The Pell Equation in India

Keith Jones* and Toke Knudsen†

August 25, 2017

Tell me, O mathematician, what is that square which multiplied by eight becomes, together with unity, a square, and what square multiplied by eleven and increased by unity becomes a square?

— Bhāskara II, the *Bījagaṇita*[1]

1 Introduction

In the mathematical tradition of India, the term *varga-prakṛti* (literally, "square-nature" or "the nature of squares") refers to methods for determining integers that satisfy given conditions involving their squares. The most prominent example is that of the equation

$$Nx^2 + 1 = y^2,$$

which is commonly known as the Pell equation in modern mathematics. A positive integer N is given, and in the solution (x, y), x and y are to be nonnegative integers.

The study of the Pell equation in Europe began with Fermat in the 17th century, and was continued by Euler and Lagrange. John Pell, after whom the Pell equation is named, never actually studied it. The attribution of the equation to Pell was actually a mistake of Euler's after Pell revised a translation of a text which discussed the equation.[2] The present-day approach to solving the Pell equation was developed by Lagrange in the 1760s. He used continued fractions, which are certain algebraic structures that have infinitely fractions nested within each other. While that approach is fascinating, we will not use continued fractions in this project, but instead examine the equation as it was studied in India, centuries before it was taken up by mathematicians in Europe.

Task 1 While the Pell equation appears simple, involving only nonnegative integers, multiplication, and addition, it can in fact be difficult or impossible to solve. In his *Bījagaṇita*, Bhāskara II

*Department of Mathematics, Computer Science & Statistics, SUNY Oneonta, Oneonta, NY, 13820; `keith.jones@oneonta.edu`.

†Department of Mathematics, Computer Science & Statistics, SUNY Oneonta, Oneonta, NY, 13820; `toke.knudsen@oneonta.edu`.

[1]Translated from the *Bījagaṇita* of Bāskara II, p.35, [Datta and Singh, 1962, Part 2, p. 155].

[2]From an August 10, 1730 letter from Euler to Goldbach.

asks "What is that number whose square multiplied by 67 or 61 and then added by unity becomes capable of yielding a square root? Tell me, O friend, if you have a thorough knowledge of the method of the square nature."[3] What equations is Bhāskara asking us to solve here?

It is highly unlikely that either problem in the preceding task would be solved by trial and error; in fact, for one of the questions, the smallest answer is 226153980, corresponding to 1766319049![4] For our initial explorations of the Pell equation, we will stick to some small values of N. In the process we will find some (hopefully interesting) solutions to the two equations posed by Bhāskara II in the quotation preceding the introduction.

Task 2 For each $N \in \{0, 1, 2, \ldots, 12\}$, try to find solutions to $Nx^2 + 1 = y^2$, using trial and error.

Task 3 For which values of N in Task 2 can you find multiple solutions? How many?

Task 4 Find a solution (a, b) that works for *all* values of N. We will call this the trivial solution. For which N are you able to find nontrivial solutions?

Task 5 If $n > 1$ is an integer and $N = n^2 - 1$, find a nontrivial solution to the Pell equation. [Hints: You may use trial and error and take inspiration from your previous work. Your answer should be given in terms of n.]

Task 6 Find a pattern to the solutions you've found that allows you to make a conjecture about a case when no nontrivial solution exists. (If necessary, continue exploring!) Can you prove this conjecture?

In fact, outside the case just mentioned, the Pell equation will always have a solution, and Bhāskara II provided an effective algorithm, called the Cyclic Method, for finding it. Before diving into the Cyclic Method, however, we will consider the works of Brahmagupta, the first Indian mathematician known to attempt to solve this equation. In many cases, the tools Brahmagupta developed enable us to easily find solutions.

2 Brahmagupta

Born in 598 CE and residing in Bhillamāla (present-day Bhinmal in Rajasthan), Brahmagupta wrote the *Brāhmasphuṭasiddhānta*, a treatise on astronomy, in 628 CE at the age of 30. However, while the bulk of the treatise is on astronomy, it contains some chapters on mathematics. In fact, at the time of Brahmagupta, independent texts on mathematics were not written in India; rather, mathematical texts were incorporated as chapters into astronomical treatises.[Plofker, 2009, p. 319]

[3]Translated from the *Bījagaṇita* of Bāskara II, p.38, [Datta and Singh, 1962, Part 2, p. 166].
[4]Exclamation point added for emphasis; there will be no factorials in this project.

2.1 Principle of Composition

The first step taken by Brahmagupta toward solving the Pell equation in general is the so-called Principle of Composition. Instead of considering the Pell equation directly, Brahmagupta considers equations of the type $Nx^2+k=y^2$, which we will call auxiliary equations.[5] That is, instead of adding 1, as is done in the Pell equation, some integer k (not necessarily nonnegative) is added instead. The Principle of Composition, stated below, shows how to combine (or compose) the solutions of two auxiliary equations into a solution for a third auxiliary equation. Note that the terms *guṇaka* (multiplier) and interpolator referred to below are, respectively, N and k in the equation $Nx^2+k=y^2$.

> Of the square of an optional number multiplied by the *guṇaka* and increased or decreased by another optional number, [extract] the square root. [Proceed] twice. The product of the first roots multiplied by the *guṇaka* together with the product of the second roots will give a [fresh] second root; the sum of their cross-products will be a [fresh] first root. The [corresponding] interpolator will be equal to the product of the [previous] interpolators.[6]

Task 7 As we noted, Brahmagupta uses the Principle of Composition to combine the solutions to two auxiliary equations, in order to find a solution to a third auxiliary equation. Suppose the first two auxiliary equations are $Nx^2+k=y^2$, with solution $(x,y)=(a,b)$, and $Nx^2+l=y^2$, with solution $(x,y)=(c,d)$. Here k and l are integers. Use the source quoted to answer the following questions:

(a) What do you think Brahmagupta means by "first root" and "second root"?

(b) In terms of the variables above, what is this third (new) equation?

(c) What does the Principle of Composition give as the "first root" of the new equation, and the "second root"?

(d) See if your formulas make sense by checking them against the numerical example with auxiliary equations $3\cdot 2^2+4=4^2$ and $3\cdot 5^2+6=9^2$. What is the new equation in this case? What should its solution be according to your interpretation of the Principle of Composition? Does that solution actually work? If not, reconsider the quotation and refine your interpretation.

Task 8 Verify directly that $31\cdot 7^2+2=39^2$ and $31\cdot 5^2+9=28^2$. Use the Principle of Composition with these auxiliary equations to find two solutions to the equation $31x^2+18=y^2$.

Brahmagupta did not provide a proof for the Principle of Composition, but it can be shown by algebra:

[5] Here we are following the language established in [Datta and Singh, 1962, Part II p. 146] and [Plofker, 2007, p. 432].

[6] *Brāhmasphuṭasiddhānta* 18, 64-65; translation by Datta and Singh in [Datta and Singh, 1962, Part 2, p. 146].

Task 9 Prove the Principle of Composition. That is, prove that if $Na^2 + k = b^2$ and $Nc^2 + l = d^2$, then $N(ad + cb)^2 + kl = (Nac + bd)^2$. [Hint: It may be helpful to rewrite the original solutions as $k = b^2 - Na^2$ and $l = d^2 - Nc^2$.]

Task 10 Note that the Principle of Composition combines solutions of two different equations, $Nx^2+k = y^2$ and $Nx^2 + l = y^2$. What if the two equations were the same? Suppose $(x, y) = (a, b)$ is a solution to $Nx^2 + k = y^2$, and use this for both auxiliary equations in the Principle of Composition to arrive at a solution to a new auxiliary equation.

Task 11 The special case we discovered in Task 11 turns out to be a very important one. Since we are applying the Principle of Composition on the same equation used twice, we will refer to this as Composition of Equals.

Composition of Equals. If $(x, y) = (a, b)$ is a solution to the equation $Nx^2 + k = y^2$, then $(x, y) = (2ab, b^2 + Na^2)$ is a solution to the equation $Nx^2 + k^2 = y^2$.

Let us apply Composition of Equals to find a new solution for $8x^2 + 1 = y^2$, which is the first of the two Diophantine equations quoted at the top of page 1. You may have already found a solution to this with small numbers in your previous work. If you haven't already, find the smallest solution with $x > 0$. Show that Composition of Equals with this solution gives rise to $(x, y) = (6, 17)$ as another solution, and then verify directly that this is indeed a solution. Can you now find yet another solution to $8x^2 + 1 = y^2$?

Task 12 By applying Composition of Equals to a Pell equation that already has at least one solution, as we did in Task 10, how many solutions can be found?

Task 13 Bhāskara adds an enlightening detail in his commentary on the Principle of Composition; he writes "Again, the difference of two cross products is a lesser root. Subtract the products of the two lesser roots multiplied by the *prakṛti* from the product of the two greater roots; (the difference) will be a great root. Here also, the interpolator is the product of the two (previous) interpolators."[7] Note that *prakṛti* refers to N. Bhāskara mentions "lesser" and "greater" roots. By comparing Bhāskara's words to Brahmagupta's, determine which variables these correspond to (using the equations from the previous task). How is Bhāskara showing us how to find another solution to third equation in the principle of composition? Check that this new solution also works for the numerical example in Task 7.

Task 14 In the previous task, we find that there are actually two solutions to the resultant equation: one using addition, the other subtraction. In Composition of Equals, we find one new solution using addition. Explain why there is no need to consider subtraction for Composition of Equals.

2.2 Brahmagupta's solution of the Pell Equation

In the continuation of the excerpt in which he stated the Principle of Composition, Brahmagupta provides an additional useful fact:[8]

[7]Translated from the Bījagaṇita p. 34; by Datta and Singh in [Datta and Singh, 1962, Part 2, p. 148].
[8]Translated from the *Brāhmasphuṭasiddhānta* 18, 65 by Datta and Singh in [Datta and Singh, 1962, Part 2, p. 150].

On dividing the two roots [of a Square-nature] by the square root of its additive or subtractive, the roots for the interpolator unity [will be found].

Task 15 What is Brahmagupta saying here? What does he mean by "additive" and "subtractive", and what does he mean when he says the roots for the "interpolator unity" will be found?

Task 16 If we are given an equation $Nx^2 + k^2 = y^2$ with solution $(x, y) = (a, b)$, what equation does Brahmagupta tell us we can solve? And what does he give as its solution? What conditions are necessary for the solution to be integer-valued? [Hint to the first question: What equation is this project about?]

Task 17 We will call the rule cited above as Brahmagupta's rule, so that we may reference it later. Prove Brahmagupta's rule.

Brahmagupta was able to combine Brahmagupta's rule with the Composition of Equals to solve the Pell equation in certain cases. First, suppose we can find a solution $(x, y) = (a_0, b_0)$ to an auxiliary equation $Nx^2 + k = y^2$. Applying Composition of Equals, we obtain a solution $(x, y) = (a_1, b_1)$ to $Nx^2 + k^2 = y^2$. If k divides both a_1 and b_1, then Brahmagupta's rule provides a solution to $Nx^2 + 1 = y^2$.

Task 18 We now find another solution to $11x^2 + 1 = y^2$. Noticing that $11 - 2 = 9$, it is tempting to use the auxiliary equation $11x^2 - 2 = y^2$. Find the positive integer solution to this equation with minimal x. Use this solution with the method described above to find a solution to $11x^2 + 1 = y^2$. How does this compare to the solution(s) you found in Task 2?

2.2.1 The cases of $k = -1$ and $k = \pm 2$

While the above approach does not guarantee a solution, Brahmagupta was able to determine a solution to $Nx^2 + 1 = y^2$ when the constant k in the auxiliary equation $Nx^2 + k = y^2$ is one of ± 1, ± 2, or ± 4.

Task 19 In the case of $k = -1$, where a solution $(x, y) = (a, b)$ is found to the auxiliary equation $Nx^2 - 1 = y^2$, how does the previous discussion lead to a solution of $Nx^2 + 1 = y^2$?

Task 20 Find a solution to $50x^2 + 1 = y^2$ by using the approach you found in in the previous task.

Task 21 In the case of $k = \pm 2$, where a solution $(x, y) = (a, b)$ is found to the auxiliary equation $Nx^2 \pm 2 = y^2$, apply the Composition of Equals and Brahmagupta's rule to show that $(x, y) = \left(ab, \frac{b^2 + Na^2}{2}\right)$ is a solution to the Pell equation $Nx^2 + 1 = y^2$.

Task 22 Verify that when $k = \pm 2$, $\frac{b^2+Na^2}{2} = b^2 \mp 1$, so that the solution found in Task 21 is an integer solution.

Task 23 Find two different solutions to $2x^2 + 2 = y^2$, and apply the above process to find the corresponding solutions to $2x^2 + 1 = y^2$.

2.2.2 The case of $k = \pm 4$

On $k = 4$, Brahmagupta states,

In the case of 4 as additive the square of the second root diminished by 3, then halved and multiplied by the second root will be the [required] second root; the square of the second root diminished by unity and then divided by 2 and multiplied by the first root will be the [required] first root [for the additive unity].[9]

Task 24 What formula is Brahmagupta providing here? *(Note that your first instinct regarding where parentheses belong may be false!)* Can you determine the formula, and try it on some examples?

Task 25 The formula for $k = 4$ can be proved using the tools Brahmagupta has provided us. Can you prove it?

On $k = -4$, Brahmagupta states,

In the case of 4 as subtractive, the square of the second is increased by three and unity; half the product of these sums and that as diminished by unity [are obtained]. The latter multiplied by the first sum less unity is the [required] second root; the former multiplied by the product of the [old] roots will be first root corresponding to the [new] second root.[10]

Task 26 What formula is Brahmagupta providing here? *(Note that your first instinct regarding where parentheses belong may be false!)* Can you determine the formula, and try it on some examples?

Task 27 The formula for $k = -4$ can be proved using the tools Brahmagupta has provided us. Can you prove it?

Task 28 Show that the formulas in the cases $k = 4$ and $k = -4$ yield integer results.

[9] Translated from the *Brāhmasphuṭasiddhānta* 18, 67 by Datta and Singh in [Datta and Singh, 1962, Part 2, p. 159].
[10] Translated from the *Brāhmasphuṭasiddhānta* 18, 68 by Datta and Singh in [Datta and Singh, 1962, Part 2, p. 160].

3 The Cyclic Method

3.1 Bhāskara II

Brahmagupta's results allow us to solve the Pell equation $Nx^2+1=y^2$ in some special cases, namely, when we are able to solve the auxiliary equation $Nx^2+k=y^2$ where k is one of -1, ± 2, or ± 4. However, if we do not have a solution for the auxiliary equation we still cannot solve the original Pell equation. As such, Brahmagupta's method is incomplete.

Centuries later, a powerful method for solving the Pell equation was developed by mathematicians in India. The method is known as *cakra-vāla*, which translates into the *Cyclic Method*, referring to the cyclic nature of the steps of the method. The mathematicians credited with the Cyclic Method are Bhāskara II (born 1114 CE) and Jayadeva (flourished in the ninth century CE). Of the latter we know next to nothing. It is likely that Jayadeva first developed the Cyclic Method, which was later refined by Bhāskara II.

Bhāskara II is perhaps the most renowned Indian mathematician and astronomer. His works are impressive not only to modern historians of science but were also impressive to his contemporaries and immediate successors. In fact, the period following Bhāskara II became a didactic one, where scholars focused on explaining the works of Bhāskara II rather than original scholarship and research.

Bhāskara II came from a long line of court scholars (mostly astronomer-astrologers) in the central-western part of India. An inscription in a temple in Maharashtra, erected by Bhāskara II's grandson Caṅgadeva, records Bhāskara II's lineage for several generations before him.[11] Bhāskara II and his father Maheśvara were natives of the city Vijjaḍaviḍa between the Tapti and Godavari rivers. According to Bhāskara II, his father taught him astronomy.

As is clear, Bhāskara II possessed the advantages of a renowned scholarly lineage and a tradition of royal and noble patronage, which doubtlessly were partly responsible for the successful dissemination and lasting popularity of his works. However, unlike earlier mathematicians, Bhāskara II wrote his own concise commentary on his treatises, including sample problems for all his rules in the arithmetic and algebra texts—and supplying solutions for all sample problems. These pedagogical aids, along with the careful organization of the works and their clear exposition, made his works stand out.

Bhāskara II is the author of six works:

1. the *Līlāvatī*, a work on arithmetic and elementary mathematics;
2. the *Bījagaṇita*, a work on algebra and higher mathematics;
3. the *Siddhāntaśiromaṇi*, an influential and famous treatise on astronomy and cosmology;
4. the *Vāsanābhāṣya*, a commentary on the *Siddhāntaśiromaṇi*;
5. the *Karaṇakutūhala*, a work on astronomy; and
6. the *Vivaraṇa*, a commentary on the *Śiṣyadhīvṛddhidatantra* of Lalla (flourished in the eight century CE);

Some scholars consider Bhāskara II's two mathematical works, the *Līlāvatī* and the *Bījagaṇita*, to be parts of his *magnum opus* the *Siddhāntaśiromaṇi*. However, they are considered two independent

[11] The inscription commemorates a grant from a local ruler, on August 9, 1207, to support Caṅgadeva's founding of a school for the study of Bhāskara II's works.

texts in the scribal tradition that copied and preserved the knowledge systems of ancient and medieval India.

3.2 The Cyclic Method

The Cyclic Method, or *cakravāla* in Sanskrit, which Bhāskara II describes in the *Bījagaṇita*, is an iterative process by which one can arrive at a solution to Pell's equation for a given N, starting with a solution to $Nx^2 + k = y^2$ for *any* integer k. Datta & Singh note "the method is so called, observes Sūryadāsa, because it proceeds in a circle, the same set of operations being applied again and again in a continuous round."[Datta and Singh, 1962, Pt. 2, p. 162].

The idea is to use a process suggested by the Principle of Composition to repeatedly generate new solutions to $Nx^2 + k = y^2$ (for various k). The process will eventually (that is, in a finite number of steps) yield a solution to $Nx^2 + 1 = y^2$. The Cyclic Method will furthermore arrive at the smallest solution (that is, the solution where x is as small as possible) in the most efficient way.[12] We begin by reading through Bāskara's statement of the Cyclic Method in its entirety,[13] before taking a look at the details set by step.

Considering the lesser root, greater root and interpolator [of a square-nature] as the dividend, addend, and divisor [in a pulverizer], the [indeterminate] multiplier of [the pulverizer] should be so taken as will make the residue of the *prakṛti* diminished by the square of that multiplier or the latter minus the *prakṛti* [as the case may be] the least. That residue divided by the [original] interpolator is the interpolator [of a new auxiliary equation]; it should be reversed in sign in case of subtraction from the *prakṛti*. The quotient corresponding to that value of the multiplier is the [new] lesser root; thence the greater root. The same process should be followed repeatedly putting aside [each time] the previous roots and the interpolator. This process is called *cakravāla* [or 'The Cyclic Method']. By this method, there will appear two integral roots corresponding to an equation with ± 1, ± 2, or ± 4 as interpolator.

Task 29 Write two or three questions or comments about Bhāskara's statement of the Cyclic Method, with at least one question and at least one comment.

To aid in interpreting the rather complicated primary source, we now break Bhāskara's statement into four parts and discuss each separately.

Considering the lesser root, greater root and interpolator [of a square-nature] as the dividend, addend, and divisor [in a pulverizer], ...

[12] Efficient in that it corresponds to the shortest possible continued fractions algorithm. Selenius. p. 177

[13] Translated from Bhāskara II's *Bījagaṇita* in Datta and Singh, Part 2, pp. 162–3.

Task 30 (a) Bhāskara II's "lesser" and "greater" roots correspond to Brahmagupta's "first" and "second" roots, respectively. In modern terms, the method begins by assuming we have been given a solution $(x, y) = (a, b)$ to a square-nature equation $Nx^2 + k = y^2$. To what variables do the terms "lesser root," "greater root," and "interpolator" refer?

(b) In Indian mathematics, the term "pulverizer" (or "grinder") refers to the process of solving linear indeterminate equations[14] (i.e., linear equations with more than one solution). In these few words, Bhāskara is telling us to consider what we would now call a linear Diophantine equation in two variables: the equation $b + am = ku$, which we must solve for integers m and u, and where the given integers a, b, and k are referred to by Indian mathematicians as the dividend, addend and divisor of the equation respectively. The Indian method for solving such equations is essentially the same as we use today; that is, application of the Euclidean algorithm. Examining this Diophantine equation, why might the terms dividend, addend and divisor terms be appropriate for these given values?

In the following, we'll need to remember that *prakṛti* refers to N in square nature equations. The term "indeterminate multiplier" corresponds to m.

> ...the [indeterminate] multiplier of it should be so taken as will make the residue of the *prakṛti* diminished by the square of that multiplier or the latter minus the *prakṛti* [as the case may be] the least.

Task 31 As we noted earlier, the pulverizer $b+am = ku$ will have more than one solution, so this portion of the quote is telling us to choose m so that a certain mathematical expression is minimized. What mathematical operation does Bhāskara say to use here, and what are its arguments? (In other words, Bhāskara is telling us to calculate what mathematical expression(s) here?) Bhāskara notes two cases that must be considered, since he requires the result of the operation be positive. Can you use modern notation to write a single mathematical expression for the quantity to be minimized?

In the next portion of Bhāskara's statement of the Cyclic Method, he tells us how to find the interpolator of the next auxiliary equation. Let's call the interpolator of the first new equation k_1, and the roots of this equation a_1 and b_1.

[14] Plofker p. 134

That residue divided by the [original] interpolator is the interpolator [of a new auxiliary equation]; it should be reversed in sign in case of subtraction from the *prakṛti*.

Task 32 According to Bhāskara, what modern formula gives us the new interpolator k_1? Explain what you think his note regarding the case when "it should be reversed in sign" means.

The quotient corresponding to that value of the multiplier is the [new] lesser root; thence the greater root. The same process should be followed repeatedly putting aside [each time] the previous roots and the interpolator. This process is called *cakravāla* [or 'The Cyclic Method']. By this method, there will appear two integral roots corresponding to an equation with ± 1, ± 2, or ± 4 as interpolator.

Task 33 In saying "the quotient corresponding to that value of the multiplier", scholars have determined that Bhāskara is telling us to find the *other* indeterminate in our Diophantine equation $b + am = ku$, for a given value of the indeterminate m. What expression do we obtain from this equation for that second indeterminate? According to Bhāskara, this will be our new lesser (i.e., first) root, a_1. (Note, we can choose a_1 to be positive since we will be squaring it.)

With a_1 and k_1, we can solve $b_1^2 = Na_1^2 + k_1$ for the new greater root b_1, and we have found a new solution to a new auxiliary equation. Finally, Bhāskara's statement of the Cyclic Method ends with a handy note referencing the mathematics covered in section 2.2.

In order to derive integral roots corresponding to an equation with the additive unity from those of the equation with the interpolator ± 2 or ± 4 the Principle of Composition (should be applied).

Task 34 Here we use the Cyclic Method to solve $67x^2 + 1 = y^2$. The calculations in the first iteration are provided. Complete the rest.

1. We start by finding any value for k that makes solving the corresponding auxiliary equation easy. Notice that if $k = -3$, then $(a, b) = (1, 8)$ solves $67x^2 + k = y^2$; that is, $67 \cdot 1^2 - 3 = 8^2$.

2. We next need to solve $b + am = ku$ for m and u in integers to determine what value to chose for m. The equation becomes $8 + m = -3u$ when substituting in the values of a, b, and k. Using the usual method involving the Euclidean algorithm, we get that $m = 1+3v$, where v is any integer, will make $\frac{a+bm}{k}$ equal to an integer. (Note, one can also apply trial an error with such small numbers.)
3. If we choose $m = 7$ (that is, take $v = 2$) then $\frac{b+a\cdot m}{k} = \frac{8+1\cdot 7}{-3} = -5$ is an integer and $|7^2 - 67|$ is minimal.
4. Compute the necessary quotients for the new interpolator and new lesser root to obtain $k_1 = \frac{(49-67)}{-3} = 6$ and $a_1 = \frac{8+1\cdot 7}{|-3|} = 5$. To find the new greater root b_1, we use $b_1^2 = Na_1^2 + k_1 = 67(5)^2 + 1 = 1681$, so $b_1 = 41$. That is, we have that $(x, y) = (5, 41)$ solves $67x^2 + 6 = y^2$.
5. Repeat the procedure with this new solution. That is, find the integer solution to $41+5m = 6u$ that minimizes $N - m^2$ and use this to compute the new interpolator k_2, lesser root a_2 and greater root b_2. You should obtain that $(x, y) = (11, 90)$ solves $67x^2 - 7 = y^2$.
6. Repeat the procedure to obtain that $(x, y) = (27, 221)$ solves $67x^2 - 2 = y^2$.
7. At this point you can use the shortcut provided by Brahmagupta to see that $(5967, 48842)$ solves $67x^2 + 1 = y^2$; or, if you do not use this shortcut, the Cyclic Method will find the final solution after 7 iterations.

Task 35 Use the Cyclic Method to solve $31x^2 + 1 = y^2$. What are some good auxiliary equations to begin with? How many iterations does your chosen auxiliary equation take to solve the Pell equation?

3.2.1 Discussion of the Cyclic Method

The description of the method provided by Bhāskara in the above translation does not give much hint as to why it works. We now have a look at a rationale for the claim that the expressions provided yield new solutions to new auxiliary equations.

Task 36 Suppose we have a non-square positive integer N and integers $a > 0$, $b > 0$, and k such that $Na^2 + k = b^2$, so that we have a solution to one auxiliary equation.

(a) Show that for any integer $m > 0$, $(x, y) = (1, m)$ solves $Nx^2 + k = y^2$ when $k = m^2 - N$. This is a second auxiliary equation, which is true for any m.

(b) Apply the Principle of Composition to combine the two above auxiliary equations. What is the new equation?

(c) Divide each term through by k^2 to obtain a new auxiliary equation. If you applied the Principle of Composition correctly, your new lesser root and interpolator should match the expressions Bhāskara provided in the algorithm. Do they? Conveniently this process also provides an expression for the new greater root as well.

Task 37 (a) It turns out in the above work that if $\frac{b+am}{k}$ and $\frac{m^2-N}{k}$ are integers, then so is $\frac{bm+Na}{k}$. To prove this, simplify the expression

$$\frac{(m(am + b) - a(m^2 - N))}{k}.$$

(b) In fact it is claimed that if $\frac{am+b}{k}$ is an integer, then so are *both* $\frac{m^2-N}{k}$ and $\frac{bm+Na}{k}$. Can you prove this?

3.3 Conclusion

The above discussion provides some justification for why the Cyclic Method provides a new solution to a new auxiliary equation, but there is much yet to be discovered. Why is the Cyclic Method guaranteed to eventually find a solution, and why the smallest? Why is this algorithm efficient?

The reader is encouraged to explore the translations of the works of Brahmagupta and Bhāskara II by Datta & Singh and Plofker. Additionally, Selenius provides some interesting history on the European interactions with, and misconceptions of, the Cyclic Method, in addition to describing the connections between this method and the modern approach of continued fractions.

References

Bibhutibhusan Datta and Avadhesh Narayan Singh. *History of Hindu mathematics: A source book. Part I: Numerical notation and arithmetic. Part II: Algebra.* Asia Publishing House, Bombay-Calcutta-New Delhi–Madras-London-New York, 1962.

Kim Plofker. Mathematics in India. In *The mathematics of Egypt, Mesopotamia, China, India, and Islam*, pages 385–514. Princeton Univ. Press, Princeton, NJ, 2007.

Kim Plofker. *Mathematics in India.* Princeton University Press, Princeton, NJ, 2009. ISBN 978-0-691-12067-6.

Clas-Olof Selenius. Rationale of the chakravāla process of Jayadeva and Bhāskara II. *Historia Math.*, 2:167–184, 1975. ISSN 0315-0860. doi: 10.1016/0315-0860(75)90143-3. URL `http://dx.doi.org/10.1016/0315-0860(75)90143-3`.

Notes to Instructors

This module is intended to replace a standard lecture on the Pell equation. It follows historical developments in India, and does not apply the more modern approach of continued fractions. Students are introduced to the methods presented by Brahmagupta and Bhāskara II, and the module culminates in discussion of *cakravāla* – the Cyclic Method for solving the Pell equation. Students completing this module should develop a strong familiarity with the Pell equation, as well as an appreciation for the Indian approach to mathematics in the time of Brahmagupta and Bhāskara II.

Only minimal background in number theory is necessary for students to take part in this project. However, the Cyclic Method does involve solving linear Diophantine equations in two variables. Students familiar with such equations and Euclid's algorithm will be most comfortable, but the relevant tasks use numbers small enough that they should also be solvable through trial and error. Most proofs amount to calculations, some of which may be lengthy. Students who are generally comfortable with the basic tools of algebra, and who are willing to work in a spirit of exploration and discovery should be able to complete most of the core tasks. Students who have had some introduction to proof should be well-positioned for the proofs that arise.

The primary source translations for this project are taken from Datta & Singh *History of Hindu Mathematics* pp. 141-173, and there is much more to be explored in this text. In particular, there is much additional commentary that the instructor and students may find interesting or useful. For further exploration, the works of Kim Plofker in *Mathematics of Egypt, Mesopotamia, China, India, and Islam* by Katz and *Mathematics in India* by Plofker provide an interesting additional perspective on this content. Additionally, in *Rationale of the Chakravala process of Jayadeva and Bhāskara II* Selenius provides a thorough investigation of the Cyclic Method, including connections to modern continued fraction approaches.

Completion Time and Sample Schedule

This module is intended to be used as a project that will take roughly one week of class time. However, because there is so much approachable and accessible mathematics, the authors erred on the side of providing too much, as opposed to too little content, and the amount of time required to complete all of the tasks would be significantly longer than one week. Moreover, a few of the tasks will be more time-consuming and some will be rather challenging for most students. The instructor who wishes to cover the module in a limited time should consider choosing a path through the material that omits some of the less essential content. The task commentary below provides some guidance on this, and a sample schedule with selected tasks is also provided. An alternative is to assign some of the early content for students to work on prior to discussing the material in class. The tasks in the introduction should work well for this, and the students might discuss their work and solutions to these tasks in small groups as the initial in-class portion of the module. Additionally, some tasks could be assigned as homework to be completed between class periods, or after the last class period in which the module is discussed. A possible selection of tasks to be completed in about one week's worth of work, considered here as three 50 minute meetings, is:

Prior to Meeting 1 Assign the Introduction and Tasks 1,2,and 4-6 for students to attempt independently.

Meeting 1 Section 2.1 covering Tasks 7, 8, and 10-12, with Task 9, time permitting.

Meeting 2 Section 2.2 and Tasks 15-17, 19, 21, and 22; with Tasks 18, 20, and 22, time permitting. Read Section 3.1 prior to Meeting 3.

Meeting 3 Section 3.2 and Tasks 29-33, with Tasks 34-35, time permitting.

Commentary on Tasks

Those tasks marked with a [C] below are considered core to the module, and those marked with an [F] are considered "further exploration". Some tasks marked as core, particularly examples, may still be omitted by those instructors wishing to complete the module more quickly, but many core tasks develop results that are used laster in the module.

The intention is that much of the knowledge should be discovered by the students through interaction with the primary source, with guidance from the module. Thus, in a few cases, students are asked to attempt to construct the modern interpretation of the work by reading the primary source and completing related tasks. In order to ensure that progress is made and the students have the correct formulas for key processes, the solutions are in some cases stated in later tasks or discussion. For the most part, these are noted in the commentary below, and instructors who do not wish for such solutions to appear later in the module are encouraged to obtain the LaTeX source from the authors and modify it as needed. It may be helpful to hold class-wide discussion between some tasks or otherwise ensure students have completed tasks successfully, and have obtained the appropriate understanding or formulas, before moving on to later tasks.

1. Introduction The Introduction provides a small amount of history on the Pell equation, and the material should be approachable to students without much guidance. Instructors wishing to cover the majority of the module content within one week may consider assigning the problems in this section as homework prior to the first class period during which the module is discussed.

[C] Tasks 1-4 These initial tasks set up consideration of the Pell equation and mostly involve exploration. They should be accessible for most students to attempt individually. Task 1 provides a "first-blush" reading of Indian mathematical writing, as translated by Datta & Singh. Tasks 2-4 ask students to explore Pell's equation, as well as define the trivial solution.

[F] Tasks 5-6 are not likely to be very challenging and provide interesting observations. In particular, Task 5 shows that values of N always yield nontrivial solutions, and Task 6 encourages students to discover and prove that when N is a perfect square, no (nontrivial) solution exists. However, these tasks are not essential for understanding the rest of the module, and instructors concerned about time may wish to omit them.

2. Brahmagupta

2.1 Principle of Composition The Principle of Composition is a key process, which is necessary for utilizing the Cyclic method.

[C] Tasks 7-8 Task 7 provides the first real student interpretation of the primary source, and might be challenging for some students. However, Instructors concerned about time may still wish to ask students to attempt it individually prior to the first class meeting; ensuring the students are primed for the initial class discussion on this task. It may be helpful for students to work on this in groups, and to have a wider discussion afterward, to ensure students have grasped the Principle of Composition. Task 8 then provides additional exercise with the Principle of Composition.

[F] Task 9 This task asks the students to prove the Principle of Composition, but also provides the answer to 7(c). Instructors who do not wish solutions to appear later in the module are encouraged to modify the LaTeX source to remove the reference to the solution. The calculation is pleasant and not difficult, but instructors wishing to save time might choose not to assign this.

[C] Tasks 10-12 These tasks ask students to discover and prove Composition of Equals. Task 12 provides the solution to Task 10, so instructors may wish to obtain the LaTeX source and remove the statement of Composition of Equals from Task 12.

[F] Tasks 13-14 These extra tasks provide additional exploration of Indian writing, observing that the principle of composition also works with subtraction, and considering what happens when we attempt the Composition of Equals with subtraction instead of addition. Instructors concerned about time may wish to omit them.

2.2 Brahmagupta's Method This section deals with Brahmagupta's approach to solving the Pell equation, which works in some cases. While the math is very pleasing and provides a nice shortcut for the Cyclic Method, the material is not necessary for the Cyclic Method; so instructors with very limited time may wish to skip this section entirely if they are content to ignore the shortcut. Moreover, even instructors who are not time-limited may wish to only assign the first one or two subsections, as the third subsection is expected to be rather challenging for any reader.

[C] Tasks 15-17 This provides students another opportunity to interpret indian mathematics and attempt develop the modern approach based on the primary source. Note that the paragraph following Task 17 explains the process Brahmagupta used, and in doing so, provides a partial solution to Task 16.

[C] Task 18 Provides an example of this approach in action.

[C] Tasks 19-23 These tasks are relatively straightforward, and show how a solution to any of $Nx - 1 = y^2$ or $Nx \pm 2 = y^2$ will yield a solution to the Pell equation.

[F] Tasks 24-28 These tasks provide additional opportunities for students to attempt to interpret Indian writing and translate it into modern mathematics. However, as it is highly likely that both students and instructors will struggle to interpret the translations of Brahmagupta's writings in the case of $k = 4$ and $k = -4$, this may be a sequence of tasks that is reserved only for further exploration. Note that the pages of Datta & Singh referenced with each quote provide very helpful interpretation leading to the full formulas and proofs.

3. The Cyclic Method

[C] Tasks 29-33 These tasks take the student through the challenging process of interpreting Datta & Singh's translation of Bhāskara's statement of the Cyclic Method. Unlike previous tasks in which students are expected to determine a key formula, the module does not provide these formulas later, so discussion may be necessary to ensure the students have successfully obtained the necessary information on the Cyclic Method to complete the tasks.

- Task 29 is intended to reinforce that students should thoroughly consider Bhāskara's statement of the Cyclic Method by asking them to come up with at least one question and at least one comment.
- Task 30 first reminds the student that when $Na^2 + k = b^2$, a is the lesser root, b is the greater root, and k is the interpolator. The second question is open-ended, asking students to consider why the terms in the pulverizer $b + am = ku$ are so named; the intention here is to encourage the students to consider the source.
- Tasks 31 and 32 have the students interpret the somewhat more straightforward next portion of the source. They are asked to come up with the fact that they need to find m making $\frac{am+b}{k}$ an integer (i.e., a solution to the Diophantine equation $uk = b + am$), which also satisfies the condition that $|m^2 - N|$ is minimal. Then the new interpolator will be $(m^2 - N)/k$.
- In Task 33, it would be quite difficult for students to determine the actual expression needed without some additional help provided. Bhāskara is stating that the expression $u = \frac{am+b}{k}$ (or $\frac{am+b}{|k|}$) will provide the new lesser root a_1.

[C] **Task 34** Includes a number of calculations which the students would benefit from working. Given the small numbers in many cases, it may be preferable for the students to solve the calculations with trial and error.

[C] **Task 35** The authors used $x = 1$ and $k = 5$ here.

[F] **Tasks 36 and 37** provide some of the fascinating justification for the use of the Cyclic Method. Applying the Principle of Composition to the auxiliary equations

$$Na^2 + k = b^2 \qquad \text{and} \qquad N(1)^2 + (m^2 - N) = m^2$$

yields the new equation

$$N(am + b)^2 + k(m^2 - N) = (Na + bm)^2$$

and dividing this through by k^2 yields

$$N\left(\frac{am + b}{|k|}\right)^2 + \frac{(m^2 - N)}{k} = \left(\frac{Na + bm}{|k|}\right)^2$$

which correspond to our new lesser root, interpolator, and greater root.

The first part of Task 37 follows Plofker's exposition in [Plofker, 2007, p. 474].

LaTeX code of this entire PSP is available from the authors by request to facilitate preparation of in-class task sheets based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Acknowledgments

The development of this project has been partially supported by the National Science Foundation's Improving Undergraduate STEM Education Program under Grants No. 1523494, 1523561, 1523747, 1523753, 1523898, 1524065, and 1524098. Any opinions, findings, and conclusions or recommendations expressed in this project are those of the author and do not necessarily reflect the views of the National Science Foundation.

The authors would like to extend their gratitude to the readers Janet Barnett and Dominic Klyve for their many great suggestions in improving this module.

For more information about TRIUMPHS, visit `http://webpages.ursinus.edu/nscoville/TRIUMPHS.html`.