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ALGEBRAIC FOUNDATIONS of COMPUTER SCIENCE
lectures by Ferucio Laurentiu Tiplea

Computational Introduction to Number Theory

Part I

Prof.dr. Ferucio Laurențiu Țiplea

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Department of Computer Science
"Alexandru Ioan Cuza" University of Iași
Iași 700506, Romania

e-mail: ferucio.tiplea@uaic.ro

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Divisibility. Prime numbers

The division theorem

The absolute value of an integer a, denoted $|a|$, is defined by:

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{otherwise.} \end{cases}$$

Theorem 1 (The Division Theorem)

For any two integers a and b with $b \neq 0$, there are unique integers q and r such that $a = bq + r$ and $0 \leq r < |b|$.

Proof.

See textbook [1], page 157-158. □

In the equality $a = bq + r$ in the division theorem, a is called the dividend, b is called the divisor, q is called the quotient, and r is called the remainder. We usually write:

q=a div b and r=a mod b

Divisibility relation

Definition 2

The binary relation $| \subseteq \mathbb{Z} \times \mathbb{Z}$ given by

$$a|b \quad \Leftrightarrow \quad (\exists c \in \mathbb{Z})(b = ac)\,,$$

for any $a, b \in \mathbb{Z}$, is called the divisibility relation on $\mathbb{Z}$.

If $a|b$ then we will say that a divides b, or a is a divisor/factor of b, or b is divisible by a, or b is a multiple of a.

Remark 3

If $a \neq 0$, then $a|b$ iff $b \bmod a = 0$.

If $a|b$ and $a \notin \{-1, 1, -b, b\}$, then a is called a proper divisor of b.

Basic properties of divisibility

Proposition 4

Let $a, b, c \in \mathbb{Z}$. Then:

1. *0 divides only 0;*
2. *a divides 0 and a;*
3. *1 divides a;*
4. *$a|b$ iff $a|-b$;*
5. *if $a|b$ and $b|c$, then $a|c$;*
6. *if $a|b+c$ and $a|b$, then $a|c$;*
7. *if $a|b$, then $ac|bc$. Conversely, if $c \neq 0$ and $ac|bc$, then $a|b$;*
8. *if $a|b$ and $a|c$, then $a|\beta b + \gamma c$, for any $\beta, \gamma \in \mathbb{Z}$;*
9. *if $a|b$ and $b \neq 0$, then $|a| \leq |b|$. Moreover, if a is a proper divisor of b, then $1 < |a| < |b|$.*

The proof is left as an exercise (see also textbook [1], page 158-159).

Prime numbers

Definition 5

A natural number $n \geq 2$ is called prime if the only positive factors of n are 1 and n. A natural number $n \geq 2$ that is not a prime is called composite.

Example 6

2, 3, 5, 7, 11 are prime numbers; 4, 6, 9 are composite.

Definition 7

Let $a_1, \ldots, a_m \in \mathbb{Z}$, where $m \geq 2$. We say that $a_1, \ldots, a_m$ are co-prime or relatively prime, denoted $(a_1, \ldots, a_m) = 1$, if the only common factors of these numbers are 1 and -1.

Example 8

$(0, 1) = 1$ (0 and 1 are co-prime) and $(4, 6, 8) \neq 1$ (4, 6, and 8 are not co-prime).

Characterization of co-prime numbers

Theorem 9

Let $a_1, \ldots, a_m \in \mathbb{Z}$, where $m \geq 2$. Then, $(a_1, \ldots, a_m) = 1$ iff there are $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}$ such that $\sum_{i=1}^{m} \alpha_i a_i = 1$.

Proof.

See textbook [1], page 159-160. □

Example 10

$1 \cdot 2 + 2 \cdot 3 + (-1) \cdot 7 = 1$, and so $(2, 3, 7) = 1$.

Characterization of co-prime numbers

Corollary 11

Let $a_1, \ldots, a_m, b \in \mathbb{Z}$, where $m \geq 2$. Then:

1. *if $(b, a_i) = 1$, for any i, then $(b, a_1 \cdots a_m) = 1$;*
2. *if $a_1, \ldots, a_m$ are pairwise co-prime and $a_i | b$, for any i, then $a_1 \cdots a_m | b$;*
3. *if $(b, a_1) = 1$ and $b | a_1 \cdots a_m$, then $b | a_2 \cdots a_m$;*
4. *if b is prime and $b | a_1 \cdots a_m$, then there exists i such that $b | a_i$.*

Proof.

See textbook [1], page 160-161. □

The fundamental theorem of arithmetic

Theorem 12 (The Fundamental Theorem of Arithmetic)

Every natural number $n \geq 2$ can be written uniquely in the form

$$n = p_1^{e_1} \cdots p_k^{e_k},$$

where $k \geq 1$, $p_1, \ldots, p_k$ are distinct prime numbers written in increasing order, and $e_1, \ldots, e_k > 0$.

Proof.

See textbook [1], page 161-162. □

Example 13

- $4 = 2^2$, $9 = 3^2$, $12 = 2^2 \cdot 3$, $36 = 2^2 \cdot 3^2$.
- $105 = 3 \cdot 5 \cdot 7$.

The prime number theorem

Theorem 14

There are infinitely many primes.

Proof.

See textbook [1], page 162. □

Theorem 15 (The Prime Number Theorem)

Let $\pi(n) = |\{p | p \text{ is a prime and } p \leq n\}|$. Then,

$$\lim_{n \to \infty} \frac{\pi(n)}{\frac{n}{\ln n}} = 1.$$

We write $\pi(n) \sim \frac{n}{\ln n}$ and say that $\pi(n)$ and $\frac{n}{\ln n}$ are asymptotically equivalent.

Values of $\pi(n)$

A few values of $\pi(n)$:

n	10^1	10^2	10^3	10^4	10^5	10^6	10^7
$\pi(n)$	4	25	168	1229	9592	78496	664579

How many 100-digit primes are there?

$$\begin{aligned}
\pi(10^{100}) - \pi(10^{99}) &\approx \frac{10^{100}}{100 \ln 10} - \frac{10^{99}}{99 \ln 10} \\
&= \frac{10^{99}}{ln\, 10}\left(\frac{1}{10} - \frac{1}{99}\right) \\
&> 0.39 \cdot 10^{98} \\
&\approx 4 \cdot 10^{97}
\end{aligned}$$

Large numbers

How large is 10^{97}? Below are a few interesting estimates and comparisons:

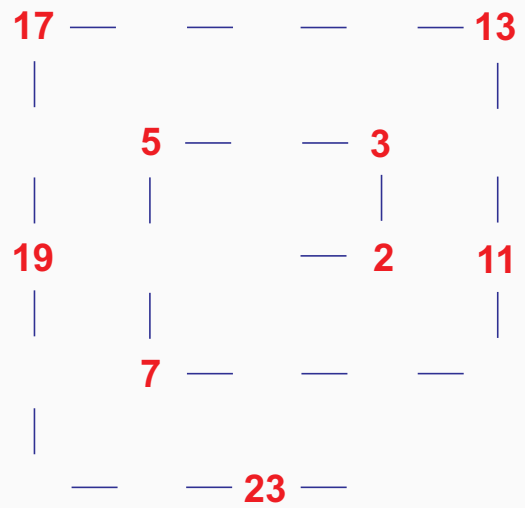
- the number of cells in the human body is estimated at 10^{14};
- the number of neuronal connections in the human brain is estimated at 10^{14};
- the universe is estimated to be $5 \cdot 10^{17}$ seconds old;
- the total number of particles in the universe has been variously estimated at numbers from 10^{72} up to 10^{87}.

Very large numbers often occur in fields such as mathematics, cosmology and cryptography. They are particularly important to cryptography where security of cryptosystems (ciphers) is usually based on solving problems which require, say, 2^{128} operations (which is about what would be required to break the 128-bit SSL commonly used in web browsers).

The prime spiral

There is no known formula for generating prime numbers in a row which is more efficient than the ancient sieve of Eratosthenes or the modern sieve of Atkin.

The Ulam spiral (or prime spiral), discovered by Stanislaw Ulam in 1963, is a simple method of graphing the prime numbers.



The prime numbers tend to line up along diagonal lines!

The greatest common divisor

The greatest common divisor

Definition 16

Let $a_1, \ldots, a_m \in \mathbb{Z}$, not all zero, where $m \geq 2$. The greatest common divisor of these numbers, denoted $gcd(a_1, \ldots, a_m)$ or $(a_1, \ldots, a_m)$, is the largest integer d such that $d|a_i$, for all i.

Proposition 17

Let $a_1, \ldots, a_m \in \mathbb{Z}$, not all zero, where $m \geq 2$. Then:

1. $(0, a_1, \ldots, a_m) = (a_1, \ldots, a_m)$;
2. $(0, a_1) = |a_1|$, *provided that* $a_1 \neq 0$;
3. $(a_1, a_2) = (a_2, a_1 \ mod \ a_2)$, *provided that* $a_2 \neq 0$.

Proof.

See textbook [1], page 164. □

Linear combination of the gcd

Theorem 18

Let $a_1, \ldots, a_m \in \mathbb{Z}$, not all zero, where $m \geq 2$. Then,

$$(a_1, \ldots, a_m) = \alpha_1 a_1 + \cdots + \alpha_m a_m$$

for some $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}$.

Proof.

See textbook [1], page 164. □

Linear combination of the gcd

Corollary 19

Let $a_1, \ldots, a_m \in \mathbb{Z}$, not all zero, where $m \geq 2$. Then, the equation

$$a_1 x_1 + \cdots + a_m x_m = b$$

has solutions in $\mathbb{Z}$ iff $(a_1, \ldots, a_m) | b$.

Proof.

See textbook [1], page 165. □

Example 20

1. $2x + 3y = 5$ has solutions in $\mathbb{Z}$ because $(2, 3) = 1$ divides 5;
2. $4x + 2y = 3$ does not have solutions in $\mathbb{Z}$ because $(4, 2) = 2$ does not divide 3.

The least common multiple

Definition 21

Let $a_1, \ldots, a_m \in \mathbb{Z}$, where $m \geq 2$. The least common multiple of these numbers, denoted $lcm(a_1, \ldots, a_m)$ or $[a_1, \ldots, a_m]$, is

- 0, if at least one of these numbers is 0;
- the smallest integer $b > 0$ such that $a_i | b$, for all i, otherwise.

Theorem 22

Let $a, b \in \mathbb{N}$, not both zero. Then, $ab = (a, b)[a, b]$.

Proof.

See textbook [1], page 166. □

The Euclidean algorithm

The Euclidean Algorithm

If $a = 0$ or $b = 0$, but not both zero, then $(a, b) = max\{|a|, |b|\}$.

Let $a > b > 0$ and

$$\begin{array}{lcll} r_{-1} & = & r_0 q_1 + r_1, & 0 < r_1 < r_0 \\ r_0 & = & r_1 q_2 + r_2, & 0 < r_2 < r_1 \\ \cdots & & & \\ r_{n-2} & = & r_{n-1} q_n + r_n, & 0 < r_n < r_{n-1} \\ r_{n-1} & = & r_n q_{n+1} + r_{n+1}, & r_{n+1} = 0, \end{array}$$

where $r_{-1} = a$ și $r_0 = b$. Then,

$$(a, b) = (r_{-1}, r_0) = (r_0, r_1) = \cdots = (r_n, 0) = r_n$$

The Euclidean algorithm

Algorithm 1: Computing gcd

input : $a, b \in \mathbb{Z}$ *not both* 0;
output: $gcd(a, b)$;

```
1 begin
2     while b ≠ 0 do
3         r := a mod b;
4         a := b;
5         b := r
6     gcd(a, b) := |a|;
```

Theorem 23 (Lamé, 1844)

Let $a \geq b > 0$ be integers. The number of division steps performed by Algorithm 1 on (a, b) does not exceed 5 times the number of decimal digits in b.

Proof.

See textbook [1], page 168. □

The extended Euclidean algorithm

The Euclidean algorithm can be easily adapted to compute a linear combination of the gcd as well. The resulting algorithm is called the Extended Euclidean Algorithm.

Given a and b there are α and β such that $(a, b) = \alpha a + \beta b$. The numbers α and β can be computed as follows:

				$V_{-1} = (1, 0)$
				$V_0 = (0, 1)$
1.	a	$=$	$bq_1 + r_1$	$V_1 = V_{-1} - q_1 V_0$
2.	b	$=$	$r_1 q_2 + r_2$	$V_2 = V_0 - q_2 V_1$
3.	r_1	$=$	$r_2 q_3 + r_3$	$V_3 = V_1 - q_3 V_2$
	$\dots$			
n.	r_{n-2}	$=$	$r_{n-1} q_n + r_n$	$V_n = V_{n-2} - q_n V_{n-1}$
$n+1$.	r_{n-1}	$=$	$r_n q_{n+1}$	

The extended Euclidean algorithm

Algorithm 2: Computing gcd and a linear combination of it

input : $a, b \in \mathbb{Z}$ *not both* 0;

output: $gcd(a, b)$ *and* $V = (\alpha, \beta)$ *s.t.* $gcd(a, b) = \alpha a + \beta b$;

```
1  begin
2      V_0 := (1, 0);
3      V_1 := (0, 1);
4      while b ≠ 0 do
5          q := a div b, r := a mod b;
6          a := b, b := r;
7          V := V_0;
8          V_0 := V_1;
9          V_1 := V − qV_1
10     gcd(a, b) := |a|;
11     V := V_0;
```

Linear Diophantine equations

The extended Euclidean algorithm can be used to compute integer solutions to linear Diophantine equations:

Algorithm 3: Computing solutions to linear Diophantine equations

input : $a, b, c \in \mathbb{Z}$ *such that not both* a *and* b *are* 0;
output: integer solution to $ax + by = c$, if it has;

```
1 begin
2     compute gcd(a, b) := αa + βb;
3     if gcd(a, b)|c then
4         c' := c/gcd(a, b);
5         x := αc';
6         y := βc'
7     else
8         "no integer solutions"
```

Congruences

Congruences

Definition 24

Let $a, b, m \in \mathbb{Z}$. We say that a is congruent to b modulo m, denoted $a \equiv_m b$ or $a \equiv b \ mod \ m$, if $m|(a-b)$.

Example 25

- $6 \equiv 0 \ mod \ 2$.
- $-7 \equiv 1 \ mod \ 2$.
- $3 \not\equiv 2 \ mod \ 2$.
- $-11 \equiv 1 \ mod \ -4$ and $-11 \equiv 1 \ mod \ 4$.

Remark 26

If $m \neq 0$, then $a \equiv b \ mod \ m$ iff $a \ mod \ m = b \ mod \ m$.

Basic properties of congruences

Proposition 27

Let $a, b, c, d, m, m' \in \mathbb{Z}$ and $f : \mathbb{Z} \to \mathbb{Z}$ be a polynomial function with integer coefficients. Then:

1. $\equiv_m$ *is an equivalence relation on* $\mathbb{Z}$;
2. *if* $a \equiv_m b$, *then* $(a, m) = (b, m)$;
3. *if* $a \equiv_m b$ *și* $c \equiv_m d$, *then* $a + c \equiv_m b + d$, $a - c \equiv_m b - d$, $ac \equiv_m bd$, *and* $f(a) \equiv_m f(b)$;
4. 4.1 *if* $ac \equiv_{mc} bc$ *and* $c \neq 0$, *then* $a \equiv_m b$;
   4.2 *if* $ac \equiv_m bc$ *and* $d = (m, c)$, *then* $a \equiv_{m/d} b$;
   4.3 *if* $ac \equiv_m bc$ *and* $(m, c) = 1$, *then* $a \equiv_m b$;
5. 5.1 *if* $a \equiv_{mm'} b$, *then* $a \equiv_m b$ *and* $a \equiv_{m'} b$;
   5.2 *if* $a \equiv_m b$ *and* $a \equiv_{m'} b$, *then* $a \equiv_{[m,m']} b$;
   5.3 *if* $a \equiv_m b$, $a \equiv_{m'} b$, *and* $(m, m') = 1$, *then* $a \equiv_{mm'} b$.

The proof is left as an exercise (see also textbook [1], page 172-173).

Residue classes of integers modulo m

Let $\mathbb{Z}_m$ be the set of all equivalence classes induced by $\equiv_m$. Then:

- $[a]_m = [a]_{-m}$, for any $a \in \mathbb{Z}$. Therefore, we may consider only $m \geq 0$;
- for any $a, b \in \mathbb{Z}$, if $a \neq b$ then $[a]_0 \neq [b]_0$. Therefore, $\mathbb{Z}_0$ has as many elements as $\mathbb{Z}$;
- for $m \geq 1$, $\mathbb{Z}_m = \{[0]_m, \ldots, [m-1]_m\}$ has exactly m elements.

Example 28

- $\mathbb{Z}_1 = \{[0]_1\}$, $\mathbb{Z}_2 = \{[0]_2, [1]_2\}$, $\mathbb{Z}_3 = \{[0]_3, [1]_3, [2]_3\}$.

Remark 29

We usually write $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ instead of $\mathbb{Z}_m = \{[0]_m, \ldots, [m-1]_m\}$, for any $m \geq 1$.

Addition and multiplication modulo m

Define the following operations on $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$:

- $a + b = (a + b) \ mod \ m$; (binary operation)
- $a \cdot b = (a \cdot b) \ mod \ m$; (binary operation)
- $-a = (m - a) \ mod \ m$, (unary operation)

for any $a, b \in \mathbb{Z}_m$.

These operations fulfill the following properties:

- $+$ and $\cdot$ are associative and commutative;
- $a + 0 = 0 + a = a$, for any a;
- $a \cdot 1 = 1 \cdot a = a$, for any a;
- $a + (-a) = 0$, for any a.

$a + (-b)$ is usually written $a - b$.

Inverses modulo m

- Additive inverse modulo m.

  We have seen that $a + (-a) = 0$, for any a. $-a$ is called the additive inverse of a modulo m (it is unique);

- Multiplicative inverse modulo m.
  - Given $a \in \mathbb{Z}_m - \{0\}$, is there any $b \in \mathbb{Z}_m$ such that $a \cdot b = 1$? That is, does any $a \in \mathbb{Z}_m$ have a multiplicative inverse modulo m?
  - Let us consider $m = 6$. There is no $b \in \mathbb{Z}_6$ such that $2 \cdot b = 1$.
  - Moreover, $\mathbb{Z}_6$ exhibits the following interesting property:

$$2 \cdot 3 = 0$$

(the product of two non-zero numbers is zero!).

Inverses modulo m and the group of units

Proposition 30

$a \in \mathbb{Z}_m$ has a multiplicative inverse modulo m iff $(a, m) = 1$.

Proof.

See textbook [1], page 175. □

The multiplicative inverse of a, when it exists, is unique and usually denoted a^{-1}.

$\mathbb{Z}_m^* = \{a \in \mathbb{Z}_m | (a, m) = 1\}$ is called the group of units of $\mathbb{Z}_m$ or the group of units modulo m.

Example 31

- $\mathbb{Z}_1^* = \{0\}$.
- $\mathbb{Z}_{26}^*$ has 12 elements: $1^{-1} = 1$, $3^{-1} = 9$, $5^{-1} = 21$, $7^{-1} = 15$, $11^{-1} = 19$, $17^{-1} = 23$, $25^{-1} = 25$.

Computing multiplicative inverses

The extended Euclidean algorithm can be easily used to compute multiplicative inverses modulo m:

Algorithm 4: Computing multiplicative inverses

input : $m \geq 1$ *and* $a \in \mathbb{Z}_m$;
output: a^{-1} *modulo* m, *if* $(a, m) = 1$;

```
1 begin
2     compute gcd(a, m) := αa + βm;
3     if gcd(a, m) = 1 then
4         a^{-1} := α mod m
5     else
6         "a^{-1} does not exist"
```

Euler's totient function

Euler's totient function

Euler's totient function: $\phi(m) = |\mathbb{Z}_m^*|$, for any $m \geq 1$.

Theorem 32

1. $\phi(1) = 1$;
2. $\phi(p) = p - 1$, *for any prime* p;
3. $\phi(ab) = \phi(a)\phi(b)$, *for any co-prime integers* $a, b \geq 1$;
4. $\phi(p^e) = p^e - p^{e-1}$, *for any prime* p *and* $e > 0$;
5. $\phi(n) = (p_1^{e_1} - p_1^{e_1-1}) \cdots (p_k^{e_k} - p_k^{e_k-1})$, *for any* $n \geq 1$, *where* $n = p_1^{e_1} \cdots p_k^{e_k}$ *is the prime decomposition of* n.

Proof.

See textbook [1], page 176-177. □

Euler's totient function: examples

Example 33

1. $\phi(5) = 4$.
2. $\phi(26) = \phi(2 \cdot 13) = 12$.
3. $\phi(245) = \phi(5 \cdot 7^2) = 168$.

Remark 34

- *It is easy to compute $\phi(n)$ if the prime decomposition of n is known.*
- *It is hard to compute the prime decomposition of large numbers (512-bit numbers (about 155 decimals) or larger).*
- *It is hard to compute $\phi(n)$ if n is large and the prime decomposition of n is not known.*

Euler's theorem

Theorem 35 (Euler's Theorem)

Let $m \geq 1$. Then, $a^{\phi(m)} \equiv 1$ mod m, for any integer a with $(a, m) = 1$.

Proof.

See textbook [1], page 177. □

Corollary 36 (Fermat's Theorem)

Let p be a prime. Then:

1. $a^{p-1} \equiv 1$ *mod* p, *for any integer* a *with* $p \nmid a$;
2. $a^p \equiv a$ *mod* p, *for any integer* a.

Proof.

See textbook [1], page 177. □

Example 37

$1359^4 \equiv 1$ *mod* 5 and $3^{168} \equiv 1$ *mod* 245.

Reading and exercise guide

Reading and exercise guide

It is highly recommended that you do all the exercises marked in red from the slides.

Course readings:

1. Pages 157-178 from textbook [1].

References

[1] Ferucio Laurențiu Țiplea. *Algebraic Foundations of Computer Science*. "Alexandru Ioan Cuza" University Publishing House, Iași, Romania, second edition, 2021.