Tema 2 - Factorizarea Cholesky

$$4x_1 + 2x_2 + 4x_3 = 12$$
$$2x_1 + 2x_2 + 2x_3 = 6$$
$$4x_1 + 2x_2 + 5x_3 = 13$$

- A - pozitiv definită
- calcul $A = LL^T$, L inf. triunghiulară
- calcul soluție sistem
- calcul A^{-1}

A pozitiv definită $\overset{def}{\Longleftrightarrow} (Ax, x) > 0 \quad \forall x \in \mathbb{R}^3 \; x \neq 0$

$$\Longleftrightarrow \quad \det \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & & & \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{vmatrix} > 0 \quad \forall r = \overline{1, n=3}$$

$$a_{11} = 4 > 0 \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 4 > 0$$

$$\begin{vmatrix} a_{11} & \dots & a_{13} \\ \vdots & & \\ a_{31} & \dots & a_{33} \end{vmatrix} = \det A > 0$$

$$\det A = \det L * \det L^T = (\det L)^2 =$$
$$(l_{11} * l_{22} \dots * l_{nn})^2 \underset{n=3}{=} (l_{11} l_{22} l_{33})^2$$

Calcul $A = LL^T$

$$A = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 2 & 2 \\ 4 & 2 & 5 \end{pmatrix} = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}}_{L^T}$$

$$= \begin{pmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ l_{21} l_{11} & l_{21}^2 + l_{22}^2 & l_{21} l_{31} + l_{22} l_{32} \\ l_{31} l_{11} & l_{31} l_{21} + l_{32} l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}$$

<u>Pas 1</u>: se calculează elementele coloanei 1 a matricei L: l_{11}, l_{21}, l_{31}

l_{11}: $a_{11} = (LL^T)_{11} \Rightarrow 4 = l_{11}^2 \Rightarrow$
$l_{11} = \pm 2$ $\boxed{l_{11} = 2}$

l_{21}: $a_{21} = (LL^T)_{21} \Rightarrow 2 = l_{21} \cdot \underbrace{l_{11}}_{=2} \Rightarrow$
$\boxed{l_{21} = 1}$

l_{31}: $a_{31} = (LL^T)_{31} \Rightarrow 4 = l_{31} l_{11} \Rightarrow \boxed{l_{31} = 2}$

<u>Pas 2</u> se calculează elementele coloanei 2 a matricei L: l_{22}, l_{32}

l_{22}: $a_{22} = (LL^T)_{22} \Rightarrow 2 = \underbrace{l_{21}^2}_{=1} + l_{22}^2$

$\Rightarrow l_{22} = \pm 1 \Rightarrow \boxed{l_{22} = 1}$

l_{32}: $a_{32} = (LL^T)_{32} \Rightarrow 2 = \underbrace{l_{31}}_{=2} \underbrace{l_{21}}_{=1} + l_{32} \underbrace{l_{22}}_{=1}$

$\Rightarrow \boxed{l_{32} = 0}$

<u>Pas 3</u>: se determină coloana 3 din L: l_{33}

l_{33}: $a_{33} = (LL^T)_{33}$ $\quad 5 = \underbrace{l_{31}^2}_{2^2} + \underbrace{l_{32}^2}_{0^2} + l_{33}^2$

$\Rightarrow l_{33} = \pm 1 \Rightarrow \boxed{l_{33} = 1}$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

Se poate verifica:

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 2 & 2 \\ 4 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculul soluției sistemului

Se rezolvă sistemele triunghiulare

$$Ly = b \rightarrow \text{soluția } y^*$$

$$L^T x = y^* \rightarrow \text{soluția } x^* \text{ este și soluția sistemului } Ax = b$$

$Ly = b$:

$$\begin{aligned} 2y_1 \quad\quad\quad &= 12 \Rightarrow y_1^* = 12/2 = 6 \\ y_1 + y_2 \quad\quad &= 6 \rightarrow y_2^* = 6 - y_1^* = 0 \\ 2y_1 \quad\quad + y_3 &= 13 \Rightarrow y_3^* = 13 - 2y_1^* = 1 \end{aligned}$$

$L^T x = y^*$:

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 6 \Rightarrow x_1^* = (6 - x_2^* - 2x_3^*)/2 = 2 \\ x_2 &= 0 \Rightarrow x_2^* = 0 \\ x_3 &= 1 \Rightarrow x_3^* = 1 \end{aligned}$$

Soluția sistemului este $x_{chol} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$