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Imperial College London

Department of Mathematics

Second-year Group Research Project

The KdV equation and solitary waves

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November 5, 2025

Abstract

Two types of analytical solutions of the KdV equation are found, firstly, assuming the boundary condition of the solitary wave to get the elementary solution and an explicit solution of two-soliton solution is found by using the inverse scattering transform method. We solve the initial value problem for the KdV equation using Newton's method and compare it to the analytical solution. A time-dependent numerical solution is found using finite differences and the Runge-Kutta method, and checked by computing conserved quantities. In addition, we study the behaviour of linearly perturbed solitons governed by the KdV equation.

Contents

1 Introduction

The Korteweg-de Vries (KdV) equation is a nonlinear partial differential equation that models waves in shallow water.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

where $u(x,t)$ is an unknown function of space and time variables x and t. This is usually written more concisely as

$$u_t + uu_x + u_{xxx} = 0.$$

One type of solution of the KdV equation has special properties which we discuss in this report. These solutions are called solitary waves (or solitons), and they were first discovered by John Scott Russell on the Union Canal in 1834. He described it as a "wave of translation" [1]. Russell carried out experiments to try and reproduce what he had seen in the canal. He noticed that solitary waves could travel long distances without losing speed or changing its shape, and also found that the speed of a solitary wave was dependent on its size. He also found that when two solitary waves collided with each other, they would pass through each other without changing shape.

Remark. In the literature, the KdV equation is most commonly given as

$$u_t - 6uu_x + u_{xxx} = 0.$$

In sections 4 and 5, a different coefficient is used for the nonlinear term uu_x. It can be shown that with a suitable transformation of x and u [1] we can obtain any coefficient for the uu_x term i.e. the solutions are qualitatively the same. Consider the equation

$$u_t + \alpha uu_x + \beta u_{xxx} = 0.$$

Using the transformation

$$x \mapsto Ax, \quad u \mapsto Bu,$$

for real constants $A, B \neq 0$, the equation becomes

$$Bu_t + \frac{\alpha B^2}{A}uu_x + \frac{\beta B}{A^3}u_{xxx} = 0$$

$$\implies u_t + \frac{\alpha B}{A}uu_x + \frac{\beta}{A^3}u_{xxx} = 0.$$

Choosing $A = 1, B = -\frac{1}{6}$ changes the coefficient of the uu_x term from -6 to 1, which gives us the equation used in sections 4 and 5.

2 Analytical solutions of the KdV equation

In this part, we will find different types of analytical solutions of the KdV equation.

2.1 Solitary wave solution

To find the solitary wave solution, we will assume that the solution u satisfies the travelling wave solution:

$$u(x,t) = u(x - ct), [1]$$

where c is a constant. Let $u(x,t) = f(\xi)$, where $\xi = x - ct$, substitute in the KdV equation gives:

$$-cf' + ff' + f''' = 0,$$

integrate:

$$-cf + \frac{1}{2}f^2 + f'' = A,$$

multiply both sides by an integrating factor f' and then integrate to yield:

$$\frac{1}{6}f^3 - \frac{1}{2}cf^2 - Af + B = -\frac{1}{2}(f')^2,$$

where A, B are arbitrary constants. After applying the boundary conditions of the solitary waves ($f, f', f'' \to 0$ as $\xi \to \pm\infty$) [1], we get $A = B = 0$, then the equation becomes:

$$f^2(f - 3c) = -3(f')^2,$$

if a real solution exists, then $(f')^2 \geq 0$, which implies $f - 3c \leq 0$. Hence, we may find the solution by separating variables:

$$\int \frac{df}{f(c - \frac{1}{3}f)^{\frac{1}{2}}} = \pm \int d\xi,$$

then substitute $f = 3c\,\mathrm{sech}^2(x)$ into the equation, we get;

$$f(x - ct) = 3c\,\mathrm{sech}^2(\frac{1}{2}c^{\frac{1}{2}}(x - ct - x_0)).$$

Note that we remove the $\pm$ because the function is even.

2.2 Two-soliton solution

In this part, we may use

$$u_t - 6uu_x + u_{xxx} = 0$$

as our KdV equation for convenience.
To explicitly compute the solutions of the two-soliton wave (which also simulate the collisions of two solitary waves), we need the following results:
For the KdV equation with initial condition $u(x,0) = f(x)$, where $f(x)$ is assumed to be well-behaved to solve the KdV equations, the corresponding Sturm-Liouville equation [1] is defined as

$$\psi_{xx} + (\lambda - u)\psi = 0, \qquad -\infty < x < \infty, \tag{1}$$

where λ is the eigenvalue and ψ is the eigenfunction with the behaviours

$$\hat{\psi} \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{as } x \to \infty \\ a(k)e^{-ikx} & \text{as } x \to -\infty, \end{cases} \quad [1] \tag{2}$$

for $\lambda > 0$, with $k = \lambda^{\frac{1}{2}}$ (continuous spectrum), and

$$\psi_n \sim c_n e^{-\kappa_n x} \qquad \text{as } x \to \infty, [1] \tag{3}$$

with $\kappa_n = (-\lambda)^{\frac{1}{2}}$ for each discrete eigenvalue ($n = 1, 2, ..., N$) where $\lambda < 0$ (discrete spectrum). Then the Marchenko equation [1] is defined as

$$K(x,z) + F(x+z) + \int_x^\infty K(x,y)F(y+z)dy = 0,$$

where

$$F(X) := \sum_{n=1}^{N} c_n^2 \exp(-\kappa_n X) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k) e^{ikX} dk.$$

The scattering data $c_n, \kappa_n, b(k)$ are called normalisation constants, discrete spectrum and reflection coefficient respectively and depends on time, where

$$\kappa_n = \text{const};\ c_n(t) = c_n(0)\exp(4\kappa_n^3 t);\ b(k;t) = b(k,0)\exp(8ik^3 t).[1]$$

Finally,

$$u(x,t) = -2\frac{\partial}{\partial x}\hat{K}(x,t), [1]$$

where $K(\hat{x,t}) = K(x,x;t)$ and $K(x,x;t)$ is the result we obtain after applying the time evolution of the scattering data and solving the Marchenko equation.

The first step of finding the two-soliton solution is to use the initial condition

$$u(x,0) = -6\,\mathrm{sech}^2 x, [1]$$

then the Sturm-Liouville equation is

$$\psi_{xx} + (\lambda + 6\,\mathrm{sech}^2 x)\psi = 0,$$

and let make the substitution $T = \tanh x$ then we will get $dT = (1-T^2)dx$, the equation then becomes

$$\frac{d}{dT}(1-T^2)\frac{d\psi}{dT} + (\frac{\lambda}{1-T^2} + 6)\psi = 0,$$

to get the only bounded solutions, we may set $\kappa_1 = 1,\ \kappa_2 = 2$ and consider the proportionality with the *associated Legendre functions* [1], which, after normalising, yields

$$\psi_1(x) = \sqrt{\frac{3}{2}}\tanh x\,\mathrm{sech}\, x; \quad \psi_2(x) = \frac{\sqrt{3}}{2}\mathrm{sech}^2 x.$$

Next, let $x \to \infty$ and compare the coefficients of asymptotic behaviour(3) above, we get $c_1(0) = 6; c_2(0) = 2\sqrt{3}$, hence

$$c_1(t) = \sqrt{6}e^{4t}; c_2(t) = 2\sqrt{3}e^{32t}$$

by previous results. Since the coefficient in our initial condition is of the form $N(N+1)$, $b(k;t) = 0$ [1] for all t and substitute in F gives

$$F(X;t) = 6\exp(8t - X) + 12\exp(64t - 2X)$$

Thus, plug into the Marchenko equation yields

$$\begin{aligned}&K(x,z;t) + 6\exp(8t - x - z) + 12\exp(64t - 2x - 2z)\\&+ \textstyle\int_x^\infty K(x,y;t)(6\exp(8t - y - z) + 12\exp(64t - 2y - 2z))dy = 0.\end{aligned}$$

By considering the dependence of z and that of y in the integral, we conclude that $K(x,z;t)$ must be in the form

$$K_1(x;t)e^{-z} + K_2(x;t)e^{-2z},$$

plug in the Marchenko equation gives

$$\begin{aligned}&K_1 e^{-z} + K_2 e^{-2z} + 6e^{(8t-x-z)} + 12e^{(64t-2x-2z)}\\&+ \textstyle\int_x^\infty (K_1 e^{-y} + K_2 e^{-2y})(6e^{(8t-y-z)} + 12e^{(64t-2y-2z)})dy = 0\end{aligned}$$

Since the coefficients of e^{-z} and e^{-2z} are zero, we get

$$\begin{array}{l} K_1 + 6e^{8t-x} + 6e^{8t}\int_x^{\infty} K_1 e^{-2y} + K_2 e^{-3y} dy = 0 \\ K_2 + 12e^{64t-2x} + 12e^{64t}\int_x^{\infty} K_1 e^{-3y} + K_2 e^{-4y} dy = 0, \end{array}$$

where the solution is

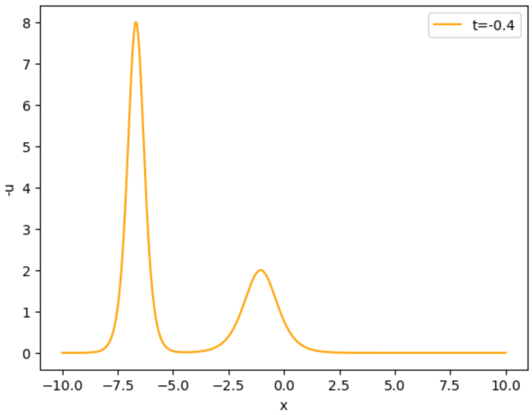
$$K_1(x,t) = \frac{6(e^{72t-5x} - e^{8t-x})}{f(x,t)}; \quad K_2(x,t) = \frac{-12(e^{64t-2x} + e^{72t-4x})}{f(x,t)}$$

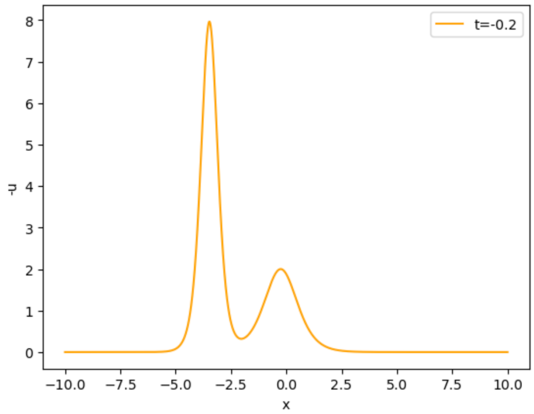
Now apply the equation

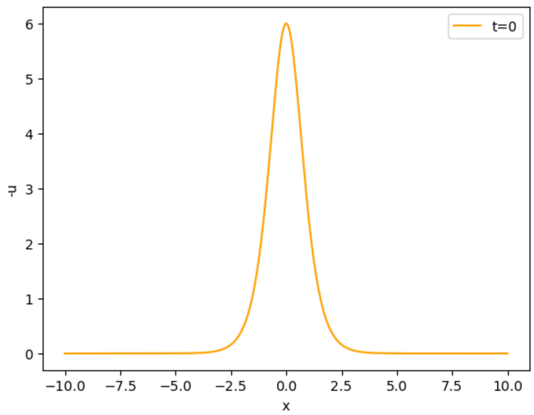
$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x;t)$$

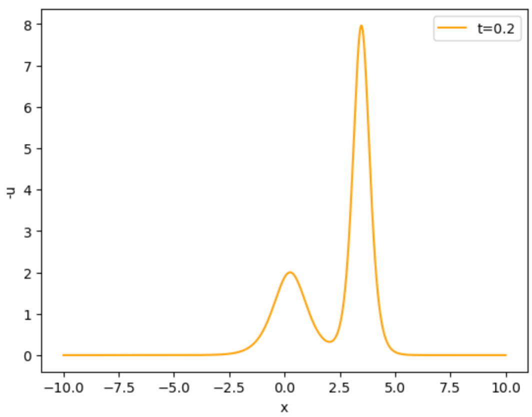
and after some algebra, we obtain

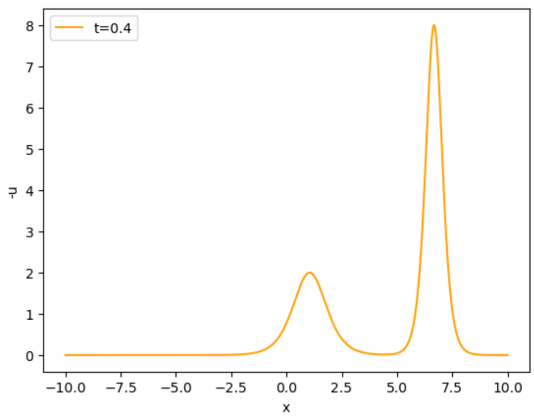
$$u(x,t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{(3\cosh(x - 28t) + \cosh(3x - 36t))^2}. [1]$$











These figures are the plots of the solutions at $t = -0.4, -0.2, 0, 0.2, 0.4$ respectively and illustrate the collision between two solitary-like waves, where the taller one moves faster and superposes with the shorter wave at $t = 0$ which is our initial condition $u(x, 0) = -6\,\mathrm{sech}^2(x)$, then the two waves re-emerge with the same shape and then move apart with same speeds as before the collision.
There is an interesting phenomenon that this interaction is nonlinear as when we look at the positions of the crests of the two waves, we could find that around $t = 0$ there is a slight forward movement of the taller wave, and backward movement for the shorter wave. It will be clear if we set $t = -0.1$, we will find that the peak of the shorter wave exceed the vertical-axis and the graph is plotted below.

We can prove this finding by substitute a fixed element $\xi = x - 16t$ into $u(x, t)$ and let $t \to \pm\infty$ then this describes a wave moving at speed 16 [1], then $u(x, t)$ becomes

$$-12\frac{3 + 4\cosh(2\xi + 24t) + \cosh(4\xi)}{(3\cosh(\xi - 12t) + \cosh(3\xi + 12t))^2},$$

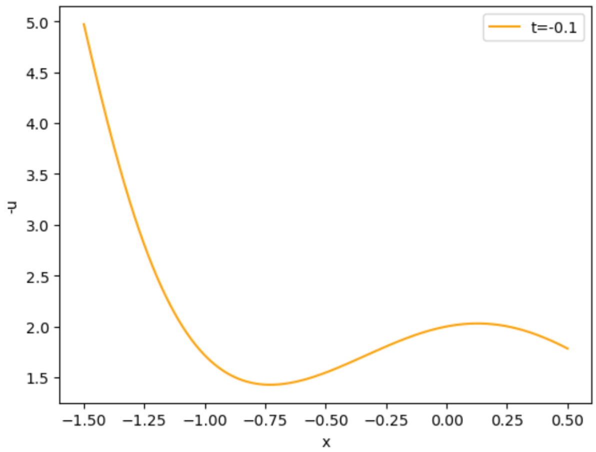


Figure 1 t=-0.1

we can simply expand it out and divide both denominator and numerator by e^{24t} to get

$$u(x,t) \sim -8\,\mathrm{sech}^2(2\xi \mp \frac{1}{2}\log 3), \quad \text{as } t \to \pm\infty.$$

Similarly, we can also define $\eta = x - 4t$ which is also held fixed to find the asymptotic behaviour

$$u(x,t) \sim -2\,\mathrm{sech}^2(\eta \pm \frac{1}{2}\log 3), \quad \text{as } t \to \pm\infty.$$

We can add these two asymptotic solutions because of the exponential behaviour of the error term to simulate a solution given as

$$u(x,t) \sim -8\,\mathrm{sech}^2(2\xi \mp \frac{1}{2}\log 3) - 2\,\mathrm{sech}^2(\eta \pm \frac{1}{2}\log 3), \quad \text{as } t \to \pm\infty$$

so that now the taller wave (which is the faster one) has a forward movement of distance $\frac{1}{2}\log(3)$ and the other one with a backward displacement of $\log(3)$.

3 Using Newton's method and finite difference to solve the initial value problem for the KdV equation

3.1 Overview

In this section we will look at solving the initial value problem for the KdV equation using Newtons method and finite difference. We will alter the step size and show that this method produces an accurate estimation for the actual solution. First we will use the solitary wave ansatz:

$$u(x,t) = \phi(z), \qquad z = x - c\,t,$$

To simplify the KdV equation:

$$u_t + 6\,u\,u_x + u_{xxx} = 0$$

From a partial differential equation into the ordinary differential equation:

$$\phi''(z) \;-\; c\,\phi(z) \;+\; 3\,\phi(z)^2 \;=\; 0,$$

Which has boundary conditions:

$$\phi(-\infty) = \phi(+\infty) = 0.$$

As discussed in [1].

3.2 Newton's Method

Newton's Method is an iterative technique for finding roots of a function $f : \mathbb{R}^n \to \mathbb{R}^n$. Starting from an initial guess X_0, it uses the function's local linear approximation to jump toward a solution of

$$f(X) = 0.$$

This is how the Newton's method works:

1. We want to find an X^* such that:

   $$f(X^*) = 0, \quad f : \mathbb{R}^n \to \mathbb{R}^n.$$

2. At a current iterate X_k, approximate f by its first-order Taylor expansion:

   $$f(X_k + \delta X) = f(X_k) + J_f(X_k)\,\delta X + O\big(\|\delta X\|^2\big),$$

   where

   $$J_f(X_k) = \frac{\partial f}{\partial X}\Big|_{X_k}$$

   is the Jacobian matrix.

3. Discard the higher order remainder and enforce the linear term to vanish:

   $$f(X_k) + J_f(X_k)\,\delta X = 0 \quad \Longrightarrow \quad \delta X = -\,J_f(X_k)^{-1}\,f(X_k).$$

4. An iterative formula is created by updating the guess as follows:

   $$X_{k+1} = X_k - J_f(X_k)^{-1}\,f(X_k).$$

5. Things to note:
   - Initialization: A good X_0 close to the true root often ensures convergence.
   - Invertibility: At each step, $J_f(X_k)$ must be non-singular.
   - Incorrect solutions: In some cases, it is possible to converge to a saddle point or a stationary point that is not a root, especially if the Jacobian provides misleading directional information. This is why it is important to analyse the behaviour of the function near critical points.

3.3 Finite Difference Approximation of the Spatial Derivatives

To numerically solve the differential equation, we breaking up the domain into a finite set of points and approximate the derivatives using finite differences (This is motivated by [1]):

1. Breaking up the domain into a finite set of points:

   We divide the interval $[-L, L]$ into N equally spaced grid points:

   $$z_j = -L + jh, \quad \text{for } j = 0, 1, \ldots, N-1,$$

   where $h = \frac{2L}{N-1}$ is the spacing between the points.

2. Approximating the Second Derivative:

   We use central difference to approximate the second derivative as follows:

   $$\phi''(z_j) \approx \frac{\phi_{j-1} - 2\phi_j + \phi_{j+1}}{h^2}.$$

   This leads to a tridiagonal matrix D_2 that approximates the second derivative operator, with the following non-zero entries:

   $$(D_2)_{j,j-1} = \frac{1}{h^2}, \quad (D_2)_{j,j} = -\frac{2}{h^2}, \quad (D_2)_{j,j+1} = \frac{1}{h^2}.$$

3. Boundary Conditions.

   To impose the Dirichlet boundary conditions $\phi(-L) = \phi(L) = 0$, we modify the first and last rows of the matrix D_2. In this case we do the following:

   $$D_2[0,:] = [1, 0, 0, \ldots, 0], \quad D_2[N-1,:] = [0, \ldots, 0, 1],$$

   which ensures that $\phi_0 = \phi_{N-1} = 0$ is enforced in the solution.

3.4 Forming the Nonlinear System

Define the discrete residual vector $F(\phi) \in \mathbb{R}^N$ by

$$F(\phi) \;=\; D_2\,\phi \;-\; c\,\phi \;+\; 3\,\phi * \phi,$$

where $*$ denotes component-wise multiplication. The boundary conditions are enforced by setting

$$F_0 = \phi_0, \quad F_{N-1} = \phi_{N-1}.$$

We then solve the nonlinear system

$$F(\phi) = 0.$$

3.5 Application of Newton's Method

Newton's iteration updates the solution via

$$\phi^{(k+1)} = \phi^{(k)} + \delta^{(k)}, \quad J\big(\phi^{(k)}\big)\, \delta^{(k)} = -\, F\big(\phi^{(k)}\big),$$

where the Jacobian matrix is

$$J(\phi) = D_2 \; - \; c\, I \; + \; 6\, \mathrm{diag}(\phi).$$

Dirichlet boundary conditions are imposed in J by overwriting its first and last rows. The iteration continues until

$$\|\delta^{(k)}\|_\infty < \text{tolerance} \quad \text{or} \quad k \geq \text{maximum no. of iterations}.$$

In the code we used tolerance = 10^{-6} as it is significantly larger than machine epsilon, yet sufficiently small to ensure convergence. Maximum no. of iterations = 10 as Newton's method typically converges quickly and in the case it does not, this prevents an infinite loop.

3.6 Choosing an Initial Guess

A convenient initial guess is the Gaussian

$$\phi^{(0)}(z_j) = \frac{c}{2}\, \exp\big(-z_j^2\big),$$

As it:

- Decays rapidly to zero at the boundaries.
- Is smooth and easy to compute.
- Hopefully places the iteration in the correct area of convergence.

Newton's method then improves upon this guess, turning it into the true solitary-wave profile.

3.7 Putting It All Together

The code works in the following manner:

1. Build the grid $\{z_j\}$ and the matrix D_2 with modified boundary rows.
2. Initialize $\phi^{(0)}$ via the Gaussian.
3. For each Newton step:
   - Assemble the residual $F(\phi^{(k)})$ and Jacobian $J(\phi^{(k)})$.
   - Solve $J\,\delta = -F$ for the update δ.
   - Update $\phi^{(k+1)} = \phi^{(k)} + \delta$.
   - Check $\|\delta\|_\infty <$ tolerance.
4. Upon convergence, output $\phi_j \approx \phi(z_j)$.

3.8 Effect of Step Size on Numerical Solution

In this section, we will investigate the relationship between the chosen step size and accuracy of the numerical solution. We solved the KdV initial value problem using four different values of step size $h = 10.0, 1.0, 0.1, 0.01$. The resulting solutions are shown overlaid in Figure 2, where we plot $\phi(z)$ against z on the interval $[-10, 10]$.

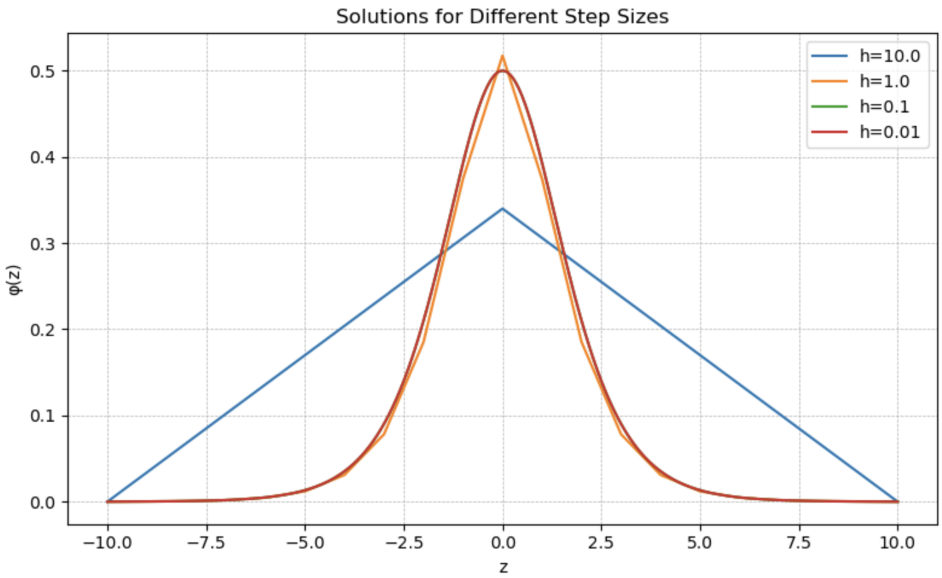


Figure 2 Solitary wave profile $\phi(z)$ computed using various step sizes h.

Observations from Figure 2:

- For large step sizes, the solution is less smooth and fails to capture the correct shape and amplitude of the solitary wave.
- As the step size is decreased, the solution becomes more accurate and resolves the peak and decay of the wave better. This demonstrates convergence consistent with the central difference approximation used.

This figure provides evidence that smaller step sizes improve the resolution and accuracy of the solution. However, smaller step sizes also increase computational cost.

3.9 Verification of Numerical Accuracy using Invariant Convergence

We can assess the accuracy of the numerical solution to the KdV equation by analysing how well it preserves the conserved quantities, known as invariants, of the equation as the time-step size h is varied. We will discuss conserved quantities in more detail in section 4.3. Figure 3 shows the relative errors in the first three conserved quantities denoted by I_1, I_2, and I_3 as a function of h on a log-log scale.

The invariants for the KdV equation are as follows [1]:

- I_1: Mass
- I_2: Momentum

- I_3: Energy

The key observations are:

- For small time-step sizes ($h \to 0$), the relative errors in all three invariants decrease significantly which indicates convergence.
- The small magnitudes of the relative errors shows that the computed solution preserves the physical invariants of the KdV equation with high accuracy.

This behaviour is expected for a well-designed numerical method applied to a PDE such as the KdV equation. Invariant quantities remain constant in the exact solution, so their convergence in the numerical results is a strong indication that Newton's method accurately solves the initial value problem. As the step size h decreases, the convergence of the KdV invariants provides compelling evidence that the method is correctly approximating the solitary wave solution. This validates both the accuracy of the finite difference approximation and that of Newton's method in solving the resulting nonlinear system.

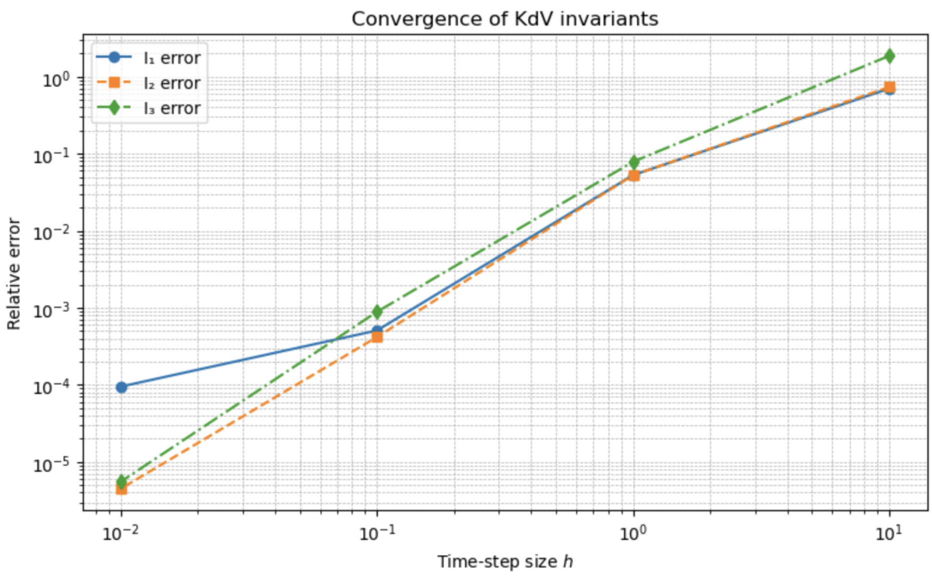


Figure 3

3.10 Comparing Numerical and Analytical solutions

To evaluate the accuracy of the numerical method, we compare the computed solitary wave profile to the exact analytical solution of the KdV equation. The analytical solution is given by:

$$\phi(z) = \frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}z\right),$$

[1]

Using a finely resolved mesh with step size $h = 0.01$, we numerically solve the nonlinear ODE using Newton's method combined with a second-order finite difference approximation. When plotted, the numerical solution aligns almost perfectly with the analytical solution across the entire domain $z \in [-10, 10]$ as shown in Figure 4. The two curves are visually indistinguishable, indicating excellent agreement in both amplitude and shape. This strong match confirms that the numerical method accurately captures the solitary wave structure of the exact solution.

To more precisely assess the error, we calculated the absolute difference between the numerical and analytical solutions as shown in Figure 5. The resulting error remains on the order of 10^{-5}, with the smallest errors occurring near the centre of the wave, getting slightly larger near the edges. This is expected, as numerical errors tend to accumulate in regions where the solution becomes very small and where round-off or boundary effects are more pronounced.

These observations provide strong numerical evidence for the accuracy and reliability of the implemented method. The finite difference method effectively approximates the spatial derivatives, and Newton's method successfully solves the resulting nonlinear system.

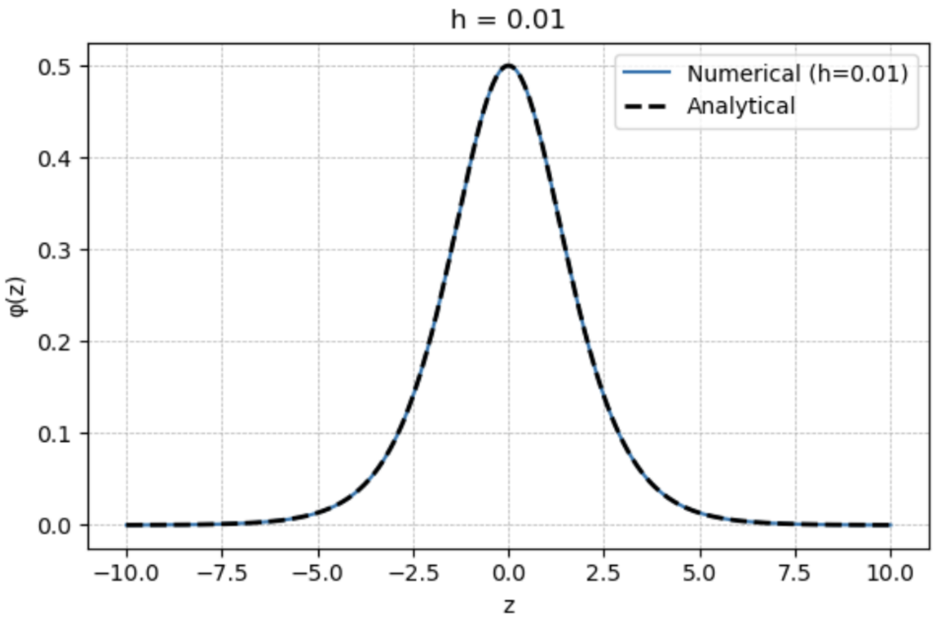


Figure 4

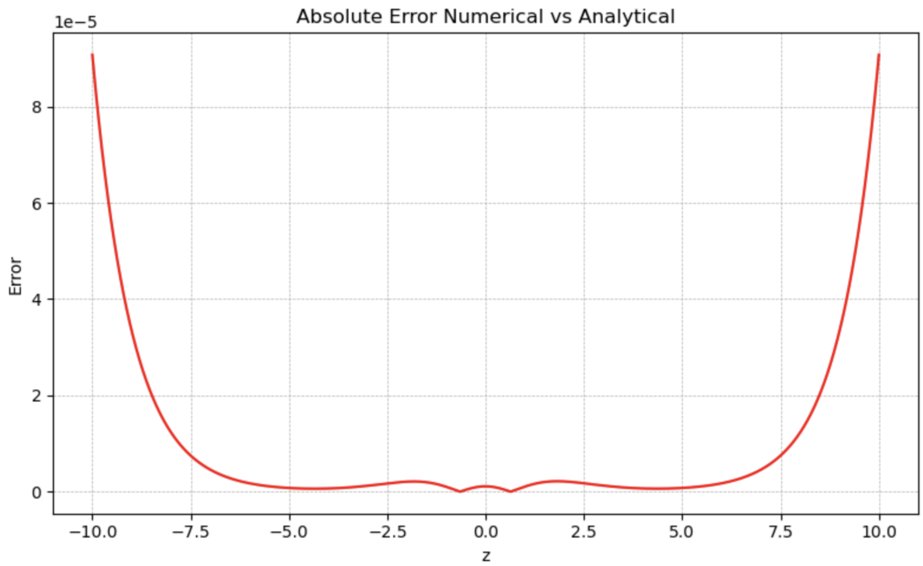


Figure 5

4 Time Evolution

In this section, we will look at solving the KdV equation numerically by discretising the equation and using iterative methods to give an approximate time-dependent solution given the initial conditions. We will then look at one way to check the validity of our numerical solution by finding conserved quantities of the equation.

4.1 Discretising the KdV equation

Firstly, we rearrange the equation so that the spatial derivative terms (u_x and u_{xxx}) are on one side.

$$u_t + uu_x + u_{xxx} = 0$$

$$\implies u_t = -uu_x - u_{xxx}.$$

We are interested in solving this numerically on an interval $[-l, l]$ for some arbitrary l. We use an evenly spaced grid of N points in the interval by defining $x_i = -l + i\Delta x$, where $\Delta x = 2l/N$. We also define $t_n = n\Delta t$. We use the notation $u_i^n = u(x_i, t_n)$ for the sake of convenience [2]. The terms on the right side of the equation can be approximated using finite differences. A central difference approximation was used for all the points in the interval except for x_0 and x_N, and these are given below.

$$u_{xxx} \approx \frac{u_{i-2}^n - 2u_{i-1}^n + 2u_{i+1}^n - u_{i+2}^n}{2\Delta x^3}, \quad u_x \approx \frac{u_{i+1}^n - u_{i-1}^n}{\Delta x}$$

Evaluating the third derivative at a point x_i using central differences requires the value of $u(x)$ at the two points on either side of it, so a different method was used at x_0, x_1, x_{N-1} and x_N. At x_0 we use a forward difference approximation

$$u_{xxx} \approx \frac{u_{i+3}^n - 3u_{i+2}^n + 3u_{i+1}^n - u_i^n}{\Delta x^3}, \quad u_x \approx \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

and at x_N we use a backward difference approximation

$$u_{xxx} \approx \frac{u_i^n - 3u_{i-1}^n + 3u_{i-2}^n - u_{i-3}^n}{\Delta x^3}, \quad u_x \approx \frac{u_i^n - u_{i-1}^n}{\Delta x}.$$

At x_1 and x_{N-1} we use central differences for u_x and forward/backward differences for u_{xxx}. From this we obtain the discretised equation (at all points except x_0, x_1, x_{N-1}, x_N)

$$\frac{du_i}{dt} = \frac{u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n}{2\Delta x^3} - u_i^n \left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right]$$

and a similar equation can be found for each of the points x_0, x_1, x_{N-1}, x_N.

4.2 Numerical Solution

We can now use numerical methods to solve for the value of u^{n+1}. We will use the 4th order Runge-Kutta method [3] to do this.

Let the step size be Δt, and u^0 be the initial conditions. For each point x_i in the interval we need to compute the value of $k_{1,i}$, $k_{2,i}$, $k_{3,i}$ and $k_{4,i}$, where

$$k_{1,i} = \frac{u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n}{2\Delta x^3} - u_i^n \left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right],$$

$$k_{2,i} = \frac{(u^n_{i+2} + \frac{\Delta t}{2}k_{1,i}) - 2(u^n_{i+1} + \frac{\Delta t}{2}k_{1,i}) + 2(u^n_{i-1} + \frac{\Delta t}{2}k_{1,i}) - (u^n_{i-2} + \frac{\Delta t}{2}k_{1,i})}{2\Delta x^3}$$
$$-(u^n_i + \frac{\Delta t}{2}k_{1,i})\left[\frac{(u^n_{i+1} + \frac{\Delta t}{2}k_{1,i}) - (u^n_{i-1} + \frac{\Delta t}{2}k_{1,i})}{2\Delta x}\right],$$

$$k_{3,i} = \frac{(u^n_{i+2} + \frac{\Delta t}{2}k_{2,i}) - 2(u^n_{i+1} + \frac{\Delta t}{2}k_{2,i}) + 2(u^n_{i-1} + \frac{\Delta t}{2}k_{2,i}) - (u^n_{i-2} + \frac{\Delta t}{2}k_{2,i})}{2\Delta x^3}$$
$$-(u^n_i + \frac{\Delta t}{2}k_{2,i})\left[\frac{(u^n_{i+1} + \frac{\Delta t}{2}k_{2,i}) - (u^n_{i-1} + \frac{\Delta t}{2}k_{2,i})}{2\Delta x}\right],$$
$$k_{4,i} = \frac{(u^n_{i+2} + \Delta t k_{3,i}) - 2(u^n_{i+1} + \Delta t k_{3,i}) + 2(u^n_{i-1} + \Delta t k_{3,i}) - (u^n_{i-2} + \Delta t k_{3,i})}{2\Delta x^3}$$
$$-(u^n_i + \Delta t k_{3,i})\left[\frac{(u^n_{i+1} + \Delta t k_{3,i}) - (u^n_{i-1} + \Delta t k_{3,i})}{2\Delta x}\right].$$

Then

$$u^{n+1} = u^n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

This completes one iteration of the Runge-Kutta method, which gives us the solution after one time step Δt. This method can be implemented in Python, which was used to produce the plot in Figure 6.

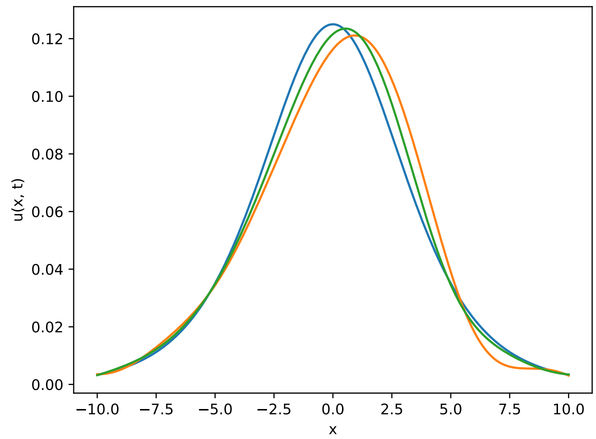


Figure 6 Numerical solution to the KdV equation on $[-10, 10]$ using RK4 at $t = 1$ (in green) and $t = 2$ (in orange) with initial conditions $u(x, 0) = \frac{1}{8}\text{sech}^2(\frac{x}{4})$ (in blue), $\Delta x = 0.1$ and $\Delta t = 10^{-3}$.

The figure shows the time evolution of the solution given solitary wave initial conditions. We can see that the wave maintains its shape as predicted by the analytical solution.

4.3 Conserved Quantities

We can use the concept of conserved quantities to verify our numerical solution. If we look at the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

we can rewrite it as

$$u_t + \left(\frac{1}{2}u^2 + u_{xx}\right)_x = 0.$$

We can then consider the boundary conditions for the equation. We require $u, u_x, u_{xx} \to 0$ as $x \to \pm\infty$. Integrating with respect to x over $\mathbb{R}$, we get

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx + \left[\frac{1}{2}u^2 + u_{xx}\right]_{-\infty}^{+\infty} = 0.$$

The boundary conditions imply that

$$\frac{d}{dt}\int_{-\infty}^{+\infty} u\,dx = 0 \implies \int_{-\infty}^{+\infty} u\,dx = \text{const}.$$

This conserved quantity can be interpreted as the conservation of mass in the system.

Naturally, we might then ask whether there are any other conserved quantities. Let us multiply all the terms of the KdV equation by u to get

$$\begin{gathered} uu_t + u^2u_x + uu_{xxx} = 0 \\ \implies \left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x + uu_{xxx} = 0. \end{gathered}$$

This time, we need to use integration by parts on uu_{xxx} as it is not in the form of the "chain rule". This gives us

$$\begin{gathered} \left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3 + uu_{xx}\right)_x - u_xu_{xx} = 0 \\ \implies \left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3 + uu_{xx} - \frac{1}{2}u_x^2\right)_x = 0 \end{gathered}$$

Integrating with respect to x and applying the boundary conditions in the same way as before, we get

$$\frac{d}{dt}\int_{-\infty}^{+\infty} \frac{1}{2}u^2\,dx = 0 \implies \int_{-\infty}^{+\infty} u^2\,dx = \text{const}.$$

In fact, there are actually infinitely many conserved quantities. We can show this using the Gardner transformation [1, 4]

$$u = w + \varepsilon w_x - \frac{1}{6}\varepsilon^2 w^2.$$

We need to substitute this into the KdV equation $u_t + uu_x + u_{xxx} = 0$. We then calculate each term:

$$\begin{gathered} u_t = w_t + \varepsilon w_{xt} - \frac{1}{3}\varepsilon^2 ww_t = \left(1 + \varepsilon\frac{\partial}{\partial x} - \frac{1}{3}\varepsilon^2 w\right) w_t \\ uu_x = (w + \varepsilon w_x - \frac{1}{6}\varepsilon^2 w^2)(w_x + \varepsilon w_{xx} - \frac{1}{3}\varepsilon^2 ww_x) \\ u_{xxx} = w_{xxx} + \varepsilon w_{xxxx} - \frac{1}{3}\varepsilon^2 (ww_x)_{xx} \end{gathered}$$

We then add the terms together and after some algebra, we get

$$u_t + uu_x + u_{xxx} = \left(1 + \varepsilon\frac{\partial}{\partial x} - \frac{1}{3}\varepsilon^2 w\right)\left(w_t + ww_x + w_{xxx} - \frac{1}{6}\varepsilon^2 w^2 w_x\right)$$

From this equation we notice that if w satisfies the equation

$$w_t + ww_x + w_{xxx} - \frac{1}{6}\varepsilon^2 w^2 w_x = 0,$$

then u is a solution to the KdV equation.

Using the same method as before, we find that

$$\int_{-\infty}^{+\infty} w\,dx = \text{const.}$$

If we rewrite w in terms of a power series in ε [1],

$$\sum_{n=0}^{\infty} \varepsilon^n w_n$$

we can compare coefficients of ε^n to find the coefficients w_n. We write out the power series expansions for w, w_x and w^2.

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ ...$$

$$w_x = w_{0x} + \varepsilon w_{1x} + \varepsilon^2 w_{2x} + ...$$

$$w^2 = (w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ ...)^2 = w_0^2 + 2\varepsilon w_0 w_1 + \ ...$$

We substitute this back into the Gardner transformation which gives

$$u = w_0 + \varepsilon(w_1 + w_{0x}) + \varepsilon^2 \left(w_2 + w_{1x} - \frac{1}{6} w_0^2 \right) + O(\varepsilon^3)$$

Comparing coefficients, we obtain the values of w_n.

$$w_0 = u$$

$$w_1 + w_{0x} = 0 \implies w_1 = -w_{0x} = -u_x$$

$$w_2 = -w_{1x} + \frac{1}{6} w_0^2 \implies w_2 = u_{xx} + \frac{1}{6} u^2 \quad \text{etc.}$$

Since $\int_{-\infty}^{+\infty} w\,dx$ = constant, we have that $\int_{-\infty}^{+\infty} w_n\,dx$ = constant, which implies that the coefficients w_n are conserved quantities of the KdV equation.

We can use numerical integration to find the values of $\int u\,dx$ and $\int u^2\,dx$ for our numerical solution. Let

$$U_1(t) = \int_{-l}^{l} u(x,t)\,dx,$$

$$U_2(t) = \int_{-l}^{l} [u(x,t)]^2\,dx.$$

We can compute the values of U_1 and U_2 for different values of t using Python. We will use Simpson's rule to integrate the solution. After that we will calculate the absolute error for U_1 and U_2

$$|U_1(t) - U_1(0)|, \quad |U_2(t) - U_1(0)|.$$

The results are given in Figure 7, which suggests that the absolute error increases over time. Increasing the size of the interval (while keeping every other variable the same) seemed to have the effect of reducing the absolute error. However, there seemed to only be a small change in the absolute error of U_2 compared to U_1 when doing this. The absolute errors are very small, which suggests that our numerical solution is a good approximation to the analytical solution.

t	$\lvert U_1(t) - U_1(0)\rvert$	$\lvert U_2(t) - U_2(0)\rvert$
0.1	5.882e-12	3.941e-10
0.2	1.707e-11	1.571e-9
0.3	6.019e-10	3.527e-9
0.4	1.375e-9	6.251e-9
0.5	2.189e-9	9.730e-9
0.6	2.971e-9	1.395e-8
0.7	3.611e-9	1.889e-8
0.8	4.013e-9	2.452e-8
0.9	8.039e-9	3.085e-8
1.0	2.208e-9	3.779e-8

Table 1 Errors on interval $[-30, 30]$

t	$\lvert U_1(t) - U_1(0)\rvert$	$\lvert U_2(t) - U_2(0)\rvert$
0.1	1.776e-16	3.932e-10
0.2	1.457e-14	1.541e-9
0.3	2.433e-14	3.527e-9
0.4	4.263e-15	6.251e-9
0.5	2.309e-14	9.730e-9
0.6	5.382e-14	1.395e-8
0.7	1.208e-13	1.889e-8
0.8	4.714e-13	2.452e-8
0.9	1.890e-12	3.084e-8
1.0	1.004e-11	3.781e-8

Table 2 Errors on interval $[-50, 50]$

Figure 7 Absolute errors of U_1 and U_2 solving the KdV equation on the intervals $[-30, 30]$ and $[-50, 50]$ using the same initial conditions as in Figure 6, with $\Delta x = 0.1$ and $\Delta t = 10^{-3}$.

5 Linear Stability Theory of Solitary Waves

In this section, we will look at the linear stability of solitary waves. Our aim is to determine the behaviour of a solitary wave $u_0(x - ct)$ after a small perturbation $u_1(x, t)$ is added.

$$u = u_0(\xi) + \epsilon u_1(x, t) + O(\epsilon^2) \quad \text{where} \quad \xi = x - ct$$

We assume that it is a small perturbation, so $\epsilon \ll 1, \quad |\epsilon u_1| \ll |u_0|$.

$$\begin{aligned} u_t &= -cu_0' + \epsilon u_{1t}, \\ u_x &= u_0' + \epsilon u_{1x}, \\ u_{xxx} &= u_0''' + \epsilon u_{1xxx}. \end{aligned}$$

Now we plug u back into the KdV equation, and we have

$$-cu_0' + \epsilon u_{1t} + (u_0' + \epsilon u_{1x})(u_0 + \epsilon u_1) + u_0''' + \epsilon u_{1xxx} = 0.$$

Group the terms with respect to ϵ, we get

$$O(1) : -cu_0' + u_0 u_0' + u_0''' = 0, \tag{4}$$
$$O(\epsilon) : u_{1t} + u_1 u_0' + u_0 u_{1x} + u_{1xxx} = 0. \tag{5}$$

In previous work, we showed that for a solitary wave $u_0(x - ct)$, it must satisfy equation (4). So we only need to think about equation (5). To simplify further, we move into a travelling frame $\xi = x - ct$, we let $u_1(x, t) = \phi(x - ct, t) = \phi(\xi, t)$. Then

$$\begin{aligned} u_{1t} &= \partial_t \phi(\xi, t) = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} = \phi_t - c\phi_\xi \\ u_{1x} &= \partial_x \phi(\xi, t) = \frac{\partial \phi}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = \phi_\xi \\ u_{1xxx} &= \phi_{\xi\xi\xi} \end{aligned}$$

Plug them into Equation (5) we have

$$\phi_t + (u_0 - c)\phi_\xi + u_0'\phi + \phi_{\xi\xi\xi} = 0$$

Now ansatz $\phi(\xi, t) = e^{\lambda t}\tilde{\phi}(\xi)$ we have

$$\lambda e^{\lambda t}\tilde{\phi} + (u_0 - c)e^{\lambda t}\tilde{\phi}_\xi + u_0' e^{\lambda t}\tilde{\phi} + e^{\lambda t}\tilde{\phi}_{\xi\xi\xi} = 0 \tag{6}$$
$$\lambda\tilde{\phi} = -(u_0 - c)\tilde{\phi}_\xi - u_0'\tilde{\phi} - \tilde{\phi}_{\xi\xi\xi} \tag{7}$$

5.1 Eigenvalues of Linear Operator

It is not an easy PDE to solve, but we can obtain a numerical result of eigenvalues. To do this, we can let D be the respective derivative matrix of $\tilde{\phi}$. By finite difference method, we have

$$D = \begin{bmatrix} 0 & 1/2h & 0 & \cdots & -1/2h \\ -1/2h & 0 & 1/2h & & \\ & \ddots & \ddots & \ddots & \\ & & -1/2h & 0 & 1/2h \\ 1/2h & 0 & \cdots & -1/2h & 0 \end{bmatrix}, \quad \tilde{\phi} = \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \vdots \\ \tilde{\phi}_{n-1} \\ \tilde{\phi}_n \end{bmatrix} = \begin{bmatrix} \tilde{\phi}(x_1) \\ \tilde{\phi}(x_2) \\ \vdots \\ \tilde{\phi}(x_{n-1}) \\ \tilde{\phi}(x_n) \end{bmatrix}$$

where x_i are points evenly spaced in closed interval$[-l, l]$. In equation 7, $(c - u_0)$ and u_0' can also be transformed to a diagonal matrix with entries the same as the value of $(c - u_0)$ and u_0' in the selected points.

$$(c - u_0) = \begin{bmatrix} c - u_0(x_1) & & & & \\ & c - u_0(x_2) & & & \\ & & \ddots & & \\ & & & c - u_0(x_{n-1}) & \\ & & & & c - u_0(x_n) \end{bmatrix}$$

$$u_0' = \begin{bmatrix} u_0'(x_1) & & & & \\ & u_0'(x_2) & & & \\ & & \ddots & & \\ & & & u_0'(x_{n-1}) & \\ & & & & u_0'(x_n) \end{bmatrix}$$

If we let the matrix L be such that $L = (c - u_0)D - u_0' - D^3$, then we successfully transform the equation into an eigenvalue problem which can be solved using Python.

$$\lambda\tilde{\phi} = \underbrace{\left[(c - u_0)D - u_0' - D^3\right]}_{L}\tilde{\phi}$$

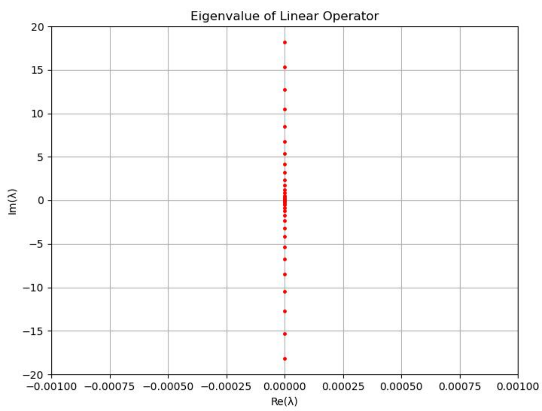


Figure 8 This is a graph showing the eigenvalues λ of matrix L in the complex plane. As the graph shows, all the eigenvalues are almost pure imaginary and many of them are close to 0.

Since real parts of the eigenvalues are all equal to 0, $|\phi|$ neither blows up to infinity nor decays to 0 forward in time. Hence, we conclude that the solitary waves are stable under linear perturbation.

5.2 Eigenfunctions of Linear Operator

The number of eigenvalues close to 0 somehow indicates the uniqueness of the eigenvalue $\lambda = 0$ and, indeed, the corresponding eigenfunction ϕ is exactly equal to u_0'. To show this, we first notice that a soliton solution of the KdV equation is translation-invariant, moving the soliton slightly in space gives another solution.

$$\tilde{u}(x,t) = u_0(x - ct + \delta)$$

This is a one-parameter family of solutions indexed by δ, the spatial shift. Taking the derivative with respective to δ, we get:

$$u_0'(\xi) = \frac{\partial}{\partial \delta} u_0(\xi + \delta)|_{\delta=0}$$

If we let $\phi = u_0'(\xi)$, then it will correspond to a infinitesimal spatial shift (or perturbation) of the soliton. Because the soliton is stable under spatial translations (i.e., small shifts do not change its shape or behaviour), the mode $\phi = u_0'$ does not grow or decay in time. Hence,

$$\phi_t = 0 \quad \text{and} \quad (u_0 - c)\phi' + u_0'\phi + \phi''' = 0$$

ϕ is the neutral mode of the linear operator. We can verify this identity by simply plugging in $\phi = u_0'$, we get

$$(u_0 - c)u_0'' + (u_0')^2 + u_0^{(4)} = 0$$

Now we use the soliton solution $u_0''' = cu_0' - u_0 u_0' \Rightarrow u_0^{(4)} = cu_0'' - (u_0')^2 - u_0 u_0''$.

$$\text{LHS} = (u_0 - c)u_0'' + (u_0')^2 + cu_0'' - (u_0')^2 - u_0 u_0'' = 0 = \text{RHS}$$

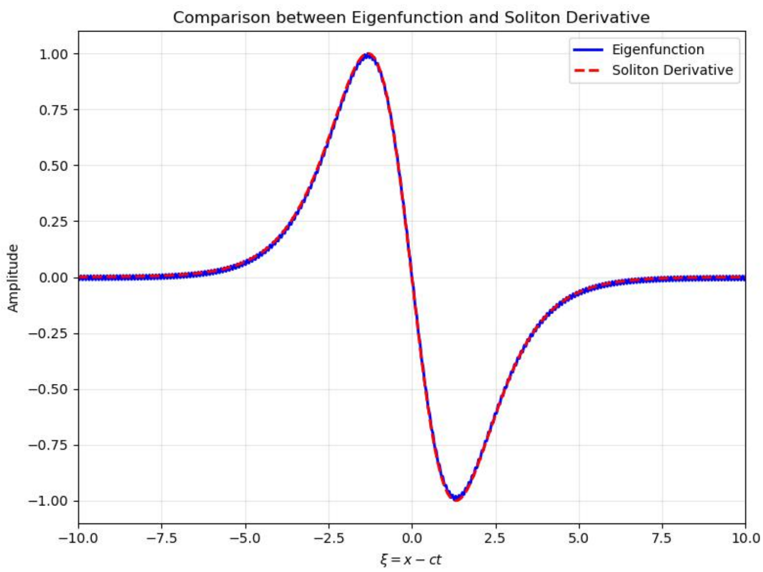


Figure 9 As the figure shows, the eigenfunction is pretty close to the derivative of u_0

Additionally, as the absolute value of the eigenvalue increases, the corresponding eigenfunctions exhibit more oscillations. Moreover, each eigenfunction is either even or odd.

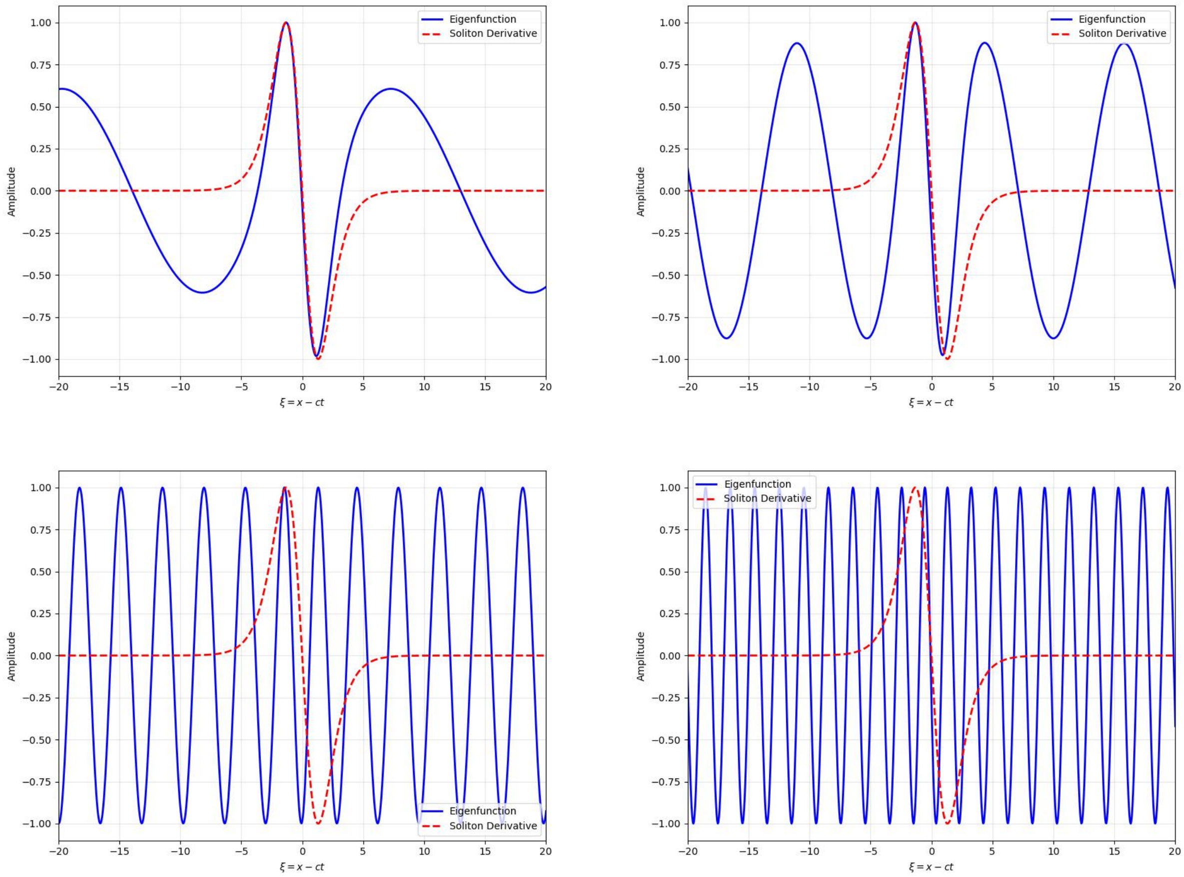
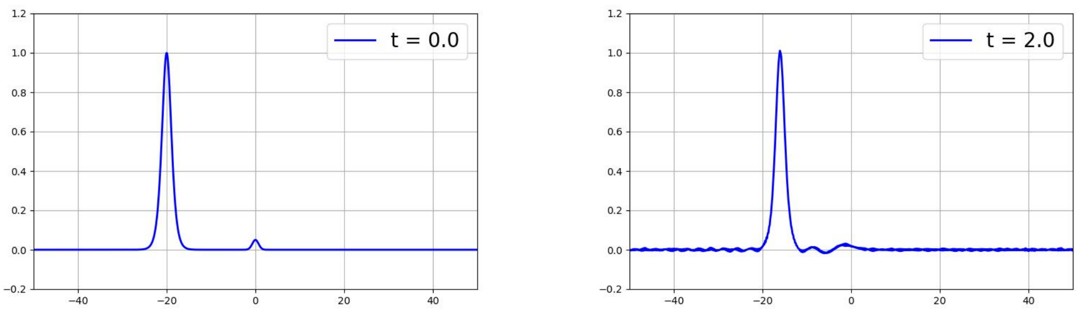


Figure 10 As the figures show, as the absolute value of the eigenvalue increases, the corresponding eigenfunctions exhibit more oscillations.

5.3 Visualization

To verify our result, we can use the Runge-Kutta method to simulate the evolution of the perturbed soliton. The perturbed soliton is $u_0(x, 0) = \text{sech}^2(\frac{\sqrt{2}}{2}(x + 20)) + 0.05e^{-x^2}$.



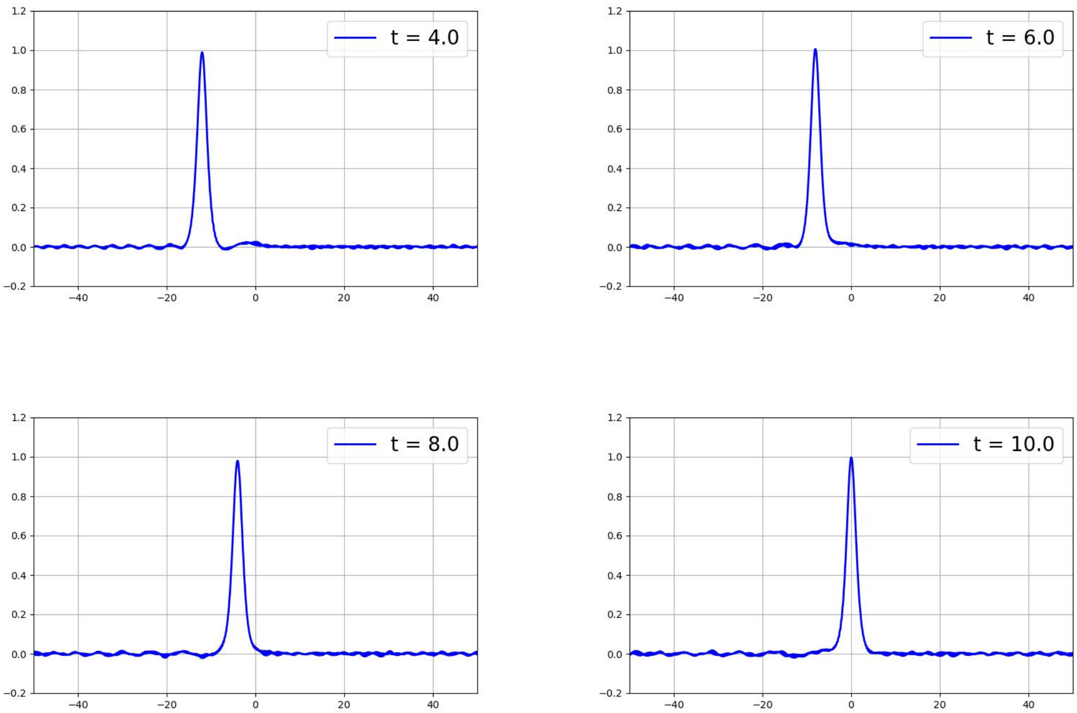


Figure 11 The 6 figures above demonstrate the time evolution from $t = 0$ to $t = 10$ of the perturbed soliton $u_0(x, 0) = \text{sech}^2(\frac{\sqrt{2}}{2}(x+20)) + 0.05e^{-x^2}$ where $0.05e^{-x^2}$ is a small perturbation at $x = 0$.

This is exactly the behaviour expected from the KdV equation's solitary wave solution. The main solitary wave remains dominant and exhibits the hallmark features of soliton dynamics.
In our simulation, the traveling speed was set to $c = 2$. The crest of the wave starts near $x = -20$ and steadily travels to the right, reaching approximately $x = 0$ by the time $t = 10$, all while maintaining its original shape. This persistence of form is a defining characteristic of solitons.
Additionally, a small perturbation added to the initial profile generates minor dispersive waves. These ripples neither blow up nor decay to zero over time but persist and gradually spread out from the main wave.

6 Conclusion

In this report, we have investigated methods of solving the KdV equation, and we mainly discussed solutions in the form of solitary waves.

In section 2, we found an analytical solution to the KdV equation in the form of a solitary wave. We then looked at the case of two-soliton solution where we considered a special initial condition which could give a solution of the interaction of two solitary-like waves, and we then showed there are a nonlinear interaction of the two solitons.

In section 3, we used a variant of Newton's method which allowed us to find the roots of a vector-valued function and we solved the non-linear system obtained by applying finite differences to the KdV equation. Following on from this we then evaluated the accuracy of the solution by investigating the convergence of the invariants and by comparing it to the analytical solution.

In section 4, we used finite differences again but with a different iterative method. Using the Runge-Kutta method we found a numerical solution that agrees with the analytical solution. We then found conserved quantities of the KdV equation which allowed us to check that our solution was a good approximation. We also used the Gardner transformation as a method to obtain conserved quantities.

Lastly, in section 5, we discussed the linear stability of solitary waves. We found that the solitary waves are stable under linear perturbation using the finite difference method and one of the special eigenfunctions of the linear operator is exactly the derivative of the original soliton solution. Using Runge-Kutta method, we successfully visualise the time evolution of perturbed soliton. The result is consistent with our analysis.

A Appendix

In Section 4 we applied the Runge-Kutta method to the discretised equation without going into any detail on the method itself. We will briefly mention it here.

Definition A.1 (4th order Runge-Kutta method)**.** Consider the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

Choose a step size $h > 0$, then

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$t_{n+1} = t_n + h,$$

where

$$k_1 = f(t, y),$$

$$k_2 = f\left(t + \frac{h}{2}, y + h\frac{k_1}{2}\right),$$

$$k_3 = f\left(t + \frac{h}{2}, y + h\frac{k_2}{2}\right),$$

$$k_4 = f(t + h, y + hk_3).$$

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