Bolzano's Definition of Continuity, his Bounded Set Theorem, and an Application to Continuous Functions.

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1 Introduction

The foundations of calculus were not yet on firm ground in early 1800's. Mathematicians such as J. L. Lagrange (1736-1813) made efforts to put limits and derivatives on firmer logical ground, but were not entirely successful.

Bernard Bolzano (1781-1848) was one of the great success stories of the foundations of analysis. He was a theologian with interests in mathematics and a contemporary of Gauss and Cauchy, but was not well known in mathematical circles. Despite his mathematical isolation in Prague, Bolzano was able to read works by Lagrange and others, and published work of his own.

This project investigates results from his important paper *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* (Prague 1817) [Bz]. In particular, his proof of the main theorem in Section 12 on a property of bounded sets inspired Weierstrass decades later, and some version of his theorem in Section 15 is found in nearly every introductory calculus text. Here are excerpts from Bolzano's preface, as translated by S. B. Russ in [Russ].

Preface.

There are two propositions in the theory of equations for which, up until recently, it could still be said that a perfectly correct proof was unknown. One is the proposition: between every two values of the unknown quantity which give results of opposite sign there must always lie at least one real root of the equation.

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We do find very distinguished mathematicians concerned with this proposition and various kinds of proof for it have already been attempted. Anyone wishing to be convinced of this need only compare the various treatments of this proposition given, for example, by Kästner ... as well as by several others.

However, a more careful examination very soon shows that none of these kinds of proof can be regarded as satisfactory. The most common kind of proof depends on a truth borrowed from geometry, namely: that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the $x-$axis somewhere at a point lying between those ordinates. There is certainly nothing to be said against the correctness, nor against the obviousness of this geometrical proposition. But it is equally clear that it is an unacceptable breach of good method to try to derive truths of pure (or general) mathematics (i.e. arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part of it, namely geometry.

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According to a correct definition, the expression that a function $f(x)$ varies according to the law of continuity for all values of x inside or outside certain limits[1] means only that, if x is any such value the difference $f(x+\omega) - f(x)$ can be made smaller than any given quantity provided ω can be taken as small as we please.

Exercise 1 *What do you think of Bolzano's philosophical criticism of geometrical proof attempts of the Preface proposition "between every two values..."? Do you agree with him?*

Exercise 2 *Rewrite Bolzano's Preface proposition "between every two values..." in your own words with modern terminology. Sketch a diagram illustrating the proposition.*

Exercise 3 *For a function $f : \mathbb{R} \to \mathbb{R}$, rephrase Bolzano's "correct definition" of continuity at x using modern $\epsilon - \delta$ terminology and appropriate quantifiers.*

Exercise 4 *Use this definition to give a modern $\epsilon - \delta$ proof of the continuity of $f(x) = 3x + 47$ at $x = 2$.*

Exercise 5 *Consider the function Bolzano discusses in his footnote. Based on the Preface proposition he is discussing, why is this an interesting example? How could you adjust the function to make it better fit the issues surrounding the Preface proposition?*

Exercise 6 *Adjust your continuity definition in Exercise 3 to include the notion of domain, so it applies to functions defined on an interval I within $\mathbb{R}$. Do you think this footnote function should be continuous at $x = 1$ and at $x = 2$? You don't need to prove this rigorously.*

Exercise 7 *Suppose a function h is continuous for all x in $[0, 4]$ and $h(3) = 6$. Show that there is a $\delta > 0$ for which $h(x) \geq 5$ for all $x \in (3 - \delta, 3 + \delta)$.*

The following exercises are not needed for the flow of Bolzano's discussion, but will sharpen your skills in working with continuity.

Exercise 8 *Define $g(x) = 3 - 5x^2$ with domain $I = [4, 7]$. Show that g is continuous at arbitrary $\alpha \in I$ using your continuity definition.*

Bonus For Exercise 8, change the domain of g to be $\mathbb{R}$. Show that g is continuous at arbitrary $\alpha \in \mathbb{R}$. You may need to adjust your proof from Exercise 8.

Exercise 9 *Suppose that functions f and g are both continuous for x in an interval I. Prove that $f - 6g$ is also continuous for x in an interval I.*

[1] There are functions which vary continuously for all values of their root, e.g., $\alpha + \beta x$. But there are others which are continuous only for values of their root inside or outside certain limits. Thus $x + \sqrt{(1-x)(2-x)}$ is continuous only for values of $x < +1$ or $> +2$ but not for values between $+1$ and $+2$.

Use the following properties of the sine and cosine functions for the exercises below.

$$\begin{aligned}\sin a - \sin b &= 2\sin\left(\left(a-b\right)/2\right)\cos\left(\left(a+b\right)/2\right) ,\\ \cos a - \cos b &= 2\sin\left(\left(b-a\right)/2\right)\sin\left(\left(a+b\right)/2\right) , \quad \left|\sin a\right| \le \left|a\right| , \quad \left|\cos a\right| \le 1 \quad \text{for } a,b \in \mathbb{R}\end{aligned}$$

Exercise 10 *Prove that* $\sin x$ *is continuous on* $\mathbb{R}$.

Exercise 11 *Prove that* $\cos x$ *is continuous on* $\mathbb{R}$.

Exercise 12 *Define* $S(x) = x\sin(1/x)$ *for* $x \neq 0$. *Find a value for* $S(0)$ *so that* S *will be continuous at* $x = 0$. *Prove your assertion.*

2 Bolzano's Bounded Set Theorem and an Application

In Sections 1-10 of paper [Bz], Bolzano discusses infinite series and their convergence. He uses his results in his Section 12 to prove a very important theorem about certain bounded sets.

§11

Preliminary note. In investigations of applied mathematics it is often the case that we learn that a definite property M applies to all values of a [nonnegative[2]] variable quantity x which are smaller than a certain u without at the same time learning that this property M does not apply to values which are greater than u. In such cases there can still perhaps be some u' that is $> u$ for which in the same way as it holds for u, all values of x lower than u' possess property M. Indeed this property M may even belong to all values of x without exception. But if this alone is known, that M does not belong to all x in general then by combining these two conditions we will now be justified in concluding: there is a certain quantity U which is the greatest of those for which it is true that all smaller values of x possess property M. This is proved in the following theorem.

§12

Theorem. If a property M does not apply to all values of a [nonnegative] variable quantity x but does apply to all values smaller than a certain u, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x possess the property M.

Let's look at some examples of this concept Bolzano is discussing.

Exercise 13 *Let M be the property "$x^2 < 3$" applied to the set $x \ge 0$.*

[2] Bolzano intends to discuss only $x \ge 0$ in this note and his Section 12 theorem statement. The term "nonegative" has been included in this project for clarity.

(a) *Find rational numbers u, u' for this example (these values are not unique). What is the value of* U *for this example?*

(b) *Let S_M be the set of ω values that possess property* M. *Sketch S_M on a number line. Are the theorem hypotheses met for this property* M*?*

(c) *Does* U *possess property* M*?*

Exercise 14 *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x$, and let $\alpha \in \mathbb{R}$ be arbitrary. Let* M *be the property "$f(\alpha + \omega) \leq f(\alpha) + 2$" applied to the set $\omega \geq 0$.*

(a) *Find rational numbers u, u' for this example (these values are not unique). What is the value of* U *for this example?*

(b) *Let S_M be the set of ω values that possess property* M. *Sketch S_M on a number line. Are the theorem hypotheses met for this property* M*?*

(c) *Does* U *possess property* M*?*

Exercise 15 *Rewrite this theorem using modern terminology and set notation.*

Bolzano's proof of the theorem is correct, based on a Cauchy sequence-like convergence assumption for infinite series. However, the proof is long and difficult, so we will omit it for this project. This theorem is crucial for Bolzano's proof of his main result for solving equations, which he gives in Section 15.

§15

Theorem. If two functions of x, fx and ϕx vary according to the law of continuity either for all values of x or for all those lying between α and β, and furthermore if $f\alpha < \phi\alpha$ and $f\beta > \phi\beta$, then there is always a certain value of x between α and β for which $fx = \phi x$.

Proof.
I. 1. Firstly assume that α and β are both positive and that (because it does not matter) β is the greater of the two, so $\beta = \alpha + i$, where i denotes a positive quantity. Now because $f\alpha < \phi\alpha$, if ω denotes a positive quantity which can become as small as we please, then also $f(\alpha + \omega) < \phi(\alpha + w)$. For because fx and ϕx vary continuously for all x lying between α and β, and $\alpha + \omega$ lies between α and β whenever we take $\omega < i$, then it must be possible to make $f(\alpha+\omega) - f\alpha$ and $\phi(\alpha+\omega) - \phi\alpha$ as small as we please if ω is taken small enough. Hence if Ω and $\Omega^{'}$ denote quantities which can be made as small as we please, $f(\alpha + \omega) - f\alpha = \Omega$ and $\phi(\alpha + \omega) - \phi\alpha = \Omega^{'}$. Hence,

$$\phi(\alpha + \omega) - f(\alpha + \omega) = \phi\alpha - f\alpha + \Omega^{'} - \Omega.$$

However, $\phi\alpha - f\alpha$ equals, by assumption, some positive quantity of constant value A. Therefore

$$\phi(\alpha + \omega) - f(\alpha + \omega) = A + \Omega^{'} - \Omega,$$

which remains positive if Ω and $\Omega^{'}$ are allowed to become small enough, i.e., if ω is given a very small value, and even more so for all smaller values of ω. Therefore it can be asserted that for all values of ω smaller than a certain value the two functions $f(\alpha + \omega)$ and $\phi(\alpha + w)$ stand in the relationship of smaller quantity to greater quantity. Let us denote this property of the variable quantity ω by M. Then we can say that all ω that are smaller than a certain one possess the property M. But nevertheless it is clear that this property M does not apply to all values of ω, namely not to the value $\omega = i$, because $f(\alpha + i) = f\beta$ which, by assumption, is not less than, but greater than $\phi(\alpha + \omega) = \phi\beta$. As a consequence of the theorem of §12 there must therefore be a certain quantity U which is the greatest of those of which it can be asserted that all ω which are less than U have the property M.

Exercise 16 *Sketch a diagram with graphs of f and ϕ that illustrates the theorem statement and label α, β and A. For an arbitrary ω possessing property M, label $\Omega^{'}$ and Ω. Also draw an ω number line and label key values i, U, and the set of values ω possessing Property M.*

Exercise 17 *Bolzano states that ω, Ω and $\Omega^{'}$ can be made "as small as we please". Explain the dependencies between these quantities. Use $\epsilon - \delta$ terminology to clarify what is going on.*

Exercise 18 *Rewrite Bolzano's claim in the first two sentences of I. 1. using modern terminology and call this Lemma 1.*

Exercise 19 *Convert Bolzano's argument in I.1. into a proof of Lemma 1 with your modern definition of continuity.*

Exercise 20 *Rewrite with symbols Bolzano's definition of Property M in the context of Section I.1. Then rephrase his statement that "all ω that are smaller than a certain one possess the property M" using set notation, and name this set S_M.*

Exercise 21 *As an example, consider the functions $f(x) = 4+(x-2)(x-4)(x-6)$ and $\phi(x) = 4$ with $\alpha = 1$ and $\beta = 7$. Informally find the set S_M and the value of U for this example.*

Exercise 22 *We can summarize the results of Section I.1. of the proof by stating a couple facts about U. First, that such a quantity exists. What else?*

Now proceed to Bolzano's Section I.2. of his proof.

2. This U must lie between 0 and i. For firstly it cannot be equal to i because this would mean that $f(\alpha{+}\omega) < \phi(\alpha{+}w)$, whenever $\omega{<}i$, and however near it came to the value of i. But in exactly the same way that we have just proved that the assumption $f\alpha < \phi\alpha$ has the consequence $f(\alpha+\omega) < \phi(\alpha+w)$, provided ω is taken small enough, so we can also prove that the assumption $f(\alpha+i) > \phi(\alpha+i)$ leads to the consequence $f(\alpha+i-\omega) > \phi(\alpha+i-\omega)$, provided ω is taken small enough. It is therefore not true that the two functions fx and ϕx stand in the relationship of smaller quantity to greater quantity for all values of x which are $< \alpha+i$. Secondly, still less can it be true that $\mathsf{U} > i$ because otherwise i would also be one of the values of ω which are $< \mathsf{U}$, and hence also $f(\alpha+i) < \phi(\alpha+i)$ which directly contradicts the assumption of the theorem. Therefore, since it is positive, U certainly lies between 0 and i and consequently $\alpha+\mathsf{U}$ lies between α and β.

Exercise 23 *Rewrite Bolzano's claim: "the assumption $f(\alpha{+}i) > \phi(\alpha{+}i)$ leads to the consequence $f(\alpha+i-\omega) > \phi(\alpha+i-\omega)$, provided ω is taken small enough" using modern terminology and call this Lemma 2.*

Exercise 24 *We can summarize this section as the claims "$0 < \mathsf{U} < i$" and "$\alpha < \alpha+\mathsf{U} < \beta$" followed by Bolzano's proof of the claim for $\mathsf{U} < i$. Rewrite his proof using your own words and modern terms, referencing the set S_M and the Section 12 theorem.*

Now read Section I.3. of Bolzano's proof.

3. It may now be asked, what relation holds between fx and ϕx for the value $x = \alpha{+}\mathsf{U}$? First of all, it cannot be that $f(\alpha+\mathsf{U}) < \phi(\alpha+\mathsf{U})$, for this would also give $f(\alpha+\mathsf{U}{+}\omega) < \phi(\alpha+\mathsf{U}+\omega)$, if ω were taken small enough, and consequently $\alpha+\mathsf{U}$ would not be the greatest value of which it can be asserted that all x below it have the property M. Secondly. just as little can it be that $f(\alpha+\mathsf{U}) > \phi(\alpha+\mathsf{U})$, because this would also give $f(\alpha+\mathsf{U}{-}\omega) > \phi(\alpha+\mathsf{U}-\omega)$ if ω were taken small enough and therefore, contrary to the assumption, the property M would not be true of all x less than $\alpha+\mathsf{U}$. Nothing else therefore remains but that $f(\alpha+\mathsf{U}) = \phi(\alpha+\mathsf{U})$, and so it is proved that there is a value of x lying between α and β, namely $\alpha+\mathsf{U}$, for which $fx = \phi x$.

Exercise 25 *Adjust your Lemmas 1 & 2 to give modern justification of the first two claims in this section.*

Exercise 26 *What property of the real numbers justifies the statement "Nothing else therefore remains but that $f(\alpha+\mathsf{U}) = \phi(\alpha+\mathsf{U})$"?*

Exercise 27 *At the beginning of the proof in I.1., Bolzano makes the assumption "that α and β are both positive". Can you find a place in the proof where he uses this assumption?*

Bolzano continues in Section 15 to address the cases α and β are both negative, one is zero, and of opposite sign. We will omit these proofs, as they are not terribly enlightening.

Exercise 28 *Use Bolzano's theorem to state and prove a result making precise the proposition "between every two values of the unknown quantity which give results of opposite sign there must always lie at least one real root of the equation" from Bolzano's preface.*

Exercise 29 *Use Bolzano's theorem to prove the following result from a standard introductory Calculus text:*

Consider an interval $I = [a, b]$ in the real numbers $\mathbb{R}$ and a continuous function $f : I \to \mathbb{R}$. If $f(a) < L < f(b)$ then there is a $c \in (a, b)$ such that $f(c) = L$.

References

[Bz] Bolzano, B., *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* (Prague 1817)

[Russ] Russ, S. B., *A Translation of Bolzano's Paper on the Intermediate Value Theorem*, Historia Mathematica 7, 1980, 156-185.