EXERCISES (TOPIC 1)

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(1) From ARE210, recall (Section 9 in Mahajan's "Handout 1") the rule for computing the distribution of certain transformations of random variables (The "inverse Jacobian rule").

Let (x, y) be independently distributed continuous random variables possessing densities f_x and f_y. Let $z = x + y$. Use the rule to obtain an expression for the distribution of z.

(2) We've discussed ways to program a convolution of random variables in a Jupyter notebook [ipynb] [datahub]. As in the notebook, consider a discrete random variable s and a continuous random variable x. Prove that the convolution of s and x (or, informally, $x + s$) has a continuous distribution, as suggested by the figure at the end of the notebook, **or** establish that the figure is wrong or misleading.

(3) Let $\boldsymbol{A}$ be an $m \times n$ matrix. A matrix $\boldsymbol{A}^-$ is a *generalized inverse* of $\boldsymbol{A}$ if $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$. Such a generalized inverse can be shown to always exist. If $\boldsymbol{A}$ is a matrix of zeros, what can we say about $\boldsymbol{A}^-$?

(4) Econometricians spend a great deal of time writing down linear regressions relating an object "Why" to an object "Ex", but sometimes use quite distinct notations to express this regression. Following our discussion in class, suggest a notation for each of the three following cases:

  (a) "Why" is a scalar random variable, while "Ex" is a vector random variable;
  (b) "Why" is a single *realization* of a scalar random variable, while "Ex" is similarly a single *realization*;
  (c) "Why" is a *vector* of N realizations, while "Ex" is similarly a *matrix* of realizations.

(5) Let $\boldsymbol{A}$ be an $m \times n$ matrix. A matrix $\boldsymbol{A}^+$ is a "Moore-Penrose" generalized inverse if:

  (a) $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}$;
  (b) $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$;
  (c) $\boldsymbol{A}^+\boldsymbol{A}$ is symmetric; and

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   (d) $\boldsymbol{A}\boldsymbol{A}^{+}$ is symmetric.
   If $\boldsymbol{A}$ is a matrix of zeros, what is $\boldsymbol{A}^{+}$?

(6) Least squares.

Definition: Let $\boldsymbol{A}$ be an $n \times m$ matrix of rank r. If $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{R}$, where $\boldsymbol{L}$ is a $n \times r$ full column rank matrix, and $\boldsymbol{R}$ is a $r \times m$ full row rank matrix, then $\boldsymbol{L}\boldsymbol{R}$ is a *full rank factorization* of $\boldsymbol{A}$.

Fact: Provided only that $r > 0$, the Moore-Penrose inverse $\boldsymbol{A}^{+} = \boldsymbol{R}^{\top}(\boldsymbol{L}^{\top}\boldsymbol{A}\boldsymbol{R}^{\top})^{-1}\boldsymbol{L}^{\top}$ exists and is unique.

Exercise: (a) Show that if $\boldsymbol{X}$ has full column rank, then $\boldsymbol{X}^{+} = (\boldsymbol{X}^{\top}\boldsymbol{X})^{-1}\boldsymbol{X}^{\top}$ (this is sometimes called the "left inverse), and $\boldsymbol{X}^{+}\boldsymbol{X} = \boldsymbol{I}$.

(b) Use the result of (a) to solve for $\boldsymbol{b}$ in the (matrix) form of the regression $\boldsymbol{y} = \boldsymbol{X}b + \boldsymbol{u}$ if $\boldsymbol{X}^{\top}\boldsymbol{u} = 0$.