S5.9 (R) Fie $\langle \cdot\,, \cdot \rangle$ un produs scalar pe $\mathbb{R}^n$ şi fie $|| \cdot ||$ norma indusă de acesta. Să se arate că $\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, au loc:

i) $||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 = 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2)$ (Euler)

ii) $||\mathbf{x}+\mathbf{y}||^2 - ||\mathbf{x}-\mathbf{y}||^2 = 4 < \mathbf{x}, \mathbf{y} >$ (Hilbert).

$\langle\ \rangle$

$$||x+y||^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$
$$= ||x||^2 + 2\langle x,y\rangle + ||y||^2 \quad (1)$$

$$||x-y||^2 = \langle x-y, x-y\rangle =$$
$$= \langle x,x\rangle - \langle y,x\rangle - \langle x,y\rangle + \langle y,y\rangle$$
$$= ||x||^2 - 2\langle x,y\rangle + ||y||^2 \quad (2)$$

(1) + (2)

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$$

(1) − (2)

$$||x+y||^2 - ||x-y||^2 = 4\langle x,y\rangle$$

Biliniaritate

$\langle\ ,\ \rangle$ este biliniară dacă este liniară în fiecare argument:

liniaritate:
primul arg: $< \alpha \bar{x} + \beta \bar{y}, \bar{z} > = \alpha < \bar{x}, \bar{z} > + \beta < \bar{y}, \bar{z} > \Leftrightarrow$

Analogie

$$(\alpha x + \beta y) \cdot z = \alpha x z + \beta y z$$

$$\begin{cases} < \alpha \bar{x}, \bar{z} > = \alpha < \bar{x}, \bar{z} > \\ < \bar{x} + \bar{y}, \bar{z} > = < \bar{x}, \bar{z} > + < \bar{y}, \bar{z} > \end{cases}$$

al doilea arg

$$< \bar{x}, \alpha \bar{y} + \beta \bar{z} > = \alpha < \bar{x}, \bar{y} > + \beta < \bar{x}, \bar{z} > \Leftrightarrow$$

$$\begin{cases} < \bar{x}, \alpha \bar{y} > = \alpha < \bar{x}, \bar{y} > \\ < \bar{x}, \bar{y} + \bar{z} > = < \bar{x}, \bar{y} > + < \bar{x}, \bar{z} > \end{cases}$$

S5.6 Se consideră spaţiul euclidian $\mathbb{R}^4$, dotat cu produsul scalar canonic. Folosind procedeul de ortonormalizare al lui Gram-Schmidt, să se afle o bază ortonormată B', plecând de la baza

$$B = \{(1, 2, -1, 0), (1, -1, 1, 1), (-1, 2, 1, 1), (-1, -1, 0, 1)\}.$$

Calculele sunt mai ușoare atunci când vectorii din-o bază sunt

ortonormare
- ortogonali = perpendiculari
  - drept unghi
- de normă 1

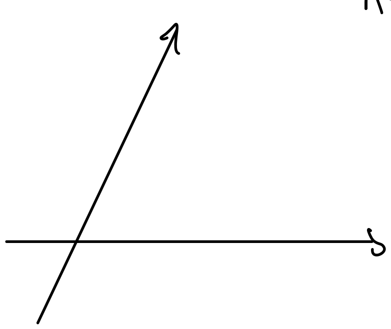
Dată o bază oarecare și un produs scalar, pot obține mereu o bază ortonormată.

A ortogonalizez vectorii

B îi fac de normă 1

A

1. Păstrez primul vector
2. Fiecare vector ulterior e o combinație de cei anterior calculați și cel corespunzător



din baza inițială a.î să fie $\perp$ pe toți cei anterior calculați.

B. Facem pe toți de normă 1.

$$B = \{\underbrace{(1,2,2)}_{\overline{b_1}}, \underbrace{(1,0,2)}_{\overline{b_2}}, \underbrace{(0,0,1)}_{\overline{b_3}}\}$$

Fac vectorii ortogonali

$\overline{b_1}' = \overline{b_1} = (1,2,2)$

$\overline{b_2}' = \overline{b_2} - \alpha \overline{b_1}'$ a.î $\overline{b_2}' \perp \overline{b_1}' \Leftrightarrow$

$$\langle \overline{b_2}', \overline{b_1}' \rangle = 0 \Rightarrow$$

$$\langle \overline{b_2} - \alpha \overline{b_1}', \overline{b_1}' \rangle = 0$$

$$\langle \overline{b_2}, \overline{b_1}' \rangle - \alpha \langle \overline{b_1}', \overline{b_1}' \rangle = 0$$

$$\Rightarrow \boxed{\alpha = \frac{\langle \overline{b_2}, \overline{b_1}' \rangle}{\langle \overline{b_1}', \overline{b_1}' \rangle}}$$

Merge în general

$$\overline{b_2}' = \overline{b_2} - \frac{\langle \overline{b_2}, \overline{b_1}' \rangle}{\langle \overline{b_1}', \overline{b_1}' \rangle} \cdot \overline{b_1}'$$

$$(1,0,2) - \frac{\langle (1,0,2), (1,2,2) \rangle}{\langle (1,2,2), (1,2,2) \rangle} \cdot (1,2,2) =$$

$$(1,0,2) - \frac{1 + 0 \cdot 2 + 2 \cdot 2}{1 \cdot 1 + 2 \cdot 2 + 2 \cdot 2} \cdot (1,2,2) =$$

$$(1,0,2) - \frac{5}{9} (1,2,2) =$$

$$(1,0,2) - \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9} \right) =$$

$$\left(\frac{4}{9}, -\frac{10}{9}, \frac{8}{9} \right)$$

$$\overline{b_3}' = \overline{b_3} - \beta \overline{b_1}' - \gamma \overline{b_2}'$$

- $\overline{b_3}' \perp \overline{b_1}' \Rightarrow \langle \overline{b_3}', \overline{b_1}' \rangle = 0 \Rightarrow$

$$\langle \overline{b_3} - \beta \overline{b_1}' - \gamma \overline{b_2}', \overline{b_1}' \rangle = 0$$

$$\langle \overline{b_3}, \overline{b_1}' \rangle - \beta \langle \overline{b_1}', \overline{b_1}' \rangle - \gamma \underbrace{\langle \overline{b_2}', \overline{b_1}' \rangle}_{0} = 0$$

$$\beta = \frac{\langle \overline{b_3}, \overline{b_1}' \rangle}{\langle \overline{b_1}', \overline{b_1}' \rangle}$$

- $\overline{b_3}' \perp \overline{b_2}' \Rightarrow \langle \overline{b_3}', \overline{b_2}' \rangle = 0$

$$\langle \overline{b_3} - \beta \overline{b_1}' - \mu \overline{b_2}', \overline{b_2}' \rangle = 0$$

$$\langle \overline{b_3}, \overline{b_2}' \rangle - \beta \underbrace{\langle \overline{b_1}', \overline{b_2}' \rangle}_{0} - \mu \langle \overline{b_2}', \overline{b_2}' \rangle = 0$$

$$\langle \overline{b_3}, \overline{b_2}' \rangle - \mu \langle \overline{b_2}', \overline{b_2}' \rangle = 0$$

$$\mu = \frac{\langle \overline{b_3}, \overline{b_2}' \rangle}{\langle \overline{b_2}', \overline{b_2}' \rangle}$$

$$\overline{b_3}' = \overline{b_3} - \frac{\langle \overline{b_3}, \overline{b_1}' \rangle}{\langle \overline{b_1}', \overline{b_1}' \rangle} \overline{b_1}' - \frac{\langle \overline{b_3}, \overline{b_2}' \rangle}{\langle \overline{b_2}', \overline{b_2}' \rangle} \overline{b_2}'$$

$$\overline{b_i}' = \overline{b_i} - \sum_{j=1}^{i-1} \frac{\langle \overline{b_i}, \overline{b_j}' \rangle}{\langle \overline{b_j}', \overline{b_j}' \rangle} \overline{b_j}'$$

$$\overline{b_3}' = (0,0,1) - \frac{\langle (0,0,1), (1,2,2) \rangle}{\langle (1,2,2), (1,2,2) \rangle} (1,2,2) -$$

$$\frac{\langle (0,0,1), (\frac{4}{9}, -\frac{10}{9}, \frac{8}{9}) \rangle}{\langle (\frac{4}{9}, -\frac{10}{9}, \frac{8}{9}), (\frac{4}{9}, -\frac{10}{9}, \frac{8}{9}) \rangle} \cdot \left(\frac{4}{9}, -\frac{10}{9}, \frac{8}{9} \right) =$$

$$(0,0,1) - \frac{2}{9}(1,2,2) - \frac{\frac{8}{9}}{{}_{20}\cancel{180}/{}_{9}\cancel{81}}\left(\frac{4}{9}, \frac{-10}{9}, \frac{8}{9}\right) =$$

$$(0,0,1) - \left(\frac{2}{9}, \frac{4}{9}, \frac{4}{9}\right) - \frac{\cancel{8}^{2}}{\cancel{9}} \cdot \frac{\cancel{9}}{\cancel{20}_{5}}\left(\frac{4}{9}, -\frac{10}{9}, \frac{8}{9}\right)$$

$$(0,0,1) - \left(\frac{2}{9}, \frac{4}{9}, \frac{4}{9}\right) - \left(\frac{8}{45}, \frac{-20}{45}, \frac{16}{45}\right)$$

$$\left(-\frac{18}{45}, \frac{0}{45}, \frac{9}{45}\right) = \left(-\frac{2}{5}, \frac{0}{5}, \frac{1}{5}\right)$$

$$\bar{b}_1'' = \frac{1}{\|\bar{b}_1'\|} \cdot \bar{b}_1' = \frac{1}{\sqrt{1^2+2^2+2^2}} \cdot (1,2,2) =$$

$$\frac{1}{3}(1,2,2) =$$

$$\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$\bar{b}_2'' = \frac{1}{\|\bar{b}_2'\|} \cdot \bar{b}_2' = \frac{1}{\sqrt{\frac{16}{81} + \frac{100}{81} + \frac{64}{81}}}\left(\frac{4}{9}, -\frac{10}{9}, \frac{8}{9}\right)$$

$$= \sqrt{\frac{\cancel{81}\,9}{\cancel{180}\,20}} \cdot \left(\frac{4}{9}, -\frac{10}{9}, \frac{8}{9}\right) =$$

$$\frac{3}{2\sqrt{5}}\left(\frac{4}{9}, -\frac{10}{9}, \frac{8}{9}\right) =$$

$$\left(\frac{2}{3\sqrt{5}}, -\frac{5}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}\right)$$

$$\overline{b_3}'' = \frac{1}{\|\overline{b_3}'\|} \cdot \overline{b_3}' = \frac{1}{\sqrt{\frac{4}{25} + 0 + \frac{1}{25}}} \cdot$$

$$\left(-\frac{2}{5}, 0, \frac{1}{5}\right) =$$

$$\frac{1}{\sqrt{\frac{5}{25}}}\left(-\frac{2}{5}, 0, \frac{1}{5}\right) =$$

$$\sqrt{5}\left(-\frac{2}{5}, 0, \frac{1}{5}\right) =$$

$$\left(-\frac{2\sqrt{5}}{5}, 0, \frac{\sqrt{5}}{5}\right)$$

S5.10 (R) Fie W un subspaţiu liniar al lui $\mathbb{R}^n$ şi $f : W \to \mathbb{R}$, o funcţie astfel încât

$$\boxed{\{\mathbf{x} \in W \mid \underline{f(\mathbf{x}) = 0}\} = \{\mathbf{0}_{\mathbb{R}^n}\}} \quad (\cdot)$$

şi

$$f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y}), \forall\, \alpha, \beta \in \mathbb{R}, \forall \mathbf{x}, \mathbf{y} \in W. \quad (\cdot\cdot)$$

Definim aplicaţia $< \cdot\,, \cdot >: W \times W \to \mathbb{R}$, prin:

$$< \mathbf{x}, \mathbf{y} >= f(\mathbf{x}) \cdot f(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in W.$$

a) Să se arate că W este un spaţiu prehilbertian.

b) Să se arate că orice două elemente ale lui W, diferite de $\mathbf{0}_{\mathbb{R}^n}$, sunt liniar dependente ($\dim(W) = 1$).

a) 1. $\langle x, x \rangle \geq 0 \qquad \langle x, x \rangle = 0 \Leftrightarrow x = 0$

2. $\langle x, y \rangle = \langle y, x \rangle$

3. biliniar

1. $\langle x, x \rangle = f(x) \cdot f(x) = \underbrace{f^2(x)}_{\in \mathbb{R}} \geq 0$

pozitivă, definire

$\langle x, x \rangle = 0 \Leftrightarrow f^2(x) = 0 \Leftrightarrow f(x) = 0 \overset{(\cdot)}{\Leftrightarrow} x = 0$

numere reale

2. $\langle x, y \rangle = f(x) \cdot f(y)$

$\qquad\qquad\qquad \parallel$

$\langle y, x \rangle = f(y) \cdot f(x)$

3. Pt că am arătat simetria, e suf să arăt că f liniară în primul arg.

$\langle \alpha x + \beta z, y \rangle = \alpha \langle x, y \rangle + \beta \langle z, y \rangle$

$$\langle \alpha x+\beta z, y\rangle = f(\alpha x+\beta z)\cdot f(y) \overset{(\cdot\cdot)}{=}$$

$$[\alpha f(x)+\beta f(z)]\cdot f(y) =$$

$$\alpha \underbrace{f(x)f(y)}_{\langle x,y\rangle} + \beta \underbrace{f(z)f(y)}_{\langle z,y\rangle} =$$

$$\alpha\langle x,y\rangle + \beta\langle z,y\rangle$$

e) Fie

$$\boxed{\alpha x+\beta y = 0_{\mathbb{R}^n}}$$

$x \neq 0_{\mathbb{R}^n}$

$y \neq 0_{\mathbb{R}^n}$

$$f(\alpha x+\beta y) = f(0_{\mathbb{R}^n}) \overset{\cdot\cdot}{\Longrightarrow}$$

$$\alpha f(x)+\beta f(y) = 0 \Rightarrow$$

$$\alpha f(x) = -\beta f(y)$$

(*)

$f(x) \neq 0$

$f(y) \neq 0$

Aleg $\beta = 1$

$$\alpha = \frac{-f(y)}{f(x)} \neq 0$$

$$\Rightarrow -\frac{f(y)}{f(x)} \cdot x + 1 \cdot y = 0_{\mathbb{R}^n}$$

$$-\underbrace{f(y)}_{\neq 0} \cdot x + \underbrace{f(x)}_{\neq 0} \cdot y = 0_{\mathbb{R}^n}$$

$\Rightarrow x, y$ lin dep.

S5.11 Care dintre mulţimile de mai jos este un subspaţiu liniar?

a) $\overbrace{\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = (x_1, x_2, \ldots, x_n), x_1 + x_n = 0\}}^{N} \subseteq \mathbb{R}^n$;

b) $\underbrace{\{A \in \mathcal{M}_2(\mathbb{R}) \mid det(A) = 0\}}_{M} \subseteq \mathcal{M}_2(\mathbb{R})$.

ştiu că e sp liniar

b)

$\forall\ \overbrace{A, B \in M}^{\Downarrow} \quad A - B \in M \quad (*)$

$\det(A) = 0$

$\det(B) = 0$

Fie $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

$$A-B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\det(A-B)=1 \Rightarrow (*)$ nu are loc

$\Rightarrow M$ nu e subspațiu liniar

$\forall A \in M \ \forall \lambda \in \mathbb{R} \quad \lambda A \in M$

$\nRightarrow$

$$\det(\lambda A) = \lambda^2 \det(A) = 0$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \lambda A = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$$

$\det A = ad - bc$

$$\det(\lambda A) = \lambda a \lambda d - \lambda b \lambda c = \lambda^2(ad-bc)$$

$$\overbrace{\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = (x_1, x_2, \ldots, x_n), x_1 + x_n = 0\}}^{N} \subseteq \underline{\mathbb{R}^n};$$

Fie $x, y \in N \quad \alpha, \beta \in \mathbb{R}$

$$\alpha x + \beta y \in N$$

$$x = (x_1, \ldots, x_n) \quad x_1 + x_n = 0$$

$$y = (y_1, \ldots, y_n) \quad y_1 + y_n = 0$$

$$\alpha x + \beta y =$$

$$\alpha (x_1, \ldots, x_n) +$$

$$\beta (y_1, \ldots, y_n) =$$

$$(\alpha x_1, \ldots, \alpha x_n) + (\beta y_1, \ldots, \beta y_n) =$$

$$(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \mathbb{R}^n$$

$$\alpha x_1 + \beta y_1 + \alpha x_n + \beta y_n =$$

$$\alpha(\underbrace{x_1+x_2}_{0})+\beta(\underbrace{y_1+y_2}_{0})=0$$

$$\Rightarrow \alpha x+\beta y \in N$$

$\Rightarrow N$ subspațiu liniar

S5.7 Se consideră sistemul de vectori $C=\{\overbrace{(1,4,3,2)}^{f_1},\overbrace{(1,1,-1,1)}^{f_2},\overbrace{(-3,0,7,6)}^{f_3}\}\subseteq\mathbb{R}^4$.

a) Să se determine $S=Sp(C)$ şi $S^\perp$.

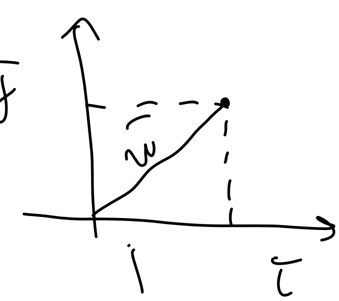
b) Să se afle proiecţiile ortogonale ale vectorului $\mathbf{w}=(14,-3,-6,-7)$ pe S şi pe $S^\perp$. Să se verifice că avem

$$||\mathbf{w}-pr_S(\mathbf{u})||\leq||\mathbf{w}-\mathbf{u}||,\forall\ \mathbf{u}\in C,$$

unde $pr_S(\mathbf{u})$ este notaţia pentru proiecţia ortogonală a vectorului $\mathbf{u}$ pe S, care, prin definiţie, înseamnă acel vector $\mathbf{v}\in S$, pentru care $\mathbf{u}-\mathbf{v}\in S^\perp$.

$Sp(C)=$ Spațiul liniar generat de C

$=Lin(C)$



Verif dacă C lin indep

$$\alpha f_1+\beta f_2+\gamma f_3=0_{\mathbb{R}^4}$$

$$\alpha\begin{pmatrix}1\\4\\3\\2\end{pmatrix}+\beta\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}+\gamma\begin{pmatrix}-3\\0\\7\\6\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

sistem liniar cu 4 ec, 3 nec

omogen

Studiez rang matrice sistem

$$\begin{bmatrix} 1 & 1 & -3 \\ 4 & 1 & 0 \\ 3 & -1 & 7 \\ 2 & 1 & 6 \end{bmatrix}$$

$$\Delta_e = \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = 1-4 = -3 \neq 0$$

$$\Delta_3' = \begin{vmatrix} 1 & 1 & -3 \\ 4 & 1 & 0 \\ 3 & -1 & 7 \end{vmatrix} = 0$$

$$\Delta_3'' = \begin{vmatrix} 1 & 1 & -3 \\ 4 & 1 & 0 \\ 2 & 1 & 6 \end{vmatrix} = -24$$

$$= \begin{vmatrix} 1 & 1 & -3 \\ 3 & 0 & 3 \\ 1 & 0 & 9 \end{vmatrix}$$

$\Rightarrow$ rang $= 3$

$\Rightarrow$ Păstrez ec 1, 2, 4 $\Rightarrow$

Rămân un sistem 3 ec, 3 nec cu det $\neq 0$

$\Rightarrow$ are sol unică $\Rightarrow$ sol nulă e unică

$\Rightarrow f_1, f_2, f_3$ l.i.

$$\text{Lin } C = \{ \delta f_1 + \varepsilon f_2 + \varphi f_3 \mid \delta, \varepsilon, \varphi \in \mathbb{R} \}$$

$$\left\{ \delta \begin{pmatrix} 1 \\ 4 \\ 3 \\ 2 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \varphi \begin{pmatrix} -3 \\ 0 \\ 7 \\ 6 \end{pmatrix} \;\middle|\; \delta, \varepsilon, \varphi \in \mathbb{R} \right\} =$$

$$\left\{ \begin{pmatrix} \delta + \varepsilon - 3\varphi \\ 4\delta + \varepsilon \\ 3\delta - \varepsilon + 7\varphi \\ 2\delta + \varepsilon + 6\varphi \end{pmatrix} \;\middle|\; \delta, \varepsilon, \varphi \in \mathbb{R} \right\}$$

$$C^{\perp} = \{ \bar{x} \in \mathbb{R}^4 \mid \bar{x} \perp c \ \forall c \in \text{Lin } C \}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{cases} \langle \bar{x}, \bar{f}_1 \rangle = 0 \\ \langle \bar{x}, \bar{f}_2 \rangle = 0 \\ \langle \bar{x}, \bar{f}_3 \rangle = 0 \end{cases}$$

$$\langle (x_1, x_2, x_3, x_4), (1, 4, 3, 2) \rangle = 0$$

$$x_1 + 4x_2 + 3x_3 + 2x_4 = 0$$

$$\langle (x_1, x_2 x_3, x_4), (1, 1, -1, 1) \rangle = 0$$

$$x_1 + x_2 - x_3 + x_4 = 0$$

$$\langle (x_1, x_2, x_3, x_4), (-3, 0, 7, 6) \rangle = 0$$

$$-3x_1 + \quad 7x_3 + 6x_4 = 0$$

$$\begin{pmatrix} 1 & 4 & 3 & 2 \\ 1 & 1 & -1 & 1 \\ -3 & 0 & 7 & 6 \end{pmatrix}$$

sistem 3 ec
cu 4 nec

rang 3

Nec pp x_1, x_2, x_4, Nec sec x_3

$x_3 = \alpha$

$$x_1 + 4x_2 + 2x_4 = -3\alpha$$
$$x_1 + x_2 + x_4 = \alpha$$
$$-3x_1 + 6x_4 = -7\alpha$$

$$x_1 = \frac{7}{3}\alpha$$

$$x_2 = \frac{-4}{3}\alpha$$

$$x_4 = 0$$

$$C^\perp = \{ (\tfrac{7}{3}\alpha, -\tfrac{4}{3}\alpha, \alpha, 0) \mid \alpha \in \mathbb{R} \}$$
$$C^\perp = \{ \alpha (\tfrac{7}{3}, -\tfrac{4}{3}, 1, 0) \mid \alpha \in \mathbb{R} \}$$
$$C^\perp = \{ \alpha (7, -4, 3, 0) \mid \alpha \in \mathbb{R} \}$$

$$\mathrm{pr}_S\, u = v$$

$$\frac{u - v \in S^{\perp}}{v \in S}$$