1=S2.1h

h) $\lim_{n\to\infty} \frac{\frac{1}{2}\ln 2 + \frac{1}{3}\ln 3 + \ldots + \frac{1}{n}\ln n}{n\sqrt{n}}. =$

$$\lim_{n\to\infty} \frac{\ln \sqrt[2]{2} + \ln \sqrt[3]{3} + \ldots + \ln \sqrt[n]{n}}{n\sqrt{n}} =$$

$$\lim_{n\to\infty} \frac{\ln \sqrt[2]{2} \cdot \sqrt[3]{3} \cdot \ldots \cdot \sqrt[n]{n}}{n\sqrt{n}} =$$

$$\lim_{n\to\infty} \frac{\ln \sqrt[2]{2}\, \sqrt[3]{3} \ldots \sqrt[n]{n}}{\sqrt{n^3}}$$

De verificat

$\ln \sqrt[2]{2} \cdot \sqrt[3]{3} \ldots \sqrt[n]{n} \longrightarrow \infty$?

$\lim_{n\to\infty} \frac{x_n}{y_n}$ Dacă $x_n \to \infty$

$y_n \to \infty$

|| Dacă $\exists$

$$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} =$$

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}\ln(n+1)}{(n+1)\sqrt{n+1}-n\sqrt{n}} =$$

$$\lim_{n\to\infty} \underbrace{\frac{\ln(n+1)}{n+1}}_{0} \cdot \underbrace{\frac{1}{(n+1)\sqrt{n+1}-n\sqrt{n}}}_{0} = 0$$

$$\lim_{n\to\infty} \frac{(n+1)\sqrt{n+1}+n\sqrt{n}}{(n+1)^3-n^3} =$$

$$\lim_{n\to\infty} \frac{(n+1)\sqrt{n+1}+n\sqrt{n}}{\cancel{n^3}+3n^2+3n+1-\cancel{n^3}} =$$

$$\lim_{n\to\infty} \frac{n^{\frac{3}{2}}\left(\left(1+\underbrace{\tfrac{1}{n}}_{\to 0}\right)\sqrt{1+\underbrace{\tfrac{1}{n}}_{\to 0}}+1\right)}{n^2\left(3+\frac{3}{n}+\frac{1}{n^2}\right)} = 0$$

$$\lim_{n\to\infty} \frac{\overbrace{\ln(n+1)}^{\to\infty}}{\underbrace{n+1}_{\to\infty}} = \lim_{n\to\infty} \frac{\ln(n+2)-\ln(n+1)}{1} =$$

$$\lim_{n\to\infty} \ln\frac{n+2}{n+1} \;\frac{\ln}{\int \text{ct cont}}\; \ln \lim_{n\to\infty} \frac{n+2}{n+1} =$$

$$\ln 1 = 0$$

2 = S2.3 c

c) $x_{n+1} = \underbrace{\sqrt{2+x_n}}_{\geq 0}, n \in \mathbb{N}, x_0 \geq -2;$

Weierstrass

$$x_{n+1} - x_n = \sqrt{2+x_n} - x_n$$

$$\sqrt{2+x_n} - x_n > 0$$

$\sqrt{2+x_n} > x_n$ | Arătăm că $x_n > 0$ pt $n \in \mathbb{N}^*$

$2+x_n > x_n^2 \Leftrightarrow$

$x_n^2 - x_n - 2 < 0 \Leftrightarrow$

$(x_n - 2)\underbrace{(x_n+1)}_{\text{pt } n \in \mathbb{N}^* > 0} < 0$

Verificăm dacă $x_n - 2 < 0$

$x_0 = 2 \quad x_1 = \sqrt{2+2} = \sqrt{4} = 2$

$x_0 = 0 \quad x_1 = \sqrt{2} \quad x_2 = \sqrt{2+\sqrt{2}}$

$x_0 = 7 \quad x_1 = 3 \quad x_2 = \sqrt{5}$

$\underline{\mathrm{I}}$ $x_0 = 2$ Dem inductiv că $x_n = 2$ $\forall n \in \mathbb{N}$

$x_{n+1} = \sqrt{2+x_n} \xrightarrow{x_n = 2} \sqrt{2+2} = \sqrt{4} = 2$

II $x_0 < 2$ Dem inductiv că $x_n < 2$
$\forall n \in \mathbb{N}$

$P(0): x_0 < 2$ A

Pp $P(n): x_n < 2$

$P(n+1): x_{n+1} < 2$

$$x_{n+1} = \sqrt{2+x_n} < 2 \Big|^2 \Leftrightarrow$$

$$2 + x_n < 4 \Leftrightarrow$$

$$x_n < 2 \text{ A}$$

$\Rightarrow x_{n+1} < 2$

$\left.\begin{array}{l} P(0) \text{ A} \\ P(n) \longrightarrow P(n+1) \end{array}\right\} \Rightarrow P(k) \text{ A } \forall n \in \mathbb{N}$

$$x_{n+1} - x_n = \sqrt{2+x_n} - x_n$$

$$x_{n+1} = \sqrt{2+x_n} \; \boxed{>} \; x_n \quad \Big|^2 \quad \Leftrightarrow$$

$$2 + x_n > x_n^2 \quad \Leftrightarrow$$

$$x_n^2 - x_n - 2 < 0 \quad \Leftrightarrow$$

$$\underbrace{(x_n - 2)}_{<0} \underbrace{(x_n + 1)}_{>0} < 0 \quad \text{A}$$

$$\left.\begin{array}{l} \Rightarrow (x_n) \text{ crescător} \\ x_0 \leq x_n < 2 \end{array}\right\} \xRightarrow{W} (x_n) \text{ convergent}$$

III $x_0 > 2$ Analog

(x_n) convergent

$\Downarrow$

$x_{n+1} = \underline{\sqrt{2+x_n}}$ $\quad \exists \lim\limits_{n\to\infty} x_n = l \Longrightarrow$

$l = \sqrt{2+l} \quad \Big|^2$

Din $x_n > 0$

$\Rightarrow l \geq 0$

$l^2 = 2 + l \quad (2)$

$l^2 - l - 2 = 0$

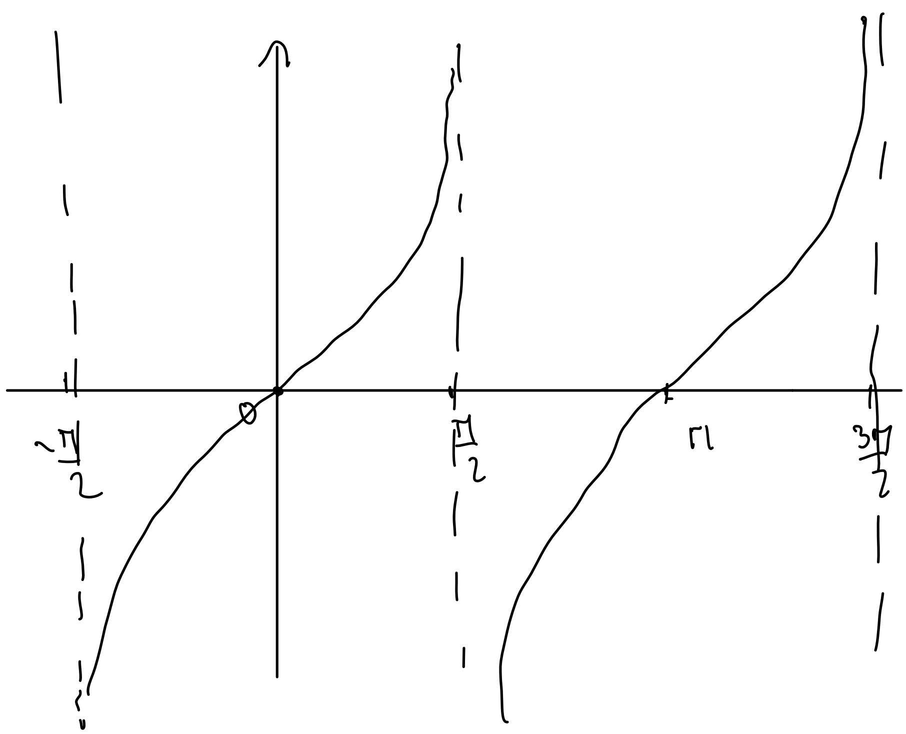
$(l-2)(l+1) = 0$

$\left.\begin{array}{l} \text{Pt } n>0,\ x_n \in [0,2) \Rightarrow l \in [0,2] \end{array}\right\} \Rightarrow$

$l = 2$

3. S2.5c

$$\text{c) } x_n = \frac{\underbrace{(-1)^n}_{n \text{ par} \quad n \text{ impar}} + n \operatorname{tg} \frac{n\pi}{4}}{n},$$



$n = 4k$

$$x_{4k} = \frac{(-1)^{4k} + 4k \operatorname{tg} \frac{\cancel{4}k\pi}{\cancel{4}}}{4k} =$$

$$\frac{1 + 4k \overbrace{\operatorname{tg}(k\pi)}^{0}}{4k} = \frac{1}{4k} \xrightarrow{k\to\infty} 0$$

$n = 4k+1$

$$x_{4k+1} = \frac{(-1)^{4k+1} + (4k+1) \operatorname{tg} \frac{(4k+1)\pi}{4}}{4k+1}$$

$$= \frac{-1 + (4k+1) \overbrace{\operatorname{tg}\left(k\pi + \frac{\pi}{4}\right)}^{1}}{4k+1} =$$

$$\frac{-1 + 4k + 1}{4k+1} = \frac{4k}{4k+1} \to 1$$

$n = 4k+2$

$$x_{4k+2} = \frac{(-1)^{4k+2} + (4k+2) \operatorname{tg} \frac{(4k+2)\pi}{4}}{4k+2} =$$

$$\frac{1 + (4k+2)\operatorname{tg}\frac{\pi}{2}}{4k+2}$$

$$\not\exists$$

$n = 4k+3$

$$x_{4k+3} = \frac{\overbrace{(-1)^{4k+3}}^{-1} + (4k+3)\operatorname{tg}\overbrace{\frac{(4k+3)\pi}{4}}^{-1}}{4k+3} =$$

0,86

$$\frac{-1 - (4k+3)}{4k+3} \longrightarrow -1$$

4

c) $\displaystyle\lim_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right)^{\frac{1}{\sin\left(\pi\sqrt{1+n^2}\right)}}$

$$x_{2k} = \underbrace{\left(1 + \frac{1}{2k}\right)}_{1}^{\frac{1}{\sin\left(\pi\sqrt{1+4k^2}\right)}}$$

$=$

$$x_{2k} = \left(1+\frac{1}{2k}\right)^{2k} \cdot \frac{1}{\sin\left(\pi\sqrt{1+4k^2}\right)\cdot 2k}$$

$$\longrightarrow e^{\frac{2}{\pi}}$$

$\sin(a-b) = \sin a \cos 2b$

$$\frac{1}{2k\sin\left(\pi\sqrt{1+4k^2}\right)} = \frac{1}{2k\sin\left(\pi\sqrt{1+4k^2}-2k\pi\right)}$$

$$\boxed{\begin{gathered}\sin(x) = \sin(x+2k\pi) \quad \forall k\in\mathbb{N} \\ \frac{\sin x}{x} \xrightarrow{x\to 0} 1\end{gathered}}$$

$$= \frac{1}{2k\,\dfrac{\sin\left(\pi\sqrt{1+4k^2}-2k\pi\right)}{\pi\sqrt{1+4k^2}-2k\pi}\cdot\left(\pi\sqrt{1+4k^2}-2k\pi\right)}$$

$$\frac{1}{\dfrac{\sin\left(\pi\sqrt{1+4k^2}-2k\pi\right)}{\pi\sqrt{1+4k^2}-2k\pi}\cdot 2k\cdot\pi\cdot\dfrac{1+\cancel{4k^2}-\cancel{4k^2}}{\sqrt{1+4k^2}+2k}} \longrightarrow$$

$$1 \qquad \frac{2k}{\sqrt{1+4k^2}+2k} \qquad \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

S2.15 Câturile împărţirii polinomului $f \in \mathbb{R}[X]$ la $X-a$ şi $X-b$ sunt respectiv X^2-3X+4 şi X^2-4X+2. Determinaţi valorile parametrilor $a, b \in \mathbb{R}$ şi polinomul f, ştiind că termenul liber al polinomului este 1.

$$f = (X-a)(X^2-3X+4) + r_1$$

$$\underbrace{\deg X-a}_{1} > \deg r_1 = 0 \Rightarrow r_1 \in \mathbb{R}$$

$$f = (X-b)(X^2-4X+2) + r_2$$

$$\underbrace{\deg X-b}_{1} > \deg r_2 = 0 \Rightarrow r_2 \in \mathbb{R}$$

$$f = X^3 - aX^2 - 3X^2 + 3aX + 4X - 4a + r_1$$

$$f = X^3 - X^2(a+3) + X(3a+4) - 4a + r_1$$

$$f = x^3 - bx^2 - 4x^2 + 4bx + 2x -$$

$$2b + r_2$$

$$f = x^3 - x^2(b+4) + x(4b+2) -$$

$$2b + r_2$$

termenul liber $= 1$

$$\begin{cases} -4a + r_1 = 1 \\ 2b + r_2 = 1 \\ a + 3 = b + 4 \\ 3a + 4 = 4b + 2 \end{cases}$$

$\Rightarrow r_1 = 1 + 4a = 25$

$r_2 = 1 - 2b = -9$

$\Rightarrow a = b + 1$

$3b + 3 + 4 = 4b + 2$

$7 = b + 2$

$\Rightarrow b = 5$

$a = 6$

$$f = x^3 - 9x^2 + 22x + 1$$

S2.17 Se consideră polinomul $f \in \mathbb{C}$, $f = (X+i)^{2020} + (X-i)^{2020}$, care are forma algebrică

$$f = a_{2020}X^{2020} + a_{2019}X^{2019} + \ldots + a_1X + a_0.$$

a) Să se calculeze $\underbrace{a_{2020}}_{2} + \underbrace{a_{2019}}_{0}$. $= 2$

b) Să se determine restul împărţirii polinomului f la $X^2 - 1$.

$$\boxed{(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}}$$

binomul lui Newton

$$(X+i)^{2020} = \sum_{k=0}^{2020} C_{2020}^k X^k i^{2020-k}$$

$k = 2020$

$$\underbrace{C_{2020}^{2020}}_{1} X^{2020} \underbrace{i^0}_{1}$$

$k = 2019$

$$\underbrace{C_{2020}^{2019}}_{2020} X^{2019} \underbrace{i^1}_{i}$$

$$(X-i)^{2020} = \sum_{k=0}^{2020} C_{2020}^k X^k (-i)^{2020-k}$$

$k = 2020$

$k = 2019$

$$\underbrace{C_{2020}^{2020}}_{1} \, X^{2020} \underbrace{(-i)^0}_{1} \quad \Bigg| \quad C_{2020}^{2019} X^{2019} \underbrace{(-i)^1}$$

$$2020 \qquad -i$$

b) $\exists$ $c(x)$ și $r(x)$

$$f(x) = (X^2-1)\, c(x) + r(x)$$

$$\deg r(x) < \deg (x^2-1) = 2$$

$$\Rightarrow r(x) = aX + b$$

$$f(x) = (X^2-1)\, c(X) + (aX+b)$$

$$f(1) = \underbrace{(1-1)}_{0} c(1) + a + b$$

$$(1+i)^{2020} + (1-i)^{2020} = a + b$$

$$(1+i)^2 = (1+i)(1+i) =$$

$$\underbrace{1+i^2}_{0} + 2i = 2i$$

$$(1-i)^2 = (1 + \underbrace{(-i)^2}_{-1} - 2i) = -2i$$

$$(1+i)^{2020} = \left((1+i)^2\right)^{1010} =$$

$$(2i)^{1010} =$$

$$2^{1010} \cdot (i^2)^{505} =$$

$$2^{1010} \cdot (-1)^{505}$$

$$= -2^{1010}$$

$$(1-i)^{2020} = (-2i)^{1010} =$$

$$2^{1010} \cdot i^{1010} =$$

$$-2^{1010}$$

$$f(1) = 2 \cdot (-2^{1010}) = -2^{1011}$$

$$f(-1) = (1-1)C(-1) - a + b$$

$$f(-1) = (-1+i)^{2020} +$$

$$(-1-i)^{2020} =$$

$$(-1+i)^2 = 1 + i^2 - 2i$$

$$(-2i)^{1010} + (2i)^{1010} =$$

$$-2^{1011}$$

$$a + b = -2^{1011}$$

$$-a + b = -2^{1011} \Rightarrow a = 0$$

$$b = -2^{1011}$$