Logic for Computer Science - Week 4
Natural Deduction

1 Introduction

In the previous lecture we have discussed some important notions about the *semantics* of propositional logic.

1. the truth value of a formula in an assignment;
2. satisfiability;
3. validity;
4. equivalence;
5. semantical consequence.

We have seen that in order to establish that two formulae are equivalent, a semantical reasoning process is necessary (meaning an argument that uses semantical notions such as truth values, assignments etc).

One of the main purposes of logic in computer science is to design mechanical models (meaning models that are suitable to computer implementation) for reasoning instead of semantical-based reasoning.

In this lecture, we will study a method for mechanizing the notion of semantical consequence. By mechnization, we understand a method for proving consequeneces with the following properties:

1. we should be able to convince someone that the consequence does take place, without the person having to following a semantical argument;
2. in particular, each proof step should be easy to check (mechanically, no RI needed);
3. in principle, the "someone" we have to convince should not necessarily be a human being, but a computer.

2 Sequents

Definition 2.1 (Sequent)**.** *A* sequent *is a pair consisting of of a set of formulae* $\{\varphi_1, \ldots, \varphi_n\}$ *and a formula* φ*, pair denoted by*

$$\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi.$$

The word *sequent* does not really exist in English. It is borrowed from the German word *Sequentz*.

Sometimes we read $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$ as φ *is a syntactical consequence of* $\{\varphi_1, \ldots, \varphi_n\}$. Many times, we will denote by $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ the set of premisses of the sequent and in this case we will write $\Gamma \vdash \varphi$. Also, the usual notion in the literature permits the writing $\varphi_1, \ldots, \varphi_n \vdash \varphi$ (without curly braces) instead of $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$, but we have to keep in mind that to the left hand side of the $\vdash$ symbol we always have a set of formulae.

Example 2.1. *Here are some examples of sequents:*

1. $\{\mathtt{p}, \mathtt{q}\} \vdash (\mathtt{p} \vee \mathtt{q})$;
2. $\{\mathtt{p}, \mathtt{q}\} \vdash (\mathtt{p} \wedge \mathtt{q})$;
3. $\{\mathtt{p}, \mathtt{q}\} \vdash (\mathtt{p} \wedge \mathtt{r})$.

Later on, we shall see that the first two sequents are valid*, and the last sequent is not.*

3 Inference Rules

Definition 3.1. *An* inference rule *is a tuple consisting of:*

1. *a set of sequents* $S_1, \ldots, S_n$*, called the* hypotheses *of the rule;*
2. *a sequent* S *that is called the* conclusion *of the rule;*
3. *a possible* condition *for applying the rule;*
4. *a name.*

An inference rule is denoted as follows:

$$\textsc{name}\ \frac{S_1 \qquad \ldots \qquad S_n}{S}\ \textit{condition}.$$

Remark 3.1. *It is possible for a sequent to have* $n = 0$ *hypotheses. Such inference rules, having* 0 *hypotheses, are called* axioms.

Remark 3.2. *Furthermore, the* condition *in an inference rule is optional.*

Example 3.1. *Here are some examples of inference rules:*

$$\wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,} \qquad \wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi'.}$$

All three rules above are sound*, in a sense that we will define later. None of the three rules above has a condition. Here is an example of a rule with $n = 0$ hypotheses, but having a condition:*

$$\textsc{Premiss}\ \frac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma.$$

Here is an example of an unsound inference rule (we will see later in what sense it is unsound):

$$\textsc{unsound rule}\ \frac{\Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi').}$$

Remark 3.3. *To be precise, the hypotheses of the inference rule, as well as the conclusion, are really* sequent schemes *and not sequents. This means that an inference rules might have several* instances*, obtained by replacing the mathematical variables $\varphi, \varphi', \Gamma$ with actual formulae. For example, here are two instances of the rule $\wedge i$ above:*

$$\wedge i\ \frac{\{\mathtt{p}, \mathtt{q}\} \vdash \mathtt{p} \qquad \{\mathtt{p}, \mathtt{q}\} \vdash \mathtt{q}}{\{\mathtt{p}, \mathtt{q}\} \vdash (\mathtt{p} \wedge \mathtt{q});} \qquad \wedge i\ \frac{\{\mathtt{p}, \mathtt{q}, \mathtt{r}\} \vdash (\mathtt{p} \wedge \mathtt{q}) \qquad \{\mathtt{p}, \mathtt{q}, \mathtt{r}\} \vdash \mathtt{p}}{\{\mathtt{p}, \mathtt{q}, \mathtt{r}\} \vdash ((\mathtt{p} \wedge \mathtt{q}) \wedge \mathtt{p}).}$$

In the first instance, we have replaced the mathematical variable Γ by the actual set of formulae $\{\mathtt{p}, \mathtt{q}\}$, the mathematical variable φ by the formula $\mathtt{p}$ *and the mathematical variable φ' by the formula* $\mathtt{q}$. *Exercise: what has each mathematical variable been replaced by in the second instance?*

Here is an example of a rule that is not an instance of the rule $\wedge i$ (exercise: explain why not):

$$\wedge i\ \frac{\{\mathtt{p}, \mathtt{q}\} \vdash \mathtt{p} \qquad \{\mathtt{p}, \mathtt{q}\} \vdash \mathtt{q}}{\{\mathtt{p}, \mathtt{q}\} \vdash (\mathtt{p} \wedge \mathtt{p}).}$$

4 Mai multe logici propoziționale

Up to this point, we have studied the propositional logic of the connectives $\neg, \wedge, \vee$, that we have denoted by *PL*. Actually, depending on the set of connectives that we need, there are several propositional logics. The logic that we have studied up to this point is $PL_{\neg,\wedge,\vee} = PL$.

Depending on the set of logical connectives allowed, we can have other interesting logics:

1. $PL_{\neg,\vee}$ is the propositional logic where the only connectives allowed are $\neg$ and $\vee$.
2. $PL_{\bot,\rightarrow}$ is the propositional logic where the only connectives allowed are $\bot$ (nullary connective) and $\rightarrow$.

   The formula $\bot$ is false in any assignment.
3. $PL_{\vee,\wedge}$ is a logic where the only connectives allowed are $\vee$ and $\wedge$.

Exercise: write the definition for the syntax of each logic above.

What do $PL, PL_{\neg,\vee}, PL_{\bot,\rightarrow}$ have in common? They are *equally expreissve*. Meaning that for any formula$\varphi \in PL$ there exists a formula $\varphi' \in PL_{\neg,\vee}$ such that $\varphi \equiv \varphi'$ (and vice-versa, for any formula $\varphi' \in PL_{\neg,\vee}$, there exista an equivalent formula in PL).

How can we show that PL and $PL_{\neg,\vee}$ are equally expressive? For one of the directions, it is sufficient to translate all possible conjunctions in PL by applying the following equivalence from left to right:

$$(\varphi \wedge \varphi') \equiv \neg(\neg\varphi \vee \neg\varphi').$$

At the end we will get an equivalent formula that has no conjunctions (and is therefore in $PL_{\neg,\vee}$). Vice-versa, any formula in $PL_{\neg,\vee}$ is already a formula in PL.

How can we show that $PL_{\neg,\vee}$ and $PL_{\bot,\rightarrow}$ are equally expressive?

We transform all disjunctions and negations using the following equivalences:

1. $(\varphi \vee \varphi') \equiv (\neg\varphi \rightarrow \varphi')$;
2. $\neg\varphi \equiv \varphi \rightarrow \bot$.

The transformation stops after a finite number of steps and the result is a formula equivalent to the formula that we started with, but which does not have any connectives other than $\bot$ and $\rightarrow$.

The logic $PL_{\vee,\wedge}$ is stricly less expressive. For example, in this logic there are no unsatisfiable formulae (exercise: explain why).

Remark 4.1. *By* propositional logic *we understand any logic that is equally expressive to PL. For example, $PL_{\neg,\wedge}$, $PL_{\neg,\vee}$, $PL_{\neg,\rightarrow}$ are all propositional logics, but $PL_{\neg}$ and $PL_{\wedge,\vee}$ are not propositional logics (they are less expressive).*

5 Proof system

Definition 5.1. *A* proof system *is a set of inference rules.*

Example 5.1. *Consider the proof system D_1, containing the following inference rules:*

$$\textsc{Premiss}\ \frac{}{\Gamma \vdash \varphi,}\ \varphi \in \Gamma \qquad \wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,}$$

$$\wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi'.}$$

6 Formal Proofs

Definition 6.1 (Formal Proof)**.** *A* formal proof *in a proof system is a list of sequents*

1. S_1;
2. S_2;

...

n. S_n,

such that each sequent S_i is justified by an inference rule in the proof system from the previous sequents ($S_1, \ldots, S_{i-1}$), in the sense that S_i is the conclusion of an instance of an inference rule in the proof system that uses hypotheses only among the sequents $S_1, \ldots, S_{i-1}$ and such that the condition of the inference rule is true (if the inference rule has a condition).

Example 6.1. *Here is an example of a formal proof in the proof system D_1 defined above:*

1. $\{\mathtt{p}, \mathtt{q}\} \vdash \mathtt{p}$; (Premiss)
2. $\{\mathtt{p}, \mathtt{q}\} \vdash \mathtt{q}$; (Premiss)
3. $\{\mathtt{p}, \mathtt{q}\} \vdash (\mathtt{p} \wedge \mathtt{q})$; ($\wedge i$, 1, 2)
4. $\{\mathtt{p}, \mathtt{q}\} \vdash (\mathtt{q} \wedge (\mathtt{p} \wedge \mathtt{q}))$. ($\wedge i$, 2, 3)

Note that each line is annotated by the name of the inference rule that was applied, plus the lines where the hypotheses of the rule are found.

Definition 6.2 (Valid Sequent)**.** *A sequent $\Gamma \vdash \varphi$ is* valid in a proof system D *if there exists a formal proof $S_1, \ldots, S_n$ in D such that $S_n = \Gamma \vdash \varphi$.*

Example 6.2. *The sequent $\{\mathtt{p}, \mathtt{q}\} \vdash (\mathtt{p} \wedge \mathtt{q})$ is valid in the proof system D_1 above, because it is the last sequent in the following formal proof:*

1. $\{\mathtt{p}, \mathtt{q}\} \vdash \mathtt{p}$; (Premiss)

2. $\{\mathtt{p}, \mathtt{q}\} \vdash \mathtt{q}$; *(*PREMISS*)*

3. $\{\mathtt{p}, \mathtt{q}\} \vdash (\mathtt{p} \wedge \mathtt{q})$. *(*$\wedge i$, 1, 2*)*

Remark 6.1. *Do not mistake the notion of* sequent valid in a proof system *for the notion of* valid formula*.*

7 Natural Deduction

Natural deduction is a proof system for $PL_{\neg,\wedge,\vee,\rightarrow,\bot}$. In this proof system, each logical connective has one or more introduction rules and one or more elimination rules.

7.1 The rules for conjunction

We have already seen the introduction and elimination rules for the connective "and":

$$\wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,} \qquad \wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi'.}$$

The proof system is called *natural* deduction because it mimics the human reasoning process:

1. The rule for introducing "and" indicates that we can prove a conjunction $\varphi \wedge \varphi'$ from Γ if we know that each conjunct, φ and φ' respectively, are consequences of Γ.

   In other words, to show that a conjunction follows from Γ, it is enough to show individually that each conjunct follows from Γ.

2. There are two elimination rules for "and". The first elimination rule indicates that if we have already established that a conjunction $(\varphi \wedge \varphi')$ follows from Γ, then the left-hand side conjunct, φ, also follows from Γ.

   The second elimination rule is symmetrical.

Here is an example of a formal proof that uses the inference rules for "and":

1. $\{(\mathtt{p} \wedge \mathtt{q}), \mathtt{r}\} \vdash (\mathtt{p} \wedge \mathtt{q})$; (PREMISS)
2. $\{(\mathtt{p} \wedge \mathtt{q}), \mathtt{r}\} \vdash \mathtt{r}$; (PREMISS)
3. $\{(\mathtt{p} \wedge \mathtt{q}), \mathtt{r}\} \vdash \mathtt{p}$; ($\wedge e_1$, 1)
4. $\{(\mathtt{p} \wedge \mathtt{q}), \mathtt{r}\} \vdash (\mathtt{p} \wedge \mathtt{r})$. ($\wedge i$, 3, 2)

Exercises:

1. $\{((\mathtt{p} \wedge \mathtt{q}) \wedge \mathtt{r})\} \vdash (\mathtt{q} \wedge \mathtt{r})$;
2. $\{((\mathtt{p} \wedge \mathtt{q}) \wedge \mathtt{r}), \mathtt{r}'\} \vdash (\mathtt{r}' \wedge \mathtt{q})$;
3. $\{((\mathtt{p} \wedge \mathtt{q}) \wedge \mathtt{r})\} \vdash (\mathtt{r} \wedge (\mathtt{q} \wedge \mathtt{p}))$.

7.2 The Rules for Implications

The rule for eliminating implications, also known as *modus ponens* in latin, is one of the most important inference rules.

$$\to e\ \frac{\Gamma \vdash (\varphi \to \varphi') \qquad \Gamma \vdash \varphi}{\Gamma \vdash \varphi'}$$

The rule tells us that, assuming that we already know that $\varphi \to \varphi'$ follows from Γ and that φ follows from Γ, then we also have that φ' follows from Γ.

Here is an example of formal proof that uses the modus ponens rule:

1. $\{(\mathtt{p} \to \mathtt{r}), (\mathtt{p} \land \mathtt{q})\} \vdash (\mathtt{p} \land \mathtt{q})$; (Premiss)
2. $\{(\mathtt{p} \to \mathtt{r}), (\mathtt{p} \land \mathtt{q})\} \vdash \mathtt{p}$; $(\land e_1,\ 1)$
3. $\{(\mathtt{p} \to \mathtt{r}), (\mathtt{p} \land \mathtt{q})\} \vdash (\mathtt{p} \to \mathtt{r})$; (Premiss)
4. $\{(\mathtt{p} \to \mathtt{r}), (\mathtt{p} \land \mathtt{q})\} \vdash \mathtt{r}$. $(\to e,\ 3,\ 1)$

This formal proof shows that the sequent $\{(\mathtt{p} \to \mathtt{r}), (\mathtt{p} \land \mathtt{q})\} \vdash \mathtt{r}$ is valid, meaning that the formula $\mathtt{r}$ is a consequence of $\{(\mathtt{p} \to \mathtt{r}), (\mathtt{p} \land \mathtt{q})\}$. Note the order of the line numbers 3 and 1 in the annotation for line 4 (they follow the order in the inference rule).

Exercises:

1. $\{((\mathtt{p} \land \mathtt{q}) \to \mathtt{r}), \mathtt{p}, \mathtt{q}\} \vdash \mathtt{r}$;
2. $\{(\mathtt{p} \to \mathtt{r}), \mathtt{p}, \mathtt{q}\} \vdash (\mathtt{q} \land \mathtt{r})$.

The rule for introducing implication is more subtle. To show that an implication $(\varphi \to \varphi')$ follows from Γ, we *a*ssume φ (in addition to Γ) and we show φ'. In other words, in the hypothesis of the rule, we add the formula φ to the formulae in Γ. The rule can be written in two equivalent ways, that differ only in their use of the convention regarding the curly braces:

$$\to i\ \frac{\Gamma, \varphi \vdash \varphi'}{\Gamma \vdash (\varphi \to \varphi'),} \qquad\qquad \to i\ \frac{\Gamma \cup \{\varphi\} \vdash \varphi'}{\Gamma \vdash (\varphi \to \varphi').}$$

What is important to note and understand for the rule for introducing implication is that the premisses of the sequent in the hypotheses are different from the premisses of the sequent in the conclusion. Whereas in the conclusion we have that the formula $(\varphi \to \varphi')$ follows from Γ, in the hypothesis we have to show that φ' follows from $\Gamma \cup \{\varphi\}$. In other words, intuitively speaking, to prove an implication $(\varphi \to \varphi')$, we assume the antecedent φ and we show the consequent φ'.

Example 7.1. *Let us show that the sequent* $\{\} \vdash (\mathtt{p} \to \mathtt{p})$ *is valid:*

1. $\{\mathtt{p}\} \vdash \mathtt{p}$; *(*Premiss*)*
2. $\{\} \vdash (\mathtt{p} \to \mathtt{p})$. $(\to i,\ 1)$

Example 7.2. *Let us show that the sequent* $\{(\mathtt{p} \to \mathtt{q})\} \vdash (\mathtt{p} \to \mathtt{q})$ *is valid. A simple proof is:*

1. $\{(\mathtt{p} \to \mathtt{q})\} \vdash (\mathtt{p} \to \mathtt{q})$. *(*Premiss*)*

A longer proof is:

1. $\{(\mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash (\mathtt{p} \to \mathtt{q})$; *(*Premiss*)*
2. $\{(\mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash \mathtt{p}$; *(*Premiss*)*
3. $\{(\mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash \mathtt{q}$; $(\to e,\ 1,\ 2)$
4. $\{(\mathtt{p} \to \mathtt{q})\} \vdash (\mathtt{p} \to \mathtt{q})$. $(\to i,\ 3)$

Example 7.3. *Let us show that the sequent* $\{(\mathtt{p} \to \mathtt{q}), (\mathtt{q} \to \mathtt{r})\} \vdash (\mathtt{p} \to \mathtt{r})$ *is valid:*

1. $\{(\mathtt{p} \to \mathtt{q}), (\mathtt{q} \to \mathtt{r}), \mathtt{p}\} \vdash (\mathtt{p} \to \mathtt{q})$; *(*Premiss*)*
2. $\{(\mathtt{p} \to \mathtt{q}), (\mathtt{q} \to \mathtt{r}), \mathtt{p}\} \vdash \mathtt{p}$; *(*Premiss*)*
3. $\{(\mathtt{p} \to \mathtt{q}), (\mathtt{q} \to \mathtt{r}), \mathtt{p}\} \vdash \mathtt{q}$; $(\to e,\ 1,\ 2)$
4. $\{(\mathtt{p} \to \mathtt{q}), (\mathtt{q} \to \mathtt{r}), \mathtt{p}\} \vdash (\mathtt{q} \to \mathtt{r})$; *(*Premiss*)*
5. $\{(\mathtt{p} \to \mathtt{q}), (\mathtt{q} \to \mathtt{r}), \mathtt{p}\} \vdash \mathtt{r}$; $(\to e,\ 4,\ 3)$
6. $\{(\mathtt{p} \to \mathtt{q}), (\mathtt{q} \to \mathtt{r})\} \vdash (\mathtt{p} \to \mathtt{r})$. $(\to i,\ 5)$

Exercise 7.1. *Show that the following sequents are valid:*

1. $\{((\mathtt{p} \wedge \mathtt{q}) \to \mathtt{r}), \mathtt{p}, \mathtt{q}\} \vdash \mathtt{r}$;
2. $\{((\mathtt{p} \wedge \mathtt{q}) \to \mathtt{r})\} \vdash (\mathtt{p} \to (\mathtt{q} \to \mathtt{r}))$;
3. $\{(\mathtt{p} \to (\mathtt{q} \to \mathtt{r}))\} \vdash ((\mathtt{p} \wedge \mathtt{q}) \to \mathtt{r})$.

7.3 Rules for Disjunctions

The connective "ore" has two introduction rules:

$$\vee i_1\ \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2),} \qquad\qquad \vee i_2\ \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2).}$$

The first rule indicates that if we already know φ_1 follows from Γ, then $(\varphi_1 \vee \varphi_2)$ must also follow from Γ, what φ_2 is. In other words, intuitively speaking, if we know φ_1, then we also know ($\varphi_1 \vee$ *whatever else*). The second introduction rule for disjunction is symmetrical.

Example 7.4. *Let us show that the sequent* $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash (\mathtt{p} \vee \mathtt{q})$ *is valid:*

1. $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash (\mathtt{p} \wedge \mathtt{q})$; *(*PREMISS*)*
2. $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash \mathtt{p}$; $(\wedge e_1,\ 1)$
3. $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash (\mathtt{p} \vee \mathtt{q})$. $(\vee i_1,\ 2)$

Another formal proof for the same sequent is:

1. $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash (\mathtt{p} \wedge \mathtt{q})$; *(*PREMISS*)*
2. $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash \mathtt{q}$; $(\wedge e_2,\ 1)$
3. $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash (\mathtt{p} \vee \mathtt{q})$. $(\vee 2_1,\ 2)$

Exercise 7.2. *Show that the sequent* $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash (\mathtt{r} \vee \mathtt{p})$ *is valid.*

The rule for eliminating disjunctions is slightly more complicated, being another rule where the set of premisses of the sequents changes from hypotheses to conclusion:

$$\vee e\ \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \qquad \Gamma, \varphi_1 \vdash \varphi' \qquad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi'}$$

The first hypothesis of the rule, $\Gamma \vdash (\varphi_1 \vee \varphi_2)$, is easy to understand: to "eliminate" a disjunction, we must have such a disjunction among the hypotheses (disjunction that we will "eliminate" in the conclusion). The following two hypotheses for disjunction elimination must be understood intuitively as follows. From the first hypothesis, we have that $(\varphi_1 \vee \varphi_2)$ follows from Γ; in other words, at least one of the formulae φ_1 and φ_2 follows from Γ. The hypotheses 2 and 3 indicate that, no matter which of φ_1 and φ_2 would hold, in any case φ' holds. Meaning that if we assume φ_1 (in addition to Γ), φ' holds, and if we assume φ_2 (in addition to Γ), φ' still holds. And then the conclusion indicates that φ' holds independently of which of φ_1 and φ_2 holds.

Example 7.5. *Let us show that the sequent* $\{(\mathtt{p} \vee \mathtt{q})\} \vdash (\mathtt{q} \vee \mathtt{p})$ *is valid:*

1. $\{(\mathtt{p} \vee \mathtt{q}), \mathtt{p}\} \vdash \mathtt{p}$; *(*PREMISS*)*
2. $\{(\mathtt{p} \vee \mathtt{q}), \mathtt{p}\} \vdash (\mathtt{q} \vee \mathtt{p})$; $(\vee i_2,\ 1)$
3. $\{(\mathtt{p} \vee \mathtt{q}), \mathtt{q}\} \vdash \mathtt{q}$; *(*PREMISS*)*
4. $\{(\mathtt{p} \vee \mathtt{q}), \mathtt{q}\} \vdash (\mathtt{q} \vee \mathtt{p})$; $(\vee i_1,\ 1)$
5. $\{(\mathtt{p} \vee \mathtt{q})\} \vdash (\mathtt{p} \vee \mathtt{q})$; *(*PREMISS*)*
6. $\{(\mathtt{p} \vee \mathtt{q})\} \vdash (\mathtt{q} \vee \mathtt{p})$. $(\vee e,\ 5,\ 2,\ 4)$

Note that way in which the set of premisses varies from one sequent to another along the formal proof, obbeying the inference rules.

Exercise 7.3. *Show that the sequent* $\{(\mathtt{p} \vee \mathtt{q}), (\mathtt{p} \rightarrow \mathtt{r}), (\mathtt{q} \rightarrow \mathtt{r})\} \vdash \mathtt{r}$ *is valid.*

Exercise 7.4. *Show that the sequent* $\{(\mathtt{p} \rightarrow \mathtt{r}), (\mathtt{q} \rightarrow \mathtt{r})\} \vdash ((\mathtt{p} \vee \mathtt{q}) \rightarrow \mathtt{r})$ *is valid.*

7.4 The Rules for Negations

The rules for introducing and respectively eliminating negations must be presented at the same time as the rules for $\bot$:

$$\neg i\ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi} \qquad \neg e\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \bot} \qquad \bot e\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

Recall that $\bot$ is a nullary logical connective (it has arity 0), meaning that $\bot$ connects 0 formulae among themselves. In other words, the connective $\bot$ is by itself a formula. The semantics of $\bot$ is that it is false in any assignment. In other words, $\bot$ is a contradiction.

The first rule above is that for introducing negations. It is easy to explain intuitively: how can we show that a formula of the form $\neg\varphi$ follows from Γ? We assume φ in addition to Γ and we show that from Γ and φ follows a contradiction ($\Gamma, \varphi \vdash \bot$). In this way, we have that $\neg\varphi$ follows from Γ.

The second rule, for eliminating negations, indicates that if both a formula, φ, and its negation, $\neg\varphi$, follow from the same set of premisses Γ, then from Γ follows a contradiction, $\bot$. A set Γ from which a contradiction follows is called *inconsistent*.

The third rule indicates that, if Γ is an inconsistent set, then any formula φ follows from Γ.

There is no rule for introducing $\bot$ (or, equivallently, the rule for eliminating negation can be considered as the rule for introducing $\bot$).

Example 7.6. *Let us show that the sequent* $\{\mathtt{p}\} \vdash \neg\neg\mathtt{p}$ *is valid:*

1. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \mathtt{p}$; (Premiss)
2. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \neg\mathtt{p}$; (Premiss)
3. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \bot$; ($\neg e$, 1, 2)
4. $\{\mathtt{p}\} \vdash \neg\neg\mathtt{p}$. ($\neg i$, 3)

Example 7.7. *Let us show that the sequent* $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \mathtt{r}$ *is valid:*

1. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \mathtt{p}$; (Premiss)
2. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \neg\mathtt{p}$; (Premiss)
3. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \bot$; ($\neg e$, 1, 2)
4. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \mathtt{r}$. ($\bot e$, 3)

Exercise 7.5. *Show that the following sequents are valid:*

1. $\{(\mathtt{p} \vee \mathtt{q})\} \vdash \neg(\neg\mathtt{p} \wedge \neg\mathtt{q})$;
2. $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash \neg(\neg\mathtt{p} \vee \neg\mathtt{q})$;

3. $\{(\neg \mathtt{p} \vee \neg \mathtt{q})\} \vdash \neg(\mathtt{p} \wedge \mathtt{q})$;
4. $\{(\neg \mathtt{p} \wedge \neg \mathtt{q})\} \vdash \neg(\mathtt{p} \vee \mathtt{q})$;
5. $\{\neg(\mathtt{p} \vee \mathtt{q})\} \vdash (\neg \mathtt{p} \wedge \neg \mathtt{q})$.

7.5 Alte reguli

Another useful rule, that is not related to any given connective is the EXTEND rule:

$$\text{EXTEND}\ \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi}$$

This rule indicates that, if φ is a consequence of a set of formulae Γ, then φ is also a consequence of $\Gamma \cup \{\varphi'\}$ (whatever φ' is). In other words, we can extend the set of premisses of a valid sequent and still get a valid sequent.

Example 7.8. *Let us show that the sequent* $\{\mathtt{p}, \neg\mathtt{q}, \mathtt{r}, (\mathtt{q}_1 \wedge \mathtt{q}_2)\} \vdash \neg\neg\mathtt{p}$ *is valid:*

1. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \mathtt{p}$; (PREMISS)
2. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \neg\mathtt{p}$; (PREMISS)
3. $\{\mathtt{p}, \neg\mathtt{p}\} \vdash \bot$; ($\neg e$, 1, 2)
4. $\{\mathtt{p}\} \vdash \neg\neg\mathtt{p}$; ($\neg i$, 3)
5. $\{\mathtt{p}, \neg\mathtt{q}\} \vdash \neg\neg\mathtt{p}$; (EXTEND, 4)
6. $\{\mathtt{p}, \neg\mathtt{q}, \mathtt{r}\} \vdash \neg\neg\mathtt{p}$; (EXTEND, 5)
7. $\{\mathtt{p}, \neg\mathtt{q}, \mathtt{r}, (\mathtt{q}_1 \wedge \mathtt{q}_2)\} \vdash \neg\neg\mathtt{p}$. (EXTEND, 6)

The rules above are natural deduction for a logic that is called *intuitionistic propositional logic*. In this lecture, we study *classical propositional logic*. The two logics have the same syntax, but their semantics are different. Intuitionist logic is important in computer science, because formal proofs in this logic are in a 1-to-1 correspondence with computer programs. Meaning that to each intuitionistic proof we can associate a program, and to each program, an intuitionistic proof. This mean essentially that each intuitionistic proof *is* a program. Intuitionistic logic is a *constructive logic*, meaning a logic where any proof gives an algorithm.

To obtain a proof system for classical propositional logic, which is the object of our study, we need one more rule:

$$\neg\neg e\ \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}$$

Example 7.9. *Let us show that the sequent* $\{(\neg \mathtt{p} \to \mathtt{q}), \neg \mathtt{q}\} \vdash \mathtt{p}$ *is valid:*

1. $\{(\neg \mathtt{p} \to \mathtt{q}), \neg \mathtt{q}, \neg \mathtt{p}\} \vdash \neg \mathtt{p}$; *(*Premiss*)*
2. $\{(\neg \mathtt{p} \to \mathtt{q}), \neg \mathtt{q}, \neg \mathtt{p}\} \vdash (\neg \mathtt{p} \to \mathtt{q})$; *(*Premiss*)*
3. $\{(\neg \mathtt{p} \to \mathtt{q}), \neg \mathtt{q}, \neg \mathtt{p}\} \vdash \mathtt{q}$; $(\to e,\ 2,\ 1)$
4. $\{(\neg \mathtt{p} \to \mathtt{q}), \neg \mathtt{q}, \neg \mathtt{p}\} \vdash \neg \mathtt{q}$; *(*Premiss*)*
5. $\{(\neg \mathtt{p} \to \mathtt{q}), \neg \mathtt{q}, \neg \mathtt{p}\} \vdash \bot$; $(\neg i,\ 4,\ 3)$
6. $\{(\neg \mathtt{p} \to \mathtt{q}), \neg \mathtt{q}\} \vdash \neg\neg \mathtt{p}$; $(\neg i,\ 5)$
7. $\{(\neg \mathtt{p} \to \mathtt{q}), \neg \mathtt{q}\} \vdash \mathtt{p}$. $(\neg\neg e,\ 6)$

Example 7.10. *Let us show that the sequent* $\{\} \vdash (p \vee \neg p)$ *is valid:*

1. $\{\neg(\mathtt{p} \vee \neg \mathtt{p}), \mathtt{p}\} \vdash \neg(\mathtt{p} \vee \neg \mathtt{p})$; *(*Premiss*)*
2. $\{\neg(\mathtt{p} \vee \neg \mathtt{p}), \mathtt{p}\} \vdash \mathtt{p}$; *(*Premiss*)*
3. $\{\neg(\mathtt{p} \vee \neg \mathtt{p}), \mathtt{p}\} \vdash (\mathtt{p} \vee \neg \mathtt{p})$; $(\vee i_1,\ 2)$
4. $\{\neg(\mathtt{p} \vee \neg \mathtt{p}), \mathtt{p}\} \vdash \bot$; $(\neg e,\ 1,\ 3)$
5. $\{\neg(\mathtt{p} \vee \neg \mathtt{p})\} \vdash \neg \mathtt{p}$; $(\neg i,\ 4)$
6. $\{\neg(\mathtt{p} \vee \neg \mathtt{p})\} \vdash (\mathtt{p} \vee \neg \mathtt{p})$; $(\vee i_2,\ 5)$
7. $\{\neg(\mathtt{p} \vee \neg \mathtt{p})\} \vdash \neg(\mathtt{p} \vee \neg \mathtt{p})$; *(*Premiss*)*
8. $\{\neg(\mathtt{p} \vee \neg \mathtt{p})\} \vdash \bot$; $(\neg e,\ 7,\ 6)$
9. $\{\} \vdash \neg\neg(\mathtt{p} \vee \neg \mathtt{p})$; $(\neg i,\ 8)$
10. $\{\} \vdash (\mathtt{p} \vee \neg \mathtt{p})$. $(\neg\neg e,\ 9)$

Exercise 7.6. *Show that the following sequents are valid:*

1. $\{\neg(\mathtt{p} \wedge \mathtt{q})\} \vdash (\neg \mathtt{p} \vee \neg \mathtt{q})$;
2. $\{\neg(\neg \mathtt{p} \vee \neg \mathtt{q})\} \vdash (\mathtt{p} \wedge \mathtt{q})$;
3. $\{\neg(\neg \mathtt{p} \wedge \neg \mathtt{q})\} \vdash (\mathtt{p} \vee \mathtt{q})$.

8 Natural Deduction

Natural deduction is the proof system containing all rules above. Here is a summary of all rules:

$$\wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,} \qquad \wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi',}$$

$$\to e\ \frac{\Gamma \vdash (\varphi \to \varphi') \qquad \Gamma \vdash \varphi}{\Gamma \vdash \varphi',} \qquad \to i\ \frac{\Gamma, \varphi \vdash \varphi'}{\Gamma \vdash (\varphi \to \varphi'),} \qquad \vee i_1\ \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2),}$$

$$\vee i_2\ \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2),} \qquad \vee e\ \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \qquad \Gamma, \varphi_1 \vdash \varphi' \qquad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi',}$$

$$\neg e\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \bot,} \qquad \neg i\ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi,} \qquad \bot e\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi,}$$

$$\textsc{Premiss}\ \frac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma, \qquad \textsc{Extend}\ \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi,} \qquad \neg\neg e\ \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi.}$$

9 Derived Rules

A *derived rule* is a proof rule that we can prove using the other proof rules in a given proof system, through a formal proof.

An example of a derived rule is the rule for introducing double negation:

$$\neg\neg i\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \neg\neg\varphi.}$$

Here is a formal proof for the rule for introducing double negation. We start with the hypotheses of the rule, and the last sequent in the proof must be the conclusion of the rule.

1. $\Gamma \vdash \varphi$; (hypothesis of the derived rule)
2. $\Gamma, \neg\varphi \vdash \varphi$; (Extend, 1)
3. $\Gamma, \neg\varphi \vdash \neg\varphi$; (Premiss)
4. $\Gamma, \neg\varphi \vdash \bot$; ($\neg e$, 2, 3)
5. $\Gamma \vdash \neg\neg\varphi$. ($\neg i$, 4)

After having been shown derived (as above), such a proof rule can be used to shorten other proofs, similarly to how we shorten code by writing subprograms (functions) in certain programming languages. Here is an example of a formal proof that uses the derived rule $\neg\neg i$:

1. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash \mathtt{p}$; (Premiss)
2. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash \neg\neg\mathtt{p}$; ($\neg\neg i$, 1)
3. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash (\neg\neg \mathtt{p} \to \mathtt{q})$; (Premiss)
4. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash \mathtt{q}$. ($\to e$, 3, 2)

In the absence of the derived rule, our proof would have been longer:

1. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash \mathtt{p}$; (Premiss)
2. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}, \neg\mathtt{p}\} \vdash \mathtt{p}$; (Extend, 1)
3. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}, \neg\mathtt{p}\} \vdash \neg\mathtt{p}$; (Ipoteza)
4. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}, \neg\mathtt{p}\} \vdash \bot$; ($\neg e$, 2, 3)
5. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash \neg\neg\mathtt{p}$. ($\neg i$, 4)
6. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash (\neg\neg \mathtt{p} \to \mathtt{q})$; (Premiss)
7. $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash \mathtt{q}$. ($\to e$, 6, 5)

Note that the lines 1–5 in the longer proof for the sequent $\{(\neg\neg \mathtt{p} \to \mathtt{q}), \mathtt{p}\} \vdash \mathtt{q}$ are an instance of the the formal proof for the derived rule.

10 Soundness and Completeness for Natural Deduction

Theorem 10.1 (Soundness of Natural Deduction)**.** *For any set of formulae Γ and and formula φ, if the sequent $\Gamma \vdash \varphi$ is valid, then $\Gamma \models \varphi$.*

Exercise: show the theorem during the tutorial.

Theorem 10.2 (Completeness of Natural Deduction)**.** *For any set of formulae Γ and any formula φ, if $\Gamma \models \varphi$ then the sequent $\Gamma \vdash \varphi$ is valid.*

The proof of the completeness theorem is beyond the level of the lecture.

Remark 10.1. *Note that, due to the soundness and completeness theorems, the relation $\vdash$ coincides with the relation $\models$, even if they have totally different definitions.*