

3.4 两个重要的极限

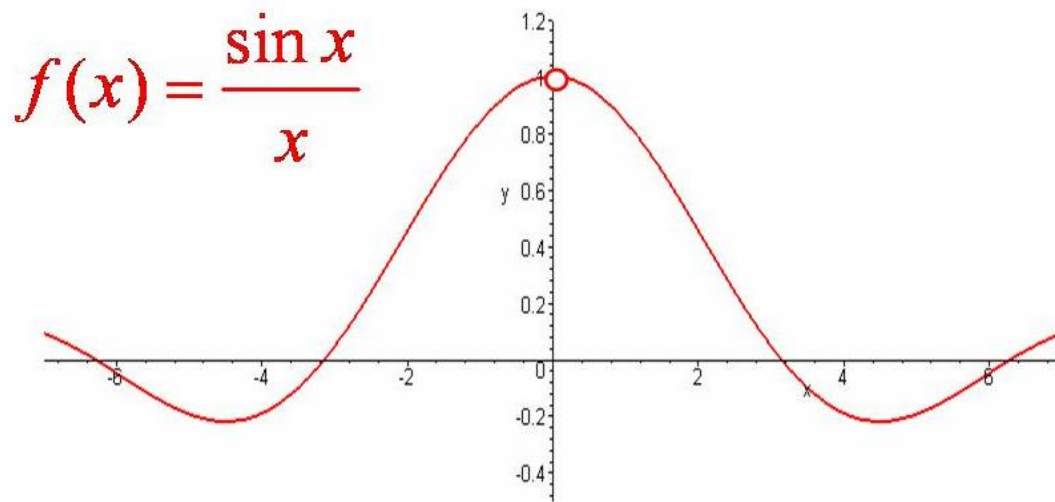


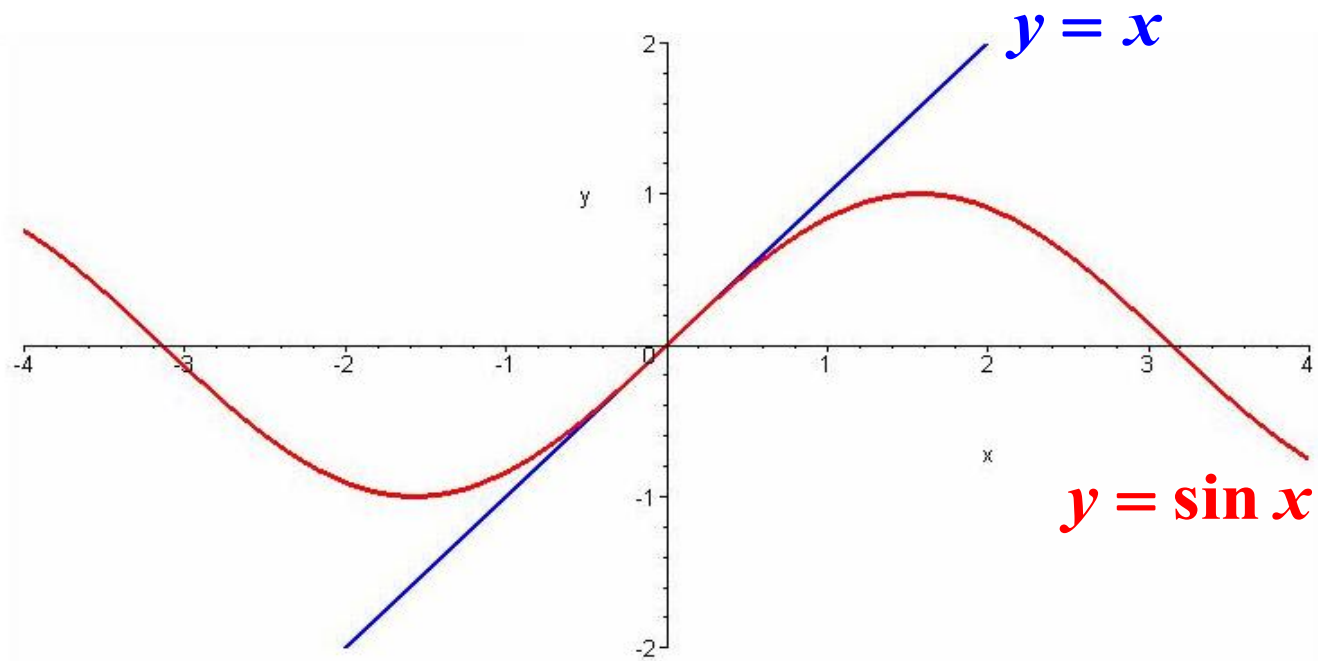
— $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

— $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$

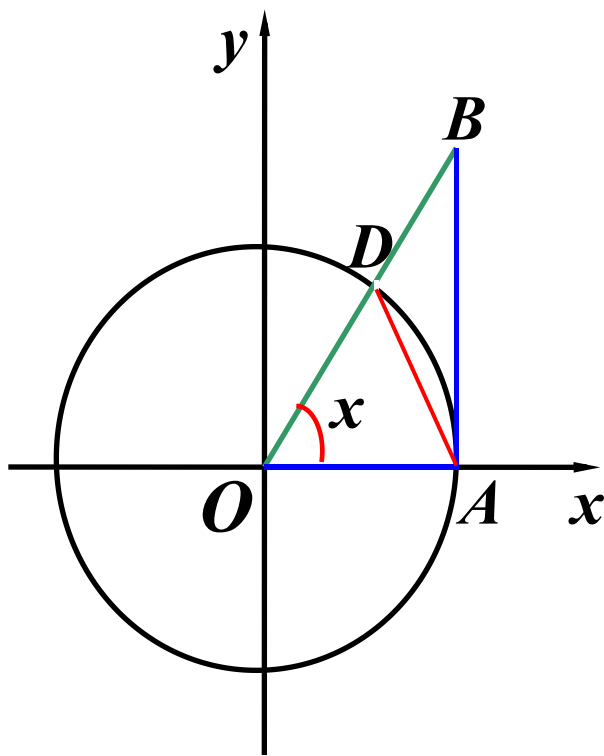
$$\text{一、} \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

x	0.1	0.01	0.001	...
$\sin x / x$	0.998 334	0.999 983	0.999 999	...





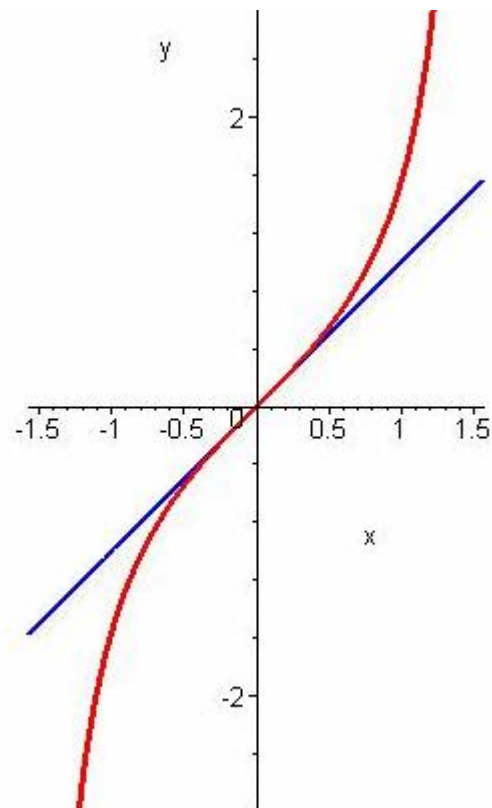
★ 证明: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$



- 当 $x \in (0, \frac{\pi}{2})$ 时,
 $\sin x < x < \tan x$;

例1、求 $\lim_{x \rightarrow 0} \frac{\tan x}{x}$.

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$



复合函数的极限运算法则

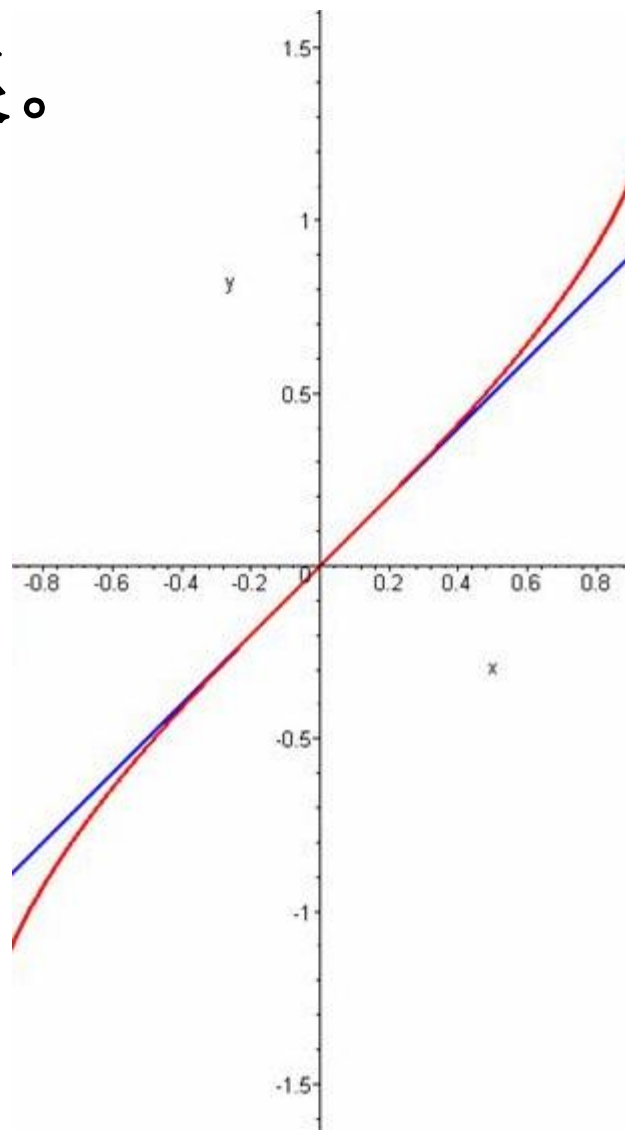
定理： 设 $\lim_{u \rightarrow u_0} f(u) = A$, $\lim_{x \rightarrow x_0} \varphi(x) = u_0$ 且在 x_0 的某去心邻域内 $\varphi(x) \neq u_0$, 则

$$\lim_{x \rightarrow x_0} f[\varphi(x)] = \lim_{u \rightarrow u_0} f(u) = A.$$

例2、求下列函数的极限。

$$(1) \lim_{x \rightarrow 0} \frac{\arcsin x}{x};$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1$$



$$(2) \lim_{x \rightarrow 0} \frac{\arctan x}{x};$$

$$\lim_{x \rightarrow 0} \frac{\arctan x}{x} = 1$$

$$(3) \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}.$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

($\frac{0}{0}$ 型)

- 一般地, 若 $\lim_{x \rightarrow x_0} \varphi(x) = 0$, 则 $\lim_{x \rightarrow x_0} \frac{\sin[\varphi(x)]}{\varphi(x)} = 1.$

例3、求下列函数的极限。

$$(1) \lim_{x \rightarrow 0} \frac{\sin 5x}{x};$$

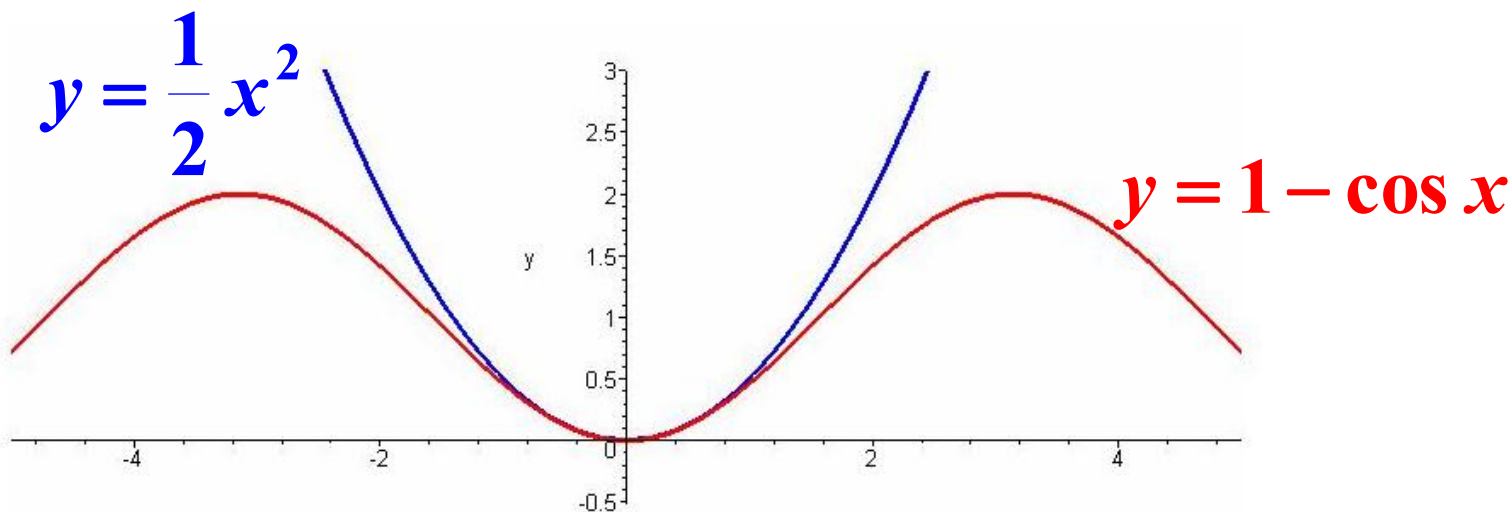
$$(2) \lim_{x \rightarrow \infty} x \sin \frac{3}{x};$$

$$(3) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}.$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

或

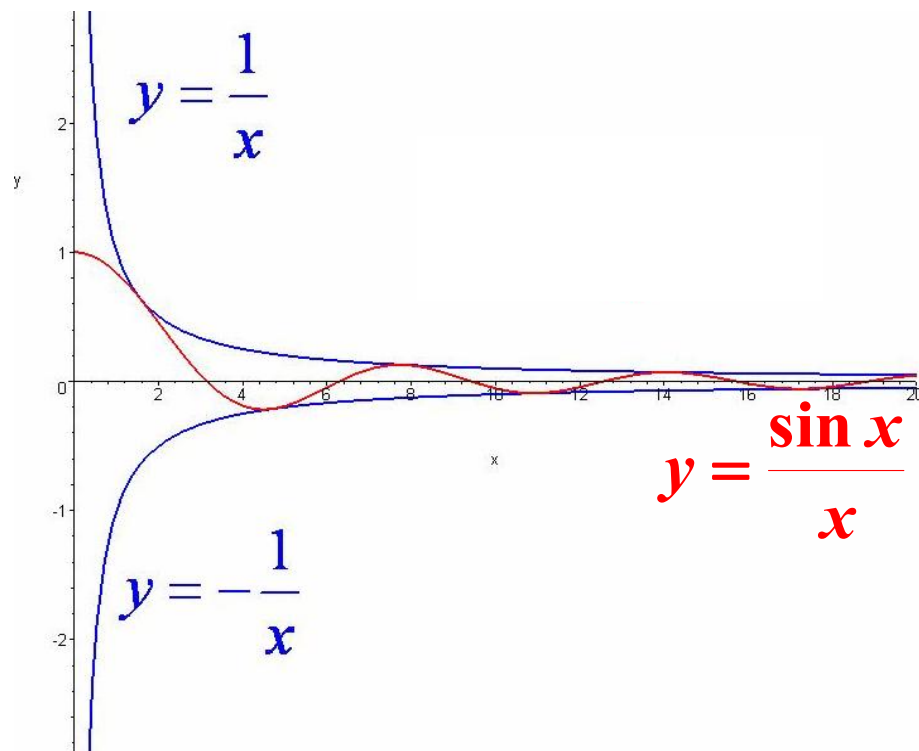
$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\frac{1}{2} x^2} = 1$$



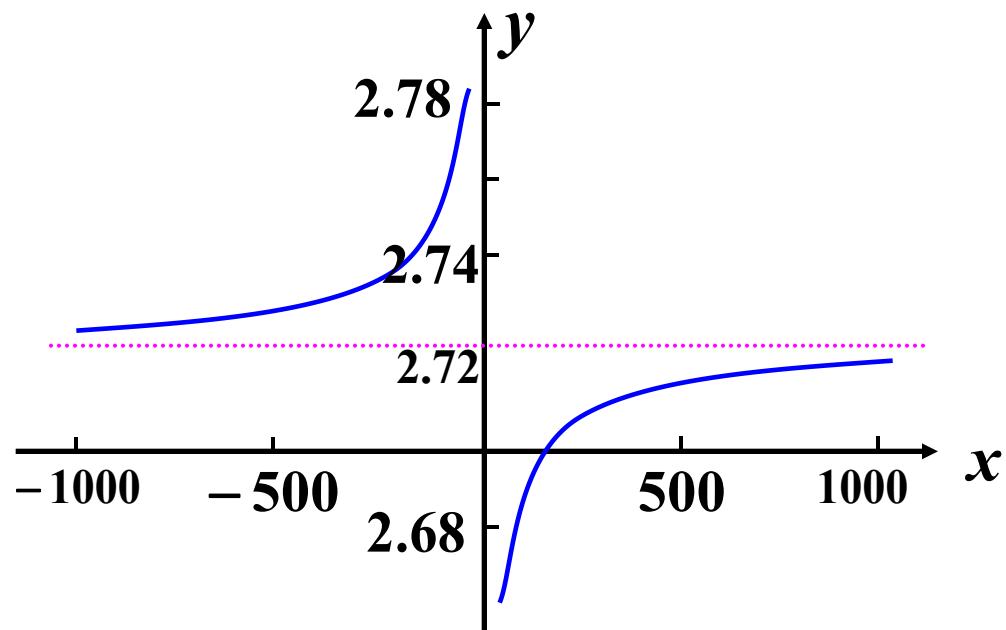
$$(4) \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3};$$

$$(5) \lim_{x \rightarrow \infty} \frac{\sin x}{x}.$$

不是重要极限



$$\text{二、} \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$



例4、求下列函数的极限。

$$(1) \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x ;$$

$$(2) \lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x ;$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e. \quad (1^\infty \text{ 型})$$

推论: (1) $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e.$

$$(2) \lim_{\varphi(x) \rightarrow 0} [1 + \varphi(x)]^{\frac{1}{\varphi(x)}} = e.$$

$$(3) \text{ 若 } \lim_x f(x) = a \ (a > 0 \text{ 且 } a \neq 1), \lim_x g(x) = b,$$

$$\text{则 } \lim_x f(x)^{g(x)} = a^b.$$

例5、求下列函数的极限。

$$(1) \lim_{x \rightarrow \infty} \left(\frac{2x-1}{2x+1} \right)^{3x};$$

$$(2) \lim_{x \rightarrow 0} (1+2x)^{\frac{1}{x}};$$

$$(3) \lim_{x \rightarrow 0} (1+x)^{\frac{2}{\sin x}};$$

$$(4) \lim_{x \rightarrow 1} x^{\frac{2}{1-x}}.$$

思考：求常数 a ，使得 $\lim_{x \rightarrow \infty} \left(\frac{x+a}{x+3} \right)^{\frac{x}{1000}} = e^2.$

例6、求 $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} - \frac{1}{n^2}\right)^n$.



作 业

习题3-4: 1 (2) (3) (9) 、

2 (2) (4) (6)