SEMINAR 8

Exerciţii recomandate: 8.1 b), 8.2 ii),v), 8.3 a), 8.4 b), 8.5 b), 8.6 a)
Rezerve: 8.1 a), 8.2 i),iii),iv), 8.3 b),c), 8.4 a), 8.5 a),c), 8.6 c), 8.7 a)

S8.1 Calculaţi următoarele limite:

a) $\lim\limits_{x\nearrow 1} \dfrac{\sin(p\arccos x)}{\sqrt{1-x^2}}$, $p\in\mathbb{R}^*$;

b) $\lim_{x\to 0}\left(\dfrac{\ln(1+3x^2)}{x^2}, \dfrac{e^{x^2}-\cos x}{x^2}\right)$;

c) $\lim_{x\nearrow\frac{\pi}{4}}\left((\operatorname{tg} x)^{\operatorname{tg}(2x)}, \dfrac{\sin(4x)}{\sqrt{\pi-4x}}, \dfrac{2^{\operatorname{tg} x}-2}{x-\frac{\pi}{4}}, \operatorname{tg}(2x)\operatorname{tg}\left(\dfrac{\pi}{4}-x\right), \dfrac{\ln(\operatorname{tg} x)}{\cos 2x}\right)$;

d)* $\lim_{x\to\infty}\left(x-\sqrt[n]{(x-a_1)(x-a_2)\dots(x-a_n)}\right)$, $n\in\mathbb{N}^*$, $a_k\in\mathbb{R}$, $k=\overline{1,n}$;

e)* $\lim\limits_{x\searrow 0}\left(\dfrac{1+\sum_{k=1}^n \ln(1+kx)}{\sum_{k=1}^n n^{kx-1}}\right)^{\frac{1}{x}}$, $n\in\mathbb{N}^*$.

S8.2 Studiaţi existenţa şi, în cazul afirmativ, determinaţi limitele iterate $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)$ şi $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} f(x,y)\right)$, limitele parţiale $\lim\limits_{x\to 0} f(x,0)$, $\lim\limits_{y\to 0} f(0,y)$ şi limita globală $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ pentru funcţiile $f:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}$ definite mai jos:

i) $f(x,y) := \dfrac{x^2-y^2}{|x|+|y|}$;

ii) $f(x,y) := \dfrac{\sqrt{1+x^2y^2}-1}{x^2+y^2}$;

iii) $f(x,y) := (x^2+y^2)^{x^2y^2}$;

iv) $f(x,y) := \dfrac{\sin(xy)}{\sqrt{x^2+y^2}}$;

v) $f(x,y) := y^2\ln(x^2+y^2)$;

vi) $f(x,y) := \dfrac{e^{\frac{1}{x^2+y^2}}}{x^6+y^6}$.

S8.3 Arătaţi că următoarele funcţii nu au limită în $(0,0)$, chiar dacă au limite iterate şi/sau limite în orice direcţie admisibilă:

a) $f:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}$, $f(x,y) := \dfrac{x^2-y^2}{x^2+y^2}$;

b) $f:\mathbb{R}^2\setminus\{(x,y)\mid x=y\}\to\mathbb{R}^2$, $f(x,y) := \left(\dfrac{x+y}{x-y}, \dfrac{x^2y}{x^4+y^2}\right)$;

c) $f:\mathbb{R}^2\setminus\{(x,y)\mid y^2=2x\}\to\mathbb{R}$, $f(x,y) := \dfrac{y^2+2x}{y^2-2x}$;

d) $f:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}^2$, $f(x,y) := \left(\dfrac{x^2y^2}{(x-y)^2+x^2y^2}, (1+|xy|)^{\frac{2}{x^2+y^2}}\right)$.

S8.4 Determinaţi următoarele limite:

a) $\lim\limits_{(x,y)\to(0,0)}\left(\dfrac{xy}{\sqrt{1+xy}-1}, \dfrac{\sin(x^3+y^3)}{\sqrt{x^2+y^2+1}-1}\right)$;

b) $\lim\limits_{(x,y,z)\to(0,0,0)}\left(\dfrac{1}{xyz}\operatorname{tg}\dfrac{xyz}{1+xyz}, (1+xyz)^{\frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}}}\right)$;

c) $\lim\limits_{(x,y,z)\to(0,0,0)}\left(\dfrac{1-\cos(1-\cos(x^2+y^2+z^2))}{(x^2+y^2+z^2)^4}, \dfrac{x^2y^2z^2}{(x-y)^2+(y-z)^2+(x-z)^2}\right)$.

S8.5 Determinaţi mulţimile punctelor de discontinuitate ale următoarelor funcţii reale:

a) $f(x,y) := \begin{cases} \dfrac{|x|}{y}\mathrm{e}^{-|x|y^{-2}}, & y\neq 0;\\ 1, & y=0;\end{cases}$

b) $f(x,y) := \begin{cases} \dfrac{xy}{x+y}, & x+y \neq 0; \\ 0, & x+y=0; \end{cases}$

c) $f(x,y,z) := \begin{cases} (x^2+y^2+z^2)^{1/3}\ln(x^2+y^2+z^2), & (x,y,z) \neq (0,0,0); \\ 1/3, & (x,y,z)=(0,0,0). \end{cases}$

S8.6 Arătaţi că următoarele funcţii sunt continue în fiecare variabilă, dar nu global continue în punctul $(0,0)$:

a) $f(x,y) := \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0); \\ 0, & (x,y)=(0,0); \end{cases}$

b) $f(x,y) := \begin{cases} \dfrac{x^4y}{x^6+y^3}, & y \neq -x^2; \\ 0, & y=-x^2; \end{cases}$

c) $f(x,y,z) := \begin{cases} \dfrac{\sin(xy+yz+xz)}{\sqrt{(x^4+y^2+z^4)}}, & (x,y,z) \neq (0,0,0); \\ 0, & (x,y,z)=(0,0,0); \end{cases}$

d) $f(x,y) := \begin{cases} \min\left\{\left|\dfrac{x}{y}\right|, \left|\dfrac{y}{x}\right|\right\}, & xy \neq 0; \\ 0, & xy=0. \end{cases}$

S8.7 Studiaţi continuitatea următoarelor funcţii:

a) $f : [-1,+\infty) \to \mathbb{R}^2$, $f(x) := (f_1(x), f_2(x))$, unde, pentru $p \in \mathbb{R}$, f_1 şi f_2 sunt definite de

$$f_1(x) := \begin{cases} 2\operatorname{tg} x \cdot \operatorname{arctg}\dfrac{1}{x}, & x \in [-1,0); \\ p, & x=0; \\ \mathrm{e}^{\frac{1-\sqrt{1+x}}{x^2 e^x}}, & x>0, \end{cases}$$

$$f_2(x) := \begin{cases} \operatorname{tg} x \cdot \sin\dfrac{1}{x}, & x \in [-1,0); \\ p, & x=0; \\ \cos\dfrac{1}{x}\mathrm{e}^{-\frac{1}{x}}, & x>0. \end{cases} \quad .$$

b) $f : A = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x^2+y^2 < \frac{\pi}{2}\right\} \to \mathbb{R}$, $\alpha \in \mathbb{R}$,

$$f(x,y) := \begin{cases} \dfrac{1-\cos\sqrt{x^2+y^2}}{\operatorname{tg}(x^2+y^2)}, & (x,y) \in A \setminus \{(0,0)\} \\ \alpha, & (x,y)=(0,0). \end{cases}$$

c) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z))$, unde, pentru $p \in \mathbb{R}$, f_1 şi f_2 sunt definite de

$$f_1(x,y,z) := \begin{cases} (x^2+y^2+z^2)^p \cos\dfrac{1}{\sqrt{x^2+y^2+z^2}}, & (x,y,z) \neq (0,0,0); \\ 0, & (x,y,z)=(0,0,0), \end{cases}$$

$$f_2(x,y,z) := \begin{cases} (x^2+y^2+z^2)^p \mathrm{e}^{\frac{1}{\sqrt{x^2+y^2+z^2}}}, & (x,y,z) \neq (0,0,0); \\ 0, & (x,y,z)=(0,0,0). \end{cases}$$

S8.8 Stabiliţi care din următoarele funcţii sunt continue pe domeniul lor de definiţie:

a) $f : (0,1) \to \mathbb{R}$, $f(x) := \sqrt{x}\sin\dfrac{2}{x}$;

b) $f : (-1,\infty) \to \mathbb{R}^2$, $f(x) := \left(\dfrac{x}{x^2+2}, \dfrac{\arcsin(x+1)}{x+1}\right)$;

c) $f : \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\} \to \mathbb{R}$, $f(x,y) := \sin\dfrac{\pi}{1-x^2-y^2}$;

d) $f : [0,\pi] \times [0,\frac{\pi}{3}] \to \mathbb{R}^2$, $f(x,y) := (\sin x + \cos y, \mathrm{e}^{-x}\operatorname{tg} y - 1)$.

S8.9 Calculaţi următoarele limite:

a) $\lim\limits_{x\to 3}\left(\dfrac{x^x-27}{x-3}, \dfrac{x\sin 3-3\sin x}{x\sin x-3\sin 3}, \dfrac{3^x-x^3}{x-3}\right)$;

b) $\lim\limits_{x\to\infty}\left(x \arcsin\dfrac{x+2}{\sqrt{x^4-x^2+1}}, \dfrac{\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x}}}}}{\sqrt{x+1}}, \left(\cos\dfrac{1}{x}+\sin\dfrac{1}{x}\right)^x\right)$;

c) $\lim\limits_{(x,y)\to(0,0)}\left((x^2+y^2)\ln(x^2+y^2), (x-y)\operatorname{arctg}\dfrac{1}{2x^2+3y^2}, (1+x^2y^2)^{\frac{1}{x^2+y^2}}\right)$;

d) $\lim\limits_{(x,y,z)\to(0,0,0)}\left(\dfrac{xy+yz+zx}{\sqrt{x^2+y^2+z^2}}, \dfrac{2x-3y}{z}, (x+y+z)\ln(1+|xyz|)\right)$.

S8.10 Analizaţi continuitatea următoarelor funcţii:

a) $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}^2$, $f(x):=(f_1(x), f_2(x))$, unde

$$f_1(x):=\begin{cases}\operatorname{tg} x, & x\in\mathbb{Q}\cap\left(0,\frac{\pi}{2}\right)\\ \operatorname{ctg} x, & x\in(\mathbb{R}\backslash\mathbb{Q})\cap\left(0,\frac{\pi}{2}\right)\end{cases} \quad \text{şi } f_2(x):=\begin{cases}x^2+1-\dfrac{\pi}{16}, & x\in\mathbb{Q}\cap\left(0,\frac{\pi}{2}\right)\\ \dfrac{\pi}{4x}, & x\in(\mathbb{R}\backslash\mathbb{Q})\cap\left(0,\frac{\pi}{2}\right)\end{cases};$$

b) $f:\mathbb{R}^2\to\mathbb{R}^2$, $f(x,y):=(f_1(x,y), f_2(x,y))$, unde

$$f_1(x,y):=\begin{cases}\dfrac{x-\sqrt{x^2-y+2}}{y^2-4}, & \text{dacă } 2\neq y\leq x^2+2\\ 2^{-3}, & \text{altfel,}\end{cases}$$

$$f_2(x,y)=\begin{cases}\dfrac{(x^4-y^2)^2}{x^6}, & \text{dacă } y^2\leq x^4\neq 0\\ 0, & \text{altfel.}\end{cases};$$

c) $f:\mathbb{R}^3\to\mathbb{R}$,

$$f(x,y,z)=\begin{cases}\alpha e^{x+y+z}, & \text{dacă } x+y+z<0\\ \beta, & \text{dacă } x+y+z\geq 0,\end{cases}$$

unde $\alpha,\beta\in\mathbb{R}$.

S8.11 Pot fi următoarele funcţii extinse prin continuitate la $\mathbb{R}^2$?

a) $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}^2$, $f(x,y):=\left(\dfrac{\ln(1+x^2|y|)}{x^2+y^2}, \left(1+\sin(x^4+y^4)\right)^{\frac{1}{x^2+y^2}}\right)$;

b) $f:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}^3$, $f(x,y)=\left(\dfrac{\sin(x^2-y^2)}{|x|+|y|}, (|x|+|y|)^{x^2+y^2}, \dfrac{xy}{\sqrt{2x^2+3y^2}}\right)$.

S8.12 Sunt următoarele funcţii continue?

a) $f:\mathbb{R}^2\to\mathbb{R}^2$,

$$f(x,y):=\begin{cases}0, & \text{dacă } (x,y)=\mathbf{0}_{\mathbb{R}^2};\\ \frac{4}{\pi}(x^2+y^2)\operatorname{arctg}\dfrac{1}{x^2+y^2}, & \text{dacă } (x,y)\in B(\mathbf{0}_{\mathbb{R}^2};1)\backslash\{\mathbf{0}_{\mathbb{R}^2}\};\\ 1, & \text{dacă } (x,y)\notin B(\mathbf{0}_{\mathbb{R}^2};1);\end{cases};$$

b) $f:\mathbb{R}^n\to\mathbb{R}$,

$$f(x,y):=\begin{cases}e^{-\frac{1}{(\|x\|-1)(2-\|x\|)}}, & \text{dacă } \|x\|\in(1,2);\\ 0, & \text{altfel.}\end{cases}.$$

S8.13

a) Arătaţi că mulţimea $A:=\left\{(x,y)\in\mathbb{R}^2 \mid x=\operatorname{arctg} t,\ y=\ln(1+t^2),\ t\in[-1,1]\right\}$ este compactă.

b) Fie $f : \mathbb{R}^3 \to \mathbb{R}^2$, definită de

$$f(x,y,z) := \begin{cases} \left((|x|+|y|+|z|) \ln\left(1 + \dfrac{1}{|x|+|y|+|z|}\right), \sqrt{2-(|x|+|y|+|z|)} \right), & (x,y,z) \in B \\ (0, \sqrt{2}), & (x,y,z) = \mathbf{0}_{\mathbb{R}^3} \\ (\ln 2, 1), & (x,y,z) \in C \end{cases},$$

unde $B := \left\{(x,y,z) \in \mathbb{R}^3 \mid |x|+|y|+|z| \le 1\right\} \setminus \{\mathbf{0}_{\mathbb{R}^3}\}$ şi $C := \mathbb{R}^3 \setminus \left\{(x,y,z) \in \mathbb{R}^3 \mid |x|+|y|+|z| \le 1\right\}$.
De asemenea, fie $A := \left\{(x,y,z) \in \mathbb{R}^3 \mid z \ge x^2 + y^2, x + y + z \le 2\right\}$.
Arătaţi că:

i) mulţimea A este compactă;

ii) $f\big|_A$ este uniform continuă;

iii) $f[A]$ este compactă.

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