Subiectul 3. (25 p.) Fie o piramidă regulată patrulateră de volum $\frac{4}{3}$. Găsiţi latura bazei $\ell > 0$ şi înălţimea $h > 0$ ale acesteia astfel încât muchia laterală a acesteia să aibă lungimea minimă (reamintim că volumul piramidei regulate este $\frac{\ell^2 h}{3}$, iar lungimea unei muchii laterale este $\sqrt{\frac{\ell^2}{2} + h^2}$).

Problemă de optimizare cu restricții: min f cu restricția g

$$V = \frac{4}{3} = \frac{l^2 h}{3} \qquad l, h > 0 \qquad \frac{\text{cu restricția } g}{g = 0}$$

$$\min \sqrt{\frac{l^2}{2} + h^2}$$

1. Problema

$$\begin{cases} f(l, h) = \sqrt{\frac{l^2}{2} + h^2} \text{ sau } \Big| \; \underline{f(l, h) = \frac{l^2}{2} + h^2} \\ g(l, h) = \frac{l^2 h}{3} - \frac{4}{3} \end{cases}$$

$$L(l, h; \lambda) = f(l, h) + \lambda g(l, h) = \frac{l^2}{2} + h^2 + \lambda \left(\frac{l^2 h}{3} - \frac{4}{3} \right)$$

2. Gradient

$$\begin{cases} \frac{\partial L}{\partial l}(l, h; \lambda) = \frac{\cancel{2} l}{\cancel{2}} + \lambda \cdot \frac{2 l h}{3} = l + \frac{2 l \lambda h}{3} \\ \frac{\partial L}{\partial h}(l, h; \lambda) = 2h + \lambda \cdot \frac{l^2}{3} \\ \frac{\partial L}{\partial \lambda}(l, h; \lambda) = \frac{l^2 h}{3} - \frac{4}{3} \end{cases}$$

3. Pct critice

$$\begin{cases} l + \lambda \cdot \frac{2lh}{3} = 0 \\ 2h + \lambda \cdot \frac{l^2}{3} = 0 \\ \frac{l^2 h}{3} - \frac{4}{3} = 0 \end{cases}$$

$\Rightarrow l\left(1 + \lambda \frac{2h}{3}\right) = 0$

$1 + \frac{2\lambda h}{3} = 0$

$\Rightarrow -3 = 2\lambda h \Rightarrow$

$h = -\frac{3}{2\lambda}$

$\lambda \neq 0$

$\cancel{2} \cdot \left(-\frac{3}{\cancel{2}\lambda}\right) + \lambda \cdot \frac{l^2}{3} = 0$

$-9 + \lambda^2 l^2 = 0$

$\Rightarrow l^2 = \frac{9}{\lambda^2}$

$$\frac{\frac{9}{\lambda^2} \cdot -\frac{\cancel{3}}{2\lambda}}{\cancel{3}} - \frac{4}{3} = 0$$

$-\frac{9}{2\lambda^3} = \frac{4}{3}$

$$\frac{1}{\lambda^3} = -\frac{8}{27}$$

$$\lambda^3 = -\frac{27}{8}$$

$$\lambda = -\frac{3}{2}$$

$h = -\frac{3}{2\lambda}$ $\qquad h = \frac{-3}{\cancel{2} \cdot -\frac{3}{\cancel{2}}} = 1$

$$l^2 = \frac{9}{\left(-\frac{3}{2}\right)^2} = \frac{9}{\frac{9}{4}} = 4$$

$l > 0$

$\Rightarrow l = 2$

Pct critic $\left(2, 1, -\frac{3}{2}\right)$

4. Hessiana

1 polinom n. deriv mixte egale

$$
\begin{cases}
\dfrac{\partial L}{\partial l}(l,h;\lambda) = \dfrac{\cancel{2}l}{\cancel{2}} + \lambda \cdot \dfrac{2lh}{3} = l + \dfrac{2l\lambda h}{3} \\[2ex]
\dfrac{\partial L}{\partial h}(l,h;\lambda) = 2h + \lambda \cdot \dfrac{l^2}{3} \\[2ex]
\dfrac{\partial L}{\partial \lambda}(l,h;\lambda) = \dfrac{l^2 h}{3} - \dfrac{4}{3}
\end{cases}
$$

$$\frac{\partial^2 L}{\partial l^2}(l,h;\lambda) = 1 + \frac{2}{3}\lambda h$$

$$\frac{\partial^2 L}{\partial h^2}(l,h;\lambda) = 2$$

$$\frac{\partial^2 L}{\partial h \partial l}(l,h;\lambda) = \frac{2}{3}\lambda l$$

$$HL\left(l,h;-\frac{3}{2}\right) = \begin{pmatrix} 1 - \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} h & \frac{2}{3} \cdot \left(-\frac{3}{2}\right) l \\[2ex] \frac{2}{3} \cdot \left(-\frac{3}{2} l\right) & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-h & -l \\ -l & 2 \end{pmatrix}$$

$$H_L\left(2, 1; -\frac{3}{2}\right) = \begin{pmatrix} 0 & -2 \\ -2 & 2 \end{pmatrix}$$

$$d^2L = -4\,dh\,dl + 2\,dh^2$$

$$f(h, l; L)$$

$$H_L\left(1, 2; -\frac{3}{2}\right) = \begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}$$

$\Delta_1 = 2$ $\qquad$ $L\left(h, l; -\frac{3}{2}\right)$

$\Delta_2 = -4$

$$\frac{l^2 h}{3} - \frac{4}{3} = 0$$

$$\frac{2}{3} l h\, dl + \frac{l^2}{3}\, dh = 0$$

În pct critic

$\frac{4}{3} dl + \frac{4}{3} dh = 0$

$dl + dh = 0$

$d^2L = -4\,dh\,dl + 2\,dh^2$

$-4\,dh \cdot (-dh) + 2\,dh^2 =$

$4\,dh^2 + 2dh^2 = 6\,dh^2$

positiv definit

$\Rightarrow (2, 1)$ pct minim

$\sqrt{\frac{l^2}{2} + h^2} \Rightarrow \sqrt{\frac{4}{2} + 1} = \sqrt{2}$

1. Problema

$$f(x,y) = \sin x + \cos y + \cos(x-y)$$

2. Deriv partiale $x, y \in [0, \frac{\pi}{2}]$

$$\frac{\partial f}{\partial x}(x,y) = \cos x - \sin(x-y)$$

$$\frac{\partial f}{\partial y}(x,y) = -\sin y + \sin(x-y)$$

$$\sin x = \sin\left(\frac{\pi}{2} - y\right) = \sin\frac{\pi}{2}\cos y - \cos\frac{\pi}{2}\sin y$$

$$\sin x = \cos y$$

3. Pct critice

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$$

$$\begin{cases} \cos x - \sin(x-y) = 0 \\ -\sin y + \sin(x-y) = 0 \end{cases} \quad (+)$$

$$\cos x - \sin y = 0, \quad \cos x = \sin y,$$

$$\Rightarrow x + y = \frac{\pi}{2}$$

$$\cos x - (\sin x \cos y - \sin y \cos x) = 0$$

$$\cos x - (\sin^2 x - \cos^2 x) = 0$$

$$\cos x - (1 - 2\cos^2 x) = 0$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$2t^2 + t - 1 = 0$$

$$t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{4} =$$

$$\frac{-1 \pm 3}{4} \begin{cases} \dfrac{-1+3}{4} = \dfrac{1}{2} \\[2ex] \dfrac{-1-3}{4} = -1 \end{cases}$$

$\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3} \qquad y = \frac{\pi}{6}$

$\cos x = -1 \Rightarrow x \notin [0, \frac{\pi}{2}]$

4.

$\frac{\partial f}{\partial x}(x, y) = \cos x - \sin(x-y)$

$\frac{\partial f}{\partial y}(x, y) = -\sin y + \sin(x-y)$

$\frac{\partial^2 f}{\partial x^2}(x, y) = -\sin x - \cos(x-y)$

$\frac{\partial^2 f}{\partial y^2}(x, -y) = -\cos y - \cos(x-y)$

$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \cos(x-y)$

$$Hf(x,y) = \begin{pmatrix} -\operatorname{sen} x - \cos(x-y) & \cos(x-y) \\ \cos(x-y) & -\cos y - \cos(x-y) \end{pmatrix}$$

$$Hf\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \begin{pmatrix} -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \end{pmatrix} =$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$Hf\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \begin{pmatrix} -\sqrt{3} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\sqrt{3} \end{pmatrix}$$

$$\Delta_1 = -\sqrt{3} < 0$$

$$\Delta_2 = 3 - \frac{3}{4} = \frac{9}{4} > 0$$

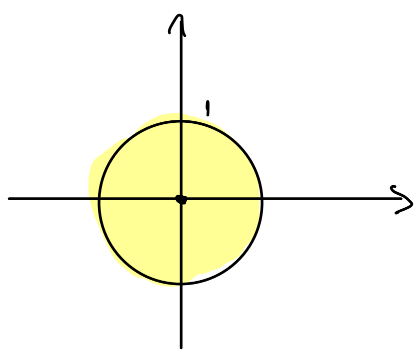
$$\Rightarrow \left(\frac{\pi}{3}, \frac{\pi}{6}\right) \text{ pt maxim}$$

a) Calculaţi

$$I = \iint_D \frac{1+x}{1+x^2+y^2}dxdy,$$

unde D este domeniul mărginit de curba $x^2+y^2=1$. **(15 p.)**

coord polare



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$r \in [0,1]$

$\theta \in [0, 2\pi]$

$$J = r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta)$$

$$I = \int_0^1\int_0^{2\pi} \frac{1+r\cos\theta}{1+r^2\cos^2\theta + r^2\sin^2\theta} \cdot r\,d\theta\,dr$$

$$= \int_0^1\int_0^{2\pi} \frac{(1+r\cos\theta)}{1+r^2} r\,d\theta\,dr$$

$$= \int_0^1 \int_0^{2\pi} \frac{r + r^2 \cos\theta}{1+r^2} \, d\theta \, dr =$$

$$\int_0^1 \int_0^{2\pi} \left(\frac{r}{1+r^2} + \frac{r^2 \cos\theta}{1+r^2} \right) d\theta \, dr =$$

$$\int_0^1 \int_0^{2\pi} \frac{r}{1+r^2} \, d\theta \, dr + \int_0^1 \int_0^{2\pi} \frac{r^2 \cos\theta}{1+r^2} \, d\theta \, dr$$

$$\int_0^1 \frac{r}{1+r^2} dr \int_0^{2\pi} d\theta + \int_0^1 \frac{r^2}{1+r^2} dr \int_0^{2\pi} \cos\theta \, d\theta$$

$$\int_0^1 \frac{\frac{1}{2} d(1+r^2)}{1+r^2} \int_0^{2\pi} d\theta + \int_0^1 \left(\frac{1+r^2}{1+r^2} - \frac{1}{1+r^2} \right) \int_0^{2\pi} \cos\theta \, d\theta$$

$1+r^2 = t$

$2r\,dr = dt \Rightarrow r\,dr = \frac{dt}{2}$

$r = 0 \Rightarrow t = 1$

$r = 1 \Rightarrow t = 2$

$$= \int_1^2 \frac{\frac{1}{2} dt}{t} \int_0^{2\pi} d\theta + \int_0^1 \left(1 - \frac{1}{1+r^2} \right) dr \int_0^{2\pi} \cos\theta \, d\theta$$

$$\frac{1}{2} \ln t \Big|_1^2 \cdot \theta \Big|_0^{2\pi} + \left(r - \operatorname{arctg} r \right) \Big|_0^1 \sin\theta \Big|_0^{2\pi} =$$

$$\frac{1}{2}\left(\ln 2 - \underbrace{\ln 1}_{0}\right)\cdot\left(2\pi - 0\right) +$$

$$\left[\left(1 - \underbrace{\operatorname{arctg} 1}_{\frac{\pi}{4}}\right) - \left(0 - \underbrace{\operatorname{arctg} 0}_{0}\right)\right]\cdot$$

$$\left(\underbrace{\sin 2\pi}_{0} - \underbrace{\sin 0}_{0}\right) =$$

$$\frac{1}{\cancel{2}}\ln 2\cdot\cancel{2}\pi = \pi\ln 2$$

Fie $f : \mathbb{R}^2 \to \mathbb{R}$ definită prin

$$f(x,y) := \begin{cases} \dfrac{xy^2}{x^4+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

a) Calculați derivata direcțională în $(0,0)$ a funcției f în direcția $(-2,1)$

b) Arătați că f este continuă în $(0,0)$

c) Integrați funcția f pe domeniul

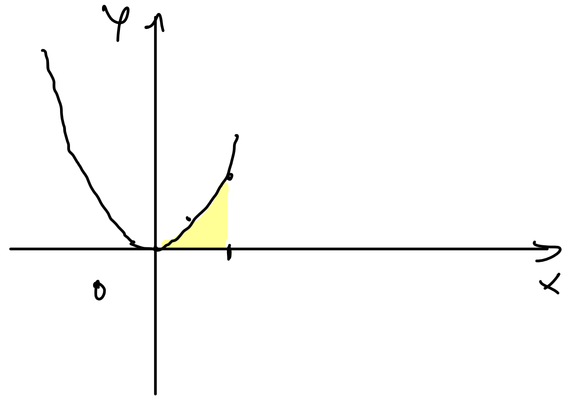
$$D := \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1],\ 0 \leq y \leq x^2\}$$

$$\lim_{t\to 0} \frac{f((0,0)+t(-2,1)) - f(0,0)}{t}$$

$$\lim_{t\to 0} \frac{f(-2t,t)}{t} =$$

$$\int_0^1 \int_0^{x^2} \frac{xy^2}{x^4+y^2}\, dy\, dx$$

$$\lim_{t\to 0} \frac{\frac{-2t\cdot t^2}{(-2t)^4+t^2}}{t} =$$



$$\lim_{t\to 0} \frac{-2t^3}{16t^5+t^3}$$

$$\lim_{t\to 0} \frac{-2}{16t^2+1} = -2$$

$$f(x,y) = \begin{cases} \sqrt{\dfrac{xy^2}{x^2+y^2}} & (x,y) \neq (0,0) \quad x \geq 0 \\ & \qquad t > 0 \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\lim_{\substack{t\to 0 \\ t>0}} \frac{f(3t,4t)}{t} = \lim_{t\to 0} \frac{\sqrt{\dfrac{3t\cdot 16t^2}{9t^2+16t^2}}}{t} =$$

$$\lim_{t \to 0} \frac{\frac{4\cancel{t}\sqrt{3t}}{5\cancel{t}}}{t} = \lim_{t\to 0} \frac{4\sqrt{3}}{5\sqrt{t}} = \infty$$

$$\frac{xy^2}{x^4+y^2}$$

$$0 \le \frac{|xy^2|}{x^4+y^2} \le \frac{|x|y^2}{y^2} \le |x| \longrightarrow 0$$

$$\Rightarrow \left|\frac{xy^2}{x^4+y^2}\right| \longrightarrow 0 \Rightarrow$$

$$\frac{xy^2}{x^4+y^2} \longrightarrow 0$$

Subiectul 2. **(45 p.)** Fie $f : \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\} \to \mathbb{R}$ definită prin

$$f(x,y) := \begin{cases} \sqrt{\dfrac{xy^2}{x^2+y^2}}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

a) Calculaţi derivata direcţională în $(0,0)$ a funcţiei f în direcţia $(3,4)$ **(15 p.)**;

b) Arătaţi că f este continuă în $(0,0)$ **(15 p.)**;

c) Integraţi funcţia f pe domeniul (sfert de disc)

$$D := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1,\ x \geq 0,\ y \geq 0\}$$ **(15 p.)**.

Subiectul 3. **(25 p.)** Fie un cilindru de volum 16π. Găsiţi raza $r > 0$ şi înălţimea $h > 0$ a cilindrul astfel încât aria sa totală să fie minimă (reamintim că volumul cilindrului este $\pi r^2 h$, aria bazei es πr^2, iar aria sa laterală este $2\pi rh$).

Puncte din oficiu: 10 p.

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xy

$\sqrt{|xy|} \le \sqrt{\frac{x^2+y^2}{2}}$

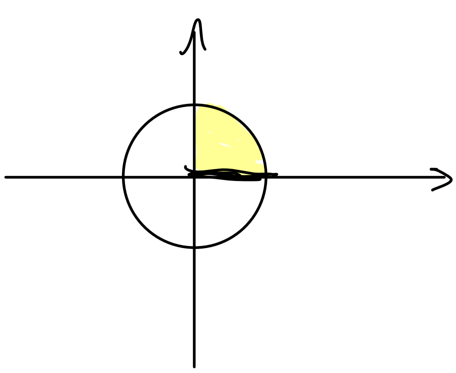
$0 \le \sqrt{\frac{xy^2}{x^2+y^2}} \le \sqrt{\frac{xy^2}{y^2}} =$

$\sqrt{\frac{|xy|}{x^2+y^2}} \le \frac{1}{2}$

$\sqrt{|y|}$

$\sqrt{x} \to 0$

$0 \le \sqrt{|y|}\sqrt{\frac{|xy|}{x^2+y^2}} \le \frac{1}{2}\sqrt{|y|} \longrightarrow 0$



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$J = r$

$r \in [0,1] \qquad \theta \in [0, \frac{\pi}{2}]$

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{\frac{r\cos\theta\, r^2 \sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}}\, r\, dr\, d\theta$$

$$\int_0^{\frac{\pi}{2}} \int_0^1 \frac{\cancel{r} \sin\theta \sqrt{r\cos\theta}}{\cancel{r}} \cdot r\, dr\, d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin\theta \sqrt{\cos\theta}\, d\theta \int_0^1 r\sqrt{r}\, dr$$

$\cos\theta = t$

$-\sin\theta\, d\theta = dt$

$\theta = 0 \to t = 0$

$\theta = \frac{\pi}{2} \to t = 0$

$$= \int_1^0 \sqrt{t}\, dt \cdot \int_0^1 r^{\frac{3}{2}} dr$$

$$\int_0^1 t^{\frac{1}{2}} dt \cdot \int_0^1 r^{\frac{3}{2}} dr$$

$$\left.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 \cdot \left.\frac{r^{\frac{5}{2}}}{\frac{5}{2}}\right|_0^1 = \frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}$$