Logic(s) for computer science - Week 11
Natural deduction
Tutorial

1 Natural Deduction Rules

$$\wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,} \qquad \wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi',}$$

$$\rightarrow e\ \frac{\Gamma \vdash (\varphi \rightarrow \varphi') \qquad \Gamma \vdash \varphi}{\Gamma \vdash \varphi',} \qquad \rightarrow i\ \frac{\Gamma, \varphi \vdash \varphi'}{\Gamma \vdash (\varphi \rightarrow \varphi'),} \qquad \vee i_1\ \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2),}$$

$$\vee i_2\ \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2),} \qquad \vee e\ \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \qquad \Gamma, \varphi_1 \vdash \varphi' \qquad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi',}$$

$$\neg e\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \bot,} \qquad \neg i\ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi,} \qquad \bot e\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi,}$$

$$\textsc{Ipoteză}\ \frac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma, \qquad \textsc{Extindere}\ \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi,} \qquad \neg\neg e\ \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi.} \qquad \forall e\ \frac{\Gamma \vdash \forall x.\varphi}{\Gamma \vdash \varphi[x \mapsto t]}$$

$$\exists e\ \frac{\Gamma \vdash \exists x.\varphi \qquad \Gamma \cup \{\varphi[x \mapsto x_0]\} \vdash \psi}{\Gamma \vdash \psi}\ x_0 \notin vars(\Gamma, \varphi) \qquad \forall i\ \frac{\Gamma \vdash \varphi[x \mapsto x_0]}{\Gamma \vdash \forall x.\varphi}\ x_0 \notin vars(\Gamma, \varphi)$$

$$\exists i\ \frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash \exists x.\varphi}$$

2 Solved exercises

1. Prove that the sequence $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(b)}$ is valid.

Proof:

1. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(a)}$; (Hypothesis)
2. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \neg\mathtt{P(a)}$; (Hypothesis)
3. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \bot$; $(\neg e,\ 1,\ 2)$
4. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(b)}$. $(\bot e,\ 3)$

2. Prove that the sequence $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{q} \vee \mathtt{p})$ is valid.

Proof:

1. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{P(a)}\} \vdash \mathtt{P(a)}$; (HYPOTHESIS)
2. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{P(a)}\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$; $(\vee i_2,\ 1)$
3. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{Q(a)}\} \vdash \mathtt{Q(a)}$; (HYPOTHESIS)
4. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{Q(a)}\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$; $(\vee i_1,\ 1)$
5. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \vee \mathtt{Q(a)})$; (HYPOTHESIS)
6. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$. $(\vee e,\ 5,\ 2,\ 4)$

3. Prove that the sequence $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ is valid.

Proof:

1. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)})$ (HYPOTHESIS)
2. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \mathtt{P(a)}$ (HYPOTHESIS)
3. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash (\mathtt{P(a)} \to \mathtt{Q(a)})$ $(\forall e, 1, \mathtt{a})$
4. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \mathtt{Q(a)}$ $(\to e, 3, 2)$
5. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ $(\exists i, 4)$

4. Prove that the sequence $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ is valid.

Proof:

1. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}\} \vdash \exists \mathtt{x}.\mathtt{P(x)}$ (HYPOTHESIS)
2. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash \mathtt{P(x_0)}$ (HYPOTHESIS)
3. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash \forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)})$ (HYPOTHESIS)
4. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash (\mathtt{P(x_0)} \to \mathtt{Q(x_0)})$ $(\forall e, 3, \mathtt{x_0})$
5. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash \mathtt{Q(x_0)}$ $(\to e, 4, 2)$
6. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ $(\exists i, 5)$
7. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ $(\exists e, 1, 6)$

5. Prove that the sequence $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(x)}\} \vdash \forall \mathtt{x}.\mathtt{Q(x)}$ is valid.

Proof:

1. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \forall \mathtt{x}.\mathtt{P(x)}\} \vdash \forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)})$ (HYPOTHESIS)
2. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \forall \mathtt{x}.\mathtt{P(x)}\} \vdash \forall \mathtt{x}.\mathtt{P(x)}$ (HYPOTHESIS)
3. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \forall \mathtt{x}.\mathtt{P(x)}\} \vdash (\mathtt{P(x_0)} \to \mathtt{Q(x_0)})$ $(\forall e, 1, \mathtt{x_0})$
4. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \forall \mathtt{x}.\mathtt{P(x)}\} \vdash \mathtt{P(x_0)}$ $(\forall e, 2, \mathtt{x_0})$
5. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \forall \mathtt{x}.\mathtt{P(x)}\} \vdash \mathtt{Q(x_0)}$ $(\to e, 3, 4)$
6. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \forall \mathtt{x}.\mathtt{P(x)}\} \vdash \forall \mathtt{x}.\mathtt{Q(x)}$ $(\forall i, 5)$

3 Proposed exercises

Are the following sequences valid?

1. $\{((\mathtt{P(a)} \land \mathtt{Q(a)}) \land \forall \mathtt{x.P(x)})\} \vdash (\mathtt{Q(a)} \land \forall \mathtt{x.P(x)});$
2. $\{((\mathtt{P(a)} \land \mathtt{Q(a)}) \land \forall \mathtt{x.P(x)}), \forall \mathtt{x.Q(x)}\} \vdash (\forall \mathtt{x.Q(x)} \land \mathtt{Q(a)});$
3. $\{((\mathtt{P(a)} \land \mathtt{Q(a)}) \land \forall \mathtt{x.P(x)})\} \vdash (\forall \mathtt{x.P(x)} \land (\mathtt{Q(a)} \land \mathtt{P(a)}));$
4. $\{((\mathtt{P(a)} \land \mathtt{Q(a)}) \rightarrow \forall \mathtt{x.P(x)}), \mathtt{P(a)}, \mathtt{Q(a)}\} \vdash \forall \mathtt{x.P(x)};$
5. $\{(\mathtt{P(a)} \rightarrow \forall \mathtt{x.P(x)}), \mathtt{P(a)}, \mathtt{Q(a)}\} \vdash (\mathtt{Q(a)} \land \forall \mathtt{x.P(x)});$
6. $\{(\mathtt{P(a)} \rightarrow \mathtt{P(b)}), (\mathtt{Q(a)} \rightarrow \mathtt{P(b)})\} \vdash ((\mathtt{P(a)} \lor \mathtt{Q(a)}) \rightarrow \mathtt{P(b)});$
7. $\{\neg(\mathtt{P(a)} \land \mathtt{Q(a)})\} \vdash (\neg \mathtt{P(a)} \lor \neg \mathtt{Q(a)});$
8. $\{\neg(\neg \mathtt{P(a)} \lor \neg \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \land \mathtt{Q(a)});$
9. $\{\neg(\neg \mathtt{P(a)} \land \neg \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \lor \mathtt{Q(a)});$
10. $\{\forall \mathtt{x.}(\mathtt{P(x)} \land \mathtt{Q(x)})\} \vdash \forall \mathtt{x.P(x)};$
11. $\{\forall \mathtt{x.Q(x)}, \mathtt{P(a)}\} \vdash \mathtt{P(a)} \land \mathtt{Q(a)};$
12. $\{\forall \mathtt{x.P(x)}, \forall \mathtt{x.Q(x)})\} \vdash \forall \mathtt{x.}(\mathtt{P(x)} \land \mathtt{Q(x)});$
13. $\{\exists \mathtt{x.}\exists \mathtt{y.P(x,y)}\} \vdash \exists \mathtt{y.}\exists \mathtt{x.P(x,y)};$
14. $\{\exists \mathtt{x.}\forall \mathtt{y.P(x,y)}\} \vdash \forall \mathtt{y.}\exists \mathtt{x.P(x,y)}$; But in the other direction: $\{\forall \mathtt{y.}\exists \mathtt{x.P(x,y)}\} \vdash \exists \mathtt{x.}\forall \mathtt{y.P(x,y)}$?
15. $\{\neg(\exists \mathtt{x.P(x)})\} \vdash \forall \mathtt{x.}\neg \mathtt{P(x)};$
16. $\{\forall \mathtt{x.}\neg \mathtt{P(x)}\} \vdash \neg(\exists \mathtt{x.P(x)});$