

1. $q \rightarrow \pi, \neg q \rightarrow \neg p, p \vdash p$ (ip)
 2. $q \rightarrow \pi, \neg q \rightarrow \neg p, p \vdash \neg p$ ($\neg i, 1$)
 3. $q \rightarrow \pi, \neg q \rightarrow \neg p, p \vdash \neg q \rightarrow \neg p$ (ip)
 4. $q \rightarrow \pi, \neg q \rightarrow \neg p, p \vdash \neg q$ ($\neg e, 3, 2$)
 5. $q \rightarrow \pi, \neg q \rightarrow \neg p \vdash \neg q$ ($\neg e, 4, 2$)
 6. $q \rightarrow \pi \vdash ((\neg q \rightarrow \neg p) \rightarrow (\neg p \rightarrow q))$ ($\rightarrow i, 5$)
 7. φ ($\rightarrow i, 6$)
 8. $\vdash ((q \rightarrow \pi) \rightarrow ((\neg q \rightarrow \neg p) \rightarrow (\neg p \rightarrow q)))$ (\neg)
- $$\frac{\Gamma \vdash \varphi_1 \rightarrow \varphi_2 \quad \Gamma \vdash \neg \varphi_2}{\Gamma \vdash \neg \varphi_1} \text{PBC} \quad \frac{\Gamma, \varphi_1 \vdash \varphi_2 \quad !}{\Gamma \vdash \varphi_1 \rightarrow \varphi_2} \rightarrow i$$

1. $\Gamma \vdash \varphi_1 \rightarrow \varphi_2$ (ip)
2. $\Gamma \vdash \neg \varphi_2$ (ip)
3. $\Gamma, \varphi_1 \vdash \neg \varphi_2$ (ext. 2)
4. $\Gamma, \varphi_1 \vdash \varphi_1$ (ip)
5. $\Gamma, \varphi_1 \vdash \varphi_1 \rightarrow \varphi_2$ (ext. 1)
6. $\Gamma \vdash \neg \neg \varphi_2$ ($\rightarrow e, 5, 4$)
7. $\Gamma, \varphi_1 \vdash \perp$ ($\neg e, 6, 3$)
8. \neg ($\neg i, 7$)

$$\frac{\Gamma' \vdash \varphi_1 \rightarrow \varphi_2 \quad \Gamma' \vdash \neg \varphi_2}{\Gamma' \vdash \varphi_2} \rightarrow e$$

$$\neg i \frac{\Gamma \vdash \varphi}{\Gamma \vdash \neg \neg \varphi}$$

1. $\Gamma \vdash \varphi$
2. $\Gamma, \neg \varphi \vdash \varphi$ (ext. 1)
3. $\Gamma, \neg \varphi \vdash \neg \varphi$ (ip)
4. $\Gamma \vdash \neg \neg \varphi$ ($\neg e, 2, 3$)
5. \neg ($\neg i, 4$)

$$\text{LEM} \frac{}{\Gamma \vdash \varphi \vee \neg \varphi}$$

1. $\Gamma, \neg(\varphi \vee \neg \varphi), \neg \varphi \vdash \neg \varphi$ (ip)
2. $\Gamma, \neg(\varphi \vee \neg \varphi), \neg \varphi \vdash \varphi \vee \neg \varphi$ ($\vee i_2, 1$)
3. $\Gamma, \neg(\varphi \vee \neg \varphi), \neg \varphi \vdash \neg(\varphi \vee \neg \varphi)$ (ip)
4. $\Gamma, \neg(\varphi \vee \neg \varphi), \neg \varphi \vdash \perp$ ($\neg e, 2, 3$)
5. $\Gamma, \neg(\varphi \vee \neg \varphi) \vdash \neg \neg \varphi$ ($\neg i, 4$)
6. $\Gamma, \neg(\varphi \vee \neg \varphi) \vdash \varphi$ ($\neg e, 5$) (PBC, 4)
7. $\Gamma, \neg(\varphi \vee \neg \varphi) \vdash \varphi \vee \neg \varphi$ ($\vee i_1, 6$)
8. $\Gamma, \neg(\varphi \vee \neg \varphi) \vdash \neg(\varphi \vee \neg \varphi)$ (ip)
9. $\Gamma, \neg(\varphi \vee \neg \varphi) \vdash \perp$ ($\neg e, 7, 8$)
10. $\Gamma \vdash \varphi \vee \neg \varphi$ (PBC, 9)

$$1. \Gamma \vdash p \vee \neg p \quad (\text{LEM})$$

Ex 1

(9)

$$1. \Gamma \vdash (p \wedge (q \vee \pi)) \quad (\text{ip})$$

$$2. \Gamma \vdash p \quad (\wedge e_1, 1)$$

$$3. \Gamma \vdash (q \vee \pi) \quad (\wedge e_2, 1)$$

$$4. \left[\begin{array}{l} \Gamma, q \vdash q \quad (\text{ip}) \\ \Gamma, q \vdash p \quad (\text{ext. 2}) \\ \Gamma, q \vdash (p \wedge q) \quad (\wedge i, 5, 4) \\ \Gamma, q \vdash ((p \wedge q) \vee (p \wedge \pi)) \quad (\vee i_1, 6) \end{array} \right.$$

$$5. \left[\begin{array}{l} \Gamma, \pi \vdash \pi \quad (\text{ip}) \\ \Gamma, \pi \vdash p \quad (\text{ext. 2}) \\ \Gamma, \pi \vdash (p \wedge \pi) \quad (\wedge i, 9, 8) \end{array} \right.$$

$$6. \left[\begin{array}{l} \Gamma, \pi \vdash ((p \wedge q) \vee (p \wedge \pi)) \quad (\vee i_2, 10) \end{array} \right.$$

$$7. \Gamma \vdash ((p \wedge (q \vee \pi)) \vdash ((p \wedge q) \vee (p \wedge \pi))) \quad (\vee e, 3, 7, 11)$$

$$\frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \quad \Gamma, \varphi_1 \vdash \varphi \quad \Gamma, \varphi_2 \vdash \varphi}{\Gamma \vdash \varphi} \rightarrow e$$

$$\frac{\Gamma \vdash \varphi_1 \quad \Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \wedge \varphi_2)} \wedge i$$

$$\varphi_1 \dashv\vdash \varphi_2$$

$$\varphi_1 \vdash \varphi_2 \quad \text{et} \quad \varphi_2 \vdash \varphi_1 \quad \text{valide.}$$