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ANLY 590

HW 2.1

Part -1 : Analytic Assignment

- $z = f(x,y) = ax + by + c$

  $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \langle a, b \rangle$

- $z = f(x) = f(x_1, x_2, \cdots, x_N) = \sum_{i=1}^{N} a_i(x_i - b_i) + S = a_1x_1 + a_2x_2 + \cdots + a_Nx_N + d$

  $\nabla f(x_1, x_2, \cdots, x_N) = \langle a_1, a_2, a_3, \cdots, a_n \rangle$

- $z = f(x,y) = A(x-x_0)^2 + B(y-y_0)^2 + c$

  $f_x(x,y) = \left(\frac{\partial f(x,y)}{\partial x}\right)_y = 2Ax - 2Ax_0$

  $f_y(x,y) = \left(\frac{\partial f(x,y)}{\partial y}\right)_x = 2By - 2By_0$

- $x^T = (3 \ 1 \ 4)$ [1x3]

  $y^T = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ [3x1]

  $B^T = \begin{pmatrix} 3 & 5 & 1 \\ 5 & 2 & 4 \end{pmatrix}$ [2x3]

$x \cdot x = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 3 \cdot 3 + 1 \cdot 1 + 4 \cdot 4 = 26$

$x \cdot y^T = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = 3 \cdot 2 + 1 \cdot 5 + 4 \cdot 1 = 15$

$x \times y = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \times (2 \ 5 \ 1) = \begin{pmatrix} 3 \cdot 2 & 3 \cdot 5 & 3 \cdot 1 \\ 1 \cdot 2 & 1 \cdot 5 & 1 \cdot 1 \\ 4 \cdot 2 & 4 \cdot 5 & 4 \cdot 1 \end{pmatrix}$

$= \begin{pmatrix} 6 & 15 & 3 \\ 2 & 5 & 1 \\ 8 & 20 & 4 \end{pmatrix}$ [3x3]

$y \times x = (2 \ 5 \ 1) \times \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = (2 \cdot 3 + 5 \cdot 1 + 1 \cdot 4)$

$= (15)$ [1x1]

$A \times x = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \cdot 3 + 5 \cdot 1 + 2 \cdot 4 \\ 3 \cdot 3 + 1 \cdot 1 + 5 \cdot 4 \\ 6 \cdot 3 + 4 \cdot 1 + 3 \cdot 4 \end{pmatrix}$

$= \begin{pmatrix} 25 \\ 30 \\ 34 \end{pmatrix}$ [3x1]

$A \times B = \begin{pmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{pmatrix}$

$= \begin{pmatrix} 4 \cdot 3 + 5 \cdot 5 + 2 \cdot 1 & 4 \cdot 5 + 5 \cdot 2 + 2 \cdot 4 \\ 3 \cdot 3 + 1 \cdot 5 + 5 \cdot 1 & 3 \cdot 5 + 1 \cdot 2 + 5 \cdot 4 \\ 6 \cdot 3 + 4 \cdot 5 + 3 \cdot 1 & 6 \cdot 5 + 4 \cdot 2 + 3 \cdot 4 \end{pmatrix}$

$= \begin{pmatrix} 39 & 38 \\ 19 & 37 \\ 41 & 50 \end{pmatrix}$ [3x2]

B. reshape(1,6) = $(3 \ 5 \ 5 \ 2 \ 1 \ 4)$ [1x6]

- Linear least squares : single variable

$$y = M(x|p) = mx + b \qquad p = (p_0, p_1) = (m, b)$$

$$L(p) = L(m,b) = \sum_{i=1}^{N} (\hat{y}_i - M(\hat{x}_i, m, b))^2$$

$$y = \beta_0 + \beta_1 x$$

$$L(p) = \sum_{i=1}^{N} (y_i - \beta_1 x_i - \beta_0)^2$$

$$\frac{\partial L(m,b)}{\partial \beta_1} = \sum_{i=1}^{N} 2(y_i - \beta_1 x_i - \beta_0)(-x_i)$$

$$= 2\sum_{i=1}^{N} (\beta_1 x_i^2 + \beta_0 x_i - x_i y_i)$$

$$\frac{\partial L(m,b)}{\partial \beta_0} = \sum_{i=1}^{N} 2(y_i - \beta_1 x_i - \beta_0)(-1) = 0$$

$$= 2\sum_{i=1}^{N} (\beta_1 x_i + \beta_0 - y_i)(-1) = 0$$

$$= 2(N\beta_1 \bar{x} + N\beta_0 - N\bar{y}) = 0$$

$$\therefore \beta_0 = \bar{y} - \beta_1 \bar{x}$$

Let $2\sum_{i=1}^{N} (\beta_1 x_i^2 + (\bar{y} - \beta_1 \bar{x}) x_i - x_i y_i) = 0$

$$\beta_1 = \frac{\sum x_i y_i - \bar{y}\sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} x_i^2 - \bar{x}\sum_{i=1}^{N} x_i} = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N} (x_i - \bar{x})}$$

$$= \frac{cov(x,y)}{var(x)}$$

- Extra Credit : Multi-variable

$$X = \begin{bmatrix} 1 & x_{12} & \cdots & x_{1n} \\ 1 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_{m2} & \cdots & x_{mn} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$L(p) = L(\beta) = \sum_{i=1}^{m} \left| y_i - \sum_{j=1}^{n} x_{ij}\beta_j \right|^2 = \|y - X\beta^T\|^2$$

Let $y - X\beta^T = 0$

$$\therefore \quad X\beta^T = y$$

$$X^T\beta^T = X^T y$$

$$(X^T X)^{-1} X^T X \beta^T = (X^T X)^{-1} X^T y$$

$$\beta^T = (X^T X)^{-1} X^T y$$