Investigations Into Bolzano's Proof of LUB existence. A Student Project with Primary Sources

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1 Introduction

The foundations of calculus were not yet on firm ground in early 1800's. Mathematicians such as J. L. Lagrange (1736-1813) made efforts to put limits and derivatives on firmer logical ground, but were not entirely successful. It took even longer for mathematicians to fully develop the notion of completeness of the real numbers.

Bernard Bolzano (1781-1848) was one of the great success stories of the foundations of analysis. He was a theologian with interests in mathematics and a contemporary of Gauss and Cauchy, but was not well known in mathematical circles. Despite his mathematical isolation in Prague, Bolzano was able to read works by Lagrange and others, and published work of his own.

This project investigates a key result from his important paper *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* (Prague 1817) [Bz]. His proof of the main theorem on a property of bounded sets, discussed in Section 2 of this project, gives some insight into the completeness of the real numbers. Bolzano's proof also inspired Karl Weierstrass decades later in his proof of what is now known as the Bolzano-Weierstrass Theorem.

2 Bolzano's Property S for series

Before getting to his main results of [Bz], Bolzano develops some preliminaries. In Sections 1-9 of [Bz], he discusses series and their convergence. He introduces partial sum notation:

$$A + Bx + Cx^2 + \cdots + Rx^r = \overset{r}{F}x \tag{1}$$

He also talks about finite geometric series and their sums. In Section 6 he creates the new sequence of partial sums

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \cdots, \overset{n}{F}x, \cdots, \overset{n+r}{F}x, \cdots \tag{2}$$

and discusses the:

"special property that the difference between its nth term $\overset{n}{F}x$ and every later term $\overset{n+r}{F}x$ (no matter how far from that nth term) stays smaller than any given quantity, provided n has first been taken large enough."

Let's use the modern subscript notation

$$F_n x = a_1 x + a_2 x + \cdots + a_n x = \sum_{k=1}^{n} a_k x^k$$

for the terms in (1).

Exercise 1 *Bolzano discusses the example series where $x = 1$ and the sequence of partial sums (2) is*

$$0.1, 0.11, 0.111, 0.1111, \ldots$$

and the "the quantity which the terms approach as closely as desired" is $1/9$.

(a) *For this example, what are the values of A, B, C, D in (1)?*

(b) *Write $F_{n+r} - F_n$ using sigma notation.*

(c) *Using your introductory calculus course knowledge, explain how Bolzano gets $1/9$ as the limit of this sequence.*

We will see later in the project that Bolzano is especially interested in using geometric series to prove his major theorem on a property of bounded sets. Earlier in his paper, he states the finite geometric series summation formula

$$ae^{n+1} + ae^{n+2} + \cdots + ae^{n+r} = ae^{n+1} \cdot \frac{1 - e^r}{1 - e} \tag{3}$$

whenever $|e| < 1$.

Exercise 2 *Use (3) for the series in Exercise 1 with $x = 1$ to find:*

(a) *$F_{n+r} - F_n$ for $n = 4$, $r = 3$*

(b) *the minimal n for which $|F_n x - F_{n+r} x| < 0.003$ holds for all $r \in \mathbb{N}$*

Part (b) of the previous exercise is an illustration of Bolzano's "special property" mentioned above, which we will now formally define as Property S.

Property S. Let $x \in \mathbb{R}$. A series with sequence of partial sums $F_1 x, F_2 x, F_3 x, \ldots$ has **Property S** if the difference between term $F_n x$ and every later term $F_{n+r} x$ stays smaller than any given quantity, provided n has first been taken large enough.

Exercise 3 *Write Property S in modern notation using quantifiers.*

Exercise 4 *Use (3) to prove that the series in Exercise 1 has Property S.*

Exercise 5 *Use (3) to prove that the series*

$$1/2 + 1/2^2 + 1/2^3 + \cdots$$

has Property S.

Exercise 6 *Use (3) to prove that any geometric series with* $|e| < 1$ *has Property S. You may assume that* $\lim e^{n+1} = 0$.

After some discussion of standard geometric series, Bolzano argues that:

> "Therefore every geometric progression whose ratio is a proper fraction can be continued so far that the increase caused by every further continuation must remain smaller than some given quantity. This must hold all the more for series whose terms decrease even more rapidly than those of a decreasing geometric progression".

We shall see in Part 2 of the project that Bolzano is particularly interested in showing the convergence of series of the form

$$u + \sum_{j=0}^{\infty} D/2^{m_j} \tag{4}$$

where $u, D > 0$ and $\{m_j\}$ is a strictly increasing sequence of integers, $m_{j+1} > m_j > m_0 \geq 1$ for all j.

For example, the series

$$7 + \frac{3}{2^1} + \frac{3}{2^{1+8}} + \frac{3}{2^{1+8+3}} + \frac{3}{2^{1+8+3+5}} + \frac{3}{2^{1+8+3+5+7}} + \frac{3}{2^{1+8+3+5+7+9}} + \cdots$$

has the form (4).

Exercise 7 *Identify* $u, D,\ m_0, m_1, m_2$ *for the example above.*

Exercise 8 *Prove that series of form (4) satisfy Property S.*

After discussing geometric series and series with Property S, Bolzano attempts to prove the following "completeness" theorem in Section 7 of his paper. While his proof is not successful, the statement is indeed true if we assume that the real number line has no "holes" in it. More formally, this theorem is true if we assume the *completeness* of $\mathbb{R}$.

Theorem. If a sequence of quantities

$$\overset{1}{F}x, \overset{2}{F}x, \overset{3}{F}x, \cdots, \overset{n}{F}x, \cdots, \overset{n+r}{F}x, \cdots$$

has the property that the difference between its nth term $\overset{n}{F}x$ and every later one $\overset{n+r}{F}x$, however far this latter term may be from the former, remains smaller than any given quantity if n has been taken large enough, then there is always a certain constant quantity, and indeed only one, which the terms of this sequence approach and to which they can come as near as we please if the sequence is continued far enough.

Exercise 9 *Explain in your own words the connection between this theorem statement and the intuitive idea of no "holes" or gaps in the real numbers.*

Exercise 10 *Rewrite this theorem using modern notation.*

One way of defining the completeness of the real numbers is to assume a version of this completeness theorem and the Archimedean Property, which Bolzano will use in the next section of the project.

We should not be too hard on Bolzano with regard to his failed proof of this completeness theorem. In fact, he was ahead of his time in realizing the completeness of the real numbers is not simply obvious, and needed some kind of justification.

For the purposes of the next section of the project, we will now distill Bolzano's discussion and results into the following completeness axiom, which we will assume as a fact, and use in the next section.

AXIOM B

Any series of the following form has a unique real number sum (limit):

$$u + \sum_{j=0}^{\infty} D/2^{m_j} \tag{5}$$

where $u, D > 0$ and $\{m_j\}$ is a strictly increasing sequence of integers, $m_{j+1} > m_j > m_0 \geq 1$ for all j.

3 Bolzano's Bounded Set Theorem

We are now ready to examine the main theorem for this project, which we will refer as the "Bounded Set Theorem".

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Preliminary note. In investigations of applied mathematics it is often the case that we learn that a definite property M applies to all values of a [nonnegative[1]] variable quantity x which are smaller than a certain u without at the same time learning that this property M does not apply to values which are greater than u. In such cases there can still perhaps be some u' that is $> u$ for which in the same way as it holds for u, all values of x lower than u' possess property M. Indeed this property M may even belong to all values of x without exception. But if this alone is known, that M does not belong to all x in general then by combining these two conditions we will now be justified in concluding: there is a certain quantity U which is the greatest of those for which it is true that all smaller values of x possess property M. This is proved in the following theorem.

[1] Bolzano intends to discuss only $x \geq 0$ in this note and his Section 12 theorem statement. The term "nonegative" has been included in this project for clarity.

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Theorem. If a property M does not apply to all values of a [nonnegative] variable quantity x but does apply to all values smaller than a certain u, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x possess the property M.

Before reading the proof, let's look at some examples of this concept Bolzano is discussing.

Exercise 11 *Let M be the property "$x^2 < 3$" applied to the set $x \geq 0$.*

(a) *Find rational numbers u, u' for this example (these values are not unique). What is the value of* U *for this example?*

(b) *Let S_M be the set of ω values that possess property* M. *Sketch S_M on a number line. Are the theorem hypotheses met for this property* M*?*

(c) *Does* U *possess property* M*?*

Exercise 12 *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x$, and let $\alpha \in \mathbb{R}$ be arbitrary. Let* M *be the property "$f(\alpha + \omega) \leq f(\alpha) + 2$" applied to the set $\omega \geq 0$.*

(a) *Find rational numbers u, u' for this example (these values are not unique). What is the value of* U *for this example?*

(b) *Let S_M be the set of ω values that possess property* M. *Sketch S_M on a number line. Are the theorem hypotheses met for this property* M*?*

(c) *Does* U *possess property* M*?*

Exercise 13 *Rewrite this theorem using modern terminology and set notation.*

Now for proof of Bolzano's Bounded Set Theorem, which he breaks into a number of steps. Begin with Step 1.

§12

Theorem. If a property M does not apply to all values of a [nonnegative] variable quantity x but does apply to all values smaller than a certain u, then there is always a quantity U which is the greatest of those of which it can be asserted that all smaller x possess the property M.

Proof.

1. Because the property M holds for all x smaller than u but nevertheless not for all x, there is certainly some quantity $V = u + D$ (where D represents something positive) of which it can be asserted that M does not apply to all x which are $< V = u + D$. If I then raise the question of whether M in fact applies to all x which are $< u + \dfrac{D}{2^m}$, where the exponent m is in turn first 0, then 1, then 2, then 3.

etc., I am sure the first of my questions will have to be answered 'no'. For the question of whether M applies to all x which are $< u + \frac{D}{2^0}$ is the same as that of whether M applies to all x which are $< u + D$, which is ruled out by assumption. What matters is whether all the succeeding questions, which arise as m gradually gets larger, will also be ruled out. Should this be the case, it is evident that u itself is the greatest value for which the assertion holds that all smaller x have property M. For if there were an even greater value, for example $u + d$, i.e. if the assertion held that also all x which are $< u + d$ have the property M, then it is obvious that if I take m large enough, $u + \frac{D}{2^m}$ will at some time be $=$ or $< u + d$. Consequently if M applies to all x which are $< u + D$, it also applies to all x which are $< u + \frac{D}{2^m}$. We would therefore not have said 'no' to this question but would have had to say 'yes'. Thus it is proved that in this case (when we say 'no' to all the above questions) there is a certain quantity U (namely u itself) which is the greatest for which the assertion holds that all x below it possess the property M.

Step 1 of Bolzano's proof has several claims and their justifications. Let's investigate them carefully.

Exercise 14 *In your own words, justify Bolzano's claim in the first sentence of Part 1 of his proof. Then illustrate the claim with the Property* M *of Exercise 11 using* $u = 3/2$ *and* $D = 8$*. For what integers* $m = 0, 1, 2, \ldots$ *does property* M *hold for all* $x < u + \frac{D}{2^m}$ *?*

Exercise 15 *Bolzano then raises some questions and states that "the first of my questions will have to be answered 'no'." Rewrite this argument in your own words and modern notation without the question/answer format.*

Bolzano makes a crucial claim in the two sentences "What matters is ... all smaller x have property M." Let's call this <u>Claim U $= u$</u> for the case "when we say 'no' to all the above questions".

Exercise 16 *Rewrite this case and Claim* U $= u$ *in your own words and modern notation as an if-then statement with appropriate quantifiers.*

Exercise 17 *In his proof Bolzano states that "it is obvious that if I take m large enough ...". Verify this claim when* $d = 1/16$ *and* $D = 8$*. More generally, if* d *and* D *are positive numbers, can you explain how to find such an* m*?*

Bolzano states that "it is obvious that if I take m large enough, $u + \frac{D}{2^m}$ will at some time be $=$ or $< u + d$". You might agree, but this actually relies on an important property of the real numbers:

Archimedean Property. For every positive real number p, there exists a natural number n for which $p > 1/n$.

Exercise 18 *Rewrite Bolzano's proof of Claim* U $= u$ *in your own words and modern notation (note this is a proof by contradiction).*

Ok, now let's read Part 2 of Bolzano's proof.

2. However, if one of the above questions is answered 'yes' and m is the particular value of the exponent for which this happens first (m can be 1 but, as we have seen, not 0), then I now know that the property M applies to all x which are $< u + \frac{D}{2^m}$ but not to all x which are $< u + \frac{D}{2^{m-1}}$. But the difference between $u + \frac{D}{2^{m-1}}$ and $u + \frac{D}{2^m}$ is $= \frac{D}{2^m}$. If I therefore deal with this as I did before with the difference D, i.e. if I raise the question of whether M applies to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$$

and here the exponent n denotes first 0, then 1, then 2, etc., then I am sure once again that at least the first of these questions will have to be answered 'no'. For to ask whether M applies to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+0}}$$

is just the same as asking whether M applies to all x which are $< u + \frac{D}{2^{m-1}}$, which had previously been denied. But if all my succeeding questions are also to be answered negatively as I gradually make n larger and larger, then it would appear, as before, that $u + \frac{D}{2^m}$ is that greatest value, or the U, for which the assertion holds that all x below it possess the property M.

3. However, if one of these questions is answered positively and this happens first for the particular value n. then I now know M applies to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$$

but not to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n-1}} \, .$$

The difference between these two quantities is $= \frac{D}{2^{m+n}}$ and I deal with this again as before with $\frac{D}{2^m}$, etc.

Let's examine some details of Bolzano's proof, Sections 2 and 3.

Exercise 19 *In your own words, justify Bolzano's claim in the first sentence of Part 2 of his proof. Then illustrate the claim with the Property* M *of Exercise 11 using $u = 3/2$ and $D = 1/2$. Mark the values $u, u + D, u + D/2^m, u + D/2^{m-1}$,* U *for Exercise 11 on a number line.*

Exercise 20 *Verify Bolzano's claim that "the difference between $u + \frac{D}{2^{m-1}}$ and $u + \frac{D}{2^m}$ is $= \frac{D}{2^m}$."*

Exercise 21 *Rewrite Bolzano's claim in the last sentence of Part 2 in your own words and modern notation as an if-then statement with appropriate quantifiers.*

Exercise 22 *Find the value of n for the Property* M *of Exercise 11 using $u = 3/2$, $D = 1/2$, and the value of m from Exercise 19. How close are $u + \frac{D}{2^m} + \frac{D}{2^{m+n-1}}$ and $u + \frac{D}{2^m} + \frac{D}{2^{m+n}}$ to the true* U *value?*

Now read parts 4,5 of Bolzano's proof.

4. If I continue this way as long as I please it may be seen that the result that I finally obtain must be one of two things.

(a) Either I find a value of the form

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

which appears to be the greatest for which the assertion holds that all x below it possess the property M. This. happens in the case when the questions of whether M applies to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r+s}}$$

are answered with 'no' for every value of s.

(b) Or I at least find that M does indeed apply to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

but not to all x which are

$$< u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r-1}}$$

Here I am always free to make the number of terms in these two quantities even greater through new questions.

5. Now if the first case occurs the truth of the theorem is already proved. In the second case we may remark that the quantity

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

represents a series whose number of terms I can increase arbitrarily and which belongs to the class described in [*Axiom B (5) in Part I of Project*]. This is because, depending on whether $m, n, ..., r$ are all $= 1$ or some of them are greater than 1, the series decreases at the same rate. or more rapidly than. a geometric progression whose ratio is the proper fraction $1/2$. From this it follows that it has the property ... that there is a certain constant quantity to which it can come as close as we please if the number of its terms is increased sufficiently. Let this quantity be U; then I claim the property M holds for all x which are $<$ U. For if it did not hold for some x which is $< U$, e.g. for $U - \delta$, then the quantity

$$u + \frac{D}{2^m} + \frac{D}{2^{m+n}} + \cdots + \frac{D}{2^{m+n+\cdots+r}}$$

must always keep at the distance δ from U because for all x that are smaller than it, the property M is to hold. Since every x that is

$$u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}-\omega,$$

however small ω is, possesses the property M, while on the other hand, M is not to apply to $x=$ U$-\delta$, it must therefore be that

$$U-\delta>u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}-\omega$$

or

$$U-\left[u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}\right]>\delta-\omega$$

Hence the difference between U and the series cannot become as small as we please, since $\delta-\omega$ cannot become as small as we please because δ does not change, while ω can become smaller than any given quantity. But just as little can M hold for all x which are $<$ U $+\,\epsilon$. For the value of the series

$$u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r-1}}$$

can be brought as close to the value of the series

$$u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}$$

as we please because the difference between the two is only $\frac{D}{2^{m+n+\cdots+r}}$. Further, the value of the latter series can be brought as close as we please to the quantity U. Therefore the value of the first series can also come as close to U as we please. So

$$u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r-1}}$$

can certainly become $<$ U $+\,\epsilon$. But now by assumption M does not hold for all x which are

$$<u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r-1}}\,;$$

so much less therefore [does M hold] for all x which are $<$ U $+\,\epsilon$. Therefore U is the greatest value for which the assertion holds that all x below it possess the property M.

Observe that in part 4 (b) of his proof, Bolzano is building a series of the form defined in Axiom B (5). Then recall from Exercise 8 that such series satisfy Property S. We are taking as an assumption that such a series will converge.

In the exercises below, you might find it convenient to rename the partial sum

$$F_r=u+\frac{D}{2^m}+\frac{D}{2^{m+n}}+\cdots+\frac{D}{2^{m+n+\cdots+r}}=u+\sum_{j=0}^{r}\frac{D}{2^{m_j}}.$$

Exercise 23 *Explain, in your own words, Bolzano's Section 4 algorithm for building his series.*

Exercise 24 *Illustrate several steps of Bolzano's algorithm using Exercises 11, 19 and $u = 3/2$, $D = 1/2$. Would you eventually stop and arrive at the true* $\mathsf{U} = \sqrt{3}$ *value, or would you need to keep adding terms indefinitely? Explain.*

In Section 5, having established the existence of the sum U of the series, Bolzano argues that "the property M holds for all x which are $< \mathsf{U}$. "

Exercise 25 *Rewrite Bolzano's argument in Section 5 as a modern proof.*

After proving his Bounded Set Theorem, Bolzano goes on to use it in his proof of the Intermediate Value Theorem, which you can explore in [I]. For this project, we will use Bolzano's Bounded Set Theorem to prove the *least upper bound property* and the *greatest lower bound property* of the real numbers. These are extremely useful and powerful properties of $\mathbb{R}$.

Greatest Lower Bound Property of $\mathbb{R}$. Every nonempty set of real numbers that has a lower bound also has a greatest lower bound in $\mathbb{R}$.

Least Upper Bound Property of $\mathbb{R}$. Every nonempty set of real numbers that has an upper bound also has a least upper bound in $\mathbb{R}$.

Here are some standard definitions for these terms.

Definition 26 *Let S be a nonempty subset of $\mathbb{R}$. The set S is* ***bounded below*** *if there exists a number $\ell \in \mathbb{R}$ for which $\ell \leq s$ for all $s \in S$. Each such number ℓ is called a* ***lower bound*** *of S. We say g is the* ***greatest lower bound*** *of S if (i) g is a lower bound of S, and (ii) $g \geq \ell$ for all lower bounds of S.*

Exercise 27 *Let S be nonempty subset of $\mathbb{R}$ such that $s > 0$ for all $s \in S$. Use Bolzano's Bounded Set Theorem to prove that S has a greatest lower bound.*

Exercise 28 *Prove the Greatest Lower Bound Property of the real numbers.*

Exercise 29 *Define the terms bounded above, upper bound, and Least Upper Bound Property by analogy with the terms in Definition 26.*

Exercise 30 *Prove the Least Upper Bound Property of the real numbers.*

4 Conclusion

The completeness property of the real numbers is a challenging topic. Some textbooks begin, like Bolzano, with our Property S and the Archmedean Property as axioms. Nowadays Property S is called the *Cauchy criterion* for historical reasons: Bolzano was largely ignored during his lifetime, even though he published his paper [Bz] before Cauchy created his version of Property S. Many mathematicians read Cauchy's work, which was very much in the mainstream of his mathematical era.

Some other books start with the least upper bound property as an axiom, and use it to prove the Cauchy criterion and the Archimedean property. It can be shown that the Cauchy criterion and the Archimedean property taken together are equivalent to the least upper bound property.

References

[Bz] Bolzano, B., *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege* (Prague 1817)

[I] Ruch, TRIUMPHS, Intermediate Value Project, this collection.