ChatGPT

1. Inner-Rim Asymptotics and Boundary Layers

Slender-Torus Expansion: In the limit of a *slender torus* (minor radius $a \ll$ major radius R), the inner equatorial region behaves like a narrow "tube" where the solution develops a **boundary-layer character**. Matched asymptotic analyses confirm this: one finds an "inner" solution around the torus cross-section and an "outer" solution for the far-field, with an overlap region for matching [1] [2]. Physically, this means the potential varies rapidly near the inner rim compared to the outer regions. The inner-rim neighborhood can be locally mapped to a simpler geometry – essentially a 2D *cylindrical* or *wedge* domain – for leading-order analysis. Shail (1979) pioneered an integral-equation approach for slender tori, treating the torus as a thin curved rod; his results (and later refinements by Koens & Lauga via slender-body theory) reduce the cross-sectional problem to something akin to an infinite cylinder or ring segment [2]. This yields asymptotic forms with *logarithmic terms*, characteristic of 2D cylinder solutions [3]. In practical terms, as the torus gets thinner, the induced surface charge and potential near the inner equator become extremely peaked and confined to a small region (a "boundary layer" on the inner side of the tube).

Behavior at the Concave Inner Rim: The torus's inner equator is a *concave* portion of the conductor, which leads to unique asymptotics. In fact, a toroidal surface approaching the limit of a *zero-hole* (a "just closed" torus with a reentrant cone at the inner equator) shows vanishing surface charge density at the very inner tip [4]. Maxwell noted that at a sharp concave corner, the surface charge goes to zero – a point of equilibrium where the field is locally weak [5]. Our torus is not that extreme, but this insight explains why the inner equatorial region is prone to **low-potential zones and steep gradients**: the geometry tends to *focus field lines* away from the concave interior. Thus, when an external point charge is placed on the axis inside the hole, the grounded torus develops a *strong induced-charge concentration just around the inner rim*, immediately facing the source, and a rapid drop-off away from that spot. One can think of this as a pseudo-"end effect": even though the torus has no ends, the presence of a nearby point charge breaks the symmetry and the inner rim closest to the charge behaves like the edge of a finite conductor, with a sharp variation (similar to end-charge accumulation on a finite cylinder) [6]. Any analytic solution would require very high-order terms to resolve this sharp inner-rim behavior, which is why a naive global series struggles there. In summary, **yes – a boundary-layer exists at the inner equator**. Locally "flattening out" the torus to a wedge/cylinder approximation is a sound approach: the inner rim can be treated as a curved wedge of angle ~2π (for the full ring) but with inner-surface field amplification and rapid decay over an $O(a)$ arc length. This is precisely the intuition behind using more localized basis functions in that region. The asymptotic message is that the standard toroidal harmonics (global eigenfunctions) have to sum to something very steep at the inner rim – suggesting we should build that behavior in *explicitly* with our basis.

High-n Modes and Inner-Rim Peaks: In toroidal-harmonic expansions, large poloidal mode numbers *n* correspond to rapid oscillations around the cross-section. For an axisymmetric problem (m=0), the potential can be expanded in associated Legendre functions $P_{n-1/2}(\cosh\xi)$/$Q_{n-1/2}(\cosh\xi)$ (the classic toroidal harmonics [7]). As *n* grows, these modes develop structure concentrated near the *ends of the poloidal interval* (i.e. around the inner and outer equators of the torus). Intuitively, $P_n(\cos\theta)$ for large n has boundary-layer peaks near $\theta=0^\circ$ and 180°. In the torus context, that means **high-n modes "live" near the inner and outer rim**. Indeed, for a highly concave case (aspect ratio approaching the slender limit), the highest-order term produces almost zero surface charge exactly at the inner tip [8] – highlighting how extreme cancellation/peaking occurs there. For more moderate aspect ratios, one finds that induced charge density tends to be **suppressed at a concave inner bend and enhanced on convex outer bends** in the static charged-conductor case [9]. However, with an *external point charge on the axis*, the situation inverts locally: the inner rim (being closer to the source) will accumulate *negative* charge, forming a pronounced *spike* in surface-charge density that decays along the surface away from that point. Accurately representing this spike via spherical or toroidal harmonics would demand a very large n – essentially a partial sum approximating a Dirac-like concentration. Instead of summing hundreds of global modes, one can attempt to **mimic the effect of high-n harmonics with a localized source**. In essence, a tight cluster of high-order toroidal harmonics creates a “bump” of potential/charge at the inner rim. This is a strong hint that we should include an explicit “bump basis” – e.g. a local ribbon or arc – rather than relying purely on global series. The literature doesn’t explicitly list “localized toroidal modes,” but the *slender-torus asymptotic solution* effectively resums the high-n contributions into a simple form (often a logarithmic variation in the surface potential [3]). That tells us the *scaling*: induced potential changes on the inner rim occur over a lengthscale on the order of the minor radius *a* (or even smaller, if the external charge is very close), and the amplitude of those variations can be orders of magnitude larger than the far-field potential. For example, Ivchenko’s recent numerical study (2021) showed that for a moderately thick torus in a uniform field, the electric field just inside the hole can even exceed that on the outside – a local maximum appears in the hole region for certain aspect ratios [10]. All these observations reinforce that **high spatial frequencies (high n) are needed near the inner rim**, and capturing them via a *few localized basis functions* is much more efficient than trying to push a global harmonic expansion to large n.

2. High-Order Toroidal/Poloidal Modes and Localization

Structure of Higher-n Poloidal Modes: The toroidal harmonics $T_{n,m}$ (using notation where n counts variation in the poloidal/cross-sectional direction and m around the azimuth) have well-understood shapes from separation-of-variables solutions. For axisymmetric ($m=0$) modes, $T_{n,0}$ essentially varies as $P_n(\cos\eta)$ around the cross-section (with η the poloidal angle) and decays/increases radially via the $Q_{n-1/2}$ or $P_{n-1/2}$ functions in the toroidal radial coordinate ξ [7]. Low-n modes are smooth: for example, $T_{1,0}$ is like a uniform potential on the torus, $T_{2,0}$ might correspond to one sign on the inner side and opposite on the outer side (a dipolar distribution aligning with the axis), etc. But as n increases, the $P_n(\cos\eta)$ term oscillates more vigorously. In fact, large-n Legendre functions concentrate their variation near $\eta=0$ and $\eta=\pi$ (the endpoints) – this is a known phenomenon of orthogonal polynomials (often one can approximate for large n: $P_n(\cos\eta) \sim \cos(n\eta + \text{phase})$, which has boundary layers at 0 and π). Therefore **higher-n poloidal modes inherently concentrate near the inner and outer equators of the torus**. They effectively create alternating positive/negative charge bands around the cross-section. For small *m* (especially $m=0$ or $m=1$), the toroidal harmonics do not oscillate in the ϕ (azimuthal) direction, so their energy is entirely focused in the poloidal direction – leading to very localized rings of charge. For instance, $T_{n,0}$ for large n has a surface charge pattern that oscillates as you go around the cross-section, with sharp peaks near the inner and outer rim. In practical terms, adding a high-n axisymmetric mode to your solution will mostly affect the inner/outer edge regions, not the sides. This matches our error maps: the largest errors are near the inner rim, which indicates that our current basis lacks sufficient high-n content there. We also see that the *outer* regions are well-captured – that’s because lower n (and our global ring modes) already represent the smooth outer behavior well. It’s the inner spikes that need those high-n shapes.

Spatial Localization and Scaling: How “localized” are the high-n contributions? We can use slender-torus results as a guide. In the slender limit, the leading-order solution had *uniform potential* on the torus to

$O(1)$, but the next-order showed a **logarithmic variation along the cross-section** [3] [11]. Essentially, the potential is slightly lower on the inner side than the outer side for a uniformly charged torus – a very slowly varying change. However, if we introduce an external field or a non-uniform boundary condition (e.g. half of the torus "active" in the phoretic analogy), then a much stronger variation appears: for example, in a "Janus" torus with one half at different potential, the solution has an $O(1)$ jump concentrated around the interface between the two halves [12] [13]. That interface region is analogous to a localized charge band. In the case of an external point charge on-axis, the induced potential on the torus surface is 0 everywhere, but just adjacent to the inner rim (nearest the charge) the free-space potential from the charge is highest – so the induced charge must spike there to cancel it. One can think of representing the effect of the point charge as a superposition of toroidal harmonics: an infinite series of $m=0$ modes that reproduces a sharp "dent" in potential at that location. Approximating a delta-like dent requires many terms, but one can also approximate it by a **localized equivalent source**. This is conceptually similar to representing a peaked function by a Gaussian instead of a Fourier series.

Approximating High-n by Local Sources: The literature on toroidal electrostatics has examples of replacing continuous distributions with discrete ones. One example is the "thin-wire" model of a torus: for a very slender torus, one treats it like a circular loop of charge (concentrated along a filament) and perhaps a continuous line of image charges along that loop [14]. Scharstein & Wilson (2005) solved a torus-in-uniform-field problem via exact toroidal harmonics and also via a filamentary ring approximation; the latter is essentially a *truncated image system* – a ring of charge whose density along the loop was adjusted to mimic the first few harmonics [15] [16]. Similarly, Hernandes & Assis (2003) modeled the surface charge on a torus carrying current by summing ring-harmonic contributions, but one could invert that sum to a local source picture. While we did not find a paper that literally proposes "short arcs" as basis functions, the **charge simulation method** in computational electrostatics often uses *discrete line segments or point charges* to approximate continuous conductors [17]. Applying that idea here: instead of an infinite series of toroidal harmonics, one could use a *small number of concentrated sources (images)* positioned near the torus. For example, placing a *line of charge* or a *ring of dipoles* just inside the inner rim could reproduce the effect of many high-n modes concentrated on the inner surface. In essence, *many high-order modes = one localized "image" source of appropriate shape*. This approach is analogous to the method of fundamental solutions – one places elementary charges near the boundary to satisfy the boundary conditions. In our context, a **short arc or patch** carrying charge can act like an elementary solution that produces a highly localized potential bump on the torus surface. Indeed, matched asymptotic analysis suggests that the inner-rim field can be reproduced by a solution of the *2D Laplace's equation in a circular wedge*, which in turn could be generated by a simple distribution like a line charge along the bisector of that wedge. All this points to using **localized primitives** to target the inner-rim errors, rather than relying solely on the global eigenfunctions.

3. Localized Basis Approximations for Toroidal Geometries

Given the above insights, it's natural to consider basis functions that are **local to a region of the torus** rather than spread over the entire surface. We did not find explicit prior art that develops a finite image system for a torus using short arcs or patches – most analytic work uses either the complete toroidal harmonic series [7] or a uniform "thin ring" approximation [16]. However, the *concept* of localized sources is well-supported by computational practices. For instance, a boundary element method would effectively use many small surface elements (which act like local charge patches). Our goal is to choose a few intelligently placed patches so we don't brute-force with thousands of elements. Here are some plausible localized basis primitives inspired by theory and analogous methods:

- **Short Arc Segments (Poloidal arcs):** Imagine a segment of the torus's circular cross-section carrying a prescribed charge distribution. This would correspond to an axisymmetric ribbon that wraps around the torus (covering all ϕ) but is confined to a small range of poloidal angle σ. For example, an "inner-rim arc" might cover the neighborhood of the inner equator $\sigma \approx \pi$ (say $\pi-\Delta \le \sigma \le \pi+\Delta$ for some half-width Δ). This basis function would deposit charge primarily on the inner side of the torus. The idea is to choose Δ small enough that the arc is **localized near the inner rim** but large enough to span the region where the error is concentrated. The asymptotic analysis and BEM error maps suggest that Δ on the order of 20°–40° (in angle) might be sufficient for a slender torus, possibly larger (50–60°) for a thicker torus where the induced charge spreads more around the inside. The charge distribution along the arc could be taken as uniform for simplicity, or shaped like a cosine lobe to smoothly taper at the ends. (In fact, a classic solution for a half-coated torus gives a leading-order surface concentration proportional to $\cos\sigma$ across the interface region [12] [13], suggesting that a cosine profile on an arc is a natural choice to approximate a localized lump of charge.) This "inner-rim arc" basis would directly target the observed spikes. From a PDE standpoint, it's like using the known *eigenfunctions of a cylindrical segment*: the arc will primarily generate high-n Fourier content in σ, which is exactly what we need. We expect the needed arc length (and the magnitude of charge on it) to scale with the minor radius *a*: roughly, the span of significant induced charge might be a fraction of the circumference of the cross-section. If the external point charge is very close to the torus, the induced charge will concentrate even more tightly (smaller Δ), whereas if the charge is farther, the distribution spreads a bit more. Thus one could choose a few arc basis functions of different widths to adapt to various source distances.

- **φ-Extended "Ribbons":** If we consider off-axis or non-axisymmetric sources, the induced charge will not be uniform around the ring – it will concentrate on the *portion of the torus closest to the source*. In that scenario, an arc that extends around the full 360° in φ is wasteful; instead, we want a patch that is localized in **both** poloidal angle *and* azimuth *φ*. A "ribbon" basis would be a short segment along the torus in the φ-direction. For example, a ribbon might span a limited azimuthal angle (say 30° or 60° of the torus's circumference) and be focused at a certain poloidal angle (like the inner equator). Such a basis function could be realized as a band of charge on the torus surface – e.g. covering ϕ from $\phi_0-\Delta\phi$ to $\phi_0+\Delta\phi$ and σ from $\pi-\Delta\sigma$ to $\pi+\Delta\sigma$, for some center location (σ_0,ϕ_0). This is essentially a local rectangular patch on the donut. The *orientation* of the ribbon can be aligned with field lines if known. For instance, if a point charge is located at some azimuth relative to the torus, you'd center the ribbon on the torus face that directly "faces" the point charge. The ribbon's ϕ-length might be chosen so that it subtends the angular size of the region significantly influenced by the source. A rough rule: if the point charge is a distance d from the torus centerline, one could take $\Delta\phi \sim \arctan(a/d)$ or similar, to cover the portion of the ring within a certain view angle of the charge. The poloidal width $\Delta\sigma$ would be set as above (perhaps on the order of a, or a fraction of it in angular terms). The surface charge on the ribbon could be uniform or perhaps peaked at the center (a 2D Gaussian profile could be used to avoid edge discontinuities). Such a ribbon effectively acts like a *little strip of charged metal* on the torus, producing a very localized field. Using a few ribbons, one could cover multiple "hot spots" – for example, if there are several external charges around the torus, each induces a hot spot on the nearest part of the torus. Each hot spot could be handled by one ribbon basis function. Notably, this approach has been used conceptually in antenna theory – e.g., representing currents on a torus by segmented loop elements.

- **Localized 2D Patches:** This is a generalization of the ribbon idea to an arbitrarily small patch. One could define a basis function that is nonzero only on a small area of the torus surface (for example, a patch covering maybe $\Delta\sigma \sim 0.2$ radians and $\Delta\phi \sim 0.2$ radians on the inner surface). This would give ultimate flexibility – essentially a “pixel” of charge. However, using many tiny patches brings us back toward a brute-force BEM, which we’re trying to avoid. Instead, a few moderately sized patches can be used. The key is to pick their positions informed by physics: likely places are the inner-upper quadrant (if the source is above the midplane) or inner-lower (if below), etc. Each patch basis could have a simple shape (constant charge density on that patch, for instance). Analytic potentials for a small patch don’t have a simple closed form, but we can compute the influence numerically or approximate it by, say, a point source or ring source at the patch’s center. In essence, each patch basis function could be treated as a small dipole or monopole distribution located near the surface. Some literature in electromagnetic engineering discusses using *equivalent surface patches* to mimic scattering – while we haven’t found a torus-specific example, the methodology would be similar (e.g. using a rectangular patch as a basis in method-of-moments).

- **Combined Patch + Ring Structures:** An interesting idea is to combine a local primitive with a global one to enforce certain constraints. For example, a local charged patch on the inner rim will induce a certain net charge and dipole moment. Our image system might need to ensure the total induced charge equals the negative of the external charge (for a grounded conductor problem). A single patch might not achieve that correct total charge or far-field behavior. So one can add a **complementary ring distribution** such that the ring handles the overall “bulk” induced charge and far-field potential, while the patch handles the local detail. In practice, this could mean: include one basis that is a full uniform ring of charge (which gives a coarse approximation of the induced effect, capturing the far-field monopole term), and another basis which is a localized patch with equal and opposite total charge, so that when added, the far field of the patch is canceled by the ring’s field, leaving a mainly localized effect. This is analogous to adding a “dipole ring” – you can think of a patch+ring pair as creating a dipole-like distribution focused at the inner rim. Another combined structure might be a **small loop or arc just off the surface**: for instance, a circular arc of wire placed inside the torus hole near the inner rim could serve as an image. (This was inspired by the method of images for toroids in some older texts: they sometimes use a ring of opposite charge inside the conductor to satisfy boundary conditions approximately [18] [19].) While not exact, such an arc-in-space can mimic the effect of a concentrated surface charge region. Designing patch+ring combinations can also help enforce orthogonality or avoid overlap with existing global modes.

In choosing these primitives, **placement and sizing rules of thumb** emerge from both asymptotic analysis and the numerical diagnostics: (a) Place localized primitives at or near the **regions of highest error/ induced charge** – for on-axis charges, this is the inner equator (if the charge is in the torus midplane) or slightly upward/downward along the inner rim (if the charge is off-center along z). Essentially, find the surface point closest to the external charge and center the local basis there. (b) Set the **angular span** of the basis to cover the zone where the potential error is elevated. For a slender torus with an on-axis charge, most of the induced charge might lie within, say, $\pm 30^\circ$ of the inner equator; for a fatter torus, it might spread to $\pm 50^\circ$. (c) Choose the **radial/vertical extent** of the basis such that it covers from the innermost edge of the torus out to perhaps the middle of the cross-section. For example, an inner-rim arc might extend from the inner surface ($\sigma = \pi$) slightly toward the top ($\sigma \approx 3\pi/4$) and bottom ($\sigma \approx 5\pi/4$) if the source is centered – or more toward the top if the source is above. (d) Use smooth **charge distribution profiles** on the primitives. Constant density is simplest, but a Gaussian or cosinusoidal profile can better concentrate the effect. For instance, a patch might have σ-dependence $\propto \exp[-((\sigma-\sigma_0)/\Delta\sigma)^2]$ and ϕ-dependence $\propto \exp[-((\phi-\phi_0)/\Delta\phi)^2]$, giving a nice localized bump. This avoids any unphysical discontinuity in surface charge, which could itself induce Gibbs-like oscillations in the solution. (e) **Scaling with geometry:** The size of a patch or arc should scale with *a* (the minor radius). It makes sense that a larger cross-section can support a broader charge patch. Indeed, for a very thin torus ($a \to 0$), the induced charge would cluster into an extremely small region – approaching a point. For a thicker torus, the induced "footprint" of charge is wider. So one might start with a patch that covers, say, 1/4 of the cross-sectional circumference for the mid-aspect ($R/a \sim 3$) case, and perhaps 1/6 of the circumference for the slender ($R/a \sim 6$–7) case, since slender torus has a tighter concentration. Similarly, the optimal patch size might depend on the source distance: a source deep inside the hole or very close to the inner wall causes a very sharp, narrow spike of charge (requiring a narrow basis function), whereas a source farther away induces a more diffuse distribution (a broader basis function).

In summary, **localized basis primitives make a lot of sense** for the torus problem. Theory tells us the inner rim behaves like a short "hot spot" – a natural target for a localized source. By introducing basis elements such as inner-rim arcs, short φ-extended ribbons, and small surface patches (optionally combined with whole-ring distributions for global balance), we can capture the missing high-n content. The payoff will be improved resolution of those pesky error spikes near the inner equator. This strategy is essentially implementing, in a finite way, what the asymptotic and numerical analyses are hinting at: *treat the inner rim with special localized functions* rather than expecting global series to do it all. We expect a dramatic drop in max error once these focused image primitives are added, as they will "kill" the inner-rim discrepancy by directly accounting for the boundary-layer behavior that was previously unrepresented.

References: The above draws on asymptotic analyses of slender tori [1] [4], toroidal harmonic expansions [7] [8], and computational studies of toroidal conductors [16] [12], all of which underline the need for localized resolution at the inner rim. These inform the proposed basis shapes and their parameterization.

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