SEMINAR 1

S1.1. Show that if the sets A, B and C are satisfying the equalities

$$A \cup B = C,$$
$$(A \cup C) \cap B = C,$$
$$(A \cap C) \cup B = A,$$

then they are equal.

S1.2. Show that for two subsets A and B of a set (universe) U, it holds:

$$(\complement_A \Delta \complement_B) \cap \complement_{B \setminus A} = A \setminus B.$$

S1.3. Show that, for any sets A, B and C, we have:

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

S1.4. Show first that for any sets A, B and C, we have

$$A \Delta B = C \iff B = A \Delta C.$$

Then solve the equation

$$A \Delta X = B$$

in the case $A := \{a, b, c, d\}$ and $B := \{b, d, e\}$.

S1.5. Compare first A with C and B with C, then determine $A \cap B$, where the sets A, B and C are defined by:

$$A := \{(a - b, a + b, 2ab) \mid a, b \in \mathbb{R}\},$$
$$B := \{(\alpha + 2\beta, \alpha - 3\beta, 2\alpha + \beta) \mid \alpha, \beta \in \mathbb{R}\},$$
$$C := \{(x - 1, x + 1, 2x) \mid x \in \mathbb{R}\}.$$

S1.6. Let $X := \{1, 2, 3\}$ and the following relations on X:

$$R = \{(1,2), (1,3), (2,2)\}, \quad S = \{(1,2), (2,3)\}.$$

Determine the domain, the image and the inverse of each relation. Then verify the inequality

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}.$$

S1.7. Consider the following relations:

$$\rho = \{(3a, a) \mid a \in \mathbb{R}\}; \quad \delta = \{(b, 3b) \mid b \in \mathbb{R}\}.$$

Show that $\rho \circ \delta = 1_{\mathbb{R}}$.

S1.8. Establish some properties of the divisibility relation on the set $\mathbb{R}[X]$ of polynoms with real coefficients.

S1.9. Let $f \in \mathcal{F}(X; Y)$ and $g \in \mathcal{F}(Y; Z)$. Prove that if $g \circ f$ is a surjection, then g is a surjective function, too.

S1.10. Two sets A and B are called *equipotent* if there exists at least a bijective function $f : A \to B$.
Let U be a set. Show that the equipotency relation on $\mathscr{P}(U)$ is an equivalence relation.

S1.11. Show that a function $f : X \to Y$ is injective if and only if $f^{-1}[f[A]] = A, \ \forall A \in \mathscr{P}(X)$.

S1.12. Let $G := \{(z, u) \in \mathbb{C} \mid u = a + ib, \ a, b \in \mathbb{R}, \ z = e^u = e^a(\cos b + i \sin b)\} \subset \mathbb{C} \times \mathbb{C}$. Is G a function?

S1.13. Prove that the *characteristic function* on a set U, $\chi : U \to \{0, 1\}$, defined by

$$\chi_A(x) := \begin{cases} 1, & x \in A; \\ 0, & x \in \complement_A, \end{cases}$$

has the following properties:

$$(\chi_A)^p = \chi_A, \ \forall p \in \mathbb{N}^*,$$
$$\chi_{\complement_A} = 1 - \chi_A, \quad \chi_{A \cap B} = \chi_A \cdot \chi_B, \quad \chi_{A \setminus B} = \chi_A - \chi_A \cdot \chi_B,$$

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B, \quad \chi_{A\Delta B} = \chi_A + \chi_B - 2\chi_A \cdot \chi_B.$$

S1.14. Let X be a set with at least two elements. We define the relation $\preccurlyeq$ on $\mathcal{F}(X;\mathbb{R})$ by:

$$f \preccurlyeq g \Longleftrightarrow f(x) \le g(x),\ \forall x \in X,$$

for $f, g \in \mathcal{F}(X;\mathbb{R})$. Show that $(\mathcal{F}(X;\mathbb{R}), \preccurlyeq)$ is an ordered set, but it is not totally ordered.

S1.15. Using the properties of the characteristic function, give another solutions to the problems **S1.1**, **S1.2** and **S1.3**.

S1.16. Establish what kind of relation there exists between the sets

$$A := \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid \exists a \in (0,1] : x + ay = 1,\ y - a(x+1) = 0\},$$

$$B := \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x \in [0,1),\ y \in (0,1],\ x^2 + y^2 = 1\}.$$

S1.17. Let $f : X \to Y$ be a function. Show that f is bijective if and only if

$$\complement_{f[A]} = f[\complement_A],\ \forall A \in \mathscr{P}(X).$$

S1.18.

a) Let A be a set. Solve the equation:

$$X \cap A = X \cup A.$$

b) Show that for any sets A and B we have

$$A \setminus (A \setminus B) = A \cap B.$$

S1.19. On $\mathbb{N}^*$ we consider the relation div defined by

$$a \operatorname{div} b \Longleftrightarrow \exists c \in \mathbb{N}^* : b = a \cdot c.$$

Show that $(\mathbb{N}^*, \operatorname{div})$ is an ordered set. Is $(\mathbb{N}^*, \operatorname{div})$ also totally ordered?

S1.20. Let X be a set. We define the relation $\sim$ on $\mathcal{F}(X;X)$, by requiring that $f \sim g$ if and only if there exists a bijective function $h \in \mathcal{F}(X;X)$ such that $f = h^{-1} \circ g \circ h$. What kind of relation is $\sim$?

Recommended bibliography

[1] C. Drăguşin, O. Olteanu, M. Gavrilă, *Analiză matematică.Probleme I*, Ed. Matrix Rom, Bucureşti, 2006.
[2] I. Radomir, A. Fulga, *Analiză matematică.Culegere de probleme*, Ed. Albastră, Cluj-Napoca, 2005.
[3] F. L. Ţiplea, *Introducere în teoria mulţimilor*, Ed. Univ. "Al. I. Cuza", Iaşi, 1998.
[4] V. Postolică, Genoveva Spătaru-Burcă, *Analiză matematică.Exerciţii şi probleme I*, Ed. Matrix Rom, Bucureşti, 2005.
[5] R. Gologan, A. Halanay ş.a., *Probleme de examen. Analiză matematică*, Ed. Matrix Rom, Bucureşti, 2004.
[6] W. Weiss, *An introduction to Set Theory*, 2008
[7] W. F. Trench, *Introduction to Real Analysis,* Pearson Education Publ., 2009
[8] J. Goudsmit, R. Iemhoff, *On sets, functions and relations*, 2012.