b) $I = \int_{-2}^{2} \min\{x(x^2-1), x+1\}\,dx$;

$x(x-1)(x+1) \ \square \ x+1$ — Compar cele 2 fct.

$(x+1)(x(x-1)-1) \ \square \ 0$ — Trec totul într-o parte și compar cu 0.

$(x+1)\underbrace{(x^2-x-1)} \ \square \ 0$ — Desfac în pol de grad I/II

$\Delta = 1 - 4\cdot 1\cdot(-1) = 5$

$x_{1,2} = \dfrac{-(-1) \pm \sqrt{5}}{2} =$

$\dfrac{1 \pm \sqrt{5}}{2}$ → $\simeq 1.618$; → $\simeq -0.5$

și studiez semnul

	-2	-1	$\frac{1-\sqrt{5}}{2}$		$\frac{1+\sqrt{5}}{2}$	2
x^2-x-1	+ +	+ +	0	− − − −	0	+ +
$x+1$	− −	0	+	+	+	
$(x+1)(x^2-x-1)$	− − − −	0	+ + 0	− − − − −	0	+

$$I = \int_{-2}^{-1} \frac{x^3 - x}{x(x^2-1)}\,dx + \int_{-1}^{\frac{1-\sqrt{5}}{2}} x+1 \; dx$$

$$+ \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} x(x^2-1)\,dx + \int_{\frac{1+\sqrt{5}}{2}}^{2} x+1 \; dx$$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_{-2}^{-1} +$$

$$\left(\frac{x^2}{2} + \cancel{x}\right)\Bigg|_{-1}^{\frac{1-\sqrt{5}}{2}} +$$

$$\left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} +$$

$$\left(\frac{x^2}{2} + x\right)\Bigg|_{\frac{1+\sqrt{5}}{2}}^{2}$$

$$\left(\frac{1}{4}-\frac{1}{2}\right)-\left(\frac{16}{4}-\frac{4}{4}\right)+$$

$$\left(\frac{\left(\frac{1-\sqrt{5}}{2}\right)^2}{2}+\frac{1-\sqrt{5}}{2}\right)-\left(\frac{1}{2}-1\right)+$$

$$\left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^4}{4}-\frac{\left(\frac{1+\sqrt{5}}{2}\right)^2}{2}\right]-\left[\frac{\left(\frac{1-\sqrt{5}}{2}\right)^4}{4}-\frac{\left(\frac{1-\sqrt{5}}{2}\right)^2}{2}\right]$$

$$+\left(\frac{4}{2}+2\right)-\left(\frac{\left(\frac{1+\sqrt{5}}{2}\right)^2}{2}+\frac{1+\sqrt{5}}{2}\right)$$

$$\int_0^{1/2} x \ln \frac{1+x}{1-x} dx + \underbrace{\int_0^1 \frac{x \operatorname{arctg} x}{\sqrt{1+x^2}} dx}_{} + \underline{\int_0^{\frac{\pi}{2}} \sin^5 x dx}.$$

$$\int x \ln \frac{1+x}{1-x} dx =$$

$$\int x \ln(1+x) - \int x \ln(1-x) dx =$$

$$\int \left(\frac{x^2}{2}\right)' \ln(1+x) dx - \int \left(\frac{x^2}{2}\right)' \ln(1-x) dx$$

$$\left(\frac{x^2}{2} \ln(1+x) - \int \underbrace{\frac{x^2}{2} \cdot \frac{1}{1+x} dx}_{} \right)$$

$$\left(\frac{x^2}{2} \ln(1-x) - \int \frac{x^2}{2} \frac{1}{1-x} dx \right)$$

$$\frac{1}{2} \int \frac{x^2}{1+x} dx = \frac{1}{2} \int \frac{x^2 + x - x - 1 + 1}{1+x} dx$$

$$\frac{1}{2} \int \frac{x(x+1) - (x+1) + 1}{1+x} dx =$$

$$\frac{1}{2}\int \frac{(x-1)(x+1)+1}{1+x}\,dx$$

$$\frac{1}{2}\int x-1+\frac{1}{1+x}\,dx =$$

$$\frac{1}{2}\left(\frac{x^2}{2}-x+\ln|1+x|\right)$$

$$\frac{1}{2}\int\frac{x^2}{1-x}\,dx = \frac{1}{2}\int\frac{x^2-x+x-1+1}{1-x}\,\cancel{dx}$$

$$= \frac{1}{2}\int\frac{x(x-1)+x-1+1}{1-x}\,dx =$$

$$\frac{1}{2}\int\frac{(1-x)(-x-1)+1}{1-x}\,dx =$$

$$\frac{1}{2}\int -x-1+\frac{1}{1-x}\,dx$$

$$\frac{1}{2}\left(-\frac{x^2}{2}-x-\log|1-x|\right)$$

$$\int_0^1 \frac{x \operatorname{arctg} x}{\sqrt{1+x^2}} dx =$$

$$\int_0^1 \operatorname{arctg} x \sqrt{1+x^2}' \, dx =$$

$$\int \frac{x}{\sqrt{1+x^2}} dx \overset{\sqrt{1+x^2}=t}{=\!=\!=}$$

$$\frac{1}{\cancel{2}\sqrt{1+x^2}} \cdot \cancel{2}x \, dx = dt$$

$$\int dt = t = \sqrt{1+x^2}$$

$$\operatorname{arctg} x \cdot \sqrt{1+x^2} \Big|_0^1 - \int_0^1 \frac{1}{1+x^2} \sqrt{1+x^2} \, dx$$

$$\operatorname{arctg} x \sqrt{1+x^2} \Big|_0^1 - \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$$

$$\operatorname{arctg} x \sqrt{1+x^2} \Big|_0^1 - \ln\left(\sqrt{1+x^2}+x\right)\Big|_0^1$$

$$\sin^5 x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin 3x = \sin(x + 2x) =$$

$$\sin x \cos 2x + \sin 2x \cos x =$$

$$\sin x \left(1 - 2\sin^2 x\right) +$$

$$2 \sin x \cos^2 x =$$

$$\sin x - 2\sin^3 x + 2\sin x(1-\sin^2 x)$$

$$\sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x$$

$$= 3\sin x - 4\sin^3 x$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\sin^5 x = \frac{1}{4}\left(3\sin x \sin^2 x - \sin 3x \sin^2 x\right)$$

$$\sin^5 x = \sin x \sin^4 x = \sin x\left(\sin^2 x\right)^2 =$$

$$\sin x\left(1-\cos^2 x\right)^2$$

$$\int \sin^5 x \, dx = \int \sin x\left(1-\cos^2 x\right)^2 dx$$

$$\frac{\underline{\cos x = t}}{-\sin x \, dx = dt} \qquad -\int (1-t^2)^2 dt =$$

$$-\int 1+t^4-2t^2\,dt=$$

$$\int -1-t^4+2t^2\,dt=$$

$$-t-\frac{t^5}{5}+\frac{2t^3}{3}=$$

$$-\cos x-\frac{(\cos x)^5}{5}+\frac{2(\cos x)^3}{3}$$

S11.3 Arătaţi că dacă $f:[0,1]\to\mathbb{R}$ este o funcţie continuă astfel încât

$$\int_0^1 f^2(x)dx \le 3\left(\int_0^1 F(x)dx\right)^2,$$

unde F este o primitivă a lui f, pentru care $F(1)=0$, atunci f este liniară.

$$f=F'$$

$$\int_0^1 f(x)dx = F(1)-F(0) = -F(0)$$

$$\int_0^1 f^2(x)dx = \int_0^1 f(x)\cdot F'(x)dx =$$

$$f(x)\cdot F(x)\Big|_0^1 - \int_0^1 F(x)\cdot f'(x)\,dx$$

$$= f(0)\cdot F(0) - \int_0^1 F(x)\cdot f'(x)\,dx$$

$$\int_0^1 f^2(x)\,dx \le 3\left(\int_0^1 F(x)\,dx\right)^2$$

$$f(0)\cdot F(0) - \int_0^1 F(x)\cdot f'(x)\,dx \le 3\left(\int_0^1 F(x)\,dx\right)^2$$

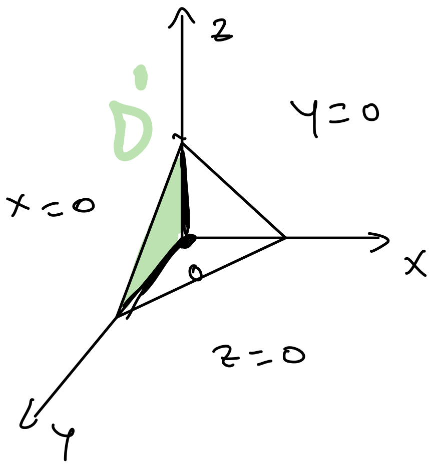
$$\int_0^1 F(x)\,dx = \int_0^1 F(x)\cdot x'\,dx =$$

$$F(x)\cdot x\Big|_0^1 - \int_0^1 f(x)\cdot x\,dx =$$

$$-\int_0^1 f(x)\cdot x\,dx$$

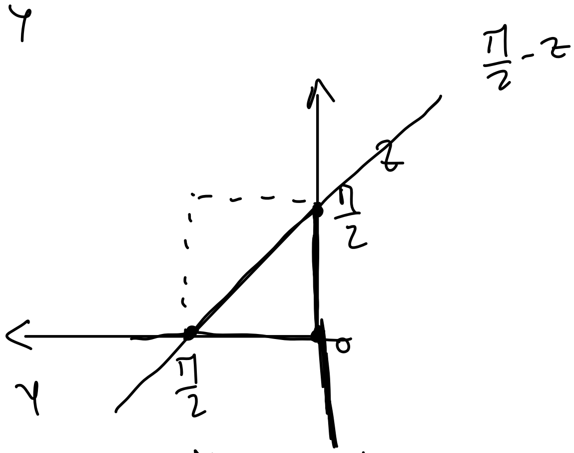
b) $\iiint_D xyz\sin(x+y+z)dxdydz$, unde D este domeniul mărginit de planele $x=0$, $y=0$, $z=0$ and $x+y+z=\frac{\pi}{2}$.

$I =$



$$0 \le x \le \frac{\pi}{2} - (y+z)$$

$$I = \iint_{D'} \int_0^{\frac{\pi}{2}-(y+z)} xyz\sin(x+y+z)\,dxdydz$$



$$0 \le z \le \frac{\pi}{2}$$

$$0 \le y \le \frac{\pi}{2} - z$$

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-z} \int_0^{\frac{\pi}{2}-(y+z)} xyz\sin(x+y+z)\,dxdydz$$

$$\int_0^{\frac{\pi}{2}} z\left(\int_0^{\frac{\pi}{2}-z} y\left(\int_0^{\frac{\pi}{2}-(y+z)} x\sin(x+y+z)\,dx\right)dy\right)dz$$

$$\int x\sin(x+ct)\,dx = -\int x(\cos(x+ct))'\,dx =$$

$\int$ fct. trig. pol

$$-\left[x\cos(x+ct) - \int\cos(x+ct)\,dx\right]$$

$$= -\left[x\cos(x+c) - \sin(x+c)\,dx\right]$$

$$= -x\cos(x+c) + \sin(x+c)\,dy$$

$$\int_0^{\frac{\pi}{2}} z \int_0^{\frac{\pi}{2}-z} y\left(-x\cos(x+y+z) + \sin(x+y+z)\right)\Bigg|_{x=0}^{x=\frac{\pi}{2}-(y+z)} dy\,dz$$

$$\int_0^{\frac{\pi}{2}} z \int_0^{\frac{\pi}{2}-z} y\Bigg(-\left(\frac{\pi}{2}-(y+z)\right)\underbrace{\cos\left(\frac{\pi}{2}-\cancel{(y+z)}+\cancel{y+z}\right)}_{0}$$

$$+\underbrace{\sin\left(\frac{\pi}{2}-\cancel{(y+z)}+\cancel{(y+z)}\right)}_{1}\Bigg) +$$

$$\underbrace{0\cos(y+z)}_{0} - \sin(y+z)\Bigg)\,dy\,dz =$$

$$\int_0^{\frac{\pi}{2}} z \int_0^{\frac{\pi}{2}-z} \left[y - y\sin(y+z)\right]dy\,dz$$

$$\int_0^{\frac{\pi}{2}} z\left(\frac{y^2}{2} - \left[-y\cos(y+z) + \sin(y+z)\right]\right)\Bigg|_{y=0}^{y=\frac{\pi}{2}-z} =$$

$$\int_0^{\frac{\pi}{2}} z\left(\frac{\left(\frac{\pi}{2}-z\right)^2}{2} + \left(\frac{\pi}{2}-z\right)\overbrace{\cos\left(\frac{\pi}{2}-\cancel{z}+\cancel{z}\right)}^{0}\right)$$

$$-\underbrace{\sin\left(\frac{\pi}{2}-\cancel{z}+\cancel{z}\right)}_{1}+\sin z\Big)\,dz$$

$$\int_0^{\frac{\pi}{2}} \frac{z\left(\frac{\pi^2}{4}+z^2-\pi z\right)}{2}-z+z\sin z\,dz$$

$$\int_0^{\frac{\pi}{2}} \frac{\pi^2 z}{8}+\underline{\frac{z^3}{2}}-\underline{\frac{\pi z^2}{2}}-z+z\sin z\,dz=$$

$$\frac{\pi^2 z^2}{16}+\frac{z^4}{8}-\frac{\pi z^3}{6}-\frac{z^2}{2}-z\cos z+$$

$$\sin z\Big|_0^{\frac{\pi}{2}}=$$

$$\frac{\pi^2\cdot\pi^2}{64}+\frac{\pi^4}{16\cdot 8}-\frac{\pi\cdot\pi^3}{8\cdot 6}-\frac{\pi^2}{8}+1$$

2)

$$\frac{\pi^4}{64}+\frac{\pi^4}{128}-\frac{\pi^4}{48}-\frac{\pi^2}{8}+1$$

3)

$$\frac{3\pi^4}{128} - \frac{8}{1}\frac{\pi^4}{48} - \frac{\pi^2}{8} + 1$$

$$\frac{\pi^4}{384} - \frac{\pi^2}{8} + 1$$

S11.8 Calculați aria mărginită de curba $(a_1 x + b_1 y + c_1)^2 + (a_2 x + b_2 y + c_2)^2 = 1$, unde $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ sunt astfel încât $a_1 b_2 - a_2 b_1 \neq 0$.

$$\iint_D dx\,dy \qquad \begin{cases} z = a_1 x + b_1 y + c_1 \\ t = a_2 x + b_2 y + c_2 \end{cases}$$

$$J = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{vmatrix} =$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\iint_{D'} (a_1 b_2 - a_2 b_1)\, dz\,dt$$

$$(a_1b_2-a_2b_1)\iint_{D'} dz\,dt =$$

$z^2+t^2=1$

$z=r\cos\theta$

$t=r\sin\theta$

$J=r$ $\quad r\in[0,1]$

$\theta\in[0,2\pi]$

$$(a_1b_2-a_2b_1)\int_0^1\int_0^{2\pi} r\,dr\,d\theta=$$

$$(a_1b_2-a_2b_1)\int_0^1 r\,dr\cdot\int_0^{2\pi} d\theta=$$

$$(a_1b_2-a_2b_1)\left.\frac{r^2}{2}\right|_0^1\cdot\left.\theta\right|_0^{2\pi}=$$

$$(a_1b_2-a_2b_1)\cdot\frac{1}{\cancel{2}}\cdot\cancel{2}\pi=$$

$$\pi(a_1b_2-a_2b_1)$$

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^4}$$

$$x_n = \frac{1}{n} \qquad y_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{x_n y_n^3}{x_n^2 + y_n^4} = \lim_{n\to\infty} \frac{\frac{1}{n}\cdot\frac{1}{n^3}}{\frac{1}{n^2}+\frac{1}{n^4}} =$$

$$\lim_{n\to\infty} \frac{\frac{1}{\cancel{n^4}}}{\frac{n^2+1}{\cancel{n^4}}} =$$

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

$$x_n = 0 \qquad y_n = \frac{1}{n}$$

$(u,v)\neq(0,0)$

$$l = \lim_{t\to 0} \frac{tu(tv)^3}{(tu)^2+(tv)^4} =$$

$$\lim_{t\to 0} \frac{t^4 u v^3}{t^2u^2+t^4v^4} =$$

$$\lim_{t\to 0} \frac{t^2 u v^3}{u^2+t^2v^4}$$

Dacă $u \neq 0$ $\quad l = \lim_{t\to 0} \dfrac{t^2uv^3}{\underbrace{u^2+t^2v^4}_{\neq 0}} = 0$

$$u=0 \Rightarrow v\neq 0$$

$$l = \lim_{t\to 0} \frac{0}{t^4v^4} = \underline{0}$$

$$\lim_{(x,y)\to(0,0)} |y| \frac{|xy^2|}{x^2+y^4} \leq$$

$$\sqrt{|x|y^2} \leq \sqrt{\frac{x^2+y^4}{2}}$$

$$|x|y^2 \leq \frac{x^2+y^4}{2}$$

$$\lim_{(x,y)\to(0,0)} |y| \cdot \frac{1}{2} = 0$$

S11.2 Fie funcţia $f : \mathbb{R} \to \mathbb{R}$, definită prin

$$f(x) = \frac{x^2 - 1}{x^4 + x^3 + 3x^2 + x + 1}, \ x \in \mathbb{R}.$$

Găsiţi o primitivă a funcţiei $f|_{\mathbb{R}_+^*}$, prin substituţia $t = x + \frac{1}{x}$. Determinaţi atunci o primitivă a lui f pe $\mathbb{R}$.

$$t^2 = \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \cdot \cancel{x} \cdot \frac{1}{\cancel{x}}$$

$$\int \frac{x^2 - 1}{x^4 + x^3 + 3x^2 + x + 1} \, dx =$$

$$t = x + \frac{1}{x}$$

$$dt = \left(1 - \frac{1}{x^2}\right) dx$$

$$\int \frac{1 - \frac{1}{x^2}}{\underbrace{x^2 + x + 3 + \frac{1}{x} + \frac{1}{x^2}}_{x + \frac{1}{x} = t, \ x^2 + \frac{1}{x^2} = t^2 - 2}} \, dx =$$

$$\int \frac{dt}{t^2 + t - 2 + 3} = \int \frac{dt}{t^2 + t + 1} =$$

$$\int \frac{dt}{t^2 + t + \frac{1}{4} + \frac{3}{4}} = \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$\frac{2}{\sqrt{3}} \operatorname{arctg} \frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t+1}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2\left(x+\frac{1}{x}\right)+1}{\sqrt{3}}$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{3xy - z^2 \sin y + 2x^3 z}{\sqrt{x^2+5y^6+4z^2}}$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{z^2 \sin y}{\sqrt{x^2+5y^6+4z^2}} =$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{z^2 \overbrace{\frac{\sin y}{y}}^{\to 1} y}{\sqrt{x^2+5y^6+4z^2}} =$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{z^2 y}{\sqrt{x^2+5y^6+4z^2}}$$

$$\frac{|xy|}{\sqrt{x^2+5y^6+4z^2}} \le \frac{|xy|}{\sqrt{x^2}} =$$

$$\frac{|x|}{|x|} \cdot |y| = |y| \to 0$$

$$\lim_{(x,y,z)\to(0,0,0)} \left| \frac{3xy - z^2 \sin y + 2x^3 z}{\sqrt{x^2+5\underline{y^6}+4z^2}} \right| \le$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{|3xy|}{\sqrt{}} +$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{|z^2 \sin y|}{\sqrt{}} +$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{2x^3 z}{\sqrt{x^2+5y^6+4z^2}}$$

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