Properties of Digital Homotopy

Article *in* Journal of Mathematical Imaging and Vision · January 2005

DOI: 10.1007/s10851-005-4780-y · Source: DBLP

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Abstract

Several recent papers have adapted notions of geometric topology to the emerging field of "digital topology." An important notion is that of digital homotopy. In this paper, we study a variety of digitally-continuous functions that preserve homotopy types or homotopy-related properties such as the digital fundamental group.

Key words and phrases: *digital image, digitally continuous, homeomorphism, retraction, homotopy, fundamental group, digital topology*

1 Introduction

Researchers wishing to characterize the properties of a digital image have turned to tools from topology. Digital versions of the *homotopy type* and the *fundamental group* have been studied in papers including [3, 5, 6, 8, 11]. These are fundamental properties of digital images. It is therefore desirable to have tools for their recognition and efficient computation. In this paper, we study several classes of digitally-continuous [10, 2] functions that preserve the digital homotopy type or properties of the digital fundamental group.

2 Preliminaries

2.1 General Properties

Let $\mathcal{Z}$ denote the set of integers. Then $\mathcal{Z}^n$ is the set of lattice points in Euclidean $n-$dimensional space. A *(binary) digital image* is a finite subset of $\mathcal{Z}^n$.

* *Journal of Mathematical Imaging and Vision* 22 (2005), 19-26. The original publication is available at www.springerlink.com.

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A variety of adjacency relations are used in the study of digital images. The following [6] are commonly used. Two points p and q in $\mathcal{Z}^2$ are $8-adjacent$ if they are distinct and differ by at most 1 in each coordinate; p and q in $\mathcal{Z}^2$ are $4-adjacent$ if they are 8-adjacent and differ in exactly one coordinate. Two points p and q in $\mathcal{Z}^3$ are $26-adjacent$ if they are distinct and differ by at most 1 in each coordinate; they are $18-adjacent$ if they are 26-adjacent and differ in at most two coordinates; they are $6-adjacent$ if they are 18-adjacent and differ in exactly one coordinate. For $k \in \{4,8,6,18,26\}$, a $k-neighbor$ of a lattice point p is a point that is $k-$adjacent to p.

We generalize 4-adjacency in $\mathcal{Z}^2$ and 6-adjacency in $\mathcal{Z}^3$ by saying $p,q \in \mathcal{Z}^n$ are $2n-adjacent$ if $p \neq q$ and p and q differ by 1 in one coordinate and by 0 in all other coordinates.

More general adjacency relations are discussed in [4]. In the following, if κ is an adjacency relation defined for an integer k on $\mathcal{Z}^n$ as one of the $k-$adjacencies discussed above, *i.e.*, if

$$(n,k) \in \{(1,2),\ (2,4),\ (2,8),\ (3,6),\ (3,18),\ (3,26)\}, \text{ or } k \ = \ 2n,$$

we refer to $\kappa-$adjacency as $k-$adjacency, $\kappa-$connectedness as $k-$connectedness, etc.

Let κ be an adjacency relation defined on $\mathcal{Z}^n$. A digital image $X \subset \mathcal{Z}^n$ is $\kappa - connected$ [4] if and only if for every pair of points $\{x,y\} \subset X$, $x \neq y$, there exists a set $\{x_0, x_1, \ldots, x_c\} \subset X$ such that $x = x_0$, $x_c = y$, and x_i and x_{i+1} are $\kappa-$neighbors, $i \in \{0,1,\ldots,c-1\}$.

Definition 2.1 ([3]; see also [10]) *Let $X \subset \mathcal{Z}^{n_0}$, $Y \subset \mathcal{Z}^{n_1}$. Let $f : X \to Y$ be a function. Let κ_i be an adjacency relation defined on $\mathcal{Z}^{n_i}$, $i \in \{0,1\}$. We say f is $(\kappa_0,\kappa_1)-$continuous if the image under f of every κ_0-connected subset of X is κ_1-connected.* ■

We will refer to a function satisfying Definition 2.1 as *digitally continuous*. A consequence of this definition is the following.

Proposition 2.2 ([3]; see also [10]) *Let X and Y be digital images. Then the function $f : X \to Y$ is (κ_0,κ_1)-continuous if and only if for every $\{x_0,x_1\} \subset X$ such that x_0 and x_1 are κ_0-adjacent, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent.* ■

Definition 2.3 [2] *Let $a,b \in \mathcal{Z}$, $a < b$. A* digital interval *is a set of the form*

$$[a,b]_{\mathcal{Z}} \ = \ \{z \in \mathcal{Z} \mid a \leq z \leq b\}$$

in which $2-$*adjacency is assumed.* ■

For example, if κ is an adjacency relation on a digital image Y, then $f : [a,b]_{\mathcal{Z}} \to Y$ is $(2,\kappa)-$connected if and only if for every $\{c, c+1\} \subset [a,b]_{\mathcal{Z}}$, either $f(c) = f(c+1)$ or $f(c)$ and $f(c+1)$ are $\kappa-$adjacent.

2.2 Digital homotopy

Intuitively, a homotopy between continuous functions is a continuous deformation of one into another over a time period.

Definition 2.4 ([3]; see also [5]) *Let X and Y be digital images. Let $f, g : X \to Y$ be (κ,λ)-continuous functions. Suppose there is a positive integer m and a function $F : X \times [0,m]_{\mathcal{Z}} \to Y$ such that*

- *for all $x \in X$, $F(x,0) = f(x)$ and $F(x,m) = g(x)$;*
- *for all $x \in X$, the induced function $F_x : [0,m]_{\mathcal{Z}} \to Y$ defined by*

$$F_x(t) \ = \ F(x,t) \text{ for all } t \in [0,m]_{\mathcal{Z}}$$

  is $(2,\lambda)-$continuous.

- *for all $t \in [0,m]_{\mathcal{Z}}$, the induced function $F_t : X \to Y$ defined by*

$$F_t(x) \ = \ F(x,t) \text{ for all } x \in X$$

  is $(\kappa,\lambda)-$continuous.

Then F is a digital $(\kappa,\lambda)-$homotopy between *f and g, and f and g are* digitally $(\kappa,\lambda)-$homotopic in Y. ■

When the adjacency relations κ and λ are understood in context, we say f and g are *digitally homotopic* to abbreviate "digitally $(\kappa,\lambda)-$homotopic in Y." We use the notation

$$f \ \simeq_{\kappa,\lambda} \ g$$

to indicate that functions f and g are digitally $(\kappa, \lambda)-$homotopic in Y.

Digital homotopy is an equivalence relation among digitally continuous functions [5, 3]. Further, composition preserves homotopy:

Proposition 2.5 [3] *If $f_0, f_1 : X \to Y$ are $(\kappa, \lambda)-$continuous functions with $f_0 \simeq_{\kappa,\lambda} f_1$ and $g_0, g_1 : Y \to Z$ are $(\lambda, \mu)-$continuous functions with $g_0 \simeq_{\lambda,\mu} g_1$, then $g_0 \circ f_0 \simeq_{\kappa,\mu} g_1 \circ f_1$.* ■

Definition 2.6 [3] *Let $f : X \to Y$ be a (κ, λ)-continuous function and let $g : Y \to X$ be a (λ, κ)-continuous function such that*

$$f \circ g \simeq_{\lambda,\lambda} 1_X \quad \textit{and} \quad g \circ f \simeq_{\kappa,\kappa} 1_Y.$$

Then we say X and Y have the same (κ, λ)-homotopy type *and that X and Y are* (κ, λ)-homotopy equivalent. ■

Definition 2.7 *A digitally continuous function $f : X \to Y$ is* digitally nullhomotopic (in Y) *if f is digitally homotopic in Y to a constant function* [3]. *A digital image X is* digitally contractible [5, 2] *if its identity map is digitally nullhomotopic.* ■

2.3 Simple Closed Curves

Let $X \subset \mathcal{Z}^n$ have an adjacency relation κ. We say X is a digital simple closed $\kappa-$curve if there is an integer $m > 3$ and a $(2, \kappa)-$continuous function $f : [0, m-1]_{\mathcal{Z}} \to X$ such that

- f is one-to-one and onto;
- $f(0)$ and $f(m-1)$ are κ-adjacent.
- for all $t \in [0, m-1]_{\mathcal{Z}}$, the only $\kappa-$neighbors of $f(t)$ in $f([0, m-1]_{\mathcal{Z}})$ are

$$f((t-1) \bmod m) \text{ and } f((t+1) \bmod m).$$

Theorem 2.8 [3] *Let $X \subset \mathcal{Z}^2$ be a digital simple closed 4-curve such that $\mathcal{Z}^2 \setminus X$ is 8-disconnected. Then X is not digitally 4-contractible.* ■

By contrast, consider the following example. Let $X \subset \mathcal{Z}^2$ be the set

$$X = \{(0,0),\ (1,-1),\ (2,0),\ (1,1)\}.$$

Then X is a digital 8-curve, $\mathcal{Z}^2 \setminus X$ is 4-disconnected, and X is 8-contractible [3].

2.4 Pointed Digital Homotopy

Definitions stated in this section are from [3]. A *pointed digital image* is a pair (X, x_0) where X is a digital image and $x_0 \in X$. A *pointed digitally continuous function* $f : (X, x_0) \to (Y, y_0)$ is a digitally continuous function from X to Y such that $f(x_0) = y_0$.

Let f and g be pointed digitally continuous functions from (X, x_0) to (Y, y_0). A digital homotopy

$$h : X \times [0, m]_{\mathcal{Z}} \to Y$$

between f and g is called a *pointed digital homotopy* between f and g if for all $t \in [0, m]_{\mathcal{Z}}$, $h(x_0, t) = y_0$. If a pointed digital homotopy between f and g exists, we say f and g *belong to the same pointed digital homotopy class.*

Membership in the same pointed digital homotopy class is an equivalence relation among pointed digitally continuous functions [3].

2.5 Digital Loops

Definition 2.9 *(See* [5]*.) A* digital $\kappa-$path *in a digital image X is a $(2, \kappa)-$continuous function $f : [0, m]_{\mathcal{Z}} \to X$. If, further, $f(0) = f(m)$, we call f a* digital $\kappa-$loop, *and the point $p = f(0)$ is the* base point *of the loop f. If f is a constant function, it is called a* trivial loop. ■

If f and g are digital $\kappa-$paths in X such that g starts where f ends, the *product* (see [5]) of f and g, written $f \cdot g$, is, intuitively, the $\kappa-$path obtained by following f by g. Formally, if $f : [0, m_1]_{\mathcal{Z}} \to X$, $g : [0, m_2]_{\mathcal{Z}} \to X$, and $f(m_1) = g(0)$, then $(f \cdot g) : [0, m_1 + m_2]_{\mathcal{Z}} \to X$ is defined by

$$(f \cdot g)(t) = \begin{cases} f(t) & \text{if } t \in [0, m_1]_{\mathcal{Z}}; \\ g(t - m_1) & \text{if } t \in [m_1, m_1 + m_2]_{\mathcal{Z}}. \end{cases}$$

We do not want homotopy classes to be restricted to loops defined on the same digital interval. The following notion of *trivial extension* allows a loop to "stretch" and remain in the same pointed homotopy class. Intuitively, f' is a trivial extension of f if f' follows the same path as f, but more slowly, with pauses for rest (subintervals of the domain on which f' is constant).

Definition 2.10 [3] *Let f and f' be $\kappa-$paths in a pointed digital image (X, x_0). We say f' is a* trivial extension of f *if there are sets of $\kappa-$paths $\{f_1, f_2, \ldots, f_k\}$ and $\{F_1, F_2, \ldots, F_p\}$ in X such that*

1. *$k \leq p$;*
2. *$f = f_1 \cdot f_2 \cdot \ldots \cdot f_k$;*
3. *$f' = F_1 \cdot F_2 \cdot \ldots \cdot F_p$; and*
4. *there are indices $1 \leq i_1 < i_2 < \ldots < i_k \leq p$ such that*
    - *$F_{i_j} = f_j$, $1 \leq j \leq k$, and*
    - *$i \notin \{i_1, i_2, \ldots, i_k\}$ implies F_i is a trivial loop.* ■

This notion allows us to compare the digital homotopy properties of loops whose domains may have differing cardinality, since if $m_1 \leq m_2$, we can obtain a trivial extension of a loop $f : [0, m_1]_{\mathcal{Z}} \to X$ to $f' : [0, m_2]_{\mathcal{Z}} \to X$ via

$$f'(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq m_1; \\ f(m_1) & \text{if } m_1 \leq t \leq m_2. \end{cases}$$

We use the following notion to define the class of a pointed loop.

Definition 2.11 *Let $f, g : [0, m]_{\mathcal{Z}} \to (X, x_0)$ be digital loops with*

$$f(0) \;=\; f(m) \;= g(0) \;=\; g(m) \;=\; x_0 \in X.$$

If $H : [0, m]_{\mathcal{Z}} \times [0, M]_{\mathcal{Z}} \to X$ is a digital homotopy such that for all $t \in [0, M]_{\mathcal{Z}}$ we have

$$H(0, t) \;=\; H(m, t) \;=\; x_0,$$

we say H holds the endpoints fixed. ■

Digital $\kappa-$loops f and g in X with the same base point p *belong to the same* $\kappa-$*loop class in* X if there are trivial extensions f' and g' of f and g, respectively, whose domains have the same cardinality, and a homotopy between f' and g' that holds the endpoints fixed [3]. Membership in the same loop class in (X, x_0) is an equivalence relation among digital $n-$loops [3].

We denote by $[f]$ the loop class of a loop f in X. The next result is used in [3] to show the product operation of our digital fundamental group is well defined.

Proposition 2.12 [3, 5] *Suppose* f_1, f_2, g_1, g_2 *are digital loops in a pointed digital image* (X, x_0), *with* $f_2 \in [f_1]$ *and* $g_2 \in [g_1]$. *Then* $f_2 \cdot g_2 \in [f_1 \cdot g_1]$. ■

2.6 Digital Fundamental Group

In this section, we discuss the digital fundamental group, derived from a classical notion of algebraic topology (see [9]).

Let (X, p, κ) be a pointed digital image. Consider the set $\Pi_1^{\kappa}(X, p)$ of κ-loop classes $[f]$ in X with base point p. By Proposition 2.12, the *product* operation

$$[f] * [g] \ = \ [f \cdot g]$$

is well-defined on $\Pi_1^{\kappa}(X, p)$. The operation $*$ is associative on $\Pi_1^{\kappa}(X, p)$ [5].

Lemma 2.13 [3] *Let* (X, p) *be a pointed digital image. Let* $\overline{p} : [0, m]_{\mathcal{Z}} \to X$ *be a constant function with image* $\{p\}$. *Then* $[\overline{p}]$ *is an identity element for* $\Pi_1^{\kappa}(X, p)$. ■

Lemma 2.14 [3] *If* $f : [0, m]_{\mathcal{Z}} \to X$ *represents an element of* $\Pi_1(X, p)$, *then the function* $g : [0, m]_{\mathcal{Z}} \to X$ *defined by*

$$g(t) = f(m - t) \text{ for } t \in [0, m]_{\mathcal{Z}}$$

is an element of $[f]^{-1}$ *in* $\Pi_1^{\kappa}(X, p)$. ■

Theorem 2.15 [3] $\Pi_1^{\kappa}(X, p)$ *is a group under the* $*$ *product operation, the* κ-fundamental group of (X, p). ■

It follows from the next result that in a connected digital image X, the choice of basepoint is immaterial in determining the digital fundamental group.

Theorem 2.16 [3] *Let X be a digital image with adjacency relation κ. If p and q belong to the same $\kappa-$component of X, then $\Pi_1^\kappa(X,p)$ and $\Pi_1^\kappa(X,q)$ are isomorphic groups.* ■

Theorem 2.17 [3] *Suppose X is a digital image that is* pointed contractible, i.e., *there exists $x_0 \in X$ and a digital homotopy $H : X \times [0,m]_{\mathcal{Z}} \to X$ such that*

- $H(x,0) \ = \ x$ *for all* $x \in X$;
- $H(x,m) \ = \ x_0$ *for all* $x \in X$; *and*
- $H(x_0,t) \ = \ x_0$ *for all* $t \in [0,m]_{\mathcal{Z}}$.

Then $\Pi_1^\kappa(X,x_0)$ is trivial (has one element, the class $[\overline{x_0}]_X$). ■

3 Loop class in the fundamental group

It is not known if digitally homotopy equivalent images must be pointed homotopy equivalent, or even if a digitally contractible image must be pointed contractible. That pointed homotopic images have isomorphic fundamental groups may be derived from Theorem 4.14 of [3]. The next result enables us to obtain a stronger result: we will show that we can drop "pointed" from the hypotheses of the statement above.

Proposition 3.1 *Let (X,p,κ) be a pointed digital image. Let $f : [0,m_0] \to (X,p)$ and $g : [0,m_1] \to (X,p)$ be κ-loops in X based at p. Then f and g represent the same member of $\Pi_1^\kappa(X,p)$ if and only if there are trivial extensions $f', g' : [0,M_1]_{\mathcal{Z}} \to X$ of f and g, respectively, and a κ-homotopy H from f' to g'.*

Proof: If $[f]_X \ = \ [g]_X$, there is a κ-homotopy H between trivial extensions f' and g', respectively, of f and g, that holds the endpoints fixed. Then H is a κ-homotopy between f' and g'.

Conversely, suppose there are trivial extensions $f', g' : [0, M_1]_{\mathcal{Z}} \to X$ of f and g, respectively, a positive integer M_2, and a κ-homotopy $H : [0, M_1]_{\mathcal{Z}} \times [0, M_2]_{\mathcal{Z}} \to X$ from f' to g'. We construct a homotopy between trivial extensions of f' and g' that holds the endpoints fixed.

We define trivial extensions F, G of f', g', respectively (hence of f, g respectively), as follows. Intuitively, we "stretch the base point" so our trivial extensions can be continuously deformed to follow paths (at both ends of the homotopy H) traced out by the base point under H. Let $F : [0, M_1 + 2M_2]_{\mathcal{Z}} \to X$ be defined by

$$F(t) = \begin{cases} p & \text{if } t \in [0, M_2]_{\mathcal{Z}}; \\ f'(t - M_2) & \text{if } t \in [M_2, M_1 + M_2]_{\mathcal{Z}}; \\ p & \text{if } t \in [M_1 + M_2, M_1 + 2M_2]_{\mathcal{Z}}. \end{cases}$$

Similarly, $G : [0, M_1 + 2M_2]_{\mathcal{Z}} \to X$ is defined by

$$G(t) = \begin{cases} p & \text{if } t \in [0, M_2]_{\mathcal{Z}}; \\ g'(t - M_2) & \text{if } t \in [M_2, M_1 + M_2]_{\mathcal{Z}}; \\ p & \text{if } t \in [M_1 + M_2, M_1 + 2M_2]_{\mathcal{Z}}. \end{cases}$$

It is clear that F and G are trivial extensions of f' (hence, of f) and of g' (hence, of g), respectively, and that

$$F(0) = F(M_1 + 2M_2) = G(0) = G(M_1 + 2M_2) = p.$$

We define the function $K : [0, M_1 + 2M_2]_{\mathcal{Z}} \times [0, M_2]_{\mathcal{Z}} \to X$ by

$$K(u, v) = \begin{cases} H(0, \min\{u, v\}) & \text{if } u \in [0, M_2]_{\mathcal{Z}}; \\ H(u - M_2, v) & \text{if } u \in [M_2, M_1 + M_2]_{\mathcal{Z}}; \\ H(M_1, \min\{M_1 + 2M_2 - u, v\}) & \text{if } u \in [M_1 + M_2, M_1 + 2M_2]_{\mathcal{Z}}. \end{cases}$$

We show K is well-defined as follows. Note $v \leq M_2$, so for $u = M_2$, the first piece of the definition of K gives

$$K(M_2, v) = H(0, \min\{M_2, v\}) = H(0, v)$$

and the second piece of the definition of K gives

$$K(M_2, v) = H(M_2 - M_2, v) = H(0, v).$$

For $u = M_1 + M_2$, the second piece of the definition of K gives

$$K(M_1 + M_2, v) = H(M_1, v)$$

and the third piece of the definition of K gives

$$K(M_1 + M_2, v) \;=\; H(M_1, \min\{M_1 + 2M_2 - (M_1 + M_2), v\}) \;=$$

$$H(M_1, \min\{M_2, v\}) \;= H(M_1, v).$$

Therefore, H is well-defined.

It is easily seen that K is a homotopy between F and G such that

$$K(0, v) \;=\; p \;=\; K(M_1 + 2M_2, v)$$

for all $v \in [0, M_2]_{\mathcal{Z}}$. Thus, in $\Pi_1^{\kappa}(X, p)$, $[f] \;=\; [F] \;=\; [G] \;=\; [g]$. ■

4 Digital maps and fundamental groups

In this section, we examine digitally continuous functions and relations they induce on fundamental groups.

4.1 Homotopy equivalences

Theorem 4.1 *Let $f : (X, \kappa) \to (Y, \lambda)$ be a (κ, λ)-homotopy equivalence. Then $\Pi_1^{\kappa}(X, x_0)$ and $\Pi_1^{\lambda}(Y, f(x_0))$ are isomorphic groups.*

Proof: By assumption, there is a (λ, κ)-continuous $g : Y \to X$ such that

$$g \circ f \simeq_{\kappa,\kappa} 1_X \text{ and } f \circ g \simeq_{\lambda,\lambda} 1_Y.$$

It follows from Propositions 2.5 and 3.1 that for $[f_0] \in \Pi_1^{\kappa}(X, x_0)$, $[g_0] \in \Pi_1^{\lambda}(Y, y_0)$,

$$(g_* \circ f_*)([f_0]) \;=\; [(g \circ f) \circ f_0] \;=\; [f_0] \text{ and } (f_* \circ g_*)([g_0]) \;=\; [(f \circ g) \circ g_0] \;=\; [g_0].$$

Hence f_* and g_* are inverse functions.

Further, f_* is a homomorphism, since

$$f_*([f_0 \cdot f_1]) \;=\; [(f \circ f_0) \cdot (f \circ f_1)] \;=\; f_*([f_0]) * f_*([f_1]).$$

Similarly, g_* is a homomorphism. The assertion follows. ■

Corollary 4.2 *Let X be a κ-contractible digital image and let $p \in X$. Then $\Pi_1^\kappa(X,p)$ is a trivial group.*

Proof: This follows from Theorem 4.1, since a contractible image has the homotopy type of a one-point image, which clearly must have trivial fundamental group. ■

4.2 Homeomorphisms

Let X be a digital image with κ-adjacency. Let Y be a digital image with λ-adjacency. Suppose $f : X \to Y$ is a (κ, λ)-continuous bijection such that the inverse function f^{-1} is (λ, κ)-continuous. Then f is called a $(\kappa, \lambda) -$ *homeomorphism* (this generalizes the definition of [2] to arbitrary adjacency relations).

Corollary 4.3 *Let $f : X \to Y$ be a (κ, λ)-homeomorphism of nonempty digital images. Then f induces a group isomorphism between $\Pi_1^\kappa(X, x_0)$ and $\Pi_1^\lambda(Y, f(x_0))$.*

Proof: A (κ, λ)-homeomorphism is clearly a (κ, λ)-homotopy equivalence, so the assertion follows from Theorem 4.1. ■

4.3 Retractions and deformation retractions

Let $A \subset X$ and let $r : X \to A$ be a digitally continuous function such that $r(a) = a$ for all $a \in A$. Such a map is called a *retraction* [1, 2]. We have the following.

Proposition 4.4 *A digital retraction $r : X \to A$ induces an epimorphism of $\Pi_1^\kappa(X,a)$ onto $\Pi_1^\kappa(A,a)$.*

Proof: Let $[f] \in \Pi_1^\kappa(A,a)$. Let $i : A \to X$ be the inclusion map. Then $[i \circ f] \in \Pi_1^\kappa(X,a)$, and

$$[f] \; = \; [r \circ i \circ f] \; = \; r_*([i \circ f]).$$

The assertion follows. ■

Following [1], we say a digital homotopy $H : X \times [0,m]_{\mathcal{Z}} \to X$ is a *deformation retraction* if

- the induced map $H(\_, 0)$ is the identity map 1_X, and

- the induced map $H(\_, m)$ is a retraction of X onto $H(X \times \{m\}) \subset X$.

The set $A = H(X \times \{m\})$ is called a *deformation retract* of X. We have the following.

Theorem 4.5 *Let A be a nonempty subset of a digital image X and let $H : X \times [0, m]_{\mathcal{Z}}$ be a κ-deformation retraction of X onto A. Then X and A have the same κ-homotopy type.*

Proof: Let $r : X \to A$ be the κ-continuous map defined by $r(x) = H(x, m)$ for all $x \in X$. Let $i : Y \to X$ be the inclusion map. Then $i \circ r = 1_Y$, and H is a homotopy between 1_X and $r \circ i$. The assertion follows. ■

Corollary 4.6 *Let A be a nonempty subset of a digital image X and let H be a κ-deformation retraction of X onto A. Then $\Pi_1^{\kappa}(X, a)$ and $\Pi_1^{\kappa}(A, a)$ are isomorphic.*

Proof: This follows from Theorem 4.5 and Theorem 4.1. ■

4.4 Digital shy maps

Definition 4.7 *Let $f : X \to Y$ be a (κ, λ)-continuous surjection of digital images. We say f is (κ, λ)-shy if*

- *For each $y \in Y$, $f^{-1}(\{y\})$ is κ-connected, and*
- *for every $y_0, y_1 \in Y$, if y_0 and y_1 are λ-adjacent, then $f^{-1}(\{y_0, y_1\})$ is κ-connected.* ■

For example, a digital homeomorphism is a shy map. On the other hand, consider the 4-connected set $S \subset Z^2$ consisting of the 8 points

$$p_0 = (1, 0),\ p_1 = (1, 1),\ p_2 = (0, 1),\ p_3 = (-1, 1),$$

$$p_4 = (-1, 0),\ p_5 = (-1, -1),\ p_6 = (0, -1),\ p_7 = (1, -1).$$

- Let $f : S \to \{0, 1\}$ be defined by $f(-1, y) = 0$, $f(x, y) = 1$ for $x > -1$. It easily seen that f is a $(4, 2)$-shy map. Note f is not a $(4, 2)$-homotopy equivalence, since S is not 4-contractible [2], but $\{0, 1\}$ is 2-contractible.

- Consider the $(2,4)$-continuous surjection $g : [0,7]_{\mathcal{Z}} \to S$ defined by $g(t) = p_t$ for all $t \in [0,7]_{\mathcal{Z}}$. Then p_0 and p_7 are 4-adjacent, but $g^{-1}(\{p_0, p_7\}) = \{0,7\}$ is not 2-connected, so g is not $(2,4)$-shy.

We have the following.

Theorem 4.8 *Let $f : X \to Y$ be a (κ, λ)-shy map of digital images. Let $y_0 = f(x_0)$. Then the induced map $f_* : \Pi_1^{\kappa}(X, x_0) \to \Pi_1^{\lambda}(Y, y_0)$ is a surjection.*

Proof: Let $[g] \in \Pi_1^{\lambda}(Y, y_0)$. Let $g : [0,m]_{\mathcal{Z}} \to Y$ be a member of $[g]$. We claim there is a positive integer M and a map $G : [0,M]_{\mathcal{Z}} \to X$ such that $G(0) = G(M) = x_0$ and $f \circ G$ is a trivial extension of g. We construct G as follows.

Let $G_0(0) = x_0$. This gives us the initial case of an inductively defined function, as follows. Suppose for some integer k satisfying $0 \leq k < m$, we have $G_k : [0,t_k]_{\mathcal{Z}} \to X$ such that $f \circ G_k$ is a trivial extension of $g|_{[0,k]_{\mathcal{Z}}}$. Now, f is shy, so $X_k = f^{-1}(\{g(k), g(k+1)\})$ is κ-connected. Thus, there is a path $p_k : [0, m_k]_{\mathcal{Z}} \to X_k$ such that

$$p_k(0) = G_k(t_k) \in f^{-1}(g(k)),$$

$$p_k([0, m_k - 1]_{\mathcal{Z}}) \subset f^{-1}(g(k)), \tag{1}$$

and

$$p_k(m_k) \in f^{-1}(g(k+1)). \tag{2}$$

Then let $t_{k+1} = t_k + m_k$ and let the map $G_{k+1} : [0, t_{k+1}]_{\mathcal{Z}} \to X$ defined by

$$G_{k+1}(t) = (G_k \cdot p_k)(t).$$

It follows from the inductive hypothesis and statements (1) and (2) that $0 \leq t < t_{k+1}$ implies $f(G_{k+1}(t)) = g(t)$, and from statement (2) that $f(G_{k+1}(t_{k+1})) = g(k+1)$. Thus, $f \circ G_{k+1}$ is a trivial extension of $g|_{[0,k+1]_{\mathcal{Z}}}$.

Our induction gives us a map $G_m : [0, t_m] \to X$ such that $G_m(0) = x_0$ and $f \circ G_m$ is a trivial extension of g. Since $f^{-1}(y_0)$ is κ-connected, there is a κ-path $h : [0,u] \to f^{-1}(y_0)$ such that $h(0) = G_m(t_m)$ and $h(u) = x_0$. Let $M = t_m + u$. Then the function $G : [0,M]_{\mathcal{Z}} \to X$ defined by

$$G(t) = (G_m \cdot h)(t)$$

represents a member of $\Pi_1^{\kappa}(X, x_0)$ such that $f \circ G$ is a trivial extension of g. Therefore, $[f \circ G] = [g]$, so f_* is a surjection. ■

5 Limitation of homotopy equivalence

In Euclidean topology, all simple closed curves are homeomorphic. However, the analogous statement is false for digital simple closed curves, as a pair of digital simple closed curves need not have the same cardinality. Indeed, this observation implies that a pair of digital simple closed curves need not have the same digital homotopy type:

Theorem 5.1 *Let $X \subset \mathcal{Z}^{n_0}$ and $Y \subset \mathcal{Z}^{n_1}$ be, respectively, κ- and λ-simple closed curves that are not contractible, such that $|X| \neq |Y|$. Then X and Y do not have the same (κ, λ)-homotopy type.*

Proof: Without loss of generality, $|X| < |Y|$. Let $f : X \to Y$ be a (κ, λ)-continuous function. Since $|X| < |Y|$, it follows that $f(X)$ is a proper subset of Y. Therefore, f is a λ-contractible map in Y. It follows that if $g : Y \to X$ is any (λ, κ)-continuous map, then $g \circ f$ is a κ-contractible map in X. Since 1_X is assumed not to be a κ-contractible map, 1_X and $g \circ f$ are not κ-homotopic in X. Since f was arbitrary, it follows that X and Y are not (κ, λ)-homotopic. ■

6 Further Remarks

We have shown that digitally homotopic loops with the same base point represent the same element of the digital fundamental group of a pointed digital image. It follows that homotopic images (they need not be pointed homotopic) have isomorphic fundamental groups. Interesting questions for further exploration:

- Let (X, x_0, κ) and (Y, y_0, λ) be connected pointed digital images. If X and Y are (κ, λ)-homotopy equivalent, must (X, x_0) and (Y, y_0) be pointed (κ, λ)-homotopy equivalent?

- Let (X, x_0, κ) be a pointed digital image. If X is κ-contractible, must (X, x_0) be pointed κ-contractible?

It is clear that a positive answer to the first of these questions implies a positive answer to the second.

We have studied a variety of digitally continuous maps, including homotopy equivalences, homeomorphisms, retractions, deformation retractions, and shy maps. We have shown that each of these preserves homotopy-related properties of digital images.

We have also discussed, in section 5, an important limitation of the digital homotopy type as a characteristic of the form of a digital image.

We thank the anonymous referees for their suggestions.

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