Vectori și valori proprii

$A \in \mathbb{R}^{n\times n}$ $\quad \lambda \in \mathbb{C}$ - valoare proprie dacă

$\exists\, u \in \mathbb{C}^n, u \neq 0$ a.î. $\quad$ u - vector propriu

$$Au = \lambda u$$

λ - valoare proprie $\Longleftrightarrow \det(\lambda I - A) = 0$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Calculul valorilor proprii:

$$\det(\lambda I - A) = 0$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & -2 \\ -2 & \lambda & -1 \\ 0 & -2 & \lambda-1 \end{vmatrix} = \lambda(\lambda-1)^2 - 8 - 2(\lambda-1) = \lambda^3 - 2\lambda^2 - \lambda - 6$$

$p_A(\lambda)$ = polinomul caracteristic al matricei A

$$= \lambda^3 - 2\lambda^2 - \lambda - 6 = (\lambda-3)(\lambda^2+\lambda+2) = 0$$

$\lambda_1 = 3 \quad \lambda_{2,3} = \dfrac{-1 \pm i\sqrt{7}}{2}$ - valorile proprii ale matricei A

Calculul vectorilor proprii asociați valorii proprii $\lambda = 3$

$$A\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 3\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

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$$u_1 + \quad + 2u_3 = 3u_1 \Rightarrow u_1 = u_3$$
$$2u_1 + \quad + u_3 = 3u_2 \Rightarrow u_1 = u_2$$
$$2u_2 + u_3 = 3u_3 \Rightarrow 0 = 0$$

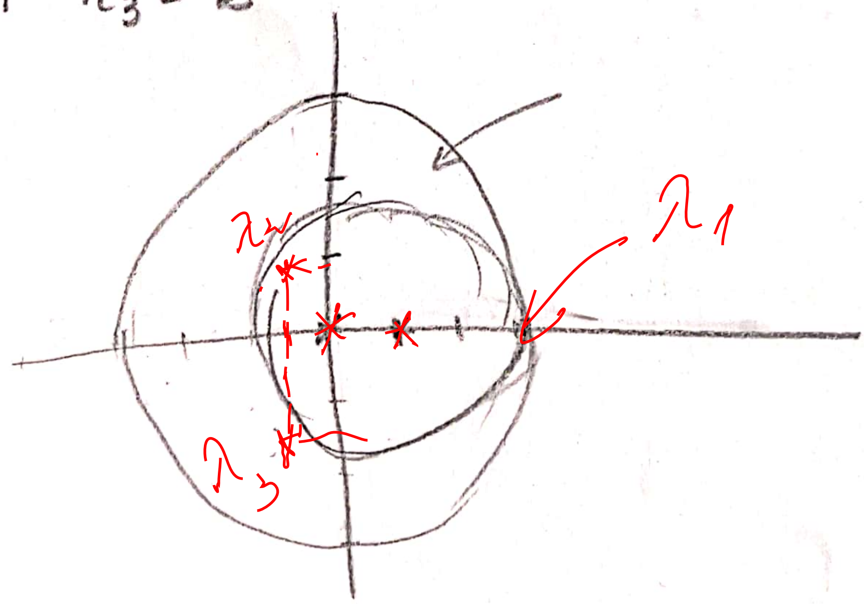
$\Rightarrow \forall u \in \mathbb{R}^3 \quad u = \begin{pmatrix} a \\ a \\ a \end{pmatrix} \quad a \in \mathbb{R} \quad a \neq 0$ este vector propriu al matricei A asociat lui $\lambda = 3$.

Teorema Gershgorin spune unde sunt plasate toate valorile proprii ale unei matrice, în spațiul $\mathbb{C}$:

$$\bigcup_{i=1}^{n} \left(\text{cerc de centru } a_{ii} \text{ și rază } \sum_{j \neq i} a_{ij}\right)$$

$a_{11} = 1 \quad r_1 = 2, \quad a_{22} = 0 \quad r_2 = 3,$

$a_{33} = 1 \quad r_3 = 2$



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Metoda puterii

(ptr găsit un vector propriu asociat valorii proprii de modul maxim, matrice simetrice)

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-1 \end{vmatrix} =$$

$$= (\lambda-1)\begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-1)[(\lambda-1)^2-1] = \lambda(\lambda-1)(\lambda-2)$$

$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = \lambda_{max} = 2$

(o matrice cu $\det A = 0$ are întotdeauna val. proprie $\lambda = 0$)

u^{max}? $\quad A u^{max} = \lambda_{max} u^{max}$

$$A\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 2\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \begin{cases} u_1 + \quad + u_3 = 2u_1 \\ \quad u_2 \quad = 2u_2 \\ u_1 + \quad + u_3 = 2u_3 \end{cases} \Rightarrow$$

$u_1 = u_3$, $u_2 = 0$ $\Rightarrow$ $\forall$ vect propriu asociat valorii proprii $\lambda = 2$ are forma

$$u^{max} = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} \text{ cu } a \in \mathbb{R} \ a \neq 0$$

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Dacă $u \in \mathbb{R}^n$ $u \neq 0$ este vect. propriu, valoarea proprie asociată se poate calcula folosind coeficientul Rayleigh.

$$r(u) = \frac{(Au, u)}{\|u\|_2^2}$$

Dacă $\|u\|_2 = 1 \Rightarrow r(u) = (Au, u)$.

Metoda puterii pentru matricea A:

$$u^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad w = Au^{(0)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda^0 = (w, u^{(0)}) = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = 1$$

$$u^{(1)} = \frac{1}{\|w\|_2} w = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \quad w = Au^{(1)}$$

$$w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix}$$

$$\lambda^1 = (w, u^{(1)}) = \left(\begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right) = 2$$

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$u^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad w = Au^{(0)} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$u^{(1)} = u^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow u^{(k)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; \forall k$

metoda puterii nu converge la u^{max} deoarece $(u^{(0)}, u^{max}) = 0$.

Metoda iterației inverse

(de găsire a unui vector propriu asociat unei valori proprii cea mai apropiată de un număr dat μ)

$\mu = -1 \qquad A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \lambda_\mu = 0$

- se aplică met. puterii pt. matricea $(A - \mu I)^{-1} = (A + I)^{-1}$

- $A + I = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \qquad (A+I)^{-1} = \begin{pmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/2 & 0 \\ -1/3 & 0 & 2/3 \end{pmatrix}$

$\det[\lambda I - (A+I)^{-1}] = (\lambda - 1/2)(\lambda - 1)(\lambda - \frac{1}{3})$

$\tilde{\lambda}_{max} = 1 = \dfrac{1}{\lambda_\mu - \mu} = \dfrac{1}{0+1}$

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- nu se calc. $(A+I)^{-1}$ ci la fiec pas se rezolvă un sist liniar.

Cu $u^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ să se calc. $u^{(1)}$ cu met iterației inverse. Rezolvarea sist. liniar să se facă cu alg de elim Gauss cu pivotare parțială.

w: soluția sist. liniar $(A+I)w = u^0$

$$\begin{aligned} 2w_1 \quad\quad + w_3 &= 1 \quad / -\tfrac{1}{2} + ec_3 \\ 2w_2 \quad\quad &= 0 \\ w_1 \quad\quad + 2w_3 &= 0 \end{aligned}$$

Pas 1: $\max\{|a_{11}|, |a_{21}|, |a_{31}|\} = 2 = |a_{11}|$

$$\begin{aligned} 2w_1 \quad\quad + w_3 &= 1 \Rightarrow w_1 = (1+\tfrac{1}{3})/2 = \tfrac{2}{3} \\ 2w_2 \quad\quad &= 0 \Rightarrow w_2 = 0 \\ \tfrac{3}{2} w_3 &= -\tfrac{1}{2} \Rightarrow w_3 = -\tfrac{1}{3} \end{aligned}$$

Pas 2: nu mai e nevoie

$$u^{(1)} = \frac{1}{\|w\|_2} \cdot w = \frac{3}{\sqrt{5}} \cdot \begin{pmatrix} 2/3 \\ 0 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} \\ 0 \\ -1/\sqrt{5} \end{pmatrix}$$

$$\lambda^{(1)} = (Au^{(1)}, u^{(1)}) = \left(\begin{pmatrix} 1/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2/\sqrt{5} \\ 0 \\ -1/\sqrt{5} \end{pmatrix} \right) = \frac{1}{5}$$

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