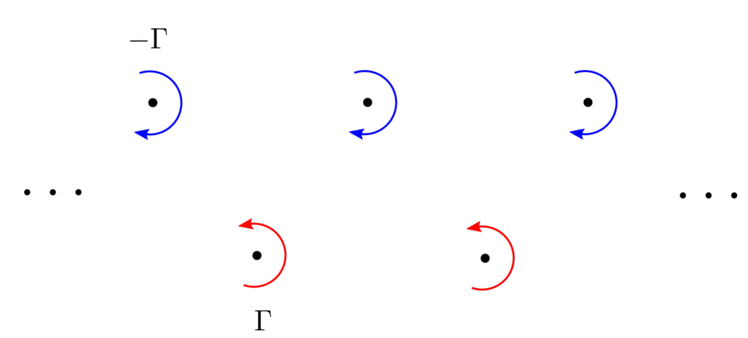
Von Karman Vortex Street

Bob Jiang 02385890

Imperial College London

Introduction

The flow past a cylinder can result in remarkable and complex dynamics in the wake behind it. Vorticity is shed off the sides and rear of the cylinder in an alternating pattern, resulting in an array of vortices travelling downstream, called the Von Karman vortex street. This poster is the process of investigating the streamlines and motion of the vortex street from the first principles



Assumption

To simplify our vortex analysis, we need to establish some assumptions and boundary conditions. Two main properties of an idealized 2D flow are **incompressibility** and **irrotationality**. Here, the flow field is defined as $\mathbf{u} = (u(x,y), v(x,y))$.

- **Incompressiblity**: A fluid is incompressible if any volume of fluid does not change volume as it moves with the flow. Mathematically, it can be expressed as
$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
- **Irrotationality**: Likewise, a flow is irrotational if the vorticity, defined by $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, is zero. For 2D flows, $\boldsymbol{\omega} = (0, 0, \omega) = \omega \mathbf{e_z}$, and so
$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Complex Potential

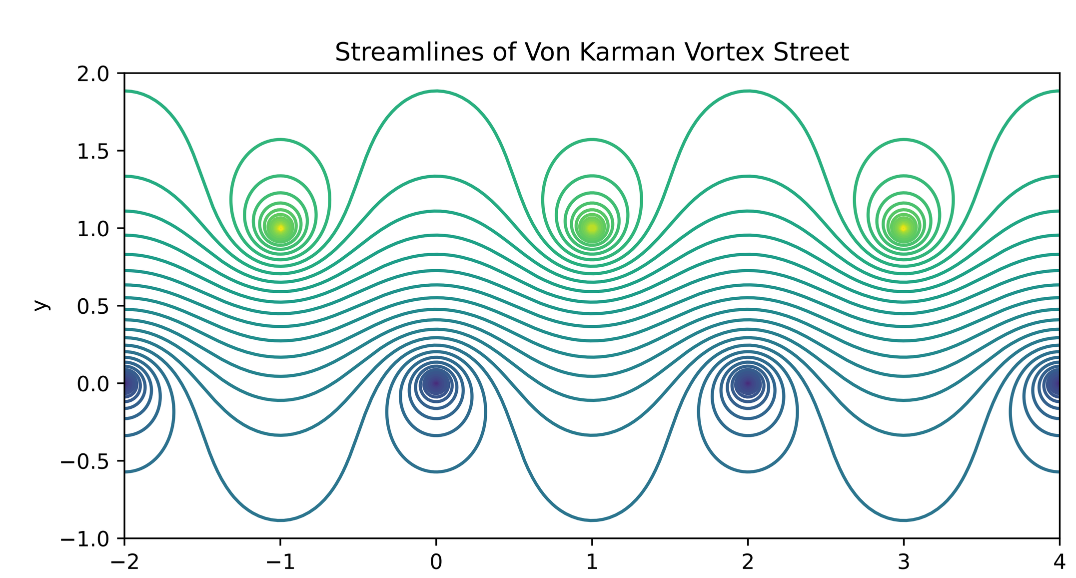
Since the flow is 2D, using complex numbers often proves practical. We can define the complex potential as
$$w(z) = \phi + i\psi$$
where ϕ is the velocity potential, with $\mathbf{u} = \nabla\phi$, ψ is the stream function, with $\psi(x, y) = c$ (c is a constant here) By differentiation, we get
$$\frac{\mathrm{d}w}{\mathrm{d}z} = \overline{V} \quad \text{where} \quad V := u + iv$$

Complex Potential of Von Karman Vortex Street

As shown in the figure in the Introduction, we can model this array of vortices as an infinite array: one set of vortices of strength Γ at $z = na$ where $n = 0, \pm 1, \pm 2, \ldots$, and one set of strength $-\Gamma$ at $z = (n + 1/2)a + ib$. We will first consider the bottom row. The complex potential due to the incompressible and irrotational vortex at $z = na$ is
$$\frac{\Gamma}{2\pi i}\log(z - na) = \frac{\Gamma}{2\pi i}\log(1 - \frac{z}{na}) + c$$
and we can simplify it by adding a constant while makes no difference to the resulting flow. Therefore, the complex potential due to the whole bottom row is equal to
$$w = \frac{\Gamma}{2\pi i}\sum_{n=-1}^{-\infty}\log(1 - \frac{z}{na}) + \frac{\Gamma}{2\pi i}\log(z) + \frac{\Gamma}{2\pi i}\sum_{n=1}^{\infty}\log(1 - \frac{z}{na}) \tag{1}$$
$$= \frac{\Gamma}{2\pi i}\log[z\prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2a^2})] \tag{2}$$
$$= \frac{\Gamma}{2\pi i}\log(\sin\frac{\pi z}{a}) + c \tag{3}$$
Similarly, we can derive the complex potential due to the whole upper row. The complex potential due to all the vortices is therefore
$$w = \frac{\Gamma}{2\pi i}\log(\sin\frac{\pi z}{a}) - \frac{\Gamma}{2\pi i}\log(\sin\frac{\pi(z - z_0)}{a}) + c \quad \text{where} \quad z_0 = \frac{1}{2}a + ib$$
It is surprising that the potential is intricately related to trigonometry [1]

Streamlines of Von Karman Vortex Street

The array cannot be stationary obviously. But what if we change into a moving frame in which the vortices appear stationary. Well, since ψ is the imaginary part of w, we can conclude that
$$\psi = -\frac{\Gamma}{2\pi}\left(\log\left|\sin\frac{\pi z}{a}\right| - \log\left|\sin\frac{\pi(z - z_0)}{a}\right|\right)$$
Here, we can express $z = x + iy$ and $z_0 = \frac{1}{2}a + ib$, and use the basic properties of trigonometry of complex variables to calculate the norms of the complex numbers. We find that
$$\psi(x, y) = \frac{\Gamma}{4\pi}\left[-\log\left(-\cos\frac{2\pi x}{a} + \cosh\frac{2\pi y}{a}\right) + \log\left(-\cos\frac{2\pi(x - \frac{a}{2})}{a} + \cosh\frac{2\pi(y - b)}{a}\right)\right]$$



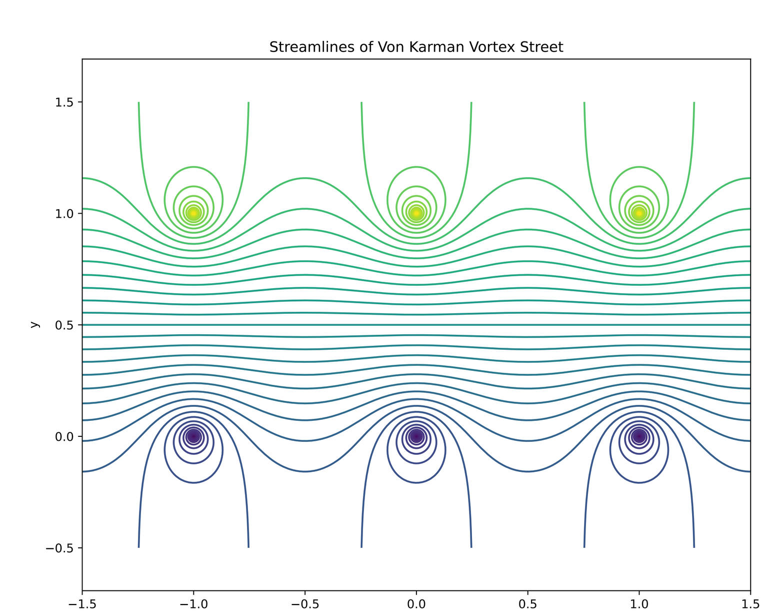
The figure above illustrates the case where $a = 2$ and $b = 1$, showing the expected symmetric and periodic pattern. [2]

Motion of Von Karman Vortex Street

Consider any vortex. We note that the local flow velocity due to the other vortices in the same row is zero because their contributions cancel in pairs. The y-components of the velocity due to those in the other row also cancel in pairs, but the x-components reinforce each other, resulting in a certain velocity to the left. Consequently, every vortex and the entire array move to the left , maintaining the same shape.
In the previous section, we derived the total complex potential of the array. To find V, we can focus on any vortex, say at $z = \frac{1}{2}a + ib$. Since the complex potentials due to vortices in the same row cancel out, the resultant complex potential, excluding the contribution from the upper row, is therefore
$$w = \frac{\Gamma}{2\pi i}\log(\sin\frac{\pi z}{a}) + c$$
Next, we differentiate w with respect to z, obtaining $\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\Gamma}{2ai}\cot\frac{\pi z}{a}$. By substituting $z = \frac{1}{2}a + ib$ into the equation, we get
$$V = \left.\frac{\mathrm{d}w}{\mathrm{d}z}\right|_{z=\frac{1}{2}a+ib} = -\frac{\Gamma}{2a}\tanh\frac{\pi b}{a}$$
The entire array of vortices therefore moves to the left with a constant speed of $\frac{\Gamma}{2a}\tanh\frac{\pi b}{a}$. When $b > 0$ is fixed, $\tanh\frac{\pi b}{a}$ is a positive decreasing function for $a > 0$, so $\frac{\Gamma}{2a}\tanh\frac{\pi b}{a}$ decreases as a increases. Furthermore, as $a \to \infty$, $V \to 0$, implying that the entire array becomes stationary. This aligns with our expectations, as a greater distance between the vortices results in less interaction between them. [1, 2]

Extension

After examining the properties of the classic vortex street, we can explore other configurations where the relative positions of the vortices differ. To do this, we simply replace $z_0 = ka + ib$ with some $k \in [0, 1)$. The figure below illustrates a special case where the vortex street is symmetric, with vortices located at $z = na$ and $z = na + ib$.



Reference

[1] David J Acheson. *Elementary fluid dynamics.* Oxford University Press, 1990.

[2] Wikibooks. *Trigonometry/Functions of complex variables.* 2011. URL: https://en.wikibooks.org/wiki/Trigonometry/Functions_of_complex_variables.