Principles of Programming Languages
Lecture 5: Semantics

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Outline

Semantics: introduction

An evaluator for IMP

Semantics: motivation

C

```
-$ cat test.c
int main()
{
  int x;
  return (x=1) + (x=2);
}
-$ gcc test.c
-$ ./a.out ; echo $?
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```

Java

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-$ cat File.java
public class File {
  ... void main(...) {
    int x = 0;
    println((x=1) + (x=2));
  }
}
-$ javac File.java
-$ java File
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GCC: 5.4.0-6 ubuntu

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Semantics

- Semantics is concerned with the meaning of language constructs
- Semantics must be unambiguous
- Semantics must be flexible

Semantic - informal

Informal semantics (examples): natural language

Rationale for the ANSI C Programming Language:

- “Trust the programmer”
- “Don’t prevent the programmer from doing what needs to be done”
- “Keep the language small and simple”
- “Provide only one way to do an operation”
- “Make it fast, even if it is not guaranteed to be portable”

Inexact! – could lead to undefined behavior in programs

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Some (formal) semantics styles:

- operational
- denotational
- axiomatic

We will focus more on operational semantics styles: Small-step SOS, Big-Step SOS

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A simple imperative language

Program:

```
n ::= 10;;
i ::= 1;;
sum ::= 0;;
while (i «= n) do
     sum ::= sum +’ i;;
     i ::= i +’ 1
end
```

Features:

- variables
- arithmetic expressions
- boolean expressions
- assignment statements
- loop statements
- decisional statements

How can we define the semantics of this language?

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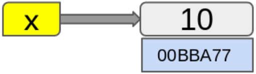
- variables can be *accessed* or *modified*
- depending on the language, they may only be able to store a specified datatype (e.g., integer, string, etc.)
- *Scope*: global, local
- Lifetime (*extent*)
- $var\_name \mapsto value$
- Environment + memory



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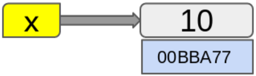

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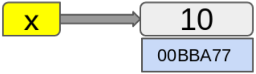

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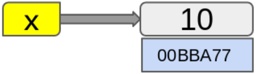

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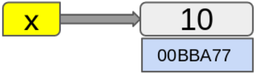

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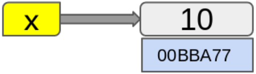

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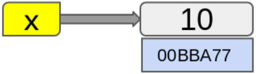

Environment

- Environment: variables mapped to values

```
Definition Env := string -> nat.
Definition env (string : string) :=
if (string_dec string "n")
                then 10
                else 0.
Compute (env n).
= 10
: nat
Compute (env i).
= 0
: nat
Check env.
env
     : string -> nat
```

- The environment is a function!

Environment update

- Update = a new function s.t. the value for `x` is updated to `v`:

```
Definition update (x : string)(v : nat)(env : Env) : Env :=
  fun y => if (string_dec x y)
             then v
             else env y.
```

Arithmetic expressions

- Syntax:

  BNF:

AExp	::=	$\mathbb{N}$
	|	AExp '+' AExp
	|	AExp '*' AExp

  ```
  Inductive AExp :=
  | anum : nat -> AExp
  | aplus : AExp -> AExp -> AExp
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  ```

- Semantics:

  ```
  Fixpoint aeval (a : AExp) : nat :=
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- Syntax:

BNF:

AExp	::=	string
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Inductive AExp :=
| var : string -> AExp
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- We need an environment to evaluate expressions with variables!

Evaluate expression with variables

- Semantics:

```
Fixpoint aeval (a : AExp) (env : Env) : nat :=
match a with
| var v => env v
| anum v => v
| aplus a1 a2 => (aeval a1 env) + (aeval a2 env)
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end.
```

- Notations and testing:

```
Coercion var : string >-> AExp.
Coercion anum : nat >-> AExp.
Notation "A +' B" := (aplus A B) (at level 49).
Notation "A *' B" := (amul A B) (at level 48).
Compute aeval (2 +' 3 *' 4) env.
= 14
: nat
Compute env n.
= 10
: nat
Compute aeval (2 +' 3 *' n) env.
= 32
: nat
```

Boolean expressions

- Syntax:

```
Inductive BExp :=
| btrue : BExp
| bfalse : BExp
| blessthan : AExp -> AExp -> BExp
| band : BExp -> BExp -> BExp
| bnot : BExp -> BExp.
```

- Semantics:

```
Fixpoint beval (b : BExp) (env : Env) : bool :=
match b with
| btrue => true
| bfalse => false
| blessthan a1 a2 => Nat.leb (aeval a1 env) (aeval a2 env)
| band b1 b2 => andb (beval b1 env) (beval b2 env)
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end.
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Assignments

- Syntax:

```
Inductive Stmt :=
| assignment : string -> AExp -> Stmt.
Notation "A ::= B" := (assignment A B) (at level 54).
Check n ::= 100 .
n ::= 100
     : Stmt
```

- Assignments modify the environment
- Semantics:

```
Fixpoint eval (s : Stmt) (env : Env) : Env :=
match s with
| assignment x a => update x (aeval a env) env
end.
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- `(eval (n ::= 100) env)` is of type Env!

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Programs: sequence of statements

- Syntax:

```
Inductive Stmt :=
...
| seq s1 s2 => eval s2 (eval s1 env)
...
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- Solution: define `eval` as a *relation*!

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