb{R}$ is continuous on closed interval $[a, b]$ and is differentiable on open interval (a, b) . If $g(a) = g(b)$, then there is at least one point c in (a, b) for which $g'(c) = 0$.

Exercise 33 Write out a proof of Theorem 32 using Lemma 23 and Theorem 27.

The first known formal proof of this result was given by Michel Rolle (1652-1719). He proved a version of this theorem in 1690 for polynomials [R]. Interestingly, the theorem was not called “Rolle’s” theorem until the 1830’s, when mathematicians were growing more interested in foundations of the calculus [Ca].

Exercise 34 Sketch a diagram demonstrating Rolle’s theorem, labeling the point c . Next sketch a diagram with the graph of the ϕ function from Serret’s proof with $a = x_0$ and $b = X$. Label the point x_1 from Serret’s proof. Compare and contrast these diagrams with the diagram for the MVT given in the project introduction.

The MVT is a great tool for proving many important results you may recall from introductory calculus, such as the following.

Theorem 35 Suppose $f : I \rightarrow \mathbb{R}$ is differentiable on interval I . Then f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.

Exercise 36 Use the MVT to prove Theorem 35 in one direction. Use the definition of derivative and Serret’s ideas with (10) in the other direction.

Recall from Section 2 that Cauchy made the very strong assumption we called Property UD in the first sentence of his proof of the Mean Value Inequality, where Property UD necessarily holds provided f' is uniformly continuous on the interval. While this is difficult to prove directly, it can be proved more easily using the Mean Value Theorem. The next exercise confirms this.

Exercise 37 Suppose that $f : [a, b] \rightarrow \mathbb{R}$ and f' is continuous on $[a, b]$. Prove that for all $\epsilon > 0$ there exists a $\delta > 0$ for which

$$\text{if } x, u \in [a, b] \text{ and } 0 < |x - u| < \delta \text{ then } \left| f'(x) - \frac{f(x) - f(u)}{x - u} \right| < \epsilon. \quad (15)$$

Exercise 38 Suppose g is differentiable and g' is continuous on $[a, b]$. Show that there is a constant K for which $|g(x) - g(y)| \leq K|x - y|$ for all $x, y \in [a, b]$.

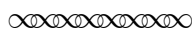
Exercise 39 Many difficult equations can be solved approximately by putting them into the form $g(x) = x$ and building a sequence of approximate solutions (x_n) defined by $x_n = g(x_{n-1})$. It can be proved that this sequence (x_n) will converge to a solution of the equation $g(x) = x$ if it is **contractive**, which means that there is a constant C , $0 < C < 1$, for which

$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}| \quad \text{for all } n \in \mathbb{N}.$$

Suppose that the function $g : [a, b] \rightarrow [a, b]$ is differentiable and $|g'(x)| \leq 0.99$ on $[a, b]$. Let $x_1 \in [a, b]$ and define sequence (x_n) by $x_n = g(x_{n-1})$. Use the MVT to prove that (x_n) is contractive.

4 Conclusion

Mathematicians learned how to define and apply the derivative well before they put the concept on a firm mathematical foundation. Important results such as Rolle's Theorem and the MVT turn out to be more difficult to prove analytically than one might think. The Mean Value Theorem is fundamental, and can be used to prove many results from an introductory Calculus course.



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5 Instructor Notes

This project is designed for a course in Real Analysis. The first part of the project is over Cauchy's version of the Mean Value Theorem. Cauchy's proof is very clever but has some hidden assumptions.

The second part of the project focuses on the standard modern version seen in Introductory Analysis texts. This standard version of this proof is apparently due to Bonnet, and recorded in his student Serret's 1868 *Cours de calcul infinitésimal*.

Project Content Goals

1. Study Cauchy's proof of the Mean Value Theorem.
2. Analyze the assumptions in Cauchy's proof and their necessity for his proof method. In particular, recognize and appreciate the relevance of *uniform* continuity and differentiability.
3. Analyze Serret's proof of the Mean Value Theorem.
4. Use Serret's proof to prove Fermat's Interior Extremum Theorem and Rolle's Theorem.
5. Use the Mean Value Theorem for applications.

Preparation of Students

The project is written for a course in Real Analysis with the assumption that students have done a rigorous study of limits, continuity and the derivative. The concept of uniform continuity appears in a crucial way in Cauchy's proof. Instructors could use this as an opportunity to motivate and define uniform continuity if students have not seen it before.

Preparation for the Instructor

This is roughly a two week project under the following methodology (basically David Pengelley's "A, B, C" method described on his website):

1. Students do some advanced reading and light preparatory exercises before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the exercises or your grading load will get out of hand! Some instructors have students write questions or summaries based on the reading.
2. Class time is largely dedicated to students working in groups on the project - reading the material and working exercises. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a "participation" grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the exercises they missed. This is usually a good incentive not to miss class!

3. Some exercises are assigned for students to do and write up outside of class. Careful grading of these exercises is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

If time does not permit a full implementation with this methodology, instructors can use more class time for guided discussion and less group work for difficult parts of the project. If students have already studied continuity in a rigorous fashion, then the first section should move very quickly and many exercises can safely be skipped.

Section 1 Introduction

Some historical perspective for the MVT is given. An exercise is designed to create some appreciation of suitable conditions under which the MVT conclusion holds.

Section 2 Cauchy

Exercise 5 is very important. It is a great chance to revisit the dependence of δ on x as well as ϵ in the definition of the derivative. The term “uniform continuity” is then explicitly mentioned later. The concept of *uniform* continuity needs to be defined or recalled here. The notion of uniform differentiability is addressed in some, but not all, modern texts. There is some debate among mathematical historians about Cauchy’s treatment and understanding of global versus local behavior. See articles in [Grab] for more on this.

Section 3 Bonnet

Most standard treatments of the MVT first prove Fermat’s theorem (if you have a local extremum, the derivative is zero there). This result is in turn used to prove Rolle’s Theorem. Finally, Rolle’s theorem is applied to a clever auxiliary function ϕ to obtain the MVT. Serret’s proof of the MVT is a bit surprising. He immediately defines the crucial auxiliary function ϕ . Serret then proves Fermat’s theorem for ϕ . Next he proves Rolle’s theorem for ϕ . Finally he shows this gives us the MVT for the original function. Having students read this proof has some pedagogical advantages, since it assumes very little and students won’t be able to track it word for word with a modern textbook. Serret is a bit vague with his inequalities (strict vs inclusive), which will force some careful reading. But once students unpack everything, they will have the standard big three results.

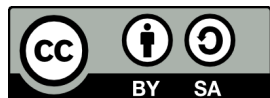
Serret doesn’t explicitly state that f has an extremum at x_1 , nor is he explicit about x_1 being *strictly* between x_0 and X . The exercises are designed to clarify these points.

The last exercises can be used as an introduction to Lipschitz functions and convergence of fixed point iterations.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of ‘in-class task sheets’ based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

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