Logic(s) for Computer Science - Week 10
The Semantics of First-Order Logic
Tutorial Exercises

December 4, 2018

All exercises work over the signature Σ, the Σ-structures S_1, S_2, S_3, S_4, S_5 and the S_1-assignments α_1, α_2 defined in the lecture notes.

1. Mark the free occurrences and respectively the bound occurrences of the variables in the formulae below:

   (a) $\phi_1 \triangleq (\forall x.P(x, x) \land P(x, y)) \land P(x, z)$;

   (b) $\phi_2 \triangleq (\forall x.P(f(x, x), i(x)) \land \exists y.(P(x, y) \land P(x, z)))$.

2. Identify the scope of the quantifiers in the formulae φ_1 and φ_2 from Exercise 1.

3. Compute the variables, the free variables and respectively the bound variables in the formulae φ_1 and φ_2 from Exercise 1.

4. Establish whether:

   (a) $S_1, \alpha_1 \models P(x_2, x_3)$;

   (b) $S_1, \alpha_1 \models \neg P(x_2, x_3)$;

   (c) $S_1, \alpha_1 \models \neg P(x_2, x_3) \land P(x_1, x_1)$;

   (d) $S_1, \alpha_1 \models \exists x_3.P(x_2, x_3)$;

   (e) $S_1, \alpha_1 \models \forall x_2.\exists x_3.P(x_2, x_3)$;

   (f) $S_1, \alpha_1 \models \exists x_3.\forall x_2.P(x_2, x_3)$;

   (g) $S_1, \alpha_2 \models \forall x_2.\exists x_3.P(x_2, i(x_3))$;

5. Find, for each of the items below, an S_2-assignment α_3 such that:

   (a) $S_2, \alpha_3 \models P(x_1, x_2)$;

   (b) $S_2, \alpha_3 \models P(f(x_1, x_2), x_3)$;

   (c) $S_2, \alpha_3 \models P(f(x_1, x_2), i(x_3))$;

(d) $S_2, \alpha_3 \models P(x, e)$;

(e) $S_2, \alpha_3 \models \exists y.P(x, i(y))$;

(f) $S_2, \alpha_3 \models \forall y.P(x, i(y))$.

6. Show that the following formulae are valid in S_2:

   (a) $\forall x.\exists y.P(x, i(y))$;

   (b) $\forall x.P(f(x, e), x)$;

   (c) $\forall x.P(x, i(i(x)))$.

7. Show that the formula $\forall x.\exists y.P(x, i(y))$ is not valid in S_3.

8. Find a formula that is satisfiable in S_1 but not in S_3.

9. Find a formula without free variables that is satisfiable in S_5 but not in S_4.

10. Show that the formula $\forall x.\exists y.P(x, y)$ is not valid (use Definition 3.8 from the lecture notes).

11. Show that the formula $(\forall x.P(x, x)) \rightarrow \exists x_2.P(x_1, x_2)$ is valid (use Definition 3.8 from the lecture notes).

12. Show that the formula $\forall x.\exists y.P(x, y)$ is not valid (use Definition 3.8 from the lecture notes).

13. Show that the formula $\forall x.\neg P(x, x)$ is satisfiable (use Definition 3.7 from the lecture notes).

14. Show that the formula $\forall x.\neg P(x, x) \wedge \exists x.P(x, x)$ is not satisfiable (use Definition 3.7 from the lecture notes).