Abel and Cauchy on on a Rigorous Approach to Infinite Series

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1 Introduction

Infinite series were of fundamental importance in the development of calculus by Newton, Euler and other mathematicians during the late 1600's and 1700's. Questions of rigor and convergence were of secondary importance in these times, but things began to change in the early 1800's. When the brilliant young mathematician Niels Abel moved to Paris in 1826 at age 24, he was aware of many paradoxes with infinite series and wanted big changes. Indeed, in a letter to his friend Holmboe he wrote that "I shall devote all my efforts to bring light into the immense obscurity that today reigns in Analysis. It so lacks any plan or system, that one is really astonished that there are so many people who devote themselves to it – and, still worse, it is absolutely devoid of rigor."

Abel was born and raised in Norway, far from the centers of mathematical activity in his time. His work was largely unrecognized during his lifetime through a series of misfortunes. Nevertheless he managed to get to Paris and attend lectures by mathematical stars such as Adrien-Marie Legendre (1752-1833) and Augustin Louis Cauchy (1789-1857). Abel was particularly taken by Cauchy and his efforts to introduce rigor into analysis, writing to Holmboe that Cauchy "is the only man who knows how mathematics should be treated. What he does is excellent".

In this project, we will read excerpts from 1820's work by Abel and Cauchy as we rigorously develop infinite series and examine some of the tough infinite series problems of their day.

Of course, the study of infinite series go back to antiquity. The Greek mathematician Archimedes used infinite series to help calculate the area under the arc of a parabola, and geometric series such as

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$$

were well known and used extensively in the development of calculus. The notion of divergent series was not clearly understood and somewhat controversial in the times leading up to Abel and Cauchy. For a simple example, here are two groupings and "sums" of the series $1-1+1-1+1-1+\cdots$

$$\begin{aligned}(1-1)+(1-1)+(1-1)+\cdots &= 0+0+0+\cdots=0\\ 1+(-1+1)+(-1+1)+(-1+1)+\cdots &= 1+0+0+0+\cdots=1\end{aligned}$$

which suggests that $1=0$! Some mathematicians in the 1700's suggested that the sum of this series should split the difference and be $1/2$, and others argued that the series did not converge and had no sum. We next read a short excerpt from Abel in another 1826 letter home to Holmboe from Paris. Abel references a much more sophisticated series example (1) that Euler had discussed in 1750. This series is quite important historically, as Joseph Fourier used it in his development of Fourier series and his model of heat transfer during the early 1800's.

Divergent series are in their entirety an invention of the devil and it is a disgrace to base the slightest demonstration on them. You can take out whatever you want when you use them, and they are what has produced so many failures and paradoxes. ... The following example shows how one can err. One can rigorously demonstrate that

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots \tag{1}$$

for all values of x smaller than π. It seems that consequently the same formula must be true for $x = \pi$; but this will give

$$\frac{\pi}{2} = \sin \pi - \frac{1}{2}\sin 2\pi + \frac{1}{3}\sin 3\pi - \text{etc.} \quad = 0.$$

On can find innumerable examples of this kind.

In general the theory of infinite series, up to the present, is very poorly established. One performs every kind of operation on infinite series, as if they were finite, but is it permissible? Never at all. Where has it been demonstrated that one can obtain the derivative of an infinite series by taking the derivative of each term? It is easy to cite examples where this is not right ... By taking derivatives [of (1)], one has

$$\frac{1}{2} = \cos x - \cos 2x + \cos 3x - \text{ etc.} \tag{2}$$

A completely false result, because this series is divergent.

Exercise 1 *Find a couple values of x less than π in absolute value which, substituted into (2), produce strange results.*

Exercise 2 *Use a CAS to graph $y = x/2$ and $y = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots + \sin(10x)/10$ together for $-2\pi \leq x \leq 2\pi$. What do you observe at $x = \pm\pi$? If you were to plot even more terms from the infinite series, the "wiggly" parts of the curve would grow even closer to straight lines. What do you think of Abel's comments about this series and its derivative?*

We won't try to tackle during this project all the issues Abel raises with this example. However, we can see why mathematicians of his time were struggling with infinite series at the same time that they were amazed by their power!

Another series that bothered Abel was the binomial theorem. Newton had discovered that the standard *finite* binomial expansion $(1+x)^m$ for positive integer m could be be generalized to an *infinite* series for *non-integer* values of m (equation (3) below), and he was able to use this series to produce a number of new results. While Newton thought this series converged only for $|x| < 1$, no one had produced a convergence proof that fully convinced Abel, and he set out to do so in an 1826 paper. here is an excerpt from the paper's introduction.

Investigations on the series:

$$1+\frac{m}{1}x+\frac{m\,(m-1)}{1\cdot 2}x^{2}+\frac{m\,(m-1)\,(m-3)}{1\cdot 2\cdot 3}x^{3}+\ldots\text{ etc.}\tag{3}$$

1.

If one subjects to a more precise examination the reasoning that one generally uses when dealing with infinite series, one will find that, taken as a whole, it is not very satisfactory, and that consequently the number of theorems concerning infinite series that may be considered rigorously based is very limited. On normally applies the operations of analysis to infinite series as if the series were finite. This does not seem to me permissible without special proof. ...

One of the most remarkable series in algebraic analysis is (3). When m is a positive whole number, one knows that the sum of this series, which in this case is finite, may be expressed as $(1+x)^{m}$. When m is not a whole number, the series becomes infinite, and it will be convergent or divergent, according to different values that one gives to m and x. In this case one writes in the same way

$$(1+x)^{m}=1+\frac{m}{1}x+\frac{m\,(m-1)}{1\cdot 2}x^{2}+\ldots\text{ etc.;}\tag{4}$$

... One assumes that numerical equality will always hold when the series is convergent; but this is what until now has not yet been proved. No one has even examined all the cases where the series is convergent ...

The aim of this memoir is to try to fill a gap with the complete solution of the following problem:

"Find the sum of the series (3) for all real or imaginary values of x and m for which the series is convergent."

2.

We are first going to establish some necessary theorems on series. The excellent work of Cauchy “Cours d’analyse de l’école polytechnique”, which must be read by every analyst who loves rigor in mathematical research, will serve as our guide.

Exercise 3 *To get a sense of the binomial series equality for whole numbers m, verify (4) with $m = 3$.*

Exercise 4 *To get a sense of Abel’s concerns about the binomial series with non-integer m, use a CAS to graph $y=\sqrt{x}$ and the first five terms of series (3) together for $-1 \le x \le 1$. Newton claimed this series converges for $|x| < 1$. Does your plot suggest this is correct?*

We will now take Abel’s advice and read Cauchy on infinite series in the next part of this project. Then we will return to Section 2 of Abel’s paper, where he develops some new infinite series results and tackles a controversial theorem of Cauchy. This work is important in its own right, independent of the binomial theorem, and will serve as the primary focus of our project.

2 Cauchy on Infinite Series

Augustin Louis Cauchy was a renowned figure in 1826 Paris. After graduating from college in 1810, he published much impressive mathematics and became a professor at École Polytechnique in Paris, a top university in all of Europe. Cauchy loved pure mathematics and was convinced of the need for a rigorous approach to analysis. He wrote his 1821 *Cours d'Analyse* as a text for his teaching, and he constructed it with his philosophy of rigor. Abel had read this text before coming to Paris, and was inspired to use its methods and spirit in his own research. One radical aspect of Cauchy's book was his study of convergence of series without necessarily finding the sum of the series, which was quite a departure from the eighteenth century tradition of focusing on series sums with little attention to convergence issues.

We now start reading Chapter 6 on infinite series of Cauchy's *Cours d'Analyse.*

6.1 General Considerations on series.

We call a series an indefinite sequence of quantities,

$$u_0, u_1, u_2, u_3, \dots,$$

which follow from one to another according to a determined law. These quantities themselves are the various terms of the series under consideration. Let

$$s_n = u_0 + u_1 + u_2 + \dots + u_{n-1} \tag{5}$$

be the sum of the first n terms, where n denotes any integer number. If, for ever increasing values of n, the sum s_n indefinitely approaches a certain limit s, the series is said to be convergent, and the limit in question is called the sum of the series. On the contrary, if the sum s_n does not approach any fixed limit as n increases indefinitely, the series is divergent, and does not have a sum. In either case, the term which corresponds to the index n, that is u_n, is what we call the general term. For the series to be completely determined, it is enough that we give this general term as a function of the index n.

One of the simplest series is the geometric progression

$$1, x, x^2, x^3, \dots,$$

which has x^n for its general term, that is to say the nth power of the quantity x. If we form the sum of the first n terms of this series, then we find

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{1}{1-x} - \frac{x^n}{1-x} \tag{6}$$

As the values of n increase, the numerical value of the fraction $\frac{x^n}{1-x}$ converges towards the limit zero, or increases beyond all limits, according to whether we suppose that the numerical value of x is less than or greater than 1. Under the first hypothesis, we ought to conclude that the progression

$$1, x, x^2, x^3, \dots,$$

is a convergent series which has $\frac{1}{1-x}$ as its sum, whereas, under the second hypothesis, the same progression is a divergent series which does not have a sum.

Cauchy's terminology and notation is essentially what we use today. His definition that a divergent sum does not have a sum was not universally accepted in his day, but is now standard. The s_n expression defined in (5) above is nowadays called the *nth partial sum.* Notice that we can form a sequence of *partial sums* $\{s_n\}$, and convergence of the **series** $u_0, u_1 u_2, u_3, \ldots$ is equivalent to convergence of the **sequence** $\{s_n\}$.

Exercise 5 *Rewrite Cauchy's definition of series convergence and the sum s in terms of the sequence of partial sums $\{s_n\}$.*

Exercise 6 *Verify the algebra in (6). This is often called the finite geometric series formula.*

Exercise 7 *When Cauchy discusses the convergence of the geometric series, note his language "whether we suppose that the numerical value of x is less than or greater than 1." Explain why the geometric series diverges for $x = -2$. In modern terminology, what do you think Cauchy means by the "numerical value of x"? Cauchy frequently uses the term "numerical value" with this meaning.*

Notice that Cauchy does not use sigma summation notation $\sum_{k=0}^{\infty} u_k$ in this 1821 work, nor does Abel use it in his 1826 paper. The sigma summation did not come into common use until later in the 1800's. It is now conventional to denote both the infinite series and its sum using the symbols $\sum u_i$ or $\sum_{i=0}^{\infty} u_i$.

Exercise 8 *Rewrite Cauchy's proof for the geometric series when $|x| < 1$ using modern notation and results from a modern treatment of sequences.*

Exercise 9 *Suppose a series $\sum a_i$ converges, and $c \in \mathbb{R}$. Prove the series $\sum (ca_i)$ converges with sum $c \sum a_i$. If $\sum a_i$ diverges, what can you say about $\sum (ca_i)$?*

Exercise 10 *Use the results above to determine convergence and sum, or divergence, of*

$$6 - \frac{2}{3} + \frac{2}{27} - \frac{2}{324} + \cdots \quad \textit{and} \quad \sum_{k=2}^{\infty} \frac{5^{k-1}}{4^{k+1}}$$

Let's return to Cauchy.

(Section 6.1 continued)
Following the principles established above, in order that the series

$$u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots \tag{7}$$

be convergent, it is necessary and it suffices that increasing values of n make the sum

$$s_n = u_0 + u_1 + u_2 + \ldots + u_{n-1}$$

converge indefinitely towards a fixed limit s. In other words, it is necessary and it suffices that, for infinitely large values of the number n, the sums

$$s_n, s_{n+1}, s_{n+2}, \ldots$$

differ from the limit s, and consequently from one another, by infinitely small quantities. Moreover, the successive differences between the first sum s_n and each of the following sums are determined, respectively, by the equations

$$\begin{aligned} s_{n+1} - s_n &= u_n \\ s_{n+2} - s_n &= u_n + u_{n+1} \\ s_{n+3} - s_n &= u_n + u_{n+1} + u_{n+2} \end{aligned}$$

....

Hence, in order for series (7) to be convergent, it is first of all necessary that the general term u_n decrease indefinitely as n increases. But this condition does not suffice, and it is also necessary that, for increasing values of n, the different sums,

$$\begin{aligned} &u_n + u_{n+1} \\ &u_n + u_{n+1} + u_{n+2} \end{aligned}$$

.........................,

that is to say, the sums of as many of the quantities

$$u_n, u_{n+1} u_{n+2}, \ldots,$$

as we may wish, beginning with the first one, eventually constantly assume numerical values less than any assignable limit. Conversely, whenever these various conditions are fulfilled, the convergence of the series is guaranteed.

Exercises!

Exercise 11 *Rewrite Cauchy's sentence beginning with "In other words" with modern $\epsilon - N$ terminology.*

Cauchy's statements that "in order for series (7) to be convergent, it is first of all necessary that the general term u_n decrease indefinitely as n increases. But this condition does not suffice" are worth a clarification, a proof and an example.

Exercise 12 *First, clarify what Cauchy means by "decrease indefinitely". Second, write the necessary condition as a theorem, and give a proof using Cauchy's equation $s_{n+1} - s_n = u_n$ and modern sequence limit laws.*

Exercise 13 *Find an example showing that the condition "u_n decrease indefinitely as n increases" is not sufficient to guarantee series convergence.*

Exercise 14 *Consider the statement "the sums of as many of the quantities*

$$u_n, u_{n+1}u_{n+2}, \ldots,$$

as we may wish, beginning with the first one, eventually constantly assume numerical values less than any assignable limit." Using finite sums, convert this statement into modern $\epsilon - N$ terminology. What is this saying about the sequence $\{s_n\}$, in modern terminology?

Exercise 15 *Consider the statement "Conversely, whenever these various conditions are fulfilled, the convergence of the series is guaranteed." What modern theorem about sequences justifies this statement?*

It is interesting that Cauchy, and many of his contemporaries, thought this last necessary and sufficient condition for convergence of a series was obvious and did not need a proof. As we shall see, Cauchy and Abel use this criterion, nowadays named after Cauchy, to prove some convergence results.

Exercise 16 *Rewrite this new "Cauchy criterion" for series convergence in modern terminology.*

Exercise 17 *Show that if an infinite series $\sum u_n$ is convergent, then there exists a $B \geq 0$ such that*

$$\sup \left\{ \left| \sum_{k=m}^{m+n} u_k \right| : m, n \in \mathbb{N} \right\} \leq B$$

for all m, n. This result will come in handy when we read Abel.

Back to Cauchy

(Section 6.1 continued)
Let us take, for example, the geometric progression

$$1, x, x^2, x^3, ... \tag{8}$$

If the numerical value of x is greater than 1, that of the general term x_n increases indefinitely with n, and this remark alone suffices to establish the divergence of the series. The series is still divergent if we let $x = \pm 1$, because the numerical value of the general term x_n, which is 1, does not decrease indefinitely for increasing values of n. However, if the numerical value of x is less than 1, then the sums of any number of terms of the series, beginning with x_n, namely:

$$\begin{aligned} & x^n, \\ x^n + x^{n+1} &= x^n \frac{1-x^2}{1-x}, \\ x^n + x^{n+1} + x^{n+2} &= x^n \frac{1-x^3}{1-x}, \\ & \ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

are all contained between the limits

$$x^n \quad \text{and} \quad \frac{1-x^n}{1-x},$$

each of which becomes infinitely small for infinitely large values of n. Consequently, the series is convergent, as we already knew.

As a second example, let us take the numerical series

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots \tag{9}$$

The general term of this series, namely $\frac{1}{n+1}$, decreases indefinitely as n increases. Nevertheless, the series is not convergent, because the sum of the terms from $\frac{1}{n+1}$ up to $\frac{1}{2n}$ inclusive, namely

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$$

is always greater than the product

$$n\frac{1}{2n} = 2$$

whatever the value of n. As a consequence, this sum does not decrease indefinitely with increasing values of n, as would be the case if the series were convergent. Let us add that, if we denote the sum of the first n terms of series (9) by s_n and the highest power of 2 bounded by $n+1$ by $2m$, then we have

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1} \\ &> 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots \\ &\quad + \left(\frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m}\right) \end{aligned}$$

and, a fortiori,

$$s_n > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + \frac{m}{2}$$

We conclude from this that the sum s_n increases indefinitely with the integer number m and consequently with n, which is a new proof of the divergence of the series.

Exercise 18 *In this part of Section 6.1, Cauchy gives a new proof that the geometric series is convergent, using the new Cauchy criterion for series convergence that you put into modern form in Exercise 16. Rewrite his proof using modern $\epsilon - N$ terminology.*

Exercise 19 *In the second example on the famous harmonic series, Cauchy again uses the new Cauchy criterion, but now for divergence. Rewrite his proof using modern $\epsilon - N$ terminology.*

Let's go back to Cauchy for another important example and a controversial theorem.

(Cauchy Section 6.1 continued)

Let us further consider the numerical series

$$1, \frac{1}{1}, \frac{1}{1\cdot 2}, \frac{1}{1\cdot 2\cdot 3}, ..., \frac{1}{1\cdot 2\cdot 3 \ldots n} \cdots \tag{10}$$

The terms of this series with index greater than n, namely

$$\frac{1}{1\cdot 2\cdot 3\ldots n}, \frac{1}{1\cdot 2\cdot 3\ldots n\,(n+1)}, \frac{1}{1\cdot 2\cdot 3\ldots n\,(n+1)\,(n+2)}, \cdots ,$$

are, respectively, less than the corresponding terms of the geometric progression

$$\frac{1}{1\cdot 2\cdot 3\ldots n}, \frac{1}{1\cdot 2\cdot 3\ldots n}\frac{1}{n}, \frac{1}{1\cdot 2\cdot 3\ldots n}\frac{1}{n^2}, \cdots ,$$

As a consequence, the sum of however many of the initial terms as we may wish is always less than the sum of the corresponding terms of the geometric progression, which is a convergent series, and so a fortiori, it is less than the sum of this series, which is to say

$$\frac{1}{1\cdot 2\cdot 3\ldots n}\frac{1}{1-\frac{1}{n}} = \frac{1}{1\cdot 2\cdot 3\ldots (n-1)}\frac{1}{n-1}\,. \tag{11}$$

Because this last sum decreases indefinitely as n increases, it follows that series (10) is itself convergent. It is conventional to denote the sum of this series by the letter e. By adding together the first n terms, we obtain an approximate value of the number e,

$$1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + ... + \frac{1}{1\cdot 2\cdot 3\ldots (n-1)}\,.$$

According to what we have just said, the error made will be smaller than the product of the nth term by $\frac{1}{n-1}$. Therefore, for example, if we let $n = 11$, we find as the approximate value of e

$$e = 2.7182818..., \tag{12}$$

and the error made in this case is less than the product of the fraction $\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}$ by $\frac{1}{10}$, that is $\frac{1}{36{,}288{,}000}$, so that it does not affect the seventh decimal place.

The number e, determined as we have just said, is often used in the summation of series and in the infinitesimal Calculus. Logarithms taken in the system with this number as its base are called Napierian, for Napier, the inventor of logarithms, or hyperbolic, because they measure the various parts of the area between the equilateral hyperbola and its asymptotes.[1]

In general, we denote the sum of a convergent series by the sum of the first terms, followed by an ellipsis. Thus, when the series

$$u_0, u_1 u_2, u_3, \ldots ,$$

is convergent, the sum of this series is denoted

$$u_0 + u_1 + u_2 + u_3 + \ldots$$

[1] Cauchy means the area under the curve $y = 1/x$, using standard terminology of his time.

By virtue of this convention, the value of the number e is determined by the equation

$$1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\ldots, \tag{13}$$

and, if one considers the geometric progression

$$1, x, x^2, x^3, ...,$$

we have, for numerical values of x less than 1,

$$1+x+x^2+x^3+...=\frac{1}{1-x}\,. \tag{14}$$

Denoting the sum of the convergent series

$$u_0, u_1 u_2, u_3, \ldots,$$

by s and the sum of the first n terms by s_n, we have

$$\begin{aligned} s &= u_0+u_1+u_2+...+u_{n-1}+u_n+u_{n+1}+\ldots \\ &= s_n+u_n+u_{n+1}+\ldots, \end{aligned}$$

and, as a consequence,

$$s-s_n=u_n+u_{n+1}+\ldots$$

From this last equation, it follows that the quantities

$$u_n, u_{n+1}u_{n+2}, \ldots$$

form a new convergent series, the sum of which is equal to $s-s_n$. If we represent this sum by r_n, we have

$$s=s_n+r_n\,,$$

and r_n is called the remainder of series (7) beginning from the nth term.

Suppose the terms of series (7) involve some variable x. If the series is convergent and its various terms are continuous functions of x in a neighborhood of some particular value of this variable, then

$$s_n, r_n \text{ and } s$$

are also three functions of the variable x, the first of which is obviously continuous with respect to x in a neighborhood of the particular value in question. Given this, let us consider the increments in these three functions when we increase x by an infinitely small quantity α. For all possible values of n, the increment in s_n is an infinitely small quantity. The increment of r_n, as well as r_n itself, becomes infinitely small for very large values of n. Consequently, the increment in the function s must be infinitely small. From this remark, we immediately deduce the following proposition:

Theorem I — *When the various terms of series (7) are functions of the same variable x, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighborhood of this particular value.*

By virtue of this theorem, the sum of series (8) must be a continuous function of the variable x between the limits $x = -1$ and $x = 1$, as we may verify by considering the values of s given by the equation

$$s = \frac{1}{1-x}.$$

Euler derived this series expression (10) for e by 1743 from his definition $e = \lim\,(1 + 1/n)^n$ using an infinitesimal argument with the binomial theorem. Cauchy is *proving* convergence of the series using an important idea: he compares the e series to a *larger* geometric series, which he shows is convergent.

Exercise 20 *Fill the algebraic details of Cauchy's argument between (10) and (11) that, for arbitrary n, the remainder r_n for the e series is less than $\frac{1}{1\cdot2\cdot3\ldots(n-1)}\frac{1}{n-1}$.*

Exercise 21 *Give a modern $\epsilon - N$ proof that the series with remainder $r_n = \frac{1}{1\cdot2\cdot3\ldots(n-1)}\frac{1}{n-1}$ converges.*

Cauchy's idea of comparing series to prove convergence is worth formalizing as a theorem.

Exercise 22 *Find a reasonable Hypothesis B so the following theorem is true, and give a modern $\epsilon - N$ proof of it.*

Theorem 23 *Suppose that $a_n \le b_n$ for all n and Hypothesis B. If $\sum b_n$ converges, then $\sum a_n$ converges.*

Exercise 24 *Use Cauchy's ideas and a calculator/CAS to find a decimal approximation s_n to e that you can prove is accurate to 12 decimal places.*

After his discussion of the e series, Cauchy develops notation of the remainder r_n and outlines a proof for his Theorem I. As we shall see in the next section of this project, Abel found a counterexample to this theorem in 1826. He did not identify the error in Cauchy's proof, but he did prove a correct variation of this theorem. Other major mathematicians worked hard during the mid-1800's to prove other variations on Theorem I. This indicates the subtlety of Cauchy's error and the difficulty involved in fixing it! For now, let's address some more modest points.

Exercise 25 *If his proof outline for Theorem I, Cauchy states that s_n is "obviously continuous with respect to x in a neighborhood of the particular value in question." Explain why this statement is correct.*

Exercise 26 *Certainly Cauchy is correct that* $s = \frac{1}{1-x}$ *is continuous for* $x \in (-1, 1)$ *. For* $x = 0.98$*, what would be a valid neighborhood of continuity of* $s = \frac{1}{1-x}$*? For an* x*-increment of* $\alpha = 0.01$*, use a calculator/CAS to find the corresponding* r_5 *and* r_{10} *increments. For an* x*-increment of* $\alpha = 0.019$*, find the corresponding* r_5 *and* r_{10} *increments. What can you say about the dependence of* r_n *on* n *and* x*?*

3 Abel's 1826 Paper

After urging us to read Cauchy in the introduction to his 1826 paper, Abel gives Cauchy's definition of series convergence for a series

$$u_0 + u_1 + u_2 + \cdots + u_m \text{ etc.},$$

and states that "in any convergent series whatever, the general term v_m always approaches zero," which we explored in Exercise 12. Abel then writes a footnote "For brevity, in this memoir we will understand by ω a quantity which may be smaller than any given quantity, however small." What exactly Abel means by this ω will become apparent in his proof of Theorem IV below.

We now return to Abel's paper, with his first two theorems.

***Theorem* I.** If one denotes a series of positive quantities by $\rho_0, \rho_1, \rho_2 \ldots$, and the quotient $\frac{\rho_{m+1}}{\rho_m}$, for ever increasing values of m, approaches a limit α greater than 1, then the series

$$\varepsilon_0\rho_0 + \varepsilon_1\rho_1 + \varepsilon_2\rho_2 + \cdots + \varepsilon_m\rho_m + \cdots ,$$

where ε_m is a quantity which, for ever increasing values of m, does not approach zero, will be necessarily divergent.

***Theorem* II.** If in a series of positive quantities $\rho_0 + \rho_1 + \rho_2 + \cdots + \rho_m + \cdots$ the quotient $\frac{\rho_{m+1}}{\rho_m}$, for ever increasing values of m, approaches a limit α smaller than 1, then the series

$$\varepsilon_0\rho_0 + \varepsilon_1\rho_1 + \varepsilon_2\rho_2 + \cdots + \varepsilon_m\rho_m + \cdots ,$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2$ etc. are quantities that are never greater than one, will be necessarily convergent.

Indeed, by assumption, one may always take m large enough that $\rho_{m+1} < \alpha\rho_m, \rho_{m+2} < \alpha\rho_{m+1}, \cdots \rho_{m+n} < \alpha\rho_{m+n-1}$. It follows from there that $\rho_{m+k} < \alpha^k\rho_m$, and consequently

$$\rho_m + \rho_{m+1} + \cdots + \rho_{m+n} < \rho_m \left(1 + \alpha + \alpha^2 + \cdots + \alpha^n\right) < \frac{\rho_m}{1-\alpha} ,$$

therefore, for all the more reason,

$$\varepsilon_m \rho_m + \varepsilon_{m+1}\rho_{m+1} + \cdots + \varepsilon_{m+n}\rho_{m+n} < \frac{\rho_m}{1-\alpha}.$$

Now, since $\rho_{m+k} < \alpha^k \rho_m$ and $\alpha < 1$, it is clear that ρ_{m+1} and consequently the sum

$$\varepsilon_m \rho_m + \varepsilon_{m+1}\rho_{m+1} + \cdots + \varepsilon_{m+n}\rho_{m+n}$$

will approach zero.

The above series is therefore convergent.

Exercise 27 *Abel does not give a proof of his first theorem, perhaps thinking it obvious. Give a modern proof of Theorem I.*

Exercise 28 *Interpret Theorems I and II in the special case where $\varepsilon_m = 1$ for all m. What name did we give these results in an Introductory Calculus course? Cauchy actually gives these results where $\varepsilon_m = 1$ in his book; Abel is generalizing it for his needs later in his 1826 paper.*

Exercise 29 *In his proof of Theorem II, Abel states that "one may always take m large enough that $\rho_{m+1} < \alpha\rho_m$". Give a modern $\epsilon - N$ justification of this statement, and clarify what values of n, k are valid in rest of the proof.*

Exercise 30 *To understand Abel's Theorem II statement, we need to remember that in 1826 we don't take the statement "where ε_0 etc. are quantities that are never greater than one" literally. To see this, set $\rho_m = 1/2^m$ and $\varepsilon_m = -2^m j$ and show that the series $\sum \rho_m \varepsilon_m$ diverges.*

Exercise 31 *Based on the exercise above, let's interpret Abel's Theorem II hypotheses about the ε_k as "the quantities $|e_k|$ are never greater than one". With this adjustment, use Abel's proof method to give a modern $\epsilon - N$ proof of Theorem II. Observe that Abel is using the Cauchy criterion!*

Exercise 32 *In the 1700's, Euler had derived the power series*

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{15}$$

Use Abel's theorems to prove this power series converges for any $|x| < 1$ and diverges for any $|x| > 1$.

Exercise 33 *For $x = 1$, the power series for $\ln(x+1)$ was known in Abel's day to converge: Leibniz gave a proof in the 1600's. Use this fact and the exercise above to show that Cauchy's Theorem I is not correct as stated.*

Exercise 34 *Generalize Theorem II to a theorem with hypothesis "$\{\epsilon_k\}$ bounded" in place of "$|e_k|$ are never greater than one". Prove your claim.*

Exercise 35 *Apply these theorems to determine convergence or divergence of the series*

$$\sum \frac{3k-1}{2^k(k+1)} \quad \text{and} \quad \sum \frac{k+1+(-1^k)}{k+1}\frac{3^{k+1}}{2^{k-1}} \quad \text{and} \quad \sum_{k=0}^{\infty} x^k/k! \text{ for arbitrary } x \geq 0.$$

Now back to Abel for his third and fourth theorems, where he proves a variant of Cauchy's flawed Theorem I.

(Abel Section 2 continued)

***Theorem* III.** On denoting by $t_0, t_1, t_2, \ldots t_m, \ldots$ a series of any quantities whatever, if $p_m = t_0 + t_1 + t_2 + \ldots t_m$ is always less than a determined quantity[2] d, one will have

$$r = \varepsilon_0 t_0 + \varepsilon_1 t_1 + \varepsilon_2 t_2 + \cdots + \varepsilon_m t_m < d\varepsilon_0$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2 \ldots$ denote positive decreasing quantities.

Indeed, one has

$$t_0 = p_0, \quad t_1 = p_1 - p_0, \quad t_2 = p_2 - p_1, \quad \text{etc.}$$

therefore

$$r = \varepsilon_0 p_0 + \varepsilon_1 (p_1 - p_0) + \varepsilon_2 (p_2 - p_1) + \cdots + \varepsilon_m (p_m - p_{m-1}) \tag{16}$$

or rather

$$r = p_0 (\varepsilon_0 - \varepsilon_1) + p_1 (\varepsilon_1 - \varepsilon_2) + \cdots + p_m (\varepsilon_m - \varepsilon_{m-1}) + p_m \varepsilon_m \tag{17}$$

But $\varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2$ are positive, so the quantity r will clearly be less than $d\varepsilon_0$.

Definition. A function $f(x)$ will be said to be a continuous function of x between the limits $x = a$ and $x = b$, if for any value of x contained between these limits, the quantity $f(x - \beta)$, for ever decreasing values of β, approaches the limit $f(x)$.

***Theorem* IV.** If the series

$$f(\alpha) = v_0 + v_1\alpha + v_2\alpha^2 + \cdots + v_m\alpha^m + \ldots$$

converges for a certain value δ of α, it will also converge for every value smaller than δ and, for this kind of series, for ever decreasing values of β, the function $f(\alpha - \beta)$ will approach the limit $f(\alpha)$, assuming that α is equal to or less than δ.

[2] Abel uses δ where we have d, but he uses δ for a different purpose in Theorem IV, so we've switched symbols to avoid confusion for the reader.

Suppose

$$\begin{array}{rcl} v_0 + v_1\alpha + v_2\alpha^2 + \cdots + v_{m-1}\alpha^{m-1} & = & \varphi(\alpha), \\ v_m\alpha^m + v_{m+1}\alpha^{m+1} + v_{m+2}\alpha^{m+2} + \text{ etc. } \cdots & = & \psi(\alpha), \end{array}$$

so

$$\psi(\alpha) = \left(\frac{\alpha}{\delta}\right)^m v_m\delta^m + \left(\frac{\alpha}{\delta}\right)^{m+1} v_{m+1}\delta^{m+1} + \text{etc.},$$

therefore, from Theorem III, $\psi(\alpha) < \left(\frac{\alpha}{\delta}\right)^m p$ where p denotes the greatest of the quantities $v_m\delta^m$, $v_m\delta^m + v_{m+1}\delta^{m+1}$, $v_m\delta^m + v_{m+1}\delta^{m+1} + v_{m+2}\delta^{m+2}$ etc. Therefore for every value of α, equal to or less than δ, one may take m large enough that one will have

$$\psi(\alpha) = \omega.$$

Now $f(\alpha) = \varphi(\alpha) + \psi(\alpha)$, so $f(\alpha) - f(\alpha - \beta) = \varphi(\alpha) - \varphi(\alpha - \beta) + \omega$. Further, $\varphi(\alpha)$ is a polynomial is α, so one may take β small enough that

$$\varphi(\alpha) - \varphi(\alpha - \beta) = \omega;$$

so also one has in the same way

$$f(\alpha) - f(\alpha - \beta) = \omega,$$

which it was required to prove.

Let's first start with Abel's Theorem III. It is correct as it stands, but we need to tweak it a bit for a modern proof of Theorem IV.

Exercise 36 *Justify the algebraic rearrangement of terms in r, between (16) and (17). This clever result is so useful that some books reference it as Abel's Lemma.*

Exercise 37 *Justify Abel's claim in his Theorem III proof that "quantity r will clearly be less than $d\varepsilon_0$".*

Exercise 38 *While reading the proof of Theorem IV, you may have noticed that Abel will use Theorem III to help show that $\psi(\alpha) = \omega$, i.e. $\psi(\alpha)$ "may be smaller than any given quantity, however small." In modern terms, this means showing that $|\psi(\alpha)| < \epsilon$ or something of the sort. This means that we need to adjust the conclusion of Theorem III to be $|r| < d\varepsilon_0$. This stronger conclusion naturally requires a stronger hypothesis, namely that we assume $|p_m| < d$ for all m. State and prove an "absolute value" version of Theorem III with the stronger hypothesis and conclusion. Abel's beautiful rearrangement of the terms in r will still be crucial for your proof!*

Exercise 39 *Abel's definition of continuity is essentially the same as Cauchy's. To make this definition consistent with our modern definition, what do you think he means by "for ever decreasing values of β"?*

Now let us turn to the statement of Theorem IV. First notice that Abel is making two claims. First, he claims that the infinite series $f(\alpha)$ will "converge for every value smaller than δ". From our previous readings, we suspect that this is not to be taken literally. For now, let's take his meaning on α to be: for every α value, $0 \leq \alpha < \delta$ where δ is positive.

Abel's second claim is that "the quantity $f(x - \beta)$, for ever decreasing values of β, approaches the limit $f(x)$." In the language of continuity, what is this saying?

Exercise 40 *Explain how Abel's Theorem IV is a variant of Cauchy's Theorem I. Be sure to compare and contrast both the hypotheses and the conclusions.*

Exercise 41 *As a modest application, consider the power series (15), which is known to converge for $x = 1$. What does Abel's Theorem IV tell us for other x values?*

Finally, let's examine the proof of Theorem IV, and think about how we can adjust it to make a modern proof.

First observe that Abel uses the symbol ω four times. Recalling his footnote "we will understand by ω a quantity which may be smaller than any given quantity, however small.", we can see that he might not mean for these ω's to be literally identical.

Second, Abel lets p denote the greatest of an infinite number of quantities. From a modern point of view, how would you criticize this? We will need to adjust the definition of p for modern readers, and justify this new p.

Third, note that the r in Theorem III involves a finite sum, but Abel uses Theorem III in his Theorem IV proof with an *infinite* sum, the remainder term $\psi(\alpha)$. This needs some clarification!

Exercise 42 *Putting aside for now the finite/infinite sum clash noted above, what expressions in the Theorem IV proof play the roles of t_i and ϵ_k in the Theorem III statement?*

To clarify and modernize Abel's proof of his first claim that the infinite series $f(\alpha)$ will converge, let's use the Cauchy criterion and introduce the notation

$$\begin{aligned}\varphi_m(\alpha) &= v_0 + v_1\alpha + v_2\alpha^2 + \cdots + v_{m-1}\alpha^{m-1} \\ \psi_{m,n}(\alpha) &= v_m\alpha^m + v_{m+1}\alpha^{m+1} + v_{m+2}\alpha^{m+2} + v_{m+n}\alpha^{m+n}\end{aligned}$$

We will use the infinite term remainder $\psi(\alpha)$ as Abel does, but take care to remember it depends on m. When this is particularly important, we can use $\psi_m(\alpha)$ in place of $\psi(\alpha)$ for emphasis.

Exercise 43 *For a fixed positive $\alpha < \delta$, use Abel's ideas and the Cauchy criterion to give a modern $\epsilon - N$ proof that the sequence of partial sums $\{\varphi_m(\alpha)\}$ converges. You will need to use your "absolute value" version of Theorem III from Exercise 38. You will also need to change Abel's definition of p in your proof and justify it. Exercise 17 will be helpful.*

Exercise 43 gives us a modern proof that the infinite series $f(\alpha)$ in Theorem IV will converge. Now we tackle the second part of Abel's proof, that f is continuous at each α less than δ. In modern terms, given $\epsilon > 0$ we need to find a $\delta' > 0$ so that $|\beta| < \delta'$ implies that $|f(\alpha - \beta) - f(\alpha)| < \epsilon$.

Exercise 44 *Use the triangle inequality to show that*

$$|f(\alpha) - f(\alpha - \beta)| \leq |\varphi(\alpha) - \varphi(\alpha - \beta)| + |\psi(\alpha)| + |\psi(\alpha - \beta)|$$

We need to convert Abel's ω statements into appropriate $\epsilon - \delta'$ statements. We need an ϵ bound on $\psi(\alpha)$ and another bound on $\varphi(\alpha) - \varphi(\alpha - \beta)$, as Abel mentions directly. Less obviously, we need a bound on $\psi(\alpha - \beta)$, which Abel absorbs into one of his ω's in the claim $f(\alpha) - f(\alpha - \beta) = \varphi(\alpha) - \varphi(\alpha - \beta) + \omega$.

It turns out that the trickiest of these three bounds is for $\psi(\alpha - \beta)$, because we need the bound to work *for all* $|\beta| < \delta'$, not just a single β. To get this bound, look at your definitions of $\psi_{m,n}(\alpha)$ and p in Exercise 43. Notice p does not depend on m, n or α! In fact observe that

$$\left|\psi_{m,n}(\alpha)\right| \leq \left(\frac{\alpha}{\delta}\right)^m p \tag{18}$$

for all n and for any $\alpha < \delta$.

Exercise 45 *For a given $\epsilon > 0$ and $\alpha < \delta$, find $N \in \mathbb{N}$ and $\delta' > 0$ so that $|\beta| < \delta'$ and $m \geq N$ imply that*

$$|\psi_m(\alpha - \beta)| < \frac{\epsilon}{3}$$

The bound (18) should be helpful!

Exercise 46 *Using your results from the past few exercises, rewrite Abel's proof that the infinite series $f(\alpha)$ will converge for $0 \leq \alpha < \delta$ using a modern $\epsilon - N$ argument.*

Exercise 47 *Adjust the Theorem IV statement to be valid for any α, $|\alpha| < \delta$, and prove your claim.*

4 Conclusion

We examined a simple counterexample of Cauchy's flawed Theorem I in Exercise 33. In his paper, Abel mentions a more complex series. Here is his footnote from the paper:

In the work by M. Cauchy one will find the following theorem: "When the various terms of series $u_0 + u_1 + u_2 + \cdots$ are functions of the same variable x, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighborhood of this particular value."

But it seems to me that this theorem admits of exceptions. For example the series

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$

is discontinuous for any value $(2m+1)\pi$, where m is a whole number. There are, as we know, many series of this kind.

We saw in the project introduction that Abel put this same series, which gave him a number of headaches, into his 1826 letter. Part of the problem was that mathematicians had not yet worked out the issues involved in an infinite series involving a variable x converging for a *single* x, versus converging for a *set* of x values. Moreover, it turns out that conditions for convergence of power series are quite a bit different than conditions for convergence of a series of trig functions. These challenges would keep mathematicians busy in the decades after Cauchy's *Cours d'analyse* and Abel's 1826 paper.

Regarding the binomial series, Abel went on in his 1826 paper to prove a number of rigorous convergence results for complex numbers x, which is outside the scope of this project.

Sadly, Abel would only have a few more years to work on mathematics, for he contracted tuberculosis while on his Paris visit, and died in 1829 at the age of 27. Nevertheless, he did an amazing amount of first class mathematics in his short lifetime and has been much celebrated for it.

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