§ 17.2　复合函数微分法

1 复合函数的求导法则

2 抽象函数求(偏)导

3 复合函数的全微分

一、复合函数的求导法则

设函数 $x=\varphi(s,t)$ 与 $y=\psi(s,t)$ 定义在 st 平面的区域 D 上, 函数 $z=f(x,y)$ 定义在 xy 平面的区域 D_1 上。若

$$\{(x,y)\,|\,x=\varphi(s,t),y=\psi(s,t),(s,t)\in D\}\subset D_1,$$

则 $z=F(s,t)=f(\varphi(s,t),\psi(s,t))$, $(s,t)\in D$

是以 (x,y) 为中间变量, (s,t) 为自变量的复合函数。

定理1: 设 $\begin{cases} x = \varphi(s,t) \\ y = \psi(s,t) \end{cases}$ 在点 $(s,t) \in D$ 可微，$z = f(x,y)$

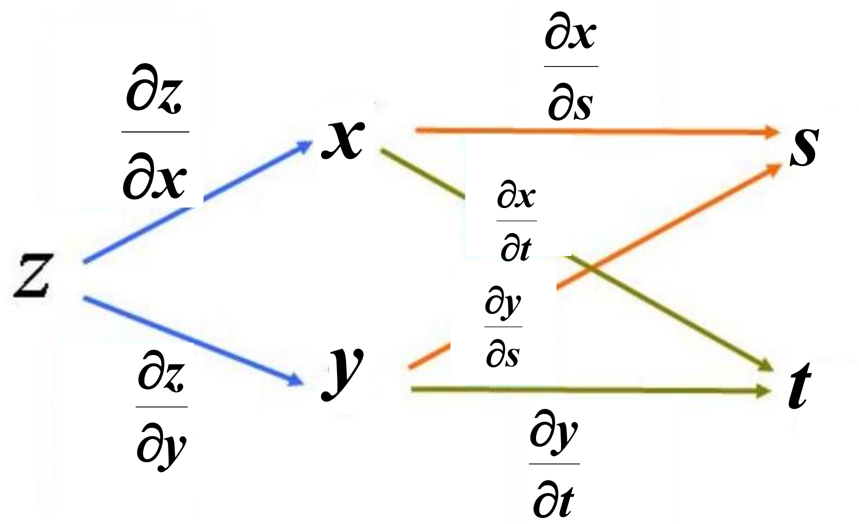
在点 $(x,y) = (\varphi(s,t), \psi(s,t))$ 可微，则复合函数

$z = f[\varphi(s,t), \psi(s,t)]$ 在点 (s,t) 可微，且

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t)} = \left.\frac{\partial z}{\partial x}\right|_{(x,y)} \cdot \left.\frac{\partial x}{\partial s}\right|_{(s,t)} + \left.\frac{\partial z}{\partial y}\right|_{(x,y)} \cdot \left.\frac{\partial y}{\partial s}\right|_{(s,t)},$$

$$\left.\frac{\partial z}{\partial t}\right|_{(s,t)} = \left.\frac{\partial z}{\partial x}\right|_{(x,y)} \cdot \left.\frac{\partial x}{\partial t}\right|_{(s,t)} + \left.\frac{\partial z}{\partial y}\right|_{(x,y)} \cdot \left.\frac{\partial y}{\partial t}\right|_{(s,t)}.$$

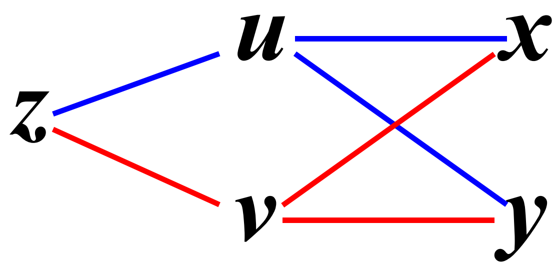
链式法则：



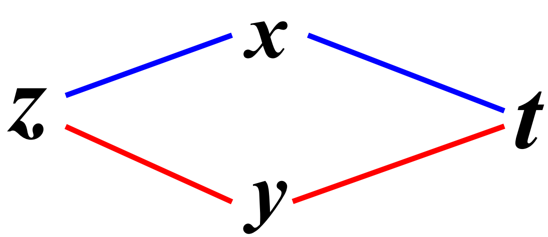
注：若只是求复合函数 $f(\varphi(s,t),\psi(s,t))$ 关于 s 或 t 的偏导数，则上述定理中 $x=\varphi(s,t)$，$y=\psi(s,t)$ 只须具有关于 s 或 t 的偏导数就够了.

例1、求下列复合函数的一阶(偏)导数。

(1) $z = u^v, u = x + y, v = x - y;$

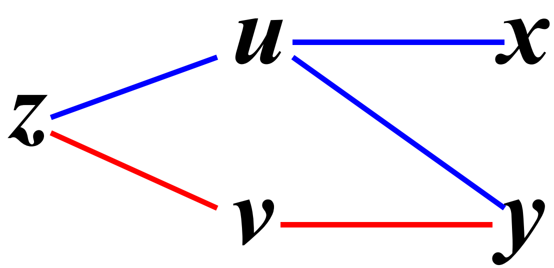


(2) $z = x\cos y, x = e^t, y = t^2;$

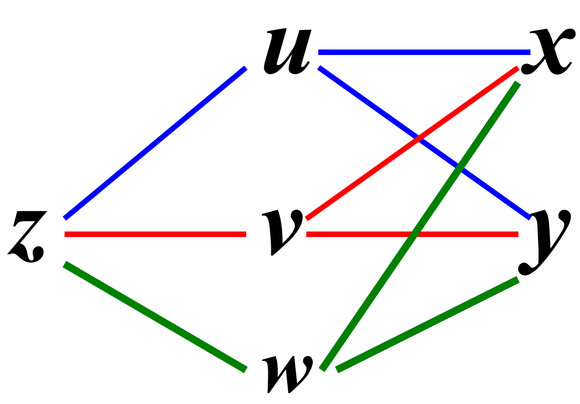


例1、求下列复合函数的一阶(偏)导数。

(3) $z = u\ln v, u = xy, v = y;$

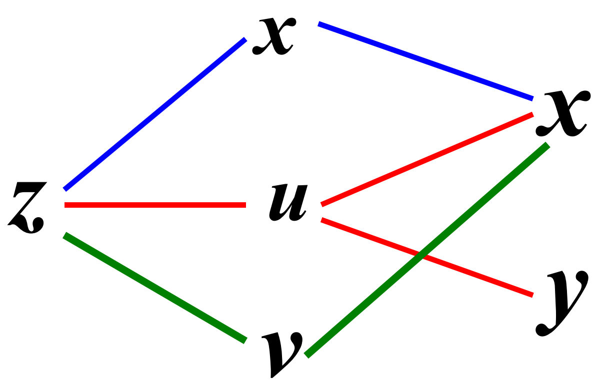


(4) $z = uvw, u = xy, v = x^2 y, w = xy^2.$



例2、求下列复合函数的一阶(偏)导数。

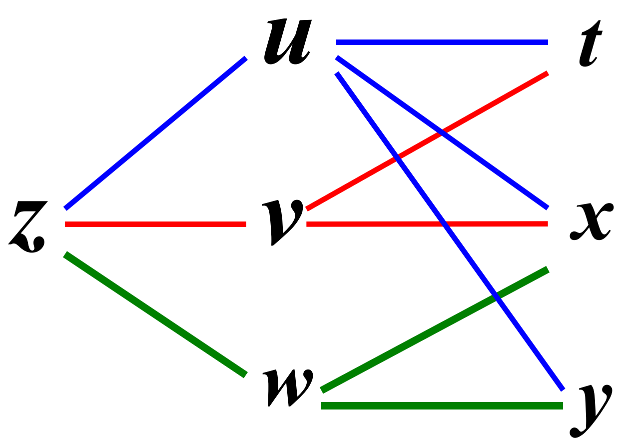
(1) $z = xuv, u = \sin xy, v = \cos x;$



(2) $z = u^2 - v, u = xyt, v = xt^2;$

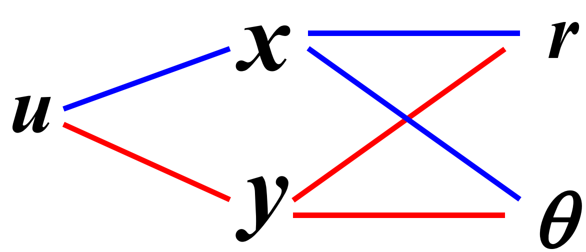
例2、求下列复合函数的一阶(偏)导数。

(3) $z = uvw, u = te^{xy}, v = xt, w = x - y$，求 $\frac{\partial z}{\partial t}$ 和 $\frac{\partial z}{\partial x}$ 在点 $(t, x, y) = (1,1,2)$ 的值.



例3、 设 $u=f(x,y)$ 可微，在极坐标变换 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

下证明：$\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2=\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2$.



二、抽象函数求(偏)导

记号：f_i：对函数 f 的第 i 个中间变量求 (偏) 导 .

例4、设 f 可微, 求下列复合函数的偏导数。

$$(1)\ u=f\left(\frac{x}{y},\frac{y}{z}\right);$$

$$(2)\ z=f(\sin x,\cos y,e^{x+y});$$

$$(3)\ u=x\,f(x^2+y^2+z^2).$$

例5、设 $z = xy + xF(u)$,而 $u = \dfrac{y}{x}$, $F(u)$ 可导,证明:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy.$$

例6、设 $z = \dfrac{y}{f(x^2 - y^2)}$,其中 $f(u)$ 可导,证明:

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{z}{y^2}.$$

三、复合函数的全微分

设 $z = f(x,y)$ 可微，且 $x=\varphi(s,t), y=\psi(s,t)$ 可微，

一方面，$\mathrm{d}z = \dfrac{\partial z}{\partial x}\mathrm{d}x + \dfrac{\partial z}{\partial y}\mathrm{d}y.$ (1)

另一方面，$\mathrm{d}z = \dfrac{\partial z}{\partial s}\mathrm{d}s + \dfrac{\partial z}{\partial t}\mathrm{d}t$

$$= \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\right)\mathrm{d}s + \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right)\mathrm{d}t$$

$$= \frac{\partial z}{\partial x}\cdot\underbrace{\left(\frac{\partial x}{\partial s}\mathrm{d}s + \frac{\partial x}{\partial t}\mathrm{d}t\right)}_{\mathrm{d}x} + \frac{\partial z}{\partial y}\cdot\underbrace{\left(\frac{\partial y}{\partial s}\mathrm{d}s + \frac{\partial y}{\partial t}\mathrm{d}t\right)}_{\mathrm{d}y}. \quad (2)$$

一阶全微分的形式不变性

例7、设 $z = e^{xy}\sin(x+y)$，求 dz 以及 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

作业

习题17-2：1(3)(4)(5)、2、4