Logic for Computer Science - Week 2
The Syntax of Propositional Logic
Tutorial Exercises

October 21, 2018

1. Recall the definition of the PL set from the lecture notes.

2. Consider the set $A = \{\mathtt{p}, \mathtt{q}, \mathtt{r}, \mathtt{p}_1, \mathtt{p}', \ldots\}$ of propositional variables. Show that the following words are *propositional formulae* (i.e., element of PL), by explaining which are the construction steps (base case, respectively one of the three inductive steps):

   (a) $\neg\mathtt{q}$; (b) $(\mathtt{p}_1 \wedge \mathtt{q})$; (c) $\neg(\mathtt{p} \vee \mathtt{q})$; (d) $(\neg\mathtt{p} \vee \neg\mathtt{q})$; (e) $\neg(\neg\mathtt{p} \vee (\mathtt{q} \wedge \neg\mathtt{q}))$.

3. Show that the following words over the alphabet L are not elements of PL (hint: show that none of the four construction rules applies):

   (a) $((\neg)\mathtt{q})$; (b) $\mathtt{q} \wedge \neg$; (c) $\mathtt{pq}$; (d) $\mathtt{p} \wedge \mathtt{q}$; (e) $(\mathtt{p}\neg\mathtt{q})$; (f) $(\mathtt{p}) \wedge (\mathtt{q})$.

4. Which of the following are formulae (in PL) and which are not?

   (a) $\mathtt{p}_1$; (b) $\mathtt{p}_1 \vee \mathtt{q}_1$; (c) $(\mathtt{p}_1 \vee \mathtt{q}_1)$; (d) $(\neg\mathtt{p}_1 \vee \mathtt{q}_1)$; (e) $((\neg\mathtt{p}_1) \vee \mathtt{q}_1)$; (f) $(\neg\mathtt{p})$?

5. Recall the recursive definition of the function $\mathit{subf} : PL \to 2^{PL}$, which computes all subformulae of a formula.

6. Compute, using the function above, the set of subformulae of the following formulae:

   (a) $((\mathtt{p} \wedge \neg\mathtt{q}) \wedge \mathtt{r})$; (b) $((\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$; (c) $\neg((\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$.

7. Recall the recursive definition of the function $\mathit{ast} : PL \to \mathit{Trees}$, which computes the abstract syntax tree of a formula.

8. Compute the abstract syntax trees of the following formulae:

   (a) $((\mathtt{p} \wedge \neg\mathtt{q}) \wedge \mathtt{r})$; (b) $((\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$; (c) $\neg((\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$; (d) $(\neg(\mathtt{p} \vee \neg\mathtt{q}) \wedge \mathtt{r})$.

9. Recall the recursive definition of the function $\mathit{height} : PL \to \mathbb{N}$, which computes, given a formula, the height of its abstract syntax tree. Compute the height of the formulae above.

10. Recall the recursive definition of the function $\mathit{size} : PL \to \mathbb{N}$, which computes, given a formula, the number of nodes of its abstract syntax tree. Compute the size of the formulae above.

11. Recall the recursive definition of the function $\mathit{prop} : PL \to 2^A$, which computes, given a formula, the set of propositional variables occuring in the formula. Compute the set of propositional variables occuring in the formulae above.

12. Show by structural induction that $\left|\mathit{prop}\Big(\varphi\Big)\right| \leq \mathit{size}\Big(\varphi\Big)$ for any formula $\varphi \in PL$.