b) $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (f_1(x), f_2(x))$, unde

$$f_1(x) = \begin{cases} -x, & x \le 0 \\ x^2 \cos \dfrac{1}{x}, & x > 0 \end{cases} \quad \text{şi } f_2(x) = \begin{cases} \sin x, & x < 0 \\ \operatorname{arctg} x, & x \ge 0 \end{cases};$$

$$f_1'(0) = \lim_{x \to 0} \frac{f_1(x) - f_1(0)}{x - 0}$$

$$f_{1s}'(0) = \lim_{x \to 0} \frac{-x - 0}{x - 0} = 1$$

$$f_{1d}'(0) = \lim_{x \to 0} \frac{x^2 \cos \frac{1}{x} - 0}{x - 0} =$$

$$\lim_{x \to 0} x \cos \frac{1}{x} = 0$$

$$f_2'(0) = \lim_{x \to 0} \frac{f_2(x) - f_2(0)}{x - 0} =$$

$$f_{2s}'(0) = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$f_2'{}_d(0) = \lim_{x \to 0} \frac{\operatorname{arctg} x}{x} = 1$$

S9.12 Fie $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = \left(x^3 + y^3 + z^3 - 3xyz, \sqrt{x^2 + y^2 + z^2}, xy + yz + zx\right)$.

a) Studiaţi derivabilitatea Gâteaux şi diferenţiabilitatea Fréchet în f pe $\ker f := \left\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\right\}$;

b) Arătaţi că matricea jacobiană a lui f există şi este singulară în orice punct al lui $\mathbb{R}^3 \setminus \{\mathbf{0}_{\mathbb{R}^3}\}$.

$f(x, y, z) = 0$?

$$\sqrt{x^2 + y^2 + z^2} = 0$$

$$f_1'((0, 0, 0), u, v, w) = 0$$

$$\frac{\partial f_1}{\partial x}(x, y, z) = 3x^2 - 3yz$$

$$\frac{\partial f_1}{\partial x}(0, 0, 0) = 0$$

$$\frac{\partial f_3}{\partial x}(x,y,z) = y+z$$

$$\frac{\partial f_3}{\partial x}(0,0,0) = 0$$

$$\lim_{t \to 0} \frac{\sqrt{t^2(u^2+v^2+w^2)}}{t} =$$

$$\sqrt{u^2+v^2+w^2}$$

$$Jf = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}$$

$$= \begin{vmatrix} 3(x^2-yz) & 3(y^2-zx) & 3(z^2-xy) \\ \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} & \dfrac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}} & \dfrac{\cancel{2}z}{\cancel{2}\sqrt{x^2+y^2+z^2}} \\ y+z & x+z & x+y \end{vmatrix}$$

$$\frac{3}{\sqrt{x^2+y^2+z^2}} \begin{vmatrix} x^2-yz & y^2-zx & z^2-xy \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix}$$

$$\frac{3}{\sqrt{x^2+y^2+z^2}} \Big[(x^2-yz)\, y(x+y) + \\ \ldots \Big] = 0$$

a) $f(x,y) = xy\sqrt{1+(x^2-y^2)^2}$,

$$xy\left\langle (y,x), (\nabla f)(x,y)\right\rangle_2 = \|(x,y)\|_2^2 \cdot f(x,y);$$

$$\frac{\partial f}{\partial x}(x,y) = y\sqrt{1+(x^2-y^2)^2} + xy \cdot \frac{\cancel{2}(x^2-y^2)\cdot 2x}{\cancel{2}\sqrt{1+(x^2-y^2)^2}} =$$

$$\frac{y\left(1+(x^2-y^2)^2 + 2x^2(x^2-y^2)\right)}{\sqrt{1+(x^2-y^2)^2}} =$$

$$\frac{y}{\sqrt{1+(x^2-y^2)^2}}\left(1+x^4+y^4-2x^2y^2 + 2x^4-2x^2y^2\right)$$

$$\frac{y}{\sqrt{1+(x^2-y^2)^2}}\left(1+3x^4+y^4-4x^2y^2\right)$$

$$\frac{\partial f}{\partial y}(x,y) = x\sqrt{1+(x^2-y^2)^2} +$$

$$xy\,\frac{\cancel{2}(x^2-y^2)\cdot(-2y)}{\cancel{2}\sqrt{1+(x^2-y^2)^2}}$$

$$\frac{x}{\sqrt{1+(x^2-y^2)^2}}\left(1 + x^4 + y^4 - 2x^2y^2 - 2x^2y^2 + 2y^4\right)$$

$$\frac{x}{\sqrt{1+(x^2-y^2)^2}}\left(1 + x^4 + 3y^4 - 4x^2y^2\right)$$

$$xy\left\langle (y,x), (\nabla f)(x,y)\right\rangle_2 = \frac{xy}{\sqrt{1+(x^2-y^2)^2}}\Big(y^2 + 3x^4y^2 + y^6 - 4x^2y^4 + x^2 + x^6 + 3y^4x^2 - 4x^4y^2\Big) =$$

$$\frac{xy}{\sqrt{1+(x^2-y^2)^2}}\left(x^2 + y^2 + x^6 + y^6 - x^2y^4 - x^4y^2 \right) =$$

$$\frac{xy}{\sqrt{1+(x^2-y^2)^2}}\left[x^2(1+x^4-y^4) + y^2(1+y^4-x^4)\right] =$$

$$\frac{\cancel{xy}}{\sqrt{}} = \left(x^2+y^2+x^6+y^6 - x^2y^4 - x^4y^2\right) =$$

$$\underline{(x^2+y^2)} \cdot \cancel{xy} \sqrt{1+(x^2-y^2)^2}$$

$$\|(x,y)\|_2^2 \cdot f(x,y); =$$

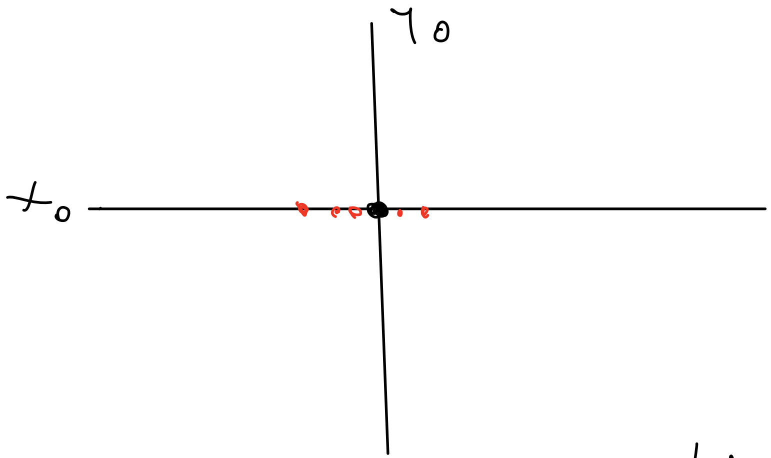
$$(x^2+y^2) \cdot xy\sqrt{1+(x^2-y^2)^2}$$

$$x^2+y^2+x^6+y^6-x^2y^4-x^4y^2$$

$$=\left(x^2+y^2\right)\left(1+\left(x^2-y^2\right)^2\right)$$

$$=x^2+y^2+\left(x^4-y^4\right)\left(x^2-y^2\right)$$

$$=x^2+y^2+x^6+y^6-x^4y^2-x^2y^4$$



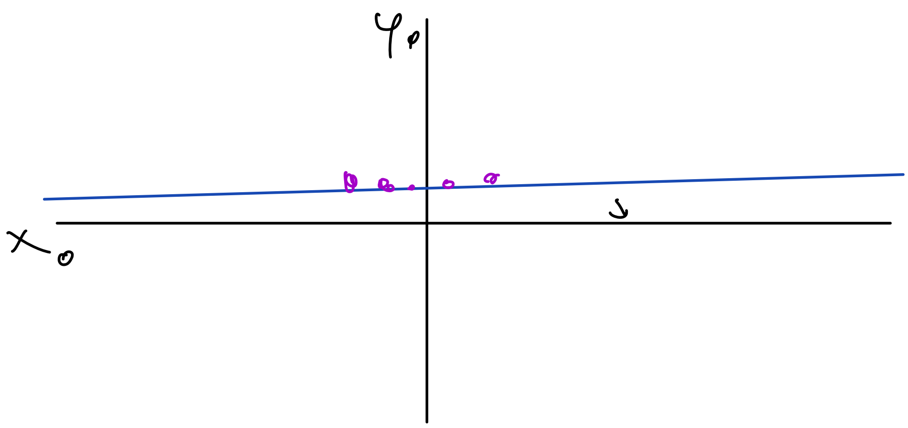
Limita parțială / derivată parțială

$$\lim_{y \to y_0} f(x_0, y)$$

Limita iterată

$$\lim_{x \to x_0} \left| \lim_{y \to y_0} f(x, y) \right|$$

x fixat aproape de x_0, dar nu x_0



Limita globală