Binary Trees (1)

Outline and Required Reading:

- Binary Trees (§ 6.3)
- Data Structures for Representing Trees (§ 6.4)

COSC 2011, Fall 2003, Section A
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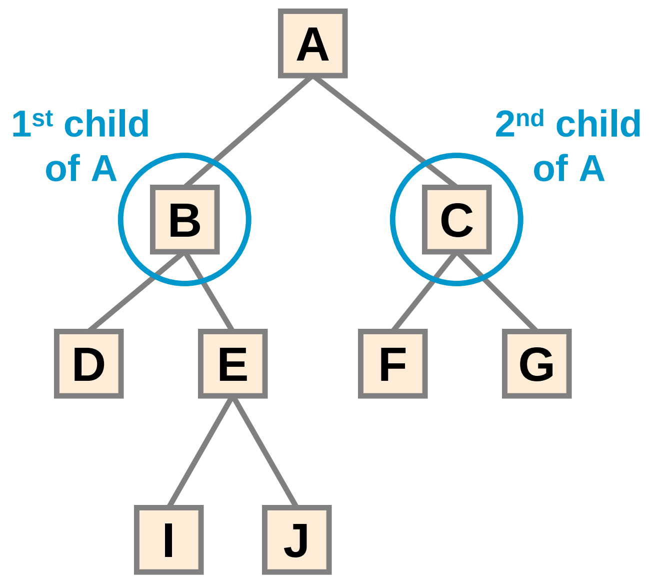
Binary Tree

Binary Tree – tree with the following properties

Proper and Ordered !

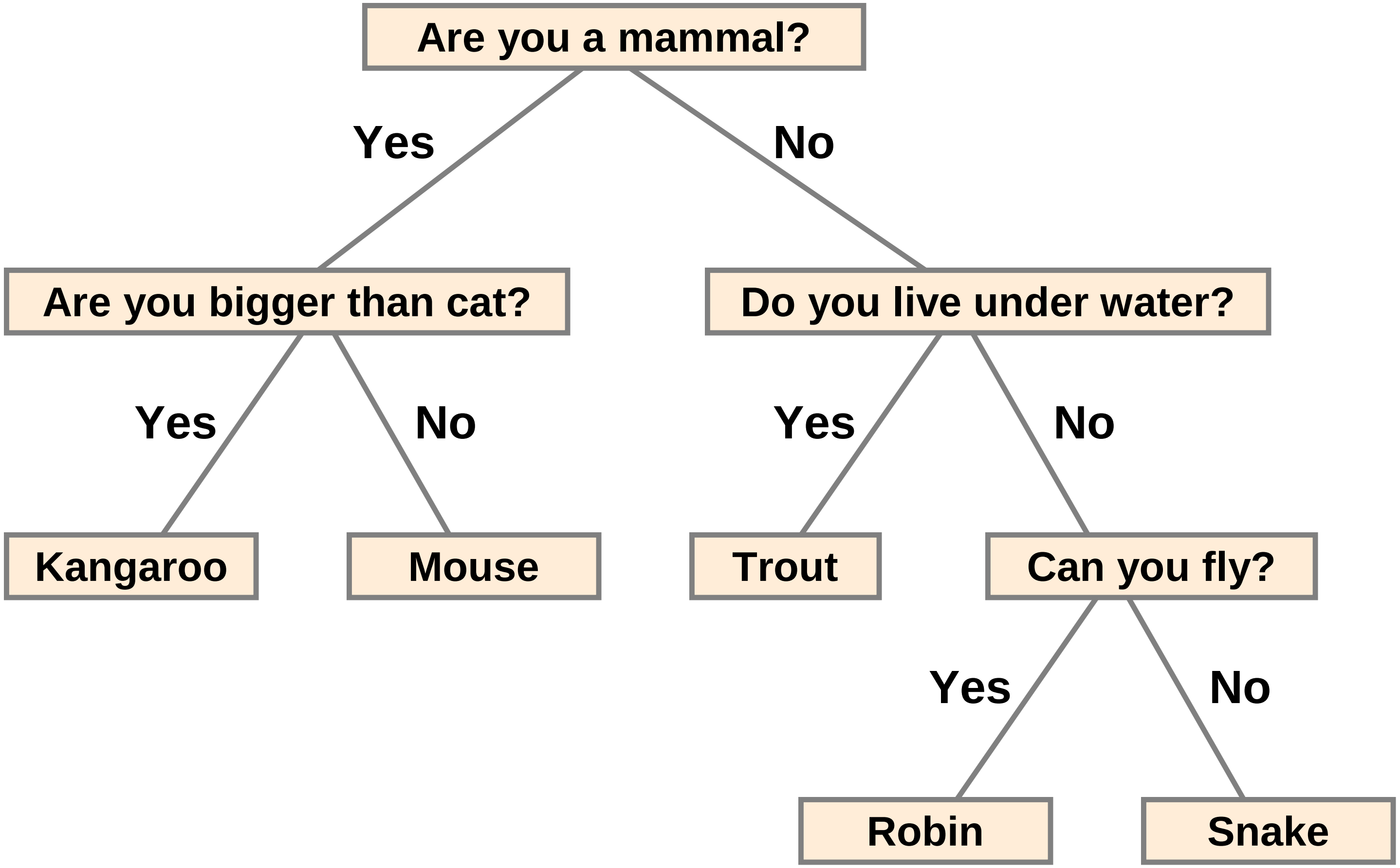
- each internal node has two children
- children of internal nodes form ordered pairs: left node – 1^{st}, right node – 2^{nd}



Application – representation of arithmetic expression, decision process, …

Binary Tree (cont.)

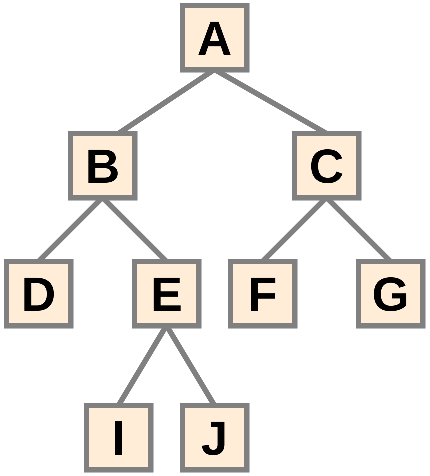
Example 1 **[binary tree for decision process]**



Properties of Binary Trees

Binary Tree Notation

- **n** – number of nodes
- **e** – number of external nodes
- **i** – number of internal nodes
- **h** – height of the tree
- **level** – set of nodes with the same depth



Property 1.1 **Level d has at most 2^d nodes.**

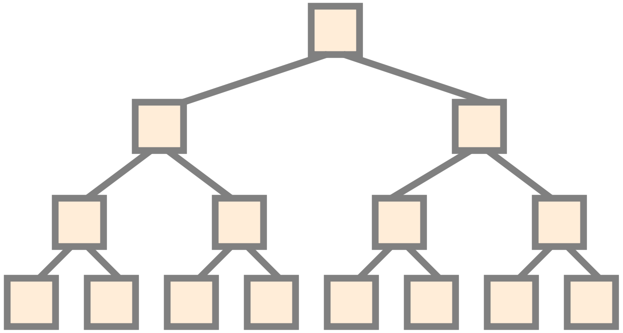
Proof **Let us annotate max number of nodes at level d with mn(d).**

Clearly, $mn(0) = 1$, **and**
$mn(d) = 2 \cdot mn(d-1)$ **for** $\forall d \geq 0$.

Hence, $mn(d) = 2 \cdot mn(d-1) = 2 \cdot 2 \cdot mn(d-2) = 2 \cdot [2 \cdot [..2 \cdot mn(0)]] = 2^d$

Properties of Binary Trees (cont.)

Property 1.2 **A full binary tree of height h has ($2^{h+1} - 1$) nodes.**



Full binary tree.

Proof

$$n = mn(0) + mn(1) + .. + mn(d) =$$

$$= 2^0 + 2^1 + 2^2 + \ldots + 2^h =$$

$$= \frac{1-2^{h+1}}{1-2} = 2^{h+1} - 1$$

Properties of Binary Trees (cont.)

Induction as a Proof Technique

Assume we want to verify the correctness of a statement (P(n)).

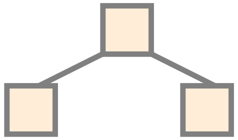
(1) First, prove that P(n) holds for n=1 (2, 3);

(2) Assume it holds for an arbitrary n and try to prove it holds for (n+1)

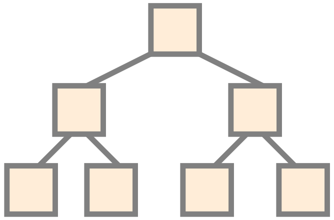
Property 2 **In a binary tree, the number of external nodes is 1 more than the number of internal nodes, i.e. e = i + 1.**

Proof

Clearly true for one node.

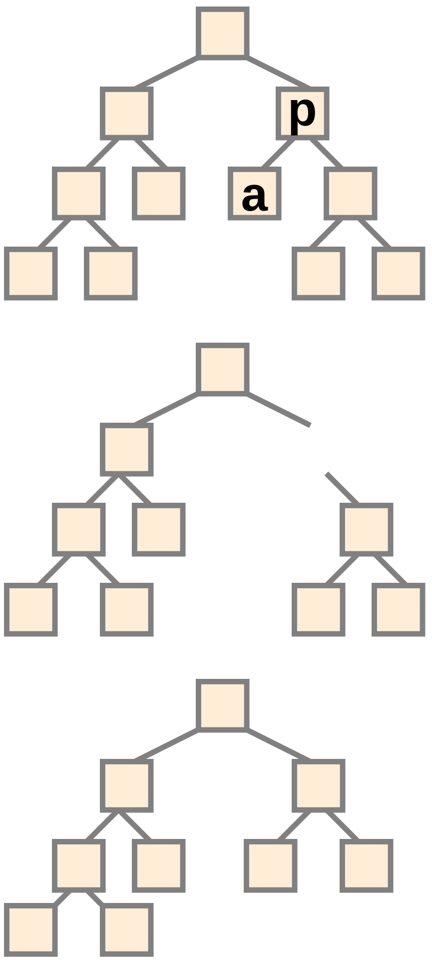


Clearly true for tree nodes.



Assume true for trees with up to n nodes.

Properties of Binary Trees (cont.)



Let T be a tree with n+1 nodes (top diagram).

1. Choose a leaf and its parent (which, of course, is internal). For example, the leaf a and parent p.

2. Remove the leaf and its parent (middle diagram).

3. Splice the tree back without the two nodes (bottom diagram).

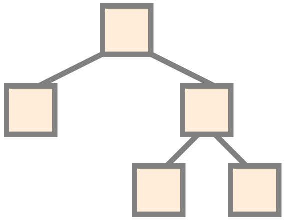
4. Since S has n-1 nodes, S satisfies initial assumption.

5. T is just S + one leaf + one internal so it also satisfies the assumption.

Properties of Binary Trees (cont.)

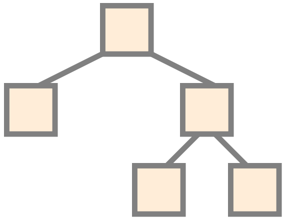
Approach (2):

Assume true for a tree with n nodes ($e = i + 1$). Now, we want to add new external nodes:



1. **Cannot add only one external – that would violate the property of proper binary tree. Hence, it cannot be: $e = i + 2$**

2. **Add two externals. In this case, one old external becomes internal, so we have:**



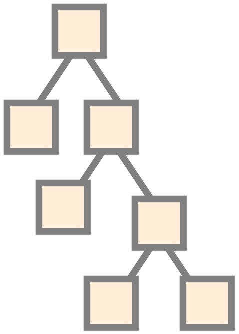
$e_{new} = (e-1)+2 = e+1 = i+1+1 = i+2$
$i_{new} = i+1$

Hence, $e_{new} = i_{new} + 1$

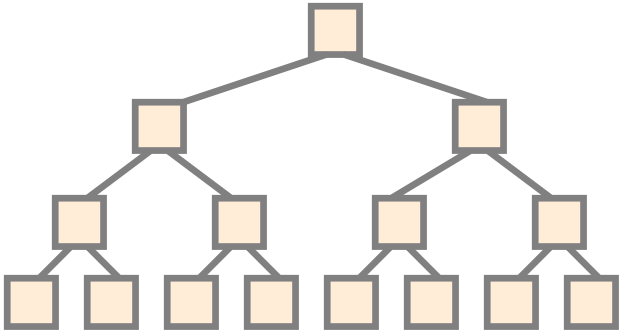
Properties of Binary Trees (cont.)

Property 3 **The number of external nodes (e) satisfies: $(h+1) \le e \le 2^h$.**

Proof



At every level (except last) there is only one internal node.



Full binary tree.

Property 4 **The number of internal nodes (i) satisfies: $h \le i \le 2^h-1$.**

Proof

Based on Property 3: $h+1 \le e \le 2^h$

Based on Property 2: $h+1 \le i+1 \le 2^h$

$h \le i \le 2^h-1$

Properties of Binary Trees (cont.)

Property 5 The total number of nodes (n) satisfies: $2h+1 \leq n \leq 2^{h+1}-1$.

Proof

Based on Property 3: $(h+1) \leq e \leq 2^h$

Based on Property 2 and n=i+e: $(h+1) \leq (n+1)/2 \leq 2^h$ …

$2h+1 \leq n \leq 2^{h+1}-1$

Property 6 The height (h) satisfies: $\log_2(n+1)-1 \leq h \leq (n-1)/2$.

Proof

Based on Property 5, the following two inequalities hold:

$2h+1 \leq n$

$h \leq (n-1)2$

$n \leq 2^{h+1}-1$

$n+1 \leq 2^{h+1}$

$\log_2(n+1) \leq h + 1$

$\log_2(n+1) - 1 \leq h$

Properties of Binary Trees (cont.)

Property 7 The height (**h**) satisfies: $\log_2(e) \leq h \leq e-1$.

Proof

Based on Property 6: $\log_2(n+1)-1 \leq h \leq (n-1)/2$

Based on Property 2: $\log_2(2e-1+1)-1 \leq h \leq (2e-1-1)/2$

$\log_2(2e)-1 \leq h \leq e-1$

$\log_2(2)+\log_2(e)-1 \leq h \leq e-1$

$\log_2(e) \leq h \leq e-1$

Properties of Binary Trees (cont.)

Summary of Properties

$$n = e + i$$

$$e = i + 1$$

All other expressions can be obtained from these three.

Number of external, internal, and overall nodes as a function of tree's height

$$(h+1) \leq e \leq 2^h$$

$$h \leq i \leq 2^h-1$$

$$2h+1 \leq n \leq 2^{h+1}-1$$

Tree's height as a function of number of external, internal, of overall nodes

$$\log_2(n+1)-1 \leq h \leq (n-1)/2$$

$$\log_2(e) \leq h \leq e-1$$

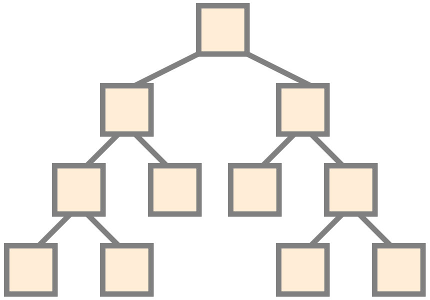
$$\log_2(i+1) \leq h \leq i$$

Binary Tree ADT: Interface

Binary Tree ADT – extends Tree ADT, i.e. inherits all its methods

Additional Methods

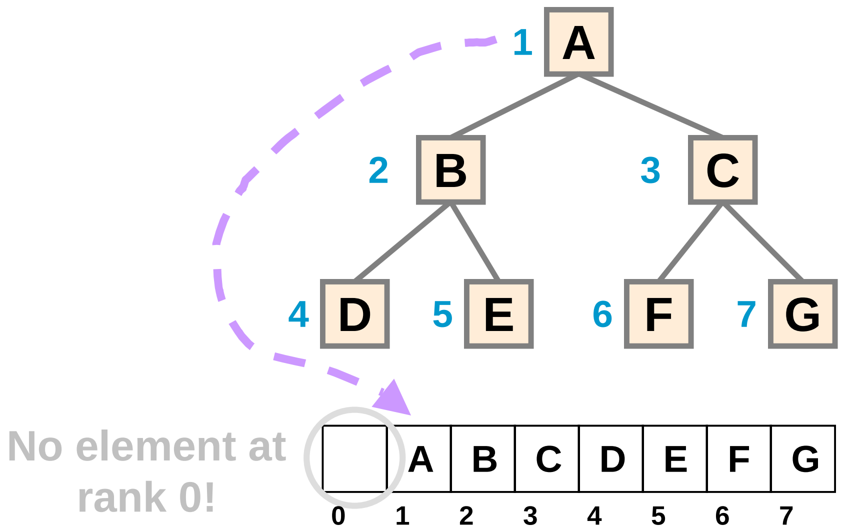
```
public Position leftChild(Position v);
/* return the left child of a node */
/* error occurs if v is an external node */

public Position rightChild(Position v);
/* return the right child of a node */
/* error occurs if v is an external node */

public Position sibling(Position v);
/* return the sibling of a node */
/* error occurs if v is the root */
```



Binary Tree: Array-Based Implementation

Indexing Scheme – for every node v of T, let its index/rank p(v) be defined as follows

- if v is the root: p(v) = 1
- if v is the left child of node u: p(v) = 2p(u)
- if v is the right child of node u: p(v) = 2p(u) + 1



Advantages – simple implementation, easy access

Binary Tree: Array-Based Implementation (cont.)

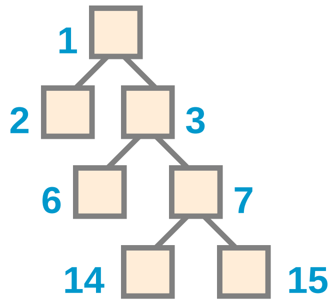
Space Complexity – let use the following notation

- n – number of nodes in T
- p_M – maximum value of p(v)
- N – array size ($N=p_M+1$), i.e. space usage

1) Best Case: full, balanced tree ⇒ all array slots occupied

$$N = p_M+1 = n+1 = O(n)$$

2) Worst Case: highly unbalanced tree ⇒ many slots empty



height: $h = (n-1)/2$

max p(v): $\boxed{p_M = 2^{h+1} - 1 = 2^{(n+1)/2} - 1}$

max p_M - as if this was full binary tree

required: array size $N = p_M+1 = 2^{(n+1)/2} = O(2^n)$

Array-Based Binary Tree: Performance

Run Times – **Good!** all methods, except positions and elements run in constant O(1) time

Method	Time
position, elements	O(n)
swapElements, replaceElement	O(1)
root, parent, children	O(1)
leftChild, rightChild, sibling	O(1)
isInternal, isExternal, isRoot	O(1)
expandExternal, removeAboveExternal	O(1)

Space Usage – **Poor!** (in general)

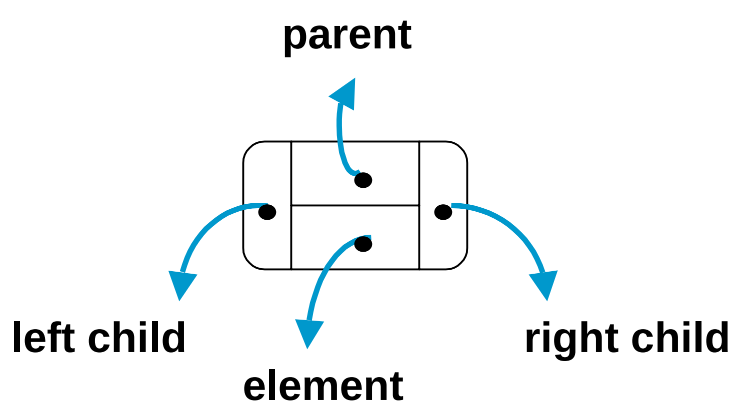
best case (full balanced tree): O(n)
worst case (highly unbalanced tree): $O(2^n)$

Link Structure - Based Implementation of Binary Tree

Node in Linked Structure for Binary Trees – object containing

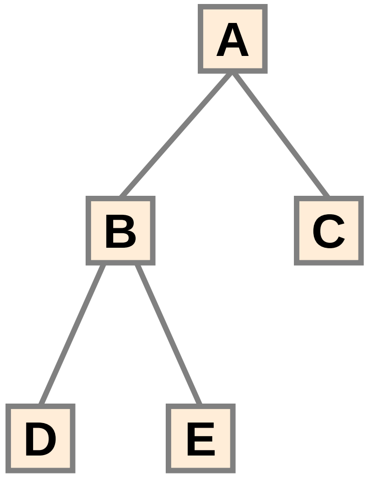
1) element
2) reference to parent
3) reference to the right child
4) reference to the left child

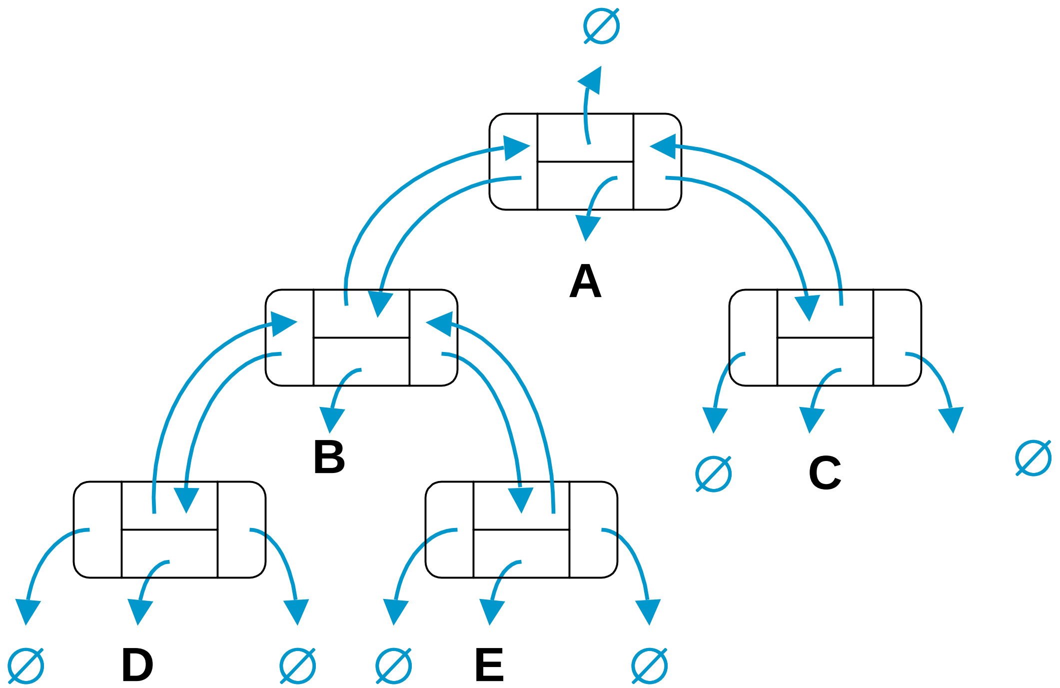
- if node is the root: reference to parent = null
- if node is external: references to children = null



Link Structure - Based Implement. of Binary Tree (cont.)

Example 4 **[binary tree and its linked list implementation]**





BTNode ADT: Implementation

BTNode Class – generalization of Position ADT, i.e. implements Position interface

```
public class BTNode implements Position {

        private Object element;
        private BTNode left, right, parent;

        public BTNode(Object o, BTNode u, BTNode v, BTNode w) {
                setElement(o);
                setParent(u);
                setLeft(v);
                setRight(w);
        }

        public Object element() { return element; }
        public void setElement(Object o) { element = o; }
```

BTNode ADT: Implementation (cont.)

```
    public BTNode getLeft() { return left; }
    public void setLeft(BTNode v) { left = v; }

    public BTNode getRight() { return right; }
    public void setRight(BTNode v) { right = v; }

    public BTNode getParent() { return parent; }
    public void setParent(BTNode v) { parent = v; }
}
```

Root Node: `BTNode root = BTNode(o,null,v, w)`

External Node: `BTNode root = BTNode(o, u, null, null)`

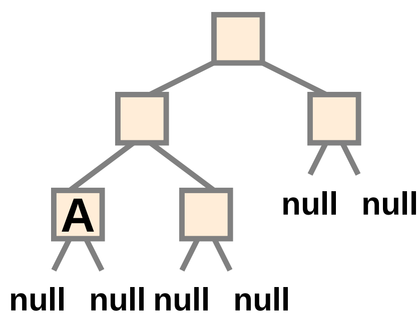
BTNode ADT: Implementation (cont.)

Null_Node – contains no elements, no children, has only reference to the parent

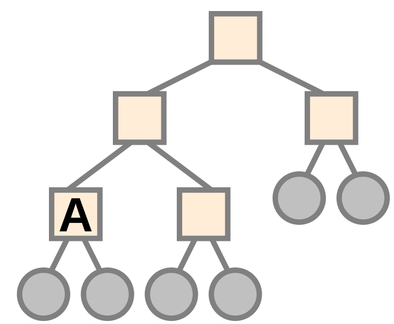
- implements Position interface !

Extended BT – every external node reference becomes reference to Null_Node

- with this approach, there is never a need to check whether a reference to a child is null
- implementation presented here, and in the textbook, does <u>not</u> employ Null_Node



~~(A.getLeft()).element();~~



(A.getLeft()).element();

LinkedBinaryTree ADT: Implementation

LinkedBinaryTree Class – implements BinaryTree interface, and also provides 2 additional methods

1) expandExternal
2) removeAboveExternal

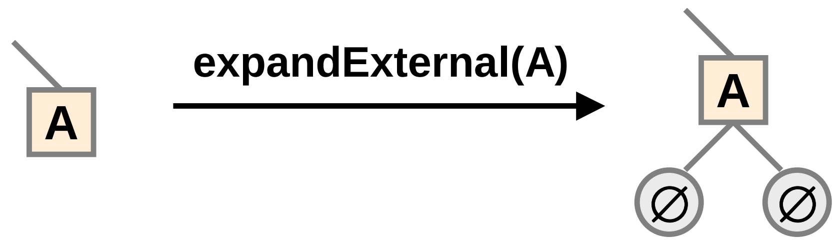
```
public class  LinkedBinaryTree implements BinaryTree {
        private Position root;     /* reference to the root */
        private int size;          /* number of nodes */

        public LinkedBinaryTree() {
                root = new BTNode(null, null, null, null);
                size = 1; }

        public void expandExternal (Position v) { … }
        public void removeAboveExternal (Position v) { … }
        …
}
```

For other methods, and their implementation details, see pp. 268-269 of the textbook.

LinkedBinaryTree ADT: Implementation (cont.)

expandExternal() Method – transforms v from external into internal node, by creating 2 new external nodes and making them the children of v

- error occurs if v is internal




```
public void expandExternal (Position v) {
        if (isExternal(v)) {
          ((BTNode) v ).setLeft(new BTNode(null, (BTNode) v, null null);
          ((BTNode) v ).setRigth(new BTNode(null, (BTNode) v, null null);
          size += 2; }
}
```

Application of expandExternal() – used for building a tree – see pp. 27

LinkedBinaryTree ADT: Implementation (cont.)

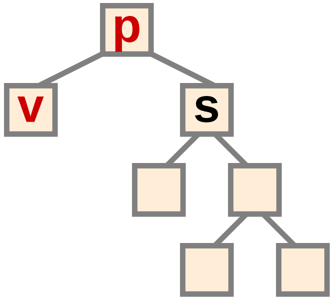
removeAboveExternal() Method – **removes external node w together with its parent v, replacing v with the sibling of w**

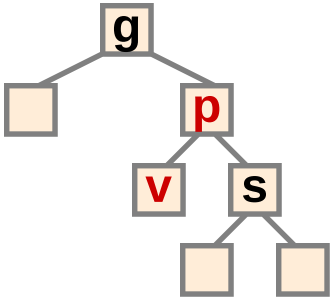
- **error occurs if w is internal**

Application of removeAboveExternal() – **used to dismantle a tree**

LinkedBinaryTree ADT: Implementation (cont.)

```
public void removeAboveExternal (Position v) {
      if (isExternal(v)) {
        BTNode p = (BTNode) parent(v);
        BTNode s = (BTNode) sibling(v);
        if (isRoot(p)) {
                s.setParent(null);
                root = s; }
        else {
                BTNode g = (BTNode) parent(p);
                if  (p == lefChild(g))  g.setLeft(s);
                else  g.setRight(s);
                s.setParent(g);
        };
        size-=2;
     }
     ...
}
```





LinkedBinaryTree ADT: Implementation (cont.)

Run Times – Good! all methods, except positions and elements run in constant O(1) time

Space Complexity – Good! only O(n), since there is one BTNode object per every node of the tree

- no empty slots as in array-based implementation

Method	Time
position, elements	O(n)
swapElements, replaceElement	O(1)
root, parent, children	O(1)
leftChild, rightChild, sibling	O(1)
isInternal, isExternal, isRoot	O(1)
expandExternal, removeAboveExternal	O(1)

Only immediate children !

LinkedBinaryTree ADT: Implementation (cont.)

Example 5 [creating a tree]

```
LinkedBinaryTree t = new LinkedBinaryTree();
t.root().setElement("Albert");
t.expandExternal(tree.root());
t.root().leftChild().setElement("Betty");
t.root().rightChild().setElement("Chris");
t.expandExternal(tree.root().leftChild());
t.root().leftChild().leftChild().setElement("David");
t.root().leftChild().rightChild().setElement("Elvis");
```