ChatGPT

=== PART A: RAW_INFO_DUMP ===

SECTION: KEY_REFERENCES - **REF1**: A. Sommerfeld (1909), *“Über die Ausbreitung der Wellen in der drahtlosen Telegraphie”*, Ann. Physik 28:665–736 – Classic paper formulating field of a source over a conducting half-space (ground); introduced integrals (Sommerfeld integrals) for layered media [1] [2]. - **REF2**: B. van der Pol (1935), *“Theory of the reflection of light from a point source by a finitely conducting flat mirror”*, Physica 2:843–853 – Early solution for a point source above a lossy (impedance) plane via a complex image; one of the first uses of complex source points for partial reflection [3] [4]. - **REF3**: I. V. Lindell & E. Alanen (1984), *“Exact image theory for the Sommerfeld half-space problem, I–III”*, IEEE Trans. Antennas Propag. 32(2):126–133; 32(8):841–847 – Derived exact image representations (continuous image distributions) for dipoles (vertical magnetic dipole in Part I, vertical electric dipole in Part II) above a dielectric half-space [5] [6]. Established that no finite number of discrete real images can exactly replace the continuous induced charge for general orientations. - **REF4**: D. G. Fang, J. J. Yang, G. Y. Delisle (1988), *“Discrete image theory for horizontal electric dipoles in a multilayered medium”*, IEE Proc. H 135(5):297–303 – Introduced a method to replace the Sommerfeld integral for N-layer media with a finite sum of discrete complex image dipoles. Demonstrated a rapidly convergent complex image series for horizontal dipoles in multilayered dielectric media [7]. - **REF5**: Y. L. Chow, J. J. Yang, D. G. Fang, G. E. Howard (1991), *“Complex images for electrostatic field computation in multilayered media”*, IEEE Trans. Microwave Theory Tech. 39(7):1120–1125 – Presented a rapidly convergent series of image charges (at complex locations) for static potentials in layered dielectrics. Gave closed-form Green’s functions for a thick microstrip substrate using a few complex image terms [8]. - **REF6**: Y. L. Chow, J. J. Yang, K. D. Srivastava (1992), *“Complex images of a ground electrode in layered soils”*, J. Appl. Phys. 71(2):569–576 – Applied the complex image method to grounding analysis; provided discrete complex image solutions for a point current source (electrode) in multilayered soil (static/quasi-static regime). Showed how image parameters vary with soil layering and resistivity. - **REF7**: M. I. Aksun (1992, 1996), *“Derivation of closed-form Green’s functions for a general microstrip geometry”*, IEEE Trans. MTT 40(11):2055–2062; and *“A robust approach for the derivation of closed-form Green’s functions”*, IEEE Trans. MTT 44(5):651–658 – Developed systematic rational approximation techniques to obtain closed-form Green’s functions in layered media. Used partial fraction expansions of spectral-domain Green’s functions, yielding complex image terms analytically [9] [10]. - **REF8**: K. A. Michalski & J. R. Mosig (1997), *“Multilayered media Green’s functions in integral equation formulations”*, IEEE Trans. Antennas Propag. 45(3):508–519 – Comprehensive review of Green’s function computation for multilayer media. Discusses Sommerfeld integral challenges and various acceleration methods (including DCIM) [11] [12]. Emphasizes that DCIM provides speed-ups but requires careful validation [13]. - **REF9**: A. Taflove et al. (2005), *“Complex Image Method”*, Wave Motion 43(1):91–97 (Taraldsen) – Historical perspective and simple derivation of the complex image method for acoustic and EM waves. Reviews how image solutions for impedance planes and layered media were rediscovered multiple times [3] [14]. Clarifies that the static limit (Laplace equation) is a special case of the Helmholtz equation ($k \to 0$). - **REF10**: P. Qin, Z. Xu, W. Cai, D. Jacobs (2009), *“Image charge methods for a three-dielectric-layer hybrid solvation model of biomolecules”*, Commun. Comput. Phys. 6(5):955–977 – Extended image charge methods to three-layer spherical and planar geometries in electrostatics. Developed least-squares approaches to fit discrete image charges for a dielectric slab (intermediate “buffer” layer) [15] [16]. Provided tables of optimized image positions and strengths for various permittivity contrasts.

SECTION: GEOMETRY_HALF_SPACE (TWO REGIONS) - **Geometry**: Single planar interface at $z=0$ separating medium 1 ($z>0$, permittivity ε_1) and medium 2 ($z<0$, permittivity ε_2). Source: point charge q at position $(0,0,z_0)$

in region 1 (distance z_0 above interface). - **Governing equation**: $\nabla\cdot(\varepsilon\,\nabla\varphi) = -q\,\delta(r - r_0)$ (electrostatic Poisson's equation). Boundary conditions at z=0: continuity of potential φ and of normal displacement $\varepsilon\,\partial\varphi/\partial z$ [17] [18]. - **Exact Image Solution** (electrostatic): The potential in region 1 can be obtained by placing an image charge q′ in region 2 and an image charge q″ in region 1, such that boundary conditions are satisfied [19] [20]. **EQ1**: $\varphi_1(r) = \frac{1}{4\pi\varepsilon_0\varepsilon_1}$

$$\frac{q}{R_1} + \frac{q\prime}{R_2}$$

with R_1 = distance from real charge ($|r - r_0|$) and R_2 = distance from image charge [21] [22]. In region 2, $\varphi_2(r) = \frac{1}{4\pi\varepsilon_0\varepsilon_2}$

$$\frac{q''}{R_1}$$

(image source q″ at the location of the real charge, used for z<0 field) [23] [24]. - **PARAM**: q′ = image charge located at $(0,0,-z_0)$ in medium 2 (mirror position of source across interface). q″ = an "image" located at the source position but contributing only in region 2's potential (accounts for field transmitted into medium 2) [23] [24]. - **Matching conditions**: Enforce φ continuous at z=0 and $\varepsilon_1\,\partial\varphi/\partial z|\{0^+\} = \varepsilon_2\,\partial\varphi/\partial z|$ [17] [25]. Solving yields **q′** = $-\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}$ q and **q″** = $\frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1}$ q [26] [27]. - **Reflection coefficient**: The factor $\rho = \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}$ emerges as the effective "reflection coefficient" for the potential. Here q′ = -ρ q and q″ = (1 - ρ) q (since $1 - \rho = 2\varepsilon_2/(\varepsilon_1+\varepsilon_2)$) [26] [28]. **FACT**: For $\varepsilon_2 > \varepsilon_1$ (e.g. point charge over water), ρ is positive, giving an induced image charge of opposite sign (attractive) [29] [28]. If $\varepsilon_2 < \varepsilon_1$, ρ is negative (image of same sign, repulsive force). If $\varepsilon_2 \to \infty$ (conducting ground), ρ → +1 and q′ → -q (recovering the classic image charge of equal-and-opposite magnitude) [28] [30]. - **Induced surface charge**: The presence of q induces a bound surface charge density on the interface z=0 given (for r from the vertical axis) by $\sigma_b(r) = -\frac{q}{2\pi\varepsilon_1}\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right)\frac{z_0}{(r^2+z_0^2)^{3/2}}$ [28] [31]. Integrating this continuous charge distribution exactly reproduces the field of the image charges above. - **Dipole generalization**: A vertical electric dipole (VED) oriented normal to the interface can be treated as two closely spaced point charges and similarly yields image dipoles. A horizontal dipole (parallel to interface) cannot be satisfied by a single image dipole; it requires a continuous image distribution or multiple images (see **REF3**, **REF4**). Lindell & Alanen derived an exact line-source image distribution for a vertical dipole [5] [6], while a horizontal dipole's field was later expressed via discrete complex images (approximate) by Fang et al. [7]. - **REF**: Analytical solutions for a dielectric half-space are discussed in advanced EM texts (e.g. Smythe, Stratton) and by Jackson. The above result is confirmed in standard references [19] [25]. The method-of-images solution for this configuration (finite number of images) is unique to the static case; for time-varying fields, an infinite spectrum is needed (no simple finite images) [32] [33].

SECTION: GEOMETRY_THREE_LAYER (PLANAR SLAB) - **Geometry**: Three homogenous regions with planar boundaries at z=0 and z=h. Region 1: z > 0, permittivity ε_1. Region 2: 0 < z < h, permittivity ε_2 (slab of thickness h). Region 3: z < h, permittivity ε_3 (half-space below). Source: point charge or dipole located in one of the regions (commonly region 1 above the slab or inside the slab). - **Sommerfeld representation**: The Green's function can be written as the free-space potential plus an image part from reflections at the two interfaces. **EQ2** (region 1 example): $\varphi_1(\rho,z) = \frac{1}{4\pi\varepsilon_0\varepsilon_1}\frac{q}{\sqrt{\rho^2+(z-z_0)^2}} + \frac{1}{4\pi\varepsilon_0\varepsilon_1}\frac{1}{2\pi}\int_0^\infty R_{eff}(k)\,e^{-k(z+z_0)}J_0(k\rho)\,dk$. Here the first term is direct Coulomb field and the second term is the contribution of all reflections, expressed as a Sommerfeld integral [34] [35]. R_eff(k) is the net reflection coefficient seen by a wave of horizontal wavenumber k incident into the slab. - **Effective reflection**

coefficient: For the 3-layer slab, $R_{eff}(k) = R_{12} + \frac{T_{12}T_{21}R_{23}e^{-2\gamma_2 h}}{1-R_{21}R_{23}e^{-2\gamma_2 h}}$. Here R_ij are Fresnel reflection coefficients between region i and j, and T_{ij} the transmission coefficients. In electrostatics (ω→0, $\gamma_2 = k$ for evanescent decay), this simplifies to a rational function of e^{-2k h}. **FACT**: as k→0 (very low spatial frequency, quasi-static fields), e^{-2k h} → 1, so R_eff tends toward $\frac{\varepsilon_2(\varepsilon_3-\varepsilon_1)}{\varepsilon_3(\varepsilon_2+\varepsilon_1)}$ independent of k (a constant). This static limit causes the integrand to have a **pole at k=0** (non-decaying term), corresponding to an infinite-range image contribution. The lack of exponential decay in static case means the image series does **not** naturally truncate. - **Infinite image series**: Each reflection at one interface produces a transmitted wave that reflects at the other interface, yielding an infinite sequence of round-trip reflections. In the static solution, this corresponds to an **infinite series of image charges**. Starting from the source q in region 1, one gets: an image q_1 in region 2 (reflection at first interface), then an image q_2 back in region 1 (after reflecting at second interface), then q_3 in region 2, etc., alternating ad infinitum [36] [37]. The image magnitudes follow a geometric progression by the product $\rho_{12}\rho_{23}$ (the reflection factors for each bounce). **EQ3**: $q_n = \rho_{12}(\rho_{21}\rho_{23})^{n-1}q$ for images in region 2, and similarly for those in region 1. Image positions alternate: e.g. for a source at $z_0 > 0$, images in region 2 at z = $-z_0$ (1st), z = $-z_0$ + 2h (3rd), z = $-z_0$ + 4h (5th), etc., and images in region 1 at z = z_0 + 2h (2nd), z_0 + 4h (4th), etc. (These positions correspond to successive “mirror” reflections across the boundaries of the slab). - **Convergence**: The series converges if |ρ_{21} ρ_{23}| < 1 (usually true for |ε contrast| < ∞). The largest term is typically the first image; subsequent images diminish by factor |ρ_{21} ρ_{23}|. However, for high-contrast interfaces (e.g. ε_2 >> ε_1, ε_3), ρ $\approx$ 1 and the series converges slowly. In practice, a finite sum of N images can approximate the field up to some error. (In extreme cases like a metal-backed dielectric ($\varepsilon_3 \to \infty$), one of ρ’s = -1 and the series becomes an alternating series; the static field inside a parallel-plate capacitor can be seen as summing an infinite image series). - **Special cases**: - If region 3 is a PEC ground ($\varepsilon_3 \to \infty$) at z = h, the series truncates: the bottom interface reflects with -1, eventually canceling further propagation. In fact, a charge above a dielectric slab over a PEC ground produces a finite or semi-infinite image series depending on ε_2. (Exact closed-form solutions exist for a charge between a dielectric slab and a conducting plane using method of images with image line charges [38] [39].) - If $\varepsilon_1 = \varepsilon_3$ (symmetric dielectric slab in air), the series can be summed in closed form by method of images (the configuration is symmetric, so effectively a periodic extension of two interfaces). The result is related to an infinite lattice of image charges spaced by 2h. - **Dipoles in layered slabs**: A dipole oriented perpendicular or parallel to the interfaces can be handled by taking derivatives of the scalar Green’s function or by using dyadic Green’s functions. The image method logic remains: each dipole induces image dipoles. For example, a vertical dipole in region 1 induces an infinite series of image dipoles in the slab and region 1. The strengths follow the same reflection coefficients as scalar charges (for the normal component of field, since Laplace’s equation decouples). Exact image formulas for dipoles in a two-layer slab generally require infinite series or complex source equivalents (no finite closed-form set of dipoles exists except in limiting cases). - **REF**: Many references provide formulas for potentials in 3-layer configurations using Sommerfeld integrals or image series [36]. Numerical convergence of the image series is discussed in literature on capacitive interfaces and multilayer electrostatics (e.g., **REF10** uses least-squares to fit 2–10 image charges for a three-layer solvent model, achieving <1% error [40] [41]).

SECTION: GEOMETRY_MULTI_LAYER (GENERAL N-LAYER) - **Geometry**: N stratified layers (permittivities ε_1,…,ε_N) with planar interfaces at z = h_1, h_2, …, h_{N-1}. Region 1 typically the top half-space (or finite layer) containing the source. Region N could be a bottom half-space or another bounded layer. - **Green’s function**: In general given by the free-space contribution plus an integral over horizontal wavenumber k (Sommerfeld integral) involving the **total reflection coefficient** R_total(k) seen by waves entering from region of source and scattering from the stratification. The reflection/transmission at each interface is accounted via a transmission-line or transfer-matrix approach [42] [43]. R_total(k) can be obtained by recursive formulas or continued fractions: - Recursion: starting from bottom region N (which if half-space has reflection = 0), compute effective reflection upward layer by layer. At each interface between layer j and j+1, use the known R_{j+1}^{eff}(k) for the semi-infinite stack below to find R_{j}^{eff}(k) for a stack starting at layer j: **EQ4**: $R_j^{eff}(k) = R_{j,j+1}(k) + \frac{T_{j,j+1}(k)T_{j+1,j}(k)\,R_{j+1}^{eff}(k)\,e^{-2\gamma_{j+1}d_j}}{1-R_{j+1,j}(k)\,R_{j+1}^{eff}(k)\,e^{-2\gamma_{j+1}d_j}}$, where d_j is thickness of layer j (if finite), and γ_{j+1} = k (static) or $\sqrt{k^2+\dots}$ (dynamic) in layer j+1. This is the generalization of the 3-layer formula. It yields R_1^{eff}(k) which is plugged into the integral for φ in region 1. - The above ensures **exact** satisfaction of boundary conditions but leaves an integral. Method-of-images aims to avoid the integral by expressing R_eff(k) (or the resultant potential) as a sum of simple terms. - **Image series**: By successive reflections, one obtains an infinite set of image charges in each layer. In principle, any source in such an N-layer environment induces bound charges at every interface, which can be represented by images that bounce indefinitely between all interfaces. For finite N with no conductors, the image sequence is doubly infinite (reflections up and down). For static fields, unless truncated by a perfectly conducting layer, the image series is infinite and typically slowly convergent if permittivity contrasts are large. - **Special scenarios**: Certain symmetric multi-layers can produce **resonant** responses (e.g., a high-ε layer can trap field lines). In static terms, this appears as image charges with very large magnitudes accumulating (denominator of R(k) → 0 for some k values indicating a trapped mode or guided mode pole [42]). These require many terms in any finite image expansion or a separate analytic treatment of the pole (e.g., a "guided mode" term corresponding to a bound charge layer). - **REF**: For detailed formulas, see e.g. **REF8** (gives Green's functions in closed-form for arbitrary N-layer via recursive algorithms) and classical multilayer electromagnetics texts (Brekhovskikh's *"Waves in Layered Media"* for spectral domain formulation). The recursive reflection coefficients approach is analogous to optical multilayer formulas (Fresnel reflections), which in electrostatics reduce to capacitance network analogs.

SECTION: SOMMERFELD_INTEGRALS - **Definition**: Sommerfeld integrals are the continuous spectral integrals representing the reflected field in layered media. In cylindrical coordinates (ρ, z), the potential of a point source can be expanded in Bessel (Hankel) functions horizontally and exponentials vertically. **EQ5** (half-space example): $\varphi(\rho,z) = \frac{q}{8\pi^2\varepsilon_0\varepsilon_1}\int_0^\infty \left(e^{-k|z-z_0|} + R(k)e^{-k(z+z_0)}\right)J_0(k\rho)\,dk.$ The term with R(k) under the integral represents the reflected contribution [44] [34]. - In electrostatics, for a dielectric interface, **R(k)** is *independent of k*: $R(k) = \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}$ (a constant) because there is no dispersion. This simplifies the integrand but actually makes the integral harder (no decay for large wavelength components). - For multilayers, R_eff(k) is generally a rational function of e^{-k d} or $\sqrt{k^2+\dots}$ factors. It introduces branch cuts (due to $\sqrt{k^2}$ in each layer's vertical propagation constant) and possibly poles (guided modes) [43] [42]. In static limit, the branch cut extends along the real k-axis from 0 to ∞. - **Branch cuts and convergence**: The static Green's function integrand decays as e^{-k(z+z_0)} for large k, ensuring the integral converges at the high-k end. However, as k→0, the integrand tends to R_eff(0) (a constant) times J_0(kρ). The J_0(kρ) ~ 1 for small k, so the integrand behaves ~ constant for k from 0 up to ~1/ρ. Thus the main contribution to the integral is from k ~ O(1/ρ). For large observation distances ρ, the integrand is dominated by small k where R_eff(0) applies – this yields the far-field behavior. - The **far-field expansion** (small k) can be obtained by expanding R_eff(k) around k=0. In static problems, R_eff(0) gives the leading term, and higher-order terms in k correspond to effective multipole corrections (e.g., an effective dipole layer at the interface). - The **near-field (large k)** corresponds to evanescent, short-range image contributions (like close to the source or interface). - **Evaluation**: Direct numerical integration of Sommerfeld integrals is challenging due to oscillatory Bessel function and slow decay. Techniques include: - Splitting into near-field and far-field parts and using series acceleration (e.g., asymptotic expansion for large k, plus numerical integration for moderate k) [45]. - Path deformation in the complex k-plane to avoid singularities (e.g., integrate slightly above the real axis to avoid branch cut) [46] [47]. - Analytical inversion via known integrals: e.g., using the identity $\int_0^\infty e^{-ka}J_0(k\rho)dk = \frac{1}{\sqrt{a^2+\rho^2}}$, one can perform integrals if R_eff(k) is written as sum of simple fractions (this is the basis for the complex image method). - **Sommerfeld identity & complex images**: If $R(k)$ or $R_{eff}(k)$ can be expressed as a sum of terms $\sum_n \frac{A_n}{k+\alpha_n}$ (or similar), then the integral can be evaluated term-by-term using tables of integrals. For example, a term like $\frac{1}{k+\alpha}$ leads to an exponential in real space: $\int_0^\infty e^{-kZ} \frac{1}{k+\alpha} J_0(k\rho) dk = e^{\alpha Z} K_0(\alpha\rho)$ (for α > 0, where K_0 is modified Bessel function). In the static case, α may be purely real or purely imaginary, yielding real exponentials or oscillatory terms. A sum of such terms yields a linear combination of known functions – which can often be interpreted as potentials of image charges at complex depths. - **REF**: Sommerfeld integrals are detailed in **REF1**, **REF8**. Classic treatments of their asymptotic evaluation and contour integration appear in Wait (1970) [48] and in textbooks by Felsen & Marcuvitz [49]. For static fields, the integrals reduce to forms involving K_0 and K_1 (modified Bessel functions) for certain simple cases (e.g., image of a line charge in a stratified medium yields such integrals). These can sometimes be summed or approximated by simpler functions (see rational approximation section).

SECTION: REFLECTION_COEFFICIENTS - **Static planar interface**: $R = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1}$. This is obtained by equating normal D = εE across the boundary in Laplace's equation. It also equals $-\frac{q'}{q}$ for the image charge ratio (with q′ placed in medium 2) [26]. **FACT**: If $\varepsilon_2 > \varepsilon_1$, then R is positive (<1), meaning the image charge has opposite sign (attractive polarization). If $\varepsilon_2 < \varepsilon_1$, R is negative, meaning the image in the second medium is like-charge (effectively repulsive boundary). - **Multiple layers**: For dynamic fields, reflection coefficient is k-dependent (Fresnel equations). For electrostatics (ω→0), it becomes purely a function of permittivities (no frequency dependence). However, for a **finite slab**, the effective reflection as seen from outside is: - **Exact form**: $R_{eff} = \frac{(\varepsilon_2 - \varepsilon_1) + (\varepsilon_3 - \varepsilon_2)\frac{\varepsilon_2}{\varepsilon_3}}{(\varepsilon_2 + \varepsilon_1) + (\varepsilon_3 + \varepsilon_2)\frac{\varepsilon_2}{\varepsilon_3}}$ in the static limit (obtained by letting k→0 in the general formula). This can be derived by treating the static field as current through capacitors (layer capacitance network). The above simplifies (for region3 half-space) to $R_{eff} = \frac{\varepsilon_2(\varepsilon_3 - \varepsilon_1)}{\varepsilon_3(\varepsilon_2 + \varepsilon_1)}$. If $\varepsilon_3 \to \infty$ (bottom a conductor), this further reduces to R_eff = +1 (meaning full reflection as conductor). - **Frequency dependence**: In quasi-static (low-frequency) analysis, R_eff(k) starts near the static value for small k, then transitions for larger k as field penetration changes. E.g., a lossy ground has R(k) transitioning from ~1 at small k (DC field sees a conductor if sufficiently thick) to a different value at high k (fields very close to source "see" the immediate medium). - **Internal reflection**: For a source inside a layer (say region 2), one can define two reflection coefficients: "upward" into region1 and "downward" into region3. The source induces fields that reflect off both top and bottom. The effective result can be expressed by multiple reflection series or by solving a linear system for image charges. Generally, the strength of images decays as powers of the product of these internal reflection coefficients. - **Impedance boundary**: If one side is a conducting surface with surface impedance Z (not perfect conductor), an effective permittivity ratio can be defined via the surface capacitance. In static limit, this is often treated as a boundary condition ∂φ/∂n = const * φ at the surface (Robin boundary). There is no simple point image for this case; instead an image line or continuous distribution is needed (analogous to the capacitive charging of the surface). - **Angular/Spectral dependence**: The reflection coefficient in spectral domain R(k) is analogous to the angular dependence of reflection in optics. In static case, "angle" is meaningless, but the dependence on k (horizontal wavenumber) corresponds to different field profiles. In full EM, R(k) for TE vs TM differ; in electrostatics, only one polarization (longitudinal E) matters. - **REF**: See Stratton's *Electromagnetic Theory* for derivation of reflection/transmission at a dielectric interface in static limit (it reduces to a simple fraction as above). **REF8** provides the general multilayer reflection coefficient in Eq. (59) and (60) for TE/TM modes [42], which in static become identical. Optical thin-film formulas (in the limit of zero frequency) also reduce to capacitance ratios giving the above results.

SECTION: COMPLEX_IMAGE_METHOD - **Principle**: The complex image method (CIM) seeks to replace the continuous induced-charge distribution with a finite set of *imaginary sources at complex coordinates*. Instead of an infinite series of real image charges, one finds a small number of charges located at depths $z_n \in \mathbb{C}$ (complex distances below the interface) with complex magnitudes q_n. These complex sources produce a potential in real space that matches the actual Green's function to a high accuracy [12] [13]. - A single complex charge at a complex depth $z' = a + ib$ corresponds to a potential $q'/(4\pi\varepsilon_0\varepsilon_1\sqrt{\rho^2 + (z - z')^2})$. For real observation z, this is generally complex, but the contributions from conjugate pairs of complex charges can be arranged to yield a real potential. **FACT**: Complex images often come as conjugate pairs $z' and \bar{z}'$ with complex conjugate strengths, so that their sum gives a real-valued field in physical region [14] [50]. - The imaginary part of a complex depth effectively provides an exponential lateral decay: e.g. an image at $z = -d \pm ib$ yields a term $\sim e^{-kb}$ in the integrand, damping the contribution for large horizontal wavenumber k. Thus, a cluster of complex images can mimic the effect of the continuous spectrum. - **History**: The concept originated from Sommerfeld's integrals (1909) and was articulated by van der Pol (1935) for electromagnetic wave reflection [3]. It was later used in acoustics (Morse & Bolt 1944) and reinvented in various contexts [51]. The method gained popularity in the 1980s in the antenna modeling community (e.g., Wait, Fung, Fang, Michalski) as the "discrete complex image method (DCIM)". It is particularly known from L. B. Felsen's work on complex source points and from its use in Green's function acceleration for Method of Moments. - **Derivation approach**: One approach is to start from the known spectral Green's function $G(k)$ and perform a **rational approximation**: find parameters {α_j, p_j} such that $R_{eff}(k) \approx \sum_{j=1}^{N} \frac{\alpha_j}{k+p_j}$ (or a similar partial-fraction form) [12]. Inverse-transforming term-by-term yields N image terms: each $\frac{\alpha_j}{k+p_j}$ corresponds to a complex point source at depth $z_j = p_j^{-1}$ (if form is 1/(k + p)). More directly, one may fit the spatial-domain Green's function (via sampling it and solving for image parameters). - **Techniques**: The Prony method / matrix-pencil method is used to fit a sum of exponentials to either the spatial field or the spectral kernel [52]. Vector fitting (Gustavsen's method) is another technique to robustly fit rational functions to frequency or k-dependent data. These yield the complex pole positions p_j and residues α_j. - **Alternate approach**: Use known asymptotic forms: e.g., at large ρ (far-field) the field looks like a dipole from an approximate image at some complex depth. At small ρ (near-field), it looks like a different distribution. This can guide choosing initial guesses for the image depths. - **Typical outcomes**: For a dielectric half-space, **N=1** complex image can already capture the main effect (especially for field points not too close to the interface). E.g., Wait (1962) found a single complex image current approximated the ground return impedance within 5% up to some distance. More commonly, **N=2 to 4** complex images are used for a half-space to achieve <1% error across all distances [13]. For a two-layer slab, **N ~ 5–10** images might be needed to span various regimes (the more layers or higher contrast, the more terms typically). - **Example**: Michalski and Mosig (1995) used DCIM with ~8–12 images to speed up microstrip Green's function calculations, achieving an order of magnitude speed-up in MoM matrix fill at the cost of ~1% error [12] [13]. Fang et al. (1988) managed to represent the field of a horizontal dipole over two interfaces with just a few image dipoles by solving for complex locations and moments [7]. - **Accuracy considerations**: DCIM lacks an *a priori* error bound – one must check the results. Typically, the error is highest near interfaces or very close to the source (where the field has singular or rapid variation that is hard to fit with few analytic terms). Also, outside the radius of convergence: for example, beyond some radial distance, the truncated set of exponentials might diverge or become unphysical (one needs to ensure images are placed such that the field decays at infinity). - **Improvements**: Adaptive DCIM chooses the number of images N based on required accuracy (e.g., keep adding image terms until the maximum error at some test points falls below tolerance). Modern approaches use optimization (least-squares fitting in space domain – see **REF10** for fitting image charges in a buffer layer) [53] [54]. They also treat separately any singular poles (e.g., guided modes): those are taken out of R(k) and added as discrete mode terms, then DCIM fits the remaining continuum part [55]. - **Interpretation**: Each complex image can be thought to represent an entire **bundle of multiple real images**. For instance, an infinite geometric series of real image charges can sum to a single complex image equivalent. This is how DCIM drastically reduces the count: instead of summing 100 real images with diminishing magnitude, one complex image in the right location can mathematically represent the sum. In practice, a few complex charges can reproduce the field of dozens of real charges. - **REF**: Comprehensive descriptions of DCIM are in papers like **REF5**, **REF8**. For electrostatics specifically, see Chow (1991) [8] for multilayer static images and Sarkar & Pereira (1995) [52] for parameter extraction. Taraldsen (2005) gives insight into why complex images appear for partial reflectors [14] [4]. Y. L. Chow's works (1991, 1992) are seminal in applying complex images to static and power-frequency problems (e.g., grounding).

SECTION: DISCRETE_COMPLEX_IMAGES_FOR_HALF_SPACE - **Electrostatic half-space**: As noted, the static half-space problem is exactly solvable by one real image charge (q′) and one transmitted charge (q″). Therefore, a complex image expansion is not needed in the static case – it's already an analytic solution. However, for time-varying fields (even quasi-static induction), one uses complex images. For example, a dipole above a lossy ground has an "image" at a complex depth corresponding to wave penetration (known as the Fresnel complex reflection method). - **DCIM usage**: In antenna modeling (full-wave), a half-space Green's function (Sommerfeld problem) is often approximated by 2–3 pairs of complex conjugate images for frequencies up to a certain range, capturing both the space wave and surface wave contributions [56] [5]. In the static limit, those images approach the real solution (one image). - **Near-boundary field**: If we restrict to static but consider other field components (like the tangential electric field or magnetic field of a current source), exact images may not exist. For instance, a horizontal current filament on a conducting half-space has no exact finite image solution; DCIM can approximate the induced current with a few complex line currents. This is analogous to the electrostatic horizontal dipole on a dielectric – requiring complex images to approximate the continuous induced charge. - **Typical complex images**: For a lossy half-space (with complex permittivity or conductivity), the image depth becomes complex naturally (Skin depth). E.g., for a ground of conductivity σ, an image for a DC current source might be placed at a depth $\sim\delta = \sqrt{(2/(\mu\sigma\omega))}$ which becomes very large as $\omega\to 0$ (hence for true DC, the method reverts to the exact image at infinite depth or effectively the real solution). - **Accuracy**: With one complex image, one can exactly match the field on the surface (z=0) at one radial distance (by choosing appropriate image parameters), but not all distances. More images improve the match uniformly. Empirically, ~3 complex images can yield <2% error for the potential everywhere above a dielectric half-space (for moderate contrasts) – these numbers come from fitting results in literature (e.g., Michalski 1995, test cases). - **Reflection coefficient approach**: Another way: approximate the constant $R = (\varepsilon_2-\varepsilon_1)/(\varepsilon_2+\varepsilon_1)$ by a rational function of k to introduce some k-dependence artificially – this can simulate the effect of a "fuzzy" interface or transition layer, yielding an effective complex image. This is sometimes done to avoid a sharp discontinuity in R at k=0 vs $k\to\infty$. - **REF**: Lindell & Alanen's "exact image theory" (**REF3**) basically showed that no finite discrete images exist for a general dipole – supporting the need for either infinite series or complex continuous image. The DCIM circumvents this by accepting complex locations (non-physical but mathematically valid). IEEE papers by Mosig and colleagues in the late 1980s demonstrate the application for half-space Green's functions.

SECTION: DISCRETE_COMPLEX_IMAGES_FOR_THREE_LAYER - **Challenges**: In a 3-layer scenario (two interfaces), the spectrum of reflections is richer (multiple poles corresponding to possible guided modes in the slab, and a branch cut). DCIM must approximate a function R_eff(k) that has a pole (if the slab can support a mode) and a step from k=0 to $k\to\infty$. Typically one separates the pole (guided mode) term and handles it analytically, then fits the remaining continuous part with exponentials [55] [57]. - **Fang's method (1988)**: Fang et al. found that for a horizontal dipole in a two-layer structure, one could place a small number of complex dipoles such that the boundary conditions at both interfaces are approximately satisfied [7]. Their approach effectively solved for images by enforcing the field match at a discrete set of points (collocation method). They reported using 2–3 complex image dipoles in each region to achieve good accuracy. - **Kipp & Chan (1994)**: Introduced a method for sources in bounded regions (like within the slab) using complex images [58]. They employed images both above and below the source layer. This required solving a system for image coefficients – essentially an eigenfunction expansion truncated to N terms. - **Empirical counts**: A 3-layer with moderate contrast (e.g. ε_2 ~ 5, $\varepsilon_1=\varepsilon_3=1$) might need ~4–6 complex images for ~1% accuracy in potential for z_0 around h/2. If ε_2 is very high (like a metal plate or water layer), more images are needed as the internal reflections are stronger. Conversely, if ε_2 is close to ε_1 and ε_3, fewer images suffice (the Green's function is closer to free-space). - **Techniques**: One effective strategy is to use the *image of an image* concept: first find complex images for the source with respect to the first interface (as if second interface weren't there), then reflect those images in the second interface, and then fit any residual difference. This can serve as initial guess for an optimization that tunes the depths and strengths. - **Example**: **REF10** (Qin et al. 2009) tackled a sphere with three dielectric regions by converting the infinite series (Legendre expansion) to a few image charges via least squares [59] [54]. When translated to planar geometry, similar math can be applied: they achieved accurate potentials with 2 images for moderate slab thickness (h) and noted error scaling on h (thinner slab easier to approximate) [60]. - **REF**: Fang 1988 (**REF4**) is a key reference for early DCIM in 3-layer. Aksun & Dural (1995) (**REF7**) provided closed-form Green's functions for stratified media, essentially solving the DCIM problem by algebraic means (they derive exact rational forms, which correspond to infinite image series or finite complex images if truncated). These are important for anyone implementing DCIM for multilayers.

SECTION: RATIONAL_APPROXIMATIONS - **Motivation**: Rational approximation of the spectral Green's function is the theoretical backbone of DCIM. By approximating $F(k) = R_{eff}(k)e^{-k(z+z_0)}$ (the integrand excluding J_0) as a sum of simple fractions, one achieves analytic integration. - **Vector fitting**: A well-known method (Gustavsen and Semlyen, 1999) to fit frequency responses with rational functions can be applied to $R_{eff}(k)$. This yields a set of poles p_j and residues α_j such that $R_{eff}(k) \approx \sum_j \frac{\alpha_j}{k-p_j}$ (for dynamic fields, these p_j correspond to complex propagation constants, in static one might use $k+p_j$ form). Once obtained, each term's inverse transform is known (usually from tables or known Fourier/Bessel transforms). - For static problems, one might fit the difference from the static limit. For example, since $R_{eff}(k \to 0) = R_0$, define $\tilde{R}(k) = R_{eff}(k) - R_0$ which goes to 0 at k=0 and typically ~ O(k^2) for small k. Fit $\tilde{R}(k)$ with $\sum \frac{\alpha_j}{k+p_j}$. This ensures the 1/k singularity at small k is avoided and the DC term is handled separately. - **Prony's method**: In spatial domain, one can sample the Green's function at a set of radial distances and fit a sum of decaying exponentials $G(\rho) \approx \sum B_n e^{-\lambda_n \rho}$. The exponents λ_n (which will be complex if oscillatory behavior is needed) are related to image depths ($\lambda_n = |Im(z_n)|$ typically). The matrix pencil method is an efficient algorithm for this and was advocated in **REF8** [52] for EM Green's functions. - **Error control**: Rational approximation allows use of known convergence results: one can increase the polynomial order until the coefficients stabilize. Tools from system identification (e.g. Akaike criterion or sigma plots) can indicate if more terms are needed. However, caution: approximating a function with a branch cut (like sqrt) by rationals over an interval is never exact; there is a trade-off between number of poles and accuracy. - **DCIM vs MoM**: In Method of Moments for antennas, rational fits of Green's functions turned out to be crucial for efficiency. By pre-fitting the Green's function, one avoids numerical integration for each matrix element. This perspective was a driver for developing DCIM in the 1990s (see Michalski & Mosig). - **Surface impedance analogy**: Sometimes instead of fitting R(k), authors fit the reflection coefficient by a Padé approximation around k=0 and k→∞. For instance, a Padé approximant of $R_{eff}(k)$ can produce a continued fraction which is effectively an L-C ladder network (each partial fraction = an image or mode). - **REF**: Sarkar & Pereira (1995) (**REF8**) is a classic reference on using the matrix pencil for complex exponentials [61]. Aksun (1996) (**REF7**) demonstrates an accurate rational fit for multilayer Green's functions, yielding closed forms without numerical integration. These methods show that many layered media Green's functions can be expressed to arbitrary precision as a sum of "simple" terms – which is essentially the theoretical justification for the method of complex images.

SECTION: ERROR_BEHAVIOR - **Near-interface field**: The hardest region to approximate is very close to an interface (within a small fraction of the source distance). Here the continuous induced surface charge has strong variations that might require many image terms to replicate. For example, the potential just above the interface ($z \to 0^+$) due to a source near the interface has a $1/\rho^2$ singularity from the induced charge distribution [31]. Approximating this singular behavior may need more images. Typically, one ensures the DCIM is exact or very accurate on the boundary (enforcing boundary conditions directly at z=0 helps). - **Large radial distance**: At $\rho >> z_0$, the field is dominated by low-order multipoles. A truncated image set can usually match the far-field by construction (e.g., ensuring total charge and dipole moment of images match the exact values yields correct $\rho \to \infty$ decay). Errors at far-field are usually small if the near-field is fitted well. - **High contrast**: If $(\varepsilon_2 - \varepsilon_1)/(\varepsilon_2 + \varepsilon_1)$ is close to ±1, the induced charge is almost as strong as the source. The image series decays slowly, and DCIM must capture a slowly varying function. This often requires more images or special treatment of the dominant image. For example, if $\varepsilon_2 >> \varepsilon_1$, the first image $q' \approx -q$ (like a conductor) and subsequent small difference cause a secondary series. One strategy is to separate the "perfect-conductor image" (which is known exactly) and only approximate the remainder (which is smaller). This improves convergence. - **Thin vs thick layers**: A very thin intermediate layer (h small) can cause rapid oscillations in R_eff as a function of k (like etalon effect). But in static limit, if h is small, one can often replace the thin layer by an equivalent interface with effective boundary condition (e.g. capacitance). DCIM can leverage this by maybe needing only one image to represent a thin layer's effect (since field lines mostly bypass it or treat it as modified interface). Conversely, a thick layer means the field experiences two well-separated interfaces; the image contributions group into two clusters (one cluster from the top interface reflection, another from bottom). In such cases, one can use a two-stage fit (fit images for top interface effect and bottom interface effect separately). - **Guided modes**: If a layer can support a guided mode (in static, a uniform field in a dielectric slab akin to a parallel plate capacitor mode), the Green's function will have a contribution that decays algebraically, not exponentially. DCIM, which uses exponentials, may struggle to approximate this component. The error will manifest as slowly decaying discrepancy at large ρ. Remedy: explicitly include the mode as a separate term (not via images, but an analytical term). For static fields, a guided mode is just a constant field in the slab – if needed, one could add a fictitious uniform sheet image to account for it. - **Error metrics**: Typically evaluated in percentage error of potential or field. DCIM papers often report max error at observation points normalized to the free-space field. For instance, a result might be: "Using 6 complex images, max error < 0.5% for $0.1 \leq z_0/h \leq 10$ and $\varepsilon_2/\varepsilon_1$ up to 10." Such data indicates the robustness range. If outside that range (e.g. extremely close source or extremely high contrast), error can spike to a few percent unless more images are added. - **REF**: **REF10** shows error distributions for a three-layer case with 2 image charges (their Fig. 5–6), indicating where the approximation is worst (usually near boundaries) [41] [62]. Mosig's review (**REF8**) notes that DCIM has no internal error estimate and must be validated by comparison to direct integration [13] [63]. Users often over-sample the Green's function after fitting to ensure error is below target.

SECTION: IMPLEMENTATION_NOTES - **Choosing initial image depths**: A practical heuristic: use the positions of the first few *real* images as starting guesses for complex ones. For example, for a slab problem, start with images at $z = -z_0$ (first interface) and $z = z_0 + 2h$ (second bounce) – then allow these to become complex during fitting. Another approach: sample the spectral function R_eff(k) and look for peaks or knee points; place initial poles near those frequencies. - **Grouping by interface**: Often it's beneficial to assign a set of images to each interface. E.g., for N-layer, you might use M images to represent reflections from the top interface, and another M for the bottom, etc. In optimization, you can constrain a group of images to lie near a certain interface (e.g., depths ~ a small imaginary part around that interface's location). This can simplify the physical interpretation: one group accounts for one reflection sequence. - **Enforcing physical constraints**: Ensure that the total induced charge equals what's required by boundary conditions. In half-space, the induced image charges should sum to - $(\varepsilon_2-\varepsilon_1)/(\varepsilon_2+\varepsilon_1) * q$ (which is exactly q′). If using multiple images, enforce $\sum_n q_n = q\prime$. This improves accuracy of the potential at large ρ (overall monopole term correct). Similarly, enforce dipole moments if possible. - **Symmetry**: Use symmetry to reduce unknowns. For a source on the axis (like above center of a disk, though here infinite plane symmetry implies azimuthal symmetry automatically), images will also be on the axis. If the problem has lateral symmetry (e.g., uniform in x direction, like a line source), one might reduce to 2D images line (method of images for line charges). - **Numerical stability**: When solving for image parameters by least squares, avoid points too close to source (singular behavior) or too far (very weak field). It's common to sample the Green's function at a set of points (ρ_i, z_j) forming a grid and solve a linear or non-linear system. Proper weighting of points (giving more weight near interfaces perhaps) yields a better balanced fit. - **Speed considerations**: Once the image expansion is obtained, evaluating the Green's function is *much* faster than integrating. For example, 10 image charges require O(10) operations vs an integral requiring adaptive quadrature each evaluation. This is crucial in simulations (like BEM or particle simulations) with many source-observation interactions [13]. - **Software**: There are known codes (e.g., AltaFEM's Green's function library, FEKO's near-ground Green's functions) that implement DCIM internally. These often come with pre-tabulated image parameters for typical substrate permittivities (for microwave circuits). In custom research, one might use symbolic math to invert small-N integrals. - **Extensions**: Complex images can also be used in time domain (by inverse Laplace transform rather than Fourier). E.g. a discharge in layered soil can be convolved with the static Green's function expanded in images – turning the convolution into a sum of exponentials (which is easier to inverse Laplace). - **REF**: Many implementation details are discussed in Antenna Propagation Magazine articles (e.g., papers by R. J. Luebbers on ground wave, or by Kipp & Chow). **REF8** provides general guidance (the need to handle guided poles, etc.) [64] [57]. **REF10** gives a modern perspective using least-squares and shows that even in complex three-layer cases, a well-chosen two-image model can be remarkably accurate [41] [65].

SECTION: OTHER_GEOMETRIES - **Sphere near a plane**: A point charge near a conducting sphere can be solved by a single image charge (Kelvin image) at a modified distance and magnitude. For a point charge q at distance D from center of a grounded conducting sphere (radius a, D > a), the exact solution is an image charge $q' = -q * (a/D)$ located at radius a^2/D inside the sphere [38] [32]. No complex images are needed (it's a closed-form image solution). However, if the sphere is dielectric (not conducting), an infinite series of image charges (of diminishing strength, inside and outside the sphere) is needed – analogous to the planar case but using Legendre expansions. - **Coated sphere (concentric dielectric layers)**: Known as the multiple shell problem – a point charge outside or inside yields an infinite series of multipole images (spherical harmonics). Method of images per se doesn't yield a few discrete charges except in special symmetry (e.g., charge at center yields a uniform polarization, equivalent to one image). - **Cylinder near a plane**: A line charge parallel to a conducting plane can be solved by an image line charge (like 2D analog of point and plane). A line charge near a dielectric half-space, however, yields an integral or infinite series of line images. - **Dipole near edges/corners**: Method of images gets complicated beyond infinite planar/spherical surfaces. For a corner or wedge, images may appear in multiples due to each surface. - **Layered sphere or cylinder**: These are more exotic, but e.g. a dielectric cylinder with coaxial layers (cable insulation) – one can use complex images along the axis (via eigenfunction expansions). Such cases show patterns similar to planar layers, but with Bessel functions in radial direction rather than exponentials. - **Periodic structures**: The method of images can handle periodicity by infinite image arrays. For example, charges in a layered medium with periodic lateral boundary conditions (say a unit cell repeated) can be handled by combining DCIM with Ewald summation [66] [67]. Essentially, one images the source across cell boundaries (real images), then uses complex images for the stratified effects. - **Summary of where images are finite**: - Finite discrete images: **Conductor planes** (1 image), **conductor spheres** (1 image), **concentric conductor**

shells (1 image per shell for symmetric internal/external charges), **two parallel conducting planes** (infinite series of real images, but can be summed by method of images analytically). - Infinite series needed: **Dielectric interfaces** (planar or spherical) – generally infinite images unless one side is a conductor. **Dielectric slab** – infinite images. - Complex images useful: practically any scenario where infinite series would appear – one trades the infinite real series for a handful of complex ones. - **REF**: Smythe's *Static and Dynamic Electricity* and Jackson's *Classical Electrodynamics* (Problem sections) cover many of these cases (sphere, cylinder, etc.) using image series or method of inversion. The image-charge method for dielectric spheres is treated in detail by (e.g.) Huang et al. (1980s, Journal of Electrostatics) and given in **REF5** for context of simulation algorithms [68]. These cases, while not directly planar, reinforce techniques like series truncation and fitting which parallel the planar layer approach.

=== PART B: MACHINE_SCHEMA ===


```
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    {
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      "representations": [
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          "type": "analytic_image",
          "equations": [
            "q' = (eps1 - eps2)/(eps1 + eps2) * q",
            "q'' = (2 * eps2)/(eps1 + eps2) * q",
            "V(r) = (1/(4π ε0 ε1)) [q/|r - r0| + q'/|r - r_img|]"
          ],
          "validity": "electrostatic, exact",
          "notes": [
            "Image charge q' placed at mirror location in medium 2; q'' is effective charge in region1 for field in region2",
            "Boundary conditions satisfied exactly with two images (one in each region) [26]"
          ]
        },
        {
          "type": "sommerfeld_integral",
          "equations": [
            "V(r) = (q/(8 π^2 ε0 ε1)) \\int_0^∞ [e^{-k|z-z0|} + R * e^{-k(z+z0)}] J0(k ρ) dk"
          ],
          "validity": "general solution, static or freq-domain",
          "notes": [
            "R = (eps2 - eps1)/(eps2 + eps1) (constant reflection coefficient) [29]",
            "Integral form with R constant can be evaluated in closed form,
```

```
yielding the same result as image method"
          ]
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        {
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          "equations": [
            "V(r) ≈ Σ_{n=1}^N a_n / √(ρ^2 + (z - z_n)^2)",
            "z_n are complex depths, a_n complex strengths"
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          "parameterization": "One complex image can emulate the continuous
interface charge; often not needed for pure static since exact real image
exists",
          "refs": ["REF3", "REF5", "REF8"]
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      ]
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      "description": "Three-region stratified medium: region1 (eps1) above z=0,
region2 (eps2) slab of thickness h, region3 (eps3) below z=h",
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      "representations": [
        {
          "type": "sommerfeld_integral",
          "equations": [
            "V1(r) = (q/(8 π^2 ε0 ε1)) \\int_0^∞ [e^{-k|z-z0|} + R_eff(k) e^{-
k(z+z0)}] J0(k ρ) dk"
          ],
          "validity": "static or frequency-domain (using k-dependent R_eff)",
          "notes": [
            "R_eff(k) = effective reflection coefficient of the slab [36] ",
            "Closed-form R_eff in static limit: R_eff = (eps2*(eps3 - eps1))/
(eps3*(eps2 + eps1)) (for half-space below) (no k dependence)",
            "Sommerfeld integral must be evaluated numerically or approximated
(source of complexity for layered Green's functions)"
          ]
        },
        {
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          "equations": [
            "V(r) = (1/(4πε0 ε1)) [q/|r-r0| + Σ_{n=1}^∞ q_n/|r - r_n| ]",
            "q_n = q * (R12 (R21 R23)^{n-1}) for images in region2, alternating
with region1"
          ],
          "validity": "electrostatic, series solution",
          "notes": [
            "Infinite series of real image charges located at mirror positions
```

```
through repeated reflections [37] ",
            "Converges if |R21 R23| < 1 (i.e., not lossless resonator). Truncation gives an approximation",
            "Used conceptually to derive or check complex image models"
          ]
        },
        {
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            "V(r) ≈ Σ_{n=1}^N a_n / √(ρ^2 + (z - z_n)^2)  (in region1)",
            "Additional sets of images in region2 and region3 if source is in region1 (to represent field in those regions)"
          ],
          "num_images_typical": [4, 6, 8],
          "parameterization": [
            "Images often come in conjugate pairs z = u ± i v for real potential",
            "Parameters fit such that boundary conditions at z=0 and z=h are approximated (via collocation or optimization) [7] "
          ],
          "refs": ["REF4", "REF7", "REF8"]
        }
      ]
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      "notes": ["Static reflection coefficient for potential (ratio of induced image charge -q' to source q) [26] "]
    },
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      "geometry": "three_layer_planar",
      "equations": [
        "R_eff(k) = R_{12} + \\frac{T_{12} T_{21} R_{23} e^{-2 k h}}{1 - R_{21} R_{23} e^{-2 k h}}"
      ],
      "parameters": ["eps1","eps2","eps3","h"],
      "notes": [
        "General k-dependent reflection for a slab (for TM polarization; static uses the same form with k as decay const) [37] ",
        "Static limit: e^{-2 k h}→1 as k→0, yielding R_eff(0) = (eps2*(eps3 - eps1))/(eps3*(eps2 + eps1))"
      ]
    }
```

```
  ],
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      ],
      "parameterizations": [
        "Usually 1 conjugate pair of images is enough for moderate accuracy (N=2) [13] ",
        "Matrix pencil or vector fit used on R(k) = const to force slight k-dependence and get a complex pole"
      ],
      "typical_N_images": [1, 2],
      "error_behavior": {
        "regimes": [
          "Near interface (z→0): small errors (< few %) since exact solution already simple",
          "Far-field ρ→∞: DCIM matches monopole term exactly, error negligible",
          "Overall, half-space static is trivial (exact); DCIM mainly relevant for time-varying or different source orientations"
        ]
      },
      "refs": ["REF5","REF8"]
    },
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"(e.g., one set ζ_m for images representing first interface, another η_n for second interface reflections)"
      ],
      "parameterizations": [
        "Solve for complex depths ζ_m, η_n by fitting boundary conditions at z=0,h (collocation at multiple ρ) [7] ",
        "Alternatively, fit R_eff(k) with partial fractions (poles) and invert transform [12] "
      ],
      "typical_N_images": [4,6,8, "depends on contrast and thickness"],
      "error_behaviour": {
        "regimes": [
          "Slow convergence if |R21*R23|≈1 (high contrast or resonant slab) – more images needed",
          "Uniformly good in mid-field; worst errors near boundaries as always",
          "Can achieve 1-2% error with ~6 images in many practical cases (e.g.
```

```
microstrip substrates) 13 "
        ]
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      "usage": [
        "Convert Sommerfeld integral to finite sum by partial fractions 12 ",
        "Perform term-by-term integration using known integrals (exponential
integrals, Bessel identities)",
        "Result is a sum of decaying exponentials or Bessel K0 functions –
interpretable as complex image contributions"
      ],
      "refs": ["REF7","REF8","REF5"]
    },
    {
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      "equations": [
        "G(ρ) = Σ_{n=1}^N C_n e^{-λ_n ρ}  (fit to sampled Green's function)"
      ],
      "usage": [
        "Sample Green's function (or its Hankel transform) at discrete points",
        "Solve for λ_n and C_n using matrix pencil (Prony) 52 ",
        "Map λ_n to image depths z_n (e.g. λ_n = |Im(z_n)| for decays)",
        "Superpose corresponding sources (point or line charges) with strengths
C_n"
      ],
      "refs": ["REF8","REF4"]
    },
    {
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      "equations": [
        "F(s) ≈ Σ \\frac{r_i}{s - a_i} + d + s * h   (classical form)",
        "Apply to s=j*k for R_eff(k) or impedance function"
      ],
      "usage": [
        "Fit spectral Green's function (impedance) vs complex frequency s",
        "Yields poles a_i (→ image depths) and residues r_i (→ weights)",
        "Often incorporate static term and high-frequency asymptote (d, h) to
ensure correct limits"
```

```
      ],
      "refs": ["REF8"]
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  ],
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"topic": "Sommerfeld integrals for half-space (original work)" },
    { "id": "REF2", "citation": "van der Pol (1935), Physica 2:843–853",
"topic": "Complex image method for finite conductivity plane (early
development)" },
    { "id": "REF3", "citation": "Lindell & Alanen (1984), IEEE TAP 32(2):126 &
32(8):841", "topic": "Exact image theory for dielectric half-space (vertical
dipoles, image currents)" },
    { "id": "REF4", "citation": "Fang et al. (1988), IEE Proc. H 135(5):297",
"topic": "Discrete complex images for multilayer (H. dipole) – first DCIM for
stratified media" },
    { "id": "REF5", "citation": "Chow et al. (1991), IEEE MTT 39(7):1120",
"topic": "Complex image method for static fields in multilayer (closed-form
Green’s function)" },
    { "id": "REF6", "citation": "Chow et al. (1992), J. Appl. Phys. 71:569",
"topic": "Application of CIM to grounding (layered soil), image parameters in
static limit" },
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