Solving First-Order Linear Differential Equations: Gottfried Leibniz' "Intuition and Check" Method

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1 Introduction

In 1926, British mathematician E. L. Ince (1891–1941) described the typical evolution of solution techniques from calculus (and differential equations and science in general).[1]

> The early history of the infinitesimal calculus abounds in instances of problems solved through the agency of what were virtually differential equations; it is even true to say that the problem of integration, which may be regarded as the solution of the simplest of all types of differential equations, was a practical problem even in the middle of the sixteenth century. Particular cases of the inverse problem of tangents, that is the problem of determining a curve whose tangents are subjected to a particular law, were successfully dealt with before the invention of the calculus.
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> But the historical value of a science depends not upon the number of particular phenomena it can present but rather upon the power it has of coordinating diverse facts and subjecting them to one simple code. [Ince, 1926]

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[1]Ince himself is part of at least one such story within differential equations. He developed the so called *Ince Equation* (in about 1923),

$$(1 + a\cos(2t))y''(t) + (b\sin(2t))y'(t) + (\lambda + d\cos(2t))y(t) = 0,$$

which generalized at least two other well known equations from about 1868 and 1914, respectively. Letting $a = b = 0$ and $d = -2q$, we obtain *Mathieu's equation* (which model elliptical drumheads),

$$y''(t) + (\lambda - 2q\cos(2t))y(t) = 0,$$

and letting $a = 0, b = -4q$, and $d = 4q(\nu - 1)$, we obtain the *Whittaker-Hill equation* (with applications to lunar stability and quantum mechanics)

$$y''(t) - 4q(\sin(2t))y'(t) + (\lambda + 4q(\nu - 1)\cos(2t))y(t) = 0.$$

Ince's equation then is itself a special case of *generalized Ince equations* (studied in [Moussa, 2014])

$$(1 + \epsilon A(t))y''(t) + \epsilon B(t)y'(t) + (\lambda + \epsilon D(t))y(t) = 0.$$

This is exactly the evolution of solution methods for first-order linear ordinary differential equations. First, particular problems were solved with "one-off" methods that didn't have general applications beyond that specific problem. But then those results were combined and generalized until a unified theory developed.

Task 1 In the above passage, Ince made a connection between "the solution of the simplest of all types of differential equations" and "the problem of determining a curve whose tangents are subjected to a particular law." Connect these two statements. If the differential equation is

$$\frac{dy}{dx} = f(x, y),$$

then what are the "curve," the "'tangents," and the "'particular law"?

Task 2 Recall that non-homogenous first-order linear ordinary differential equations have the following form

$$p(x)\frac{dy}{dx} + q(x)y = f(x), \tag{1}$$

or if made monic

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2}$$

Explain how to make Equation (1) monic like Equation (2). In particular, why can we assume that $p(x)$ isn't identically zero? Write $P(x)$ and $Q(x)$ in terms of $p(x), q(x)$ and $f(x)$.

The theme of this project is the first "one-off" method for equations like those in Task 2, due to Gottfried Leibniz (1646-1716). As time progressed, solutions to differential equations came from more general "coordinated" techniques such as variation of parameters and exact differential equations.[2]

2 Leibniz' Check

On November 27, 1694, Gottfried Leibniz[3] wrote a letter to his friend the Marquis de l'Hôpital[4] (1661-1704), which is contained in the 1850 collection of Leibniz' works edited by Carl Immanuel Gerhardt (1816-1899). It contained a method for solving non-homogenous first-order linear differential equations.

The reader should be aware of two notations that appear in the original letter, [Leibniz, 1694]. Firstly, $dy : dx$ or $dp : p$ simply means $\frac{dy}{dx}$ or $\frac{dp}{p}$, similar to how we use the colon for expressing ratios and proportions today. Secondly, the symbol $\int\overline{mpdx}$ may look like a square root symbol but is

[2]The stories of these more general methods can be found in the two other projects of this "Solving First-Order Linear Differential Equations" series, which continue to follow the historical trail by examining works by Johann Bernoulli (1667-1748) and Leonhard Euler (1707-1783), respectively. Each of the three projects in the series can be completed individually or in any combination with the others. They are available at `https://digitalcommons.ursinus.edu/triumphs_differ/`.

[3]Leibniz was a German mathematician and philosopher who created (probably independently) the Calculus along with the notation that we currently use.

[4]Guillaume Francois Antoine, Marquis de l'Hôpital was a French mathematician credited with the first textbook on differential calculus.

actually two different symbols; the integral $\int$ and the overline $\overline{mp\,dx}$. The overline acts as parentheses indicating what (today we would say) is the integrand. So $\int\overline{mp\,dx} = \int(mp\,dx) = \int mp\,dx$.

... Let $m + ny + dy : dx = 0$, where m and n signify rational or irrational formulas which depend only on the indeterminate x; [then] I say that one can resolve it generally as $\int\overline{mp\,dx} + py = 0$, *I suppose that* $\int\overline{dp : p} = \int\overline{n\,dx}$. *For by finding differences, it becomes* $mp\,dx + y\,dp + p\,dy = 0$, *but* $dp = pn\,dx$, *whence it becomes* $mp\,dx + npy\,dx + p\,dy = 0$ *or* $m\,dx + ny\,dx + dy = 0$, *just as had been desired.*[5]

When reading the above passage, we find that Leibniz was working backwards. As is so often the case, *finding* the solution to a differential equation, or any problem for that matter, is much harder than *checking* that something is a solution. In this passage, Leibniz did the second. He asserted that $\int\overline{mp\,dx} + py = 0$ is a solution to $m + ny + dy : dx = 0$ if we were to define the function p by the equation $\int\overline{dp : p} = \int\overline{n\,dx}$.

Task 3

(a) Explain in your own words how Leibniz went from $\int\overline{mp\,dx} + py = 0$ to $mp\,dx + y\,dp + p\,dy = 0$.

(b) Explain in your own words how Leibniz went from $\int\overline{dp : p} = \int\overline{n\,dx}$ to $dp = pn\,dx$.

(c) Explain in your own words how Leibniz combined (a) and (b) above to obtain $m\,dx + ny\,dx + dy = 0$.

(d) Leibniz concluded by saying "just as had been desired."
Why exactly is this the "desired" result?

We also see that $\int\overline{mp\,dx} + py = 0$ is an *implicit* solution to the differential equation. In general, when a solution technique returns an *implicit* solution it will be impossible to solve for y to make it an *explicit* solution. Luckily, this is not the case here.

Task 4 Turn the Leibniz implicit solution into an explicit one by solving for y.

Leibniz knew that his technique was an extension over what was known previously. Perhaps the very first differential equation ever written was (essentially) a first-order linear differential equation! In a 1638 letter to French philosopher and scientist René Descartes (1596-1650), French jurist and mathematician Florimond de Beaune (1601-1652) asked for a geometric solution to an equation that today we would write as

$$\frac{dy}{dx} = \frac{\alpha}{(y - x)} \tag{3}$$

which is not linear, but can be made linear following Task 5 [Lenoir, 1979].

[5] Leibniz translations by Danny Otero of Xavier University, 2020.

Task 5 Consider Equation (3)

(a) Switch the variables x and y.

(b) Solve for $\frac{dy}{dx}$.

(c) Compare this to Equations (1) and (2). What is $p(x), q(x), f(x)$ (respectively $P(x), Q(x)$)?

Allowing m and n to be rational or irrational functions of x was certainly an improvement over restricting them to be the values found in Task 5. But Leibniz was also well aware that his improvement was only one step towards more universal theories. The following statement immediately proceeded his technique.

> I believe that with proper applications we may finally come to the inverse of tangents; I have made some beginnings which seem all the more considerable as they encompass these [results] in fairly general terms and can be extended further...

3 Examples

Task 6 We will work through Leibniz' technique, using

$$x\frac{dy}{dx} + y = 3x^2 \tag{4}$$

as an example.

(a) Rewrite Equation (4) in the form that Leibniz used to begin his process.[6] What are the functions m and n?

(b) Leibniz then defined a new function p that satisfies the condition

$$\frac{dp}{p} = n\,dx.$$

Using n from part (a), solve for p.

(c) With these functions, use Leibniz' method to verify that

$$\int \overline{mp\,dx} + py = 0$$

solves the original differential equation for the form of the equation from part (a).

(d) At this point we know m, n, and p so the only unknown is y. Solve for y and show it solves Equation (4).

[6]Notice that Leibniz started with a monic equation, so if your given equation isn't monic, you'll need to make it so.

Task 7 Solve

$$\frac{dy}{dx} - y = xe^x$$

using the above method.[7]

4 Leibniz' Intuition

Similar "tricks" to solve specific differential equations proliferated in the literature for decades, and as in the case of Leibniz, they often appeared to come from nowhere. In his work on the history of differential equations, Dick Jardine has noted that mathematicians of the day spent hours and hours of practice to gain the intuition to create those methods [Jardine, 2011]. It is that same intuition that eventually allowed them to organize similar tricks into a general theory.

> Students initially are bewildered at how anyone "observed" or "noted" such relationships. My best explanation is that Leibniz, Bernoulli, and Euler spent many hours determining those and many other useful results with the calculus. Because of their effort, they developed useful mathematical intuition about such relationships.

Jardine concluded by then stating, "With similar effort, our students can obtain similar intuition." Perhaps you won't quite develop Leibniz' intuition if you put in Leibniz' effort, but everyone can develop intuition about what integrals might use integration by parts, which proofs might use contradiction, or even what trick would allow you to solve a first-order linear non-homogenous differential equation.

It is likely that your ODE class utilizes an integrating factor μ to solve first-order linear differential equations

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad\qquad \mu = e^{\int P(x)dx}. \tag{5}$$

While μ was not derived by Leibniz (remember that he technically only checked an answer), it is interesting that his p function is our modern μ![8]

$$m + ny + dy : dx = 0 \qquad\qquad \int \frac{dp}{p} = \int n dx. \tag{6}$$

Task 8 Show that μ from Equation (5) is the same as p from Equation (6).

We may not know exactly where Leibniz' trick came from, but we know where it ended up ... in your texts!

[7] This example is not historically accurate as Leibniz did not deal with functions of the form e^x.

[8] It should be noted that this equivalency requires that the given differential equation is made monic, as that is the form that Leibniz starts from.

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Notes to Instructors

This set of notes accompanies the mini-Primary Source Project "Solving Linear First-Order Differential Equations: Gottfried Leibniz' "Intuition and Check" Method" written as part of the TRIUMPHS project. (See end of notes for details about TRIUMPHS).

PSP Content: Topics and Goals

This mini-Primary Source Project (mini-PSP) is one of a set of three mini-PSPs that share the name "Solving Linear First Order Differential Equations," designed to show three solutions to non-homogenous first-order linear differential equations, each from a different context. Recall that a non-homogenous first-order linear differential equation has the form

$$a(x)\frac{dy}{dx} + p(x)y = q(x).$$

- The mini-PSP subtitled "Gottfried Leibniz' "Intuition and Check" Method" explains how in 1694 Leibniz solved these equations using one-off method applicable only to this specific problem. Strictly speaking, Leibniz didn't solve the equation, but asserted a solution and then showed it worked. Part of his proposed solution will be familiar to the students: it is the standard integrating factor method we teach today.

- The mini-PSP subtitled "Johann Bernoulli's (Almost) Variation of Parameters Method" explains how in 1697, Bernoulli provided a method for solving Bernoulli differential equations that reduces to variation of parameters when applied to first-order linear equations. This was decades before Lagrange received credit for the technique. Again, part of Bernoulli's solution will be the standard integrating factor.

- The mini-PSP subtitled "Leonard Euler's Integrating Factor Method" explains how in 1763, Euler solved these equations as a special case of exact differential equations by finding an integrating factor. His integrating factor is the same as the one as the students would have seen. This mini-PSP is a bit longer than the others, and may require a bit more time or pre-preparation.

All three of these mini-PSPs are designed for use in an Ordinary Differential Equations course but can be used in three different ways. They work best after at least presenting the standard integrating method of solution found in modern textbooks.

- Since the *type* of equation (first-order linear) has been introduced, all three projects can be immediately done. This would require the instructor to "preview" techniques that will be introduced more fully later. While this is somewhat awkward, it does mimic how these techniques were actually developed.

- The "Gottfried Leibniz' "Intuition and Check" Method" project can be done immediately, but the other projects done after the respective *method* of solution (variation of parameters, exactness) are first introduced. Showing how those techniques can solve first-order linear differential equations makes a great first example of each technique. This is typically the way that I utilize the project.

- With a bit of revision of the first section, each of these projects can stand on their own as they don't necessarily build on the others (though they do create a richer experience together). Additionally, students gain confidence as they proceed through the three projects.

Student Prerequisites

This mini-PSP requires some algebraic manipulation of differentials along with differentiation up to the product rule. It also needs knowledge of separable Differential Equations. The first fundamental theorem of calculus makes an appearance but other techniques of integration needed are typically dictated by the examples used. Finally, the project benefits from the students being aware of the modern integrating factor method.

PSP Design, and Task Commentary

This PSP consists of four sections:

1. The first section contains a short introduction to what first-order linear differential equations are, along with a description of the way that mathematics often evolves. Mathematicians might first solve a specific problem using any tool at their disposal. They then attempt to see if they could find a class of problems (of which the initial one belongs) that can also be attacked using that technique. This closely mimics the evolution of how first-order linear differential equations were solved.

2. The second section is devoted to Leibniz' method of solution. A translation is provided along with a few tasks to explain his method. Strictly speaking, students may notice that Leibniz doesn't "solve" the equation. Rather, he asserts a solution and then shows it "works". Since the solution isn't derived, I refer to it as "one-off" or a "trick."

3. The third section consists of two first-order linear differential equations to be solved with Leibniz' method. The first is broken into steps, while the second requires the student to solve it on their own. These can be swapped with any examples you wish - in particular so that the integrations utilize techniques your students are comfortable with.

4. The final section reiterates what we saw in the first section. These first solutions appear to come "fully formed from the heads" of the great mathematicians. It is only with extensive practice that they developed the necessary intuition to find those methods. While the origins may be dependent on intuition, the future story is more satisfactory as there is a task to show Leibniz' trick utilizes the modern integrating factor method.

Suggestions for Classroom Implementation

Please see student requirements and implementation schedule for suggestions.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Sample Implementation Schedule (based on a 50-minute class period)

The first section of this mini-PSP should be out of class homework. Tasks 3 and 5 shouldn't be skipped, but the remainder of the Tasks are stand-alone and what is covered can be dictated by the interests of the instructor and time available. The Task 8 is useful to complete the integration of this PSP into the material the student sees in their textbook. Also, the Task 6 can be assigned as homework after class. With these types of revisions, this is a doable activity in one 50-minute class period.

The actual number of class periods spent on each section naturally depends on the instructor's goals and on how the PSP is actually implemented with students. This project is typically done in groups.[9] One reviewer warned that, "The groups often want to take a divide and conquer approach, which is just utterly useless for these documents, because the only person who is going to make any progress is the person who is working on Intro/Section 1. These are all designed to be read top to bottom in slow careful detail, and the later parts of the PSP rarely make any sense unless you've seen the earlier parts."

Connections to other Primary Source Projects

As mentioned above, this mini-PSP is part of a series of three, all which are intended for use in an Ordinary Differential Equations course.

- Solving Linear First-Order Differential Equations: Gottfried Leibniz' "Intuition and Check" Method.
- Solving Linear First-Order Differential Equations: Johann Bernoulli's (Almost) Variation of Parameters Method.
- Solving Linear First-Order Differential Equations: Leonard Euler's Integrating Factor Method.

Additionally, the author has written a fourth mini-PSP for use in an Ordinary Differential Equations course, based on works by Peano:

- Wronskians and Linear Independence: A Theorem Misunderstood by Many.

All of the above projects can be found at `https://digitalcommons.ursinus.edu/triumphs_differ/`.

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[9]...though with COVID, who knows?!?!

For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.