# Instructor Notes for PSP Investigating Difference Equations

David Ruch
MSU Denver

February 21, 2017

This project is aimed at use in a Discrete Mathematics course that also makes the transition to higher mathematics, has a Calculus II prerequisite and a programming course prerequisite. The calculus background is needed because the project uses infinite series, particularly geometric series, and some partial fractions for the De Moivre section. While this project should improve students' proof reading and writing skills, it will flow more smoothly if students have already seen and written some proofs earlier in the Discrete Math course. In particular, two exercises in the Bernoulli section use strong induction.

As with most student projects, instructors have considerable flexibility in how to handle the exercises. Some of the more difficult core exercises can be done through class discussion to ensure that all students grasp the fundamental ideas. Indeed, the project is designed to be undertaken with considerable interaction between student and instructor and/or other students over the course of several weeks. It is also assumed that less central exercises can be skipped entirely, assigned only to exceptional students, or perhaps be reserved for a more advanced audience, such as a senior History of Mathematics course. Some more specific comments on these issues are given below.

1. The first four exercises are straightforward, making sure the students are comfortable with the basic definition, terms and notation for recurring sequences. The example sequences will reappear later in the project as we develop new techniques, so make sure the students get these right. De Moivre doesn't discuss here the arbitrary nature of the first couple terms in each sequence and it is not crucial to dwell on this here, but it might be worth a comment or question to the class.

2. In Lemma II, students may not be familiar with the term "viz." Of course you can handle this as you see fit, telling them in advance, or asking them to look up "viz." As with the $xx$ notation, $\overline{1-x}\,|^{2}$ was used by some authors at that time before falling out of favor. Cajori's *A History of Mathematical Notation* (Vol I pg 367) is a great place to read about $xx$ vs. $x^2$ and other notational topics.

3. De Moivre never really needs to discuss convergence of his infinite series, as he treats all his series as formal algebraic objects. Today we call these series generating functions rather than recurring series. Students don't need to worry about convergence for this project. Nevertheless, some students may ask what De Moivre meant by "continually decreasing" in his Lemma

II or why he even mentioned it. I don't actually know the answer to this question, but I can speculate that he actually meant the coefficients or entire terms are decreasing in absolute value for some set of values for $x$. Perhaps this was a hedge toward guaranteeing convergence of the series. This is really an open-ended question aimed at the various convergence tests of Calculus II, not really central to the project themes.

4. The exercises immediately after the proof of Lemma 2 give students a chance to generalize both theorem statements and proofs. Moving from 2-scale to 3 or 4 scale results should be pretty straightforward for most students. One nice feature of the proofs is that they are constructive and can be used to actually discover the sum formulas. The 3-scale result is used near the very end of the project. The Optional exercise is not conceptually any harder, but requires much more facility with notation. Finally, we ask students to use De Moivre's proof technique to find the sum $a + ax + ax^2 + ax^3 + \cdots$, which is of course a geometric series. This series also reappears in a crucial way later in the project.

5. In the exercises immediately following Proposition I, students will hopefully make the connection to partial fractions from their calculus courses. Proving that De Moivre's formula is correct requires making the connection between the coefficients and roots: $(x - r)(x - p) = x^2 - ex + f$, which may necessitate some hints to students. After this, the proof is computational. One exercise examines an example with repeated roots. We will investigate the repeated root case thoroughly in the Bernoulli section of the project.

6. In Proposition III De Moivre uses the term "unity", which may confuse some students. Processing the meaning of this proposition may be challenging for some students. The next couple exercises right after the proposition are routine, meant to clear up any confusion in the proposition statement using example trinomials they have seen already. The "golden ratio" appears for the first time here, as a root of the quadratic associated with the Fibonacci sequence. We then ask students to prove Proposition III using geometric series in the next exercise. Once again convergence is not a consideration, so the geometric series ratio $r$ may exceed 1 in magnitude. While some students may be able to handle this proof on their own, some hints or intermediate steps may be needed for other students. This is left to the discretion of the instructor.

7. At this point in the project, students are asked to link the concepts of recurring series for rational functions, and infinite series expansions for rational functions via geometric series. Working this out with the Fibonacci sequence keeps things concrete, but students may need considerable help with the last three exercises. Nevertheless, spending a lot of time and effort is worthwhile here, as the linkage is the key to making De Moivre's method work. If students have some programming experience, they could be encouraged to code up the algorithm in the "putting it all together" Exercise 30. While De Moivre's formulas are a bit of a pain, they are quite suitable for a simple program. The $m, n$ here are given to be consistent with $m, n$ in Lemma II.

8. While De Moivre's method may not be the quickest way to solve linear difference equations, it does introduce students to the important concept of generating functions. Depending on the course and instructor's interests, this project could be a stepping stone to more exploration of this topic.

BERNOULLI

9. The general results in Bernoulli's Section 5, 6 may be difficult to read. For this reason, we ask the students to work through a couple familiar examples first to make sure the notation is clear.

10. Note that Bernoulli's solution (11) is shifted by one in the index from De Moivre's, unfortunate but not too hard to reconcile. This is addressed in a exercise.

11. Several of the exercises go deeper into the theory that Bernoulli stated but did not give proofs for. The instructor and perhaps some students may recognize Bernoulli's approach is the standard method we see in most textbooks today. In particular, Exercises 41 and 42 are fairly standard results in many Discrete Math textbooks. The proofs of 41 and 47 use strong induction, and the proof of 42 for the general case of $n$ roots is much tougher and needs significant linear algebra.

12. The exercises requiring a CAS are not central and can be omitted.

# References

[1] Bernoulli, D. 1728. Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* Bd. III: 85-100. Translation by Stacy Langton, personal communication.

[2] De Moivre, A. 1718. *The Doctrine of Chances, or, a Method of Calculating the Probability of Events in Play.* London: W. Pearson.

[3] De Moivre, A. 1722. De fractionibus algebrais radicalitate immunibus ad fractiones simpliciores reducendis, de que summandis terminis quarundam serierum aequali intervallo a se distantibus. *Philisophical Transactions* 32: 162-178.

[4] Euler, L. 1988. *Introduction to Analysis of the Infinite, Book I.* New York: Springer-Verlag.