d) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (f_1(x,y), f_2(x,y))$, cu

$$f_1(x,y) = \begin{cases} \dfrac{1-\cos\left(x^3+y^3\right)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \quad \text{şi } f_2(x,y) = \begin{cases} \dfrac{x^2 y}{x^6+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

$$\lim_{(x,y)\to(0,0)} f_1(x,y) =$$

$$1 - \cos\alpha = 2\sin^2\frac{\alpha}{2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2\sin^2\frac{x^3+y^3}{2}}{\left(\frac{x^3+y^3}{2}\right)^2} \cdot$$

$$\frac{\left(\frac{x^3+y^3}{2}\right)^2}{x^2+y^2} =$$

$$\underbrace{\lim_{z\to 0} \cancel{2}\frac{\sin^2 z}{z^2}}_{1} \cdot \lim_{(x,y)\to(0,0)} \frac{\left(x^3+y^3\right)^2}{\frac{\cancel{4}}{2}\left(x^2+y^2\right)}$$

$$\lim_{(x,y)\to(0,0)} \frac{(x^3+y^3)^2}{2(x^2+y^2)} \leq$$

$$\lim_{(x,y)\to(0,0)} \frac{(|x|^3+|y|^3)^2}{2(x^2+y^2)} =$$

$$\frac{1}{2}\lim_{(x,y)\to(0,0)} \frac{\left[(|x|+|y|)(x^2-|xy|+y^2)\right]^2}{x^2+y^2} =$$

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

$$\lim_{(x,y)\to(0,0)} \frac{(|x|+|y|)^2}{2(x^2+y^2)} \cdot (x^2-|xy|+y^2)^2$$

$$\frac{|x|+|y|}{2} \leq \sqrt{\frac{x^2+y^2}{2}}$$

$$\frac{|x|+|y|^2}{2} \leq x^2+y^2$$

$-$

$$\leq \lim_{(x,y)\to(0,0)} \qquad x^2 - |xy| + y^2 = 0$$

$$f_2(x,y) = \begin{cases} \dfrac{x^2y}{x^6+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} .$$

$$\frac{y^6}{x^2+y^2} \leq \frac{x^6}{\cancel{x}^2}$$

$$\frac{(x^3+y^3)^2}{x^2+y^2} =$$

$$\frac{x^6+y^6+2x^3y^3}{x^2+y^2} =$$

$$\frac{x^6}{x^2+y^2} + \frac{y^6}{x^2+y^2} + \frac{2|x|^3|y|^3}{x^2+y^2}$$

$$\leq x^4 + y^4 + \frac{2|x|^3|y|^3}{x^2}$$

$$\text{i } f_2(x,y) = \begin{cases} \dfrac{x^2 y}{x^6+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} .$$

$$x_n = \frac{1}{n}$$

$$y_n = \frac{1}{n^3}$$

$$\lim_{n\to\infty} f_2(x_n, y_n) = \frac{\frac{1}{n^2}\cdot\frac{1}{n^3}}{\frac{1}{n^6}+\frac{1}{n^6}}$$

$$= \frac{\frac{1}{n^5}}{\frac{1}{n^6}} =$$

$$n = \infty \neq f(0,0)$$

$\Rightarrow f$ nu e dif Fréchet în $(0,0)$

$$f_1(x,y) = \begin{cases} \dfrac{1-\cos\left(x^3+y^3\right)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T(a,b) = \langle \nabla f_1(0,0), (a,b) \rangle$$

$$\frac{\partial f_1}{\partial x}(x,y) = \lim_{x \to 0} \frac{f_1(x,0) - f_1(0,0)}{x-0}$$

$$= \lim_{x \to 0} \frac{\frac{1-\cos x^3}{x^2}}{x} =$$

$$\lim_{x \to 0} \frac{1-\cos x^3}{x^3} =$$

$$\lim_{x \to 0} \frac{2 \sin^2 \frac{x^3}{2}}{\frac{x^6}{4}} \cdot \frac{x^3}{4} = 0$$

$$\frac{\partial f_1}{\partial y}(x, y) = 0$$

$$\lim_{(x,y) \to (0,0)} \frac{f(x,y) - f(0,0) - T(x,y)}{\|(x,y)\|}$$

$$\lim_{(x,y) \to (0,0)} \frac{\frac{1-\cos(x^3+y^3)}{x^2+y^2}}{\sqrt{x^2+y^2}} =$$

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^3+y^3)}{(x^2+y^2)^{\frac{3}{2}}} =$$

$$\lim_{(x,y)\to(0,0)} \frac{2\sin^2\left(\frac{x^3+y^3}{2}\right)}{\left(\frac{x^3+y^3}{2}\right)^2} \cdot$$

$$\frac{\left(\frac{x^3+y^3}{2}\right)^2}{(x^2+y^2)^{\frac{3}{2}}}$$