ChatGPT

1. Functional-Analytic Setup for the Torous Dirichlet Problem

1.1 Function Spaces and Operators: Let $\Gamma\subset\mathbb{R}^3$ be the smooth closed surface of the torus (major radius $R=1$, minor radius $a=0.35$). We consider the exterior domain $D_{\text{ext}} =\mathbb{R}^3\setminus \Gamma$, with the conductor interior regarded as a void (Dirichlet condition on $ \Gamma$). For potentials $V(x)$ in D_{ext} that decay at infinity, a natural function space is the energy space $H^1(D_{\text{ext}})$ of functions with square-integrable gradient (and suitable decay). On the boundary, Dirichlet trace spaces and their duals are used: the space of boundary potentials (Dirichlet data) is $H^{1/2}(\Gamma)$, and the space of surface charge densities (Neumann data for Laplace's equation, or single-layer source densities) is $H^{-1/2}(\Gamma)$. These spaces are paired via the $L^2(\Gamma)$ duality.

We use classical boundary integral operators from potential theory. The **single-layer potential operator** S maps a surface charge density $\sigma(p)$ (with $p\in\Gamma$) to a harmonic potential in D_{ext} given by $$ (S\sigma)(x)\;=\;\int_{\Gamma} \frac{1}{4\pi|x-z|}\,\sigma(z)\,dS_z~, $$ for $x\in D_{\text{ext}}$. Here $G_0(x,z)=1/(4\pi|x-z|)$ is the free-space Newtonian kernel. The single-layer potential $S\sigma$ is a harmonic function in D_{ext} (and also in the interior $\mathbb{R}^3\setminus D_{\text{ext}}$) except at the surface where the layer is distributed. Importantly, $S\sigma$ is *continuous* across Γ (no jump in potential value), although its normal derivative has a jump proportional to $ \sigma$. We also have the **double-layer potential** $D\mu$, defined for a dipole density $\mu(p)$ on $ \Gamma$ by $$ (D\mu)(x)\;=\;\int_{\Gamma} \frac{\partial}{\partial n_z}\left(\frac{1}{4\pi|x-z|}\right)\mu(z) \,dS_z, $$ where $\partial/\partial n_z$ is differentiation in the direction of the outward normal at z. $D\mu$ is also harmonic in D_{ext}, but it is *discontinuous* across Γ (its value jumps, while the normal derivative is continuous). In our setup, we will mainly use the single-layer formulation since the exterior Dirichlet problem is conveniently solved by a single-layer ansatz.

On the boundary Γ, one can define boundary integral operators that map densities to their induced boundary potentials or fluxes. We denote by $V: H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$ the *single-layer boundary operator*, defined as the trace on Γ of $S\sigma$ (often $V(p) = \int_\Gamma \frac{1}{4\pi|p-z|}\sigma(z)\,dS_z$ for $p\in\Gamma$). Likewise, K or K' can denote the double-layer boundary operators (which we do not detail here). For Laplace's equation on a closed surface, V is a symmetric, positive-definite operator relating surface charge to the resulting surface potential [1]. In particular V is a compact operator (Green's kernel is smooth on $\Gamma\times\Gamma$ except the mild singularity) and has a discrete spectrum of eigenvalues accumulating at 0. (We note V is sometimes called the *capacitor operator* in electrostatics context.)

1.2 Boundary Integral Equation (BIE) Formulation: The exterior Dirichlet problem can be solved via a boundary integral equation. We seek $V(x)$ harmonic in D_{ext}, decaying at infinity, with $V=0$ on Γ, and with a singularity corresponding to a unit point source at $y\in D_{\text{ext}}$. Physically, this $V(x)$ is the potential produced by the point source plus the induced surface charge on the conductor. We represent V as a single-layer potential induced by some surface charge density σ that must be

determined. That is, we write $$ V(x) \;=\; \frac{1}{4\pi|x-y|} \;+\; (S\sigma)(x), $$ where the first term $1/(4\pi|x-y|)$ is the *incident* potential of the point source (free-space Green's function centered at y), and $(S\sigma)(x)$ is the *induced* potential from surface charge σ on Γ. The boundary condition $V(p)=0$ for all $p\in\Gamma$ leads to the BIE: $$ (S\sigma)(p) \;=\; -\,\frac{1}{4\pi|p-y|}, \qquad p\in\Gamma~. $$ Here the right-hand side is the negative incident potential on the boundary. In operator form, using $V\sigma$ to denote the single-layer boundary operator as above, we have $$ V\,\sigma \;=\; f, \qquad \text{with } f(p) = -\,\frac{1}{4\pi|p-y|}~\text{ on }\Gamma~. $$ We define $T: H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$ as this boundary integral operator $T := V$ (the single-layer operator on Γ). Then the BIE is $T\sigma = f$. For a simply-connected closed surface in 3D (like our torus) and for the Laplace equation, $T = V$ is known to be *Fredholm of the second kind* with a one-dimensional nullspace consisting of constant densities (physically corresponding to the fact that a constant σ produces a constant potential on Γ) [1]. However, because the conductor is *grounded* (held at 0 potential), any constant-offset in potential is fixed, and one can enforce $\int_\Gamma \sigma = 0$ (zero net charge) to break the nullspace degeneracy. In practice, for a grounded conductor, T is invertible on the subspace of mean-zero densities. We thus assume T can be (uniquely) inverted to find σ. The solution for the density is $\sigma = T^{-1}f$. Once σ is found, the potential in the exterior is given by the representation $V(x) = G(x,y) = \frac{1}{4\pi|x-y|} + (S\sigma)(x)$. This $G(x,y)$ is by definition the **Green's function** for the exterior Dirichlet problem: it satisfies $\Delta_x G(x,y) = \delta(x-y)$ in D_{ext} (in the distributional sense), $G(x,y)=0$ for $x\in\Gamma$, and $G(x,y)\to 0$ as $|x|\to\infty$. It is also symmetric in x,y (by reciprocity for Laplace's equation and the self-adjointness of T).

1.3 Green's Function via Boundary Operators: Using the operator T^{-1}, we can formally express the Green's function kernel. Let $\mathcal{G}(p,y) := G(p,y)$ for p on Γ (the *Dirichlet-to-Neumann* kernel is related to $\partial G/\partial n$, but here we consider the Dirichlet trace which is zero). The induced density solving $T\sigma_y = -G_0(\cdot,y)$ (where $G_0(p,y)=\frac{1}{4\pi|p-y|}$) can be written as $\sigma_y = -\,T^{-1}[\,G_0(\cdot,y)\,]$. Then the Green's function in the exterior can be written as $$ G(x,y) \;=\; G_0(x,y) \;-\;\int_{\Gamma} G_0(x,z)\,\big(T^{-1}[G_0(\cdot,y)]\big)(z)\,dS_z~, $$ for $x\in D_{\text{ext}}$. This is an exact representation. In operator language, the kernel $G(x,y)$ for $x,y\in D_{\text{ext}}$ can be seen as a *resolvent kernel* of the boundary-integral operator: indeed one can verify that $G(x,y)$ restricted to $x\in D_{\text{ext}}, y\in D_{\text{ext}}$ is the integral kernel of the *Poisson operator* (solving Laplace's equation with Dirichlet data concentrated at a point), or dually of the inverse of the *Dirichlet-to-Neumann operator*. For our purposes, the main point is that $G(x,y)$ arises from the combination of the fundamental solution $G_0(x,y)$ and an induced layer potential on Γ. Throughout, we will treat $G(x,y)$ as a two-variable function (kernel) that is harmonic in x for fixed y and harmonic in y for fixed x, except for the singularity on the diagonal $x=y$.

2. Finite-Basis Green's Function as Finite-Rank Approximation

2.1 Finite-Basis Ansatz: We formalize the notion of a *finite-basis Green's function*. Suppose we have a fixed collection of N candidate potential functions ${\Phi_k(x)}*{k=1}^N$. These $\Phi_k(x)$ are independent of the source location and are typically chosen to be solutions of Laplace's equation in D* (each might be an "image potential" due to some hypothetical source arrangement chosen a priori). We then seek to represent the Green's function (or the solution for a given source) as a linear combination of these basis potentials. That is, for each source position y, we attempt $$ G(x,y)\;\approx\; G_N(x,y) \;:=\; \sum_{k=1}^N \alpha_k(y)\,

\Phi_k(x)~, $$ where $\alpha_k(y)$ are scalar coefficient functions depending on y. If this holds }**exactly** for all x in the domain (and, say, for all y in some subset of the source region), then we have an *exact finite-rank representation* of the Green's function kernel. More typically, we consider this as an approximation, aiming for the right-hand side to approximate $G(x,y)$ to small error in a chosen norm. In our numerical context, $N=6$ and Φ_k consist of the 2 ring potentials and 4 point-charge potentials fixed near the torus (as described), and $\alpha_k(y)$ are the weights $w_k(z)$ obtained by a least-squares fit for each source $y=y(z)$ on the axis.

2.2 Finite Rank and Operator Kernel Approximations: The above ansatz can be viewed in terms of operator theory. The Green's function $G(x,y)$ is the kernel of an integral operator that takes a source distribution (in y) to the resulting potential (in x). A *finite-rank kernel* means that $G(x,y)$ can be written as a finite sum of *separable* terms $f_k(x)g_k(y)$. In our notation, $f_k(x)=\Phi_k(x)$ and $g_k(y)=\alpha_k(y)$. Equivalently, the mapping $y \mapsto G(\cdot,y)$ (which takes a source location to the function $G(x,y)$ as a function of x) would lie in an N-dimensional subspace spanned by ${\Phi_1,\dots,\Phi_N}$. In operator language, consider an operator K that takes a source at y (treated as a Dirac measure $ \delta_y$ or as boundary data induced by that source) to the solution potential $G(\cdot,y)$. The kernel $G(x,y)$ being finite-rank is tantamount to saying the range of K is finite-dimensional. For example, one might look at the *Dirichlet-to-Neumann operator* or the *Poincaré-Steklov operator* for this domain: if that operator had finite rank, it would imply a finite number of modes suffice to describe all solutions. Typically, K (and similarly the Green's function kernel) is *not* finite rank for a nontrivial geometry; rather, one seeks *approximate* finite-rank representations (low-rank approximations).

In numerical analysis, a classic concept is a **degenerate kernel approximation**: one approximates an integral kernel $K(x,y)$ by a sum $\sum_{k=1}^N f_k(x)g_k(y)$. This is essentially what we attempt, restricted to a subset of source locations. Such approximations are ubiquitous: for example, the free-space Green's function $1/|x-y|$ admits the *Laplace expansion* in spherical harmonics, which is an infinite series but can be truncated for an approximate low-rank separation when x and y are well-separated [2] [3]. In our case, the sources and observation points are not asymptotically far, but we are empirically observing that the 2-variable function $G(x,y)$, when restricted to y on the axis interval, has most of its variability captured by a few separable components.

2.3 Possibility of Exact Finite-Rank Representations: A key theoretical point is that for a generic smooth bounded conductor, the Dirichlet Green's function **cannot** be represented *exactly* by a finite sum of separable terms. Equivalently, the operator T^{-1} (or the Dirichlet-to-Neumann map) is an infinite-dimensional operator with an infinite spectrum of eigenfunctions, so its kernel $G(x,y)$ does not decompose into finitely many product functions. In classical terms, $G(x,y)$ is the *influence matrix* for boundary data of arbitrary shape, and the space of harmonic functions in a domain is infinite-dimensional. Only in very special geometries can one obtain an exact closed-form image solution with a finite number of image sources – and even in those cases, that usually applies *for a single source configuration at a time*, not as a uniform representation for all source locations. For example, a grounded infinite plane can replace a point charge by one image charge, and a grounded sphere of radius R can replace an external point charge by a single image charge located along the line through the center [4] [5]. These are classical *method of images* solutions, but crucially, the location and strength of the image charge depend on the source position (distance D of the source from the center, etc.), i.e. the representation is *not using a fixed source location*. If one attempted to represent all possible source positions outside a sphere by a fixed point charge inside the sphere, that would fail – the image must move as y moves. Thus, no fixed finite set of source points inside a sphere can generate the exact Green's function for all source locations (except by

taking an infinite continuum of image points forming a distribution). In general convex geometries (and a torus is even less symmetric), the solution space is spanned by an infinite set of *harmonic basis functions* (spherical harmonics for a sphere, toroidal harmonics for a torus, etc.). The classical expansions for Green's functions in separated coordinates (spherical, cylindrical, toroidal, etc.) involve summing an infinite series of harmonic functions [3] [6]. Therefore, standard potential theory tells us that *exact* finite-rank kernels $G(x,y)$ do not exist for a generic 3D shape – the only exceptions are essentially trivial or highly symmetric cases that allow a conformal mapping to a simpler domain.

In summary, a "finite-basis Green's function" in the strict sense (exact representation with N terms) is ruled out by spectral theory for the torus. The Green's function is a typical *Hilbert–Schmidt kernel* on Γ, with an infinite discrete spectrum of eigenfunctions (e.g. the Steklov eigenfunctions of the domain) – only an infinite sum of separable terms (or an integral) can represent it exactly. Thus any finite N can at best yield an approximation. We interpret the numerical findings in this light: the success of $N=6$ images in approximating the Green's function for axis sources implies that the kernel has *effective low-rank structure* when restricted to that family of sources, not that it is truly finite-rank.

3. Axis Source Family and Solution Manifold M_{axis}

3.1 Definition of the Axis Family: We focus on sources located on the symmetry axis of the torus. Let $y(z) = (0,0,z)$ in Cartesian coordinates (so y lies on the z-axis). We take z in an interval $I\subset \mathbb{R}$ chosen such that $y(z)$ is in the "hole" of the torus – meaning $y(z)$ lies along the axis passing through the center of the torus, between the top and bottom of the donut, not intersecting Γ. For example, $I=[0.4,0.9]$ (units presumably matching the geometry) lies within the hole. For any such z, $y(z)\in D_{\text{ext}}$ and defines a Dirac source at that location.

By symmetry of the torus and the source placement, note that the physical solution for a source on the axis will be **axisymmetric**: when y is on the z-axis, the conductor and source are invariant under rotations about the z-axis, so the induced potential $G(x,y(z))$ is independent of the azimuthal angle ϕ of the field point x (provided we also place x such that we consider the full 3D field). In other words, $G(x,y(z))$ for an axis source depends only on $r= \sqrt{x^2+y^2}$ and z (in cylindrical coordinates), not on the angle ϕ of x. The surface charge density σ_z induced on Γ will likewise be symmetric around the axis. This reduces the effective complexity of the problem (we have essentially a 2D axisymmetric PDE in the r-z meridian plane, albeit on a complicated domain cross-section).

3.2 Solution Manifold M_{axis}: We define $$ M_{\text{axis}} \;:=\; {\, G(\cdot,\,y(z)) : z\in I\,}~, $$ the set of Green's function solutions (viewed as functions of x) for sources along the axis in the interval I. Each $G(\cdot,y(z))$ is a harmonic function in $x\in D_{\text{ext}}$ (with a $1/|x-y(z)|$ singularity at $x=y(z)$ and satisfying the boundary condition on Γ). We can think of M_{axis} as a subset of the infinite-dimensional function space of admissible harmonic potentials (e.g. as a subset of $H^1(D_{\text{ext}})$ or of $C^\infty(D_{\text{ext}}\setminus{y(z)})$). Alternatively, one can consider the traces or the surface charge densities: one could consider ${\sigma(z)\in H^{-1/2}(\Gamma): T\sigma(z) = -G_0(\cdot,y(z))}$, the set of solutions in density space. In any case, M_{axis} is parameterized by a single real parameter $z\in I$. Under mild assumptions, this parameterization is smooth; indeed as we vary z, the boundary data $f(z,p) = -\frac{1}{4\pi|p-y(z)|}$ changes smoothly (in fact *real-analytically*) with z

for each fixed $p\in\Gamma$. Since the operator T^{-1} is linear and continuous, the mapping $z\mapsto \sigma(z) = -T^{-1}[G_0(\cdot,y(z))]$ is also smooth (indeed analytic) in z. Likewise $G(x,y(z)) = G_0(x,y(z)) + (S\sigma(z))(x)$ will depend smoothly on z for each fixed field point x (away from the singular case $x=y$). In more intuitive terms, as the source moves along the axis, the induced charge distribution and resulting potential change continuously and without singular behavior. We can thus regard M_{axis} as a *one-dimensional manifold* (in fact an analytic curve) embedded in the function space of harmonic solutions.

Formally, one might say $M_{\text{axis}} = {u(z): z\in I}$ where $u(z)\in H^1(D_{\text{ext}})$ and $u(z)$ satisfies the PDE and BC for source at $y(z)$. We can differentiate under the integral sign in the representation $S\sigma$ to see analyticity: $f(p;z) = -\frac{1}{4\pi|p-y(z)|}$ is analytic in z (for $p\not=$ the axis), hence $\sigma(z)=T^{-1}f(\cdot;z)$ inherits analyticity in z, and $u(z)=G(\cdot,y(z))$ inherits it too. This aligns with general results on analytic dependence of PDE solutions on parameters (here the parameter enters the source term smoothly) – one can in principle expand $u(z)$ in a power series around a given z_0 and solve for the series coefficients by differentiating the BIE or PDE, guaranteeing a radius of convergence etc., as long as $y(z)$ stays a fixed distance away from Γ to avoid singular limits.

3.3 Manifold Properties: Because M_{axis} is essentially the image of an interval under an analytic map into an infinite-dimensional space, it is indeed a 1-dimensional submanifold (locally diffeomorphic to an interval). All the solution functions in M_{axis} share qualitative properties: they are axisymmetric harmonic functions vanishing on Γ and decaying at infinity, with a singularity on the axis inside the hole. As z varies, one can think of the solution's *shape* morphing in the function space. Importantly, although intrinsically M_{axis} has dimension 1, its *extrinsic* dimension (the dimension of the linear span required to represent it) need not be 1. In fact, generically the span of ${u(z): z\in I}$ will be infinite-dimensional, since an analytic curve in a function space can meander through infinitely many independent directions. But if the curve M_{axis} is *nearly flat* in some sense or well-approximated by a low-dimensional plane, we may find that a low-dimensional subspace captures it with small error. This is precisely what the numerical evidence of a 6-dimensional basis suggests.

3.4 Relation to Boundary Data and Analytic Dependence: Each $u(z)\in M_{\text{axis}}$ corresponds uniquely (via the BIE) to a surface charge $\sigma(z)$ on Γ. The map $z\mapsto \sigma(z)$ is analytic, as argued. We can think in terms of expansions: for example, one might expand the boundary data $f(p;z) = -1/(4\pi|p-y(z)|)$ in powers of z around some center z_0. Because $1/|p-(0,0,z)|$ can be expanded in Legendre polynomials in z (for fixed p outside or inside certain regions), $\sigma(z)$ will have a power series whose coefficients are determined by applying T^{-1} to those expansions. In classical terms, the induced potential can be expanded in *toroidal harmonic functions* (for axisymmetric situations, these are essentially Legendre functions of half-integer degree in the toroidal coordinate parameter) [3]. One can show that $G(\cdot,y(z))$ for axis sources admits an expansion in an *infinite* series of basis functions (e.g. "poloidal" harmonics of increasing order). Thus, M_{axis} is infinite-dimensional *as a set* (uncountably many distinct functions) but only one-dimensional *as a manifold*. The question is how large a linear space is needed to approximate it well – that ties into the notion of its curvature in function space and the decay of those expansion coefficients, which we address next.

4. Numerical Evidence and Interpretation in Terms of Rank

4.1 Discrete Weight Matrix: In the numerical experiments, for each source $y(z_i)$ (with $z_i \in {0.4,0.5,\dots,0.9}$ for example), an approximate representation $$ V_{\text{img}}(x;z_i)\;=\; \sum_{k=1}^6 w_k(z_i)\,\Phi_k(x) $$ was obtained by solving a regularized least-squares problem. The coefficients $w_k(z_i)$ are the computed weights $\alpha_k(y(z_i))$. We can collect these results for M sample points along I in a $6\times M$ matrix: $$ W \;=\; \begin{bmatrix} | & & | \\ w(z_1) & \cdots & w(z_M) \\ | & & | \end{bmatrix}, $$ where $w(z_j) = (w_1(z_j),\dots,w_6(z_j))^T \in \mathbb{R}^6$ is the weight vector for the j-th source sample. Each column of W thus corresponds to one source location and lists the combination of the 6 basis functions used. According to the reported high-res BEM results, for all these sampled z values the *mean relative error* (an average pointwise error on a certain r–z grid inside the domain – see below) is on the order of 5–10%. In particular, with a fixed geometry (the basis Φ_k from trial02), the errors were: e.g. $\text{mean_rel}\approx 7.3\%$ at $z=0.5$, 5.5% at $z=0.7$, 7.1% at $z=0.9$, etc. The fact that all these errors are of similar magnitude $\mathcal{O}(10^{-1})$ suggests that *for every source $y(z)$ in the interval, there is a choice of weights yielding a solution within ~10% error of the true $G(x,y)$.* In other words, the entire manifold M_{axis} lies within a small distance (10% in the metric induced by that error measure) of the 6-dimensional linear subspace spanned by ${\Phi_1,\dots,\Phi_6}$.

This is evidence that M_{axis} is an *approximately low-dimensional* set. If we denote by $V = \operatorname{span}{\Phi_1,\ldots,\Phi_6}$, then the statement is roughly: $\sup_{z\in I} \inf_{v\in V}| G(\cdot,y(z)) - v|$ is about 0.1 (in the relative error norm). Here the norm is not a standard L^2 or H^1 norm, but the discrete mean relative error on a grid; still, it indicates a uniform approximation quality. Thus, M_{axis} is contained in the 0.1-neighborhood of a 6-dimensional subspace. Geometrically, this means M_{axis} is very “flat” in some 6-dimensional direction, with only small deviations.

4.2 Effective Rank via SVD of Weights: To delve deeper, one can perform a singular value decomposition (SVD) of the weight matrix W. Write $W = U \Sigma V^T$, where $U\in\mathbb{R}^{6\times 6}$, $\Sigma\in \mathbb{R}^{6\times M}$, $V\in\mathbb{R}^{M\times M}$ (assuming $M\ge 6$). The singular values $\sigma_1,\sigma_2,\dots,\sigma_6$ (the diagonal entries of Σ) quantify the orthogonal directions of largest variation among the weight vectors ${w(z_j)}$. If one singular value σ_r is significantly smaller than σ_1, it means that the data $w(z)$ lies mostly in an $r-1$ dimensional subspace. For example, if we find σ_5,σ_6 are very small compared to $\sigma_1,\ldots,\sigma_4$, that would indicate that effectively the 6 basis functions’ usage is dominated by 4 combinations – i.e. $w(z)$ varies in (approximately) a 4-dimensional subspace of $\mathbb{R}^6$. In other words, the *effective rank* r_{eff} of the mapping $z\mapsto w(z)$ could be less than 6. Since W is essentially a sampling of the map $z\mapsto w(z)$, its singular values are discrete analogs of the spectrum of that map considered as an operator on functions of z. In the limit of fine sampling, singular values of W relate to the proper orthogonal decomposition (POD) modes of the family ${G(\cdot,y(z))}$.

Conceptually, a low r_{eff} means that the solution manifold M_{axis} can be well-approximated by a subspace of dimension r_{eff}, which is $\leq$ 6 here. The SVD would reveal the “principal components” of M_{axis}. For instance, U’s columns are orthonormal combinations of the original Φ_k that align with principal directions of the manifold, and V’s columns (or rows) give

the corresponding parametric variation. If $\Sigma = \operatorname{diag}(\sigma_1,\dots,\sigma_6)$ has $\sigma_6 \approx 0$, that actually indicates the 6th basis function was almost redundant across all z. If $\sigma_1 \gg \sigma_2 \gg \dots$, then one combination of images explains most of the solution (perhaps the ring basis alone captures the bulk, etc.).

We expect, based on typical behavior of analytic families of solutions, that the singular values will decay, although not to zero unless infinitely many bases were used. In a continuous viewpoint, one can consider the set ${w(z):z\in I}$ as an image of a 1D continuum under a smooth map into $\mathbb{R}^6$. The SVD of W (with sufficiently fine sampling) approximates the **Kolmogorov n-width** of the manifold in the ambient space. The n-width d_n of a set M in a normed space is roughly the smallest maximum deviation of M from some n-dimensional subspace. Equivalently $d_n = \inf_{\dim(V)=n} \sup_{f\in M} \inf_{v\in V}|f-v|$. If the singular values of the sampled snapshot matrix decay rapidly, it implies small d_n, meaning the manifold is *highly compressible*. In our context, a small d_6 (with errors ~5-10%) means a 6-dimensional subspace does a good job. If one computed d_n for increasing n, one could see how the approximation error falls – this is linked to the singular value decay of the appropriate linear operator spanning M_{axis}. In fact, if we had an orthonormal basis expansion of $G(\cdot,y(z))$ in some basis, the coefficients' behavior in n would indicate how d_n decays.

4.3 Approximation Numbers and Maps: Another way to formalize this: consider the mapping $\Psi: I \to H^1(D_{\text{ext}})$ given by $\Psi(z) = G(\cdot,y(z))$. This is a smooth (analytic) curve in the function space. We can linearize around some point and also consider the *tangent space* at a given z (which is spanned by $\partial G/\partial z$ at that point, a single direction since z is one-dimensional). But as z varies over an interval, the set of all these solution functions may span a larger space. One can consider the closure of $\operatorname{span}{\Psi(z): z\in I}$ – in principle this could be infinite-dimensional. However, for an analytic $\Psi(z)$ on a finite interval, it is known in approximation theory that the set can be approximated with exponential convergence by a sequence of finite-dimensional spaces (this is related to the concept of **analytic manifolds have small Kolmogorov widths**, often exploited in model order reduction). The singular values of the discrete snapshot matrix W are proxies for the **approximation numbers** of the compact set M_{axis}. If we denote by a_n the nth approximation number (the error of the best rank-n approximation in L^2 norm, say), then a_n will typically decay to 0 as $n\to\infty$. A truly finite-rank situation would be $a_N = 0$ for some N, which we do not expect here. Instead, we expect a fast *decay* but not abrupt truncation.

In summary, the numerical evidence of uniformly ~5-10% error with $N=6$ suggests that $d_6(M_{\text{axis}})$ in the relevant norm is ~0.05–0.1. The SVD of W would likely show a couple of singular values significantly larger than the rest, indicating an effective rank perhaps of order 3–4 (this is speculative without the actual SVD computation). This implies that although we used 6 basis functions, the *intrinsic* dimensionality of the solution family's image in that basis might be lower. We interpret $r_{\text{eff}} = \dim({w(z)})$ (approximately) as the number of independent modes excited as z runs through [0.4,0.9].

4.4 Connection to Manifold Approximation: The concept of **Kolmogorov n-width** mentioned is central: it asks how well an n-dimensional linear space can approximate the set M_{axis}. Our 6 images attained an approximation error ~7%. If one increased N, presumably the error would drop. If M_{axis} were truly a smooth curve, one might expect d_n to scale like $\sim C n^{-p}$ or even exponentially small in n if the curve is analytic. In many parametric PDE contexts, d_n decays exponentially because the solution as a function of parameter can be expanded in power series (or other

analytic basis) [7]. In practice, a small set of basis functions (like 6) capturing ~90-95% of the solution is a hint of either rapid decay of parametric modes or perhaps that the chosen basis functions align very well with the first few modes of variation. For instance, Φ_1,Φ_2 might correspond to the two lowest-order "poloidal" harmonics which capture a large part of the axisymmetric field, while Φ_{3-6} (point charges) fine-tune local features. If we had instead chosen a random set of 6 basis functions, the error would be much worse – the success here indicates these particular images span the same subspace as the dominant physical solution modes.

In short, the near low-rank structure is quantitatively reflected by the weight matrix W. Analyzing W via SVD would yield the principal axes of M_{axis} in the chosen image basis, and likely show that the solution family is mostly confined to a few-dimensional subset of that basis space.

5. Feasibility of a True Finite-Basis Green's Function

5.1 Structure of $G(x,y)$ for Smooth Boundaries: The Green's function $G(x,y)$ for Laplace's equation exterior to a smooth compact boundary is a smooth function of x and y (for $x\neq y$) that satisfies harmonic conditions. It can be viewed as the kernel of the inverse of a boundary-integral or differential operator, and as such it has the typical structure of an *integral kernel with a singularity on the diagonal*. Usually, one expands $G(x,y)$ in multipole or eigenfunction series. For example, for a spherical boundary, one can express $G(x,y)$ in spherical harmonic series (Legendre polynomials) which sums an infinite series (though in certain special source arrangements that series can be summed in closed form). For a torus, one natural expansion uses **toroidal harmonics**, which are based on the toroidal coordinate system. As the Wikipedia reference indicates, the free-space Green's function $1/|x-x'|$ can be expanded in toroidal harmonics requiring an infinite summation over mode index m (the azimuthal index) [3], or alternatively as an infinite integral over a continuous spectral parameter [8]. When one imposes the Dirichlet condition on the torus boundary, the solution will involve those same toroidal harmonic modes to satisfy the boundary condition. We can thus safely say $G(x,y)$ on a torus involves an infinite series of basis functions (associated Legendre functions $Q_{m-1/2}(\chi)$ etc., as in the cited formula). No finite truncation of these series can produce the exact Green's function for all x,y.

Another perspective: if $G(x,y)$ were exactly of rank N, it could be written $G(x,y)=\sum_{k=1}^N f_k(x)g_k(y)$. Consider this as a matrix in the (x,y) continuum. A fundamental result in analysis is that such a finite-rank kernel has a very special form: it implies a certain factorization through a finite-dimensional space. In effect, it would mean the space of harmonic functions with those boundary conditions is N-dimensional for all possible point sources – which is false except in trivial cases. In fact, the only harmonic functions exterior to a torus that decay at infinity are those generated by an infinite series of multipoles or distributed sources; there is no finite basis spanning all possibilities. The completeness of eigenfunctions of the Dirichlet-to-Neumann map (the **Steklov eigenfunctions**) ensures that an arbitrary boundary perturbation (like the field from an arbitrary source) requires an infinite series. For a generic analytic surface (torus included), the eigenvalues of the Dirichlet-to-Neumann operator form an unbounded sequence (the Steklov spectrum), implying an infinite number of independent "shape functions" on the boundary (and corresponding harmonic extension fields) [1]. Thus a finite linear combination cannot represent the response to all source locations exactly.

5.2 Spectral Considerations and Symmetry Exceptions: In linear operator terms, $G(x,y)$ being finite rank would mean the operator mapping Dirichlet data to solutions (or the inverse of that mapping) has a finite-dimensional range. But in 3D, the Dirichlet boundary-value problem for Laplace's equation is equivalent to finding a harmonic function given boundary values, and since there are infinitely many spherical harmonics (or toroidal harmonics) needed to describe an arbitrary boundary function, the solution space is infinite-dimensional. Only if the boundary admitted only a finite number of harmonics (which would imply a very special shape, essentially a degenerate case) could G be finite rank. In practice, special geometries allow a simpler form of Green's function, but not strictly finite-rank in the sense of a finite sum of separable products valid for all sources: - **Infinite plane:** $G(x,y)$ for a grounded infinite plane has an "image charge" representation $G(x,y) = \frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-y^|}$, with $y^$ the mirror of y across the plane. This is an exact closed form with two terms. However, notice those two terms are both singular functions coupling x and y; one cannot rewrite this as $f_1(x)g_1(y)+f_2(x)g_2(y)$ because each term is not separable in x and y (they are functions of $x-y$). In an expansion sense, the plane's Green's function can be expanded in plane waves or Fourier modes – infinitely many – but the symmetry allowed us to find a closed form. Yet, if we restrict to sources on a line perpendicular to the plane, one could argue a single image always directly opposite works for each, but again the image *moves with the source*. No fixed finite set of image locations works for all source positions on that line. - **Sphere:** As noted, a grounded sphere of radius R with an external source q at distance D has an exact solution given by one image charge $q'=-q R/D$ placed at radius $b=R^2/D$ inside [4]. This again is a case of one image per source. But if we attempted to cover all possible source positions outside the sphere with a *fixed* set of images, we would fail. In effect, the sphere has an infinite set of spherical harmonic modes Y_{ℓ}^m – the image method exploits the special form of the fundamental solution's expansion in those harmonics to sum the infinite series into a simple rational form for that particular source. However, from the kernel perspective, the Green's function for the sphere can be written in closed form but not as a finite sum of separated x-only and y-only functions. (If one restricts to sources on the axis of the sphere, by symmetry only $m=0$ modes contribute, but still an infinite sum over ℓ is generally needed – except that in the sphere's case that sum yields a simple closed form for the combined effect.) - **Other symmetric domains:** Infinite cylinder, bispherical coordinates, etc., allow separation of variables and thus infinite series expansions [1]. None yield a finite truncation unless the domain is effectively a trivial geometry (half-space or sphere being the most symmetric in electrostatic sense).

Therefore, known spectral theory *rules out* the existence of a globally valid finite-basis Green's function for a generic shape like a torus. The only potential loophole is if one restricts the source locations to a subset and allows some approximation.

5.3 The Case of the Torus: A torus does not possess a simple image-charge principle. Its symmetry (rotational about an axis and a mirror symmetry) is not sufficient to collapse the infinite series of toroidal harmonics into a finite combination. Indeed, the toroidal harmonic expansion of the free-space kernel involves an infinite sum over m (azimuthal index) [3]. For axisymmetric situations ($m=0$), one still has a continuous spectrum or infinite series in the "toroidal" index (related to the Legendre function degree). There is no indication of any truncation in known solutions. Thus, an **exact** finite-rank representation of $G(x,y)$ for the torus would defy conventional understanding.

The numerical evidence we have – a 6-term ansatz giving ~5-10% error on the axis – should be interpreted as capturing the *first few terms of an infinite expansion*. The ring basis functions are essentially low-order terms in an axisymmetric multipole expansion (they correspond to certain $m=0$ modes concentrated in the hole), and the four point charges near the surface likely emulate higher-order localized contributions.

The fact that adding more point charges and/or rings and optimizing geometry improved the approximation (from 54% error down to ~5%) supports the view that as N increases, one can systematically improve the approximation by spanning more of the harmonic content. But we did not reach machine precision or anything close – even at best ~5% mean error remained, and notably the *maximum pointwise relative error* was large (10^4, which occurs where the true solution is very small and the approximation has some finite error). This indicates that there remain fine details uncaptured by the 6-term basis.

In conclusion, a *true* finite-basis Green's function (exact representation) for the torus does not exist except as an infinite series. The observed low-dimensional behavior along the axis is likely due to the fact that the axisymmetric Green's functions are dominated by the first few toroidal harmonics for these source positions. It is a case of an **approximation** being very good in a restricted context, rather than a fundamental finite rank property. Standard potential theory would strongly suggest that if we broaden the range of sources or demand higher accuracy, more basis functions will be needed – in principle without bound as accuracy demands increase. We are seeing an instance of an *effectively low-rank* kernel over a subset of inputs and to limited precision, not an actually degenerate kernel.

6. Quantitative Analysis of Axis Sweep Data

6.1 Error Metrics and Norms: The reported "mean_rel" and "inner_mean_rel" are discrete metrics calculated on an r–z grid in a cross-sectional plane (exploiting axisymmetry). Specifically, mean_rel is the arithmetic mean of pointwise relative errors $|V_{\text{img}}-V_{\text{ref}}|/(|V_{\text{ref}}|+10^{-12})$ over all grid points【user's note】. This is essentially an L^1 average of relative error in the domain (with a tiny stabilization term). "inner_mean_rel" is the same average but restricted to an inner region of the cross-section (closer to the inner side of the torus). These metrics are not standard norms but give a sense of average performance in the field. They do not directly measure boundary error or energy norm error. Nonetheless, their magnitudes (5–10% on average) indicate a reasonably good approximation in most of the domain, while not extremely high accuracy. The maximum relative error ("max_rel") being on the order of 10^4% (i.e. 10000%) means there are points where the approximate solution has essentially lost accuracy – likely in regions where the true solution V_{ref} is extremely small (perhaps a potential null or a point far away) so that a small absolute error translates to huge relative error. This is common: relative error blows up where the field is nearly zero. It suggests our basis might fail to cancel the field to the same tiny level as the true solution in some cancellation region, but that might not be physically significant if both true and approximate potentials are near zero there.

From the data: at $z=0.70$, mean_rel 5.51%, inner_mean_rel 7.37%. At $z=0.50$, mean_rel 7.29%, inner_mean_rel 17.78%. So, the inner region error was sometimes higher (e.g. 17.78% at $z=0.5$) even when overall mean was moderate. This could indicate that for some source positions, the basis struggles a bit more to match the field geometry near the inner side of the torus (maybe the field there has finer features or higher harmonics needed). The variation with z is not monotonic but stays in the same order of magnitude. The best was around the middle ($z=0.7$ gave ~5.5%), while $z=0.8$ spiked to ~10% mean. This non-uniformity suggests that the chosen basis (which was tuned around mid-range z) has a "sweet spot" of optimal performance and degrades slightly toward the ends of the interval. However, even the worst-case (10.3% mean at $z=0.8$) is still in the same ballpark.

One can interpret this as the **Kolmogorov 6-width** not being exactly constant across the interval. Perhaps at $z=0.7$ the 6 chosen basis functions align extremely well (maybe by design, since the basis was found by optimizing around mid-range), whereas at the edges ($z=0.4$ or 0.8) the manifold M_{axis} might start to curve out of the span a bit more, giving a larger distance. If we were to allow a slightly larger basis or optimize it differently, we might reduce those edge errors. But overall, the fact that error remained $\leq$10% for all indicates that a single 6-dimensional subspace works *globally* over I reasonably – it's not just a local fit at one point.

6.2 Interpretation in Terms of Manifold Approximation: The roughly uniform error across I suggests that the 6-dimensional space is capturing the dominant shape of all those solution functions. If the error varied drastically, it would mean that perhaps different parts of the manifold required different subspaces (i.e. one subspace can't approximate both ends well). The modest variation (5% to 10%) implies that the manifold M_{axis} lies relatively near a single linear plane over the whole range. Perhaps the first few basis functions (rings, maybe two of the point charges) adjust coefficients with z in such a way as to cover the main effects, and the remaining differences are small second-order effects.

The inner_mean_rel being higher in some cases (e.g. 17.8%) indicates that near the inner boundary of the torus (the surface facing the hole), the approximation had more trouble. This could be due to needing higher spatial frequency components to match the rapid potential variation in that region. Possibly, when the source is closer to the inner side (like $z=0.5$ might be relatively closer to the torus surface in the hole), the induced charge distribution might have a sharper peak that the basis couldn't fully reproduce with only 6 elements.

From a *n-width* standpoint, one could say: the 6-dimensional subspace achieved an error of ~0.05 in an averaged norm. If one demanded an order of magnitude smaller error (say 0.5% = 0.005), likely a larger N would be needed. Perhaps $N=10$ or $N=15$ would push mean_rel down significantly. We do see that max_rel remained large always – indicating the approximation never was uniformly good everywhere (somewhere it always had a significant relative discrepancy). This is not surprising since relative error is a stringent measure (it punishes zeros). It suggests that while the approximation captures the bulk of the field, it doesn't capture some fine cancellation patterns exactly.

6.3 Further Experiments Outlook: To fully characterize the approximate rank, one would ideally sample more densely in z (say every 0.01 from 0.4 to 0.9) and perform an SVD on that dense sample matrix W. This would reveal whether, for instance, one or two singular values dominate or all 6 are comparable. Also, sampling beyond [0.4,0.9] (closer to 0 or further to 1.2 if possible) would test if the same basis still works or if error blows up outside the range – likely it would blow up if source gets too close to the torus or too far because the basis wasn't optimized for those regimes.

Additionally, one could do a *leave-one-out test*: use weights solved at certain z points to predict the solution at an intermediate z without re-fitting weights, to see if interpolation of $w(z)$ works. If $w(z)$ is smooth, then a simple interpolation might give a near-optimal approximation at untried z. This would indicate that no fundamentally new directions are needed as z varies – consistent with a low-dimensional manifold.

Finally, cross-section error distributions would tell us *where* the approximation is failing. Perhaps the basis does well in most regions but less so in localized spots. If one finds, for example, a systematic

underrepresentation of some mode (like a higher-order toroidal harmonic manifesting as a pattern of error on the inner ring), one could add a basis function targeting that.

In summary, the axis sweep data confirms that a fixed 6-basis spans the solution manifold to within ~10% uniformly. It also hints that the approximation is not *spectacular* (not 1% level or better) and that there are spatial regions (inner torus surface vicinity) where error is larger. This implies that while M_{axis} is largely low-dimensional, it isn't exactly contained in a 6D subspace – higher dimensions contribute smaller but non-negligible effects, especially evident when finer accuracy is sought or specific local features are considered.

7. Theoretical Questions and Conjectures

7.1 Levels of Finite-Basis Approximation Conjectures:

- **(A) Finite-basis for Axis Sources (Approximate):** *Conjecture:* There exists a relatively small N (possibly $N=6$ as found, or another of that order) and a fixed set of harmonic basis functions ${\Phi_k(x)}*{k=1}^N$ such that for all source points y on the symmetry axis within a certain interval inside the torus, the Dirichlet Green's function $G(x,y)$ can be approximated with ≤10% error (in appropriate norm, e.g. mean L^1 relative error in the domain) by $G_N(x,y)$ = \sum^N \alpha_k(z) \Phi_k(\cdot)| \le \varepsilon |G(\cdot,y(z))|$, with $\varepsilon \approx 0.1$ and $N=6$." One could even conjecture that ε could be made smaller (say 0.01) if N is allowed to increase modestly (say to 10 or 12).}^N \alpha_k(y)\Phi_k(x)$. This conjecture is essentially affirming what the numerical experiment suggests. We might refine it: "for all $y(z)$ with $z\in[0.4,0.9]$, there exist coefficients $\alpha_k(z)$ such that $|G(\cdot,y(z)) - \sum_{k=1

- **(B) Finite-basis for Near-Axis Sources (Local 3D region):** *Conjecture:* For sources not strictly on the axis but in a neighborhood thereof (e.g. points whose cylindrical coordinates $(r_{\text{src}}, \phi_{\text{src}},z_{\text{src}})$ satisfy r_{src} small, z_{src} in some range), one can still find a finite set of basis functions (perhaps larger N than for the axis) that approximate $G(x,y)$ well for all such sources. Essentially this extends (A) to a small volume of source positions (not just a line). When off-axis, azimuthal symmetry is broken, so likely more basis functions (including ones that break symmetry) are needed. Perhaps the 6 basis found (which includes points off certain angles) might already handle mild off-axis displacements with some increased error. This conjecture is plausible in the sense of continuity: if the axis-case manifold is low-dimensional, moving slightly off-axis should produce a solution close to those, plus maybe one new mode accounting for the ϕ-variation. One might need to include some rotated versions of existing images or additional ones to capture ϕ dependence.

- **(C) Finite-basis for Arbitrary Sources (Global 3D):** *Conjecture (likely false):* There exists some fixed N (independent of source position) such that for *all* source locations y in the exterior region (or a large region thereof), $G(x,y)$ can be represented or approximated by $\sum_{k=1}^N \alpha_k(y) \Phi_k(x)$ to high accuracy. We believe this is false except if N is effectively infinite. Standard theory indicates one cannot cover all directions and distances with a fixed finite number of images except as a coarse approximation. If one allowed *variable accuracy*, a weaker statement could be: for any compact region of source locations not too close to Γ, there is an N that yields some moderate approximation, but as that region grows or accuracy demands grow, N must increase.

Each of these conjectures could be studied. (A) is nearly confirmed by current evidence (for one particular torus geometry). (B) would require new experiments: e.g. sample sources at $(r=0.1, z=0.7)$ etc., and see if the same images (or slightly expanded set) fit them. (C) is a theoretical extreme – essentially asking if the Green's function kernel is globally low-rank, which we argued is not.

7.2 Analytical Tools to Address These Conjectures:

- For (A), one could attempt to analyze the **toroidal harmonic expansion** for axisymmetric sources. If one expands $G(x,y(z))$ in toroidal harmonics (with $m=0$ due to symmetry), one might see that the first few modes (say the first 6 modes of degree/order) contribute, say, 90-95% of the boundary charge distribution or field energy, for sources in that range. If that could be shown, it would analytically substantiate why a 6-term truncation works. Tools: separation of variables in toroidal coordinates, asymptotic analysis of Legendre functions $Q_{n-1/2}(\chi(z))$ as n grows. Perhaps for moderate aspect ratio, higher-order modes decay in influence. One might also use **parametric analyticity** and the theory of *exponential convergence of reduced-basis for analytic families* [7] : since $z\mapsto G(\cdot,y(z))$ is analytic, the best N-term approximation error should in theory decay exponentially in N. Thus for some $N=6$ one expects a certain error, and increasing N should rapidly reduce the error. So proving (A) to arbitrary small ε would require letting N grow (not a fixed finite basis for arbitrary accuracy, but a sequence as N increases). But if we fix a tolerance like 5%, existence of some finite N achieving it is almost guaranteed by continuity; the nontrivial part is that N remains small.

To rigorously prove something like "there is a 6-dimensional subspace giving $\leq$10% error for all $z\in[0.4,0.9]$" is difficult analytically, but one might bound the Kolmogorov width by constructing explicit approximations (maybe using the first few terms of an analytic expansion) and estimating the remainder.

- For (B), one would need to consider non-axisymmetric perturbations. Perhaps one could do a multipole expansion: off-axis source can be expanded in image charges or spherical harmonics about the axis. Likely one needs to include $m=1$ Fourier modes in ϕ and so on. Analytical tool: treat the source off-axis as the sum of an axisymmetric source plus a small perturbation series in r_{src} times associated Legendre functions P_ℓ^m with $m=1$. Possibly show that the space spanned by a given set of images (like 6 original plus some rotated copies for $m=1$) can approximate those.

- For (C), to prove or disprove a finite-rank property, one can use spectral theory arguments we discussed: show that if N were finite the operator's spectrum would collapse which doesn't happen. Or use uniqueness: if ${\Phi_k}$ were such that $\sum_k \alpha_k(y)\Phi_k(x)$ equals $G(x,y)$, then by analytic continuation in y one could derive that each $\Phi_k(x)$ must itself produce boundary conditions that span all needed harmonics, which is impossible with finite many. A direct contradiction approach: assume $G(x,y)=\sum_{k=1}^N f_k(x)g_k(y)$. Because $\Delta_x G = \delta(x-y)$, each $f_k(x)$ must be harmonic in D_{ext} (with no singularity except possibly something that can superpose to delta), and the combination yields a delta at $x=y$. That imposes conditions that likely require infinite N. So (C) is false in the strict sense.

7.3 Hierarchy of Notions – Exact vs Effective Finite Basis:

We propose to distinguish: - **Finite-rank Green's function (exact):** There exists an equality $G(x,y) = \sum_{k=1}^N \Phi_k(x)\, \alpha_k(y)$ for all x,y in the domain of interest (with convergence in distribution or appropriate sense). This requires a true degenerate kernel of rank N. Based on theory, for a torus this does not happen for any finite $N>0$. - **Effectively finite-rank up to ε (uniform approximation):** There exists an N and a subspace $V=\text{span}{\Phi_1,\dots,\Phi_N}$ such that $ \sup_{y\in \Omega_{\text{src}}} \frac{|G(\cdot,y) - P_V G(\cdot,y)|}{|G(\cdot,y)|} \le \varepsilon$, where P_V is orthogonal projection onto V (or some approximation procedure). Here ε is a tolerance like 0.05 (5%). In our case, Ω_{src} could be the axis segment. The evidence suggests that for $\Omega_{\text{src}} = {(0,0,z): z\in[0.4,0.9]}$, $N=6$ achieves $\varepsilon\approx0.1$. If we allowed a bit larger N or optimized better, maybe ε could be a few percent. This notion is essentially a small Kolmogorov width. - **Moderately low-rank (non-uniform or local):** One might say that for each fixed source, very few images are needed but the images might have to change with source (this is trivial – method of images does that). Or that for each source one can find a representation with N terms but not with the same basis for all sources. This is a weaker notion and not of main interest here, since we want a *fixed* basis.

Our numerical evidence places us in the second category: "effectively finite-basis up to ε". We have $ \varepsilon\sim0.05$–$0.1$ with $N=6$ on the axis interval. If the question is whether we might have an exact finite basis, the theory says no. If the question is whether the manifold of axis solutions has small n-width, the evidence says yes, fairly small for n=6. It would be valuable to quantify how ε scales with N: e.g. does ε drop to 1% by $N=10$? That would strongly reinforce an exponential convergence expected from analytic dependence. If it plateaus, that would indicate missing something.

Thus, I would conjecture: the axis Green's function manifold is an analytic curve in function space, thus its Kolmogorov n-width decays exponentially as $n\to\infty$. In practical terms, a 6-term approximation gave ~5% error; a 10-term approximation might give ~0.5% error; a 15-term ~0.01%, etc., eventually saturating at machine precision up to the error inherent in BEM itself. This is speculative but fits typical reduced-basis results for analytic parametric PDEs [7].

In conclusion of this section: The current evidence strongly supports the *effective low-rank hypothesis* for axis sources. It does not support a true finite-rank property (nor was that expected). The interesting theoretical challenge is to explain quantitatively *why* the first few toroidal harmonics/images capture so much – likely due to the geometry and source being such that higher-order modes are weaker (perhaps related to the source being not too close to the boundary, giving a convergent expansion). Formulating and proving a theorem along these lines would be a significant step, perhaps showing something like *"For a torus of given aspect ratio, the family of axisymmetric Dirichlet Green's functions has Kolmogorov width $d_n = O(q^n)$ for some $q<1$."* Numerical singular value analysis could estimate q.

8. Guidance for Further Numerical Work

8.1 Data Collection for Informative Analysis:

- **Dense Sampling of Axis Sources:** Increase the number of sample points z in [0.4,0.9] (e.g. step 0.05 or 0.02) and compute the weight vectors $w(z)$ for each. This will allow construction of a refined matrix W capturing the continuous variation.

- **Singular Value Decomposition (SVD) of W:** Compute the SVD of this matrix. Given $W\in\mathbb{R}^{6\times M}$, obtain singular values ${\sigma_i}_{i=1}^6$. Look at the magnitude drop-off. If, say, σ_1,σ_2 are much larger than σ_5,σ_6, that means effectively rank 4 or less. If they decay roughly geometric (e.g. each is 1/3 of previous), that indicates exponential convergence and supports the analytic parametric theory [7] . If they are all of similar magnitude, then each basis is actively used (which would be surprising given some intuition that maybe one or two ring modes dominate).
- **Principal Component Analysis of Basis Usage:** The SVD's left singular vectors U will give new orthonormal basis functions $U_{:,i}$ which are linear combos of the original 6 images, ranked by significance. We can inspect those. For example, $U_{:,1}$ might correspond to a dominant "mode" of the solution (perhaps a combination of the two rings, representing an overall dipole-like pattern), $U_{:,2}$ next, etc. Projecting the solution snapshots onto fewer modes (like first 4) and checking error would validate how many modes are really needed.
- **Off-Axis and Other Family Studies:** Take sources slightly off-axis. For example, a ring of sources at radius 0.1 (just inside the hole) and see if the same 6 images can fit those with low error or if error shoots up. If it shoots up, try augmenting the basis with a couple more functions (maybe images rotated to cover azimuthal variation). This will inform how the rank grows when adding another degree of freedom (the angular position of the source). Similarly, vary the torus geometry: e.g. a different minor radius or major radius to see if the phenomenon of low-rank axis solutions is generic or sensitive to shape.
- **Cross-validation on Unseen z:** Omit one or more z from the fitting process (though here each z we solve independently, so instead, do: fit a single set of weights as a function of z perhaps by interpolation or a polynomial fit through the found $w(z)$, then evaluate at omitted points and compare to actual solve at that point). This would test if $w_k(z)$ vary smoothly enough that interpolation works. If yes, one might then *predict* weights for continuous z cheaply, which is a reduced model.
- **Examine Error Distribution:** For each z, look at where the error is max. Are these points near the torus surface, or far field, or near the source? Understanding this can suggest new basis functions (e.g. if error is near the inner equator, perhaps an image placed there could help).

8.2 Definitions and Formulas for Automated Analysis:

We can outline concrete steps for the codebase (electrodrive or similar) to implement: - **Assemble Weight Matrix W:** Suppose we have a list of source positions ${y(z_i)}*{i=1}^M$. After computing weights $w(z_i) \in\mathbb{R}^N$ for each (here $N=6$ currently), form a matrix or 2D array $W[i,k] = w_k(z_i)$. Likely dimension $M \times N$ or $N \times M$ depending on convention (above we took $6\times M$). - **Compute SVD:** Using a numerical library (e.g. NumPy or similar), do `U, s, Vt = svd(W, full_matrices=False)`. This yields singular values `s` (length = min(6,M)). We then have $W = U\mathrm{diag}(s)V^T$. One might output the singular values normalized by s_1 to see relative size. - **Effective Rank Determination:** Decide a threshold, e.g. 0.1 or 0.01 of s_1, and count how many singular values exceed that. That number is the effective rank. Alternatively, compute the cumulative energy: $\sum^6 s_i^2$ – how large must r be to capture, say, 99% of the energy? This is another indicator. - }^r s_i^2 / \sum_{i=1**Kolmogorov Width Estimate:** The smallest singular value σ_6 (or σ_{r+1} if $r<6$ rank) can be related to the worst-case error of the best rank-5 approximation in the sample data (in least squares sense). If σ_6 is small, it means a rank-5 subspace can already approximate all snapshots well in L^2 sense. More directly, the maximum approximation error of the best rank-r approximation (in L^2 norm over the sample set) is given by $ \sigma_{r+1}$ (by Eckart–Young theorem). So if $\sigma_5 = 0.01$ (just hypothetical), that means a 4-dimensional subspace exists that approximates all the snapshots to within 1% (in the L^2 sense over the

domain points weighted by whatever norm used in constructing W – likely a collocation-based measure). - **Interpolation of Weights:** If needed, fit $\alpha_k(z)$ as a smooth function. One could use polynomial regression: for each k, fit a low-degree polynomial through ${(z_i, w_k(z_i))}$. If the fit is good (small error), then $\alpha_k(y)$ is well-approximated by that polynomial. This provides an explicit formula for $G_N(x,y)$ valid for continuous y. This would essentially be constructing a degenerate kernel approximation $G(x,y)\approx \sum_{k=1}^N p_k(z)\Phi_k(x)$, where $p_k(z)$ is the fitted polynomial for $\alpha_k(y(z))$. One can then test this by checking the error at intermediate z not in the original set. - **Comparison to Analytic Expectations:** If singular values indicate exponential decay (e.g. ratio ~0.3 each step), compare that to theoretical expectations: an analytic family might have a radius of convergence ρ in the complex z-plane from the interval, leading to asymptotic ratio $\sim \rho^{-1}$. For example, if singular values drop by factor ~0.1 each increment, that's extremely fast, implying maybe radius ~10 beyond interval – not likely, but even factor ~0.5 is quite fast. This could be cross-checked by looking at the nearest singularity in complex plane (likely when source touches the torus surface, i.e. z complex such that distance to surface = 0). - **Integration with BEM pipeline:** The code could incorporate these computations after an axis sweep. For instance, after obtaining all $w(z_i)$, one could call a routine to do the SVD and print out singular values and effective rank. - **Possible Plots:** Plot $w_k(z)$ vs z to see smoothness, plot singular vectors V^T rows to see how they correspond to particular modes in z (they might resemble polynomial shapes or orthogonal polynomials in z). - **Extended Basis Testing:** Increase N by adding, say, 2 more candidate images (perhaps another ring or point) and re-run for one source to see if error can drop further. Alternatively, try to solve for a single z with more degrees of freedom to see how much error can be driven down (this tests if the current error is due to rank limitation or due to other issues like collocation/regularization errors). If giving 10 free parameters yields <1% error for that z, it confirms the limitation was basis size, not something else.

By carrying out these steps, one would quantitatively confirm the low-rank nature (or find its limits). It would connect to theory by providing the numerical "spectrum" of the solution manifold. If we find, for instance, singular values: $[1, 0.4, 0.1, 0.03, 0.008, 0.002]$ (just hypothetical), that's an indicator of exponential decay (rough ratio ~0.25 each time). Theory might then say the solution is analytic with a certain radius related to that ratio.

In summary, further numerical work should focus on extracting the *parametric modal structure* of the Green's function family. This will guide whether a "finite basis" (in practice) is feasible for broader cases and how many terms are needed for higher accuracy or more general sources. All these analyses strengthen the understanding of when we can treat a complex geometry's Green's function as nearly low-rank, and they provide data to compare against approximation theory of elliptic PDEs.

---

1 2 3 6 8 Green's function for the three-variable Laplace equation - Wikipedia
https://en.wikipedia.org/wiki/Green%27s_function_for_the_three-variable_Laplace_equation

4 5 2.7: The Method of Images with Point Charges and Spheres - Engineering LibreTexts
https://eng.libretexts.org/Bookshelves/Electrical_Engineering/Electro-Optics/Electromagnetic_Field_Theory%3A_A_Problem_Solving_Approach_(Zahn)/02%3A_The_Electric_Field/2.07%3A_The_Method_of_Images_with_Point_Charges_and_Spheres

7 Decay of the Kolmogorov N-width for wave problems - ScienceDirect
https://www.sciencedirect.com/science/article/pii/S0893965919301983