Metode iterative – exerciții

$Ax = b$, $\det A \neq 0$,

x^* soluția exactă

$A = B - C$, $\det B \neq 0$

B – "ușor" inversabilă

$x^{(0)}$ – dat $\quad x^{(k+1)} = Mx^{(k)} + d$

M matricea iterației $\quad M = B^{-1}C$

$d = B^{-1}b$

Convergența: $x^{(k)} \longrightarrow x^*$?

$$x^{(k)} \longrightarrow x^* \iff \rho(M) < 1$$

$\rho(M) = \max\{|\lambda| ; \lambda$ - valoare proprie a matricei $M\}$

dacă $\exists \|\cdot\|$ a.î. $\|M\| < 1 \implies$

$$x^{(k)} \longrightarrow x^*$$

$\|\cdot\|$ normă matricială naturală

Metoda Jacobi $\quad B = \operatorname{diag} A$

Metoda Gauss-Seidel

$$A = L + D + U$$

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ a_{31} & a_{32} & \cdots & 0 \\ \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{pmatrix} \qquad U = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots \\ 0 & 0 & \cdots & & 0 \end{pmatrix}$$

$$D = \operatorname{diag} A$$

$$B = L + D$$

Metodele relaxării succesive
(se pot aplica și pe matrici oarecare)

$$B = L + \frac{1}{\omega} D \quad , \omega \in (0,2)$$

Pb: Fie sistemul liniar:

$$\begin{cases} x_1 + \quad + x_3 = 2 \\ 4x_2 + 2x_3 = 6 \\ x_2 + x_3 = 2 \end{cases}$$

Să se calculeze matricea iterației M cu metoda Jacobi / Gauss-Seidel / relaxării succesive cu $\omega = 0.5$. Să se studieze convergența metodei respective. Pentru $x^{(0)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, să se calculeze $x^{(1)}$.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad \det A = 2 \neq 0, \; a_{ii} \neq 0 \; \forall i$$

$$b = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$$

<u>Metoda Jacobi</u>:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M = B^{-1}C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1/2 \\ 0 & -1 & 0 \end{pmatrix}$$

Convergența: $\rho(M) < 1$?

Valori proprii pentru M: $\det(\lambda I - M) = 0$

$$\det(\lambda I - M) = \begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 1/2 \\ 0 & 1 & \lambda \end{vmatrix} = \lambda\left(\lambda^2 - \frac{1}{2}\right) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = \frac{1}{\sqrt{2}} \quad \lambda_3 = -\frac{1}{\sqrt{2}} \Rightarrow \rho(M) = \frac{1}{\sqrt{2}}$$

$\rho(M) < 1 \Rightarrow$ metoda Jacobi construiește un șir convergent la soluție.

$x^{(1)}$:

$$d = B^{-1} b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3/2 \\ 2 \end{pmatrix}$$

$$M x^{(0)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1/2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -3/2 \\ -2 \end{pmatrix}$$

$$x^{(1)} = M x^{(0)} + d = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Metoda Gauss-Seidel

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\det B = 4 \neq 0$

B^{-1}: coloanele lui B^{-1} se calculează rezolvând sistemele $Bx = e_k$, $k = 1, 2, 3$ (ca la Tema 2)

se pot fol. și euristica la rez. sist.

x_1	$=$	1	$x_1 = 1$	0	$x_1 = 0$	0	$x_1 = 0$
$4x_2$	$=$	0	$x_2 = 0$	1	$x_2 = 1/4$	0	$x_2 = 0$
$x_2 + x_3$	$=$	0	$x_3 = 0$	0	$x_3 = -1/4$	1	$x_3 = 1$

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1/4 & 1 \end{pmatrix}$$

$$M = B^{-1}C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1/4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1/2 \\ 0 & 0 & 1/2 \end{pmatrix}$$

Convergența: $\rho(M) < 1$?

$$\det(\lambda I - M) = \begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 1/2 \\ 0 & 0 & \lambda - 1/2 \end{vmatrix} = \lambda^2 \left(\lambda - \frac{1}{2}\right)$$

Valorile proprii ale lui M: $\lambda_{1,2} = 0$, $\lambda_3 = \frac{1}{2}$

$\rho(M) = \frac{1}{2} < 1 \Rightarrow$ șirul construit cu metoda Gauss-Seidel converge la soluția sistemului x^*.

$$x^{(1)} : \quad M x^{(0)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1/2 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -3/2 \\ 3/2 \end{pmatrix}$$

$$d = B^{-1}b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1/4 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3/2 \\ 1/2 \end{pmatrix} \Rightarrow x^{(1)} = Mx^{(0)} + d = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

Metoda relaxării succesive $\omega = 0.5$

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = L + \frac{1}{\omega} \cdot D = L + 2D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$C = B - A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det B = 32 \neq 0$$

$$B^{-1}: \quad \begin{array}{lcl|l|l} 2x_1 & = & 1 \quad x_1 = 1/2 & 0 \quad x_1 = 0 & 0 \quad x_1 = 0 \\ 8x_2 & = & 0 \quad x_2 = 0 & 1 \quad x_2 = 1/8 & 0 \quad x_2 = 0 \\ x_2 + 2x_3 & = & 0 \quad x_3 = 0 & 0 \quad x_3 = -1/16 & 1 \quad x_3 = 1/2 \end{array}$$

$$B^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & -1/16 & 1/2 \end{pmatrix}$$

$$M = B^{-1}C = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & -1/16 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1/2 & -1/4 \\ 0 & -1/4 & 5/8 \end{pmatrix}$$

Convergența: $\rho(M) < 1$?

$$\det(\lambda I - M) = \begin{vmatrix} \lambda - 1/2 & 0 & 1/2 \\ 0 & \lambda - 1/2 & 1/4 \\ 0 & 1/4 & \lambda - 5/8 \end{vmatrix} =$$

$$\left(\lambda - 1/2\right)\left(\lambda^2 - \frac{9}{8}\lambda + \frac{1}{4}\right) = 0$$

$$\lambda_1 = 1/2 \quad , \quad \lambda_{2,3} = \frac{9 \pm \sqrt{17}}{16}$$

$$\rho(M) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} = \frac{9+\sqrt{17}}{16} < 1$$

$\rho(M) < 1 \Rightarrow$ șirul construit cu metoda relaxării succesive cu $\omega = \frac{1}{2}$ este convergent

$$z^{(1)} : \quad M z^{(0)} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1/2 & -1/4 \\ 0 & -1/4 & 5/8 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1/4 \\ 11/8 \end{pmatrix}$$

$$d = B^{-1} b = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/8 & 0 \\ 0 & -1/16 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3/4 \\ 5/8 \end{pmatrix}$$

$$z^{(1)} = M z^{(0)} + d = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$