1 = S 3. 2.g

g) (R) $\sum_{n=1}^{\infty} \underbrace{\frac{1!+2!+...+n!}{(n+2)!}}_{x_n}$;

Se poate anterior $\lim_{n\to\infty} x_n \xrightarrow[S-C]{\text{Dacă } \exists} \lim_{n\to\infty} \frac{(n+1)!}{(n+3)!-(n+2)!}$

$= \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!(n+3-1)} =$

$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)!\,(n+2)(n+2)} = \lim_{n\to\infty} \frac{1}{(n+2)^2} = 0$

Propunere:

1) $x_n \qquad y_n = \frac{1}{n^2} \quad , \quad z_n = \frac{n!}{(n+2)!}$

$\frac{1!+2!+...+n!}{(n+2)!} > \frac{n!}{(n+2)!} = \underbrace{\frac{1}{n(n+1)}}$

2) Gauss:

$\frac{x_{n+1}}{x_n} = \frac{\frac{1!+2!+...+n!+(n+1)!}{(n+3)!}}{\frac{1!+2!+...+n!}{(n+2)!}} =$

$\frac{1!+2!+...+n!+(n+1)!}{(n+3)!} \cdot \frac{\cancel{(n+2)!}}{1!+2!+...+n!} =$

$$\frac{1!+2!+\dots+n!+(n+1)!}{(n+3)(1!+2!+\dots+n!)}$$

3) Raportului

$$\overbrace{\frac{1!+2!+\dots+n!+(n+1)!}{(n+3)(1!+2!+\dots+n!)}}^{a_n} =$$

$$\overline{\frac{1}{n+3}\cdot\frac{1!+2!+\dots+n!+(n+1)!}{1!+2!+\dots+n!}}$$

$$\frac{1}{n+3}\cdot\left(1+\frac{(n+1)!}{1!+2!+\dots+n!}\right)$$

$$\ln\frac{1}{n+3}\cdot\left(1+\frac{(n+1)!}{1!+2!+\dots+n!}\right) =$$

$$\ln\frac{1}{n+3}+\ln\left(1+\frac{(n+1)!}{}\right)$$

$$\lim_{n\to\infty} a_n =$$

$$\lim_{n\to\infty} \frac{(n+2)!}{(n+4)\cdot(1!+2!+\dots+n!+(n+1)!)-(n+3)(1!+\dots+n!)}$$

$$\lim_{n\to\infty} \frac{(n+2)!}{1!+2!+\dots+n!+(n+4)(n+1)!}$$

Raabe - Duhamel

$$n\left(\frac{x_n}{x_{n+1}}-1\right)=n\left(\frac{(n+3)(1!+2!+\dots+n!)}{1!+2!+\dots+(n+1)!}-1\right)$$

$$=n\left(\frac{(n+3)(1!+2!+\dots+n!)-1!-2!-\dots-(n+1)!}{1!+2!+\dots+(n+1)!}\right)$$

$$=n\left(\frac{(n+2)(1!+2!+\dots+n!)-(n+1)!}{1!+2!+\dots+(n+1)!}\right)$$

$$=n\left(\frac{(n+2)(1!+2!+\dots+(n-1)!)+(n+2)\cdot n!-(n+1)!}{1!+2!+\dots+(n+1)!}\right)$$

$$n\left(\frac{(n+2)(1!+2!+\dots+(n-1)!)+n!(\cancel{n}+2-\cancel{n}-1)}{1!+2!+\dots+(n+1)!}\right)$$

$$n\underbrace{\left(\frac{(n+2)(1!+2!+\dots+(n-1)!)+n!}{1!+2!+\dots+(n+1)!}\right)}=$$

Puteti încerca și altfel

$$\frac{n[(n+2)(1!+2!+\dots+(n-1)!)+n!]}{(n+1)!}\cdot\underbrace{\frac{(n+1)!}{1!+2!+\dots+(n+1)!}}_{\downarrow\, 1}$$

$$\to 2$$

$$\lim_{n\to\infty}\frac{(n+1)!}{1!+2!+\dots+(n+1)!}\ \overset{\text{Dacă }\exists}{\underset{CS}{=}}$$

$$\lim_{n\to\infty}\frac{(n+2)!-(n+1)!}{(n+2)!}=$$

$$\lim_{n\to\infty}\frac{\cancel{(n+1)!}(n+2-1)}{(n+2)\cancel{!}}=\lim_{n\to\infty}\frac{n+1}{n+2}=1$$

$$\lim_{n\to\infty}\frac{n[(n+2)(1!+2!+\dots+(n-1)!)+n!]}{(n+1)!}=$$

$$\lim_{n\to\infty}\left[\underbrace{\frac{n(n+2)}{n(n+1)}}_{\downarrow\atop 1}\ \underbrace{\frac{1!+2!+\dots+(n-1)!}{(n-1)!}}_{\downarrow\atop 1} + \underbrace{\frac{n\cdot n!}{(n+1)!}}_{\downarrow\atop 1}\right] = 2$$

$\underset{\text{Raabe}}{\Longrightarrow}$ C

2 = 53.2 ℓ

$$\sum_{k=0}^{n} \arcsin\frac{1}{k\sqrt[3]{k}+5}$$

$$\sum \frac{1}{n}$$

$$\frac{1}{n}\longrightarrow 0$$

1) $\displaystyle\sum_{n=1}^{\infty} \arcsin \underbrace{\frac{1}{n\sqrt[3]{n}+5}}_{\downarrow\atop 0}$

$\sim$

$$a_n = \arcsin \underbrace{\frac{1}{n\sqrt[3]{n}+5}}_{\downarrow\atop 0}$$

$$b_n = \frac{1}{n\sqrt[3]{n}+5}$$

$$\frac{a_n}{b_n} = \frac{\arcsin\frac{1}{n\sqrt[3]{n}+5}}{\frac{1}{n\sqrt[3]{n}+5}} \longrightarrow 1 \overset{CC\,III}{\Longrightarrow}$$

$$\sum a_n \sim \sum b_n$$

$$c_n = \frac{1}{n^{\frac{4}{3}}}$$

$$\frac{b_n}{c_n} = \frac{\frac{1}{n\sqrt[3]{n}+5}}{\frac{1}{n^{\frac{4}{3}}}} = \frac{n\sqrt[3]{n}}{n\sqrt[3]{n}+5} \longrightarrow 1 \overset{CC\,III}{\Longrightarrow}$$

$\sum b_n \sim \sum c_n$ $\quad C \Rightarrow$

serie armonică generalizată cu $\alpha = \frac{4}{3}$

$\sum b_n C \Rightarrow \sum a_n C$

3.

S3.2 o) (R) $\underbrace{\sum_{n=1}^{\infty} \frac{1}{e \cdot \sqrt{e} \cdot \sqrt[3]{e} \cdot \ldots \cdot \sqrt[n]{e}}}_{x_n}$;

$$x_n = \frac{1}{e \cdot \sqrt{e} \cdot \sqrt[3]{e} \cdot \ldots \cdot \sqrt[n]{e}} =$$

$$\frac{1}{e^{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}}$$

S2.8* Să se arate că şirul cu termenul general

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n, \ \forall\, n \in \mathbb{N}^*$$

este convergent în $\mathbb{R}$ (limita sa fiind aşa numita constantă a lui Euler, $c = 0,577215... \in \mathbb{R} \setminus \mathbb{Q}$).

$$y_n = \frac{1}{e^{\ln n}} = \frac{1}{n}$$

$$\frac{x_n}{y_n} = \frac{\frac{1}{e^{1+\frac{1}{2}+\ldots+\frac{1}{n}}}}{\frac{1}{e^{\ln n}}} = e^{\ln n - \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)} \longrightarrow e^{-c}$$

$\overline{CCIII}$

$\Rightarrow \sum x_n \sim \sum y_n \; D \Rightarrow \sum x_n \; D$

$$\sum x_n \qquad \frac{\ln \frac{1}{x_n}}{\ln n}$$

$$\frac{\ln \left(e^{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}\right)}{\ln n} = \frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{\ln n}$$

$$\frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n}-1+1$$

$$\underbrace{\frac{1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n}{\ln n}}_{\substack{0 \\ \downarrow \\ 1}}+1 =$$

4

S3.2 e) $\sum_{n=2}^{\infty} \underbrace{\frac{1}{(\ln n)^{\ln n}}}_{x_n}$;

$$\frac{\ln\left[(\ln n)^{\ln n}\right]}{\ln n} = \frac{\cancel{\ln n}\ \ln(\ln n)}{\cancel{\ln n}} \to \infty$$

$$\Rightarrow \sum x_n \ C$$

q) $\sum_{n=1}^{\infty} \left(\frac{1}{n}+\ln\frac{n}{n+1}\right)$;

Studiem șirul sumelor parțiale

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{k}+\ln\frac{k}{k+1}\right) =$$

$$\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}\ln\frac{k}{k+1} =$$

$$\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}\ln k-\sum_{k=1}^{n}\ln(k+1) \overset{k+1=l}{=}$$

$$\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}\ln k-\sum_{l=2}^{n+1}\ln l =$$

$$\sum_{k=1}^{n} \frac{1}{k} + \underbrace{\ln 1}_{0} - \ln(n+1) =$$

$$\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) =$$

$$\underbrace{\sum_{k=1}^{n} \frac{1}{k} - \ln n}_{\downarrow \atop C} + \underbrace{\ln \frac{n}{n+1}}_{\downarrow \atop 0} \longrightarrow C$$

$$\Rightarrow \sum C$$

3.6

$$\arccos \frac{n(n+1) + \sqrt{(n+1)(n+2)(3n+1)(3n+4)}}{(2n+1)(2n+3)} = \left(\arccos \frac{n}{2n+1} - \arccos \frac{n+1}{2n+3} \right)$$

$$\rightarrow \frac{1+\sqrt{9}}{2 \cdot 2} = 1$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b =$$

$$\cos a \cos b + \sqrt{1-\cos^2 a}\sqrt{1-\cos^2 b}$$

$a = \arccos x$

$b = \arccos y$

$$\cos(\arccos x - \arccos y) = \cos(\arccos x)\cos(\arccos y) + \sqrt{1-\cos^2(\arccos x)}\sqrt{1-\cos^2(\arccos y)}$$

$$\cos(\arccos x - \arccos y) = xy + \sqrt{(1-x^2)(1-y^2)}$$

$$\arccos(\cos(\arccos x - \arccos y)) = \arccos\left(xy + \sqrt{(1-x^2)(1-y^2)}\right)$$

$$\arccos x - \arccos y = \arccos\left(xy + \sqrt{(1-x^2)(1-y^2)}\right) =$$

$$\arccos\left(xy + \sqrt{(1-x)(1+x)(1-y)(1+y)}\right)$$

$$xy = \frac{n(n+1)}{(2n+1)(2n+3)}$$

$$(1-x)(1+x)(1-y)(1+y) = \frac{(n+1)(n+2)(3n+1)(3n+4)}{(2n+1)(2n+3)} =$$

$$\frac{(2n+1-n)(2n+3-(n+1))(\underline{2n+1+n})(2n+3+n+1)}{(2n+1)(2n+3)} =$$

$$\left(1-\frac{n}{2n+1}\right)\left(1+\frac{n}{2n+1}\right)\left(1-\frac{n+1}{2n+3}\right)\left(1+\frac{n+1}{2n+3}\right)$$

$$x = \frac{n}{2n+1} \qquad y = \frac{n+1}{2n+3}$$

$$S_n = \sum_{k=1}^{n} \arccos \frac{k(k+1) + \sqrt{(k+1)(k+2)(3k+1)(3k+4)}}{(2k+1)(2k+3)} =$$

$$\sum_{k=1}^{n} \left(\arccos \frac{k}{2k+1} - \arccos \frac{k+1}{2k+3} \right) = \quad k+1=l$$

$$\sum_{k=1}^{n} \arccos \frac{k}{2k+1} - \sum_{l=2}^{n+1} \arccos \frac{l}{2l+1} =$$

$$\arccos \frac{1}{3} - \arccos \frac{n}{2n+3} =$$

$$\arccos \frac{1}{3} - \underbrace{\arccos \frac{1}{2}}_{\frac{\pi}{3}}$$

$$\arccos \frac{1}{3} - \frac{\pi}{3}$$

$$\to \sum C$$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{\operatorname{arctg}(n\alpha)}{(\ln 3)^n}, \alpha \in \mathbb{R};$

$e \simeq 2.718$

$3 > e \Rightarrow$

$\ln 3 > \ln e = 1$

$$x_n = \frac{\operatorname{arctg}(n\alpha)}{(\ln 3)^n} \to 0$$

$(\ln 3)^n \to \infty$

$$-\frac{\pi}{2} \le \operatorname{arctg} n\alpha \le \frac{\pi}{2}$$

$$0 \le |\operatorname{arctg} n\alpha| \le \frac{\pi}{2} \to \frac{|\operatorname{arctg} n\alpha|}{(\ln 3)^n} \le \frac{\frac{\pi}{2}}{(\ln 3)^n}$$

$$\sum \frac{\frac{\pi}{2}}{(\ln 3)^n} \quad C$$

R) $\sum_{n=1}^{\infty}\underbrace{(\sqrt{n^4+3n^2+1}-n^2)}_{x_n}$.

$$x_n = \frac{\cancel{n^4}+3n^2+1-\cancel{n^4}}{\sqrt{n^4+3n^2+1}+n^2} = \frac{3n^2}{\sqrt{n^4+3n^2+1}+n^2} \;\Big|\, n^2 \to \frac{3}{2}$$

$$= \frac{3}{\sqrt{1+\underbrace{\frac{3}{n^2}}_{\downarrow 0}+\underbrace{\frac{1}{n^4}}_{\downarrow 0}}+1}$$

$$\to \sum x_n \quad D$$

$P \longrightarrow Q$ echivalente

Dacă afară e cald, eu mănânc îngheţată

$\neg Q \longrightarrow \neg P$

Dacă nu mănânc îngheţată, nu e cald afară

Nu rezultă că $Q \longrightarrow P$

Dacă $\sum x_n$ C $\Rightarrow x_n \to 0$

Dacă $x_n \not\to 0 \Rightarrow \sum x_n$ D

~~Nu rezultă $x_n \to 0 \Rightarrow \sum x_n$ C~~

$$\sum_{n=1}^{\infty} \underbrace{\frac{\ln n}{n^2}}_{x_n};$$

log:

$$\frac{\ln \frac{1}{x_n}}{\ln n} = \frac{\ln\left(\frac{n^2}{\ln n}\right)}{\ln n} =$$

$$\frac{\ln n^2 - \ln(\ln n)}{\ln n} =$$

$$\frac{2\ln n - \ln(\ln n)}{\ln n} =$$

$$2 - \frac{\ln(\ln n)}{\ln n} \to 2$$

C

<u>condensare:</u> $\sum x_n \sim \sum 2^n x_{2^n} =$

$$\sum_{n=1}^{\infty} \cancel{2^n} \frac{\ln 2^n}{(2^n)^2} =$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{n \ln 2}{2^n}}_{y_n}$$

$$\frac{y_{n+1}}{y_n} = \frac{\frac{(n+1)\cancel{\ln 2}}{2^{\cancel{n+1}}}}{\frac{n \cancel{\ln 2}}{\cancel{2^n}}} = \frac{n+1}{2n} \to \frac{1}{2} \ \cancel{<1}$$

$$\sum y_n \text{ C} \Leftrightarrow \sum x_n \text{ C}$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{2^n + 3^{n+1} - 6^{n-1}}{12^n}}_{x_n};$$

$$x_n = \frac{1}{6^n} + 3 \cdot \frac{1}{4^n} - \frac{1}{12} \cdot \frac{1}{2^{n-1}}$$

Șirul sumelor parțiale

$$S_n = \sum_{k=1}^{n} \frac{1}{6^k} + 3 \cdot \frac{1}{4^k} - \frac{1}{12} \cdot \frac{1}{2^{k-1}} =$$

$$\frac{1}{6}\cdot\frac{1-\frac{1}{6^n}}{1-\frac{1}{6}}+\frac{3}{4}\cdot\frac{1-\frac{1}{4^n}}{1-\frac{1}{4}}-\frac{1}{12}\cdot\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}$$

$$\longrightarrow \frac{1}{\cancel{6}}\cdot\frac{1}{\frac{5}{\cancel{6}}}+\cancel{\frac{3}{4}}\cdot\frac{1}{\cancel{\frac{3}{4}}}-\frac{1}{\cancel{12}\,6}\cdot\frac{1}{\frac{1}{\cancel{2}}}=$$

$$\frac{1}{5}+1-\frac{1}{6}=$$

$$=\frac{31}{30}$$

$$\sum_{n=2}^{\infty}\underbrace{\left(\sqrt{n+1}-\sqrt{n}\right)^{a}\ln\left(\frac{n+1}{n-1}\right)}_{x_n},\ a\in\mathbb{R};$$

$$x_n=\left(\frac{n+1-n}{(\sqrt{n+1}+\sqrt{n})}\right)^{a}\ln\left(1+\frac{2}{n-1}\right)$$

$$x_n=\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)^{a}\ln\left(1+\frac{2}{n-1}\right)$$

$$x_n=\ln\underbrace{\left(1+\frac{2}{n-1}\right)^{\frac{n-1}{2}}}_{\downarrow\atop e}\cdot\frac{2}{(n-1)}\cdot\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)^{a}$$

$$= \ln\left(1+\frac{2}{n-1}\right)^{\frac{n-1}{2}} \cdot \frac{2}{(n-1)(\sqrt{n+1}+\sqrt{n})^{a}}$$

$1+\frac{1}{2}\cdot a > 0 \Rightarrow$

$\frac{1}{2}a > -1 \Rightarrow$

$a > -2$ Condiție necesară pt

care $a > -2$

x_n $y_n = \frac{1}{n^{\frac{a}{2}}}$

$$\frac{x_n}{y_n} = \frac{\frac{1}{(\sqrt{n+1}+\sqrt{n})^{a}} \cdot \ln\left(1+\frac{2}{n-1}\right)}{\frac{1}{n^{\frac{a}{2}}}}$$

$$= 2\ln\left(1+\frac{2}{n-1}\right) \longrightarrow 0$$

Dacă $\frac{a}{2} > 1 \Leftrightarrow a > 2$

$\Rightarrow \sum x_n\ C$

pt $a \in (-2, 2]$?

$$\left(\sqrt{n+1} - \sqrt{n}\right)^a \ln\left(\frac{n+1}{n-1}\right)$$

$$\frac{1}{\left(\sqrt{n+1} + \sqrt{n}\right)^a} \ln\left(\frac{n+1}{n-1}\right)$$

$a = 0$

$$\sum \ln\left(\frac{n+1}{n-1}\right)$$

$$S_n = \sum_{k=2}^{n} \ln\left(\frac{k+1}{k-1}\right) =$$

$$\sum_{k=2}^{n} \ln(k+1) - \sum_{k=2}^{n} \ln(k-1) =$$

$a = k+1 \qquad b = k-1$

$$\sum_{a=3}^{n+1} \ln a - \sum_{b=1}^{n-1} \ln b =$$

$\ln(n+1) + \ln n - \ln 2 - \ln 1 \to \infty$

Dacă $a < 0$

$$\frac{1}{\left(\sqrt{n+1} + \sqrt{n}\right)^{a}} = \left(\sqrt{n+1} + \sqrt{n}\right)^{-a} > 1$$

$$\Rightarrow \frac{1}{\left(\sqrt{n+1} + \sqrt{n}\right)^{a}} \ln\left(\frac{n+1}{n-1}\right) > \ln\left(\frac{n+1}{n-1}\right)$$

$$\sum \ln\left(\frac{n+1}{n-1}\right) D$$

$\Rightarrow \sum x_n \; D$

$$\Rightarrow \begin{cases} \alpha \leq -2 & x_n \not\to 0 \;\; \sum x_n D \\ \alpha \in [-2, 0] & \sum x_n D \\ \alpha \in (0, 2] & ? \\ \alpha > 2 & \sum x_n C \end{cases}$$

$\alpha = 2$

$$x_n = \frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)^2} \ln\left(\frac{n+1}{n-1}\right)$$