S7.1 i) Fie $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ definită de

$$g_1(\mathbf{x}, \mathbf{y}) = x_1y_1 + 5x_2y_2 + x_3y_3 + x_1y_2 + 3x_1y_3 + x_2y_1 + x_2y_3 + 3x_3y_1 + x_3y_2; \qquad {}_2,$$

pentru $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$.

a) Arătaţi că aplicaţia g este o formă biliniară simetrică pe $\mathbb{R}^3$.

b) Găsiţi matricea lui g în raport cu baza canonică a lui $\mathbb{R}^3$. Determinaţi discriminantul lui g şi rang g.

c) Determinaţi $\mathrm{Ker}(g)$.

B

d) Găsiţi matricea lui g în raport cu baza $\{(1, 1, 1), (2, -1, 2), (1, 3, -3)\}$.

e) Scrieţi forma pătratică h corespunzătoare lui g şi stabiliţi o formă normală a lui h. Determinaţi signatura lui h şi deduceţi forma biliniară corespunzătoare formei normale a lui h.

f) Determinaţi o bază a lui $\mathbb{R}^3$ în raport cu care h are forma normală de mai sus. Caracterizaţi dintr-un punct de vedere geometric nucleul lui h.

Repetaţi acest exerciţiu pentru:

ii) $g_1(\mathbf{x}, \mathbf{y}) = x_1y_1 + 5x_2y_2 + x_3y_3 + x_1y_2 + 3x_1y_3 + x_2y_1 + x_2y_3 + 3x_3y_1 + x_3y_2$.

a) Fie $A_{g_1} = (a_{ij})$ a_{ij} = coef lui $x_i y_j$

$$A_{g_1} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$g_1(x, y) = x^T A_{g_1} y$$

$\Rightarrow g_1$ biliniară

$$A_{g_1}^T = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} = A_{g_1} \Rightarrow g_1 \text{ simetrică}$$

b) discriminant $g_1 = \det A_{g_1}$

$$\det A_{g_1} = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix} \quad \frac{C_2 = C_2 - C_1}{C_3 = C_3 - 3C_1}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & -2 \\ 3 & -2 & -8 \end{vmatrix} =$$

$$1 \begin{vmatrix} 4 & -2 \\ -2 & -8 \end{vmatrix} = -36$$

$$\text{rang } g_1 = \text{rang } A_{g_1} \underset{\det A_{g_1} \neq 0}{=} 3$$

c) $\text{Ker}(g_1) = \{(0,0,0)\}$

g_1 simetrică $\quad \text{Ker}(g_1) = \text{Ker}(g_1)_s = \text{Ker}(g_1)_d$

$\text{Ker } g_1 \quad A Y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\det A \neq 0 \Rightarrow \exists A^{-1}$

$$y = A^{-1}\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Alternativ

formă liniară

$\operatorname{Ker} g_1 = \operatorname{Ker}(A_{\varphi})$

$\operatorname{def}(A_{\varphi}) = \dim \operatorname{Ker} =$

$3 - \operatorname{rang} A = 0$

$\Rightarrow \operatorname{Ker} g_1 = \{(0,0,0)\}$

d) $A_{g_1}^{B} = S_{CB}^{T} \cdot A_{g_1} \cdot S_{CB}$

$$S_{CB} = \begin{pmatrix}1 & 2 & 1\\ 1 & -1 & 3\\ 1 & 2 & -3\end{pmatrix}$$

$B = \{(1,1,1), (2,-1,2), (1,3,-3)\}.$

$$S_{CB}^{T} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & -3 \end{pmatrix}$$

$$A_{g_1} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$$A_{g_1}^{B} = \begin{pmatrix} 17 & 13 & 11 \\ 13 & 29 & -17 \\ 11 & \underline{-17} & 25 \end{pmatrix}$$

e) $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$$h(x) = g_1(x, x)$$

$$g_1(\mathbf{x}, \mathbf{y}) = x_1y_1 + 5x_2y_2 + x_3y_3 + x_1y_2 + 3x_1y_3 + x_2y_1 + x_2y_3 + 3x_3y_1 + x_3y_2;$$

Met Gauss

$$h(x) = x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 6x_1x_3 + 2x_2x_3$$

$$h(x) = \underbrace{x_1^2}_{a^2} + \underbrace{2x_1}_{2a}\underbrace{x_2}_{b} + 2\underbrace{x_1}_{a}\underbrace{3x_3}_{c} + 5x_2^2 + x_3^2 + 2x_2x_3$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$h(x) = x_1^2 + 2x_1x_2 + 2x_1 3x_3 + (3x_3)^2 + x_2^2 + 6x_2x_3 - 9x_3^2 - x_2^2 - 6x_2x_3 + 5x_2^2 + x_3^2 + 2x_2x_3$$

$$h(x) = (x_1 + x_2 + 3x_3)^2 + 4x_2^2 - 8x_3^2 - 4x_2x_3$$

$$h(x) = (x_1 + x_2 + 3x_3)^2 + 4\left(x_2^2 - x_2x_3 + \frac{x_3^2}{4}\right) - \underbrace{4 \cdot \frac{x_3^2}{4}}_{x_3^2} - 8x_3^2 \qquad 2x_2 \cdot \frac{-x_3}{2}$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$h(x) = (x_1 + x_2 + \cancel{3}x_3)^2 + 4\left(x_2 - \frac{x_3}{2}\right)^2 - 9x_3^2$$

$$\begin{cases} y_1 = x_1 + x_2 + 3x_3 \\ y_2 = x_2 - \frac{x_3}{2} \\ y_3 = x_3 \end{cases} \qquad S_{CB}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$h(y) = y_1^2 + 4y_2^2 - 9y_3^2$$

$$\begin{cases} z_1 = y_1 \\ z_2 = 2y_2 \\ z_3 = 3y_3 \end{cases} \qquad S_{B'B''}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$h(z) = z_1^2 + z_2^2 - z_3^2$$

Met Jacobi

$$A_{g_1} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

$\Delta_1 = 1$

$\Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4$

$\Delta_3 = \det A_{g_1} = -36$ $\qquad \Delta_0 = 1$

$$h(y) = \frac{\Delta_0}{\Delta_1} y_1^2 + \frac{\Delta_1}{\Delta_2} y_2^2 + \frac{\Delta_2}{\Delta_3} y_3^2$$

$$\det\left(A_{g_1} - \lambda I_3\right) = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} =$$

$$-\lambda^3 + 7\lambda^2 - 36$$

$$-\lambda^3 + 7\lambda^2 - 36 = 0$$

	λ^3	λ^2	λ	1
	-1	7	0	-36
-2	-1	$2+7=9$	$-2\cdot 9+0=-18$	$-2\cdot -18+(-36)=0$

$$-\lambda^3 + 7\lambda^2 - 36 = (-\lambda^2 + 9\lambda - 18)(\lambda+2)$$

$$-(\lambda^2 - 9\lambda + 18)(\lambda+2)$$

$$\lambda_{1,2} = \frac{9 \pm \sqrt{81-72}}{2} = \frac{9 \pm 3}{2} \begin{cases} 3 \\ 6 \end{cases}$$

$\lambda_1 = 3 \quad \lambda_2 = 6 \quad , \lambda_3 = -2$

$$\Rightarrow D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad h(y) = 3y_1^2 + 6y_2^2 - 2y_3^2$$

$$h(z) = z_1^2 + z_2^2 - z_3^2$$

Signatura lui $h = (2,1,0)$

$$\tilde{g}(x,y) = \frac{1}{2}\left[h(x+y) - h(x) - h(y)\right]$$

$$\tilde{g}(x,y) = \frac{1}{2}\left[(x_1+y_1)^2 + (x_2+y_2)^2 - (x_3+y_3)^2 - (x_1^2 + x_2^2 - x_3^2) - (y_1^2 + y_2^2 - y_3^2)\right] =$$

$$\frac{1}{2}(x_1^2 + y_1^2 + 2x_1y_1 + x_2^2 + y_2^2 + 2x_2y_2 - x_3^2 - y_3^2 - 2x_3y_3$$

$$-x_1^2 - x_2^2 + x_3^2$$

$$-y_1^2 - y_2^2 + y_3^2) =$$

$$x_1y_1 + x_2y_2 - x_3y_3$$

$$S_{CB'}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_{B'B''}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$S_{CB''} = ?$$

$$S_{CB''} = S_{CB'} \; S_{B'B''} \Big)^{-1}$$

$$\left(S_{CB''}\right)^{-1} = \left(S_{B'B''}\right)^{-1} \cdot \left(S_{CB'}\right)^{-1}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3 = \frac{1}{3} L_3]{L_2 = \frac{1}{2} L_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right) \quad L_1 = L_1 - L_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right) \quad L_2 = L_2 + \frac{L_3}{2}$$

$$\begin{pmatrix} 1 & 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$L_1 = L_1 - \frac{3}{2} L_3$$

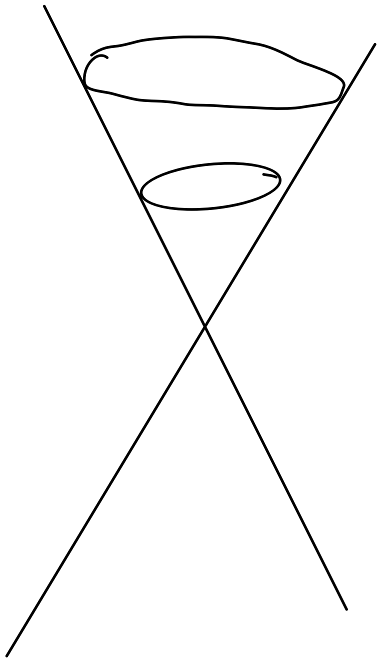
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right)$$

$$B'' = \left\{ (1, 0, 0), \left(-\frac{1}{2}, \frac{1}{2}, 0\right), \left(-\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right) \right\}$$

$$q(z) = z_1^2 + z_2^2 - z_3^2$$

$$z_1^2 + z_2^2 = z_3^2$$

$\to$ cono circular



a) $f(x, y, z) = xy + xz + yz$, pentru $xyz = 1$, $x > 0$, $y > 0$, $z > 0$;

Găsiți extremele ariei cutiei de volum 1

$g(x,y,z) = xyz - 1$

$L(x,y,z;\lambda) = f(x,y,z) + \lambda g(x,y,z)$

Deriv parțiale $= xy + xz + yz + \lambda xyz - \lambda$

$\frac{\partial L}{\partial x}(x,y,z;\lambda) = y + z + \lambda yz$

$\frac{\partial L}{\partial y}(x,y,z;\lambda) = x + z + \lambda xz$

$$\frac{\partial L}{\partial z}(x,y,z;\lambda) = x+y+\lambda xy$$

$$\frac{\partial L}{\partial \lambda}(x,y,z;\lambda) = g(x,y,z) = xyz-1$$

Gradient

$$\nabla L(x,y,z;\lambda) = (0,0,0,0)$$

$$\begin{cases} y+z+\lambda yz = 0 \quad \Big|\cdot x \\ x+z+\lambda xz = 0 \quad \Big|\cdot y \\ x+y+\lambda xy = 0 \\ g(x,y,z) = xyz-1 = 0 \end{cases}$$

$\ominus$ $y-x+z\lambda(y-x)=0$

$\cdot y$ $(y-x)(1+z\lambda)=0$

$$\boxed{\begin{aligned} &yx+xz+\lambda = 0 \\ &xy+yz+\lambda = 0 \;\ominus \\ &x=y \end{aligned}}$$

I $x=y$

$$2x+\lambda x^2 = 0$$

$$x(2+\lambda x) = 0 \Rightarrow 2+\lambda x = 0$$

$$\lambda = -\frac{2}{x}$$

$$x + z + -\frac{2}{\cancel{x}} \cdot \cancel{x} \cdot z = 0$$

$$x + z - 2z = 0 \Rightarrow x - z = 0$$

$$x = z$$

$$x^3 - 1 = 0 \Rightarrow x = y = z = 1$$

$$\lambda = -2$$

$$\text{II} \quad 1 + z\lambda = 0 \Rightarrow$$

$$\lambda = -\frac{1}{z}$$

$$x + z + -\frac{1}{\cancel{z}} \cdot x \cdot \cancel{z} = 0$$

$$z = 0 \quad F$$

Pct critic $(1, 1, 1, -2)$

Studiem dacă $d^2L(1,1,1,-2)$ ← formă pos/neg def pătratică

$$H_L(x,y,z;-2)=$$

Derivate parțiale

$$\frac{\partial L}{\partial x}(x,y,z;\lambda)=y+z+\lambda yz$$

$$\frac{\partial L}{\partial y}(x,y,z;\lambda)=x+z+\lambda xz$$

$$\frac{\partial L}{\partial z}(x,y,z;\lambda)=\underline{x+y+\lambda xy}$$

$$\frac{\partial L}{\partial \lambda}(x,y,z;\lambda)=g(x,y,z)=xyz-1$$

$$\frac{\partial^2 L}{\partial x^2}(x,y,z;\lambda)=0$$

$$\frac{\partial^2 L}{\partial y^2}(x,y,z;\lambda)=0$$

$$\frac{\partial^2 L}{\partial z^2}(x,y,z;\lambda)=0$$

L polinom $\to$ deriv mixte egale

$$\frac{\partial^2 L}{\partial y \partial x}(x, y, z; \lambda) = 1 + \lambda z$$

$$\frac{\partial^2 L}{\partial y \partial z}(x, y, z; \lambda) = 1 + \lambda x$$

$$\frac{\partial^2 L}{\partial x \partial z}(x, y, z; \lambda) = 1 + \lambda y$$

$$H_L^2(1,1,1;-2) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$d^2L(1,1,1,-2) = -2(dx\,dy + dy\,dz + dx\,dz) =$$

$$dx^2 + dy^2 + dz^2 \geq 0$$

cu o pt $dx = dy = dz$

$$xyz = 1$$

$$yz\,dx + xz\,dy + xy\,dz = 0$$

Pct critic $\quad dx + dy + dz = 0$

$$(dx + dy + dz)^2 = 0$$

$$dx^2+dy^2+dz^2+$$
$$2(dx\,dy+dy\,dz+$$
$$dz\,dx)=0$$

$$-2(dx\,dy+dy\,dz+$$
$$dz\,dx)=$$
$$dx^2+dy^2+dz^2$$

Pos. def $\Rightarrow$ $(1,1,1)$ pct de minim local

Fie fct.

$$f(x,y,z)=\begin{cases}\dfrac{xyz^2}{x^2+y^2+z^4} & (x,y,z)\neq(0,0,0)\\[2ex] 0, & (x,y,z)=(0,0,0)\end{cases}$$

→ limite parțiale în $(0,0,0)$

$$\lim_{x\to 0} f(x,0,0) = \lim_{x\to 0} \frac{x\cdot 0\cdot 0}{x^2+0+0} =$$

$$\lim_{x\to 0} 0 = 0$$

→ limita globală a lui f în $(0,0,0)$

• limita direcțională $(u,v,w) \neq (0,0,0)$

$f(tu, tv, tw)$

$$l = \lim_{t\to 0} \frac{tu\, tv\, t^2w^2}{t^2u^2+t^2v^2+t^4w^4}$$

$$\lim_{t\to 0} \frac{\cancel{t^4}^{\,t^2} uvw^2}{\cancel{t^2}(u^2+v^2+t^2w^4)}$$

Dacă $u^2+v^2 \neq 0$ $\quad l = 0$

Dacă $u^2+v^2 = 0 \Rightarrow u = v = 0$

$\Rightarrow w \neq 0$

$$l = \lim_{t\to 0} \frac{0}{\underbrace{t^4w^4}_{\neq 0}} = 0$$

$$\frac{|xyz^2|}{x^2+y^2+z^4} = \frac{\sqrt[3]{|xyz^2|^2}}{x^2+y^2+z^4}\cdot\sqrt[3]{|xy|z^2} \le$$

$$\left[\sqrt[3]{|x||y||z^2|} \le \sqrt{\frac{x^2+y^2+z^4}{3}}\right]^2$$

$$\sqrt[3]{|xy|z^2}^{\,2} \le \frac{x^2+y^2+z^4}{3}$$

$$\frac{\sqrt[3]{|xyz^2|}^{\,2}}{x^2+y^2+z^4} \le \frac{1}{3}$$

$$\le \frac{1}{3}\sqrt[3]{|xy|z^2} \longrightarrow 0$$

$$\Rightarrow \lim_{(x,y,z)\to(0,0,0)} \frac{xyz^2}{x^2+y^2+z^4} = 0$$

$$f(x,y,z) = \frac{xyz^2}{x^2+y^2+z^4} \quad (x,y,z) \neq (0,0,0)$$

pt $(x,y,z) \neq (0,0,0)$

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{yz^2(x^2+y^2+z^4) - xyz^2 \cdot 2x}{(x^2+y^2+z^4)^2}$$

$$= \frac{\cancel{x^2yz^2} + y^3z^2 + yz^6 - \cancel{2}x^2yz^2}{(x^2+y^2+z^4)^2}$$

$$= \frac{yz^2(-x^2+y^2+z^4)}{(x^2+y^2+z^4)^2}$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{xz^2(-y^2+x^2+z^4)}{(x^2+y^2+z^4)^2}$$

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{2x^3yz + 2xy^3z - 2xyz^5}{(x^2+y^2+z^4)^2}$$

$$= \frac{2xyz(x^2+y^2-z^4)}{(x^2+y^2+z^4)^2}$$

$$\frac{\partial f}{\partial x}(0,0,0) = \lim_{x \to 0} \frac{f(x,0,0) - f(0,0,0)}{x} =$$

$$\lim_{x \to 0} \frac{0-0}{x} = 0$$

$$\nabla f(x,y,z) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\nabla f(x,y,z) = \begin{cases} (0,0,0) & \text{pt } (x,y,z) = (0,0,0) \\ \left(\frac{yz^2(-x^2+y^2+z^4)}{(x^2+y^2+z^4)^2}, \frac{xz^2(-y^2+x^2+z^4)}{(x^2+y^2+z^4)^2}, \frac{2xyz(x^2+y^2-z^4)}{(x^2+y^2+z^4)^2} \right) & \text{pt } (x,y,z) \neq (0,0,0) \end{cases}$$