Investigations Into d'Alembert's Definition of Limit

David Ruch
MSU Denver

February 17, 2017

1 Introduction

The modern definition of a limit evolved over many decades. One of the earliest attempts at a precise definition is credited to Jean-Baptiste le Rond d'Alembert (1717 – 1783), a French mathematician, philosopher and physicist. Among his many accomplishments, he argued in a 1754 article of the Encyclopédie that the theory of limits should be put on a firm foundation.

2 D'Alembert's Limit Definition

By 1754 mathematical techniques were quite advanced. d'Alembert won a 1747 prize for his work in partial differential equations, but became embroiled in arguments with Euler and others over methodology and foundational issues. Perhaps these squabbles led to his interest in clearing up the foundations of limits and convergence.

Here is d'Alembert's limit definition from the Encyclopédie [1]:

Limit. One says that one quantity is the limit of another quantity, when the second can approach nearer to the first than any given quantity, however small; nevertheless, without the quantity which is approaching ever being able to surpass the quantity which it approaches; so that the difference between a quantity and its limit is absolutely inassignable.

For example, consider two polygons, one inscribed and the other circumscribed about a circle, it is evident that one can increase the sides however one wishes; and in this case, each polygon will more and more closely approach the circumference of the circle, the contour of the inscribed polygon increasing, and that of the circumscribed polygon decreasing; but the perimeter or the contour of the first will never surpass the length of the circumference, and that of the second will never be smaller than the same circumference; the circumference of the circle is thus the limit of the increase of the first polygon, and the decrease of the second.

[Claim] 1st. If two values are the limit of the same quantity, these two quantities are equal to one another.

[Claim] 2nd. Let A $\times$ B be the product of two quantities A, B. We suppose that C is the limit of the quantity A, and D the limit of the quantity B; I say that C $\times$ D, the product of the limits, will necessarily be the limit of A $\times$ B, the product of the two quantities A, B.

....

Properly speaking, the limit never coincides, or is never equal to the quantity of which it is the limit; but it is approached more and more, and can differ by as little as one wants. The circle, for example, is the limit of the inscribed and circumscribed polygons; because it never merges with them, though they can approach it ad infinitum.

Note that this definition is lacking in precise, modern mathematical notation. Also observe that the polygon/circle example is for the limit of a *sequence*. Here is a standard first year calculus book definition of sequence:

First Year Calculus Definition. A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

Let's examine some examples.

Exercise 1 *Use modern subscript notation for an appropriate sequence to rewrite d'Alembert's inscribed polygon $\to$ circle limit example. Assume for simplicity that the inscribed polygons are regular with n sides centered at the circle's center. These polygons have perimeter formula*

$$perimeter = 2n \cdot radius \cdot \sin(\pi/n).$$

As a bonus, derive the perimeter formula, and use Calculus to confirm this limit.

Exercise 2 *Consider d'Alembert's polygon $\to$ circle limit example and his definition. For "given quantity" 0.1 and a circle of radius 1, how many sides for the inscribed polygon are needed to guarantee the "second can approach nearer to the first than" given quantity 0.1? Technology will be helpful! How many sides are needed for given quantity 0.01?*

Exercise 3 *Consider the sequence $\{a_n\}$ with $a_n = \dfrac{n}{2n+1}$. Find its limit by any means. For "given quantity" 0.01, suppose we want a_n and its limit to "differ by as little as" 0.01. What is "sufficiently large" for n to guarantee that a_n and its limit differ by 0.01 or less?*

Now let us generalize this example a bit, replacing "given quantity" 0.01 by a generic value c.

Exercise 4 *For sequence $\left\{\dfrac{n}{2n+1}\right\}$, let c be an arbitrary small positive number. Suppose we want a_n and its limit to differ by less than c. In terms of c, what is "sufficiently large" for n?*

Exercise 5 *Look closely at d'Alembert's phrase "Properly speaking, the limit never coincides, or is never equal to the quantity of which it is the limit" and notice that it does not appear in the First Year Calculus definition. Can you think of a simple convergent sequence that violates this requirement of d'Alembert's limit definition?*

Exercise 6 *Consider d'Alembert's phrase "without the quantity which is approaching ever being able to surpass the quantity which it approaches" and notice that it does not appear in the First Year Calculus definition. Find a simple convergent sequence that violates this requirement of d'Alembert's limit definition.*

Exercise 7 *Use modern notation to help rewrite d'Alembert's limit definition for sequences using quantifiers "for all", and "there exists". The First Year Calculus Definition and a graph of the sequence $\{a_n\}$ should be helpful! You should introduce a variable c to bound the distance between the quantities, and another variable M to measure n being "sufficiently large".*

As we have seen, d'Alembert's limit definition doesn't fully apply to some types of sequences studied by today's mathematicians. The First Year Calculus definition avoids them, but is too vague for actually constructing complex proofs.

Exercise 8 *Use quantifiers "for all", and "there exists" to rewrite the First Year Calculus limit definition for sequences.*

Exercise 9 *Use your definition from Exercise 8 to prove that sequence* $\left\{\dfrac{n}{2n+1}\right\}$ *converges.*

3 Limit Properties

D'Alembert makes a couple assertions about limit properties. One quality of a good modern definition is that it should be useful in constructing proofs of a concept's properties. Let's investigate d'Alembert's first uniqueness claim and his proof.

Exercise 10 *Write d'Alembert's Claim 1st for sequences in modern notation.*

Here is his proof of uniqueness.

Given Z and X, limits of the same quantity Y, I say that X = Z, because if there is some difference between them, such as V, it would be X = Z ± V. By hypothesis, the quantity Y can approach Z as closely as we desire. That is to say that the difference between Y and X can be as small as wished. Therefore, since Z differs from X by the quantity V, it follows that Y cannot approach Z any closer than the quantity V, and consequently, that of Z is not the limit of Y, which is contrary to the hypothesis.

Exercise 11 *Rewrite this uniqueness proof using your modern definition from Exercise 8.*

The proof of d'Alembert's Claim 2nd is harder and d'Alembert does not give one in his article.

Exercise 12 *Write d'Alembert's Claim 2nd for sequences in modern notation.*

The next exercise investigates a proof for a special case of the second claim on the product of sequences to give you some appreciation of the challenges. It may give you ideas for writing a general proof!

Exercise 13 *Suppose you know a sequence* $\{a_n\}$ *is within* 0.01 *of its limit* $C = 5$ *if* n *is larger than some integer* $N_1 = 47$. *Also suppose you know a sequence* $\{b_n\}$ *is within* 0.01 *of its limit* $D = 3$ *if* n *is larger than some integer* $N_2 = 92$. *Can you determine how far you must go with sequence* $\{a_n b_n\}$ *to get close to the product of limits* $CD = 15$? *How little difference between* $a_n b_n$ *and* CD *can you* ***guarantee*** *if you go out far enough?*

References

[1] Alembert, Jean le Rond d' and La Chapelle, Jean-Baptiste de, Limite. In *Dictionnaire encyclopedique des mathematiques*, Paris: Hotel del Thou, 1789. Translation: http://quod.lib.umich.edu/d/did/did2222.0000.123/–limit?rgn=main;view=fulltext;q1=Mathematics.