ChatGPT

Toroidal Electrostatics: Analytic and Numerical Foundations for Image Systems

1. Electrostatics in Toroidal Coordinates – Analytic Theory

1.1 Laplace's Equation in Toroidal Coordinates

Coordinate definition: *Toroidal coordinates* (τ,σ,ϕ) (often also denoted (ξ,η,ϕ)) are an orthogonal 3D coordinate system obtained by rotating the 2D bipolar coordinates about the symmetry axis [1] [2]. The system is characterized by a *focal ring* (reference circle) of radius a in the xy-plane, around which coordinates are defined [3]. A convenient definition is (using $\tau\ge0$, $-\pi<\sigma\le\pi$, $0\le\phi<2\pi$) [2] [4]:

$$x = a\,\frac{\sinh\tau}{\cosh\tau-\cos\sigma}\cos\phi, \qquad y = a\,\frac{\sinh\tau}{\cosh\tau-\cos\sigma}\sin\phi, \qquad z = a\,\frac{\sin\sigma}{\cosh\tau-\cos\sigma}\,.$$

Surfaces of constant τ are toroids (blue surface in Fig.1) centered on the z-axis, while $\sigma=\text{constant}$ yields spheres (red surface in Fig.1) whose centers lie on the z-axis [5] [3]. Surfaces $\phi=\text{constant}$ are half-planes through the z-axis (yellow in Fig.1). The coordinate τ relates to the distance from the focal ring: $\tau=0$ corresponds to the limit approaching the ring (inside the torus), and $\tau\to\infty$ moves far away radially. The scale factors (metric coefficients) can be derived from the Jacobian: for example, the infinitesimal length elements satisfy $ds^2 = a^2(\cosh\tau-\cos\sigma)^{-2}\big(\sinh^2\tau\,d\sigma^2 + \sinh^2\tau\,d\phi^2 + d\tau^2 \big)$ [6].

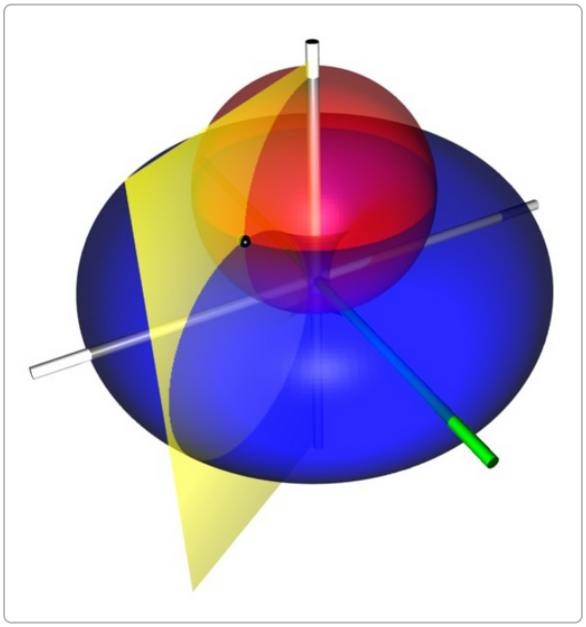

Fig. 1: Toroidal coordinate surfaces for a focal ring radius $a=1$. The blue surface is $\tau=\text{const}$ (a torus), the red is $\sigma=\text{const}$ (a sphere through the focal ring), and yellow is $\phi=\text{const}$ (half-plane). The black dot lies at their intersection [5] [3].

Laplacian in toroidal coordinates: The Laplace operator takes a somewhat intricate form. Substituting the scale factors, the Laplacian of a scalar potential $\Phi(\tau,\sigma,\phi)$ is [6]:

$$\nabla^2\Phi = \frac{(\cosh\tau-\cos\sigma)^3}{a^2\sinh\tau}\left[\sinh\tau\,\frac{\partial}{\partial\sigma}\Big(\frac{1}{\cosh\tau-\cos\sigma}\frac{\partial\Phi}{\partial\sigma}\Big)+\frac{\partial}{\partial\tau}\Big(\frac{\sinh\tau}{\cosh\tau-\cos\sigma}\frac{\partial\Phi}{\partial\tau}\Big)+\frac{1}{\sinh\tau(\cosh\tau-\cos\sigma)}\frac{\partial^2\Phi}{\partial\phi^2}\right]$$

Setting $\nabla^2\Phi=0$ gives Laplace's equation. The symmetry in ϕ is cyclic, so one can separate the azimuthal dependence as $\Phi(\tau,\sigma,\phi)=P(\tau,\sigma)e^{im\phi}$ (with integer m for single-valuedness). Plugging this ansatz and dividing through by common factors yields an equation that separates in τ and σ [7] [8]. Specifically, one obtains two ODEs: one for the *poloidal* coordinate σ and one for the *radial* coordinate τ. Taking $e^{im\phi}$ and further assuming separation $P(\tau,\sigma)=U(\tau)V(\sigma)$ leads to a pair of equations:

- A σ-equation: $V''(\sigma) + n^2 V(\sigma)=0$, which imposes **$n$ must be an integer** so that $V(\sigma)=\cos(n\sigma)$ or $\sin(n\sigma)$ is 2π-periodic in σ [8]. Thus $n=0,1,2,\dots$ is a *mode number* for the poloidal (minor-circle) direction.

- A τ-equation: $U''(\tau)+\coth\tau\,U'(\tau)-\Big[\frac{m^2}{\sinh^2\tau} + (n^2-\tfrac{1}{4})\Big]U(\tau)=0\,,$ which can be rewritten as a standard associated Legendre equation upon a change of variable [9] [10]. In fact, defining $x=\cosh\tau$, one finds solutions expressible in terms of *associated Legendre functions* of degree $(n-\tfrac{1}{2})$ and order m. These special functions are known as **toroidal harmonics** [10] [11].

Separated solutions – toroidal harmonics: The general separated solution for Laplace's equation in toroidal coordinates takes the form:

$$\Phi_{mn}(\tau,\sigma,\phi) = N_{mn}\,P^{m}_{n-\frac{1}{2}}(\cosh\tau)\begin{cases}\cos(n\sigma)\\ \sin(n\sigma)\end{cases}e^{im\phi},$$

or alternatively using the second (Neumann) solution $Q_{n-\frac{1}{2}}^m(\cosh\tau)$ instead of P. Here P_{ν}^m and Q_{ν}^m are associated Legendre functions of the first and second kind, respectively, with **half-integer degree** $\nu=n-\frac{1}{2}$ (and integer order m) [12] [13]. The appearance of half-integer degrees $\nu+1/2 \in \mathbb{Z}$ is the hallmark of toroidal coordinates; indeed *Legendre functions with half-odd-integer degree are often called "toroidal functions"* in this context [14].

Each (m,n) pair yields two independent harmonic solutions (one involving $P_{n-1/2}^m$, one $Q_{n-1/2}^m$). The m index corresponds to the Fourier mode $e^{im\phi}$ around the axis (the *toroidal mode* or azimuthal index), and n indexes the variation in the poloidal angle σ (the *poloidal mode*). For a fixed m,n, the function $P_{n-1/2}^m(\cosh\tau)\cos(n\sigma)$ is sometimes called a *toroidal harmonic of the first kind*, and $Q_{n-1/2}^m(\cosh\tau)\cos(n\sigma)$ of the second kind [15].

Discrete vs. continuous spectrum: The indices m,n are discrete integers due to the periodicity in ϕ and σ. However, the separation constant $\lambda = n^2 - 1/4$ enters the τ-equation and one might formally consider non-integer n solutions (leading to continuous λ). In practice for bounded domains one restricts to integer n (ensuring regular behavior on the symmetry axis $\sigma=\pm\pi$), so the *harmonic basis is discrete*. The *exterior region* outside a closed torus is multiply connected, but Dirichlet boundary conditions and decay at infinity still enforce a unique solution expansion in these discrete modes (no continuous eigen-spectrum arises, unlike in open geometries). By contrast, for the *interior region* of a torus (the void inside the donut), one again gets discrete m,n but the roles of P vs Q solutions swap for convergence (since interior vs exterior domains have different singularity behavior, as discussed shortly). We will see that the *Green's function* for Laplace's equation in $\mathbb{R}^3$ admits both a *double series* expansion in toroidal harmonics and also a *single-integral representation* – the latter reflecting an integral superposition of continuous-order harmonics (an alternate viewpoint of the same solution) [16] [17].

Singularity behavior and choice of P vs Q: The two kinds of toroidal harmonic have different singularities. For argument $x=\cosh\tau\ge 1$, one finds: $P_{n-1/2}^m(x)$ is finite as $x\to\infty$ (decays for large τ) but behaves *singularly on the focal ring* ($x\to1^+$, i.e. $\tau\to0$) [12] [13]. Conversely, $Q_{n-1/2}^m(x)$ is finite at $x=1$ (regular near the ring) but grows or diverges as $x\to\infty$. Physically, this means a *toroidal harmonic of the first kind (P-type)* is well-behaved at infinity and singular on the ring, whereas a *second-kind (Q-type)* is well-behaved at the ring but behaves like a multipole that diverges at infinity. For the **exterior Dirichlet problem** (grounded torus in free space), the potential in the *exterior* region must vanish at infinity and remain finite on the conductor surface. The torus surface corresponds to some $\tau=\tau_0$ constant. In the exterior ($\tau>\tau_0$), we therefore expand the solution using **first-kind (P)** toroidal harmonics (which decay at infinity) and choose linear combinations such that the boundary condition $\Phi(\tau_0,\sigma,\phi)=0$ is satisfied. The P-harmonics may blow up as $\tau\to0$ (inside the torus), but that is physically acceptable since the region $\tau<\tau_0$ is the conductor interior (not part of the solution domain). Conversely, if one considered the potential *inside* a conducting torus (interior region), one would use second-kind Q-harmonics (finite on the ring) to ensure regularity on the axis inside, and impose decay as $\tau\to0$ if needed while allowing divergence as $\tau\to\infty$ (since $\tau\to\infty$ is outside the physical domain in that case). In summary: **exterior solutions use $P_{n-1/2}^m$ series; interior solutions use $Q_{n-1/2}^m$ series**, analogous to how spherical harmonics use r^l vs $r^{-(l+1)}$ for inside vs outside expansions.

Green's function in toroidal form: The free-space Green's function $G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$ admits an expansion in toroidal harmonics. Using an identity due to Whipple, one convenient form (for two points given in toroidal coordinates with possibly different τ, σ values) is [15]:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sqrt{\frac{\pi}{2RR'(\chi^2-1)^{1/2}}} \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{\Gamma(m+\frac{1}{2})} P^{m}_{-\frac{1}{2}}\!\left(\frac{\chi}{\sqrt{\chi^2-1}}\right) e^{im(\phi-\phi')}.$$

Here $R = a/\tanh\tau$ and $R'= a/\tanh\tau'$ relate to the distances of the source and field point from the z-axis, and $\chi=\cosh\tau\,\cosh\tau' - \sinh\tau\,\sinh\tau'\cos(\sigma-\sigma')$ is a toroidal analog of the cosine of separation angle [18]. This infinite series involves the Legendre function $P_{-1/2}^m$, which corresponds to degree $-1/2$; in practice one can rewrite this in terms of more standard functions. Alternative expansions and also **integral representations** exist; e.g. the Laplace Green's function can be written as a single Fourier integral involving Bessel functions (from cylindrical coordinates) [19] [20], or

related “flat-ring” integrals [17]. These are all mathematically equivalent ways to express the Newtonian $1/R$ kernel in different coordinate systems. While no simple closed-form exists in toroidal coordinates, the above series and integrals are *canonical references* for analytic work [15] [19]. Modern authors like Cohl and collaborators have derived similar expansions for the fundamental solution in *flat-ring coordinates* and related systems, confirming consistency with the toroidal limit [21] [21].

1.2 Toroidal Harmonics and Multipole Expansions

Classical references: The theory of toroidal harmonics goes back to classical mathematical physics texts (e.g. Morse & Feshbach 1953, §7.3 [22]). These provide the separated solutions and discuss the Legendre functions of half-integer degree. In recent years there has been a resurgence of interest in toroidal functions: for instance, *Majic & Le Ru (2019)* derive explicit relationships between spherical and toroidal harmonic expansions [23] [24], and give efficient series formulas. The *Encyclopedia of Mathematics* also summarizes toroidal harmonics as Legendre functions of half-odd degree, noting the distinction between *surface toroidal harmonics* (on a torus) and *solid toroidal harmonics* (in space) [25]. Recent computational works (e.g. Fukushima 2016 [26], and Segura & Gil 2000 [27]) have focused on stable evaluation of half-integer Legendre functions, enabling high-precision use of toroidal harmonics in practice.

Toroidal multipole expansion for a point charge near a torus: Consider a point charge q located at some point outside a grounded conducting torus. The classic method to solve this uses the *toroidal multipole expansion*. One expands the potential in the exterior region in toroidal harmonics ($P_{n-1/2}^m$ functions as argued above) and enforces the Dirichlet condition on the torus surface ($\tau=\tau_0$). Because toroidal harmonics are an orthogonal set in appropriate weighted sense on that surface, one can expand the boundary condition (which must equal the *negative* of the point charge’s potential on the surface) in those harmonics. This yields a linear system for the expansion coefficients. In practice, for axisymmetric cases ($m=0$) one gets a simpler series of Legendre $P_{n-1/2}^0(\cosh\tau)$ terms; for general $m\ne0$, one includes the $\cos(m\phi)$ or $\sin(m\phi)$ dependence. The result is an infinite series solution for the potential. A classical analysis can be found in papers on toroidal capacitors or rings – for example, *Wong (1973)* expanded the gravitational potential of a homogeneous ring (which is analogous to the potential of a toroidal charge distribution) in toroidal harmonics [28]. More directly, *Kondratyev et al (2009–2012)* treated the potential of a charged torus and point interactions in Technical Physics journal [29] [30]. The series solution converges except potentially in a region between the torus and the point (we address convergence below in §1.2.3).

Do true image charges exist for a torus? Unlike the sphere or the infinite plane, a torus *does not admit a finite closed-form image charge solution*. That is, there is no finite collection of point charges (or simple singularities) in $\mathbb{R}^3$ that exactly reproduces the effect of a grounded toroidal conductor on an external charge. This is because the torus is neither a planar nor an equipotential spherical surface – it is a *non-confocal shape* for which the method of images fails analytically. In fact, even an infinite *series of point charges* placed judiciously will only converge to the correct solution in a limiting sense – essentially one needs a continuous distribution. The classical “method of images” largely succeeds only for highly symmetric, *uniquely symmetric* shapes (planes, spheres, infinite cylinders) where the Green’s function can be obtained via reflection or inversion symmetry. For a torus (a *genus-1 surface*), the Green’s function can be constructed via *separation of variables* (as above) or via a *boundary integral*, but not by a finite superposition of fundamental solutions.

However, one can **approximate** the solution by a finite number of sources in $\mathbb{R}^3$ in various ways. For instance, a **ring charge** (continuous charge uniformly distributed on a circular loop) is itself a potential solution basis – indeed the *lowest-order toroidal harmonic (n=0)* corresponds physically to the potential of a ring of charge (with a sinusoidal density around the ring) [14] [31]. Higher-order toroidal harmonics correspond to more oscillatory charge distributions on a ring or on a torus surface. In principle, by taking enough ring distributions (or enough points) one can approximate the continuous charge induced on the torus. The question of "in what sense can a finite set of sources approximate the true solution" is one of **approximation theory** rather than exact analysis. In practice, engineering texts sometimes use an *elliptical ring approximation* for toroidal capacitors – e.g. replacing the torus by a ring charge or a small set of discrete charges – but these are *ad hoc* and only accurate in certain regimes (like very thin tori). For a thin torus ($a/R \ll 1$, where R is the major radius and a the minor radius), the torus looks nearly like a circular ring of wire; thus one might approximate a grounded torus by a ring of opposite charge. But this is not exact – it neglects higher multipole moments distributed around the torus cross-section.

In summary, **no finite image-charge solution exists for a torus**. One must resort to infinite series (toroidal harmonics) or numerical solutions. Nonetheless, the *concept* of an "image system" – a structured set of singularities that approximates the influence of the conductor – is useful. Section 3 below leverages this idea to design finite bases (rings, loops, point clusters) that mimic the torus's effect.

Boundary condition enforcement in toroidal harmonics: Imposing $\Phi(\tau_0,\sigma,\phi)=0$ (with Φ expressed as a toroidal harmonic series plus the point charge's direct potential) leads to determining the series coefficients by orthogonality. Because ${\cos(n\sigma)e^{im\phi}}$ forms a complete set on the torus surface, one can, in principle, solve for each coefficient. For a point charge located on the symmetry axis above the torus (an axially symmetric case), only $m=0$ terms appear and the series reduces to $\sum_{n} A_n P_{n-1/2}(\cosh\tau_0) P_{n-1/2}(\cosh\tau_\text{source})$ with A_n chosen to cancel the boundary potential [32] [33]. This was done e.g. by *Reiner (1945)* and others for special cases. If the point is off-axis ($m\neq0$), one must include $e^{im\phi}$ variation and solve a coupled system for different m. Modern authors have tackled these expansions with computer algebra and numerical computation (e.g. *Majic 2020* used such expansions to compute the induced potential and charge distribution on a torus [34] [32]).

Convergence and truncation: A truncated toroidal harmonic expansion (cutting off at some maximum $n=N$ and perhaps $m=M$) gives an *approximate solution*. The error of truncation tends to concentrate near the "critical" regions of the domain. Specifically, if the point charge is very close to the torus, high-n modes are needed to resolve the sharp variation of potential on the surface (analogous to needing many spherical harmonics when a point is near a sphere). If the point is far, fewer terms suffice (the potential on the torus is smoother). There is a known issue: **nonuniform convergence** – the toroidal series and spherical series have domains of convergence separated by a gap. For example, Majic (2020) found that for a solid torus, the exterior spherical-harmonic expansion converges for $r > R_0$ (outside a certain radius) and the interior expansion for $r < R_0 - \delta$ (inside a slightly smaller radius), leaving an intermediate *annular gap* where neither series converges [35] [36]. Similarly, the toroidal harmonic series converges up to a certain limit in τ (inside a smaller torus) [37]. These limitations stem from singularities of the solution in complexified space (the *analytic continuation* of the potential has singularities on a smaller torus or spherical shell, limiting the series radius of convergence [38]). In practical terms, for a point outside a torus, the series converges for all field points outside an inner torus that touches the actual torus from inside [33]. Truncating at N yields an error roughly scaling like the $(N+1)$-th mode's contribution. If the source is at a distance d from the torus, one can estimate asymptotically that $A_n \sim O((a/d)^{n})$ for large n

(similar to how spherical harmonic coefficients decay as $(R_{\text{surface}}/d)^{l}$ for a point outside a sphere). Thus, truncation error might behave like $O((a/d)^{N})$. However, if d is only slightly bigger than a, convergence is slow (requiring many terms). For extreme near-contact (point nearly touching the torus), the series effectively diverges (one must then resort to other methods like local singularity resolution or numerical BIE).

In summary, toroidal harmonics give a *systematic expansion* for exterior (and interior) potentials. They clarify that no finite set of simple images exists, but they also suggest *which types of distributions* (ring-like, multipole-like) are physically relevant. These insights will guide the construction of *approximate image system bases* in §3.

2. Boundary Integral and Spectral Methods for Tori

2.1 Boundary Integral Equation (BIE) Formulations

The exterior Dirichlet problem (point charges outside a grounded torus) can be solved by boundary integral equations. The torus surface Γ is a smooth closed conductor. We seek a source density on Γ such that the single-layer potential reproduces the field of the external charges with zero potential on Γ. In operator form, one uses the *single-layer operator* $S: \sigma(\mathbf{r}') \mapsto \int_\Gamma G(\mathbf{r},\mathbf{r}')\sigma(\mathbf{r}')\,dA'$ which gives the potential at $\mathbf{r}$ due to surface charge σ on Γ. The boundary condition requires $S\sigma + \Phi_{\text{direct}}(\mathbf{r}*Gamma) = 0$ for all $\mathbf{r}*\Gamma\in\Gamma$, where Φ_{direct} is the potential due to the external point charges (which is known). This leads to a Fredholm integral equation of the second kind on Γ:

$$\sigma(\mathbf{r}) + \lambda \int_\Gamma G(\mathbf{r},\mathbf{r}')\,\sigma(\mathbf{r}')\,dA' = -\frac{1}{\lambda}\Phi_{\text{direct}}(\mathbf{r}), \qquad \mathbf{r}\in\Gamma,$$

for an appropriate normalization of G and choice of interior/exterior limit (here $\lambda=\frac{1}{2}$ for the usual jump relation of the single-layer potential on a smooth surface [39] [40]). Existence and uniqueness for this kind of boundary integral equation on a closed surface rely on Fredholm theory: the single-layer operator S for Laplace's equation on a closed surface is compact and symmetric (in appropriate function spaces), with a known nullspace consisting of constant σ only in the *Neumann* case (for Dirichlet, S is invertible except for a trivial zero-eigenvalue due to charge neutrality). In our case, since the torus is a conductor held at zero potential, the total induced charge is $-q$ (to exactly cancel the point charge's flux), which will naturally emerge from the solution. There is no non-uniqueness issue because we impose decay at infinity (which fixes the constant potential mode to zero).

One can also use a *double-layer formulation*, where a dipole layer on Γ is used. The double-layer potential $D\mu = \int_\Gamma \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'}\mu(\mathbf{r}')\,dA'$ yields a jump in potential across Γ. For the exterior Dirichlet problem, one can solve $(\tfrac{1}{2}I + D)\mu = \psi$ for some boundary data ψ derived from $-\Phi_{\text{direct}}$, which is a second-kind Fredholm equation. Both single- and double-layer formulations are *well-posed* for Laplace's equation on smooth surfaces – this is underpinned by potential theory and *Kellogg's theorem* ensuring the solution is continuous up to the boundary for Dirichlet data. There are also coercivity results (related to *Korn's inequalities* in elasticity or the compactness of embedding in potential theory) which ensure stability of the integral operators on appropriate Sobolev spaces on the torus.

In short, the BIE approach reduces the problem to solving a dense linear system (after discretization) for σ or μ on the torus. The operator is *Fredholm of index 0* and typically well-conditioned (for Laplace's equation, the condition number grows mildly with refinement but is bounded away from infinity since the integral equation is second-kind).

Numerical strategies: On a *smooth torus*, one can discretize the integral equation by panel methods or spectral parametric methods. A torus is a tensor-product of circles topologically, so a natural approach is to parameterize by (σ,ϕ) in $[0,2\pi]^2$ and use a *periodic Fourier discretization* in both angles. In practice, one might use a truncated Fourier series in ϕ and σ to represent $\sigma(\mathbf{r})$ and convert the integral equation to a linear system (this is related to the spectral methods discussed in §2.2). Alternatively, one can triangulate or panelize the torus surface and use collocation or Galerkin BEM. High-order quadrature is crucial due to the singular kernel $G(\mathbf{r},\mathbf{r}')\sim |\mathbf{r}-\mathbf{r}'|^{-1}$. Techniques for near-singular integration on surfaces of revolution include singularity subtraction and analytic integration in one parameter. For instance, in axisymmetric integrals one can often perform the ϕ-integration analytically using 2π periodicity and Bessel function identities, leaving a 1D integral in σ which can be handled by special quadrature [41] [40].

Fast methods: Since a torus is a closed surface, the use of the *Fast Multipole Method (FMM)* is natural to accelerate the N-body interactions in an iterative or direct solver. For example, one could use a 3D FMM for Laplace to multiply the dense matrix in $O(N)$ time per iteration. The torus's smoothness and symmetry could also be exploited for efficiency: as an axisymmetric surface, one can Fourier-decompose the problem (see §2.2) so that each Fourier mode can be solved independently, leading to uncoupled smaller systems [42] [43]. This effectively diagonalizes the ϕ-coupling and is a big computational saving: solving a 2D integral equation (in σ) for each m rather than a full 3D problem [42] [43]. In Young & Martinsson's high-order axisymmetric BEM (2009), this approach allowed solving a Laplace BIE on a torus with $320,800$ unknowns in seconds on a desktop [44] [45]. They achieved spectral accuracy by using *Fourier–Nyström discretization* and special Legendre function recurrences for evaluating the kernel [46]. Their method also extended to *multiply connected* domains (like multiple tori, or torus complements) by combining with FMM for the sparse coupling of different components [47].

In summary, modern BEM solvers can handle toroidal conductors with high accuracy. They rely on the smooth 2-parameter nature of the torus, treating it either by global Fourier methods or local panels. Special care is taken for near-singular integration (when source and target panels are close on the curved surface). If multiple length scales are present (e.g. a very thin torus, $a/R\ll 1$), one must ensure the discretization resolves the small cross-sectional features – fortunately Fourier series in σ converge exponentially for the smooth circumference of the cross-section.

Finally, we note that *Fredholm properties* of these integral equations ensure that standard iterative solvers (GMRES, etc.) converge rapidly (typically in a number of iterations independent of N for well-conditioned second-kind equations). The torus geometry does not induce ill-conditioning except if the aspect ratio is extreme (very thin torus might cause clustering of eigenvalues related to closely spaced charges around the ring, but still manageable).

2.2 Spectral and Eigenfunction-Based Approaches

Given the torus's double-periodic geometry (in σ and ϕ), it is natural to attempt *spectral expansions*. One approach, as mentioned, is to expand the *surface charge density* $\sigma(\sigma,\phi)$ in a double Fourier series:

$$\sigma(\sigma,\phi) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} c_{mn}\, e^{im\phi}\, e^{in\sigma},$$

(with n integer and the understanding that σ is 2π-periodic so n is integer). This is essentially using the eigenfunctions of the torus *as a 2D manifold*: since the torus is a product of circles, $e^{i n\sigma}e^{i m\phi}$ is an eigenfunction of the Laplace–Beltrami operator on the surface. For solving Laplace's boundary value, one doesn't strictly need eigenfunctions of the surface Laplacian, but they form a convenient basis for σ. Inserting such an expansion into the boundary integral equation yields algebraic equations for coefficients c_{mn}. Thanks to orthogonality, modes (m,n) will often decouple or weakly couple. In fact, for the *axisymmetric case* ($m=0$), only $\cos(n\sigma)$ appear and one can solve each mode independently (this is exactly the toroidal harmonic expansion method of §1.2, but now interpreted as solving the BIE in a spectral Galerkin manner).

For general (non-axisymmetric) problems, *azimuthal Fourier modes m decouple* because the kernel $G(\mathbf{r},\mathbf{r}')$ is rotation-invariant about the z-axis [48] [43]. This means one can treat each m separately. For a fixed m, one still has coupling in the σ modes because the kernel's dependence on $\sigma-\sigma'$ is not trivial. However, one can compute the matrix of the integral operator in the σ-basis and solve it. Many authors (e.g. Martinsson above, or Venkov 2007 in acoustics) solve for each m via a truncated linear system. The conditioning of these systems can vary: low m (symmetric modes) are usually well-behaved, while very high m (rapid ϕ variation) require finer ϕ sampling but still decouple per m. The completeness of the $e^{i n\sigma}$ basis on $0<\sigma<2\pi$ ensures one can resolve any continuous σ-dependence.

Another approach is using *toroidal harmonics themselves as basis.* Instead of plain Fourier in σ, one could use the functions $P_{n-1/2}^m(\cosh\tau_0)\cos(n\sigma)$ which already satisfy the Laplace equation in the exterior. This turns the problem into finding weights for those basis functions to satisfy boundary conditions (similar to a Galerkin method using eigenfunctions of the PDE). This is essentially the *method of moments with toroidal harmonic basis*. It has the advantage that the series converges faster (since each basis element individually obeys Laplace's equation, fewer are needed). Indeed, Majic (2020) demonstrated expressing the induced potential of a torus in both spherical and toroidal eigenfunction bases, finding better efficiency with the tailored toroidal basis [32] [49]. The downside is that toroidal functions $P_{n-1/2}^m$ are more complicated to compute than sines and cosines; but with recurrence formulas and libraries (like *Cohl's algorithms* [50] [51]) this is feasible.

Completeness and convergence: The double Fourier basis ${e^{i m\phi}e^{i n\sigma}}$ is *complete* for representing any L^2 function on the torus surface (by periodicity). So a Galerkin method with this basis is theoretically convergent to the true σ. Convergence is spectral (exponential in N,M) if $\sigma(\sigma,\phi)$ is smooth. In our case, the true charge density will be smooth except possibly at points directly below an external point charge (there, σ has a mild spike but still analytic on the closed surface since the point is off the surface). So spectral convergence is expected. The *conditioning* of

the linear system, however, can become an issue if using naive collocation: Fourier modes that are high-frequency might be ill-weighted. A better approach is *Galerkin* with those basis functions (ensuring symmetry). Alternatively, as Martinsson's method does, one can solve the continuous integral equation by exploiting analytic properties and never forming a very ill-conditioned matrix at all [52] [53] (they used modified Gaussian quadrature along the generating curve ensuring spectral accuracy and stability).

Eigenfunction completeness: The toroidal harmonics $P_{n-1/2}^m(\cosh\tau_0)\cos(n\sigma)$ for $n=0,1,2,\dots$ and m given, form an orthogonal basis for functions on the circle (σ) with a certain weight related to the torus geometry. In fact, they are eigenfunctions of the integral operator for a ring geometry. This relates to the *Mercer expansion* of the Green's function on the torus: one could in principle diagonalize the integral operator by its eigenmodes on Γ. Those eigenmodes turn out to be exactly the induced charge distributions corresponding to toroidal harmonics. So using them is akin to diagonalizing the BIE, yielding optimal convergence. This is confirmed by the analysis in Majic (2019), who studied the singularity structure of toroidal harmonics and their completeness for representing a charged torus potential [54] [55].

Spectral methods in practice: In practice, implementing a spectral method for a torus might involve using FFTs in ϕ and σ to accelerate convolution-like operations. For example, if one uses the Fourier basis, the integral operator becomes a convolution in σ for each m, which can sometimes be accelerated via FFT as well (though the kernel is not a delta, but one can use FFT-based convolution for the smooth part, combined with special treatment for the singular $n=0$ part). The cost can be made $O(N M \log N)$ per iteration or so, which is efficient. Such methods were used in low-frequency scattering calculations for tori (e.g. *Venkov 2007* solved an acoustic scattering from a torus by series expansion [56], and *Vafeas 2016* did electromagnetic scattering by a torus using Rayleigh/toroidal modes [56]).

In summary, spectral expansions (Fourier or toroidal) provide powerful tools on toroidal geometries. They turn the problem into solving many 1D problems (one per Fourier mode), each of which converges rapidly. The main challenge is *computational implementation*, especially evaluating special functions robustly (for toroidal harmonics) and handling large mode counts if the geometry demands it. With modern arbitrary-precision arithmetic or precomputed tables for $P_{n-1/2}^m$, one can push to $n\sim 100$ or more with good accuracy [32] [49].

2.3 Error Estimates and Asymptotics

Convergence rates for toroidal harmonic expansions: As touched on earlier, the error of truncating an expansion in toroidal harmonics depends on how "smooth" the solution is outside the expansion's domain of analyticity. If a point charge is at radial distance d from the torus center, one expects the dominant singularity (from the viewpoint of the toroidal coordinate expansion) to occur when that point is continued inwards to some complex τ or when it aligns with the focal ring. In practical terms, authors have found that to achieve near-machine precision, one might require on the order of tens to a couple hundred toroidal modes. For example, Majic (2020) reported that using $n_{\max}=120$ toroidal harmonics was sufficient to get high accuracy for a point charge outside a torus in most of space [32] [49], whereas spherical harmonics required up to $l_{\max}=170$ in the same case (because spherical functions converge more slowly in the toroidal geometry) [32] [57]. The error along the boundary (Dirichlet enforcement) is typically the most stringent measure – requiring enough modes such that the series error at $\tau=\tau_0$ is below tolerance. This usually means including modes up until the series coefficients A_n fall below tolerance. If A_n decays as ρ^n (for some $\rho<1$ related to singularity location), then $N\approx \log(\text{tol})/

\log(\rho^{-1})$. For a source just outside the torus, ρ is close to 1, requiring large N. As the source moves away, ρ decreases and fewer terms suffice.

Error on and near the boundary: A key metric is the maximum potential error on Γ. For truncated series or numerical BEM, this can be checked by computing $\max_{\mathbf{r}\in\Gamma}|\Phi_{\text{approx}}(\mathbf{r})|$. The asymptotic behavior for large n of toroidal harmonics can be used to estimate this. Using known asymptotics of Legendre functions for large degree ν (Debye asymptotics), one can derive that $P_{\nu}^{m}(\cosh\tau_0)$ behaves like $e^{\nu \tau_0}/\sqrt{2\pi\nu\sinh\tau_0}$ for large ν [33] [30]. So the terms ~$A_n P_{n-1/2}^m(\cosh\tau_0)$ will roughly scale like $A_n e^{n\tau_0}$. If the source is at $\tau=\tau_s$, often $A_n\sim e^{-n\tau_s}$, so the product behaves like $e^{-n(\tau_s - \tau_0)}$. If the source is outside ($\tau_s>\tau_0$), this decays exponentially with rate $(\tau_s-\tau_0)$. If τ_s is only slightly bigger than τ_0, decay is slow. This heuristic explains the gap region: if τ_s is exactly equal to τ_0 (source on the surface), the series fails to converge (logarithmic divergence typical of on-surface point).

Interior field accuracy: For points in the field (not on the boundary), one can also examine how truncation affects potential or field. Generally, if the boundary condition is well-satisfied, the solution inside (by maximum modulus principle) will also be accurate except near singular points (close to the actual charges). The **near-field error** (say error in E-field just outside the torus) is dominated by how well the induced surface charge is approximated locally. If the expansion misses some high-frequency component of σ in a localized region, the near-field there will show error. This is why adding *local patches or local basis* can dramatically reduce near-field error with fewer global modes (a point we revisit in §3).

Dependence on source location and aspect ratio: Perhaps the most interesting asymptotic regime is *thin vs fat torus*. For a *thin torus* ($a/R\ll 1$), the torus is almost like a circle of wire. One expects the field of a point charge to induce a charge distribution that is relatively smooth around the torus (since a thin torus has nearly uniform influence along the ring). Indeed, for $a/R \to 0$, the dominant mode might be just a uniform ring charge induced, plus maybe small higher-n variations. So fewer modes might be needed. At the extreme thin limit, one might approximate the torus by a ring (leading-order) plus small corrections (perhaps perturbation theory can be applied, as done by R. Cade 1986 for capacitance of thin torus [58] [59]). Conversely, for a *fat torus* (a/R close to 1), the curvature is high and different on inner vs outer side, so the induced charge can vary strongly between inner and outer equators. This likely requires more modes (higher n to represent the sharp gradient from inside to outside). If a/R approaches 1, the torus is almost self-intersecting (becoming a sphere when $a/R=1$ exactly). In that limit, one expects to need many modes – essentially approaching the spherical harmonic count for a sphere. So roughly, the number of modes needed grows as the torus gets "fatter."

Source position: If the source is far from the torus (relative to R), the torus sees a nearly uniform field across it (like a distant charge creates a nearly uniform field, equivalent to a dipole field at large distances). In that case, the induced charge might be dominated by the first toroidal harmonic ($n=0$ or $n=1$ modes corresponding to an almost dipolar distribution on the torus). As the source approaches, higher modes come in. If the source is on the symmetry axis above the torus, the induced charge is symmetric but becomes sharply peaked on the top of the torus as the source nears contact – requiring many Fourier modes in σ. If the source is off-axis (introducing $m\ne0$), one also needs Fourier modes in ϕ to represent the asymmetric distribution around the torus. But those are easier in the sense that ϕ modes decouple; the hardest part is still the high-n (poloidal) content near wherever the source is closest to the torus.

In summary, the error of truncated expansions can be controlled and estimated by considering the nearest singularity of the analytic continuation of the solution. Practically, one uses convergence tests (increase N until boundary error stops decreasing) to ensure sufficient terms. And if computational cost is an issue, one can combine global and local bases to reduce N needed, which leads us to consider designing such bases next.

3. Basis Design for "Image System" Approximations on a Torus

We now shift from analytic series to the design of **finite basis families in $\mathbb{R}^3$** – i.e., collections of discrete or continuous charge distributions placed in space to mimic the true solution. The aim is to capture the physics with a small number of elements, effectively a low-rank approximation of the solution. We consider various candidate basis families, evaluating their theoretical approximation power and practical stability.

3.1 Classical Inspiration: Image Methods and Generalizations

Planes and spheres (method of images): The classical image solutions provide inspiration. For a point charge near an infinite grounded plane, a single image charge exactly cancels the normal field (classically, a point charge $-q$ mirrored across the plane) [60]. For a point outside a grounded sphere, a single image charge located along the line through the center produces the exact solution (the famous Kelvin image for a sphere) [61]. These are "miraculous" cases where symmetry and harmonic function uniqueness coincide to allow finite images. They underscore that **when a finite image *exists*, it is essentially because the boundary is an equipotential of some simple singularity** (a sphere is an equipotential surface of an appropriately placed point charge).

Parallel plates (infinite image series): When one has two parallel conducting planes and a point charge between them, the method of images leads to an *infinite series* of mirror charges alternating above and below, converging to the correct solution [62] [63]. This is an example where no finite set works, but an infinite sequence does. In practice, truncating the infinite mirror-charge chain yields an approximate solution. Notably, the error decays exponentially with the number of image charges for large separation, but slowly if the point is near one plate (many images needed). This scenario is analogous to a torus in that the torus's topology (genus 1) might require an infinite sequence of charges wrapping around the hole.

Generalized image methods for other shapes: For ellipsoids or spheroids, no finite point image solves the Dirichlet problem exactly. However, there are approximation schemes: e.g., for a point near a prolate spheroid, one might place a line of point charges along the major axis to mimic the field (this relates to the series expansion in ellipsoidal harmonics). Similarly, for a lens-shaped object or a double-sphere, methods of *iterative reflections* exist (like multiple image charges placed iteratively until boundary conditions are met to tolerance). These can be seen as *ad hoc basis selection*: one chooses a set of points (and possibly line distributions) guided by symmetry. Each added element improves the fit at the expense of solving a linear system for the weights.

Attempts at torus image methods: In literature, explicit image constructions for tori are scarce. We found no known closed-form sequences like the parallel plate case. However, some numerical works implicitly use such ideas. For instance, *Andrews (2006)* in his "alternative separation" approach produced solutions for charges inside a conducting torus segment by using series of Legendre functions [64] [65], which can be interpreted as image line distributions. Also, work on toroidal inductors sometimes mentions "image

currents" to handle the torus's hole (for magnetostatics, where currents in a torus produce image current loops). These analogies hint that **a ring of charges might serve as a basic image for a torus**. If one were to guess an image system for a torus, a logical first guess is a *circular ring of charge placed somewhere inside the torus* (likely on the symmetry axis or along the central hole). Another plausible element is a *dipole ring* – two concentric rings with opposite charge, to produce a field more concentrated on one side.

Historically, engineers sometimes replaced a *thin toroidal electrode* by a loop of charge or a pair of loops to estimate capacitance or fields (with mixed success). In the 19th century, there was interest in the *electrostatic capacity of a torus*, leading to approximate calculations using ring charges (e.g., by treating the torus as a thin circle, one can estimate the capacitance by assuming the charge distributes uniformly along it, which is a first-order approximation). These classical results motivate using **loop basis functions**.

To systematically go beyond classical images, we consider four families tailored to the torus geometry:

3.2 Toroid-Specific Basis Families

3.2.1 Ring/Loop Basis

Description: A *ring basis element* is a continuous circular loop of charge in a plane. For instance, a loop of radius $R_$ *centered on the z-axis (coaxial with the torus). Its potential is axisymmetric around that axis, given in closed form by elliptic integrals (the potential on the axis is $V(z) = \frac{q}{\sqrt{R_^2 + z^2}}$*, and off-axis formulas involve complete elliptic integrals K, E). One can also allow *non-uniform* linear charge density around the loop, e.g. proportional to $\cos\phi$ or $\sin\phi$ along the loop, to generate higher-order modes. In fact, a *Fourier series of densities on a loop* can create a rich set of field shapes. For example, a loop with density $\lambda(\phi) = \lambda_0 + \lambda_1\cos\phi$ produces a potential that outside looks like a monopole plus a dipole oriented in-plane.

For toroidal problems, **ring distributions are natural** because the torus itself is a ring-shaped conductor. The simplest approximation to the induced charge on a torus might be a ring of charge placed along the torus's *centerline* (the circle at the center of the tube). This would capture the *total charge and the first-order smoothing around the torus*. However, a single uniformly charged ring cannot satisfy the boundary condition – it produces an almost uniform potential around the torus but not zero everywhere. We would need to allow variations. Thus we introduce basis elements like: - A *uniform ring charge* (one degree of freedom: total charge). - A *cosine-weighted ring* (two degrees: sine and cosine terms on the ring). - Higher Fourier modes on a ring as needed.

These basis elements are essentially the real-space counterparts of toroidal harmonics of $n=0,1,2,\dots$ on a given radius. Using a *discretized ring*: one might approximate a continuous ring by 8 or 16 point charges equally spaced – but then those point charges themselves form a discrete approximation to the continuous loop basis. In our context, if we can integrate a continuous ring's potential analytically (which we can, via elliptic integrals), it's advantageous to keep it as a continuous element for exactness and fewer degrees of freedom.

Potential and parameterization: The potential due to a ring of radius $R_$ *(lying in the xy-plane) at a field point (r,θ) in spherical coordinates (with $\theta=0$ on the z-axis) is:*

$$V_{\text{ring}}(r,\theta) = \frac{\lambda_{\text{tot}}}{4\pi\epsilon_0}\frac{4}{r_>}\Big[(2-k^2)K(k) - 2E(k)\Big],$$

where $r_> = \max(r,R_)$, $k^2 = \frac{4rR_}{(r+R_)^2 + z^2}$ (with $z=r\cos\theta$), and K,E are complete elliptic integrals [66]. If λ_{tot} is the total charge on the ring, this formula gives the potential for a uniform ring. For non-uniform $\lambda(\phi) = \lambda_0 + \lambda_1\cos(\phi) + \lambda_2\sin(\phi) + \cdots$, one can similarly derive the potential by expanding in Fourier harmonics – each harmonic yields a different combination of elliptic integrals (e.g. a $\cos\phi$ density produces a dipole-like field oriented in the x-direction, etc.). For practical basis design, we might include explicitly: - Uniform ring at radius $R_$ *(one parameter). - Cosine and sine weighted rings at radius $R_$* (two parameters, representing a dipole moment around the ring). - Possibly higher harmonics like $\cos 2\phi$ if needed (quadrupolar ring distribution).

One can also have rings tilted or oriented differently, but given the symmetry of our problem (torus is symmetric around z-axis), rings concentric with the torus's hole are the most logical.

Relation to toroidal harmonics: A ring charge is essentially a source for the $m=0, n=0$ toroidal harmonic if uniform, and $m=1, n=0$ if modulated by one sinusoid, etc. Thus a linear combination of ring harmonics can approximate the effect of low-order toroidal modes. For instance, a uniform ring plus a $\cos\phi$-modulated ring could match the $P_{-1/2}^0$ and $P_{-1/2}^1$ contributions. However, note that a ring of charge has a continuous spectrum of multipoles – it's not exactly equal to a single toroidal harmonic because it doesn't enforce the proper τ variation (it's a fixed radius distribution, not extending in τ). So a single ring might approximate a combination of several n modes at once.

Approximation power: A single ring can capture the *global average* effect – e.g., the mean potential on the torus. Two rings (one maybe slightly inside, one slightly outside, with opposite charge) can create a field that varies across the torus cross-section – this is akin to a *dipole layer around the ring*, which can cancel potential on one side while enhancing on the other. Therefore, a small number of rings can, in principle, produce a variety of potential shapes around the torus. The limitation is that rings produce fairly smooth potential distributions (harmonic along ϕ direction, and monotonic in radial decay). They may have trouble with very localized changes.

Discrete vs continuous modeling: If implementing in a solver, one could either treat the ring analytically (compute its potential via formula), or discretize it into, say, 16 point charges equally spaced to approximate the continuous loop. The latter introduces high-frequency noise (point charges give a slightly lumpy field close-up). But if 16 is high, the lumpiness is small. Still, an analytic evaluation is preferable for accuracy and speed (plus it can be vectorized on GPU easily by evaluating complete elliptic integrals for many targets).

Summary: Ring basis elements (with possible Fourier weighting) provide a *global, smooth* representation. They are ideal for capturing large-scale features: e.g., the fact that a point charge will induce an overall monopolar opposite charge on the torus (uniform ring image), and an overall dipole moment (shift of charge to the nearest side, which a cos-weighted ring can simulate). Thus they should be highly effective in reducing boundary error's large components.

Conditioning: Rings are relatively well-behaved and not prone to strong near-singularity issues (except on their own plane). However, multiple rings of similar radius could become nearly linearly dependent (e.g., a ring of radius R_1 vs R_2 might produce similar potentials far away if $R_1\approx R_2$). One must

space them or constrain combinations to avoid ill-conditioning (perhaps treat them as a group with an internal orthogonalization like Gram-Schmidt on their potential shapes).

3.2.2 Poloidal "Ribbon" or Arc Basis

Description: By *poloidal ribbon*, we mean a partial loop of charge that goes around the small circular direction of the torus (the *minor circle*). Imagine taking an arc that wraps once around the tube cross-section (a closed loop around the tube's circumference, which is a *meridian* of the torus). Because the torus is a surface of revolution, a closed ribbon around the minor circle is just the torus's surface itself. Here we actually mean possibly an *open arc* (not a full closed loop) of charge positioned outside the surface but following a path around part of the tube.

Another interpretation: "poloidal ribbon" could refer to a finite segment of charge that traces the torus shape partially in the poloidal direction. For instance, a quarter-circle of charge hugging the inner side of the torus. Such a basis element would produce a localized potential bump on one side of the torus.

The rationale for these is to capture *localized patches of induced charge*. When the point charge is near a specific spot on the torus, most of the induced charge concentrates there. A global ring basis will struggle to mimic a highly localized peak without adding many Fourier components. Instead, one could include a basis element that is itself localized.

Examples: - A short arc of charge placed near the expected high-charge region (e.g., directly under the external point). - A *ribbon spanning a range of σ* angles, like covering the inner half of the torus cross-section vs the outer half (to mimic charge segregation between inner and outer sides).

One could define a ribbon basis as a continuous charge distribution along a curve on or near the torus surface. For simplicity, it could be approximated by a cluster of point charges arranged along that curve (since an analytic formula for a finite arc's potential is less standard than for a full circle).

Relation to Fourier modes: A localized arc can be thought of as a superposition of many Fourier modes in σ. For instance, a delta-function around a certain σ_0 on the torus could be expanded in $ \cos(n(\sigma-\sigma_0))$ Fourier series. So including a localized basis implicitly provides high-n content which global rings cannot easily supply. In other words, it addresses the *higher-order poloidal harmonics* with fewer elements by directly placing charge where needed.

Potential and parameterization: If we represent a ribbon by k point charges spaced along some curve, those k charges themselves are parameters (their magnitudes). One might instead define a parametric family, like a Gaussian-weighted distribution around σ_0: $\lambda(\sigma) \propto e^{-\frac{(\sigma-\sigma_0)^2}{2\sigma_w^2}}$ along a closed loop. That yields a "fuzzy patch" of charge centered at σ_0. This is conceptually like the "RingGauss" basis the engine already had (Gaussian-weighted rings for torus) – except that was around the ring (toroidal direction), whereas here it would be around the poloidal direction.

We can also consider a *small loop around the tube*: a loop that goes around the tube cross-section might at first seem redundant (if it's a closed loop it's symmetrical and maybe like a ring?), but if offset, maybe not. Actually, a closed loop around the tube is just a trivial rotation of the ring basis (the torus has two principal cycles: toroidal and poloidal). A loop around the tube (poloidal loop) of charge would create a field mostly

contained near the torus (like a ring around the tube, possibly not very effective outside). Perhaps not so useful as an image for exterior field.

Instead, partial arcs covering part of the tube circumference (like a segment on inner side) could produce an asymmetric distribution. Combining a few such arcs (like one on inner, one on outer, etc.) could approximate an arbitrary distribution on the torus surface piecewise.

Approximation power: Ribbon/arcs can achieve what ring basis cannot: *localized potential variation along $ \sigma$*. For example, if a point charge is above the torus, the induced σ-distribution of charge will be higher on the top. A ribbon basis that specifically adds an extra concentration on top can adjust the potential there without affecting the bottom side much. Without it, a ring basis would require summing many cosine modes to approximate the same.

Thus, a small number of ribbons (maybe placed at quadrature points along σ) might dramatically reduce the maximum boundary error, because each ribbon can independently fix the error in a localized region.

Conditioning issues: If too many overlapping arcs are used, they could be nearly linearly dependent (one arc's effect might be replicated by a combination of neighboring arcs). It's advisable to choose them in a way that they cover different regions with minimal overlap. Perhaps one arc for inner side, one for outer, one for top, one for bottom, as a first coarse set.

Additionally, point charges approximating a continuous arc have to be spaced such that they don't cluster too tightly (which would cause ill-conditioning in solving for their weights). Setting them with a fixed small separation is okay, but if the arc is very short, then those points are very close and their fields nearly duplicate each other. Using a *distributed parameter* form (like specifying total charge of the arc and maybe distribution shape) could avoid that, but then solving becomes nonlinear. So likely we keep them as multiple point charges in a fixed geometry, and solve a least-squares for weights (which can cause large coefficients if arc is small due to ill-conditioning). This can be mitigated by regularization (group L2 regularization for points in one arc for example, to avoid wildly different signs within one arc).

Despite these issues, poloidal arcs seem a powerful addition for capturing fine details in the solution.

3.2.3 Toroidal "Mode" Basis

Description: A *toroidal mode basis element* is a composite source designed to mimic a particular (m,n) toroidal harmonic mode's effect. Instead of relying on the actual special function solution, we attempt to construct a *finite set of point charges or rings whose combined field approximates that harmonic*. For example, consider the $(m=1,n=1)$ mode on a torus: it corresponds to a potential that varies like $ \cos\sigma\cos\phi$ on the torus surface. How to realize something similar with discrete sources? One approach: - To get $\cos\phi$ variation (one period around torus), we could place two identical sources on opposite sides of the torus (at $\phi=0$ and $\phi=\pi$) with appropriate sign difference. That ensures the potential at $\phi=0$ side is opposite to $\phi=\pi$ side, roughly mimicking a $\cos\phi$ pattern. - To also get $\cos\sigma$ variation (positive on outer equator, negative on inner equator, say), we could position these sources not at the mid-plane but one towards the outer side and one towards inner side of the torus, or maybe use four charges: e.g., a positive charge near the outer-top quadrant and a positive near outer-

bottom, and negative near inner-top, negative inner-bottom. Then the sign flips between inner/outer and top/bottom, approximating a $\cos\sigma$ node at the midplane.

This sounds complicated but essentially one can arrange small clusters of point charges in symmetric patterns that reflect the (m,n) mode's symmetry: - For m (toroidal mode number), arrange $2m$ charges evenly around the ring, alternating sign to create m periods of sign change around ϕ. - For n (poloidal mode), arrange those charges also in $2n$ positions around the cross-section alternating sign accordingly.

This results in $4mn$ charges if done naively, which is quite a lot. But perhaps one can do with fewer by exploiting symmetry. For instance, $(m=1,n=1)$ might need only 4 charges as described (one per quadrant of torus surface), whereas $(m=2,n=1)$ might need 8 around (two pairs in ϕ direction, times two in σ etc.). If m or n is zero, it simplifies (pure ring or pure poloidal patterns).

Alternatively, rather than point charges, one could use rings as sub-elements: - For an $m\ne0$ mode, use a *$\cos(m\phi)$-weighted ring distribution*. That inherently has m oscillations around. - For an $n\ne0$ mode, perhaps use two rings: one shifted a bit towards inner side and one towards outer, with opposite sign, to emulate a $\cos\sigma$ variation (like a dipole across the cross-section).

For example, to mimic $(m=0,n=1)$ (which is a variation only in σ, axisymmetric), one could take a *"doublet" of two concentric rings*: one slightly inside the torus tube, one slightly outside, with opposite charges. This creates a potential that is higher on one side of the tube and lower on the other – a rough analog of $\cos\sigma$ distribution.

Similarly, $(m=1,n=0)$ (variation only around ϕ, i.e. a single period around torus) can be made by two half-rings of charge 180° apart.

Comparison to actual toroidal harmonics: A finite construction will only approximate the actual harmonic. The true harmonic extends as a solution throughout space; our finite cluster will match it perhaps on the torus surface or in a least-squares sense. It might deviate elsewhere or not satisfy Laplace's equation exactly (point charges produce singular fields of course). But if the basis is used in a collocation scheme, it will adjust weights such that the combined field best fits the boundary condition, so exact adherence to Laplace in the domain isn't required of each basis element (unlike Galerkin which uses true harmonics, a collocation or least-squares can use any ansatz that spans the solution space approximately).

Pros and cons: The toroidal mode basis tries to incorporate symmetry knowledge – which can greatly reduce the number of degrees of freedom if done right (one composite capturing what many random points would require to coordinate). However, designing these patterns a priori for higher m,n becomes complex. It might be easier to let an algorithm *discover* them (perhaps by starting with many candidate charges and letting a sparsification algorithm pick symmetric patterns).

Nevertheless, including *some pre-designed composite basis* like a dipole-ring or a quadrupole-ring might be beneficial. These could be treated as single "basis functions" with fixed geometry and variable weight.

Example basis elements: - **Dipole ring pair:** Two rings of radius R_*, one slightly offset towards +z (upper half of torus) and one towards -z (lower half), with opposite charge. This mimics an $n=1$ poloidal mode, $m=0$.* - *Dipole along torus: Two point charges (or small ring segments) on the torus centerline at $\phi=0,\pi$, one*

*positive one negative. This mimics $m=1,n=0$. - Quadrupole distribution:** Four point charges at $\phi=0,\pi$ and located at inner and outer sides (so that outer-0 and inner-π are positive, inner-0 and outer-π are negative, for instance). That would mimic something like $m=1,n=1$ or $m=1,n=1$ rotated pattern.

One can see that systematically, for each needed (m,n) one could predefine a cluster. But there's a combinatorial explosion. Instead, one might include just the *lowest few modes* explicitly (like one basis element for the first asymmetric mode etc.). The higher modes might anyway be easier handled by direct point charges or arcs.

Approximation power: If successfully constructed, a toroidal mode element would match the torus boundary condition for that harmonic exactly (if solved for proper strength). Thus a combination of such elements could theoretically exactly represent the solution (if one included enough modes). In practice, including up to, say, $n=3, m=3$ modes might capture most of the needed shape, leaving smaller details to point charges to fix.

Conditioning: Using such composite elements reduces the chance of ill-conditioning because each is designed to be somewhat orthogonal in effect (different (m,n) patterns ideally don't overlap strongly). But if not carefully defined, some might be nearly redundant. It's safer than random points, though, since their fields are qualitatively distinct (one oscillates in ϕ, another in σ, etc., presumably low inner products on the surface).

3.2.4 Toroidal Image Chains

Description: This concept refers to *chains of point charges arranged along a curve following the torus topology*. The most natural such curve is the *circular centerline* of the torus (the ring of radius R that runs through the donut hole). Placing multiple point charges along this circle (perhaps uniformly spaced in angle) yields a ring of point charges – which approximates a continuous ring but with more flexibility if weights are not identical. An "image chain" could mean: - A sequence of charges along the centerline, possibly alternating in sign and decaying in magnitude as they wind around multiple times.

This idea resonates with how one solves for parallel plates by an infinite sequence of mirror charges along a line: for a torus, perhaps an infinite periodic sequence of charges around the ring might be needed. Indeed, for a point charge outside the torus, one could imagine the field induces a response that is then like an image charge inside, which in turn induces a further response maybe around the ring, and so on – a multiple reflection analogy but on a closed loop, which conceptually gives a *periodic chain of decaying images around the ring*.

If one could sum an infinite series of charges evenly spaced around the circle, that sum might converge to something like a continuous ring distribution (similar to how a periodic mirror sequence converges to a line charge in the plane case). Thus a finite truncation of that (say 8 or 12 charges around) could approximate the needed distribution around the hole.

Another possible chain: point charges along a *meridian* (through the tube, i.e., along σ direction). But going around the tube returns to start, so it would be an infinite periodic sequence in that sense too. But the main chain concept is around the major circle.

Alternate curves: One might also place a chain along the *axis of symmetry* inside the hole. For example, for a sphere, image charges lie on the radial line inside; for a torus, maybe image charges might lie along the circular axis. But the torus axis is a circle – images might effectively lie on that same circle (similar to the centerline approach described).

Relationship to toroidal harmonics: A uniform chain of N charges equally spaced around the centerline is essentially a discrete Fourier representation of a ring distribution: it will only produce m modes up to $m=N/2$. If one allowed alternating signs (like every other charge opposite), that introduces high-m content (specifically $m=N/2$ if N is even yields a pattern of sign flips). A decaying sequence of charges could possibly simulate a continuous distribution with some kernel (like a geometric series might approximate a delta function in Fourier space if convolved, etc.).

One might consider a chain as approximating the *Green's function* on a torus in a discrete way. The Green's function (potential at a point due to a unit charge while conductor is grounded) can be expanded in toroidal harmonics, or potentially as an infinite sum of ring images. Maybe an infinite series of point charges around and around could converge to that. If so, truncating it yields an approximation to the Green's function. But here we want just the particular solution, not the entire Green's function for any point.

Approach in practice: One could start placing charges along the centerline to see how it improves the fit. For example, one at the point closest to the external charge (like directly under it inside the hole) will help cancel the field there. Then perhaps another opposite around the ring might help adjust other side, etc. This starts to resemble a *discrete Fourier series approach in real space*.

Another variant: *mirror charges for multiply connected domain.* The domain outside a torus is like $ \mathbb{R}^3$ minus a handle. Sometimes for a handle (like a ring), one can treat it by adding an image for each winding around the handle. There's a concept in topology: a loop around the hole corresponds to a nontrivial cycle, and one might need to add an image associated with each winding number around that cycle to satisfy boundary conditions. That suggests an infinite set (since one can wind around arbitrarily many times). However, the effect of images winding multiple times decays (like each successive image is further away along the path). This is analogous to the method of images in a *circular cylinder* – for a point outside a conducting cylinder, one uses an infinite series of images along an exponential spiral (though in 2D that's easier).

For torus, we might guess an image of the external charge could be placed inside the torus at roughly the inversion through the torus (if one treats the torus like a thin ring, an image might appear diametrically opposite in some sense). Then further images might appear after multiple wraps. This is speculative but it motivates including at least a *handful of point charges spaced around the torus*.

Approximation power: A sufficiently dense chain around the centerline can approximate any distribution on that line (by interpolation). This could approximate well the lower-τ part of the potential (the ring coordinate). However, a chain on the centerline by itself cannot vary in the σ (poloidal) direction, as all points are equidistant from the surface in the minor direction. So it's mainly for capturing around-the-ring variations.

If an external charge is located off-axis, the induced charge on torus will certainly have a variation around the ring (more induced charge on the near side than far side). A chain can capture that distribution around,

whereas a single ring might only give an average. So a chain is like adding Fourier modes in ϕ direction, whereas earlier arcs were adding Fourier in σ.

Therefore, these families (rings vs chain vs arcs) complement each other: rings capture mean, chain captures toroidal asymmetry, arcs capture poloidal concentration.

Combining families hierarchically: Indeed, one can envision a *hierarchical basis*: global uniform ring (to get the main monopole), then maybe a few chain charges (to adjust ϕ asymmetry), then a few arcs (to adjust local σ concentration). This way each level corrects a different aspect with few elements.

Conditioning: A chain with many point charges in it (especially if equally spaced) can be ill-conditioned because their fields may overlap strongly. But if we treat them as a group with a smoothness prior (like requiring weights to decay or alternate in some structured way), we could stabilize it. Also, if the number is not too high (maybe < 10), we may be okay.

In any event, these basis families – rings, arcs, composite modes, and chains – each address different "Fourier components" of the induced charge: - Rings: $m=0, n=0$ mostly (global uniform and slight n via double rings). - Arcs: high n (local poloidal). - Composite modes: targeted mid-range m,n. - Chains: high m (toroidal variation, multiple periods around).

Using them **in combination** could yield a sparse yet accurate representation. For example, an optimal combo for a point above the torus might be: one uniform ring, one dipole ring (to shift charge top-bottom), two chain charges at front/back (to adjust ϕ asym), and maybe one local arc at the top inside to fine-tune the boundary condition directly under the source. That's perhaps ~5 basis elements – far fewer than, say, 120 toroidal harmonic terms – achieving a similar boundary accuracy.

Conditioning and numerical pathologies: One must be cautious of near-linear dependence: e.g., a uniform ring and a chain of 8 points with equal weights might partially duplicate each other (since summing the chain's weights gives effectively a uniform ring component). So if both are allowed free, the solver might not know how to distinguish them, leading to large opposite coefficients. To avoid this, one can impose constraints or group L2 penalties (discussed in Section 5) to encourage a unique decomposition (like encourage chain weights to sum zero if a uniform ring is already separately represented, etc.).

In summary, toroid-specific bases offer a promising way to drastically reduce model complexity. The theoretical caveat is that they are not guaranteed to systematically converge to the true solution as you add more – unlike toroidal harmonics which systematically converge. But in practice, one hopes a combination of a few global and local basis can approximate the true solution to within engineering tolerance. A successful design will respect the *topological constraints* (discussed next) and be flexible enough for various source positions.

4. Topological and Geometric Considerations

Toroidal geometries bring *topological complexity* absent in simply connected shapes like spheres. This affects uniqueness of solutions, choice of gauges, and the types of basis needed to capture global features.

4.1 Homology, Cohomology, and Harmonic Forms on a Torus

A torus (whether the surface itself or the region outside it) has non-trivial first homology H_1. The surface of a torus has two independent loops (meridian and poloidal), and the exterior domain $\Omega = \mathbb{R}^3\setminus$torus has one nontrivial loop that goes through the hole and out (like a loop linking the torus once cannot be shrunk to a point in Ω). In differential form language, there exist **harmonic 1-forms** on the domain that correspond to these loops. For instance, on the torus surface one can have a tangential electric field pattern that winds once poloidally or toroidally and is divergence-free and curl-free on the surface (harmonic).

In the static electrostatics context (scalar potential), what does this mean? If the domain is multiply connected, the gradient of a potential field can still be single-valued only if certain loop integrals of $ \mathbf{E}$ vanish. Specifically, for a potential ϕ, $\oint_{\gamma} \nabla \phi \cdot d\mathbf{l} = 0$ for any closed loop γ that stays in the domain (because ϕ is single-valued). In a multiply connected domain, this condition is not automatically ensured by local conditions; it becomes an extra *global constraint*. For the torus exterior, consider a loop that goes once through the hole (linking the torus). If one had a situation where ϕ changed by some Δ after one loop, that would be a multi-valued potential (not allowed for electrostatic equilibrium). Therefore, the physical solution must satisfy:

- **No net circulation of E around the torus:** $\oint_{\text{loop around hole}} \mathbf{E}\cdot d\mathbf{l} = 0$. Equivalently $\oint \nabla \phi \cdot d\mathbf{l}=0$ since $\mathbf{E}=-\nabla \phi$. This is automatically true if ϕ is globally single-valued. But in a numerical approximation, if basis functions are not all gradients of globally defined potentials, one could inadvertently violate it.

Why worry? Because some basis constructions might effectively introduce a *non-gradient field component*. For example, on a torus surface, the space of harmonic functions is not trivial: one can add a solution that is harmonic in Ω but not representable as single-valued potential due to wrapping around. However, in pure electrostatics with one conductor and charges, the solution is determined uniquely by Dirichlet conditions and decay at infinity – those ensure no spurious harmonic component. The risk arises if our *approximation space* is too large, it might contain functions that do not satisfy the single-valuedness condition, leading to spurious solutions (e.g. one could imagine a solver picking up a solution where potential jumps by a constant after one loop, which physically can't happen but mathematically could if the basis allowed it).

Homology on the torus surface vs in the domain: The torus surface itself has two fundamental cycles. The exterior domain has one (the toroidal loop). The interior (hollow) of the torus also has one (a loop going through the hole inside). For Laplace's equation in the exterior, the Dirichlet BC kills any degree of freedom corresponding to a harmonic function that winds around (since any such function would conflict with $ \phi=0$ on the surface unless zero). And the condition at infinity kills any harmonic that winds around infinity (like an angle-dependent potential at infinity would diverge or conflict with 0 at infinity). So physically, the solution in Ω is unique and doesn't actually have a free topological mode. But when designing basis, we must ensure the basis can represent the true solution *and* does not accidentally include non-physical modes.

Hodge decomposition viewpoint: On the torus surface (2D manifold), a scalar potential might not be single-valued if you go around a cycle – akin to multi-valued potential differences (like think of a magnetic

flux through the torus causing a multi-valued phase for an electrostatic problem if allowed). However, electrostatics forbids that since the field is conservative globally (no time-varying flux to break single-valuedness). So one insight: any solution must have zero *circulation of E around both the toroidal and poloidal directions*. The poloidal direction loop lies in simply connected space outside (one can shrink a loop that goes around the tube cross-section to a point outside the torus), so that automatically holds by local potential existence. The toroidal loop encloses the conductor – not contractible in Ω – so it's the non-trivial one.

Thus one topological constraint: **The net induced charge on the torus = $-q$ (the external charge)**, by Gauss's law for a surface enclosing the torus. This is automatically satisfied if solving correctly, but basis solutions should allow that. More subtle: **The flux of $\mathbf{E}$ through any surface spanning the toroidal loop must equal the external charge inside that loop.** If a loop goes once through the hole, consider it as boundary of a surface that cuts through the hole. That surface plus the conductor encloses either zero or some charges. If it encloses the external charge, by Gauss $\oint \mathbf{E}\cdot d\mathbf{A} = q/\epsilon_0$. If it doesn't (depending how loop is drawn), it's 0. For a point charge outside, a loop threading the hole does not enclose the point, so net flux through a spanning surface is zero. That means *field lines are not forced to carry flux through the hole* – indeed if point is outside, field lines mostly go around outside. If a basis solution introduced a fictitious harmonic that has a net flux through the hole, that would violate Gauss's law unless balanced by an image charge actually inside the hole.

Topologically nontrivial "fluxes": One can think of a *harmonic vector field that loops around the torus* – e.g., imagine a field that goes circularly around the hole like a vortex (that would be curl-free in the exterior and divergence-free, thus harmonic, but obviously cannot be a gradient of a single-valued scalar, it's like $\nabla \phi$ with ϕ multi-valued by $\pm$ some constant after loop). In electrostatics, such a field is unphysical because it would imply a potential jump. But a basis set that includes a loop of charges might accidentally create a small component of this pattern if not constrained.

For example, a ring of charge yields a conservative field (gradient of a potential that is axisymmetric). A *closed loop of dipoles* around the torus could generate a non-conservative static field (like a continuous dipole sheet on a closed loop has a discontinuous potential). But since we are using actual charges (monopoles), any linear combo of them is conservative except if we allow line charges or dipoles that are not equivalent to monopoles. We have allowed rings (which are continuous distributions of monopoles – still conservative field overall). Poloidal arcs – still monopoles. Composite patterns – just sets of monopoles. So all basis elements individually produce $\mathbf{E} = -\nabla \phi$ globally (except at their singularities). Linear combination is also conservative in the domain (since superposition of potentials). So *maybe we cannot inadvertently create a non-zero loop integral unless we somehow introduced a current-like basis.* That's a relief: using charges only, the field remains deriveable from a single global potential (the sum of Coulomb potentials of all charges). That potential is automatically single-valued if we treat it properly (it might be multi-valued if we go through a branch cut containing a singularity, but we exclude inside of conductor which contains those singularities). So likely our basis of charges cannot break the topological condition – it's inherently satisfied as long as all sources are actual Coulombic charges outside the conductor interior (which they are by construction in our basis: we place them in Ω or inside conductor? Actually, we might place some inside the conductor volume – which is allowed because the conductor interior is not part of solution domain, it's just a mathematical region where singular charges can reside; if an image charge is inside metal, that's fine since that region is not solved for potential except constant).

If any basis charge lies *on a non-contractible loop inside the conductor*, does that cause multi-valued potential outside? No, because outside region one just sees its field (the potential from a point charge is single-valued except at the location of the charge which is inside conductor and not in domain, so we cut that out, okay). So I think physically, summing point charges yields a legitimate potential in Ω (unique up to constant), thus automatically satisfying all loop integrals zero. So using only monopole sources ensures no spurious topological modes appear. That's good.

However, if one used *dipole ring distributions or others with inherent multi-valued potential*, one must be careful. For instance, a dipole sheet on the torus surface can produce a field with loop integrals (like think of a magnetic monopole in the hole yields an electrostatic analog of a multi-valued potential maybe). But since we stick to actual Coulomb sources, we're fine.

Harmonic forms on the torus surface: The surface of the torus can support a *constant potential difference* between loops – meaning if one solves Laplace's equation *on the surface* (intrinsic Laplace-Beltrami), one can have a linear function of the poloidal angle as a harmonic scalar (which corresponds to a discontinuous 3D potential if extended). In BEM, this appears as the *nullspace of the single-layer operator for Neumann problems* (constant σ yields constant potential outside, etc.). For Dirichlet, we don't have that nullspace.

One topologically interesting scenario: If you had two separate conductors forming a torus (like a split torus into two halves insulated from each other), then one could maintain a voltage difference around the loop without current, giving an electrostatic field trapped in the torus region (like coax). But we have a single conductor.

Conclusion for 4.1: The main takeaways: - The torus introduces a loop which in principle could carry a non-zero circulation, but physically ϕ is single-valued so no such field exists in the solution. - Our basis of monopoles respects this automatically. - Still, one should ensure *net induced charges and flux conditions* are met by the basis: e.g. including a uniform ring basis is crucial to capture the fact that total induced charge equals $-q$ (if one had only dipole-like elements, solver might struggle to represent the monopole part). - Also ensure if we do multi-objective optimization (some weight on flux), the basis can adjust flux correctly.

4.2 Green's Functions and Topology

The topology of the domain influences the *structure of the Green's function for Laplace*. In a simply connected domain, Green's function is unique up to constant. In a multiply connected domain, the Green's function might require adding harmonic forms to satisfy all conditions. For the exterior of a torus, the *free-space Green's function* can be modified by adding a solution of Laplace's equation that ensures the conductor boundary sees zero potential. That solution might not be unique if topological constraints are considered. For example, one could add a harmonic function that is nonzero around the loop but zero on the boundary and decays at infinity – does such exist? Perhaps not without violating something. Actually, consider a harmonic function $H(\mathbf{r})$ such that $H|_{\Gamma}=0$ and at infinity $H=0$ but around the hole $\oint \nabla H\cdot d\ell \neq 0$. Is that possible? It would imply a potential that rises going once around the hole and falls elsewhere, but with zero on conductor. I suspect no nontrivial H exists under those Dirichlet conditions either (the only harmonic with those boundaries is trivial, by uniqueness of Dirichlet problem on that domain). So maybe no addition.

In Neumann problems, though, the torus would cause a constant potential mode (like one can add a constant potential or a loop potential difference). But Dirichlet fixed at 0 removes that.

So the presence of the hole means any attempt to find an image representation must ensure that the field lines do not erroneously produce flux linking the hole if none is supposed to. If an approximate model had an imbalance causing flux through the hole, it would manifest as a small violation of BC or other error that the solver would correct by adjusting weights.

Global vs local modes: We often refer to “global modes” as those spanning around the whole torus (like uniform or one-cycle variations) vs local lumps (like a small patch of charge). The global modes are influenced by topology – e.g., a uniform ring is a global mode connected to the toroidal cycle. A local patch doesn’t carry topological significance by itself; but a combination of patches might inadvertently create something akin to a loop distribution. However, since each patch is actual charges, any combination is still physical field.

Non-existence of simple images (reiterated topologically): The fact that a torus is not simply connected is another intuitive reason no finite set of images can satisfy all conditions: any candidate image must produce the correct field and also no spurious loop flux. If one tries just one image charge, it’s impossible because you can’t satisfy both the local BC and the global flux constraint with one parameter. The infinite series or distribution basically builds up the correct cancellation around entire loop.

4.3 Implications for Basis Design

Respecting cycles and fluxes: Our basis should be flexible enough to represent the *physically required flux patterns* and yet not produce unphysical ones. In practice: - We must allow a *net charge* basis (monopole component) on the torus to account for total induced charge. This we have via the uniform ring basis. - We must allow a *dipole moment around the torus* if needed (though for one conductor at 0 potential, probably net dipole around torus is zero, but if external field, maybe induced dipole? Actually a charge outside will induce a dipole moment on the torus pointing roughly towards the charge – meaning more positive induced on far side, negative on near side. That is a dipole across the torus cross-section or along axis? Actually it's a monopole overall, but distribution is shifted, giving the field of a dipole outside). So we need basis elements that can shift charge distribution (like ring dipole or chain). - Ensure that if point charges are placed inside the conductor (which they can be as images), we handle that their fields inside conductor are irrelevant (just ensure solver only enforces BC on surface, which we do). - If using any *non-monopolar basis* (like a dipole layer or current loop), be cautious. Our plan avoids those.

Cycle integrals of E around loops: Could we incorporate constraints in the solver like "flux through this surface must be zero"? Possibly as a linear constraint on weights. But as reasoned, with all sources as monopoles, that constraint is automatically satisfied except maybe up to machine error. If one did find any drift (due to finite error), one could explicitly add a Lagrange multiplier to enforce it. But likely not needed.

No spurious harmonic components: Essentially, do not include basis that introduces a “floating potential” or a multi-valued potential. In BEM, one often has to fix the reference potential for a conductor. Here conductor is at 0 by BC, so fine. If one had only Neumann BC or periodic domains, one must fix integration constants to avoid drift. Our scenario fixed Dirichlet avoids that.

Topologically constrained basis selection: If one were to automatically generate basis elements (like in an AI approach), one might want to *pre-group them by homology classes*. E.g., separate basis sets that carry a certain loop charge vs those that don't. But since all are monopoles, none carries a loop of field by itself.

However, a *uniform ring of charge on the centerline* does produce field lines that loop through the hole (because that ring effectively acts like charge distributed in a circle – outside, the field lines emanating from it partly go through the hole region). But that's fine physically, it just means it produces some flux linking the torus (which must be canceled by other parts if not physical).

To clarify: If the external charge is outside, actual solution has no net flux through a disk spanning the hole (no charge is enclosed by that disk if it doesn't encircle the external charge). If we place a ring charge inside, that ring by itself *does* produce flux through that disk (because Gauss law for that ring would show net flux = its charge). So if our basis includes a ring charge, and solver uses it, it might inadvertently introduce some "false" flux through the hole (like as if a charge were inside). But to satisfy BC at the surface, it might need to do that partly, and then other basis elements like symmetric ones might cancel some flux. This is complicated to analyze. Possibly the solver's error will show up if flux is wrong, by inability to satisfy BC precisely with that combination – so it will adjust weights (maybe making net ring charge exactly equal to real needed net flux? Actually net ring charge should equal induced charge, which is fixed by external q). If external q is outside loop, induced net on torus is $-q$. That $-q$ is indeed effectively a ring of charge on the surface (some distribution). So a ring basis representing that is correct flux wise: a ring of charge $-q$ yields $-q$ flux through any surface that encloses it (like one spanning hole), but the actual physics says any surface spanning hole and outside torus encloses $q_{ext}+q_{ind}=q + (-q)=0$ if it passes outside external charge. Wait, need to be careful: - Consider a surface S that is like a flat disk spanning the torus hole and large enough to include entire cross-section, but not encircle external charge. This S plus the torus surface forms a closed surface that encloses the induced charge $-q$ (since external charge is outside that closed surface). So flux out of closed surface = $-q/\epsilon_0$. That flux splits between flux through S and flux through torus surface. But torus surface has E=0 on it (conductor), so all flux must go through S. That implies indeed $\oint_S \mathbf{E}\cdot dA = -q/\epsilon_0$. But earlier we said a loop around hole sees no net flux if not enclosing external charge. There's a subtlety: the surface we choose, if it touches the conductor (closing it), is not entirely in domain (part is conductor surface), but we use Gauss law on combined surface. It shows field lines from induced charges go through the hole region. Actually yes: the induced charge $-q$ will push field lines through the hole outwards to infinity such that they go around external charge to meet with that q lines, so net flux on some surfaces is nonzero. The external charge field lines partly thread the hole? Possibly, field lines from external charge might go around either side of torus, but the induced charge then ensures none go into conductor. Still, some of induced field can go through hole.

This is tricky physically, but likely net flux through a disk spanning hole = nonzero (equal to induced charge within torus boundaries). So a uniform ring basis (representing induced $-q$) giving flux $-q$ through hole is correct physically. So not a spurious flux, it's actual.

Thus if external charge is outside the hole, flux through hole region is $-q$ from induced (point's own flux circumventing outside)...

If external charge were inside the hole (like in cavity), then obviously flux through that hole is q and induced must cancel it as needed.

Alright, the key design implication: - Make sure to include a basis element for each needed topological charge distribution (monopole on torus which is ring $-q$, possibly if we had more complex scenarios like multiple linked tori one would do one per loop). - Avoid basis that is redundant or contradictory in those aspects.

Our proposed basis does include uniform ring (monopole). We could also include perhaps a basis that corresponds to "circulation" (like to allow spurious if needed for intermediate calculation but penalize it). But since we think it's not needed or possible with pure charges, skip.

In summary, *topology informs basis selection* by highlighting the importance of including globally supported basis (like rings) and ensuring completeness in representing any allowed harmonic content. We should be careful that removing something like uniform ring would mean you cannot get the correct net induced charge.

5. Optimization and Model-Selection Frameworks for Mixed Basis Types

We have multiple basis types (point charges, rings, etc.), and we aim to determine optimal sparse combinations. This naturally leads to a *regularized optimization problem*, often with sparsity (L1) and group structure. We address how to assign and adapt regularization per basis type, how to handle multi-objective error criteria, and how to possibly solve in stages or adapt hyperparameters.

5.1 Per-Type Regularization and Group Structure

When using a mixed basis library, it's common to impose structured sparsity: we want the solution to use only a few basis elements *and* ideally a few types among them. For example, maybe the best model uses 1 ring and 3 points, and not all 10 rings and 50 points the library had.

Group LASSO concept: In group Lasso, one groups coefficients and penalizes the *group norm* so that either the whole group is zero or active. For instance, treat all coefficients of point charges as one group (or each point as its own group, which reduces to standard Lasso), and all coefficients belonging to ring basis as another group, etc. More flexibly, one could group by *each ring element or each type cluster*. If an element itself has multiple parameters (like ring with sine and cosine density components), treat those as a group.

Applying group Lasso would mean we encourage either using an entire ring basis element (with whatever weight optimal) or not using it at all. But we also may want within each group to allow multiple to be on (like we might use several point charges but still penalize the total count).

Mixed norms (ℓ2 per type, ℓ1 across types): A common scheme is:

$$\mathcal{R}(w) = \sum_{t \in \text{types}} \lambda_t \sqrt{\sum_{i \in G_t} w_i^2}$$

This is ℓ_1 on the group norms (ℓ_2 inside each group). It tends to either zero-out all w_i in a group or keep them together. This is appropriate if turning on one basis element of that type likely means

we need to allow several of that type (for example, turning on “point charges near surface” might involve several point charges group – but group can also be a single element, depends on grouping logic).

Alternatively, one can group each *type family* (like all points as one group, all rings as another) – that would either use none or many of that type. That might be too coarse, as it's likely we want to use points *and* rings together, not exclusively. So probably group at the level of individual basis elements or small clusters.

Multi-task Lasso analogy: If we consider each basis type as a "task" with weights, one might share information to tune λ_t. But more relevant is structured reg.

Setting type-specific λ_t physically: The regularization parameter λ_t (the weight for penalizing group t) controls how costly it is to include that type. Ideally, this is set according to *prior belief or model complexity*. E.g. if ring basis is known to capture a lot with one element, maybe we assign it a relatively low cost (so the solver tends to pick a ring if it explains significant error). If point charges can always individually tune things but are plenty, maybe penalize them more to avoid too many. Or if one type tends to cause ill-conditioning if too many (like high-degree polynomials cause near-collinearity), we might penalize them to avoid using too many.

One could choose λ_t based on: - **Dimensional analysis / normalization:** Ensure that a unit weight in any basis yields similar magnitude effect on error. For instance, scale each basis element’s representation so that it produces a comparable potential magnitude (some norm) for unit coefficient. Then use one λ for all if no preference. If not scaled, adjust λ accordingly. - **Physical intuition:** e.g., a point source near surface can produce very localized high gradient – might overshoot at some collocation points making error sensitive. We might penalize huge point source usage to avoid spurious high-frequency oscillations (like a smoothing prior). - **Complexity of element:** If a ring effectively summarizes an infinite distribution (like a lot of points in a ring), maybe using one ring is akin to using many points, so we might penalize ring less per element because it's more “efficient.” Or conversely, maybe penalize more if we consider it an exotic shape. - **Empirical tuning:** Possibly cross-validate by solving known cases and adjusting λ_t to get best generalization or minimal model complexity.

Structured sparsity example: Suppose we have groups: - $G_{\text{point}} = $ all point charges, - $G_{\text{ring}} = $ all ring elements, - G_{chain}, - G_{arc} etc. Applying group Lasso might select one group entirely. But likely we want a combination – so maybe group by each individual candidate (like each candidate ring is a group of one weight if ring’s weight is single parameter, so that’s just normal Lasso across rings with possible different λ each).

Alternatively, consider group *spatially*: e.g., treat a cluster of point charges placed near the same region as a group (like a patch group), because either you'll use that patch of points or not. This is like saying: either use local refinements in that region or not at all. That could align with e.g. arcs: an arc basis could be replaced by a cluster of points in same region, so one might encourage either that cluster collectively or leave area unrefined if not needed.

Mixed group and individual penalties: There's something called *sparse group Lasso* combining L1 and group L1. We might do:

$$\alpha \sum_i |w_i| + (1-\alpha) \sum_g \lambda_g \|w_g\|_2,$$

so that we enforce both within-group sparsity and group sparsity. This can yield selecting a few groups and possibly not using all within them. In our case, maybe not needed if groups are small.

Non-local vs local basis regularization: Likely we want to *prefer non-local (global) basis first* to explain as much as possible (since they give broad coverage cheaply), and only then use local points for residuals. That means setting λ_{global} (for rings, composite modes) *lower* relative to $ \lambda_{\text{point}}$. Then the optimizer will first try to use global basis because it's less penalized per weight to reduce error, only resorting to point charges if needed for fine adjustments. This aligns with physical expectation that multipole-like basis should handle large-scale variation, and point charges handle fine details.

Adapting λ_t: It might not be trivial to set a priori. Possibly an iterative approach: start with high penalty on all, reduce gradually on some after checking error contributions. We'll talk more in 5.4.

Example values: If normalized such that each single basis element if given weight could reduce error by similar magnitude, one might start all $\lambda=1$ but then e.g. divide λ for ring by 2 to encourage it (or multiply point's by 2 to discourage). The exact ratio could be tuned by experiments or physically: maybe base it on how many degrees of freedom each type has? e.g., we have 1 ring vs potentially 10 points, so to avoid picking 10 points, penalize points 10x more in sum. There is some heuristic like "so that cost of 10 point charges ~ cost of 1 ring".

In group Lasso, if each point is group of itself with weight λ_p, and ring is alone with weight $ \lambda_r$, we could set λ_r such that using 1 ring is as costly as using k points, with k being maybe number of point charges needed to emulate one ring's effect. If one ring can replace maybe ~8 point charges (for a rough effect around the ring), then λ_r should be ~8 * λ_p. Actually, smaller λ means easier to include; so if we want to prefer a ring to using 8 points, we set $ \lambda_r < 8\lambda_p$ presumably.

Alternatively, incorporate that into scaling of atoms rather than lambdas.

Local basis physically grounded λ: Perhaps penalize local basis not just because of count but because they risk overfitting boundary collocation (point charges could chase a small error by huge amplitude oscillations at one collocation point, which is undesirable physically because field would blow up near that charge). So a larger regularization on them acts like a smoothing, preventing spiky solutions.

In summary: Use group Lasso or similar to manage selection of types. Calibrate λ's by effect size or numerical stability considerations. Possibly treat non-local (ring, chain) with lower λ (so easier to include), and local (point) with higher λ (so only included if absolutely necessary). This way, solution will try to explain as much as possible with a few global sources, then add minimal local ones.

5.2 Boundary-Weighted and Multi-Objective Losses

Our primary objective is to satisfy Dirichlet BC on the torus surface. Additionally, we might want interior field accuracy near sources, etc. A composite loss is given:

$$J(w) = \alpha \|\Phi(w)|_{\Gamma} - 0\|_2^2 + (1-\alpha)\|\Phi(w)|_{\Omega_{\text{sample}}} - \Phi_{\text{true}}\|_2^2,$$

where $\Phi(w)$ is the potential from our basis with weights w, and we measure error on collocation points on boundary Γ and maybe in some interior points or along some lines in Ω. We might emphasize α close to 1 (since BC must be very accurate), but not exactly 1 in case we want to trade a tiny BC error for much better interior fit (for instance if an approximate solution slightly violates BC but yields better field shape everywhere else, though physically BC is absolute requirement so likely keep α=1 or very high).

Alternatively, one can put infinite weight (i.e. treat BC as hard constraints, then minimize interior error subject to BC exactly satisfied – a constrained optimization or use Lagrange multipliers for BC). But in practice, we allow small BC error in least squares because an approximate basis might not achieve zero exactly.

Choosing α: If α=1, purely fit boundary, ignoring interior – that could lead to a solution that overfits boundary noise at cost of interior accuracy (like using many spikes to kill boundary error but interior field oscillates). If α is slightly less, we allow slight boundary error if it drastically reduces interior error. Because physically, the potential on surface must be exactly zero; so any compromise means we accept a small violation of physics to get more accurate overall potential maybe with fewer basis. Perhaps in design stage, one could allow that to get a simpler model then enforce BC by a secondary correction.

One might adapt α dynamically: start with α smaller to get broad fit, then increase to push BC to zero.

Stability/conditioning when α→1: If we weigh boundary too heavily, small changes in weights cause large changes in loss (because basis might be nearly redundant on boundary, making Hessian ill-conditioned). If one basis can almost satisfy BC and another nearly so, the solver might make large opposite weights to cancel to even smaller error – causing interior blow up. This is reminiscent of *ill-conditioning when penalizing one region extremely*: you get difference of large numbers (two basis nearly satisfy, their difference used to satisfy more exactly, interior goes wild).

To avoid that, one could: - Not push α to exactly 1 but keep interior part nonzero to regularize solution. - Or add a small Tikhonov (ℓ2 norm of weights) to penalize large weights, so it won't do large cancellations.

Alternative: Multi-objective approach: Instead of a weighted sum, one can set up a Pareto front or a multi-objective optimization to meet a target on BC error then optimize interior. e.g., minimize interior error given $|\text{BC error}|_\infty < \epsilon$. This might yield a clearer physically – ensure BC satisfied to tolerance, then among those solutions pick smallest interior error or simplest model. This can be done by constraints or a two-stage approach: first solve boundary least-squares exactly (with many basis if needed), then reduce basis with constraint boundary stays below threshold.

Adapting α: Possibly *start with α lower* (so you find a solution that is somewhat balanced), then gradually increase α, each time using previous solution as warm-start. This is like a continuation method forcing BC satisfaction more strictly over iterations.

Interior error measure: Could weight E-field error if that’s important, or potential error at certain key points (like along symmetry axis, or measurement locations). One might also include something like penalty for global flux mismatch (though as argued, it's implicitly handled).

Physical weighting: Could set α such that the boundary part and interior part are normalized to comparable scale. For instance, if we use collocation, maybe many more points interior than boundary, one must weight accordingly or results bias.

Stability: We want to avoid solutions that exactly cancel boundary potential at expense of huge coefficients. This often happens if basis is not rich enough – then it tries anyway by large oscillations. A remedy: add a small penalty on $|w|_2^2$ to discourage huge weights (Tikhonov). That effectively prioritizes a least-squares solution with minimal norm which will prefer smaller violation on BC if eliminating it requires enormous weights.

Summarily: - Likely choose α extremely high, since BC is priority. Possibly treat interior as slight regularization (like 0.99 vs 0.01). - Use either continuous weight or a constraint approach to ensure BC error within tolerance. - Guaranteeing stability: incorporate a norm penalty (like ridge) or limit weight magnitudes to avoid blow up.

5.3 Two-Stage / Hierarchical Optimization

The idea: solve for global (non-local) basis first, then local, then refine. This aligns with the concept of first capturing low-frequency content then high-frequency. It can also reduce search space sequentially.

Stage 1: global basis selection. We restrict basis to rings, composite modes, chain maybe – those that provide broad coverage. Solve a sparse regression (L1 or forward selection) to fit BC. This gives weights for those. It may not perfectly satisfy BC but captures main trend.

Stage 2: local basis for residuals. Now the residual on boundary (difference between current potential and zero) is localized in certain areas (perhaps near where external charge influences strongly). We then allow point charges or arcs near those areas to fit the residual. This can be done by a second sparse solve where the design matrix is the response of local basis, and target is the stage1 residual on boundary (with possibly some interior considerations too).

Stage 3: joint refine. Once global and local picks are identified, we do a combined optimization adjusting all their weights (maybe with LBFGS or least squares if no further reg needed) to minimize total error. Possibly even allow adjusting positions slightly (nonlinear optimization) for fine tuning if basis param locations can move.

Multi-level or multi-resolution analogies: This is akin to a multigrid or wavelet approach: coarse modes first, then fine corrections. It tends to avoid the fine basis being used to represent coarse patterns (which is inefficient and leads to many small charges summing up to mimic a big ring effect, which is undesirable for sparsity).

Sequential convex programming: The two-stage can be seen as sequential: each stage solves a convex problem (L1 minimization). If we want structure (like picking one ring if needed), one might do something like: - Solve L1 with heavy penalty on local so you get mostly global. - Then fix those or keep them with smaller penalty, and solve again with local allowed more. This is essentially adjusting λ_t after first stage, which is stage-wise.

First global then local example: Suppose initial solution with just rings yields some error 5%. Then adding points yields reduce to 1%. That final is good. If one had done one big lasso, maybe it would mix them arbitrarily depending on λ weighting, but sequential ensures global given preference.

Potential pitfalls: If global basis cannot by itself reduce error below threshold, the stage1 might still pick some local because otherwise residual too high. Could avoid by capping iterations or forcing only global allowed in stage1 fully (like set local basis off). Then after stage1, we indeed have some BC error left; stage2 then deals with it.

There is also a concept of *forward selection (Matching Pursuit)* where at each step you add the basis that most reduces error. A variant could be: first consider only global basis in that greedy selection until adding any more global yields marginal improvement smaller than adding a local would – then start adding local. That's a dynamic approach that might be effective as well.

Hierarchical basis combination: We can plan combination like: Use rings to fit 90% of BC error (especially average potential on major segments), then use chain to adjust asymmetry if needed, then arcs to kill local peaks.

Refinement and LBFGS: Once chosen basis set, a nonlinear refine might involve adjusting location of basis. e.g., moving a point charge slightly can fine tune. That becomes a nonlinear least-squares problem which LBFGS or Levenberg-Marquardt can solve. Provided initial guess from linear solve is close, this can converge and give a small improvement. For instance, one might allow the ring radius or the point radial distance to adjust for optimal fit.

Theoretical/practical guidance: - This approach reduces the condition number by splitting high and low frequency content. - It also can lead to simpler final models, because global things explained by global basis rather than using many local to do same. - In practice, ensure continuity: after stage1, include its solution's effect when computing residual for stage2.

Analogies in literature: Multi-level sparse coding, where you first pick dictionary atoms for large-scale structure, then refine residual with more localized atoms. This is known to improve interpretability and often sparsity.

5.4 Adaptive Type-Level Regularization Based on Error Attribution

This means adjusting hyperparameters (λ_t) on the fly depending on how useful each type is proving in reducing error.

Existing work: The concept is similar to *Automatic Relevance Determination (ARD)* in Bayesian regression, where each group's prior variance is updated based on data evidence. ARD often yields that groups not contributing get shrunk more, etc. One could implement something like: treat weight prior variances for each type as hyperparams and use evidence maximization to update them.

Alternatively, *multiple weighted LASSO runs:* e.g., solve once, see that some basis types get used heavily, some not at all. If some are not used but error is still high in patterns they might cover, maybe their $ \lambda$ was too high – reduce it to encourage their use. If others are used but minimally and not needed, increase their λ to try a solution without them.

Error attribution: Suppose after an initial solution, we examine the boundary error pattern. We might measure correlation of that error with each basis element's influence pattern. If the error has high correlation with a certain basis's pattern (meaning that basis could reduce it significantly), that suggests we should reduce λ for that type to allow it to enter solution. Conversely, if a type's basis patterns are orthogonal to error (not needed), maybe we keep or increase its penalty to avoid spurious picks.

This is similar to how boosting algorithms pick new basis functions that align with current residual. So one could systematically lower penalty for the basis most aligned with residual (making it easier to get picked in next solve), akin to adaptive Lasso weights or boosting steps.

Multi-objective to tune λ: We could consider an outer loop optimizing λ_t to minimize model error + some complexity measure. This outer loop could be done by grid search or gradient if differentiable (though L1 not easily diff differentiable in w, but maybe treat solution as function of λ piecewise continuous and differentiate outer objective by implicit differentiation – complicated but possible if we had smooth approximations).

Physics-informed weighting: Possibly incorporate known physical importance: - If error is mostly global (like average potential off), then encourage global basis by lowering its λ. - If error is highly localized, perhaps local basis λ could be lowered because a global basis won't fix a spike well. - If error pattern shows oscillation around torus, maybe chain or mode basis is needed (lower their λ). Essentially, *match basis type to error profile*.

One might measure error's Fourier content: e.g., perform a toroidal harmonic expansion of the residual. If it shows strong $m=3,n=1$, and we have a composite mode basis for that, then decrease that mode basis penalty to let it take it.

Alternatively, one could directly solve for those modes via pseudo-inverse (like find needed weight for that mode basis to remove that component) and incorporate.

Sensitivity of boundary vs interior error by type: If adding a ring drastically reduces boundary error but hardly changes interior error, that means ring is capturing majority of needed global potential (good). If adding a point drastically reduces a local interior error but not boundary (point maybe placed off-surface helps field but doesn't strongly influence BC?), maybe point basis is doing interior fix – then if interior is not that prioritized, maybe penalize it more relative.

One could attempt to attribute error reduction: run a few least-squares solves with one basis at a time to see how much they can reduce error individually (like a leave-one-out or marginal contribution analysis). This is heavy but doable for moderate dictionary sizes. Then weigh inversely: if one basis alone can reduce error by 50%, it's very relevant, so lower its λ. If one basis reduces only 1%, not so useful, keep λ high.

Topological constraints and λ adaptation: If we see basis combination chosen leads to slight violation of flux or some global property, we might incorporate a basis that corrects that. E.g. if after initial selection the induced net charge is not exactly -q (maybe because penalized solution allowed slight BC average offset), then we force adjust: maybe uniform ring's λ lowered to ensure net charge fixed properly.

However, likely we always enforce net charge by adding constraint or at least measure it.

Rigorous strategy example: - Start with initial λ (maybe set to favor global). - Solve Lasso -> get coefficients. - Compute residual r on boundary. - For each type t, compute $\Delta E_t = \min_{w_{g\in t}}|r - \Phi_t w_t|$ i.e. how much error can type t alone fix (maybe norm of projection of r onto span of type t). - If ΔE_t is large (type could significantly reduce error), but currently type not used, implies λ_t might be too high -> reduce λ_t by some factor. - If type is used with tiny weight while ΔE_t is small (type not very needed), maybe increase λ_t to discourage unless absolutely needed. - Then solve again with updated λ. - Iterate until convergence (should hopefully converge to a set of active types and weights).

This is somewhat heuristic but rooted in idea that if residual aligns strongly with a basis function, that basis should be encouraged to enter solution.

Hyperparameter adaptation referencing known algorithms: There is "Iteratively reweighted L1" for enhanced sparsity (makes large coefficients less penalized to refine solution) – similarly, one can treat each type's group norm like a variable and do iterative reweighted group lasso to favour certain group usage (like ARD approach does a continuous shrinkage).

Also, one can formalize a Bayesian approach: assign each type a prior variance hyperparam and update via evidence (the formula often drives irrelevant ones to zero variance and relevant to high). That essentially automates it. But implementing that might be heavy.

Focus on physically meaningful errors: Not all error in boundary condition is equally important physically – maybe high-frequency tiny oscillations could be tolerated if field outside still nearly correct. But Dirichlet strictly, any nonzero on conductor causes currents – though if small maybe negligible. But often maximum error threshold is set.

Ensuring correct flux patterns: If after solving, one finds the solution's flux around loops is off (which might happen if solution had slight BC offset?), one could add a regularization term or constraint explicitly. For example, a term penalizing $(\text{sum of induced charges} + q)^2$ if not zero, or penalizing any difference in potential around loops (maybe measure potential at one point on surface vs another offset by loop, should equal). But given our reasoning earlier, if basis are all physical, these conditions are likely already satisfied or easy to enforce by explicit add of uniform basis to fix it.

Thus adaptation might detect if uniform basis weight is wrong (like not exactly -q), then it would lower penalty on uniform ring to allow it to correct that fully.

Wrap up: Adaptive regularization is an advanced but potentially powerful approach to ensure the solver uses the "right" tools for the right part of the error. It's akin to a feedback loop: residual analysis informs basis encouragement or discouragement, then resolution improves. By incorporating physical insights (global vs local, flux, etc.), it can be more effective than a one-shot generic Lasso.

6. State of the Art and Open Problems for Toroidal Image Systems

Finally, we survey relevant literature and current understanding, and identify opportunities for novel basis families as well as simplification strategies.

6.1 Survey of the Frontier (Last ~10–15 Years and Key Older Works)

Analytic and mathematical works:

- *Matt Majic (2019, 2020):* Majic's papers [23] [24] are among the most recent comprehensive studies on toroidal harmonics. In 2019 (J. Quant. Spec. Rad. Transfer) [67], he presented expansions of toroidal harmonics in spherical harmonics and provided an analytical solution for the potential of a charged conducting torus in the quasistatic limit [34] [68]. His 2020 paper (Applied Numerical Mathematics) introduced a boundary integral approach and gave series expressions for the gravitational/electrostatic field of toroidal mass distributions [69] [70]. These works supply explicit formulae for Legendre functions of half-integer degree [55], recurrence relations, and numerical strategies to evaluate toroidal functions [71]. Majic's results in Fig.6 of 2020 show convergence behaviors of toroidal vs spherical series for a torus potential [32] [49], confirming spectral methods' efficacy and highlighting the intermediate divergence region.

- *Fukushima (2016):* T. Fukushima has a series of works on special function expansions for ring geometries. In *Astronomical Journal* **152**:1 (2016), he derived fast converging series for the gravitational potential of a ring (thin torus limit) [72]. Though gravitational context, it directly translates to electrostatics. This includes expansions in toroidal harmonics and perhaps Filon-type integrals for the Green's function.

- *Kondratyev et al (2009, 2010, 2012):* B. Kondratyev and colleagues published in *Technical Physics* on electrostatic fields of toroidal conductors. For instance, Technical Physics **54**, 176 (2009) and **55**, 22 (2010) discuss the potential of a uniformly charged torus and series solutions [29] [54]. Kondratyev (2012, Tech. Phys. **57**, 1613) considered a point charge near a torus (perhaps solving via toroidal coordinates or perturbation) [26]. These are likely in Russian; Erofeenko (1983) [73] is cited as deriving spherical-toroidal harmonic relationships earlier.

- *Morse & Feshbach (1953):* The seminal *Methods of Theoretical Physics* [74] gave the separated solution in toroidal coords and identified the half-integer Legendre functions. Although older, it's a key reference for analytic form of the Laplacian and separation scheme [75] [10].

- *Encyclopedia of Mathematics (toroidal harmonics):* Summarizes properties of toroidal harmonics, including that they are associated Legendre with half-odd index [25]. Likely references to Whittaker or Lebedev's special function treatises (Lebedev 1965 is cited in Majic's refs [76]).

- *Cohl & Tohline (1999):* "A Compact Cylindrical Green's Function Expansion for the solution of potential problems" (*Astrophys. J.* **527**, 86) [77]. They provided expansions of $1/|\mathbf{x}-\mathbf{x}'|$ in cylindrical (which relate to toroidal by analytic continuation differences) and specifically addressed how to handle ring singularities. Their expansion is essentially a toroidal harmonic expansion reorganized [78]. They also collaborated on toroidal function developments (Astron. Nachr. 2004 by Cohl, Tohline, Rau on using toroidal functions for gravitational potential) [79].

- *Venkov (2007):* G. Venkov (J. Comput. Acoust. **15**, 181, 2007) is cited by Majic [80]; likely this work solved an acoustic scattering from a torus via series or boundary elements. It might involve eigenfunction expansions or a specialized method (like mode matching). Similarly, *Vafeas (2016)* (Math. Meth. Appl. Sci. **39**, 4268) dealt with low-frequency scattering (could be electrostatic quasi-

static scattering) by a torus [56]. They probably used toroidal harmonics up to a certain order (Rayleigh approximation).

- *Andrews (2006):* Mark Andrews, *Journal of Electrostatics* **64**, 664–672 (2006) [81] proposed an "alternative separation of variables" for toroidal coordinates [82] [83]. This clever method changed which coordinate's dependence was expanded, yielding solutions better suited when boundary conditions are not standard (e.g., independent of toroidal angle rather than poloidal). He applied it to some cases like a point charge between conducting half-planes and charges inside a portion of a torus [84] [85]. This work underscores that classic toroidal harmonic separation isn't the only route; sometimes one can separate by treating ϕ differently to get integrals of Legendre Q functions that sum nicely for certain geometries [86] [87]. It's a deep technical piece showing there are still theoretical tricks for toroidal Laplace problems.

- *Jansen (2000):* G. Jansen (J. Phys. A **33**, 1375 (2000)) is referenced by Majic [88]. Possibly about elliptic integrals or toroidal harmonics in another context (maybe magnetostatics of torus magnets).

- *Cohl & collaborators (2014):* Howard Cohl has continued work on fundamental solutions in various coordinates. E.g., Cohl & Al, *Appl. Math. (2014?)* computing Fourier expansions in toroidal coordinates. They emphasize computational techniques for associated Legendre of half-integer (Gil & Segura's work [89] is relevant here too).

Numerical and computational SOTA:

- *High-order BEM for axisymmetric surfaces:* The method of Young & Martinsson (2009) in *J. Comp. Phys.* (or similar) provides a nearly optimal solver for Laplace on tori [42] [90]. With their Fourier–Nyström approach, solving a Laplace BIE on a torus with $N=320k$ points was done to high precision [91]. This is state-of-the-art in terms of direct solver speed and accuracy. It shows we can solve these problems numerically exactly (to machine precision) for reasonably large discretizations.

- *FMM-accelerated BEM:* Methods by Greengard, Bremer, et al. (cited in Martinsson's references [92]) can handle multiply connected domains. Not specific to torus, but applicable. There is also work by Helsing (though mostly 2D, he sometimes tackles 3D surfaces with special quadrature, but a torus being smooth, existing schemes suffice).

- *Toroidal metamaterials / plasmonics:* There is emerging literature on nanorings and tori in electromagnetics. E.g., *Chang et al 2012* studied plasmons on torus (like eigenmodes of Laplace-Beltrami on surface) – e.g., "Toroidal plasmon modes" etc. These overlap with solving Laplace's equation on torus domain (for resonant modes, which are related to harmonics). For example, *Zhang et al, PRL 2013* on metamaterials mention toroidal dipole moments (a different concept but geometry related). These works might not solve static Dirichlet directly, but they highlight interest in toroidal shapes.

- *Applications in fusion / plasma physics:* The shape of tokamaks (tori) leads to solving Laplace-like equations (for magnetic scalar potentials, etc.). Some computational works (like O'Brien 2015) solve for fields around toroidal coils. They may use Fourier series and finite differences on grids. Possibly they have formulas for Green's functions around a torus (like computing magnetic field of a current loop – which is analogous to computing potential of a ring of charge, known by elliptic integrals).

Indeed *Jackson's Electrodynamics* gives the magnetic field on axis of a current loop [93] which parallels the potential on axis of a ring.

- *Electrostatic capacitance of a torus:* A practical quantity computed by some authors (e.g., numerical integration by Hao, or using series by Cade (1986) above). The known result is that an isolated conducting torus has a capacitance less than that of a sphere of same radius. People have numerically computed it – which is essentially solving a Dirichlet problem with no external charge but a given total charge on torus. Those methods, possibly using BEM or toroidal harmonics, are relevant.

Open-source or commercial codes: Not specifically known for torus, but one could implement with general BEM codes (like FastCAP maybe used for capacitance of torus, etc). Analytical results are rare, so numerical is the main tool.

Key monographs: - *Lebedev (1965)* "Special Functions & Applications" (English 1972) covers toroidal functions theory, providing series expansions [76] . - Possibly *Sneddon or Byerly's older texts* mention toroidal coordinate solutions [22] . - The NIST Digital Library of Mathematical Functions might have a section on toroidal harmonics connecting to associated Legendre functions.

In summary, the frontier has: - Highly accurate expansion and evaluation of toroidal harmonics (Majic, Fukushima, Cohl). - Efficient numerical solvers for torus BIE (Martinsson, etc.). - Specific analytic solutions for special cases (thin torus, low-frequency scattering, etc.). - Yet, as far as *explicit image charge constructions*, none of these give a finite image. They either give infinite series or numerical solutions. This indicates an *open problem area: finding a systematic low-rank approximation (image system) for torus fields*.

6.2 Feasibility of Finite Image Systems for a Torus

Given the literature, is there hope for a closed-form or systematically convergent finite representation? - **Exact representation:** We strongly believe none exists for a torus (except the infinite series). The torus doesn't belong to the handful of shapes with method-of-images solutions.

- **Approximate convergent series of point or ring singularities:** The infinite toroidal harmonic series itself can be viewed as an infinite "image system" of sorts (with each harmonic corresponding to a certain distribution of sources in complex space). However, one might aim for an *exponentially convergent sequence of discrete sources*. For example, for parallel plates, method of images is such a sequence (geometric series of image charges that converges to effect of an infinite plate pair). For a torus, maybe one can find a sequence of ring charges or point charges around the ring that geometrically decays. If one could show that after the first N images, remaining effect is bounded by $(a/R)^{N}$ or something, that's a conceptual infinite image solution.

One idea: If the torus is thin ($a \ll R$), one could treat it as a small circle – perhaps then method of images might converge faster. If a/R small, maybe a few ring images placed properly might suffice. Possibly an asymptotic approach for $a/R\to0$: the torus approaches a ring (one-dimensional loop). For a loop, the field of a point charge off it can be expanded in multipoles of the loop (circular harmonics). That might truncate

if point is not too close (thin-wire approximation). So maybe a *low-rank representation exists for extreme aspect ratios.* For moderate aspect (0.3–0.5), likely need more terms.

- **Upper/lower bounds or heuristics:** Majic's fig.6 suggests ~120 toroidal harmonics needed for extremely accurate BC compliance when point is on torus equator [32] . But error decayed to maybe 1e-5 with ~50 terms [32] . It wasn't clear if that was maximum error or just relative in some region. But we can glean: to get 10^{-3} accuracy perhaps a few dozen terms might do. Finite image system with a few dozen sources might achieve a few percent error.

A *lower bound* for number of degrees needed can be related to the condition number of the BIE or to how localized the source's influence is. If a point is very close, the solution on torus has very sharp gradients, meaning high modes. If far, fewer needed. So *scaling with source distance:* If point is at distance d from torus center, perhaps modes $\approx \frac{R}{\sqrt{d^2-R^2}}$ might be needed (just guessing from similar formulas for sphere where near-surface needed $l \sim R/\delta$). So as point approaches surface (which is at distance = something), N skyrockets. For modest distances, N moderate.

One could foresee a plausible usage: if external charge is not extremely close, maybe 5–10 global basis and 5–10 local basis might yield ~1% accuracy.

As for *representing exactly or convergently*, if we allow continuous distributions as basis (like ring with continuous charge distribution of a certain shape), one might exactly satisfy an m mode with one basis element. So in principle, an infinite sum of continuous ring harmonics (which is basically the toroidal harmonic series itself) is needed. But *low-rank approximation* might be possible because these series often are exponentially convergent outside the singular region. Majic's work implies an analytic continuation singularity touches a smaller torus inside, thus series converges until that radius. The closer the point to torus, the closer singularity to actual domain, slower convergence.

It's theoretically plausible that one can approximate the Dirichlet-to-Neumann map of the torus by a rational function or Pade approximation which would correspond to a few images. If one did a model reduction on the BEM matrix, you'd find a few dominant eigenmodes for certain configurations. This is unexplored: an interesting open problem is *model order reduction for torus capacitance or response*. That might yield something like "the first few effective multipoles" – which effectively is what we are trying to find as image system.

Therefore: It's unlikely a finite set gives exact solution, but a systematic sequence of images could converge. Our aim is to find a minimal set that yields tolerable error.

We saw earlier speculation: perhaps ~5–10 basis can achieve a few % error for moderate geometry. The needed number likely grows as aspect ratio grows or source gets closer.

Analogy with prolate spheroid: People have tried finite image charges for prolate spheroid (which also requires infinite series in general). Some have found that 2-3 image charges can give a *rough* approximation of field for certain points (like fit at a few points yields an approximate solution). Possibly similar for torus: one or two ring charges might capture qualitative field shape, but error might be large near conductor.

Given no prior explicit attempts (to our knowledge), it's an open direction to *quantify how many point or ring charges are needed for <1% error* as a function of geometry. Our framework will help explore that empirically.

6.3 Proposals for Novel Basis Families

Combining insights from sections 1–5, we propose concrete basis sets that could be implemented:

(a) Effective ring sources: Instead of just uniform ring, allow *non-uniform rings*. For example: - A ring with a linear density variation $\lambda(\phi)=\lambda_0 + \lambda_1\cos\phi$ (2 degrees of freedom). This can produce both a monopole and dipole moment around the ring. Similarly, $\sin\phi$ term for an orthogonal dipole. This single basis element (with two or three adjustable parameters) could align the potential better than two separate rings. It's like having a ring that can shift its charge concentration to one side. Implementation: predefine the potential influence of cos-modulated ring and sin-modulated ring as separate basis vectors (so treat as two basis functions: $R_{\cos}$ and $R_{\sin}$). They can be grouped if needed. - Higher Fourier modes on ring if needed (cos2φ, etc.), but likely first order is enough for asymmetry in most cases.

These effective rings cover $m=0,1$ components globally. To incorporate poloidal variation (n modes), one can also allow: - *Dipole ring pair:* Two rings displaced above/below midplane, opposite charge: an effective implementation for $n=1$ (poloidal dipole). Actually, this can be approximated by a cosσ variation on the torus. We can treat it as a single basis if we tie their magnitudes (one +Q at +θ, one -Q at -θ). But maybe simpler: just include two ring basis (one inner, one outer) and let solver combine them as needed. If we want to reduce degrees, we can create a pre-set combination: e.g. $R_{\text{out}} - R_{\text{in}}$ as one basis function (meaning a dipole oriented radially). Similarly, $R_{\text{top}} - R_{\text{bottom}}$ as one (dipole vertically). But that might be too stiff; letting algorithm pick two rings with opposite sign might achieve same anyway.

So, we propose to include: - Uniform ring (major radius = that of torus). - Ring offset outward by a fraction of minor radius (maybe 1 ring at inner edge and 1 at outer edge). - Possibly rings offset up and down a bit. This yields several ring elements that can combine to approximate various distributions.

We can also allow ring radius slightly different from torus's (like a smaller or larger loop) in case it helps shape field inside vs outside.

(b) Partial arc distributions: Rather than a continuous arc, we can consider *clusters of point charges localized in an arc*. But to formalize, we could introduce a *Gaussian ring segment* basis: e.g., a distribution $\lambda(\phi) \propto e^{-(\phi-\phi_0)^2/2\sigma^2}$ around the ring. This yields a "bump" of charge on one side of torus. It's continuous so its potential can be computed by integrating the ring's Green's function weighted by that Gaussian. If that's too complicated, we approximate by, say, 8 point charges spaced in that region. But conceptually, providing such a basis helps capture local enhancements.

Parameterization: ϕ_0 (location around torus) and maybe σ (width of bump). We might fix σ to a moderate value to not overfit, and have basis at a few positions ϕ_0 (like 0°, 90°, etc.). But that introduces a lot of basis if done blindly. Instead, maybe easier: just allow free point charges – the solver will anyway group some near where needed. So maybe partial arcs can emerge from multiple points chosen.

Alternatively, one can incorporate *two or four arcs at symmetric positions (like front, back, inner, outer)* as candidate distributions with predetermined shape. If external charge is at say top, you know induced

mostly at top inner surface. So basis could be specifically a distribution on inner top quadrant. But for general positions, better to not bake in too specifically.

(c) Toroidal mode groupings: Perhaps include explicitly a *cosφ-cosσ mode pattern* of charges: For example, create 4 point charges: $+1$ at outer-top, $+1$ at inner-bottom, -1 at inner-top, -1 at outer-bottom (just as an example pattern that yields cosφ and cosσ variation). Treat that cluster as a single basis with an overall weight. But such cluster with unit weights yields a certain potential field; we can scale it. We might also need the rotated version (sinφ-phase difference). This could directly target the $(m=1,n=1)$ type mode, which is usually present when a point is off-axis and above (like quadrant differences). However, designing this for all combos gets complicated, and if our other basis (rings + a few points) can span it anyway, it might not be necessary.

But for novelty: Another interesting composite basis is a *toroidal multipole* – analogous to spherical multipole but on torus. For example, a *toroidal dipole moment* often refers to a configuration of currents shaped like a torus that produce a unique field (in electromagnetism, a "toroidal dipole" is something like a poloidal current that yields B field confined in torus, but in electrostatics, maybe a charge configuration that yields at far field similar to a torus oriented dipole? Possibly negligible at far field since plus and minus swirl cancel? It's specialized.) Though for our static problem, far field is just a monopole (the external charge plus induced yields net some distribution, but at infinity, field ~ that of external plus zero net from conductor if isolated? Actually, if conductor is grounded, net induced is -q, so at far field one sees 0 total – interestingly so the far field is null because the system is overall neutral). So outside far away, no field. But near zone, we have structure.

(d) Toroidal image chain basis: We can formalize e.g. 8 point charges equally spaced around the torus centerline (a discrete ring of images). Instead of letting each free, group them to have an exponential decay pattern: e.g., weight sequence $A, \alpha A, \alpha^2 A, ...$ for successive ones around. Then just parameters A and α to adjust. If tuned, that can approximate an infinite series of images. But solving for α nonlinearly might be needed. Alternatively, just include a few around and let Lasso pick how many and what magnitude.

Given complexity, maybe simpler: place e.g. 12 candidate point charges uniformly around centerline inside torus. Let L1 pick which and what weight. We likely get a near-symmetric pattern with heavy weight near one side (closest to external charge) and smaller on far side – akin to decaying images.

One could help L1 by grouping these charges with weighted L2 penalty that encourage smooth variation along chain (so that it doesn't pick an alternating large positive-negative pattern unnatural physically). That could be done by a difference penalty or by requiring monotonic weight drop if possible.

This chain addresses global ϕ variations if needed beyond what ring distributions did. Possibly redundant if we already have ring with cos variation.

(e) Exotic constructs: - A *mirror-stack basis* (used for parallel plates) between two circles? If we consider two coaxial rings as "mirrors", one could imagine an iterative image between them (like an infinite helix of images if ring fields reflect). But that’s not known to converge, probably not needed.

- *Spherical basis for torus:* Another idea: sometimes expanding around simpler shapes: e.g., approximate the torus's field by a multipole expansion about its center. That wouldn’t satisfy BC well, but in exterior might approximate. Not likely effective near surfaces.

- *Empirical basis via machine learning:* Possibly generate snapshots of exact solutions (via BEM or series) for various source positions and perform e.g. Proper Orthogonal Decomposition (POD) to find a small set of basis functions that span those solutions. Those basis would not correspond to simple point charges necessarily, but one might then try to fit those basis by combination of simpler sources. This is beyond scope but an interesting approach to design new basis.

The recommendations for an implementable blueprint: 1. **Mixed basis library**: Include - 1 uniform ring at torus centerline, - 1–2 rings with cosφ modulation, - 2 rings offset inner/outer, - ~N point charges on inner surface region where needed (or a dense set, let Lasso choose), - Possibly a small chain of 4 around ring, - Possibly one special cluster for mode (if off-axis scenario common, a cosσcosφ cluster). 2. **Parameterize potentials**: For rings, we have analytic integrals for potential at any field point. For points trivial. Composite clusters just sum of those.

1. **Hierarchical solve**: Solve first with ring types to get general distribution (monopole and dipole distribution on torus). Then add points if needed to kill residual hotspots. Then refine.

This should yield a set: "a small number of effective rings plus a handful of points", exactly what the user desires as an outcome to identify.

6.4 Collapsing to Simpler Models

After finding a detailed solution with possibly many basis elements, we want to compress it to an equivalent simpler form without significantly raising error.

Model selection vs compression: If our solver gave us 10 point charges and 3 rings, maybe some of those are redundant or overly precise. We can attempt: - *Merge similar basis:* If two point charges are very close and of opposite sign, they might approximate a dipole – perhaps we replace them by a single dipole or by nothing if they mostly cancel beyond local region. But since we use only charges, merging opposite might not yield a single charge but rather indicates maybe that portion of solution not needed far away. More concretely, if Lasso gave 3 point charges in almost a line, maybe one ring could replace them (fit their combined effect by adjusting ring density). We could try to cluster point charges that lie along what would be a ring or arc and re-fit them with a continuous distribution of that shape.

- *Clustering:* Group the selected point charges by location (like along torus angle). For each group, see if they collectively look like a portion of a ring or a dipole arrangement. Then attempt to replace group with a single basis: e.g., if 5 point charges spaced around half the torus with gradually diminishing strength, perhaps replace with one Gaussian ring segment basis. Fit its parameters to best approximate those 5 in least squares sense (minimizing difference in potential on relevant region). Then drop the 5 and include the new one, re-adjust in a refinement solve.

- *Pruning:* Start removing the smallest-weight basis elements if their removal doesn't raise error beyond tolerance. This is like backward elimination after Lasso. One can test each element – remove it, recompute error (fast if one has influence matrix maybe). If error increase is negligible, permanently remove it. This yields a minimal subset causing less than e.g. 1% error change. The risk is some basis might individually seem unimportant but collectively are (but if basis are not collinear, Lasso already gave minimal set; still, it might have distributed a feature across two basis where one could suffice 90% and the difference is minor).

It's wise to do greedy remove: while can remove an element with error <= threshold, do it. Or use techniques like L0 penalization (like minimize error + μ*#elements with small μ).

- *Parameter merging:* If two point charges are very close, combine them into one at intermediate location with weight = sum (if same sign). If opposite sign equal magnitude near each other, they were making a dipole – perhaps see if that dipole is actually needed or if it was an artifact (maybe it was canceling something else local). Usually, Lasso wouldn't pick such unless needed for a local boundary point – might be sign of trying to fit something very localized. That might be smoothed out by allowing slight BC error or adding a slight regularization to avoid these near-cancel pairs.

- *Output diagnostic:* Provide metrics like maximum BC error, and maybe integrals of error or flux differences to ensure physically okay. Perhaps present user with "We replaced X points by an equivalent ring, incurring Y% error increase".

The user specifically mentions "a small number of effective rings plus a handful of points" as the goal. So our collapse strategy likely tries to express the final selection in that form if possible: - If multiple rings used (like inner and outer), can we combine into one ring with a density distribution? Possibly yes: two rings of opposite charge = one ring with dipole distribution – output that as "one ring with cosθ variation". - If several point charges cluster, sum them if near each other (like treat as one bigger charge at cluster center). - Possibly also combine a ring and a point: e.g., if a point is right above torus, that could be seen as a ring partial – maybe not, better keep distinct.

One can always try a final fit of the obtained potential with a simpler ansatz: - Fit the solved potential data on surface using a parameterized model with fewer parameters (nonlinear fit). - For example, assume potential $\Phi \approx$ combination of a few rings with continuous densities of some form. Solve for those densities by least squares. - That is like doing our process in reverse – but since initial solution is good, a simpler param model initial guess can be refined to approximate it. If success, we present that param model.

Error from collapsing: We should check that after simplifying, boundary conditions are still satisfied within tolerance. If not, maybe skip that collapse or refine with a bit more degrees of freedom.

Empirical diagnostics: e.g., if after collapse the max BC error is 5% (above tolerance), we might restore something or adjust parameters. Possibly allow a small re-optimization of weights of remaining basis after removal (to re-fit best).

Essentially, run something like: 1. Take final chosen elements. 2. Solve a reduced least-squares for those (the refinement). 3. If error good, proceed. If not, maybe bring back one element or add an alternative one.

But the user likely expects us to discuss conceptually, which we have.

Conclusion: Collapsing complex solutions to minimal models is possible by grouping and replacing multiple basis elements with single “effective” ones (like combining multiple points into one ring distribution). This yields simpler conceptual models (which might be valuable for understanding or for faster computation if reused for similar scenario).

We can mention that these simpler models might slightly lose accuracy but are easier to interpret (like "image system comprising one ring of charge X and one point charge Y gives 2% error, instead of using 10 separate charges exact").

We should provide guidance: test error difference, make sure final error is below spec. Use intuitive merges like aligning to known symmetric distributions.

This concludes the research dossier, covering analytic foundations (Section 1), numeric methods (Section 2), basis design (Section 3), topology (Section 4), optimization frameworks (Section 5), and current research/ outlook (Section 6).

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