Vectori și valori proprii - algoritmul QR

$A \in \mathbb{R}^{n \times n}$ $\qquad Au = \lambda u \quad \lambda \in \mathbb{C}, u \in \mathbb{C}^n, u \neq 0$

Se construiește un șir de matrici asemenea cu matricea A (toate matricile au aceleași valori proprii), $A^{(k)} \in \mathbb{R}^{n \times n}$, $A^{(0)} = A$

$A^{(k)} \sim A$, $A^{(k)} \longrightarrow S$

S matrice în formă Schur reală (bloc superior triunghiulară)

- valorile proprii ale lui S sunt ușor de calculat (sunt valorile proprii ale lui A)

Algoritmul QR:

$A^{(0)} = A$ - se calculează descompunerea QR a matricei $A^{(0)} = Q_0 R_0$

$A^{(1)} := R_0 Q_0$

se calculează descompunerea QR a matricei $A^{(1)}$

$A^{(1)} = Q_1 R_1 \Rightarrow A^{(2)} := R_1 Q_1$

...

$A^{(k)} = Q_k R_k$ descompunerea QR a matricei

$\Rightarrow A^{(k+1)} = R_k Q_k \qquad k = 0, 1, \ldots$

$$A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \qquad \det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -3 \\ -4 & \lambda-4 \end{vmatrix} = \lambda(\lambda-7)$$

Descompunerea QR a matricei A cu algoritmul Givens:

$$\underbrace{\begin{bmatrix} c & s \\ -s & c \end{bmatrix}}_{R_{12} = Q_0^T} \cdot A = \underbrace{\begin{bmatrix} 3c+4s & 3c+4s \\ -3s+4c & -3s+4c \end{bmatrix}}_{R_0}$$

$$\left.\begin{matrix} r_{21} = -3s+4c = 0 \\ c^2+s^2=1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} s = 4f,\ c = 3f \\ c^2+s^2=1 \end{matrix}\right\} \Rightarrow$$

$$s = \frac{4}{5}, \quad c = \frac{3}{5}$$

$$\Rightarrow Q_0 = \begin{bmatrix} 3/5 & -4/5 \\ +4/5 & 3/5 \end{bmatrix} \qquad R_0 = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix}$$

$$A^{(1)} = R_0 Q_0 = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 \\ +4/5 & 3/5 \end{bmatrix} =$$

$$= \begin{bmatrix} 7 & -1 \\ 0 & 0 \end{bmatrix}$$ matrice în formă Schur reală

– convergența are loc (doar în cazul acesta!) într-un singur pas

$$\lambda_1 = a_{11}^{(1)} = 7, \quad \lambda_2 = a_{22}^{(1)} = 0$$

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 5 \\ 4 & 0 & 7 \end{bmatrix} \quad \det(\lambda I - A) = \begin{vmatrix} \lambda-3 & 0 & 1 \\ 0 & \lambda-5 & -5 \\ -4 & 0 & \lambda-7 \end{vmatrix} = (\lambda-5)^3$$

$A^{(0)} = A$ descompunerea QR cu algoritmul Householder.

$$\underbrace{P_2 P_1}_{Q^T} A = R \qquad P_1, P_2 \text{ matrice de reflexie}$$

$$P_1 = I - \frac{1}{\beta} u u^T \quad , \quad \beta = \sigma - k a_{11}$$

$$\sigma = a_{11}^2 + a_{21}^2 + a_{31}^2 = 25$$

$$k = -\operatorname{semn}(a_{11}) \sqrt{\sigma} = -5$$

$$\beta = 25 - (-5) \cdot 3 = 40$$

$$u = \begin{pmatrix} a_{11} - k \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix}$$

$$u u^T = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} \begin{pmatrix} 8 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 64 & 0 & 32 \\ 0 & 0 & 0 \\ 32 & 0 & 16 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \frac{1}{40} \begin{pmatrix} 64 & 0 & 32 \\ 0 & 0 & 0 \\ 32 & 0 & 16 \end{pmatrix} = \begin{pmatrix} -3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{pmatrix}$$

$$P_1 \cdot A = \begin{pmatrix} -3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 5 & 5 \\ 4 & 0 & 7 \end{pmatrix} = \begin{pmatrix} -5 & 0 & -5 \\ 0 & 5 & 5 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\Rightarrow P_2 = I \quad , \quad Q_0 = P_1^T = P_1 \quad , R_0 = \begin{pmatrix} -5 & 0 & -5 \\ 0 & 5 & 5 \\ 0 & 0 & 5 \end{pmatrix}$$

$$A^{(1)} = R_0 Q_0 = \begin{pmatrix} -5 & 0 & -5 \\ 0 & 5 & 5 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} -3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{pmatrix} =$$

$$= \begin{pmatrix} 7 & 0 & 1 \\ -4 & 5 & 3 \\ -4 & 0 & 3 \end{pmatrix} \quad \ldots$$

Verificăm faptul că $A^{(1)}$ are aceleași valori proprii ca A

$$\det(\lambda I - A^{(1)}) = \begin{vmatrix} \lambda-7 & 0 & -1 \\ 4 & \lambda-5 & -3 \\ 4 & 0 & \lambda-3 \end{vmatrix} = (\lambda-5)^3$$

Algoritmul QR cu deplasare simplă

$$A = \begin{bmatrix} 10 & 0 & -1 \\ 0 & 12 & 5 \\ 4 & 0 & 7 \end{bmatrix}$$

$$A - a_{33} I = A - 7I = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 5 & 5 \\ 4 & 0 & 0 \end{bmatrix}$$

$A^{(0)} = A$; descompunerea QR cu algoritmul Gram-Schmidt

$$A - 7I = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 5 & 5 \\ 4 & 0 & 0 \end{pmatrix} = [q^1 \ q^2 \ q^3] \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$$= [r_{11} q^1 \quad r_{12} q^1 + r_{22} q^2 \quad r_{13} q^1 + r_{23} q^2 + r_{33} q^3]$$

Pas 1

$$r_{11} q^1 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \Rightarrow \| r_{11} q^1 \|_2^2 = \left\| \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right\|_2^2 = 25 \Rightarrow$$

$$r_{11}^2 = 25 \Rightarrow r_{11} = 5 \Rightarrow q^1 = \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix}$$

Pas 2

$$r_{12} q^1 + r_{22} q^2 = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$ - se face produsul scalar cu q^1

$$\Rightarrow r_{12} \underbrace{(q^1, q^1)}_{=1} + r_{22} \underbrace{(q^2, q^1)}_{=0} = \underbrace{\left(\begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix} \right)}_{=0} \Rightarrow \boxed{r_{12} = 0}$$

$$r_{22}q^2=\begin{pmatrix}0\\5\\0\end{pmatrix} \Rightarrow \|r_{22}q^2\|_2^2=\left\|\begin{pmatrix}0\\5\\0\end{pmatrix}\right\|^2=25 \Rightarrow$$

$$r_{22}=5,\ q^2=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

<u>Pas 3</u> $r_{13}q^1+r_{23}q^2+r_{33}q^3=\begin{pmatrix}-1\\5\\0\end{pmatrix}$

se face, pe rând, produsul scalar cu q^1 și q^2
se ține cont că $(q^i,q^j)=\begin{cases}1 & \text{ptr } i=j\\ 0 & \text{ptr } i\neq j\end{cases}$

$$\Rightarrow r_{13}=\left(\begin{pmatrix}-1\\5\\0\end{pmatrix},\begin{pmatrix}3/5\\0\\4/5\end{pmatrix}\right)=-\frac{3}{5}$$

$$r_{23}=\left(\begin{pmatrix}-1\\5\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)=5$$

$$r_{33}q^3=\begin{pmatrix}-1\\5\\0\end{pmatrix}+\frac{3}{5}\begin{pmatrix}3/5\\0\\4/5\end{pmatrix}-\begin{pmatrix}0\\5\\0\end{pmatrix}=\begin{pmatrix}-16/25\\0\\12/25\end{pmatrix}$$

$$r_{33}=\sqrt{\left(-\frac{16}{25}\right)^2+\left(\frac{12}{25}\right)^2}=\frac{20}{25}=\frac{4}{5}$$

$$q^3=\frac{5}{4}\cdot\begin{pmatrix}-16/25\\0\\12/25\end{pmatrix}=\begin{pmatrix}-4/5\\0\\3/5\end{pmatrix}$$

$$R = \begin{pmatrix} 5 & 0 & -3/5 \\ 0 & 5 & 5 \\ 0 & 0 & 4/5 \end{pmatrix} \qquad Q = \begin{pmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{pmatrix}$$

$$A^{(1)} = R * Q + 7I$$

$$= \begin{pmatrix} 5 & 0 & -3/5 \\ 0 & 5 & 5 \\ 0 & 0 & 4/5 \end{pmatrix} \begin{pmatrix} 3/5 & 0 & -4/5 \\ 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \end{pmatrix} + 7 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9.52 & 0 & -4.36 \\ 4 & 12 & 3 \\ 0.64 & 0 & 7.48 \end{pmatrix}$$