# Probabilități și Statistică

## - Seminar 5 -

## Repartiții comune

### Exercițiul VI (Seminar 4)

Fie X și Y cu următoarea repartiție comună:

Determinați rep. individuale ale lui X și Y

| X \ Y | -3 | 2 | 4 | |
|---|---|---|---|---|
| 1 | 0.1 | 0.2 | 0.2 | 0.5 |
| 3 | 0.3 | 0.1 | 0.1 | 0.5 |
| | 0.4 | 0.3 | 0.3 | |

$P(X=1) = 0.1+0.2+0.2 = 0.5$

$P(\cancel{X}=-3) = 0.1+0.3 = 0.4$

$P(Y=4) = 0.2+0.1 = 0.3$

$P(Y=-3)+P(Y=2)+P(Y=4) = 1 \Rightarrow P(Y=2) = 1-0.4-0.3 = 0.3$

$P(X=1)+P(X=3) = 1 \Rightarrow P(X=3) = 1-0.5 = 0.5$

$? \Leftrightarrow P(X=3 \cap Y=2) = 0.3-0.2 = 0.1$

sau

$= 0.5 - 0.3 - 0.1 = 0.1$

$\Rightarrow P(X=3 \cap Y=2) = 0.1$

$X: \begin{pmatrix} 1 & 3 \\ 0.5 & 0.5 \end{pmatrix} \quad Y: \begin{pmatrix} -3 & 2 & 4 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$

### Exercițiul II.3

Fie $X: \begin{pmatrix} -2 & -1 & 1 & 2 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix}$ și $Y = X^2$

a) Repartiția lui $\cancel{X}$ și repartiția comună a celor două variabile

$Y = X^2 \qquad Y: \begin{pmatrix} 1 & 4 \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix}$

$P(Y=1) = P(X^2=1) = P(\{X=-1\} \cup \{X=1\}) = P(\{X=-1\}) + P(\{X=1\})$

$= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

$P(Y=4) = P((X=-2) \cup (X=2)) = \frac{1}{4} + \frac{3}{8} = \frac{5}{8}$

Verificare: $P(Y=1) + P(Y=4) = 1$ $\left( \frac{3}{8} + \frac{5}{8} = 1 \right.$ - adevărat$)$

$$X: \begin{pmatrix} -2 & -1 & 1 & 2 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix} \qquad Y: \begin{pmatrix} 1 & 4 \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix}$$

| X \ Y | 1 | 4 | |
|---|---|---|---|
| -2 | $r_{11}$ | $r_{12}$ | 1/4 |
| -1 | $r_{21}$ | $r_{22}$ | 1/8 |
| 1 | $r_{31}$ | $r_{32}$ | 1/4 |
| 2 | $r_{41}$ | $r_{42}$ | 3/8 |
| | 3/8 | 5/8 | |

$r_{ij} = P(X = x_i \cap Y = y_j)$

$r_{11} = P(\{X=-2\} \cap \{Y=1\}) = P(\emptyset) = 0$

(nu este posibil $\left.\begin{matrix} Y = X^2 \\ X = -2 \end{matrix}\right| \Rightarrow Y = 4 \neq 1$ )

$r_{12} = P(\{X=-2\} \cap \{Y=4\}) = \frac{1}{4}$

$r_{21} = P(\{X=-1\} \cap \{Y=1\}) = \frac{1}{8}$

$r_{22} = P(\{X=-1\} \cap \{Y=4\}) = 0$

$r_{31} = P(\{X=1\} \cap \{Y=1\}) = \frac{1}{4}$

$r_{32} = P(\{X=1\} \cap \{Y=4\}) = P(\emptyset) = 0$

$r_{41} = P(\{X=2\} \cap \{Y=1\}) = 0$

$r_{42} = P(\{X=2\} \cap \{Y=4\}) = \frac{3}{8}$

| X \ Y | 1 | 4 | |
|---|---|---|---|
| -2 | 0 | 1/4 | 1/4 |
| -1 | 1/8 | 0 | 1/8 |
| 1 | 1/4 | 0 | 1/4 |
| 2 | 0 | 3/8 | 3/8 |
| | 3/8 | 5/8 | |

Verificare: $r_{11} + r_{21} + r_{31} + r_{41} = \frac{3}{8}$

$\left(0 + \frac{1}{8} + \frac{1}{4} + 0 = \frac{3}{8} \text{ - adevărat}\right)$

$r_{12} + r_{22} + r_{32} + r_{42} = \frac{5}{8}$ (A)

(b) $Cov(X,Y)$, $\rho(X,Y)$

Covarianța X, Y

$$\boxed{Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]}$$

$$\boxed{Cov(X,Y) = \mathbb{E}(X \cdot Y) - \mathbb{E}(X)\mathbb{E}(Y)}$$

Corelația X, Y

$$\rho(X,Y) = \frac{cov(X,Y)}{StDev(X)\, StDev(Y)}$$

unde $StDev(X) = \sqrt{Var(X)}$ ← deviația standard

Cum $X:\begin{pmatrix} -2 & -1 & 1 & 2 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix}$ $\quad Y:\begin{pmatrix} 1 & 4 \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix}$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

Întâi determinăm repartiția lui $XY$

$$XY:\begin{pmatrix} -8 & -4 & -2 & -1 & 1 & 2 & 4 & 8 \\ \frac{1}{4} & 0 & 0 & \frac{1}{8} & \frac{1}{4} & 0 & 0 & \frac{3}{8} \end{pmatrix}$$

$P(XY=-8) = P(X=-2 \cap Y=4) = \frac{1}{4}$

$P(XY=-4) = P(X=-1 \cap Y=4) = 0$

$P(XY=-2) = P(X=-2 \cap Y=1) = 0$

$P(XY=-1) = P(X=-1 \cap Y=1) = 1/8$

$P(XY=1) = P(X=1 \cap Y=1) = \frac{1}{4}$

$P(XY=2) = 0$

$P(XY=4) = 0$

$P(XY=8) = 3/8$

Așadar $XY:\begin{pmatrix} -8 & -1 & 1 & 8 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} \end{pmatrix}$

$E(XY) = -8\cdot\frac{1}{4} - 1\cdot\frac{1}{8} + 1\cdot\frac{1}{4} + 8\cdot\frac{3}{8} = \frac{-16-1+2+24}{8} = \frac{9}{8}$

$E(X) = -2\cdot\frac{1}{4} - 1\cdot\frac{1}{8} + 1\cdot\frac{1}{4} + 2\cdot\frac{3}{8} = \frac{-4-1+2+6}{8} = \frac{3}{8}$

$E(Y) = 1\cdot\frac{3}{8} + 4\cdot\frac{5}{8} = \frac{23}{8}$

$Cov(X,Y) = {}^{8/}\frac{9}{8} - \frac{3}{8}\cdot\frac{23}{8} = \frac{72-69}{8^2} = \frac{3}{8^2}$

$\rho(X,Y) = \frac{Cov(X,Y)}{St\,dev(X)\,StDev(Y)}$

$StDev(X) = \sqrt{Var(X)} = \sqrt{E(X^2) - E(X)^2} = \sqrt{E(Y) - E(X)^2} = \sqrt{\frac{23}{8} - \left(\frac{3}{8}\right)^2} = \sqrt{\frac{23\cdot 8 - 9}{8^2}}$

$= \sqrt{\frac{175}{8^2}} = \frac{5\sqrt{7}}{8}$

$X^2:\begin{pmatrix} 1 & 4 \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix} = Y$

$StDev(Y) = \sqrt{Var\,Y} = \sqrt{E(Y^2) - E(Y)^2} = \sqrt{\frac{3}{8} + 16\cdot\frac{5}{8} - \left(\frac{23}{8}\right)^2} = \sqrt{\frac{83}{8} - \left(\frac{23}{8}\right)^2} = \frac{1}{8}\sqrt{135} =$

$Y^2:\begin{pmatrix} 1 & 16 \\ \frac{3}{8} & \frac{5}{8} \end{pmatrix}$

$\rho(X,Y) = \frac{\frac{3}{8^2}}{\frac{5\sqrt{7}\,\sqrt{135}}{8^2}} = \frac{3}{5\sqrt{7}\cdot\sqrt{135}}$



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## Exercițiul II.6

Urnă: 4 bile albe (2 num. cu 1 și 2 num. cu 2)
3 bile negre (2 num cu 1 și 1 num cu 2)

Se extrage succesiv și fără întoarcere 2 bile.

X = "nr. bile albe obținute" $X = \{0, 1, 2\}$

Y = "nr. bile numerotate cu 2". $Y = \{0, 1, 2\}$

Notăm ~~$A_i$ = "a bila extrasă~~

$A_i$ = " la extragerea i se obține bilă albă"

$$P(X=0) = P(\bar{A}_1 \cap \bar{A}_2) = P(\bar{A}_1) P(\bar{A}_2 | \bar{A}_1) = \frac{3}{7} \cdot \frac{2}{6} = \frac{1}{7}$$

sau $$P(\bar{A}_1 \cap \bar{A}_2) = \frac{C_4^0 C_3^2}{C_7^2} = \frac{3}{\frac{7 \cdot 6}{2}} = \frac{1}{7}$$

$$P(X=1) = P((\bar{A}_1 \cap A_2) \cup (A_1 \cap \bar{A}_2)) = \frac{3}{7} \cdot \frac{4}{6} + \frac{4}{7} \cdot \frac{3}{6} = \frac{24}{7 \cdot 6} = \frac{4}{7}$$

$$= \frac{C_4^1 C_3^1}{C_7^2} = \frac{4 \cdot 3}{\frac{7 \cdot 6}{2}} = \frac{4}{7}$$

$$P(X=2) = \frac{C_4^2 C_3^0}{C_7^2} = \frac{3 \cdot 4 \cdot 2}{2 \cdot 21} = \frac{6}{21} = \frac{2}{7}$$

$$X : \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{7} & \frac{4}{7} & \frac{2}{7} \end{pmatrix}$$

Notez $B_i$ = " la extragerea i se obține bila numerotată cu 2 "

Consider U: 4 bile num cu 1 și 3 bile num cu 2

Pt calculul repartiției lui Y nu ne interesează culoarea bilelor.

$$P(Y=0) = P(\bar{B}_1 \cap \bar{B}_2) = \frac{C_4^2 \cdot C_3^0}{C_7^2} = \frac{\frac{4!}{2!\,2!}}{\frac{7!}{5!\,2!}} = \frac{6}{21} = \frac{2}{7}$$

$$P(Y=1) = \frac{C_4^1 C_3^1}{C_7^2} = \frac{4 \cdot 3}{21} = \frac{4}{7}$$

$$P(Y=2) = \frac{C_4^0 C_3^2}{C_7^2} = \frac{1}{7}$$

$$Y : \begin{pmatrix} 0 & 1 & 2 \\ \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \end{pmatrix}$$

Repartiția comună

$P(X=0 \cap Y=0) = r_{11}$

|   |   | Y |   |   |   |
|---|---|---|---|---|---|
|   |   | 0 | 1 | 2 |   |
| X | 0 | $\frac{1}{21}$ | $2/21$ | 0 | $1/7$ |
|   | 1 | $4/21$ | $6/21$ | $2/21$ | $4/7$ |
|   | 2 | $1/21$ | $4/21$ | $1/21$ | $2/7$ |
|   |   | $2/7$ | $4/7$ | $1/7$ |   |

Interpretare:

$(X=0 \cap Y=0)$ = "0 b. albe și 0 num cu 2"

$\overset{sau}{=}$ "2 b. negre num cu 1"

$$P(\underbrace{\bar{A}_1 \cap \bar{A}_2} \cap \underbrace{\bar{B}_1 \cap \bar{B}_2}) = P(\bar{A}_1)P(\bar{A}_2|\bar{A}_1)\cdot P(\bar{B}_1)\cdot P(\bar{B}_2|B_1) = \underbrace{\frac{3}{7}\cdot\frac{2}{6}}\cdot\frac{2}{3}\;\frac{1}{2} = \frac{1}{21}$$

$$P(X=0 \cap Y=2) = P(\bar{A}_1 \cap \bar{A}_2 \cap B_1 \cap B_2) = \frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{3}\cdot\frac{0}{3} = 0$$

(nu se poate să am 2 bile negre num cu 2)

$$P(X=0 \cap Y=1) = 1/7 - \frac{1}{21} - 0 = \frac{2}{21}$$

sau

$$P(X=0 \cap Y=1) = P((\bar{A}_1 \cap \bar{A}_2 \cap B_1 \cap \bar{B}_2) \cup (\bar{A}_1 \cap \bar{A}_2 \cap \bar{B}_1 \cap B_2)) =$$

$$= \frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{3}\cdot\frac{2}{2} + \frac{3\cdot 2}{7\cdot 6}\cdot\frac{2}{3}\cdot\frac{1}{2} = \frac{2}{21}$$

$$P(X=2 \cap Y=0) = P(\underbrace{A_1 \cap A_2 \cap \bar{B}_1 \cap \bar{B}_2}_{\text{2 albe num cu 1}}) = \frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{4}\cdot\frac{1}{3} = \frac{1}{21}$$

$$P(X=2 \cap Y=2) = P(A_1 \cap A_2 \cap B_1 \cap B_2) = \frac{4\cdot 3}{7\cdot 6}\cdot\frac{2}{4}\cdot\frac{1}{3} = \frac{1}{21}$$

$$P(X=1 \cap Y=0) = \frac{2}{7} - \frac{1}{21} - \frac{1}{21} = \frac{6-2}{21} = \frac{4}{21}$$

$$P(X=2 \cap Y=1) = P(A_1 \cap A_2 \cap B_1 \cap \bar{B}_2) + P(A_1 \cap A_2 \cap \bar{B}_1 \cap B_2) =$$

$$= \frac{\cancel{4}\cdot 3}{6\cdot 7}\,\frac{\cancel{2}}{\cancel{4}}\cdot\frac{2}{3} + \frac{4\cdot 3}{6\cdot 7}\,\frac{2}{4}\cdot\frac{2}{3} = \frac{4}{21}$$

$$P(X=1 \cap Y=1) = \frac{4}{7} - \frac{2}{21} - \frac{4}{21} = \frac{12-6}{21} = \frac{6}{21} = \frac{2}{7}$$

$$P(X=1 \cap Y=2) = \frac{1}{7} - \frac{1}{21} = \frac{2}{21}$$

b) Sunt var. X și Y indep

Rezultat:

- Dacă $X, Y$ independente $\Rightarrow cov(X,Y) = 0$.

sau

- Dacă $cov(X,Y) \neq 0 \Rightarrow X, Y$ dependente

Sau • $X, Y$ independente dacă $P(X=x_i \cap Y=y_j) = P(X=x_i)P(Y=y_j)$ $\forall i,j$

$cov(X,Y) = ?$

$$XY: \begin{pmatrix} 0 & 1 & 2 & 4 \\ \frac{8}{21} & \frac{6}{21} & \frac{6}{21} & \frac{1}{21} \end{pmatrix}$$

$P(XY=0) = P(X=0 \cap Y=0) + P(X=0 \cap Y=1) + P(X=0 \cap Y=2) + P(X=1 \cap Y=0)$
$+ P(X=2 \cap Y=0) = \frac{2}{7} + \frac{2}{21} = \frac{8}{21}$

$P(XY=1) = P(X=1 \cap Y=1) = \frac{6}{21}$

$P(XY=4) = \frac{1}{21}$

$P(XY=2) = P(X=1 \cap Y=2) + P(X=2 \cap Y=1) = \frac{2}{21} + \frac{4}{21} = \frac{6}{21}$

Verif: $\frac{8+6+6+1}{21} = \frac{21}{21}$ ok.

$E(XY) = \frac{6}{21} + \frac{12}{21} + \frac{4}{21} = \frac{22}{21}$

$cov(X,Y) = \frac{22}{21} - \underbrace{\left(0\cdot\frac{1}{7} + 1\cdot\frac{4}{7} + 2\cdot\frac{2}{7}\right)}_{E(X)}\underbrace{\left(0\cdot\frac{2}{7} + 1\cdot\frac{4}{7} + \frac{2}{7}\right)}_{E(Y)}$

$= \left(\frac{22}{21} - \frac{8}{7}\cdot\frac{6}{7}\right) \neq 0$

! Sau, mai ușor

$\underbrace{P(X=0 \cap Y=0)}_{\frac{1}{21}} \neq P(X=0)P(Y=0)$ $\Rightarrow X, Y$ sunt dependente

$\frac{1}{21} \neq \frac{1}{7}\cdot\frac{2}{7}$

## Exercițiul II.9

Se aruncă monedele A și B de trei ori

P("stema pe moneda B") = 0.4

X = "nr. apariții steme pe moneda A"

Y = "nr. apariții steme pe moneda B"

a) X și Y sunt independente?

Da, rezultatul de la aruncarea monedei A nu este influențat de moneda B.

Tabelele de repartiție:

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \qquad Y : \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{27}{5^3} & \frac{54}{5^3} & \frac{36}{5^3} & \frac{8}{5^3} \end{pmatrix}$$

Notez $A_i$ := "stema obținută la aruncarea i a monedei A"

$B_i$ := " —— || —— a monedei B"

$$P(X=0) = P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3) = P(\bar{A}_1)P(\bar{A}_2)P(\bar{A}_3) = \frac{1}{8} \quad (A_1, A_2, A_3 \text{ indep})$$

$$P(X=3) = P(A_1 \cap A_2 \cap A_3) = \frac{1}{8}$$

$P(X=1) = P($" din cele 3 aruncări apare stema de exact o dată "$)$

$$= C_3^1 \left(\frac{1}{2}\right)^1 \left(1-\frac{1}{2}\right)^{3-1} = \frac{3}{8}$$

$$P(X=2) = C_3^2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = \frac{3}{8}$$

$$P(Y=0) = C_3^0 \cdot (0.4)^0 (0.6)^3 = (0.6)^3 = \left(\frac{3}{5}\right)^3 = \frac{27}{5^3}$$

$$P(Y=1) = C_3^1 (0.4)^1 (0.6)^2 = 3 \cdot \frac{2}{5} \cdot \left(\frac{3}{5}\right)^2 = \frac{54}{5^3}$$

$$P(Y=2) = C_3^2 (0.4)^2 (0.6)^1 = 3 \cdot \left(\frac{2}{5}\right)^2 \cdot \frac{3}{5} = \frac{36}{5^3}$$

$$P(Y=3) = C_3^3 (0.4)^3 = \left(\frac{2}{5}\right)^3 = \frac{8}{5^3}$$

b) Repartitia comună

| | Y: 0 | 1 | 2 | 3 | |
|---|---|---|---|---|---|
| X: 0 | $\frac{27}{10^3}$ | $\frac{54}{10^3}$ | $\frac{36}{10^3}$ | $\frac{8}{10^3}$ | $1/8$ |
| 1 | $\frac{3\cdot 27}{10^3}$ | $\frac{54\cdot 3}{10^3}$ | $\frac{36\cdot 3}{10^3}$ | $\frac{24}{10^3}$ | $3/8$ |
| 2 | $\frac{3\cdot 27}{10^3}$ | $\frac{54\cdot 3}{10^3}$ | $\frac{36\cdot 3}{10^3}$ | $\frac{24}{10^3}$ | $3/8$ |
| 3 | $\frac{27}{10^3}$ | $\frac{54}{10^3}$ | $\frac{36}{10^3}$ | $\frac{8}{10^3}$ | $1/8$ |
| | $\frac{27}{5^3}$ | $54/5^3$ | $36/5^3$ | $8/5^3$ | |

$$P(X=0\cap Y=0) \overset{X,Y \text{ indep}}{=} P(X=0)\cdot P(Y=0) = \frac{1}{8}\cdot\frac{27}{5^3} = \frac{27}{10^3}$$

$$P(X=0\cap Y=1) = \frac{54}{5^3}\cdot\frac{1}{8} = \frac{54}{10^3}$$

. . .

c) $P(X=Y) = P(X=0\cap Y=0) + P(X=1\cap Y=1) + \dots + P(X=3\cap Y=3)$

$$= \frac{27}{10^3} + \frac{54\cdot 3}{10^3} + \frac{36\cdot 3}{10^3} + \frac{8}{10^3} = \frac{305}{10^3}$$ ← suma el. de pe diagonală

$P(X>Y) = P(X=1\cap Y=0) + \dots + P(X=3\cap Y=2)$

$$= \frac{3\cdot 27}{10^3} + \frac{54\cdot 3}{10^3} + \frac{36}{10^3} + \frac{3\cdot 27}{10^3} + \frac{54}{10^3} + \frac{27}{10^3} = \frac{441}{10^3}$$

$P(X+Y\geq 4) = P((X=1\cap Y=3)\cup(X=2\cap Y=2)\cup(X=2\cap Y=3)\cup(X=3\cap Y=1)$
$\cup(X=3\cap Y=2)\cup(X=3\cap Y=3)))$

$$= \frac{24}{10^3} + \frac{36\cdot 3}{10^3} + \frac{24}{10^3} + \frac{54}{10^3} + \frac{36}{10^3} + \frac{8}{10^3}.$$

## Exercițiul II.11

Meci tenis cu jucatorii $P_1$ și $P_2$. Învinge primul ce câștigă 2 seturi.

$P(\text{"}P_1 \text{ câștigă"}) = \frac{1}{3}$

X = "nr seturi jucate de $P_1$"
Y = "nr seturi câștigate de $P_2$"

> vezi Seminar 4, ex I.15*

$$X: \begin{pmatrix} 2 & 3 \\ \frac{5}{9} & \frac{4}{9} \end{pmatrix} \qquad Y: \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{9} & \frac{4}{27} & \frac{20}{27} \end{pmatrix}$$

Notez $A_i$ = "juc. $P_1$ câștigă setul i"

$P(X=2) = P((A_1\cap A_2)\cup(\bar{A_1}\cap\bar{A_2})) = P(A_1\cap A_2)\cdot P(\bar{A_1}\cap\bar{A_2}) = \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3} = \frac{5}{9}$

a) Repartiția comună

| | | Y | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | |
| X | 2 | $\frac{1}{9}$ | 0 | $\frac{4}{9}$ | $5/9$ |
| | 3 | 0 | $4/27$ | $8/27$ | $4/9$ |
| | | $1/9$ | $4/27$ | $20/27$ | |

$P(X=2 \cap Y=0) = ?$

$X=2 \cap Y=0 \Rightarrow$ "2 seturi jucate de $P_1$ și 0 seturi câștigate de $P_2$"

$P(X=2 \cap Y=0) = P(A_1 \cap A_2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

$P(X=2 \cap Y=2) = P(\bar{A}_1 \cap \bar{A}_2) = \frac{4}{9}$

$P(X=2 \cap Y=1) = P(\emptyset) = 0$

$P(X=3 \cap Y=0) = \frac{1}{9} - \frac{1}{9}$

$P(X=3 \cap Y=1) = \frac{4}{27} - 0 = \frac{4}{27}$

$P(X=3 \cap Y=2) = \frac{20}{27} - {}^{3/}\frac{4}{9} = \frac{8}{27}$

Verific pe linie că e ok

$\frac{1}{9} + 0 + \frac{4}{9} = \frac{5}{9}$ ok

$0 + \frac{4}{27} + \frac{8}{27} = \frac{4}{9} = \frac{12}{27}$ ok

b) $Cov(X,Y) = ?$

$X, Y$ indep? $\Rightarrow$ NU $\left( P(X=2 \cap Y=0) \neq P(X=2) P(Y=0) \right)$

$\frac{1}{9} \neq \frac{5}{9} \cdot \frac{1}{9}$

$X \cdot Y : \begin{pmatrix} 0 & 2 & 3 & 4 & 6 \\ \frac{1}{9} & 0 & \frac{4}{27} & \frac{4}{9} & \frac{8}{27} \end{pmatrix} \Rightarrow XY : \begin{pmatrix} 0 & 3 & 4 & 6 \\ \frac{1}{9} & \frac{4}{27} & \frac{4}{9} & \frac{8}{27} \end{pmatrix}$

$P(XY = 6) = P(X=3 \cap Y=2) = \frac{8}{27}$

$E(XY) = 0 \cdot \frac{1}{9} + 3 \cdot \frac{4}{27} + 4 \cdot \frac{4}{9} + 6 \cdot \frac{8}{27} = \frac{12}{27} + \frac{16}{9} + \frac{48}{27} = \frac{60 + 48}{27} = \frac{108}{27}$

$E(Y) = \frac{4}{27} + 2 \cdot \frac{20}{27} = \frac{44}{27}$

$E(X) = \frac{10}{9} + \frac{12}{9} = \frac{22}{9}$

$cov(X,Y) = \frac{108}{27} - \frac{44}{27} \cdot \frac{22}{9} \neq 0 \Rightarrow X, Y$ sunt dependente

$\boxed{\text{II.12 + II.14 Încercați voi!}}$

# Inegalitățile lui Markov și Cebâșev

## Ineg lui Markov

Fie $X \geq 0$ o variabilă aleatoare cu $E(X) = \mu$. Atunci

$$\boxed{P(X \geq t) \leq \frac{\mu}{t}, \ \forall t > 0}$$

## Ineg lui Cebâșev

Fie $X$ o var. aleatoare cu $E(X) = \mu$ și $Var(X) = \sigma^2$. Atunci are loc

$$\boxed{P(|X-\mu| \geq k\cdot\sigma) \leq \frac{1}{k^2}, \ \forall k > 0}$$

## Exercițiul III.2

Fie $X \geq 0$ $E(X) = Var(X) = 1$. Majorați, minorați $P(X \geq 2)$, $P(|X-1| \geq 2)$ și $P(X \leq -3)$

Soluție:

- $\underset{t}{P(X \geq 2)} \underset{\text{Markov}}{\leq} \frac{1 \leftarrow E(X)}{2 \leftarrow t)}$ ; $P(X \geq 2) \leq P(|X-1| \geq \underset{\sigma \cdot k}{1}) \leq \frac{1}{1}$

- $P(|X - \underset{\mu}{1}| \geq \underset{k\cdot\sigma}{2}) \overset{\text{Cebâșev}}{\leq} \frac{1}{4 \to k^2}$

$\mu = 1$
$\sigma^2 = Var(X) = 1 \Rightarrow \sigma = 1$

- $P(X \leq -3) \leq P(|X-1| \geq 4) \leq \frac{1}{4^2} = \frac{1}{16}$

## Exercițiul III.6

P("apare stema pe moneda falsificată") = 0.3

Arunc de 300 ori

Majorați Prob ca stema să apară de cel puțin 100 de ori

X = "nr apariții steme"

$X \sim B(\underbrace{300}_{n}, \underbrace{0.3}_{p}) \Rightarrow E(X) = n\cdot p = 300 \cdot \frac{3}{10} = 90$

$Var(X) = np(1-p) = 90\cdot(1-0.3) = 90 \cdot \frac{7}{10} = 9\cdot 7 = 63$

- continuare III.6 -

Aşadar, $\mu = \mathbb{E}(X) = 90$

$\sigma^2 = 63$

Ne interesează $\underbrace{P(X \geq \underbrace{100}_{t})} \overset{\text{Markov}}{\leq} \frac{90}{100} = \underbrace{\frac{9}{10}}$

Cebîşev

$$P(X \geq 100) \leq P(|X - \underset{\mu}{90}| \geq \underset{k \cdot \sigma}{10}) \leq \frac{1}{\left(\frac{10}{3\sqrt{7}}\right)^2}$$

$k \cdot \sigma = 10$

$\sigma^2 = 63 \Rightarrow \sigma = \sqrt{63} = 3\sqrt{7}$ $\Big| \Rightarrow k = \frac{10}{\sigma} = \frac{10}{3\sqrt{7}}$

$\Rightarrow P(X \geq 100) \leq \frac{63}{100}$

## Exercițiul III.7

A = "apare stema pe moneda măsluită"

$P(A) = 0.2$

X = "nr apariții steme" $X \sim B(n, 0.2)$

$\Rightarrow \mathbb{E}(X) = n \cdot 0.2 = \frac{n}{5}$ $\Rightarrow \boxed{\mu = \frac{n}{5}}$

$Var(X) = \frac{n}{5}(1 - 0.2) = \frac{n}{5} \cdot \frac{8}{10} = \frac{4n}{25}$ $\Rightarrow \boxed{\sigma^2 = \frac{4n}{25}}$

$P(X \geq \frac{n}{2}) \leq \frac{\frac{n}{5}}{\frac{n}{2}} = \frac{2}{5}$ $\Rightarrow \boxed{P(X \geq \frac{n}{2}) \leq \frac{2}{5}}$

$\mu$

cel puțin 50% din n

## Exercițiul III.8

Arunc 2 monede de 25 ori $\Rightarrow n = 25$

$A_i$ = "stema pe moneda 1 la aruncarea i" $P(A_i) = 0.25 = \frac{1}{4}$

$B_i$ = "stema pe moneda 2 la aruncarea i" $P(B_i) = 0.8 = \frac{4}{5}$

X = "nr apariții stema pe ambele monede"

$X \sim B(25, p)$

Observ că nu contează aici aruncarea i, puteam să notez A, B

$p = P(\text{"stema pe ambele monede"}) = P(A_i \cap B_i) = \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$

$\mathbb{E}(X) = 25 \cdot \frac{1}{5} = 5$ $Var(X) = 5 \cdot (1 - \frac{1}{5}) = 5 \frac{4}{5} = 4$

$$P(X \geq \underbrace{10}_{t}) \underset{\substack{\downarrow \\ \text{Markov}}}{\leq} \frac{\overset{E(X)}{5}}{10} \Rightarrow P(X \geq 10) \leq \frac{1}{2}$$

$$P(X \geq 10) \leq P(|X-5| \geq \underbrace{5}_{k\cdot\sigma}) \underset{\substack{\downarrow \\ \text{Cebâşev}}}{\leq} \frac{1}{\frac{25}{4}} \Rightarrow P(X \geq 10) \leq \frac{4}{25}$$

$k \cdot \sigma = 5$

$\sigma^2 = 4 \Rightarrow \sigma = 2$ $\Big| \Rightarrow k = \frac{5}{2}$