# Connecting Connectedness

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Connectedness has become a fundamental concept in modern topology. The concept seems clear enough – a space is connected if it is a "single piece." Yet the definition of connectedness we use today is not what was originally written down. As we will see, today's version of connectedness is a classic example of a definition that took decades to evolve. The first definition of connectedness was given by Georg Cantor. Cantor (1845–1918) is best known for his work in set theory. His work in set theory, however, began with questions concerning Fourier series in an 1872 paper [Cantor, 1872]. In his study of Fourier series, Cantor was interested in finding conditions for when a function has a unique Fourier expansion. This study compelled him to define for the first time some purely topological concepts, including the concepts of a point-set, a neighborhood, and a derived set. Cantor's early investigations were the precursors to a series of six papers entitled *Ueber unendliche, lineare Punktmannichfaltigkeiten (On infinite, linear point sets)* which were themselves part of his work on set theory. It is with his fifth paper in this series that we begin this project on connectedness. Given this history, one could trace not only the origins of modern set theory back to Cantor, but also the origins of modern point-set topology.

## 1 Cantor: A continuum

We begin our investigation into connectedness with Cantor's 1883 paper [Cantor, 1883]. His reason for writing this paper, however, does not appear to have been a consideration of connectedness. Cantor wrote[1]:

> The point of this examination is to establish a precise criterion – but one that is simultaneously as generally applicable as possible – for designating $P$ as a continuum.

Let's put ourselves into Cantor's shoes. What comes to mind when you think of a continuum? A straight line? A closed disk? An open disk? The irrational numbers? Definitions in mathematics tend to be driven by our examples; that is, what do all the examples of objects that you think of as a continuum have in common? Furthermore, what do all the examples that you think of as violating a continuum have in common? The goal is to abstract away the particulars and find the common properties that a continuum satisfies. As the quote above indicates, this is what Cantor desired to do in this paper.

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[1] Translations of Cantor, Jordan, and Schoenflies excerpts are the author's own translations.

**Task 1** Consider a straight line, a closed disk, an open disk, and the irrational numbers. Which of these do you consider a continuum? Why?

One concept that Cantor had at his disposal was that of the **derived set**.

For the sake of brevity I call a given finite or infinite number of ... points on the line a *point set*. If a point set is given in a finite interval, a second point set is generally given along with it, and with the latter a third, etc., which are essential to understanding the first set. In order to define these point sets, we must begin with the concept of a *limit point of a set*. I define a "limit point of a point set $P$" to be a point of the line situated in such a way that each neighborhood of it contains *infinitely* many points of $P$, and it may happen that the point itself belongs to the set. By a "neighborhood of a point" I mean any interval that has the point *in its interior*. ... Every point of the line is now in a definite relation to a given set $P$, either being a limit point of $P$ or not, and thereby along with the point set $P$ the set of limit points of $P$ is *conceptually* given, a set which I wish to denote by $P'$ and call the *first derived point set of* $P$. Unless the point set $P'$ contains only a finite number of points, it also has a derived set $P''$, which I call the *second derived point set of* $P$. By $v$ such transitions one obtains the concept of the $v$th derived set $P^{(v)}$ of $P$. ... [If] $P$ is such that the derivation process produces no change:

$$P = P^{(1)}$$

and therefore

$$P = P^{(\gamma)}$$

[then] such sets $P$ I call **perfect** point sets.

**Task 2** Using Cantor's definition of limit point and derived set, determine which of the following sets are perfect in $\mathbb{R}$.

(a) $[0, 1]$

(b) $(a, b)$, $a < b \in \mathbb{R}$

(c) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

(d) $\{\frac{1}{n} : n = 1, 2, 3, \ldots\}$

(e) $\mathbb{Q}$

(f) $\mathbb{R}$

(g) $[\mathbb{Q} \cap (0, 1)] \cup [-4, 2)$

You have have noticed seen that, for the above examples, each set that you would intuitively think of as constituting a continuum is a perfect set, and each one that you would not think of as a continuum is not perfect. This raises the question: should a perfect set be defined as a continuum?

**Task 3** Is the set $[0,1] \cup [2,3]$ a perfect set? Should it be a continuum?

As the example above illustrates, being perfect might be a necessary, but not a sufficient condition for a set to be a continuum. The missing piece that Cantor added to the condition of being perfect in order to complete his definition of a continuum was that the set also be **connected.** Cantor wrote:

A [closed and bounded] point set $T$ is **connected** if for every two of its points $t$ and $t'$, and arbitrary given positive number $\epsilon$, there always exists a finite number of points $t_1, t_2, \ldots, t_n$ of $T$ such that the distances $tt_1, t_1t_2, \ldots, t_nt'$ are smaller than $\epsilon$.

**Task 4** Rewrite this definition using modern notation (and possibly terminology), and use that definition to prove that $[0,1] \cup [2,3]$ is not connected. Then prove that $[0,1]$ is connected.

**Task 5** According to Cantor's definition, what can be said about the set $[0,1]-\left\{\frac{1}{2}\right\}$ in terms of connectedness?

## 2 Jordan: Distance between sets

Cantor's definition of connected appealed to a notion of distance, which leads to a setting that is not as general as that which one encounters in topology today. This approach to the definition was nevertheless taken up by French mathematician Camille Jordan (1838–1922) in his *Cours D'analyse (Course on analysis)* [Jordan, 1893].

Let $E, E'$ be two [closed] sets with no point in common. The distance between the various points $p$ in $E$ to the various points $p'$ in $E'$ form a set of nonnegative numbers. It is therefore bounded below, and it admits a minimum $\Delta$, positive or $0$, we call the **distance** between the sets $E, E'$. If the distance is greater than $0$, we say that the sets $E, E'$ are **separated**.

**Task 6** Using Jordan's definition, compute the distance between two closed, disjoint disks in $\mathbb{R}^2$.

Jordan immediately made the following statement and provided a proof.

If two bounded and closed sets $E, E'$ which have no points in common have distance $\Delta$, then they will contain at least a couple of points whose mutual distance is precisely $\Delta$ ....

**Task 7** Show that this is not necessarily true if the sets in question are not closed.

We are finally ready for Jordan's definition. Jordan wrote as follows:

We say that a closed and bounded set $E$ is a **component** if it cannot decompose into several separated sets.

**Task 8** Using Jordan's definition of a component, prove that a closed and bounded set $E$ in $\mathbb{R}^n$ is connected if and only if $E$ itself is a component.

## 3 Schoenflies: No distance required

We have noted that both Cantor and Jordan defined connectedness by appealing to a distance. The German mathematician Schoenflies (1853–1928), who is best known for the Jordan-Schoenflies theorem, instead gave a purely set-theoretic definition of connectedness which did not appeal to the concept of distance [Schoenflies, 1904]. His definition was simple and elegant.

> A perfect set T is called **connected** if it can not be decomposed into subsets, each of which is perfect.

**Task 9** Rewrite Schoenflies' definition of connected with modern terminology and notation.

**Task 10** Does Schoenflies' definition give us any better indication of whether or not $[0,1]-\left\{\frac{1}{2}\right\}$ is connected?

Although there are other things that Schoenflies did with this definition, our main purpose here is to note that he abstracted away any need for distance, yet kept the definition very much in the spirit of Jordan and Cantor, as the following task illustrates.

**Task 11** Prove that if a set satisfies the Jordan definition, then it must satisfy the Schoenflies definition. Why is the converse not necessarily true?

## 4 Lennes: The modern definition

Finally, we turn to Nels Johann Lennes (1874–1951). Lennes was a Norwegian-born mathematician who earned his Ph.D. at the University of Chicago and lived out most of his mathematics career at Montana State University. Lennes was aware of the mathematical thought on connectedness up to this point. He wrote [Lennes, 1911]:

> It is apparent that in a geometry possessing linear order and continuity curves and limit curves exist independently of metric properties. . . . Schoenflies uses metric hypotheses in the proof of practically every important theorem dealing with curves and the regions defined by them. . . his treatment makes full use of metric properties. . . .

Lennes then gave his own definition of limit point.

A point $\ell$ is a **limit-point** of a set of points $P$ if there are points of $P$ other than $\ell$ within every [neighborhood] of which $\ell$ is an interior point.

**Task 12** Compare Cantor's definition of limit point with that of Lennes. Are they equivalent? If so, prove it. If not, give a counterexample.

Lennes was interested in the Jordan Curve Theorem, one of the most important and difficult theorems of late $19^{\text{th}}$ and early $20^{\text{th}}$ century mathematics. This theorem states:

Let $J$ be a closed curve in $\mathbb{R}^2$ which does not self-intersect.
Then $\mathbb{R}^2 - J$ is disconnected with exactly two open, connected components.

Although easy to state and intuitively obvious, a rigorous and satisfying proof of this fact eluded mathematicians for many years. In order to attempt a rigorous proof, Lennes needed a careful and precise definition of connectedness.

A set of points is a **connected set** if at least one of any two complementary subsets contains a limit point of points in the other set.

Lennes' definition turns out to be equivalent to the modern definition. To substantiate our claim, here is a definition from a modern classic book on point-set topology [Kelley, 1975, p. 53].

A topological space is **connected** iff $X$ is not the union of two non-void separated [open] subsets.

**Task 13** Show that this definition and the one given by Lennes are equivalent.
Then determine whether or not $[0,1] - \left\{\frac{1}{2}\right\}$ is connected.

## 5 Conclusion

We have seen how the definition of connectedness, starting with Cantor, has evolved into the modern definition. For Cantor, connectedness was somewhat of a side note – it needed to be defined in order to properly understand the definition of a continuum. Jordan thought this definition of connected interesting enough to take it up in his own work, and studied it as a concept in its own right. Schoenflies then realized that there was no need to appeal to a notion of "distance" to give a coherent definition of connected. Finally, Lennes tweaked Schoenflies' definition sufficiently to obtain one that is accepted by mathematicians today. So remember this the next time you see a definition in a textbook. The crisp, clean, and pithy definition may have taken some of the world's greatest mathematicians years to arrive at!

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## Notes to Instructors

The purpose of this project is two-fold. First, it is meant to show where definitions come from. The second purpose is to show how a definition can change over time. The first goal is mostly achieved in this mini-PSP in the section on Cantor, in two different ways. The first way is by illustrating the process through which mathematicians make precise an intuitive idea. The discussion on a continuum is meant to imitate the working out of a definition. We all have in our minds paradigmatic examples of sets that we do and don't consider to be continua. Let us take all those examples, abstract away the particulars, and be left with what they all have in common – the essence of what it means to be a continuum. A good way to accomplish this in class is to ask the students to give examples of what they think are and are not continua, and to write their responses in two columns on the board. This should generate a discussion of some of the properties that students do and don't consider to be essential for a set to be a continuum. There may be students for whom nothing comes to mind when they hear "continuum." That is okay. But by the end of this task, the class as a whole should have a somewhat unified, even if still vague, idea of what 'continuum' means.

The next step in this process is then attempting to precisely formulate the meaning of a continuum in a definition. In order to tease this out, students are asked to recall the definition of a perfect point set. After a little reflection, it seems that our intuitive idea of what ought and ought not constitute a continuum coincides exactly with that of a perfect set. (In fact, I have had classes where those sets which the class considered a continuum were precisely the perfect point sets, while sets which were not considered a continuum were precisely those which are not perfect.) Task 3, however, serves as an example to show why equating the two concepts is not appropriate. This leads into the second way in which the project shows where definitions come from, by considering the question: what property needs to be added to "perfect" in order to exclude examples like the one in Task 3 from being considered a continuum? There is then a need to define this additional property, and the concept that seems to work well is that of being "connected." Students are then asked to wrestle with Cantor's original formulation of the definition of connectedness.

The section on Jordan begins to address the second project goal of showing how a definition can evolve, either in its verbal formulation or in its point of view. While Cantor defined what it means for a set to be connected, Jordan added the viewpoint of "separation" as a way to look at connectedness. That is, now we have a positive definition of this concept (connected) as well as a negative concept (separation). The concept of separation can furthermore be quantified in the sense that if a separation exists, we can sometimes assign a number to it (e.g. Task 6). The culmination of Section 2 is in Task 9 where students are asked to show that the two definitions, given by Cantor and Jordan respectively, are equivalent.

The Schoenflies section is brief and meant to be a bit ambiguous. What did he mean by "decomposed into"? A partition? Only two sets? A finite number of sets? Students will wrestle with this question and justify an answer that makes the most sense relative to what Schoenflies was trying to do in Task 10. However, it should be noted that Schoenflies gave a definition that is not equivalent to those given by Cantor and Jordan, since there is no appeal to a metric in Schoenflies' definition. The final definition due to Lennes is easily motivated with a word about the Jordan Curve Theorem, and it is satisfying for the students to see that this is the first time that we see the current definition being used.

Furthermore, there is a running example of determining whether or not the set $[0,1]-\left\{\frac{1}{2}\right\}$is connected or not. For both Cantor and Jordan, the concept of connectedness only applied to closed and bounded sets, so that this question would not even make sense to them, a seemingly major drawback of the definition. Depending on how students interpret Schoenflies, students may or may not find the set connected according to his definition. However, it should once again be satisfying to see this simple example, unclear for many years, now easily shown to be disconnected using the Lennes definition.

This project does assume some background familiarity with point-set topology. In particular, the students should have been exposed to the concept of derived sets. Otherwise, the definition of a perfect set in Section 1 comes seemingly out of left field.

The entire project should take about two 50-minute class periods in total. As mentioned above, the first day implementation of the project can begin with the professor posing the question "what are some examples and non-examples of a continuum?" to the class. This can then be discussed either by the class as a whole or in small groups with responses shared with the class as a whole. Such an approach does not require the student to do any reading or work before class. After the class shares their ideas, the students can then read through the excerpts and work on the tasks in small groups, sharing their responses with the class or as part of homework to be turned in.

This mini-Primary Source Project (mini-PSP) is one of six available for use in a topology course: *Topology from Analysis, Connecting Connectedness, The Cantor set before Cantor, A Compact introduction to a generalized extreme value theorem, From sets to metric spaces to topological spaces,* and *The closure operation as the foundation of topology.* Instructors who wish to use this pedagogical approach more extensively may also be interested in the following two full-length PSPs: *Nearness without distance* and *Connectedness: its evolution and applications.* The latter project is a fuller development of the ideas found in this mini-PSP. Each of the above projects can be obtained from the author or from the TRIUMPHS website.


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