TRIUMPHS Student Projects: Detailed Descriptions
TRansforming Instruction in Undergraduate Mathematics via Primary Historical Sources

F 01. A Genetic Context for Understanding the Trigonometric Functions

In this project, we explore the genesis of the trigonometric functions: sine, cosine, tangent, cotangent, secant, and cosecant. The goal is to provide the typical student in a pre-calculus course some context for understanding these concepts that is generally missing from standard textbook developments. Trigonometry emerged in the ancient Greek world (and, it is suspected, independently in China and India as well) from the geometrical analyses needed to solve basic astronomical problems regarding the relative positions and motions of celestial objects. While the Greeks (Hipparchus, Ptolemy) recognized the usefulness of tabulating chords of central angles in a circle as aids to solving problems of spherical geometry, Hindu mathematicians, like Varahamahira (505–587), in his *Pancasiddhantika* [75], found it more expedient to tabulate half-chords, from whence the use of the sine and cosine became popular. We examine an excerpt from this work, wherein Varahamahira described a few of the standard modern relationships between sine and cosine in the course of creating a sine table. In the eleventh century, the Arabic scholar and expert on Hindu science Abu l-Rayhan Muhammad al-Biruni (973–1055) published *The Exhaustive Treatise on Shadows* (c. 1021) [61]. In this work, we see how Biruni presented geometrical methods for the use of sundials; the relations within right triangles made by the gnomon of a sundial and the shadow cast on its face lead to the study and tabulation of values of the tangent and cotangent, secant and cosecant. Biruni also worked out the relationships that these quantities have with the sines and cosines of the angles. However, the modern terminology for the standard trigonometric quantities was not established until the European Renaissance. Foremost in this development is the landmark *On Triangles* (1463) by Regiomontanus (Johannes Müller) [53]. Regiomontanus exposed trigonometry in a purely geometrical form and then applies the ideas to problems in circular and spherical geometry. We examine a few of the theorems that explore the trigonometric relations and which are used to solve triangle problems.

This project is intended for courses in pre-calculus, trigonometry, the history of mathematics, or as a capstone course for teachers. Author: Danny Otero.

F 02. Determining the Determinant

This project in linear algebra illustrates how the mathematicians of the eighteenth and nineteenth centuries dealt with solving systems of linear equations in many variables, a complicated problem that ultimately required attention to issues of the notation and representation of equations as well as careful development of the auxiliary notion of a “derangement” or “permutation.” Colin Maclaurin (1698–1746) taught a course in algebra at the University of Edinburgh in 1730 whose lecture notes includeds formulas for solving systems of linear equations in 2 and 3 variables; an examination of these lecture notes [51] illustrate the forms of the modern determinant long before the notion was formally crystalized. In 1750, Gabriel Cramer (1704–1752) published his landmark *Introduction a l’Analyse des Lignes Courbes algébriques* (*Introduction to the Analysis of Algebraic Curves*) [28]. In an appendix to this work, Cramer tackles the solution of linear systems more systematically, providing a formula for the solution to such a system, today known as Cramer’s Rule. More significantly, he pointed out the rules for formation of the determinantal expressions that appear in the formulas for the solution quantities, using the term “derangement” to refer to the complex permuting of variables and their coefficients that gives structure to these expressions. These ideas reach maturity in an 1812 memoir by Augustin-Louis Cauchy (1789–1857) entitled *M’emoire sur les functions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu’elles renferement* (*Memoir on those functions which take only two values, equal but of opposite sign, as a result of transpositions performed* *on the variables which they contain*) [24]. In this work, Cauchy provided a full development of the determinant and its permutational properties in an essentially modern form. Cauchy used the term "determinant" (adopted from Gauss (1777–1855)) to refer to these expressions and even adopted an early form of matrix notation to express the formulas for solving a linear system.

This project is intended for courses in linear algebra. Author: Danny Otero.

F 03. Solving a System of Linear Equations Using Ancient Chinese Methods

Gaussian Elimination for solving systems of linear equations is one of the first topics in a standard linear algebra class. The algorithm is named in honor of Carl Friedrich Gauss (1777–1855), but the technique was not his invention. In fact, Chinese mathematicians were solving linear equations with a version of elimination as early as 100 CE. This project has the students study portions of Chapter 8 Rectangular Arrays in *The Nine Chapters on the Mathematical Art* [59] to learn the technique known to the Chinese by 100 CE. Students then read the commentary to Chapter 8 of *Nine Chapters* given in 263 by Chinese mathemtician Liu Hui (fl. 3rd century CE) and are asked how his commentary helps understanding. The method of the *Nine Chapters* is compared to the modern algorithm. The similarity between the ancient Chinese and the modern algorithm exemplifies the sophisticated level of ancient Chinese mathematics. The format of the *Nine Chapters* as a series of practical problems and solutions reinforces the concept that mathematics is connected to everyday life.

This project is appropriate for an introductory linear algebra class, and may be used in a more advanced class with appropriate choice of the more challenging exercises. Author: Mary Flagg.

F 04. Investigating Difference Equations

Abraham de Moivre (1667–1754) is generally given credit for the first systematic method for solving a general linear difference equation with constant coefficients. He did this by creating and using a general theory of recurrent series, the details of which appeared in his 1718 *Doctrine of Chances* and a second manuscript written that same year. While de Moivre's methods are accessible to students in a sophomore/junior discrete math course, they are not as clear or straightforward as the methods found in today's textbooks. Building on de Moivre's work, Daniel Bernoulli (1700–1782) published a 1728 paper, Observations about series produced by adding or subtracting their consecutive terms which are particularly useful for determining all the roots of algebraic equations, in which he laid out a simpler approach, along with illuminating examples and a superior exposition. The first part of the project develops de Moivre's approach with excerpts from original sources. The second part gives Bernoulli's 1728 methodology, no doubt more attractive to most students. Ideally, this project will help students understand and appreciate how mathematics is developed over time, in addition to learning how to solve a general linear difference equation with constant coefficients.

This project is intended for courses in discrete mathematics. Author: David Ruch.

F 05. Quantifying Certainty: The p-value

The history of statistics is closely linked to our ability to quantify uncertainty in predictions based on partial information. In modern statistics, this rather complex idea is crystallized in one concept: the p-value. Understanding p-values is famously difficult for students, and statistics professors often have trouble getting their students to understand the rather precise nuances involved in the definition. In this project, students work to build a robust understanding of p-values by working through some early texts on probability and certainty. These include the famous text *Statistical Methods for Research Workers* by Sir Ronald Fisher (1890–1962), as well as earlier attempts that came very close to the modern concept, such as Buffon's *Essai d'Arithmtique Morale* [64].

This project is intended for courses in statistics. Author: Dominic Klyve.

F 06. The Exigency of the Parallel Postulate

In this project, we examine the use of the parallel postulate for such basic constructions as the distance formula between two points and the angle sum of a triangle (in Euclidean space). Beginning with Book I of Euclid's (c. 300 BCE) *Elements* [36], we witness the necessity of the parallel postulate for constructing such basic figures as parallelograms, rectangles and squares. This is followed by Euclid's demonstration that parallelograms on the same base and between the same parallels have equal area, an observation essential for the proof of the Pythagorean Theorem. Given a right triangle, Euclid constructed squares on the three sides of the triangle, and showed that the area of the square on the hypotenuse is equal to the combined area of the squares on the other two sides. The proof is a geometric puzzle with the pieces found between parallel lines and on the same base. The project stresses the ancient Greek view of area, which greatly facilitates an understanding of the Pythagorean Theorem. This theorem is then essential for the modern distance formula between two points, often used in high school and college mathematics, engineering and science courses.

The project is designed for courses in geometry taken both by mathematics majors and secondary education majors. Author: Jerry Lodder.

F 07. The Failure of the Parallel Postulate

This project develops the non-Euclidean geometry pioneered by János Bolyai (1802–1860), Nikolai Lobachevsky (1792–1856) and Carl Friedrich Gauss (1777–1855). Beginning with Adrien-Marie Legendre's (1752–1833) failed proof of the parallel postulate [66], the project begins by questioning the validity of the Euclidean parallel postulate and the consequences of doing so. How would distance be measured without this axiom, how would "rectangles" be constructed, and what would the angle sum of a triangle be? The project continues with Lobachevsky's work [9], where he stated that in the uncertainty whether there is only one line though a given point parallel to a given line, he considered the possibility of multiple parallels, and continued to study the resulting geometry, limiting parallels, and proprieties of triangles in this new world. This is followed by a discussion of distance in hyperbolic geometry from the work of Bolyai [47] and Lobachevksy [9]. The project shows that all triangles in hyperbolic geometry have angle sum less than 180°, with zero being the sharp lower bound for such a sum, as anticipated by Gauss [49, p. 244]. The project continues with the unit disk model of hyperbolic geometry provided by Henri Poincaré (1854–1912) [81], and, following the work of Albert Einstein (1879–1955) [35], closes with the open question of whether the universe is best modeled by Euclidean or non-Euclidean geometry.

This project is designed for courses in geometry taken both by mathematics majors and secondary education majors. Author: Jerry Lodder.

F 08. Richard Dedekind and the Creation of an Ideal: Early Developments in Ring Theory

As with other structures in modern Abstract Algebra, the ring concept has deep historical roots in several nineteenth century mathematical developments, including the work of Richard Dedekind (1831–1916) on algebraic number theory. This project draws on Dedekind's 1877 text *Theory of Algebraic Numbers*[33] as a means to introduce students to the elementary theory of commutative rings and ideals. Characteristics of Dedekind's work that make it an excellent vehicle for students in a first course on abstract algebra include his emphasis on abstraction, his continual quest for generality and his careful methodology. The 1877 version of his ideal theory (the third of four versions he developed in all) is an especially good choice for students to read, due to the care Dedekind devoted therein to motivating why ideals are of interest to mathematicians by way of examples from number theory that are readily accessible to students at this level.

The project begins with Dedekind's discussion of several specific integral domains, including the example of $\mathcal{Z}[\sqrt{-5}]$ which fails to satisfy certain expected number theoretic properties (e.g., a prime divisor of a product should divide one of the factors of that product). Having thus set the stage for his eventual introduction of the concept of an *ideal*, the project next offers students the opportunity to explore the general algebraic structures of a ring, integral domain and fields. Following this short detour from the historical story — rings themselves were first singled out as a structure separate from ideals only in Emmy Noether's later work—the project returns to Dedekind's exploration of ideals and their basic properties. Starting only with his formal definition of an ideal, project tasks lead students to explore the basic concept of and elementary theorems about ideals (e.g., the difference between ideals and subrings, how properties of subrings and ideals may differ from the properties of the larger ring, properties of ideals in rings with unity). Subsequent project tasks based on excerpts from Dedekind's study of principal ideals and divisibility relationships between ideals conclude with his (very modern!) proofs that the least common multiple and the greatest common divisor of two ideals are also ideals. The project closes by returning to Dedekind's original motivation for developing a theory of ideals, and considers the sense in which ideals serve to recover the essential properties of divisibility — such as the fact that a prime divides a product of two rational integer factors only if it divides one of the factors — for rings like $\mathcal{Z}[\sqrt{-5}]$ that fail to satisfy these properties.

No prior familiarity with ring theory is assumed in the project. Although some familiarity with elementary group theory can be useful in certain portions of the project, it has also been successfully used with students who had not yet studied group theory. For those who have not yet studied group theory (or those who have forgotten it!), basic definitions and results about identities, inverses and subgroups are fully stated when they are first used within the project (with the minor exception of Lagrange's Theorem for Finite Groups which is needed in one project task). The only number theory concepts required should be familiar to students from their K-12 experiences; namely, the definitions (within $\mathcal{Z}$) of *prime*, *composite*, *factor*, *multiple*, *divisor*, *least common multiple*, and *greatest common divisor*.

This project is suitable for use in either a general abstract algebra courses at the introductory level, or as part of a junior or senior level courses in ring theory. Author: Janet Heine Barnett.

F 09. Primes, Divisibility, and Factoring

Questions about primality, divisibility, and the factorization of integers have been part of mathematics since at least the time of Euclid (c. 300 BCE). Today, they comprise a large part of an introductory class in number theory, and they are equally important in contemporary research. In this project, students investigate the development of the modern theory of these three topics by reading a remarkable 1732 paper by Leonhard Euler (1707–1783). This, Euler's first paper in number theory, contains a surprising number of new ideas in the theory of numbers. In a few short pages, he provided for the first time a factorization of $2^{2^5} + 1$ (believed by Fermat to be prime), discussed the factorization of $2^n - 1$ and $2^n + 1$, and began to develop the ideas that would later lead to the first proof of what we now call Fermat's Little Theorem. In this work, Euler provided few proofs. By providing these, students develop an intimacy with the techniques of number theory, and simultaneously come to discover the importance of modern ideas and notation in the field.

This project is intended for courses in number theory. Author: Dominic Klyve.

F 10. The Pell Equation in Indian Mathematics

The Pell Equation is the Diophantine Equation

$$x^2 - Ny^2 = 1 \tag{1}$$

where N is a non-square, positive integer. The equation has infinitely many solutions in positive integers x and y, though finding a solution is not trivial.

In modern mathematics, the method of solving the Pell equation via continued fractions was developed by Lagrange (1736–1813). However, much earlier, Indian mathematicians made significant contributions to the study of the Pell equation and its solution. Brahmagupta (b. 598 CE) discovered that the Pell equation (1) can be solved if a solution to

$$x^2 - Ny^2 = k \tag{2}$$

where $k = -1, 2, -2, 4, -4$ is known. Later a method, a cyclic algorithm known in Sanskrit as *cakravāla*, to solve the Pell equation was developed by Jayadeva (fl. ninth century CE) and Bhāskara II (b. 1114 CE). While the project touches on the Pell equation in modern mathematics, the main focus is on its solution in Sanskrit mathematical texts. This approach will not only familiarize the students with the Pell equation and how it can be solved, but also expose them to significant mathematical work from a nonwestern culture.

This project is intended for a number theory course. Authors: Toke Knudsen and Keith Jones.

F 11. The Greatest Common Divisor: Algorithm and Proof

Finding the greatest common divisor of two integers is a foundational skill in mathematics, needed for tasks from simplifying fractions to cryptography. Yet, the best place to look for a simple algorithm for finding the greatest common divisor is not in a modern textbook, but in the writings of the ancient Chinese and the *Elements* [36] of Euclid (c. 300 BCE) in ancient Greece. In this project, students explore how the mutual subtraction algorithm evolved in ancient China, starting from a text dated c. 200 BCE, to the version of the algorithm in *The Nine Chapters on the Mathematical Art* [59], to the explanation of the *Nine Chapters* algorithm given by Liu Hui (fl. 3rd century BCE). They then explore the algorithm of Euclid and examine his careful proof. Parallel to the story of the development of the algorithm is a beautiful illustration of the history of proof. Proof in ancient China was not based on prepositional logic, but on demonstrating the correctness of an algorithm. Euclid was the pioneer of logical proof, yet his proof has flaws when examined in the light of modern rigor. Therefore, the project finishes by explicitly stating the properties of integers assumed in the proof of Euclid, and justifying the correctness of Euclid's iterative method using the power of inductive proof.

The project is suitable for a introduction to proof class, including junior level courses in algebra, discrete math or number theory. Author: Mary Flagg.

F 12. The Möbius Inversion Formula

It is often easier to find a formula for the divisor sum, $\sum_{d|n} f(d)$, of an arithmetic function, $f(n)$, than it is to directly find a formula for $f(n)$. *Möbius Inversion* can then be used to find a formula for $f(n)$ itself. The first time you see this in action it's as cool as the first time you see Möbius' more well-known, but equally cool, Möbius strip. A typical first application of his inversion formula in a number theory class is to find a formula for Euler's ϕ function, the number of integers between 1 and n relatively prime to n.

But, how and why did Möbius develop this technique and the associated *Möbius function*? In this project, we'll read Möbius' *Über eine besondere Art von Umkehrung der Reihen* [74] from 1832 to see the start of the story. We'll also study the applications Möbius provided. We'll continue by reading from the work of Dedekind, Laguerre, Mertens and Bell [5, 30, 63, 73] to follow the topic's development to its modern presentation.

This project is intended for introductory number theory courses. It could also be used in a discrete math course or a combinatorics course. Author: Carl Lienert

F 13. Bolzano on Continuity and the Intermediate Value Theorem

The foundations of calculus were not yet on firm ground in early 1800's. Students read from 1817 paper [8] by Bernard Bolzano (1781–1848) in which he gave a definition of continuity and formulated a version of the least upper bound property of the real numbers. Students then read Bolzano's proof of the Intermediate Value Theorem.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 14. Rigorous Debates over Debatable Rigor: Monster Functions in Introductory Analysis

Although students in an introductory analysis course will have already encountered the majority of concepts studied in such a course during their earlier calculus experience, the study of analysis requires them to re-examine these concepts through a new set of powerful lenses. Among the new creatures revealed by these lenses are the family of functions defined by $f_\alpha(x) = x^\alpha \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f_\alpha(0) = 0$. In the late nineteenth century, Gaston Darboux (1842–1917) and Giuseppe Peano (1858–1932) each used members of this function family to critique the level of rigor in certain contemporaneous proofs. Reflecting on the introduction of such functions into analysis for this purpose, Henri Poincaré (1854–1912) lamented in [81]: "Logic sometimes begets monsters. The last half-century saw the emergence of a crowd of bizarre functions, which seem to strive to be as different as possible from those honest [honnêtes] functions that serve a purpose. No more continuity, or continuity without differentiability, etc. What's more, from the logical point of view, it is these strange functions which are the most general, [while] those which arise without being looked for appear only as a particular case. They are left with but a small corner. In the old days, when a new function was invented, it was for a practical purpose; nowadays, they are invented for the very purpose of finding fault in our father's reasoning, and nothing more will come out of it." Yet in [11], Émile Borel (1871–1956) proposed two reasons why these "refined subtleties with no practical use" should not be ignored: "[O]n the one hand, until now, no one could draw a clear line between straightforward and bizarre functions; when studying the first, you can never be certain you will not come across the others; thus they need to be known, if only to be able to rule them out. On the other hand, one cannot decide, from the outset, to ignore the wealth of works by outstanding mathematicians; these works have to be studied before they can be criticized."

In this project, students come to know these "monster" functions directly from the writings of the influential French mathematician Darboux and one of the mathematicians whose works he critiqued, Guillaume Houël (1823–1886). Project tasks based on the sources [29, 46] prompt students to refine their intuitions about continuity, differentiability and their relationship, and also includes an optional section that introduces them to the concept of uniform differentiability. The project closes with an examination of Darboux's proof of the theorem that now bears his name: every derivative has the intermediate value property. The project thus fosters students' ability to read and critique proofs in modern analysis, thereby enhancing their understanding of current standards of proof and rigor in mathematics more generally.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: Janet Heine Barnett.

F 15. An Introduction to Algebra and Geometry in the Complex Plane

In this project, students sudy the basic definitions, as well as geometric and algebraic properties, of complex numbers via Wessel's 1797 paper *On the Analytical Representation of Direction. An attempt Applied Chiefly to Solving Plane and Spherical Polygons* [95], the first to develop the geometry of complex numbers.

This project is suitable for a first course in complex variables, or a capstone course for high school math teachers. Authors: Diana White and Nick Scoville.

F 16. Nearness without Distance

Point-set topology is often described as "nearness without distance." Although this phrase is intended to convey some intuitive notion of the study of topology, the student is often left feeling underwhelmed after seeing this idea made precise in the definition of a topology. This project follows the development of topology, starting with a question in analysis, into a theory of nearness of points that took place over several decades. Motivated by a question of uniqueness of a Fourier expansion [17], Cantor (1845–1918) developed a theory of nearness based on the notion of limit points over several papers written over a decade, beginning in 1872 [18, 19, 20, 21, 22, 23]. Borel then took Cantor's ideas and began to apply them to a more general setting. Finally, Hausdorff (1868–1842) developed a coherent theory of topology in his famous 1914 book *Grundzüge der Mengenlehre* [50]. The purpose of this project is to introduce the student to the ways in which we can have nearness of points without a concept of distance by studying these contributions of Cantor, Borel, and Hausdorff.

This project is intended for courses in point-set topology or introductory topology. Author: Nick Scoville.

F 17. Connectedness–Its Evolution and Applications

The need to define the concept of "connected" is first seen in an 1883 work of Cantor (1845–1918) where he gives a rigourous definition of a continuum. After its inception by Cantor, definitions of connectedness were given by Jordan (1838–1922) and Schoenflies (1853–1928), among others, culminating with the current definition proposed by Lennes (1874–1951) in 1905. This led to connectedness being studied for its own sake by Knaster and Kuratowski. In this project, we trace the development of the concept of connectedness through the works of these authors [22, 57, 62, 67, 89], proving many fundamental properties of connectedness along the way.

This project is intended for courses in point-set topology or introductory topology. Author: Nick Scoville.

F 18. Construction of the Figurate Numbers

This project is accessible to a wide audience, requiring only arithmetic and elementary high school algebra as a prerequisite. The project opens by studying the triangular numbers, which enumerate the number of dots in regularly shaped triangles, forming the sequence 1, 3, 6, 10, 15, 21, etc. Student activities include sketching certain of these triangles, counting the dots, and studying how the nth triangular number, T_n, is constructed from the previous triangular number, T_{n-1}. Further exercises focus on tabulating the values of T_n, conjecturing an additive pattern based on the first differences $T_n - T_{n-1}$, and conjecturing a multiplicative pattern based on the quotients T_n/n. The triangular numbers are related to probability by enumerating the number of ways two objects can be chosen from n (given that order does not matter). Other sequences of two-dimensional numbers based on squares, regular pentagons, etc. are studied from the work of Nicomachus (c. 60–120 CE) [76].

The project continues with the development of the pyramidal numbers, P_n, which enumerate the number of dots in regularly shaped pyramids, forming the sequence 1, 4, 10, 20, 35, etc. Student activities again include sketching certain of these pyramids, tabulating the values of P_n, conjecturing an additive pattern based on the first differences $P_n - P_{n-1}$, and conjecturing a multiplicative pattern based on the quotients P_n/T_n. The pyramidal numbers are related to probability by counting the number of ways three objects can be chosen from n. Similar exercises are provided for the four-dimensional (triangulo-triangular) numbers and the five-dimensional (triangulo-pyramidal) numbers. The multiplicative patterns for these figurate numbers are compared to those stated by Pierre de Fermat (1601–1665), such as "The last number multiplied by the triangle of the next larger is three times the collateral pyramid" [69, p. 230f], which, when generalized, hint at a method for computing the n-dimensional figurate numbers similar to an integration formula.

This project is designed for a general education course in mathematics. Author: Jerry Lodder.

F 19. Pascal's Triangle and Mathematical Induction

In this project, students build on their knowledge of the figurate numbers gleaned in the previous project (F 18). The material centered around excerpts from Blasie Pascal's (1623–1662) "Treatise on the Arithmetical Triangle" [78], in which Pascal employs a simple organizational tool by arranging the figurate numbers into the columns of one table. The nth column contains the n-dimensional figurate numbers, beginning the process with $n = 0$. Pascal identifies a simple principle for the construction of the table, based on the additive patterns for the figurate numbers. He then notices many other patters in the table, which he calls consequences of this construction principle. To verify that the patterns continue no matter how far the table is constructed, Pascal states verbally what has become known as mathematical induction. Students read Pascal's actual formulation of this method, discuss its validity, and compare it to other types of reasoning used in the sciences and humanities today. Finally, students are asked to verify Pascal's twelfth consequence, where he identifies a pattern in the quotient of two figurate numbers in the same base of the triangle. This then leads to the modern formula for the combination numbers (binomial coefficients) in terms of factorials.

This project is designed for a general education course in mathematics. Author: Jerry Lodder.

F 20. The French Connection: Borda, Condorcet and the Mathematics of Voting Theory

Voting theory has become a standard topic in the undergraduate mathematics curriculum. Its connection to important issues within a democratic society and the accessibility of its methods make a unit on voting theory especially well-suited for students in liberal studies program, as well as for students at the high school level. The *piéce de resistance* of such a unit is a somewhat startling theorem known as Arrow's Impossibility Theory, named in honor of economist and Nobel Prize laureate Kenneth Arrow (1921–2017) who was the first to state it, in his 1951 doctoral dissertation [4].

In essence, Arrow's Impossibility Theory asserts that there is no fair voting system for elections involving three or more candidates. Unpacking what this means by exploring the relationship between different methods for determining election results (called Methods of Voting) and different notions of fairness (called Fairness Criteria) is the primary objective of the standard undergraduate treatment of voting theory. The study of specific voting methods and their drawbacks actually dates back well before Arrows twentieth-century work. Indeed, Iain McLean has remarked that "the theory of voting has in fact been discovered four times and lost three times" [72, p. 99]. Arrow, of course, was responsible for the fourth discovery. McLean's 1990 article examines the first discovery, made at the hands of two medieval thinkers, Ramon Lull (c. 1235–1315) and Nicolas of Cusa (1401–1464), within the context of ecclesiastical elections. More recently, McLean has written about the third discovery by Charles Dodgson (18321898), the British mathematician more widely known as Lewis Carroll, who was motivated to write on the topic as a result of certain election decisions made by the faculty at Christ Church, Oxford [71]. As those familiar with today's treatment of voting theory will know, however, none of the names of Lull, Cusa or Dodgson/Carroll are generally associated with the topic.

In contrast, the second time that this discovery was made involved two late eighteenth-century French mathematicians for whom certain key ideas of voting theory are now named: Jean Charles, Chevalier de Borda (1733–1799) and Marie-Jean-Antoine-Nicolas de Caritat, Marquis de Condorcet (1743–1794). This project leads students through an exploration of the temporarily "lost" texts that explain the attachment of their names to those ideas:

- Borda's "Memoire sur les lections au scrutin" ("Memoir on elections by ballo"), published in 1784; and
- Condorcet's *Essai sur LApplication de LAnalyse a la Probabilit des Dcisions Rendues à la Pluralit des Voix* (*Essay on the Application of the Analysis of Probabilities to Decisions Rendered by a Plurality of Votes*), published in 1785.

Through their engagement with select excerpts from these two sources ([10, 27]), students are introduced to all of the content contained in a standard textbook treatment of Voting Theory, including the Plurality, Plurality with Elimination, Borda Count and Pairwise Comparison Methods of Voting; the Majority, Condorcet, Independence of Irrelevant Alternatives and Monotonicity Fairness Criteria; and the use of a Preference Schedule as a means to organize voter ballots. By drawing on Condorcets rich discussion of his own motivations for studying the problem of collective decision making, the project also goes beyond a standard textbook treatment in terms of its investigation of why Arrow's Impossibility Theorem, and voting more generally, matters to their own lives. An optional appendix is also provided for instructors who choose to have students read more about the historical context in which Borda and Condorcet lived and worked, perhaps as part of an interdisciplinary unit with colleagues from history or the social sciences.

This project is intended for "Math for the Liberal Arts". It is also suitable for use at the high-school level. Author: Janet Heine Barnett.

F 21. An Introduction to a Rigorous Definition of Derivative

Cauchy (1789–1857) is generally credited with being among the first to define and use the derivative in a near-modern fashion. This project is designed to introduce the derivative with some historical background from Newton (1643–1727), Berkeley (1685–1783) and L'Hôpital (1661-1704). Students then read Cauchy's definition and examples from [25], and explore relevant examples and basic properties.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 22. Investigations into Bolzanos Bounded Set Theorem

Bernard Bolzano (1781–1848) was among the first mathematicians to rigorously analyze the completeness property of the real numbers. This project investigates his formulation of the least upper bound property from his 1817 paper [8]. Students read his proof of a theorem on this property, a proof that inspired Karl Weierstrass (1815–1897) decades later in his proof of what is now known as the Bolzano-Weierstrass Theorem.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 23. The Mean Value Theorem

The Mean Value Theorem has come to be recognized as a fundamental result in a modern theory of the differential calculus. Iin this project, students read from the efforts of Cauchy (1789–1857) in [?] to rigorously prove this theorem for a function with continuous derivative. Later in the project, students explore a very different approach that was developed some forty years after Cauchy's proof, by mathematicians Serret and Bonnet [90].

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 24. Abel and Cauchy on a Rigorous Approach to Infinite Series.

Infinite series were of fundamental importance in the development of the calculus. Questions of rigor and convergence were of secondary importance early on, but things began to change in the early 1800s. When Niels Abel (1802–1829) moved to Paris in 1826, he was aware of certain paradoxes concerning infinite series and wanted big changes. In this project, students read from the 1821 *Cours d'Analyse* [15], in which Cauchy (1789–1857) carefully defined infinite series and proved some properties. Students then read from the paper [1], in which Able attempted to correct a flawed series convergence theorem from Cauchy's book.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 25. The Definite Integrals of Cauchy and Riemann

Rigorous attempts to define the definite integral began in earnest in the early 1800s. One of the pioneers in this development was Augustin-Louis Cauchy (1789–1857). In this project, students read from his 1823 study of the definite integral for continuous functions [?]. They then read from the 1854 paper [87], in which Bernard Riemann (1826–1846) developed a more general concept of the definite integral that could be applied to functions with infinite discontinuities.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

F 26. Gaussian Integers and Dedekind's Creation of an Ideal: A Number Theory Project

In the historical development of mathematics, the nineteenth century was a time of extraordinary change during which the discipline became more abstract, more formal and more rigorous than ever before. Within the subdiscipline of algebra, these tendencies led to a new focus on studying the underlying *structure* of various number (and number-like) systems related to the solution of various equations. The concept of a *group*, for example, was singled out by Évariste Galois (1811–1832) as an important algebraic structure related to the problem of finding all complex solutions of a general polynomial equation. Two other important algebraic structures — *ideals* and *rings* — emerged later in that century from the problem of finding all integer solutions of various equations in number theory. In their efforts to solve these equations, nineteenth century number theorists were led to introduce generalizations of the seemingly simple and quite ancient concept of an integer. This project examines the ideas from algebraic number theory that eventually led to the new algebraic concepts of an 'ideal' and a 'ring' via excerpts from the work of German mathematician Richard Dedekind (1831–1916).

A key feature of Dedekind's approach was the formulation of a new conceptual framework for studying problems that were previously treated algorithmically. Dedekind himself described his interest in solving problems through the introduction of new concepts as follows [34, p. 16]:

> The greatest and most fruitful progress in mathematics and other sciences is through the creation and introduction of new concepts; those to which we are impelled by the frequent recurrence of compound phenomena which are only understood with great difficulty in the older view.

In this project, students encounter Dedekind's creative talents first hand through excerpts from his 1877 *Theory of Algebraic Integers* [33]. The project begins with Dedekind's description of the number theoretic properties of two specific integral domains: the set of rational integers $\mathbf{Z}$, and the set of Guassian integers $\mathbf{Z}[i]$. The basic properties of Gaussian integer divisibility are then introduced, and connections between Gaussian Primes and number theory results such as The Two Squares Theorem are explored. The project next delves deeper into the essential properties of rational primes in $\mathbf{Z}$ — namely, the Prime Divisibility Property and Unique Factorization — to see how these are mirrored by properties of the Guassian Primes in $\mathbb{Z}[i]$. Concluding sections of the project then draw on Dedekind's treatment of indecomposables in the integral domain $\mathbf{Z}[\sqrt{-5}]$, in which Prime Divisibility Property and Unique Factorization both break down, and briefly consider the mathematical after-effects of this 'break down' in Dedekind's creation of an *ideal*.

This project is intended for junior level courses in number theory, and assumes no prior knowledge of abstract algebra. Author: Janet Heine Barnett.

F 27. Otto Hölder's Formal Christening of the Quotient Group Concept

Today's undergraduate students are typically introduced to quotient groups only after meeting the concepts of equivalence, normal subgroups and cosets. Not surprisingly, the historical record reveals a different course of development. Although quotient groups implicitly appeared in work on algebraic solvability done by Galois (1811–1832) in the 1830s, that work itself pre-dated the development of an abstract group concept. Even the 1854 paper by Cayley (1821–1895) which marks the first appearance of a definition of an abstract group was premature, and went essentially ignored by mathematicians for decades. Permutation groups were extensively studied during that time, however, with implicit uses of quotient groups naturally arising as part of those studies. Camille Jordan (1838–1922), for example, used the idea of congruence of group elements modulo a subgroup to produce a quotient group structure [55, 56]. Thus, when Otto Hölder (1859–1937) gave what is now considered to be the first "modern" definition of quotient groups in 1889, he was able to treat the concept as neither new nor difficult [52]. This project examines Hölder's own treatment of the quotient group concept, leading up to a statement of the Fundamental Homomorphism Theorem. The evolution of the concept of abstract quotient groups within the context of earlier work done by Jordan and others who paved the way for Hölder is also treated in optional appendices to the project.

This project is intended is intended for introductory courses in abstract algebra or group theory. Author: Janet Heine Barnett.

F 28. Roots of Early Group Theory in the Works of Lagrange

This project studies works by one of the early precursors of abstract group, French mathematician J. L. Lagrange (1736–1813). An important figure in the development of group theory, Lagrange made the first real advance in the problem of solving polynomial equations by radicals since the work of Cardano (1501–1576) and his sixteeth-century contemporaries. In particular, Lagrange was the first to suggest a relation between permutations and the solution of equations of radicals that was later exploited by the mathematicians Abel (1802–1829) and Galois (1811–1832). Lagrange's description of his search for a general method of algebraically solving all polynomial equations is a model of mathematical research that make him a master well worth reading even today. In addition to the concept of a permutation, the project employs excerpts from Lagrange's work on roots of unity to develop concepts related to finite cyclic groups. Through their guided reading of excerpts from Lagrange, abstract algebra students encounter his original motivations while develop their own understanding of these important group-theoretic concepts.

This project is intended is intended for introductory courses in abstract algebra or group theory. Author: Janet Heine Barnett.

F 29. The Radius of Curvature According to Christiaan Huygens

Curvature is a topic in calculus and physics used today to describe motion (velocity and acceleration) of vector-valued functions. Many modern textbooks introduce curvature via a rather opaque definition, namely the magnitude of the rate of change of the unit tangent vector with respect to arc length. Such a definition offers little insight into what curvature was designed to capture, not to mention its rich historical origins. This project offers Christiaan Huygens's (1629–1695) highly original work on the radius of curvature and its use in the construction of an isochronous pendulum clock. A perfect time-keeper, if one could be constructed to operate at sea, would solve the longitude problem for naval navigation during the Age of Exploration.

Amazingly, Huygens identified the path of the isochrone as a cycloid, a curve that had been studied intensely and independently during the seventeenth century. To force a pendulum bob to swing along a cycloidal path, Huygens constrained the thread of the pendulum with metal or wooden plates. He dubbed the curve for the plates an evolute of the cycloid and described the evolutes of curves more general than cycloids. Given a curve and a point B on this curve, consider the circle that best matches the curve at B. Suppose that this circle has center A. Segment AB became known as the radius of curvature of the original curve at B, and the collection of all centers A as B varies over the curve form the evolute. Note that the radius of curvature AB is perpendicular to the original curve at B. For an object moving along this curve, AB helps in the identification of the perpendicular component of the force necessary to cause the object to traverse the curve. This is the key insight into the meaning of curvature.

This project is intended is intended for courses in multivariable or vector calculus. Author: Jerry Lodder.

F 30. A Proof and Application of Cotes's Theorem

The goal of this project is to develop and prove a theorem due to English mathematician Robert Cotes (1682–1716). Because no proof of Cotes's Theorem from the pen of Cotes himself is known today, the project instead follows the paper [95] by Caspar Wessel (1745-1818), a Danish surveyor by trade who made significant contributions to mathematics.

This project is suitable for a first course in complex variables, or a capstone course for high school math teachers. Authors: Diana White and Nick Scoville.

F 31. Cross Cultural Comparisons: The Art of Computing the Greatest Common Divisor

Finding the greatest common divisor between two or more numbers is fundamental to basic number theory. There are three algorithms taught to pre-service elementary teachers: finding the largest element in the intersection of the sets of factors of each number, using prime factorization and the Euclidean algorithm. This project has students investigate a fourth method found in *The Nine Chapters on the Mathematical Art* [59], an important text in the history of Chinese mathematics that dates from before 100 CE. This project asks students to read the translated original text instructions for finding the gcd of two numbers using repeated subtraction. Then students are asked to compare this method with the other modern methods taught. Students are led to discover that the Chinese method is equivalent to the Euclidean algorithm.

The project is well-suited to a basic algebra course for pre-service elementary and middle school teachers. Author: Mary Flagg.

F 32. A Look at Desargues' Theorem from Dual Perspectives

Girard Desargues (1591–1661) is often cited as one of the founders of Projective Geometry. Desargues was, at least in part, motivated by perspective drawing and other practical applications. However, this project focuses on Desargues' Theorem from a mathematical point of view. The theorem that today goes by his name is central to modern Projective Geometry. This porject, in fact, starts with a modern statement of Desargues' Theorem in order to more quickly appreciate the elegant beauty of the statement. Desargues' own proof of the theorem is, perhaps ironically, buried at the end of the treatise [12], which was written by his student Abraham Bosse (164–1676). The primary focus of this project is to understand Desargues' proof of the theorem from a classical perspective. To achieve this goal we read the proof given by Bosse, which requires a visit to other results of Desargues in his more famous work on conics [93], to classical results of Euclid (c. 300 BCE) from the *Elements* [36], and to a result of Menelaus (c. 100 CE) which we find both in Desargues' own colorful writings [93] and in those of Ptolemy (c. 100 CE) [94]. The project concludes with a view of Desargues' Theorem from a modern perspective. We also use the work of Jean Victor Poncelet (1788–1867) to reexamine Desargues' Theorem with the assumption that parallel lines meet at a point at infinity and with the principle of duality [82].

The development of the project is intended to both convey the geometrical content and help students learn to *do* math. It is meant to be accessible to students at the "Introduction to Proofs" level. Many of the exercises explicitly go through a read-understand-experiment-prove cycle. Some experience proving theorems in the spirit of Euclid would be helpful, but not absolutely necessary. A few optional exercises (whose answers could easily be found in a modern text) are left more open.

This project is designed for students in a Modern Geometry course or an Introduction to Proofs course. Author: Carl Lienert.

F 33. Solving Equations and Completing the Square: From the Roots of Algebra

This project seeks to provide a deep understanding of the standard algebraic method of completing the square, the universal procedure for solving quadratic equations, through the reading of selections from *The Compendious Book on Calculation by Restoration and Reduction* [2, 85], written in the ninth century in Baghdad by Muḥammad ibn Mūsā al-Khwārizmī (c. 780–850 CE), better known today simply as al-Khwārizmī. At the same time, students become acquainted with a sense of how algebraic problem solving was successfully carried out in its earliest days even in the absence of symbolic notation, thereby conveying the importance of modern symbolic practices.

Future high school mathematics teachers who will be responsible for teaching algebra courses in their own classrooms will be well-served by working through this classroom module. It is also suitable for use in a general history of mathematics course, and is of value to instructors of higher algebra courses who are interested in conveying a sense of the early history of the theory of equations to their students. Author: Danny Otero.

F 34. Argand's Development of the Complex Plane

Complex numbers are a puzzling concept for today's student of mathematics. This is not entirely surprising, as complex numbers were not immediately embraced by mathematicians either. Complex numbers showed up somewhat sporadically in works such as those of Cardano (1501–1576), Tartaglia (1499–1557), Bombelli (1526–1572), and Wallis (1616–1703), but a systematic treatment of complex numbers was given in an essay titled *Imaginary Quantities: Their Geometrical Interpretation* [3], written by Swiss mathematician Jean-Robert Argand (1768–1822). This project studies the basic definitions, as well as geometric and algebraic properties, of complex numbers via Argand's essay.

This project is suitable for a first course in complex variables, or a capstone course for high school math teachers. Authors: Diana White and Nick Scoville.

F 35. Riemann's Development of the Cauchy-Riemann Equations

This project examines the Cauchy-Riemann equations (CRE) and some consequences from Riemann's perspective, using excerpts from his 1851 Inauguraldissertation. Students work through Riemann's argument that satisfying the CRE is equivalent to the differentiability of a complex function $w = u(x, y) + iv(x, y)$ of a complex variable $z = x + iy$. Riemann also introduces Laplace's equation for the u and v components of w, from which students explore some basic ideas on harmonic functions. Riemann's approach with differentials creates some challenges for modern readers, but works nicely at an intuitive level and motivates the standard modern proof that the CRE follow from differentiability. In the final section of the project, students are introduced to the modern definition of derivative and revisit the CRE in this context.

This project is suitable for a first course in complex variables. Author: Dave Ruch.

F 36. Gauss and Cauchy on Complex Integration

This project begins with an short excerpt from Gauss on the meaning of definite complex integrals and a claim about their path independence. Students then work through Cauchy's detailed development of a definite complex integral, culminating in his parameterized version allowing for evaluation of these integrals. Students then apply Cauchy's parametric form to illustrate Gauss's ideas on path independence for certain complex integrals.

This project is suitable for a first course in complex variables. Author: Dave Ruch.

F 37. Representing and Interpreting Data from Playfair

With the proliferation of data in all aspects of our lives, understanding how to present and interpret visual representations is an essential skill for students to develop. Using the seminal work of William Playfair in his *Statistical Breviary* [80], this project introduces students to the bar graph, pie chart, and time series graphs, asking them to interpret real data from the late 1700s and early 1800s. Compound bar graphs, compound time series, and visual depictions incorporating both bar graphs and time series graphs are also included.

This project is intended for use in an introductory statistics or data science course at the undergraduate level. However, it could also be used in courses for pre-service teachers, mathematics for liberal arts courses, professional development courses/workshops for teachers, or in history of mathematics courses. It is also potentially suitable for use at the high-school level. Authors: Diana White, River Bond, Joshua Eastes, and Negar Janani.

F 38. Runge-Kutta 4 (and Other Numerical Methods for ODEs)

Just as there are numerical methods for integration (e.g., left-hand rule, trapezoidal rule, Simpsons method), we also have numerical methods that allow us to calculate $y(x_1)$ for the initial value problem

$$\frac{dy}{dx} = f(x, y) \qquad\qquad y(x_0) = y_0.$$

While the simplest of these numerical methods is due to Euler in 1768, it wasn't until 1901 that Wilhelm Kutta placed Euler's method, along with several other numerical methods, into a unifying context. This PSP describes Kutta's method, carrying out the calculations up to order 3 approximations. We derive Euler's method, the Improved Euler method and several other numerical methods, something that is rarely done in a standard ODE text. And while the Runge-Kutta 4th order approximations (RK4) may hold a special place in today's textbooks, the actual appearance of the RK4 method is simply one of five examples that Kutta gave for order 4 approximations.

This project is intended for a course in differential equations. Author: Adam Parker.

F 39. Stitching Dedekind Cuts to Construct the Real Numbers

As a fledgling professor and mathematician, Richard Dedekind (1831–1916) was unsatisfied with the lack of foundational rigor with which differential calculus was taught, and in particular, with the way the set of real numbers and its properties were developed and used to prove the most fundamental theorems of calculus. His efforts to rectify this situation resulted in his 1872 monograph *Continuity and Irrational Numbers* [31], which was later published (in 1901) in a longer compilation entitled *Essays on the Theory of Numbers* [32]. This project guides the students through the development of the real numbers through the examination of Dedekind's own words in translation. The real numbers are formed through Dedekind cuts, which are pairs of subsets of the set of rational numbers that represent a real number. The properties of the real numbers emerge out of corresponding properties of the rationals. The project tasks ask the students to interpret, scrutinize and reflect on the source text. They also challenge them to fill in details that Dedekind had decided to leave out.

This project is intended for courses in introductory real analysis or introduction to proofs. Author: Michael P. Saclolo

F 40. The Fermat-Torricelli Point of a Triangle and Cauchy's Gradient Descent Method

The Fermat-Torricelli point of a triangle is the point that achieves the minimum possible sum of distances to the three vertices of the triangle. The problem of finding this point was posed by Pierre de Fermat (1607–1665) and then solved by Evangelista Torricelli (1608–1647) using very geometric techniques. Today, one can apply the standard optimization techniques of multivariable calculus to achieve the same result. This problem is incredibly historically significant, as it served as somewhat of a base case for operations research — imagine, for example, a shipping company trying to place a warehouse in a way that minimizes the sum of distances to delivery sites. For larger instances of this problem, finding an exact solution is extremely difficult, and researchers instead often rely on an iterative approximation technique like the gradient descent technique proposed by Augustin-Louis Cauchy (1789–1857). This project walks the student through two solutions to the Fermat-Torricelli problem (one via geometry and one via multivariable calculus), as well as Cauchy's gradient descent method.

This project is intended for use in multivariable calculus courses. Author: Kenneth M Monks.

F 41. Stained Glass and Windmills: An Exploration of Green's Theorems

In his relatively short life, George Green (1793–1841) accomplished many things. He was the first to create a mathematical theory of electricity and magnetism. His work paved the way for developments by James Clerk Maxwell (1831–1879) and William Thomson (1824–1907), better known as Lord Kelvin. His ideas about light waves anticipated quantum mechanics. And he is memorialized in Westminster Abbey alongside Isaac Newton. Green accomplished all this despite being largely self-taught. The one thing Green did not do was write the theorem that now bears his name! In this project, students develop a thorough understanding of that theorem by working through the primary sources [48, 88, 26]:

- *An essay on the application of mathematical analysis to the theories of electricity and magnetism*, written in 1828 by George Green;
- "Sur les intégrales qui s'étendent à tous les points d'une courbe fermée", written in 1846 by Augustin-Louis Cauchy (1789–1857); and
- "Foundations for a general theory of functions of a complex variable," written in 1851 by Bernhard Riemann (1826–1866).

Drawing on ideas contained in all three sources, students prove Green's theorem and consider several applications. Along the way, they solidify their understanding of partial derivatives, multiple integrals, line integrals, vector fields, and more.

This project is intended for use in multivarible calculus courses. Author: Abe Edwards.

F 42. Finding Exact Sums of Infinite Series

Students typically encounter the theory of infinite series in their second semester course in calculus, in which the focus is the determination of the *convergence* of series. They generally conclude the course realizing that very few of the infinite series which they find are convergent are easy to determine, save geometric series and telescoping series. This project is designed to provide students an immersive experience in determining exact sums of a number of infinite series by following the work of Jakob Bernoulli (1655–1705) in his *Tractatus de Seriebus Infinitis* [6]. In this work, Bernoulli found exact sums of series of the forms

$$\sum_{n=1}^{\infty} \frac{c}{bd^{n-1}} \binom{n+k-1}{k} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{c}{bd^{n-1}} n^k,$$

where $b \neq 0$ and c are arbitrary, $|d| > 1$ and $k = 1, 2$ or 3. The project guides the student through Bernoulli's technique of finding these sums by splitting the given series into an infinite number of convergent geometric series, the sum of which in turn produces a series that can be summed exactly using a result from an earlier paragraph of the treatise. Optional sections then invite the student to use a more modern approach to explore cases of these series corresponding to integer values of $k > 3$. This project thus provides a wealth of examples of convergent series for which the sums *can* be found exactly.

This project is intended for use in second-semester calculus courses. Authors: Danny Otero and James Sellers

F 43. Sum of Four Squares

Which numbers (non-negative integers) can be written as a sum of two squares? three squares? With a little experimentation, you'll discover the answer is, at any rate, not all. The French mathematician Claude Gaspar Bachet (1581–1638) conjectured, on the other hand, that *all* numbers can be written as a sum of four squares, and it seems likely that others before him would have suspected the same. He didn't have a proof, but he was in good company; others including Descartes (1596–1650), Fermat (1601–1665), and Euler (1707–1783) worked on the problem but also were unable to provide a proof. The first proof that all numbers can be written as a sum of four squares was given by Lagrange (1736–1813) who used a *method of descent*. In this project we'll study a different proof, one given by Carl Gustav Jacob Jacobi (1804–1851) in his 1829 *Fundamenta Nova Theoriae Functionum Ellipticarum* [54]. Jacobi's proof is remarkable for at least two reasons. First, his proof also provided a formula which determines the number of different ways a given number can be written as a sum of four squares. Second, Jacobi introduced *theta series* in *Fundamenta nova*, a tool that has become important in many areas of contemporary mathematics including the study of modular forms and of elliptic curves. In the full (anticipated 6 days) version of this project we'll spend a little time studying the topic that motivated Jacobi: *elliptic integrals* and *elliptic functions* and then see how two different series expansions for a particular elliptic integral are used in Jacobi's proof of the four squares theorem. In the short (anticipated 2 days) version of the project we'll use the series expansions as the starting point.

This project is primarily intended for an introductory number theory course but could be used by any students who have seen infinite series, say, in a second semester Calculus course. With a little modification in the more proof oriented tasks it could be used in a Calculus II course as an application of infinite series. Author: Lienert

F 44. Deciphering the Calculations on Some Old Babylonian Tablets

This project explores the Old Babylonian numeration system and geometric calculations through three ancient tablets dating to around 1700 BCE. Students read directly from copies of these tablets, which are written in cuneiform script. The first tablet is a "multiplication table" that introduces the Babylonians' base-sixty (sexagesimal) number system; this text deals primarily with whole numbers, but a catch-line at the end invites exploration of numbers with fractional parts. The second tablet contains the Old Babylonian calculation of the area of a circle, which is based on the circumference rather than the radius and uses three as an approximation for pi. This tablet builds further understanding of sexagesimal fractions. The third tablet calculates the length of a square's diagonal, using an extremely precise approximation for the square root of two. Students must now work with sexagesimal numbers that have multiple "sexagesimal places," including 60ths, 3600ths and 216000ths. Encountering the square root of two in this very early geometric context raises questions about who deserves credit for the "Pythagorean Theorem." Taken as a whole, this set of tablets invites pre-service teachers to consider familiar mathematical concepts in an unfamiliar setting, strengthening their understanding of place-value notation in particular.

This project is intended for use in courses for pre-service elementary teachers. It is also suitable for use in history of mathematics courses. Author: Zoë Misiewicz

Mini-Primary Source Project Descriptions

M 01. Babylonian numeration

Rather than being taught a different system of numeration, students in this project discover one for themselves. Students are given an accuracy recreation of a cuneiform tablet from Nippur with no initial introduction to Babylonian numerals. Unknown to the students, the table contains some simple mathematics – a list of the first 13 integers and their squares. Their challenge is threefold: to deduce how the numerals represent values, to work out the mathematics on the tablet, and to decide how to write the number "seventy two" using Babylonian numerals.
The Notes to Instructors for the project also suggests the small optional extension of asking students to compare the good and bad traits of several numeration systems.

This project is intended for "Math for the Liberal Arts" and Elementary Education courses. Author: Dominic Klyve.

M 02. L'Hópital's Rule

Students of the calculus learn quickly that this grand collection of theoretical ideas and problem solving tools that center on the concepts of derivative and integral ultimately find their justification in the careful computation of limits. And while many of the limits students encounter are trivially determined as applications of the continuity of the underlying functions involved (wherein

$\lim_{x\to a} f(x) = f(a)$), quite a few are not. "Indeterminate forms" are identified as the chief obstacle to the evaluation of such limits, and L'Hôpital's Rule is the standard remedy for resolving these forms. This project introduces students to this important Rule, as it appeared in the first book to expose the entirety of the "new" calculus, *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (*Analysis of the Infinitely Small for the Understanding of Curved Lines*) [14], published in 1696 by the French nobleman Guillaume François Antoine, Marquis de l'Hôpital, based on notes he took from private lessons given him by Jakob Bernoulli. Students also see a justification of the Rule, a few of its major variants, and some applications.

This project is intended for first-year courses in calculus. Author: Danny Otero.

M 03. The Derivatives of the Sine and Cosine Functions

Working through the standard presentation of computing the derivative of $\sin(x)$ is a difficult task for a first-year mathematics student. Often, explaining "why" cosine is the derivative of sine is done via ad-hoc handwaving and pictures. Using an older definition of the derivative, Leonhard Euler (1707–1783) gave a very interesting and accessible presentation of finding the derivative of $\sin(x)$ in his *Institutiones Calculi Differentialis* [39]. The entire process can be mastered quite easily in a day's class, and leads to a deeper understanding of the nature of the derivative and of the sine function.

This project is intended for a Calculus 1 course. Author: Dominic Klyve.

M 04. Beyond Riemann Sums

The purpose of this project is to introduce the method of integration developed by Fermat (1601–1665), in which he essentially used Riemann sums, but allowed the width of the rectangles to vary. Students work through Fermat's text [41], with the goal of better understanding the method of approximating areas with rectangles.

This project is intended for a Calculus 1 course. Author: Dominic Klyve.

M 05. Fermat's Method for Finding Maxima and Minima

In his 1636 article "Method for the Study of Maxima and Minima" [40], Pierre de Fermat (1601–1665) proposed his method of *adequality* for optimization. In this work, he provided a rather cryptic sounding paragraph of instructions regarding how to find maxima and minima. Afterwards, he claimed that "It is impossible to give a more general method." Here, we trace through his instructions and see how it ends up being mostly equivalent to the standard modern textbook approach of taking a derivative and setting it equal to zero.

This project is intended for a Calculus 1 course. Author: Kenneth M Monks.

M 06. Euler's Calculation of the Sum of the Reciprocals of the Squares

This project introduces students to p-series via a proof of the divergence of the harmonic series in *Quaestiones super Geometriam Euclidis* [77], written by Nicole Oresme (c. 1325–1382) in approximately 1350. It continues with the proof via an infinite product formula for $\sin(s)/s$ that was given by Leonhard Euler (1707–1783) in his 1740 "De summis serierum reciprocarum" [37], showing that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

This project is intended for a Calculus 2 course. Author: Kenneth M Monks.

M 07. Braess' Paradox in City Planning: An Application of Multivariable Optimization

On December 5, 1990, The New York Times published an article titled *What if They Closed 42nd Street and Nobody Noticed?* Two of the early paragraphs in this article summarize what happened:

> "On Earth Day this year, New York City's Transportation Commissioner decided to close 42nd Street, which as every New Yorker knows is always congested. 'Many predicted it would be doomsday,' said the Commissioner, Lucius J. Riccio. 'You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem."

But to everyone's surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed."

This very counterintuitive phenomenon, in which the removal of an edge in a congested network actually results in improved flow, is known as Braess' Paradox. This paradox had actually been studied decades earlier not by rocket scientists, but by mathematicians. In the 1968 paper "On a paradox of traffic planning" [16], Dietrich Braess (1938–) described a framework for detecting this paradox in a network. In this project, we see how the examples he provided can be analyzed using standard optimization techniques from a multivariable calculus course.

This project is suitable for a course in multivariable calculus, as well as a course in combinatorial optimization/network flows. Author: Kenneth M Monks.

M 08: The Origin of the Prime Number Theorem

Near the end of the eighteenth century, Adrien-Marie Legendre (1752–1833) and Carl Friederich Gauss (1777–1855) seemingly independently began a study of the primes—more specifically, of what we now call their *density*. It would seem fairly clear to anyone who considered the matter that prime numbers are more rare among larger values than among smaller ones, but describing this difference mathematically seems not to have occurred to anyone earlier. Indeed, there's arguably no *a priori* reason to assume that there is a nice function that describes the density of primes at all. Yet both Gauss and Legendre managed to provide exactly that: a nice function for estimating the density of primes. Gauss claimed merely to have looked at the data and seen the pattern (His complete statement reads "I soon recognized that behind all of its fluctuations, this frequency is on the average inversely proportional to the logarithm.") Legendre gave even less indication of the origin of his estimate. In this project, students explore how they may have arrived at their conjectures, compare their similar (though not identical) estimates for the number of primes up to x, and examine some of the ideas related to different formulations of the Prime Number Theorem. Using a letter written by Gauss, they then examine the error in their respective estimates.

This project is intended for courses in number theory. Author: Dominic Klyve.

M 09–10. How to Calculate π

Most students have no idea how they might, even in theory, calculate π. Demonstrating ways that it can be calculated is fun, and provides a useful demonstration of how the mathematics they are learning can be applied. This set of mini-projects, either of which can be completed in one class period, leads students through different ways to calculate π. For a capstone or honors course, an instructor may choose to have students study both methods, and then compare their efficiency. The sources on which the projects are based include [64, 70].

The intended course for each mini-project is indicated below. Author of both mini-PSPs: Dominic Klyve.

M 09. **How to Calculate π: Machin's Inverse Tangents** In This project, students rediscover the work of John Machin (1681–1751) and Leonhard Euler (1707–1783), who used a tangent identity to calculate π by hand to almost 100 digits.

M 10. **How to Calculate π: Buffon's Needle** This project explores the clever experimental method for calculating π by throwing a needle on a floor on which several parallel lines have been drawn developed by Georges-Louis Leclerc, Comte de Buffon (1707–1788). It is available in two versions, as described below. Basic notions of geometric probability are introduced in both versions of the project.

M 10.1 **How to Calculate π - Buffon's Needle (Non-Calculus Version)** This version requires some basic trigonometry, but uses no calculus. It is suitable for use with students who have completed a course in pre-calculus or trigonometry.

M 10.2 **How to Calculate π - Buffon's Needle (Calculus Version)** This calculus-based version requires the ability to perform integration by parts. It is suitable for use in Calculus 2, capstone courses for secondary teachers and history of mathematics.

M 11. Bhāskara's Approximation and Mādhava's Infinite Series for Sine

The idea of approximating a transcendental function by an algebraic one is most commonly taught to today's calculus students via the machinery of power series. However, that idea goes back much much further! In this project, we visit 7th century India, where Bhāskara I (c. 600–c. 680) gave an incredibly accurate approximation to sine using a rational function in his work *Māhabhāskarīya* (*Great Book of Bhāskara*) [60]. Though there is no surviving account of how exactly he came up with the formula, we guide the student through one plausible approach.

The more familiar power series formula for the sine function has been attributed to Mādhava of Sangamagrama (c. 1350–c. 1425). Though there are no surviving writings from Mādhava's own hand, the Kerala school astronomer Keḻallur Nilakantha Somayaji (1444–1544) published Mādhava's sine series in the *Tantrasamgraha* in 1501 [84]. The student will translate Mādhava's formula, as stated in words by Nilakantha Somayaji, into more modern notation to construct the power series for sine. The project concludes by asking the student to apply Taylor's Error Theorem to compare the accuracy of various formulas for sine. First, the student compares the error in Bhāskara's and Mādhava's formulas. Second, the student is asked to construct a sine power series centered at $\pi/2$ for comparison with Bhāskara's approximation.

This project is intended for a Calculus 2 course. Author: Kenneth M Monks.

M 12. Fourier's Proof of the Irrationality of e

Few topics are as central to the ideas of the calculus sequence as the infinite geometric series formula, the power series for e^x, and arguing via comparison (direct or limit). Joseph Fourier's (1768–1830) short and beautiful proof that e is irrational combines exactly those three ideas! This project walks the student through the first written account of this argument, which appeared in *Mélanges d'analyse algébrique et de géométrie* by Janot de Stainville (1783 – 1828) [92].

The key idea in Fourier's proof was later leveraged by Joseph Liouville (1809–1882) in *Sur l'Irrationnalite du nombre* $e = 2.718\ldots$ [68] to show that e^2 is irrational as well. The project uses excerpts from Liouville's work to point students towards the contrasting behavior of $\sqrt{2}$ (which becomes rational upon squaring) versus e (which does not), as a stepping stone towards the idea a transcendental number.

This project is intended for a Calculus 2 course. Author: Kenneth M Monks.

M 13–15. Gaussian Guesswork

Just prior to his nineteenth birthday, the mathematical genius Carl Friederich Gauss (1777–1855) began a "mathematical diary" in which he recorded his mathematical discoveries for nearly 20 years. Among these discoveries was the existence of a beautiful relationship relationship between three particular numbers: the ratio of the circumference of a circle to its diameter (π), a specific value (ϖ) of the elliptic integral $u = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$; and the Arithmetic-Geometric Mean of 1 and $\sqrt{2}$. Like many of his discoveries, Gauss uncovered this particular relationship through a combination of the use of analogy and the examination of computational data, a practice referred to as "Gaussian Guesswork" by historian Adrian Rice in his *Math Horizons* article "Gaussian Guesswork, or why 1.19814023473559220744... is such a beautiful number" [86].

This set of three mini-projects, based on excerpts from Gauss' mathematical diary [45] and related texts, introduces students to the power of numerical experimentation via the story of his discovery of this beautiful relationship, while also serving to consolidate student proficiency of the following traditional topics from a Calculus 2 course:

- **M 13: Gaussian Guesswork: Elliptic Integrals and Integration by Substitution**
- **M 14: Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve**
- **M 15: Infinite Sequences and the Arithmetic-Geometric Mean**

Each of the three mini-PSPs can be used either alone or in conjunction with any of the other three.

All three of these mini-PSPs are intended for Calculus 2. Author: Janet Heine Barnett.

M 16. The Logarithm of -1

Understanding the behavior of multiple-valued functions can be a difficult mental hurdle to overcome in the early study of complex analysis. Many eighteenth-century mathematicians also found this difficult. This one-day project looks at excerpts from letters (taken from [13]) in the correspondence between Euler (1707–1783) and Jean Le Rond d'Alembert in which they argued about the value of $\log(-1)$. This argument between Euler and d'Alembert not only set the stage for the rise of complex analysis, but helped to end a longstanding friendship.

This project is intended for a course in complex variables. Author: Dominic Klyve.

M 17. Why be so Critical? Nineteenth Century Mathematical and the Origins of Analysis

The seventeenth century witnessed the development of calculus as the study of curves in the hands of Newton and Leibniz, with Euler (1707–1783) transforming the subject into the study of analytic functions in the eighteenth century. Soon thereafter, mathematicians began to express concerns about the relation of calculus (analysis) to geometry, as well as the status of calculus (analysis) more generally. The language, techniques and theorems that developed as the result of the critical perspective adopted in response to these concerns are precisely those which students encounter in an introductory analysis course — but without the context that motivated nineteenth-century mathematicians. This project employs excerpts from the texts [1, 7, **?**, 31], written by Abel (1802–1829), Bolzano (1781–1848), Cauchy (1789–1857) and Dedekind (1831–1916) respectively, as a means to introduce students to that larger context in order to motivate and support development of the more rigorous and critical view required of students for success in an analysis course.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: Janet Heine Barnett.

M 18. Topology from Analysis: Making the Connection

Topology is often described as having no notion of distance, but a notion of nearness. How can such a thing be possible? Isn't this just a distinction without a difference? In this project, students discover the notion of "nearness without distance" by studying the work of Georg Cantor [17] on a problem involving Fourier series. In this work, they see that it is the relationship of points to each other, and not their distances per se, that is essential. In this way, students are led to see the roots of topology organically springing from analysis.

This project is intended for a course in point-set topology. It is also suitable for use in a a course in Introductory Analysis. Author: Nick Scoville.

M 19. Connecting Connectedness

Connectedness has become a fundamental concept in modern topology. The concept seems clear enough-a space is connected if it is a "single piece." Yet the definition of connectedness we use today was not what was originally written down. In fact, today's definition of connectedness is a classic example of a definition that took decades to evolve. The first definition of this concept was given by Georg Cantor in an 1872 paper [17]. After investigating his definition, the project traces the evolution of the definition of connectedness through works of Jordan [58] and Schoenflies [89], culminating with the modern definition given by Lennes [67].

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 20. The Cantor Set before Cantor

A special construction used in both analysis and topology today is known as the Cantor set. Cantor used this set in a paper in the 1880s. Yet a variation of this set appeared as early as 1875, in the paper *On the Integration of Discontinuous Functions* [91] by the Irish mathematician Henry John Stephen Smith (1826–1883). Smith, who is best known for the Smith-normal form of a matrix, was a professor at Oxford who made great contributions in matrix theory and number theory. This project explores the concept of nowhere dense sets in general, and the Cantor set in particular, through his 1875 paper.

This project is intended for a course in point-set topology. It is also suitable for use in a a course in Introductory Analysis. Author: Nick Scoville.

M 21. A Compact Introduction to a Generalized Extreme Value Theorem

In a short paper published just one year prior to his thesis, Maurice Frechet (1878–1973) gave a simple generalization of what we today call the Extreme Value Theorem: continuous real-valued funtions attain a maximum and a minimum on a closed bounded interval. Developing this generalization was a simple matter of coming up with "the right" definitions in order to make things work. In This project, students work through Frechet's entire 1.5-page long paper [43] to give an extreme value theorem for a more general topological spaces: those which, to use Frechet's newly-coined term, are compact.

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 22. From Sets to Metric Spaces to Topological Spaces

One of the significant contributions that Hausdorff made in his 1914 book *Grundzüge der Mengenlehre* (*Fundamentals of Set Theory*) [50] was to clearly lay out for the reader the differences and similarities between sets, metric spaces, and topological spaces. It is easily seen how metric and topological spaces are built upon sets as a foundation, while also clearly seeing what is "added" to sets in order to obtain metric and topological spaces. In this project, we follow Hausdorff as he builds topology "from the ground up" with sets as his starting point.

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 23. The Closure Operation as the Foundation of Topology.

The axioms for a topology are well established- closure under unions of open sets, closure under finite intersections of open sets, and the entire space and empty set are open. However, in the early twentieth century, multiple systems were being proposed as equivalent options for a topology. Once such system was based on the closure property, and it was the subject of Polish mathematician K. Kuratowski's doctoral thesis. In this mini-project, students work through a proof that today's axioms for a topology are equivalent to Kuratowski's closure axioms by studying excerpts from both Kuratowski and Hausdorff.

This project is intended for a course in point-set topology. Author: Nick Scoville.

M 24. Euler's Rediscovery of e.

The famous constant e appears periodically in the history of mathematics. In this mini-project, students read Euler (1707–1783) on e and logarithms from his 1748 book *Introductio in Analysin Infinitorum* [38], and use Euler's ideas to justify the modern definition: $e = \lim_{j\to\infty} (1 + 1/j)^j$.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: David Ruch.

M 25. Henri Lebesgue and the Development of the Integral Concept

The primary goal of this project is to consolidate students' understanding of the Riemann integral, and its relative strengths and weaknesses. This is accomplished by contrasting the Riemann integral with the Lebesgue integral, as described by Lebesgue himself in a relatively non-technical 1926 paper [65]. A second mathematical goal of this project is to introduce the important concept of the Lebesgue integral, which is rarely discussed in an undergraduate course on analysis. Additionally, by offering an overview of the evolution of the integral concept, students are exposed to the ways in which mathematicians hone various tools of their trade (e.g., definitions, theorems). In light of the project's goals, it is assumed that students have studied the rigorous definition of the Riemann integral as it is presented in an undergraduate textbook on analysis. Familiarity with the Dirichlet function is also useful for two project tasks. These tasks also refer to pointwise convergence of function sequences, but no prior familiarity with function sequences is required.

This project is intended for introductory courses in analysis (i.e., advanced calculus). Author: Janet Heine Barnett.

M 26. Generating Pythagorean Triples via Gnomons

This project is designed to provide students an opportunity to explore the number-theoretic concept of a Pythagorean triple. Using excerpts from Proclus' *Commentary on Euclid's Elements* [83], it focuses on developing an understanding of two now-standard formulas for such triples, commonly referred to as 'Plato's method' and 'Pythagoras' method' respectively. The project further explores how those formulas may be developed/proved via figurate number diagrams involving gnomons. It is available in two versions, as described below.

- **M 26.1 Generating Pythagorean Triples via Gnomons: The Methods of Pythagoras and of Plato via Gnomons**

  In this less open-ended version, students begin by completing tasks based on Proclus' verbal descriptions of the two methods, and are presented with the task of connecting the method in question to gnomons in a figurate number diagram only after assimilating its verbal formulation. *This version of the project may be more suitable for use in lower division mathematics courses for non-majors or prospective elementary teachers.*

- **M 26.2 Generating Pythagorean Triples via Gnomons: A Gnomonic Exploration**

  In this more open-ended version, students begin with the task of using gnomons in a figurate number diagram to first come up with procedures for generating new Pythagorean triples themselves, and are presented with Proclus' verbal description of each method only after completing the associated exploratory tasks. *This version of the project may be more suitable for use in upper division courses in number theory and discrete mathematics, or in capstone courses for prospective secondary teachers.*

Although more advanced students will naturally find the algebraic simplifications involved in certain tasks to be more straightforward, the only mathematical content pre-requisites are required in either version is some basic arithmetic and (high school level) algebraic skills. The major distinction between the two versions of this project is instead the degree of general mathematical maturity expected. Both versions include an open-ended "comparisons and conjectures" penultimate section that could be omitted (or expanded upon) depending on the instructor's goals for the course.

Author: Janet Heine Barnett.

M 27. Seeing and Understanding Data

Modern data-driven decision-making includes the ubiquitous use of visualizations, mainly in the form of graphs or charts. This project explores the parallel development of thinking about data visually and the technological means for sharing data through pictures rather than words, tables, or lists. Students are provided the opportunity to consider both the data and the construction methods along with impact that broadening access to data has had on social concerns. Beginning with a tenth-century graph that was hand-drawn in a manuscript, students experience data displays printed with woodcuts and plates through those generated by digital typesetting and dynamic online or video-recorded presentations of data. Early uses of bar charts, pie charts, histograms, line charts, boxplots, and stem-and-leaf plots are compared with modern thoughts on graphical excellence.

This project is intended for courses in statistics, and is also well-suited to use in courses for general education and elementary education audiences that treat graphical displays of data. Authors: Beverly Wood and Charlotte Bolch.

M 28. Completing the Square: From the Roots of Algebra

This project seeks to provide a deep understanding of the standard algebraic method of completing the square, the universal procedure for solving quadratic equations, through the reading of selections from *The Compendious Book on Calculation by Restoration and Reduction* [2, 85], written in the ninth century in Baghdad by Muḥammad ibn Mūsā al-Khwārizmī (c. 780–850 CE), better known today simply as al-Khwārizmī.

Future high school mathematics teachers who will be responsible for teaching algebra courses in their own classrooms will be well-served by working through This project. It is also suitable for use in a general history of mathematics course, and is of value to instructors of higher algebra courses who are interested in conveying a sense of the early history of the theory of equations to their students. Author: Danny Otero.

M 29. Euler's Square Root Laws for Negative Numbers

Students read excerpts from Euler's *Elements of Algebra* on square roots of negative numbers and the laws $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$, $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ when a and/or b is negative. While some of Euler's statements initially appear false, students explore how to make sense of the laws with a broader, multivalued interpretation of square roots. This leads naturally to the notion of multivalued functions, an important concept in complex variables.

This project is suitable for a first course in complex variables. Author: Dave Ruch.

M 30. Investigations Into d'Alembert's Definition of Limit

The modern definition of a limit evolved over many decades. One of the earliest attempts at a precise definition is credited to d'Alembert (1717–1783). This project is designed to investigate the definition of limit for sequences, beginning with d'Alembert's definition and a modern Introductory Calculus text definition. *Two versions of this project are available, for very different audiences, as described below. Author: David Ruch.*

- **M 30.1 Investigations Into d'Alembert's Definition of Limit - Calculus Version**

  This version of the project is aimed at Calculus 2 students studying sequences for the first time. In this version, project tasks first lead students through some examples based on dAlemerts completely verbal definition. Students are next asked to find examples illustrating the difference between the modern conception of limit and that of d'Alembert. An optional section then examines these differences in a more technical fashion by having students write definitions for each using inequalities and quantifiers.

- **M 30.2 Investigations Into d'Alembert's Definition of Limit - Real Analysis Version**

  This longer version of the project is aimed at Real Analysis students. D'Alembert's definition is completely verbal, and project tasks first lead students through some examples and a translation of this definition to one with modern notation and quantifiers. Students are also asked to find examples illustrating the difference between the modern and d'Alembert definitions. This version of the project then investigates two limit properties stated by d'Alembert, including modern proofs of the properties.

M 31. Playfair's Introduction of Bar Graphs and Pie Charts to Represent Data

With the proliferation of data in all aspects of our lives, understanding how to present and interpret visual representations is an essential skill for students to develop. Using the seminal work of William Playfair in his *Statistical Breviary* [80], this project introduces students to the bar graph (including compound bar graphs) and pie chart, asking them to interpret real data from the late 1700s and early 1800s. Students are also exposed to a modern 3-D misleading pie chart. This project is intended to be usable in a single class period.

This project is intended for use in an introductory statistics or data science course at the undergraduate level. However, it could also be used in courses for pre-service teachers, mathematics for liberal arts courses, professional development courses/workshops for teachers, or in history of mathematics courses. It is also potentially suitable for use at the high-school level. Authors: Diana White, River Bond, Joshua Eastes, and Negar Janani.

M 32. Playfair's Introduction of Time Series to Represent Data

With the proliferation of data in all aspects of our lives, understanding how to present and interpret visual representations is an essential skill for students to develop. Using the seminal work of William Playfair in his *Statistical Breviary* [80], this project introduces students to the time series including compound time series, asking them to interpret real data from the late 1700s and early 1800s. This project is intended to be usable in a single class period.

This project is intended for use in an introductory statistics or data science course at the undergraduate level. However, it could also be used in courses for pre-service teachers, mathematics for liberal arts courses, professional development courses/workshops for teachers, or in history of mathematics courses. It is also potentially suitable for use at the high-school level. Authors: Diana White, River Bond, Joshua Eastes, and Negar Janani.

M 33. Playfair's Novel Visual Displays of Data

With the proliferation of data in all aspects of our lives, understanding how to present and interpret visual representations is an essential skill for students to develop. Using the seminal work of William Playfair in his *Statistical Breviary* [80], this project exposes students to the visual displays of information that combine compound time series and compound bar graphs, asking them to interpret real data from the late 1700s and early 1800s. This project is intended to be usable in a single class period.

This project is intended for use in an introductory statistics or data science course at the undergraduate level. However, it could also be used in courses for pre-service teachers, mathematics for liberal arts courses, professional development courses/workshops for teachers, or in history of mathematics courses. It is also potentially suitable for use at the high-school level. Authors: Diana White, River Bond, Joshua Eastes, and Negar Janani.

M 34. Regression to the Mean

Over a century ago, Francis Galton (1822–1911) noted the curious fact that tall parents usually have children shorter than they, and that short parents, in turn, have taller children. This observation was the beginning of what is now called "regression to the mean" – the phenomenon that extreme observations are generally followed by more average ones. In this project, students engage with Galton's original work on the subject [44], and build an understanding of the underlying causes for this sometimes non-intuitive phenomenon.

This project is intended for classes in Statistics, and would also be useful in a general education class on quantitative reasoning. Author: Dominic Klyve.

M 35-37. Solving Linear First Order Differential Equations

A first order linear differential equation can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

and is often the first non-separable differential equation that students encounter. The problem of solving this linear first order differential equation was first proposed in print in 1695 by Jacob Bernoulli (1655–1705), as a challenge problem in *Acta Eruditorum*. This series of mini-PSPs examines three solution methods that have become core topics in courses on differential equations, proposed by Johann Bernoulli (1646–1716), Gottfried Leinbiz (1646–1716) and Leonard Euler (1701–1783) respectively:

- **M 35: Solving First-Order Linear Differential Equations: Gottfried Leibniz' "Intuition and Check" Method**
- **M 36: Solving Linear First Order Differential Equations: Johann Bernoullis Variation of Parameters**
- **M 37: Solving Linear First Order Differential Equations: Leonard Eulers Integrating Factor**

Each of these projects can be used either alone or in conjunction with either of the other two. *All three of these mini-PSPs are intended for courses in differential equations. Author: Adam Parker.*

M 38: Wronskians and Linear Independence: A Theorem Misunderstood by Many

Wronskiana are often presented to students in a differential equations class, during the discussion of fundamental sets of solutions. The name "Wronskian" was first used in this connection by Thomas Muir (1834–1934), in his 1882 *Treatise on the Theory of Determinants*. Muir also gave the first definition of the Wronskian with which we are familiar today:

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_3 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots & \frac{dy_n}{dx} \\ \frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \cdots & \frac{dy_n^2}{d^2x} \end{vmatrix}$$

For years, respected mathematicians took for it for granted that a zero Wronskian implied linear dependence for the functions $y_1, y_2 \ldots, y_n$, and even provided proofs for this claim. The first person to realize that it was not true appears to have been Giuseppe Peano (1858–1932). Yet even after Peano provided an elementary counterexample, mathematicians had difficulty understanding the subtlety of the situation. This project uses excerpts from Peano's works to help students understand the subtle connection between the Wronskian and fundamental solutions of differential equations.

This project is intended for a course in differential equations, and is also well-suited for use in linear algebra or Introduction to Proof courses. Author: Adam Parker.

M 39: Leonhard Euler and Johann Bernoulli on Solving Higher Order Linear Differential Equations with Constant Coefficients

We can really only explicitly solve higher order ($n > 1$) linear differential equations

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

when the coefficient functions are either constants (the theme of this PSP), or monomials c_ix^i (called Cauchy-Euler equations). In the constant case, the important observation involves the relationship between the above differential equation and the algebraic equation

$$a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 = 0.$$

This relationship was first noted by Euler in an 1739 letter to Johann Bernoulli (though perhaps expectedly, Bernoulli claimed to have already known it). The argument contained in the correspondence is unfortunately incomplete. However in 1743, Euler published a complete classification of the relationship between constant coefficient linear differential equations and polynomials. This PSP works through Eulers classification, then concludes by revisiting the original correspondence to consider two examples that Euler and Bernoulli attempted to solve.

This project is intended for a course in differential equations. Author: Adam Parker.

M 40: Fourier's Heat Equation

Joseph Fourier is credited with being the first to postulate the greenhouse effect. He did so in his 1827 paper *On the Temperatures of the Terrestrial Sphere and Interplanetary Space* (translated in [79]) in the excerpt shown below.

> The Earth is heated by solar radiation. . . Our solar system is located in a region of the universe of which all points have a common and constant temperature, determined by the light rays and the heat sent by all the surrounding stars. This cold temperature of the interplanetary sky is slightly below that of the Earths polar regions. The Earth would have none other than this same temperature of the Sky, were it not for . . . causes which act . . . to further heat it.

The mathematical basis for this argument came five years earlier, in Fourier's highly influential work *Théorie analytique de la chaleur* (*The Analytical Theory of Heat*) [42]. A selected tour of this work fits beautifully in an undergraduate introductory course on ordinary differential equations. Newton's Law of Cooling is already a standard introductory example in such a course, since it is solvable by so many of the standard methods of solving first-order ODEs (separation of variables, integrating factors, and power series, to name a few). Fourier uses Newton's Law of Cooling as a starting point to determine how heat propagates throughout various types of objects (thin rods, cylinders, rectangles, etc). Through this journey, the student will get to see an application of a very standard second-order linear differential equation, as well as sneak peeks into more advanced topics in differential equations, including Fourier series and PDEs.

This project is intended for a course in differential equations. Author: Kenneth M. Monks.

M 42. Finding Exact Sums of Infinite Series

Students typically encounter the theory of infinite series in their second semester course in calculus, in which the focus is the determination of the *convergence* of series. They generally conclude the course realizing that very few of the infinite series which they find are convergent are easy to determine, save geometric series and telescoping series. This project is designed to provide students an immersive experience in determining exact sums of a number of infinite series by following the work of Jakob Bernoulli (1655–1705) in his *Tractatus de Seriebus Infinitis* [6]. In this work, Bernoulli found exact sums of series of the forms

$$\sum_{n=1}^{\infty} \frac{c}{bd^{n-1}} \binom{n+k-1}{k} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{c}{bd^{n-1}} n^k,$$

where $b \neq 0$ and c are arbitrary, $|d| > 1$ and $k = 1, 2$ or 3. The project guides the student through Bernoulli's technique of finding these sums by splitting the given series into an infinite number of convergent geometric series, the sum of which in turn produces a series that can be summed exactly using a result from an earlier paragraph of the treatise. Optional sections then invite the student to use a more modern approach to explore cases of these series corresponding to integer values of $k > 3$. This project thus provides a wealth of examples of convergent series for which the sums *can* be found exactly.

This project is intended for use in second-semester calculus courses. Authors: Danny Otero and James Sellers

M 43. Sum of Four Squares

Which numbers (non-negative integers) can be written as a sum of two squares? three squares? With a little experimentation, you'll discover the answer is, at any rate, not all. The French mathematician Claude Gaspar Bachet (1581–1638) conjectured, on the other hand, that *all* numbers can be written as a sum of four squares, and it seems likely that others before him would have suspected the same. He didn't have a proof, but he was in good company; others including Descartes (1596–1650), Fermat (1601–1665), and Euler (1707–1783) worked on the problem but also were unable to provide a proof. The first proof that all numbers can be written as a sum of four squares was given by Lagrange (1736–1813) who used a *method of descent*. In this project we'll study a different proof, one given by Carl Gustav Jacob Jacobi (1804–1851) in his 1829 *Fundamenta Nova Theoriae Functionum Ellipticarum.*[54]. Jacobi's proof is remarkable for at least two reasons. First, his proof also provided a formula which determines the number of different ways a given number can be written as a sum of four squares. Second, Jacobi introduced *theta series* in *Fundamenta* *nova*, a tool that has become important in many areas of contemporary mathematics including the study of modular forms and of elliptic curves. In the full (anticipated 6 days) version of this project we'll spend a little time studying the topic that motivated Jacobi: *elliptic integrals* and *elliptic functions* and then see how two different series expansions for a particular elliptic integral are used in Jacobi's proof of the four squares theorem. In the short (anticipated 2 days) version of the project we'll use the series expansions as the starting point.

This project is primarily intended for an introductory number theory course but could be used by any students who have seen infinite series, say, in a second semester Calculus course. With a little modification in the more proof oriented tasks it could be used in a Calculus II course as an application of infinite series. Author: Lienert

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