541 g) (R) $\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)^{n+1}}{n^{n+2}}}_{a_n}$;

$$y_n = \frac{(n+1)^{n+1}}{n^{n+2}} = |a_n|$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(n+2)^{n+2}}{(n+1)^{n+3}}}{\frac{(n+1)^{n+1}}{n^{n+2}}} = \frac{(n+2)^{n+2}}{(n+1)^{n+3}} \cdot \frac{n^{n+2}}{(n+1)^{n+1}} =$$

$$\frac{(n^2+2n)^{n+2}}{(n+1)^{2n+4}} = \frac{(n^2+2n)^{n+2}}{(n^2+2n+1)^{n+2}} =$$

$$\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+2} = \left(1-\frac{1}{n^2+2n+1}\right)^{n+2}$$

$$\underbrace{\left(1-\frac{1}{n^2+2n+1}\right)^{n^2+2n+1 \cdot \frac{n+2}{n^2+2n+1}}}_{}$$

$$\left(\frac{1}{e}\right)^0 \longrightarrow 1$$

Pasul inițial, se poate sări

Propunere : Leibniz

$$|a_n| = \frac{(n+1)^{n+1}}{n^{n+2}} = \left(\frac{n+1}{n}\right)^{n+1} \cdot \frac{1}{n} =$$

$$\underbrace{\left(1+\frac{1}{n}\right)^{n+1}}_{\substack{\downarrow \\ e}} \cdot \underbrace{\frac{1}{n}}_{\substack{\downarrow \\ 0}} \longrightarrow 0$$

deci e marg

$$\left(1+\frac{1}{n}\right)^{n} \longrightarrow e$$

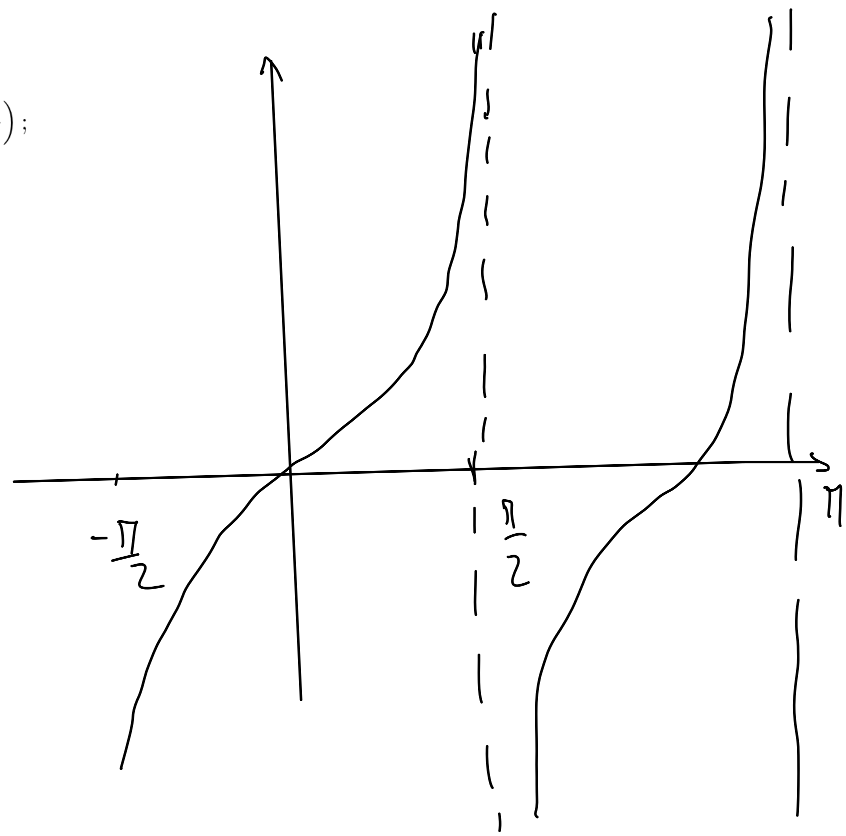
$$\left(1+\frac{1}{n}\right)^{n+1\cdot\frac{n}{n+1}} \longrightarrow e^{1}$$

$$\left.\begin{matrix}\frac{|a_{n+1}|}{|a_n|} < 1 \Rightarrow (|a_n|) \searrow \\ |a_n| \to 0\end{matrix}\right\} \Rightarrow \text{Leibniz}$$

$$\sum (-1)^n \cdot |a_n| \; C$$

n) $\sum_{n=1}^{\infty} \underbrace{\operatorname{tg}^n\left(a+\frac{b}{n}\right)}_{x_n}, a, b \in \left(0, \frac{\pi}{2}\right)$;

x_n



$$0 < a < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} - a > 0$$

$$0 < b < \frac{\pi}{2}$$

$$\frac{b}{n} \to 0 \quad \forall \varepsilon > 0 \ \exists n_\varepsilon \text{ a.î. } \forall n \geq n_\varepsilon$$

$$\frac{b}{n} < \varepsilon$$

$$\varepsilon = \frac{\pi}{2} - a \quad \exists n_\varepsilon \text{ a.î. } \forall n \geq n_\varepsilon$$

$$\frac{b}{n} < \frac{\pi}{2} - a$$

$$a + \frac{b}{n} < \frac{\pi}{2}$$

De la un rang încolo $a + \frac{b}{n} < \frac{\pi}{2}$

$$\Rightarrow x_n > 0 \ \forall n \geq n_\varepsilon$$

$$\sqrt[n]{x_n} = \operatorname{tg}\left(a + \frac{b}{n}\right) \to \operatorname{tg} a$$

$\frac{b}{n} \to 0$

tg cresc

$$\operatorname{tg} a \begin{cases} \text{Dacă } a < \frac{\pi}{4} \to \Sigma\, C \\ \text{Dacă } a > \frac{\pi}{4} \to \Sigma\, D \end{cases}$$

Dacă $a = \frac{\pi}{4}$? Studiem

$$x_n = \operatorname{tg}^n\left(\frac{\pi}{4} + \frac{b}{n}\right) =$$

$\operatorname{tg}(a+b) = \dfrac{\operatorname{tg} a + \operatorname{tg} b}{1 - \operatorname{tg} a \operatorname{tg} b}$

$$\left(\frac{\operatorname{tg}\frac{\pi}{4} + \operatorname{tg}\frac{b}{n}}{1 - \operatorname{tg}\frac{\pi}{4} \cdot \operatorname{tg}\frac{b}{n}}\right)^n =$$

$$\left(\frac{1 + \operatorname{tg}\frac{b}{n}}{1 - \operatorname{tg}\frac{b}{n}}\right)^n =$$

$$\left(\frac{1 - \operatorname{tg}\frac{b}{n} + 2\operatorname{tg}\frac{b}{n}}{1 - \operatorname{tg}\frac{b}{n}}\right)^n =$$

$$\left(1 + \frac{2\operatorname{tg}\frac{b}{n}}{1 - \operatorname{tg}\frac{b}{n}}\right)^{\frac{1 - \operatorname{tg}\frac{b}{n}}{2\operatorname{tg}\frac{b}{n}} \cdot \frac{2\operatorname{tg}\frac{b}{n}}{1 - \operatorname{tg}\frac{b}{n}} \cdot n}$$

$$\underbrace{\left(1+\frac{2\,\mathrm{tg}\,\frac{b}{n}}{1-\mathrm{tg}\,\frac{b}{n}}\right)^{\frac{1-\mathrm{tg}\,\frac{b}{n}}{2\,\mathrm{tg}\,\frac{b}{n}}}}_{\downarrow \atop e} \cdot \underbrace{\frac{2\,\mathrm{tg}\,\frac{b}{n}}{\frac{b}{n}}}_{\downarrow \atop 2} \cdot \underbrace{\frac{\frac{b}{\cancel{n}}}{1-\mathrm{tg}\,\frac{b}{n}}}_{\downarrow \atop b} \cdot \cancel{n}$$

$$\longrightarrow e^{2b} > 0$$

$$\Rightarrow x_n \not\to 0 \Rightarrow \sum x_n \ D$$

Sugestie

$$\frac{\ln \dfrac{1}{\mathrm{tg}^n\left(\frac{\pi}{4}+\frac{b}{n}\right)}}{\ln n} =$$

$$\frac{\ln 1 - \ln \mathrm{tg}^n\left(\frac{\pi}{4}+\frac{b}{n}\right)}{\ln n} =$$

$$- \frac{n \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{b}{n}\right)}{\ln n}$$

Prea complicat

541. k) (R) $\sum_{n=1}^{\infty} \underbrace{\frac{n+1}{n} \cdot \frac{\sin \frac{n\pi}{6}}{\sqrt{n^3+1}}}_{x_n}$;

$$|x_n| = \frac{n+1}{n\sqrt{n^3+1}} \cdot \underbrace{\left|\sin \frac{n\pi}{6}\right|}_{\leq 1}$$

Var 1

$$|x_n| \leq \underbrace{\frac{n+1}{n} \cdot \frac{1}{\sqrt{n^3+1}}}_{z_n} = \frac{n+1}{\underbrace{n\sqrt{n^3+1}}_{\sqrt{n^5+n^2}}}$$

$$-n^2 < \frac{1}{n^2}$$

$$y_n = \frac{1}{n^{\frac{3}{2}}}$$

$$\frac{z_n}{y_n} = \frac{\frac{n+1}{n\sqrt{n^3+1}}}{\frac{1}{n^{\frac{3}{2}}}} = \frac{n+1}{n} \cdot \frac{\sqrt{n^3}}{\sqrt{n^3+1}} \to 1 \xrightarrow{\text{LCIII}}$$

$$\sum z_n \sim \sum y_n \ C \to \sum z_n \ C$$

$$\left.\begin{array}{l} |x_n| < z_n \\ \sum z_n \ C \end{array}\right\} \xrightarrow{\text{CCI}} \sum |x_n| \ C$$

Var 2:

$$z_n = \frac{n+1}{n\sqrt{n^3+1}}$$

$$w_n = \sin \frac{n\pi}{6}$$

$$S_n \sum_{k=1}^{n} \sin \frac{k\pi}{6} = \sum_{k=1}^{12\left[\frac{n}{12}\right]} \sin \frac{k\pi}{6} + \sum_{12\left[\frac{n}{12}\right]}^{n} \sin \frac{k\pi}{6} =$$

$$\left(\sin \frac{\pi}{6} + \sin \frac{2\pi}{6} + \sin \frac{3\pi}{6} + \sin \frac{4\pi}{6} + \right.$$

$$\sin \frac{5\pi}{6} + \sin \frac{6\pi}{6} + \sin \frac{7\pi}{6} + \sin \frac{8\pi}{6}$$

$$\left. + \sin \frac{9\pi}{6} + \sin \frac{10\pi}{6} + \sin \frac{11\pi}{6} + \sin \frac{12\pi}{6} \right)$$

$$+ \left(\sin \frac{13\pi}{6} + \sin \frac{14\pi}{6} + \dots + \sin \frac{24\pi}{6} \right)$$

$$+ \sum_{12\left[\frac{n}{12}\right]}^{n} \sin \frac{k\pi}{6} =$$

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0 \right.$$

$$\left. -\frac{1}{2} + \frac{-\sqrt{3}}{2} + -1 + \frac{-\sqrt{3}}{2} + -\frac{1}{2} + 0 \right)$$

$$+ \left(\qquad \right) + \dots + \sum_{12\left[\frac{n}{12}\right]}^{n} \sin \frac{k\pi}{6} =$$

$$\sum_{12\left[\frac{n}{12}\right]}^{n} \sin \frac{k\pi}{6}$$

$$|S_n| \leq 11$$

$$z_n \longrightarrow 0$$

$$z_n = \frac{n+1}{n\sqrt{n^3+1}} > 0$$

$$z_{n+1} = \frac{n+2}{(n+1)\sqrt{(n+1)^3+1}}$$

$$\frac{z_{n+1}}{z_n} = \frac{n+2}{(n+1)\sqrt{(n+1)^3+1}} \cdot \frac{n\sqrt{n^3+1}}{n+1} =$$

$$\frac{(n^2+2n)\sqrt{n^3+1}}{(n^2+2n+1)\sqrt{n^3+3n^2+3n+2}} \leq 1$$

$$\frac{n^2+2n}{n^2+2n+1}\sqrt{\frac{n^3+1}{n^3+3n^2+3n+2}} < 1$$

$z_n \searrow 0$

w_n șirul sumelor parțiale mărg. (Dirichlet)

$\sum w_n z_n$ C

j) (R) $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{\sqrt[n+1]{\ln(n+1)}}}_{x_n}$;

$\ln(1+x) < x$

$\ln(n+1) < n$

$\sqrt[n+1]{\ln(n+1)} < \sqrt[n+1]{n}$

$$\frac{1}{\sqrt[n+1]{\ln(n+1)}} > \frac{1}{\sqrt[n+1]{n}} = 1$$

$z_n = \sqrt[n+1]{n}$

$$\lim_{n\to\infty} \sqrt[n+1]{n} \underset{\exists}{\overset{\text{Dacă}}{=}} \lim_{n\to\infty} \frac{n+1}{n} = 1$$

$x_n > \frac{1}{2n} \longrightarrow 1$

$\Rightarrow x_n \not\longrightarrow 0 \Rightarrow \sum x_n \, D$

$\frac{1}{\sqrt[n+1]{\ln(n+1)}}$

$\lim\limits_{n\to\infty} \sqrt[n+1]{\ln(n+1)} \overset{\text{Dacă } \exists}{=} \lim\limits_{n\to\infty} \frac{\ln(n+2)}{\ln(n+1)} \overset{?}{=} 1$

Considerăm $f: \mathbb{R}_+ \longrightarrow \mathbb{R}$

$f(x) = \frac{\ln(x+2)}{\ln(x+1)}$

$\lim\limits_{x\to\infty} f(x) \xrightarrow[l'H]{\text{Dacă } \exists} \lim\limits_{x\to\infty} \frac{\frac{1}{x+2}}{\frac{1}{x+1}} =$

1

$x_n \longrightarrow 1 \Rightarrow \sum x_n \, D$

$$\sum \frac{1}{\sqrt[n+1]{n+1}} \qquad \ln$$

$$\frac{\ln\left(\sqrt[n+1]{\ln(n+1)}\right)}{\ln n} =$$

$$\underbrace{\frac{1}{n+1}}_{\downarrow\, 0} \ \underbrace{\frac{\ln(\ln(n+1))}{\ln n}}_{\text{mărg}} \longrightarrow 0$$

mărg

$$0 < \frac{\ln(\ln(n+1))}{\ln n} \le \frac{\ln(n+1)+1}{\ln n} =$$

$e^{x+1} \ge x$

$x+1 \ge \ln x$

$$\underbrace{\frac{\ln(n+1)}{\ln n}}_{\downarrow\, 1} + \underbrace{\frac{1}{\ln n}}_{\downarrow\, 0}$$

Un şir x_n $\quad f: \underbrace{\mathbb{N}}_{\text{discretă}} \longrightarrow \mathbb{R}$

$f(n) = x_n$

Derivarea se aplică pe mulț. de tip interval $\Rightarrow$ Nu pot deriva şiruri

$$x_n = \sin(2\pi \cdot n) = 0 \longrightarrow 0$$

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \sin(2\pi x)$

$$\nexists \lim_{x \to \infty} f(x)$$

Dar dacă f-ț. are limită la $\infty \Rightarrow$ şirul are limită

532.g

$$\sum_{n=0}^{\infty}(-1)^n \frac{1}{3^{\frac{n}{2}}\sqrt{1+n^2}}\operatorname{tg}^n x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right);$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^n} \cdot (\operatorname{tg} x)^n \cdot \frac{1}{\sqrt{1+n^2}} =$$

$$\sum_{n=0}^{\infty} \underbrace{\left(\frac{-\operatorname{tg} x}{\sqrt{3}}\right)^n}_{z} \cdot \underbrace{\frac{1}{\sqrt{1+n^2}}}_{a_n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\sqrt{1+(n+1)^2}}}{\frac{1}{\sqrt{1+n^2}}} = \frac{\sqrt{1+n^2}}{\sqrt{1+n^2+2n+1}} \longrightarrow 1$$

$$\sum a_n z^n \begin{cases} AC & pt\ |z|<1 \\ D & pt\ |z|>1 \\ D & z=1 \\ SC & z=-1 \end{cases}$$

$|z|=1$ $z=1$

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{1+n^2}}$$

$$b_n = \frac{1}{n}$$

$$\frac{a_n}{b_n} = \frac{n}{\sqrt{1+n^2}} \longrightarrow 1 \overset{CC\,\overline{II}}{\Rightarrow} \sum a_n \sim \sum b_n\ D$$

$\to \sum a_n$ D

$z = -1$

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{1+n^2}}$$

a_n

$a_n \longrightarrow 0$

$1+n^2$ cresc $\to \frac{1}{\sqrt{1+n^2}}$ descresc. $\Big\}$ leibniz

$$\sum_{n=0}^{\infty} (-1)^n a_n \ C$$

$\sum a_n z^n \begin{cases} A\ C \text{ pt } |z|<1 & \text{pt } x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \\ D \text{ pt } |z|>1 & \text{pt } x \in \left(-\frac{\pi}{2}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{3}, \frac{\pi}{2}\right) \\ D_{SC} \quad z=1 \quad x = -\frac{\pi}{3} \\ \quad\quad\ \ z=-1 \quad x = \frac{\pi}{3} \end{cases}$

I $\quad \frac{-\operatorname{tg} x}{\sqrt{3}} = 1$

$-\operatorname{tg} x = \sqrt{3}$

$\operatorname{tg} x = -\sqrt{3}$

II $\quad -\frac{\operatorname{tg} x}{\sqrt{3}} = -1$

$\Rightarrow x = \frac{\pi}{3}$

$$x = \operatorname{arctg}\left(-\sqrt{3}\right) \Rightarrow$$

$$x = -\operatorname{arctg}(\sqrt{3}) =$$

$$-\frac{\pi}{3}$$

III $$\left|-\frac{\operatorname{tg} x}{\sqrt{3}}\right| < 1$$

$$-1 < -\frac{\operatorname{tg} x}{\sqrt{3}} < 1$$

$$\begin{cases} -\dfrac{\operatorname{tg} x}{\sqrt{3}} + 1 > 0 \\[2ex] \dfrac{\operatorname{tg} x}{\sqrt{3}} + 1 > 0 \end{cases} \Rightarrow$$

$$x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$$

g) (R) $\sum_{n=1}^{\infty} \underbrace{\frac{1!+2!+...+n!}{(n+2)!}}_{x_n}$;

$$\lim_{n\to\infty} x_n \overset{\text{Dacă } \exists}{\underset{CS}{=}} \lim_{n\to\infty} \frac{(n+1)!}{(n+3)!-(n+2)!} =$$

$$\lim_{n\to\infty} \frac{\cancel{(n+1)!}}{(n+2)!(n+3-1)} =$$

$$\lim_{n\to\infty} \frac{1}{(n+2)^2} = 0$$

$$\frac{x_{n+1}}{x_n} = \frac{1!+2!+...+n!+(n+1)!}{(n+3)!} \cdot \frac{\cancel{(n+2)!}}{1!+2!+...+n!}$$

$$= \frac{1!+2!+...+n!+(n+1)!}{1!+2!+...+n!} \cdot \frac{1}{n+3}$$

$$\frac{1!+2!+\dots+n!+(n+1)!}{(n+1)!} \cdot \frac{n!}{1!+2!+\dots+n!}$$

$$\cdot \frac{n+1}{n+3} \to 1$$

$$n\left(\frac{x_n}{x_{n+1}}-1\right)=$$

$$n\left((n+3)\cdot\frac{1!+2!+\dots+n!}{1!+2!+\dots+n!+(n+1)!}-1\right)=$$

$$n\left(\frac{(n+3)(1!+2!+\dots+n!)-(1!+2!+\dots+n!+(n+1)!)}{1!+2!+\dots+n!+(n+1)!}\right)=$$

$$n\left(\frac{(n+2)(1!+2!+\dots+n!)-(n+1)!}{1!+2!+\dots+n!+(n+1)!}\right)=$$

$$n\left(\frac{(n+2)(1!+2!+\dots+(n-1)!+n!(n+2)-(n+1)!}{1!+2!+\dots+n!+(n+1)!}\right)$$

$$= \frac{n(n+2)\left(1!+2!+\dots+(n-1)!\right)}{1!+2!+\dots+(n+1)!} +$$

$$n \frac{n!\cancel{(n+1)} + n! - \cancel{(n+1)}!}{1!+2!+\dots+(n+1)!} =$$

$$= \frac{n\left[(n+2)\left(1!+2!+\dots+(n-1)!\right) + n!\right]}{(n+1)!} \cdot$$

$$\frac{(n+1)!}{1!+2!+\dots+(n+1)!} =$$

$$\left(\frac{n+2}{n+1} \cdot \frac{1!+2!+\dots+(n-1)!}{(n-1)!} + \frac{n}{n+1}\right)$$

$$\cdot \frac{(n+1)!}{1!+2!+\dots+(n+1)!}$$

$\to 1$

$\to 2$

$\to 1$

1

$$\lim_{n\to\infty} \frac{1!+2!+\dots+(n-1)!}{(n-1)!} \quad \underline{\underline{\text{Dacă } \exists}}$$

$$\lim_{n\to\infty} \frac{n!}{n!-(n-1)!} = \lim_{n\to\infty} \frac{n!}{\cancel{(n-1)!}\cdot(n-1)} =$$

$$\lim_{n\to\infty} \frac{n}{n-1} = 1$$

$\Rightarrow \sum C$

(R) $\sum_{n=0}^{\infty} \underbrace{\frac{a^n+\operatorname{sh} n}{3^n}\cdot b^n}_{x_n}, a,b\in\mathbb{R};$

$$x_n = \frac{a^n + \frac{e^n - e^{-n}}{2}}{3^n}\cdot b^n$$

$$x_n = \left(\frac{a\cdot b}{3}\right)^n + \frac{1}{2}\left(\frac{e\cdot b}{3}\right)^n - \frac{1}{2}\left(\frac{b}{3e}\right)^n$$

$$s_n = \sum_{k=0}^{n} \left(\frac{a \cdot b}{3}\right)^k + \frac{1}{2} \sum_{k=0}^{n} \left(\frac{e \cdot b}{3}\right)^k -$$

$$\frac{1}{2} \sum_{k=0}^{n} \left(\frac{b}{3e}\right)^k =$$

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} \quad \text{pt } x \neq 1$$

pt $ab \neq 3$ $\quad eb \neq 3$ $\quad b \neq 3e$

$$\frac{\left(\frac{a \cdot b}{3}\right)^{n+1} - 1}{\frac{ab}{3} - 1} + \frac{1}{2} \frac{\left(\frac{eb}{3}\right)^{n+1} - 1}{\frac{eb}{3} - 1} -$$

$$\frac{1}{2} \frac{\left(\frac{b}{3e}\right)^{n+1} - 1}{\frac{b}{3e} - 1}$$

Dacă $\frac{ab}{3} < 1 \iff ab < 3$

$\frac{eb}{3} < 1 \iff eb < 3$ +

$b < \frac{3}{e} < 3e$

$$\to \frac{-1}{\frac{a\ell}{3}-1} + \frac{1}{2}\frac{-1}{\frac{e\ell_0}{3}-1} -$$

$$\frac{1}{2}\frac{-1}{\frac{\ell_e}{3e}-1}$$