

Algorithm Design: Domain Specific Algorithms - Strings

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- 1 Problem Domain
- 2 Boyer-Moore Algorithm
- 3 Boyer-Moore Algorithm Revised
- 4 Algoritmul Knuth-Morris-Pratt
- 5 Regular Expressions

Plan

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Alphabet, string

- ① **Alphabet** = nonempty set A de **characters**

$$A_1 = \{A, a, B, b, C, c, \dots, Z, z\}$$

$$A_2 = \{0, 1\}$$

$$A_3 = \mathbb{N}$$

- ② **string** of characters: a finite sequence of characters in A

$$\text{formal: } s : \{0, 1, \dots, n-1\} \rightarrow A$$

- ③ **empty string**: $\varepsilon : \emptyset \rightarrow A$

- ④ the **length** $|s|$ of a string s (or $length(s)$ or $s.length()$): the number of characters of the string

$$|\varepsilon| = 0$$

$$\text{if } s : \{0, 1, \dots, n-1\} \rightarrow A \text{ then } |s| = n$$

- ⑤ $s[i]$ - character on **position** i ($0 \leq i \leq |s|$);

- ⑥ **substring**: $s[i..j] = s[i]s[i+1] \dots s[j]$ ($i \leq j$)

The set of strings A^*

- **concatenation (product)** of two strings s_1 and s_2 : is the string s_1 immediately followed by s_2

$$s_1 s_2 = s_1[0] \dots s_1[|s_1| - 1] s_2[0] \dots s_2[|s_2| - 1]$$

$$|s_1 s_2| = |s_1| + |s_2|$$
- $s\varepsilon = \varepsilon s = s$
- A^* as the set of strings over alphabet A
 A^* the monoid freely generated by A
 if $A = \{a_1, \dots, a_m\}$, then $A^* = L((a_1 + \dots + a_m)^*)$ (we define later the regular expressions and their language)
- a **language** is a subset $L \subseteq A^*$

Factor, subsequence

- x is a **factor** of s iff there exists u, v . s.t. $s = uxv$
factor and *substring* are equivalent
 x is a **proper factor** iff $x \neq s$ (equivalently, $uv \neq \epsilon$)
- x is a **prefix** of s iff there exists v . s.t. $s = xv$
write $x \leq_{pref} s$
- x is a **suffix** of s if there exists u . s.t. $s = ux$
write $x \leq_{suff} s$
- x is **subsequence** of s iff there exists $|x| + 1$ strings $w_0, w_1, \dots, w_{|x|}$
s.t. $s = w_0x[0]w_1x[1] \dots x[|x| - 1]w_{|x|}$ (i.e., x is obtained by erasing
 $|y| - |x|$ characters in s); write $x \leq_{sseq} s$

Exercise

Are there \leq_{pref} and \leq_{suff} partial orders?

Lexicographic order

- lexicographic order:

Assume a total order \leq on A . Extend \leq to $A^* \times A^*$ as follows:

$s_1 \leq s_2$ iff $s_1 \leq_{pref} s_2$ or there exists $u, v, w \in A^*$ and $a, b \in A$ s.t.
 $s_1 = uav$, $s_2 = ubw$ and $a < b$.

Exercise. Write in Alk a function that decides the lexicographic order between two strings.

Occurrence

- x **occurs** in s if x is un factor of s
- x has an occurrence at **start position** i in s if $s[i] \dots s[i + |x| - 1] = x$
- x has an occurrence at **end position** j in s if $s[j - |x| + 1] \dots s[j] = x$
- **first occurrence** of x in s is the smallest start position (if there exists)

String Searching (Matching) Problem

Input Two strings: $s = s[0] \dots s[n-1]$, called **subject** or **text**, and $p = p[0] \dots p[m-1]$, called **pattern**.

Output The first occurrence of the pattern p in the text s , if any; -1 , otherwise.

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Variant: find all occurrences:

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Variant: find all occurrences:

Input Two strings: $s = s[0] \dots s[n-1]$, called **subject or text**, and $p = p[0] \dots p[m-1]$, called **pattern**.

Output A set M consisting of all the occurrences of p in s .

The naive algorithm: helper

```

/* @input: strings s[0..n-1], p[0..m-1],
        a position i, 0 <= i < n
   @output: true,  if p <=_pref s[i..n-1]
            false, otherwise */
occAtPos(s, p, i) {
    n = s.size();
    m = p.size();
    for (j = 0; j < m; ++j) {
        if (i + j >= n || s[i + j] != p[j]) {
            return false;
        }
    }
    return true;
}

```

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            return false;
        }
    }
    return true;
}

```

Execution time: $O(m)$ in the worst case

The naive algorithm

```
/*
  @input:  strings s[0..n-1], p[0..m-1]
  @ouput:  the first occurence of p in s, if any
           -1, otherwise
*/
firstOcc(s, p)
{
    n = s.size();
    m = p.size();
    for (i = 0; i < n; ++i) {
        if (occAtPos(s, p, i)) {
            return i;
        }
    }
    return -1;
}
```

The naive algorithm: the worst case analysis

Time: $O(n \cdot m)$ in the worst case,

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Exercise: 1. What is the worst case?

Remains the algorithm correct if we replace $i < n$ by $i < n - m$ in the loop for? What about the execution time?

The naive algorithm: expected time

Assume that A has d characters, $d \geq 2$.

$$X_{ij} = \begin{cases} 1 & , s[i-1] \text{ and } p[j-1] \text{ are compared} \\ 0 & , \textit{otherwise} \end{cases}$$

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the **expected** number of comparisons $= E(\sum_{i=1}^{n+1-m} \sum_{j=1}^m X_{ij})$
 $\leq 2(n+1-m)$

Question: are there algorithms requiring $O(n)$ for the worst case?

A bit of history

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- 1980: R.M. Karp and M.O. Rabin design an algorithm on a hashing-based idea (see the seminar)

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Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
V	I	S	U	L		U	N	E	I		N	O	P	T	I		D	E

≠

I	A	R
---	---	---

19	20	21	22	23	24
	I	A	R	N	A

1

Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
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Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
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Example

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V	I	S	U	L		U	N	E	I		N	O	P	T	I		D	E

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Example

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V	I	S	U	L		U	N	E	I		N	O	P	T	I		D	E

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Example

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
V	I	S	U	L		U	N	E	I		N	O	P	T	I		D	E

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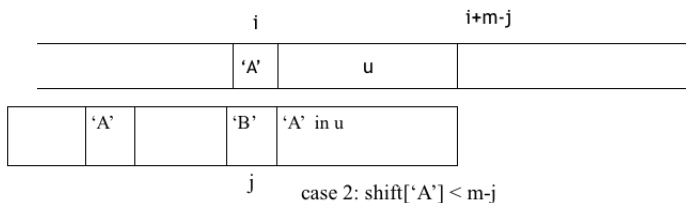
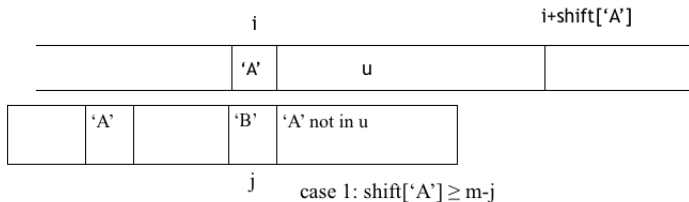
Bad character shift rule 1/3

Avoids unsuccessful comparisons with characters from the subject that do not occur in the pattern or in a (maximal) suffix of it.

$$\text{shift}[C] = \begin{cases} (m - 1) - \text{the last occurrence position} \\ \quad \text{of } C \text{ in pattern} & , \text{ if } C \text{ occurs in pattern} \\ m & , \text{ otherwise} \end{cases}$$

(alternatively, $\text{shift}(C) = \max(\{0\} \cup \{i < m \mid p[i] = C\})$)

Bad character shift rule 2/3



First case: $i = i + \text{shift}[s[i]];$

Second case: $i = i + m - j;$

Bad character shift rule 3/3

If $p[j] \neq s[i] = C$,

- ① if the rightmost occurrence of C in p is $k < j$, $p[k]$ and $s[i]$ will be aligned ($i = i + \text{shift}[s[i]]$)
- ② if the rightmost occurrence of C in p is $k > j$, p is shifted to right with one position ($i = i + m - j$)
- ③ if C does not occur in p , the pattern p is aligned with $s[i + 1..i + m]$ ($i = i + m$). Become a particular case of the first one if $\text{shift}[s[i]] = m$.

Boyer-Moore Algorithm (version 1)

```

BM(s, p, shift) {
  n = s.size();
  m = p.size();
  i = m-1; j = m-1;
  repeat
    if (s[i] == p[j]) {
      i = i-1;
      j = j-1;
    }
    else {
      if ((m-j) > shift[s[i]]) i = i+m-j;
      else i = i+shift[s[i]];
      j = m-1;
    }
  until (j<0 or i>n-1);
  if (j<0) return i+1;
  else return -1;
}

```

Analysis

Worst case: $O(m \cdot n)$.

Expected time is much better.

We will see later that.

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Motivation

Bad character shift rule is inefficient if the alphabet is small (binary, for example).

In such cases the algorithm could be more efficient if uses the information gained from compared suffixes.

This case is called **the good suffix rule**.

The good suffix rule: case 1

0	1	2	3	4	5	6	7	8	9	1	1	1	1	1	1	1		1	1	1	2	2	2	2	2
										0	1	2	3	4	5	6		7	8	9	0	1	2	3	4
a	c	c	a	c	b	b	c	a	b	b	a	c	b	a	b	a	b	b	a	b	b	c	b	a	b

a	a	b	a	b	b	c	c	a	b
0	1	2	3	4	5	6	7	8	9

0	1	2	3	4	5	6	7	8	9	1	1	1	1	1	1	1		1	1	1	2	2	2	2	2
										0	1	2	3	4	5	6		7	8	9	0	1	2	3	4
a	c	c	a	c	b	b	c	a	b	b	a	c	b	a	b	a	b	b	a	b	b	c	b	a	b

a	a	b	a	b	b	c	b	a	b
0	1	2	3	4	5	6	7	8	9

$i-1 = j-1 = 7$, $m = 10$, $s[i-1] \neq p[j-1]$, $p[j..m-1] = s[i..i+m-j-1]$
 $p[1..2] = p[8..9]$, $p[0] \neq p[7]$ and $p[1..2]$ is closest to $p[8..9]$ having this property

The good suffix rule: case 1, formally

Case 1:

if $p[j-1]$ does not match and p includes a copy of $p[j..m-1]$ preceded by a character $\neq p[j-1]$, then shift to the closest copy from left with this property.

The good suffix rule: case 2

0	1	2	3	4	5	6	7	8	9	1	1	1	1	1	1		1	1	1	2	2	2	2	2	
										0	1	2	3	4	5	6		7	8	9	0	1	2	3	4
a	b	c	a	c	b	b	c	a	b	b	a	c	b	a	b	a	b	b	a	b	b	c	b	a	b

a	b	b	a	b	c	b	c	a	b
0	1	2	3	4	5	6	7	8	9

0	1	2	3	4	5	6	7	8	9	1	1	1	1	1	1	1		1	1	1	2	2	2	2	2
										0	1	2	3	4	5	6		7	8	9	0	1	2	3	4
a	b	c	a	c	b	b	c	a	b	b	a	c	b	a	b	a	b	b	a	b	b	c	b	a	b

a	b	b	a	b	c	b	c	a	b
0	1	2	3	4	5	6	7	8	9

$i-1 = j-1 = 5$, $m = 10$, $p[j..m-1] = s[i+m-j-1]$,
 $p[0..1] = s[8..9]$ is the longest prefix of p that is a suffix of $s[0..9]$

The good suffix rule: case 2, formally

Case 2:

- if Case 1 is not applicable, then do the smallest shift such that the suffix of $s[0..i+m-j-1]$ is matched by a prefix of p
- if longest suffix of $s[0..i+m-j-1]$ matched by a prefix of p is the empty string, then shift with m positions
- but “suffix of $s[0..i+m-j-1]$ is matched by a prefix of p ” is equivalent to “a suffix of p is matched by a prefix of p ”
- proper factor that is prefix and suffix is called **border**

goodSuff(j) - definition (Case 1)

goodSuff(j) = the end position of the occurrence of $p[j..m-1]$ closest to j and it is not preceded by $p[j-1]$.

If such a copy does not exist, $\text{goodSuff}(j) = 0$.

We have $0 \leq \text{goodSuff}(j) < m - 1$.

Proposition

The values of *goodSuff(j)* can be computed in $O(m)$ time.

The proof at seminar.

Preprocessing in Case 2

$lp(j)$ = the length of the longest prefix of p that is suffix of $p[j..m-1]$.

We have

Proposition

$lp(j)$ can be computed in $O(m)$ time.

The proof at seminar.

Good Suffix Rule

Assume that $p[j - 1]$ does not match (after $p[j..m - 1]$ matched).

- ① if $goodSuff(j) > 0$, the shift with $m - goodSuff(j)$ positions (case 1)
- ② if $goodSuff(j) = 0$, the shift with $m - lp(j)$ (case 2)

If $p[m - 1]$ matches, then $j = m$ and the shift is correct.

Boyer-Moore Algorithm (version 2)

```

BM(s, n, p, m, goodSuff, lp) {
    k = m-1;
    while (k < n) {
        i = k;  j = m-1;
        while (j > 0 && p[j] == s[i]) {
            i = i-1;
            j = j-1;
        }
        if (j < 0) return i+1;
        otherwise p[j] does not match and shift with the maximum of the
        values returned by the bad character shift rule and the good suffix rule
    }
}

```

Boyer-Moore Algorithm: summary

- $O(n + m)$ time if the pattern p does not occur in the text; otherwise it remains $O(m \cdot n)$
- however, with a simple modification (Galil rule, 1979) $O(n + m)$ time can be obtained in all the cases
- the original algorithm of Boyer-Moore (1977) uses a simplified form of the good suffix rule
- the first proof for $O(n + m)$, when the pattern p does not occur in the text, was given by Knuth, Morris and Pratt (1977); a different proof was independently given by Guibas and Odlyzko (1980)
- Richard Colen (1991) established a limit of $4n$ (with a easier proof), then a limit of $3n$ (with a more complex proof)

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Naive Algorithm¹

<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>
				<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>				

¹Exemplu din [CLRS]

Naive Algorithm¹

<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>
				=										
				<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>				

¹Exemplu din [CLRS]

Naive Algorithm¹

<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>
				=	=									
				<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>				

¹Exemplu din [CLRS]

Naive Algorithm¹

<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>
				=	=	=								
				<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>				

¹Exemplu din [CLRS]

Naive Algorithm¹

<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>
				=	=	=	=							
				<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>				

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<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>
				=	=	=	=	=						
				<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>				

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				=	=	=	=	=	≠					
				<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>				

¹Exemplu din [CLRS]

Intuition²

a	b	c	b	a	b	a	b	a	a	b	c	b	a	b
				=	=	=	=	=	≠					
				a	b	a	b	a	c	a				
					a	b	a	b	a	c	a			
						a	b	a	b	a	c	a		
							a	b	a	b	a	c	a	

²Exemplu din [CLRS]

Intuition³

?	?	?	?	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	?	?	?	?	?	?
				=	=	=	=	=	≠					
				<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	?	?				
					<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	?	?	?			
						<i>a</i>	<i>b</i>	<i>a</i>	?	?	?	?		

³Exemplu din [CLRS]

Intuition³

?	?	?	?	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	?	?	?	?	?	?
				=	=	=	=	=	≠					
				<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	?	?				
					<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	?	?	?			
						<i>a</i>	<i>b</i>	<i>a</i>	?	?	?	?		

For the pattern *ababaca*, if at position i exact the first 5 characters match, then there is no chance that the pattern match at position $i + 1$. But we have chance at $i + 2$. Why?

³Exemplu din [CLRS]

Ideea

?	?	?	?	x_0	x_1	x_2	x_3	x_4	x_5	x_6	?	?	?	?	?	?	?	?
				=	=	=	=	=	=	=	\neq							
				x_0	x_1	x_2	x_3	x_4	x_5	x_6	?							
								=	=	=								
								x_0	x_1	x_2	x_3	x_4	x_5	x_6				?

Idea

?	?	?	?	x_0	x_1	x_2	x_3	x_0	x_1	x_2	?	?	?	?	?	?	?	?
				=	=	=	=	=	=	=	\neq							
				x_0	x_1	x_2	x_3	x_0	x_1	x_2	?							
								=	=	=								
								x_0	x_1	x_2	x_3	x_0	x_1	x_2	?			

Idea

?	?	?	?	x_0	...	x_{k-1}	...	x_0	...	x_{k-1}	?	?	?	?
				=	=	=	=	=	=	=	≠			
				x_0	...	x_{k-1}	...	x_0	...	x_{k-1}	?			

Idea

?	?	?	?	x_0	...	x_{k-1}	...	x_0	...	x_{k-1}	?	?	?	?
				=	=	=	=	=	=	=	≠			
				x_0	...	x_{k-1}	...	x_0	...	x_{k-1}	?			

We are interested to find the largest k s.t. $x_1 \dots x_k$ is both prefix and suffix of the matched prefix.

Notations

- reminder: **border (frontier)** of a string t - un factor that is both prefix and suffix of t
- notation: $\text{maxFr}(k)$ - the maximum border of $p[0..k-1]$ that is proper factor of ($\neq p[0..k-1]$)
 $f[k] = |\text{maxFr}(k)|$ (the length of the longest border of $p[0..k-1]$)
- example: $p =$

a	b	a	b	a	c	a
0	1	2	3	4	5	6

k	$\text{maxFr}(k)$	$f[i]$
1	ε	0
2	ε	0
3	a	1
4	ab	2
5	aba	3
6	ε	0

Notations

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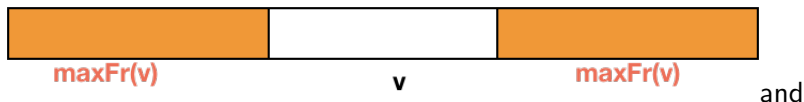
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0	1	2	3	4	5	6

k	$\text{maxFr}(k)$	$f[i]$
1	ε	0
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3	a	1
4	ab	2
5	aba	3
6	ε	0

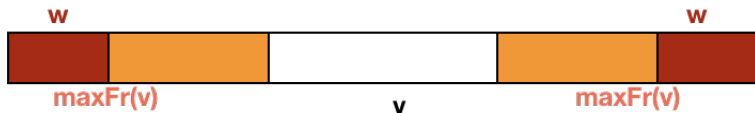
- notation: $u \leq_{fr} v$ iff $u \leq_{pref} v$ and $u \leq_{suff} v$

Reasoning problem domain 1/4

- formal definition of $\text{maxFr}(v)$:
 $\text{maxFr}(v) <_{fr} v$



$(\forall w) w <_{fr} v$ implies $w \leq_{fr} \text{maxFr}(v)$;
 i.e., the maximum border is maximum relative to \leq_{fr} as well.



Reasoning problem domain 2/4

- notation: $\text{maxFr}^0(v) = v$, $\text{maxFr}^{j+1}(v) = \text{maxFr}(\text{maxFr}^j(v))$

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 $\text{maxFr}^0(v) = v$



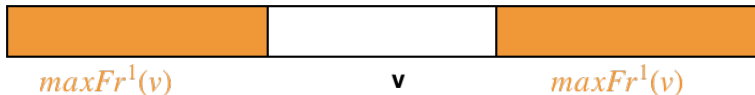
$$v = \text{maxFr}^0(v)$$

Reasoning problem domain 2/4

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$$v = \text{maxFr}^0(v)$$

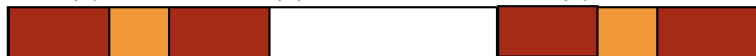


Reasoning problem domain 2/4

- notation: $\maxFr^0(v) = v$, $\maxFr^{j+1}(v) = \maxFr(\maxFr^j(v))$
- we have:
 $\dots <_{fr} \maxFr^{j+1}(v) <_{fr} \maxFr^j(v) <_{fr} \dots <_{fr} \maxFr^1(v) <_{fr} \maxFr^0(v) = v$



$$v = \maxFr^0(v)$$


 $\maxFr^1(v)$
 v
 $\maxFr^1(v)$
 $\maxFr^2(v)$ $\maxFr^2(v)$ $\maxFr^2(v)$ $\maxFr^2(v)$

 $\maxFr^1(v)$
 v
 $\maxFr^1(v)$

Reasoning problem domain 3/4

Theorem

$u \leq_{fr} v$ dif there exists $j \geq 0$ s.t. $u = \max Fr^j(v)$.

Reasoning problem domain 3/4

Theorem

$u \leq_{fr} v$ if there exists $j \geq 0$ s.t. $u = \max Fr^j(v)$.

Corollary

$u <_{fr} v$ if there exists $j > 0$ s.t. $u = \max Fr^j(v)$.

Reasoning problem domain 4/4

Theorem

$$|maxFr^j(p[0..k-1])| = f^j[k].$$

Since the borders of $v = p[0..k-1]$ are

$$\dots <_{fr} maxFr^{j+1}(v) <_{fr} maxFr^j(v) <_{fr} \dots <_{fr} maxFr^1(v) <_{fr} maxFr^0(v) = v$$

it follows that their lengths satisfy the relation

$$\dots < f^{j+1}[k] < f^j[k] < \dots < f[k] < f^0[k] = k$$

and the "shift" from $maxFr^{j+1}(v)$ to $maxFr^j(v)$ is equal to $f^{j+1}[k] - f^j[k]$

Example 1/6

Here is an example how $f[i]$ is used for an efficient searching:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a	b	c	b	a	b	a	b	a	b	a	b	a	c	a
=	=	≠												
a	b	a	b	a	c	a								
0	1	2	3	4	5	6								

 $f =$

0	1	2	3	4	5	6
a	b	a	b	a	c	a
-1	0	0	1	2	3	0

- failure at position $i = k = 2$
- $f[k] = f[2] = 0$
- shift with $k - f[k] = 2 - 0 = 2$ positions
- the next position to be compared: $i = 2, k = f[k] = 0$

Example 2/6

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a	b	c	b	a	b	a	b	a	b	a	b	a	c	a

\neq														
a	b	a	b	a	c	a								
0	1	2	3	4	5	6								

$f =$

0	1	2	3	4	5	6
a	b	a	b	a	c	a
-1	0	0	1	2	3	0

- failure at $i = 2, k = 0$
- $f[0] = ?$
- shift $k - f[k] = 0$, so $f[0] = -1$
- the next position to be compared: $i = i + 1 = 3, k = f[k] + 1 = 0$

Example 3/6

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a	b	c	b	a	b	a	b	a	b	a	b	a	c	a

\neq						
a	b	a	b	a	c	a
0	1	2	3	4	5	6

$$f =$$

0	1	2	3	4	5	6
a	b	a	b	a	c	a
-1	0	0	1	2	3	0

- failure at $i = 3, k = 0$
- $f[0] = -1$
- shift with $k - f[k] = 0 - f[0] = 1$ positions
- the next position to be compared: $i = i + 1 = 4, k = f[k] + 1 = 0$

Example 4/6

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a	b	c	b	a	b	a	b	a	b	a	b	a	c	a
				=	=	=	=	=	≠					
				a	b	a	b	a	c	a				
				0	1	2	3	4	5	6				

$$f =$$

0	1	2	3	4	5	6
a	b	a	b	a	c	a
-1	0	0	1	2	3	0

- failure at $i = 9, k = 5$
- $f[5] = 3$
- shift $k - f[k] = 5 - f[5] = 2$ positions
- the next position to be compared: $i = 9, k = f[k] = 3$

Example 5/6

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>
									=	=	≠			
						<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>		
						0	1	2	3	4	5	6		

$$f =$$

0	1	2	3	4	5	6
<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>
-1	0	0	1	2	3	0

- failure at $i = 11, k = 5$
- $f[5] = 3$
- shift $k - f[k] = 5 - f[5] = 2$ positions
- the next position to be compared: $i = 11, k = f[k] = 3$

Example 6/6

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
<i>a</i>	<i>b</i>	<i>c</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>
											=	=	=	=
								<i>a</i>	<i>b</i>	<i>a</i>	<i>b</i>	<i>a</i>	<i>c</i>	<i>a</i>
								0	1	2	3	4	5	6

- the first occurrence of the pattern matching

Rules

- if $p[k] == s[i]$ and $k = m - 1$, then we have an occurrence of the patterns at the start position $i - m + 1$;
- if $p[k] \neq s[i]$ then k becomes $f[k]$ ($k = f[k]$), i.e., we test the next border;
- if $k == -1$ then increment both i and k ;
- if $p[k] == s[i]$ and $k < m - 1$ then increment both i and k ;

KMP Algorithm in Alk

```

KMP(s, p, f) {
    n = s.size();
    m = p.size();
    i = 0;
    k = 0;
    while (i < n) {
        while ((k != -1) && (p[k] != s[i]))
            k = f[k];
        // k == -1 or p[k] == s[i]
        if (k == m-1)
            return i-m+1; /* gasit p in s */
        else {
            i = i+1;
            k = k+1;
        }
    }
    return -1; /* p nu apare in s */
}

```


Execution Time

1 Remarks:

- 1 for any k , $-1 \leq f[k] < k$.

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2 Conclusion: the execution time for KMP is $O(n)$

Failure function f : introduction

- since f is used when a comparison fails, f is also called **failure function**
- usually denoted by π (e.g., in [CLR])
- recall that $f[i] = |\text{maxFr}(p[0..i-1])|$ (the length of the maximum border of $p[0..i-1]$)
- example:

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- A naive implementation with $O(m^3)$ time is possible (exercise)
- Question: if $f[0..i-1]$ is already computed, how $f[i]$ can be efficiently computed?

Failure function f : computation

- recall that the borders of $v = p[0..i - 1]$ are:
 $\cdots <_{fr} \max Fr^{j+1}(v) <_{fr} \max Fr^j(v) <_{fr} \cdots <_{fr} \max Fr^1(v) <_{fr}$
 $\max Fr^0(v) = v$
 and $f^j[i] = |\max Fr^j(p[0..i - 1])|$

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 and $f^j[i] = |\max Fr^j(p[0..i - 1])|$
- it follows that $f[i] = f^k[i - 1] + 1$, where k is the smallest integer having the property $p[f^k[i - 1] + 1] = p[i - 1]$

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- i.e., we look at the prefixes of p that are suffixes of $p[0..i-2]$ and take the largest for that the next character is equal to $p[i-1]$
- but the values $f^j[i-1]$, $j = 0, 1, \dots$ are in $f[0..i-1]$! (which is already computed)

Failure function f : Alk description

```

f[0] = -1; f[1] = 0;
k = 0;
for (i = 2; i < m; ++i) {
    // invariant: k = f[i-1]
    while(k >= 0 && p[k] != p[i-1])
        // invariant: there exists j cu k = f^j[i-1] si
        // j is cel mai mic cu p[f^j[i-1]+1] != p[i-1]
        k = f[k];
    k = k + 1;
    f[i] = k;
}

```

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Execution time: $\Theta(m)$.

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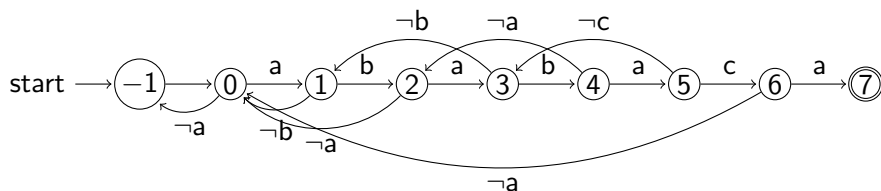
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    f[i] = k;
}

```

Execution time: $\Theta(m)$.

The analysis is similar to that of KMP.

Failure function as an automaton



An automaton consists of:

- input alphabet (e.g. a, b, c, \dots)
- state (e.g., $-1, 0, 1, \dots, 7$)
- initial state (-1 in the example)
- accepting (final) state (7 in the example)
- instantaneous transitions: (e.g., $-1 \rightarrow 0$)
- labelled transitions: ($0 \xrightarrow{a} 1, 1 \xrightarrow{b} 2, 2 \xrightarrow{a} 3, \dots, 0 \rightarrow -1, 1 \rightarrow 0, \dots$)

Plan

- 1 Problem Domain
- 2 Boyer-Moore Algorithm
- 3 Boyer-Moore Algorithm Revised
- 4 Algoritmul Knuth-Morris-Pratt
- 5 Regular Expressions**

Motivation: patterns in many text editors (e.g., Emacs)

From documentation (Emacs):

Pattern	Matches
.	Any single character except newline (" <code>\n</code> ").
\.	One period
[0-9]+	One or more digits
[^ 0-9]+	One or more non-digit characters
[A-Za-z]+	one or more letters
[-A-Za-z0-9]+	one or more letter, digit, hyphen
[_A-Za-z0-9]+	one or more letter, digit, underscore
[-_A-Za-z0-9]+	one or more letter, digit, hyphen, underscore
[:ascii:]+	one or more ASCII chars. (codepoint 0 to 127, inclusive)
[:nonascii:]+	one or more none-ASCII characters (For example, Unicode charac
[\n\t]+	one or more {newline character, tab, space}.

Demo cu Emacs

(Mathematical) Definition

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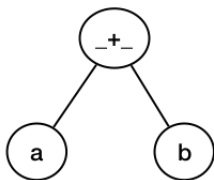
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- if e_1, e_2 are regular expressions, then $e_1 + e_2$ is a regular expression;
- if e is a regular expression, then e^* is a regular expressions.

Often we use parentheses to show how the above rules were applied; e.g., $(a + b)^*$

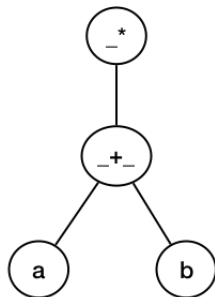
Abstract Syntactic Tree (AST)



a



a + b



(a + b)*

Relationship with the package <regex> in C++, Emacs

<regex>	regular expression in math notation
[abc]	$a + b + c$
\d sau [[:digit:]]	$0 + 1 + \dots + 9$
[[:digit:]]*	$(0 + 1 + \dots + 9)^*$
[[:digit:]]+	$(0 + 1 + \dots + 9)(0 + 1 + \dots + 9)^*$

The language defined by a regular expression 1/2

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The language defined by a regular expression 1/2

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The language defined by a regular expression 1/2

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Remark. The operator $_*$ is called **Kleene star** or **Kleene closure**.

The language defined by a regular expression 2/2

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The language defined by a regular expression 2/2

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...

$$L((ab)^*) = \bigcup_{k \geq 0} L((ab)^k) = \{\varepsilon, ab, abab, ababab, \dots\}$$

The end

The next lecture:

- the AST of regular expression in ALK
- parsing algorithm
- the automaton associated to a regular expression
- searching using the automaton