I. Pe $\mathbb{Z}$

$x R y \Leftrightarrow y = x - 2 \quad \forall x, y \in \mathbb{Z}$

- Este reflexivă?

Fie $x \in \mathbb{Z}$ $\quad x - 2 \neq x \Rightarrow$

nu are loc $x R x \Rightarrow$

R nu e reflexivă

- Este simetrică?

Fie $x, y \in \mathbb{Z}$ a.î. $x R y \Rightarrow$

$y = x - 2 \Rightarrow x = y + 2$

$\Rightarrow x \neq y - 2$ nu are loc

$y R x \Rightarrow R$ nu este simetrică

- Este antisimetrică?

Fie $x, y \in \mathbb{Z}$ a.î. $x R y$ $\Rightarrow$
$y R x$

$x = y - 2$ $\Rightarrow$ $x = y - 2 = x - 2 - 2 =$

$y = x - 2$ $\qquad x - 4$

$x = x - 4$ absurd

$\Rightarrow$ Nu am perechi cu xRy și yRx

$\Rightarrow$ relația este antisimetrică.

(Falsul implică orice)

• Este tranzitivă?

Fie $x, y, z \in \mathbb{Z}$ a.î.

$xRy \Rightarrow x = y - 2$

$yRz \Rightarrow y = z - 2$

$x = z - 2 - 2 \Rightarrow$

$x = z - 4$

Deci $x \neq z - 2$,

nu are loc xRz, deci R nu e tranzitivă

11. $\sum \frac{(-1)^{n-1}}{4^n n^\alpha}(1+x)^n$

i. Studiați natura seriei ... de α

Var I. $\frac{(-1)}{4}(1+x) = z$ Var II

$z = 1+x$

$a_n = \frac{-1}{n^\alpha}$

$a_n = \frac{(-1)^{n-1}}{4^n n^\alpha}$

$\Rightarrow |a_n| = \frac{1}{4^n n^\alpha}$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{4^{n+1}(n+1)^\alpha}}{\frac{1}{4^n n^\alpha}} =$$

$$\frac{1}{4} \cdot \frac{n^\alpha}{(n+1)^\alpha} \longrightarrow \frac{1}{4}$$

$$R = 1 \Rightarrow \sum a_n z^n \left\{ \begin{array}{l} \text{AC pt } |z|<1 \\ \text{D pt } |z|>1 \\ \text{? pt } |z|=1 \end{array} \right.$$

$$z = 1 \quad \sum \frac{(-1)^{n-1}}{\cancel{1^n}\, n^\alpha} \cdot \cancel{1^n}$$

$$\sum \frac{(-1)^{n-1}}{n^\alpha} =$$

$$-1 \sum \frac{(-1)^n}{n^\alpha} =$$

$$-\sum (-1)^n \cdot \underbrace{\frac{1}{n^\alpha}}_{b_n} \quad \Big\}\ \text{Leibniz}$$

pt $\alpha > 0$ $n^\alpha \nearrow$

$\to \frac{1}{n^\alpha} \searrow$

$\frac{1}{n^\alpha} \longrightarrow 0$

$- \sum (-1)^n \frac{1}{n^\alpha}$ C pt $\alpha > 0$

$z = -4 \quad \sum \frac{(-1)^{n-1}}{n^\alpha 4^n} \cdot (-4)^n$

$\sum \frac{(-1)^{n-1}}{n^\alpha \cancel{4^n}} \cdot (-1)^n \cdot \cancel{4^n}$

$\sum -\frac{1}{n^\alpha}$ $\begin{cases} D \text{ dacă } \alpha \leq 1 \\ C \text{ dacă } \alpha > 1 \end{cases}$

$$\sum a_n z^n \begin{cases} AC \text{ pt } |z|<4 \\ D \text{ pt } |z|>4 \\ AC \text{ pt } |z|=4 \text{ și } \alpha>1 \\ SC \text{ pt } z=4 \text{ și } \alpha>0 \\ D \text{ pt } z=4 \text{ și } \alpha=0 \\ D \text{ pt } z=-4 \text{ și } \alpha\in[0,1] \end{cases}$$

$1+x=4 \rightarrow x=3$

$1+x=-4 \rightarrow x=-5$

$-4<1+x<4$

$-5 < x < 3$

$$\sum a_n (1+x)^n \begin{cases} AC \text{ pt } x\in(-5,3) \\ D \text{ pt } x\in\mathbb{R}\setminus[-5,3] \\ AC \text{ pt } x\in\{3,-5\} \text{ și } \alpha>1 \\ SC \text{ pt } x\in 3 \text{ și } \alpha\in(0,1] \\ D \text{ pt } x=3 \text{ și } \alpha=0 \\ D \text{ pt } x=-5 \text{ și } \alpha\in[0,1] \end{cases}$$

$$\sum \frac{(-1)^{n-1}}{4^n n^\alpha}(1+x)^n$$

$x = -\frac{5}{2} \quad \alpha = 0$

$$\sum \frac{(-1)^{n-1}}{4^n \cdot n^0} \cdot \left(1 - \frac{5}{2}\right)^n =$$

$$\sum \frac{(-1)^{n-1}}{4^n} \cdot \left(-\frac{3}{2}\right)^n =$$

$$-\sum \frac{(-1)^n \cdot (-1)^n}{4^n} \cdot \left(\frac{3}{2}\right)^n =$$

$$-\sum_{n=1}^{\infty} \left(\frac{3}{8}\right)^n = -\frac{3}{5}$$

$$S_n = -\sum_{k=1}^{n} \left(\frac{3}{8}\right)^k = -\frac{3}{5}$$

$$x + x^2 + x^3 + \dots + x^n =$$

$$\| \quad \frac{x^n - 1}{x - 1} \cdot x$$

$$x(1 + x + x^2 + \dots + x^{n-1}) =$$

$$x \cdot \frac{x^n - 1}{x - 1} \xrightarrow[n \to \infty]{|x|<1}$$

$$\frac{-x}{x-1}$$

$$= \frac{\frac{3}{8}}{-\frac{5}{8}} = \frac{3}{5}$$

2. $x_n = \dfrac{\overbrace{3^\alpha + 5^\alpha + \dots + (2n+1)^\alpha}^{a_n}}{\underbrace{n^{\alpha+1}}_{b_n}}$ $\qquad b_{n+1} = (n+1)^{\alpha+1}$

$n^{\alpha+1}$ monoton nemărg

$\alpha \in \mathbb{N} \Rightarrow \alpha + 1 > 0$ $\qquad a_{n+1} = 3^\alpha + 5^\alpha + \dots + (2n+1)^\alpha + \underline{(2n+3)^\alpha}$

$\Rightarrow n^{\alpha+1} \longrightarrow \infty$

$n^{\alpha+1} \nearrow$

$$\lim_{n\to\infty} x_n \underset{\text{Data}}{\overset{CS}{=}} \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} =$$

$$\lim_{n\to\infty} \frac{(2n+3)^\alpha}{(n+1)^{\alpha+1} - n^{\alpha+1}} =$$

$$(n+1)^{\alpha+1} = n^{\alpha+1} + C^1_{\alpha+1} n^\alpha + C^2_{\alpha+1} n^{\alpha-1} + \dots$$

$$\lim_{n\to\infty} \frac{(2n+3)^\alpha}{\cancel{n^{\alpha+1}} + (\alpha+1)n^\alpha + \dots + 1 - \cancel{n^{\alpha+1}}}$$

$$\lim_{n \to \infty} \frac{(2n)^{\alpha} + C_{\alpha}^{1}(2n)^{\alpha-1} + \dots}{(\alpha+1)n^{\alpha}} =$$

$$\frac{2^{\alpha}}{\alpha+1}$$

III

$$A_{B_C} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{9} \\ -3 & 0 & -\frac{1}{3} \\ 9 & -3 & 0 \end{pmatrix}$$

1. $$T\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{9} \\ -3 & 0 & -\frac{1}{3} \\ 9 & -3 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{1}{3} \\ 6 \\ -21 \end{pmatrix}$$

2. $$\det(A - \lambda I_3) = \begin{vmatrix} -\lambda & -\frac{1}{3} & \frac{1}{9} \\ -3 & -\lambda & -\frac{1}{3} \\ 9 & -3 & -\lambda \end{vmatrix}$$

$$\underline{\underline{C_2 = 3C_2}}$$

$$C_3 = 9C_3 \quad \frac{1}{27}\begin{vmatrix} -\lambda & -1 & 1 \\ -3 & -3\lambda & -3 \\ 9 & -9 & -9\lambda \end{vmatrix}$$

$$L_3 = \frac{1}{9}L_3$$

$$L_2 = \frac{1}{3}L_2 \quad \begin{vmatrix} -\lambda & -1 & 1 \\ -1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} \qquad \underline{\underline{C_1 = C_1 - C_3}}$$

$$\begin{vmatrix} -\lambda-1 & -1 & 1 \\ 0 & -\lambda & -1 \\ 1+\lambda & -1 & -\lambda \end{vmatrix} =$$

$$(1+\lambda)\begin{vmatrix} -1 & -1 & 1 \\ 0 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix}$$

$$L_1 = L_1 + L_3$$

$$(1+\lambda)\begin{vmatrix} 0 & -2 & 1-\lambda \\ 0 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} =$$

$$(1+\lambda)\begin{vmatrix} -2 & 1-\lambda \\ -\lambda & -1 \end{vmatrix} =$$

$$(1+\lambda)(2+\lambda(1-\lambda))$$

$$(1+\lambda)(2+\lambda-\lambda^2) =$$

$$(1+\lambda)(2+2\lambda-\lambda-\lambda^2) =$$

$$(1+\lambda)(2(1+\lambda)-\lambda(1+\lambda)) =$$

$$(1+\lambda)^2(2-\lambda)$$

$\lambda_1 = -1 \quad m_1 = 2$

$\lambda_2 = 2 \quad m_2 = 1$

Subspații proprii

$\lambda_1 = -1$

$$(A + I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{9} \\ -3 & 1 & -\frac{1}{3} \\ 9 & -3 & 1 \end{pmatrix}}_{\det 0}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$C_2 = \frac{C_1}{-3}$ $\quad \Delta = 1$

$\Rightarrow$ rang $= 1$

$C_3 = \frac{C_1}{9}$

x_1 nec pp

$x_2 = \alpha$

$x_3 = \beta$ nec sec

$x_1 - \frac{1}{3}\alpha + \frac{1}{9}\beta = 0$

$x_1 = \frac{1}{3}\alpha - \frac{1}{9}\beta$

$V_{\lambda_1} = \{ (\frac{1}{3}\alpha - \frac{1}{9}\beta, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R} \}$

O bază

$\alpha = 3 \quad \beta = 0 \quad v'_{\lambda_1} = (1, 3, 0)$

$\alpha = 0 \quad \beta = 9 \quad v_{\lambda_1}^2 = (-1, 0, 9)$

$\dim V_{\lambda_1} = 2$

$\lambda_2 = 2$

$$(A - 2I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & -\frac{1}{3} & \frac{1}{9} \\ -3 & -2 & -\frac{1}{3} \\ 9 & -3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D = \begin{vmatrix} -2 & -\frac{1}{3} \\ -3 & -2 \end{vmatrix} = 4 - 1 = 3 \neq 0$$

x_1, x_2 nec pp, $x_3 = \alpha$ nec sec

$$\begin{pmatrix} -2 & -\frac{1}{3} \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\alpha \begin{pmatrix} \frac{1}{9} \\ -\frac{1}{3} \end{pmatrix}$$

$$-2x_1 - \frac{1}{3}x_2 = -\frac{\alpha}{9} \quad \Big|\cdot 3$$

$$-3x_1 - 2x_2 = \frac{\alpha}{3} \quad \Big|\cdot 2$$

$$-6x_1 - x_2 = -\frac{\alpha}{3}$$

$$-6x_1 - 4x_2 = \frac{2\alpha}{3} \qquad -$$

$$3x_2 = -\alpha$$

$$x_2 = -\frac{\alpha}{3}$$

$$-3x_1 + \frac{2\alpha}{3} = \frac{\alpha}{3}$$

$$-3x_1 = -\frac{\alpha}{3}$$

$$x_1 = \frac{\alpha}{9}$$

$$V_{\lambda 2} = \{ \left(\frac{\alpha}{9}, -\frac{\alpha}{3}, \alpha \right) |$$

$\alpha \in \mathbb{R} \}$

$v_{\lambda_2} = (1, -3, 9)$

$\alpha = 9$

$\dim V_{\lambda_2} = 1$

c) Este T diag?

$m_{\lambda_1} + m_{\lambda_2} = 2 + 1 = 3 = \dim \mathbb{R}^3$ ✓

λ_1 $\quad \dim V_{\lambda_1} = m_{\lambda_1}$?

$$\begin{array}{ccc} \dim V_{\lambda_1} & = & m_{\lambda_1} \\ \| & & \| \\ 2 & & 2 \end{array} \checkmark$$

λ_2 $\quad \dim V_{\lambda_2} = m_{\lambda_2}$?

$$\begin{array}{ccc} \dim V_{\lambda_2} & = & m_{\lambda_2} \\ \| & & \| \\ 1 & & 1 \end{array} \checkmark$$

$\Rightarrow T$ diagonalizabil $\Rightarrow$

$$A_{B_D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$B_D = \{(1,3,0), (-1,0,9), (1,-3,9)\}$$