Logic for Computer Science - Week 5
Normal Forms

1 Introducere

In the rest of the course, we will allow to write formulae without explicit brackets, by following the following priority order of the logical connectives:

$$\bot, \neg, \wedge, \vee, \rightarrow, \leftrightarrow .$$

For example, by

$$\neg\neg \mathtt{p} \vee \bot \wedge \mathtt{p} \rightarrow \neg \mathtt{p} \wedge \mathtt{q} \leftrightarrow \mathtt{q}$$

we will understand the formula

$$(((\neg\neg \mathtt{p} \vee (\bot \wedge \mathtt{p})) \rightarrow (\neg \mathtt{p} \wedge \mathtt{q})) \leftrightarrow \mathtt{q}).$$

We will also write $((\varphi_1 \vee \varphi_2) \vee \varphi_3)$ as $\varphi_1 \vee \varphi_2 \vee \varphi_3$ (meaning that a series of $\vee$ will be implicitly left-associated).

We will also write $((\varphi_1 \wedge \varphi_2) \wedge \varphi_3)$ as $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ (meaning that a series of $\wedge$s will also be implicitly left-associated).

Example 1.1. *We write*

$$\mathtt{p} \wedge \mathtt{q} \wedge \mathtt{r} \vee \neg \mathtt{p} \wedge \neg \mathtt{q} \wedge \neg \mathtt{r}$$

for the formula

$$(((\mathtt{p} \wedge \mathtt{q}) \wedge \mathtt{r}) \vee ((\neg \mathtt{p} \wedge \neg \mathtt{q}) \wedge \neg \mathtt{r})).$$

We write

$$\mathtt{p}_1 \wedge \mathtt{p}_2 \wedge \mathtt{p}_3 \wedge \mathtt{p}_4$$

for the

$$(((\mathtt{p}_1 \wedge \mathtt{p}_2) \wedge \mathtt{p}_3) \wedge \mathtt{p}_4).$$

We write

$$\mathtt{p}_1 \vee \mathtt{p}_2 \vee \mathtt{p}_3 \vee \mathtt{p}_4$$

for the formula

$$(((\mathtt{p}_1 \vee \mathtt{p}_2) \vee \mathtt{p}_3) \vee \mathtt{p}_4).$$

2 Replacement Theorem

We present a theorem that we have used implicitly so far:

Theorem 2.1 (The Replacement Theorem)**.** *Let φ, φ' be two formulae such that $\varphi \equiv \varphi'$.*

Let φ_1 be a formula that contains φ as a subformula.

Let φ_2 be the formula obtained from φ_1 by replacing one occurence of φ by φ'.

Then $\varphi_1 \equiv \varphi_2$.

In other words, $\equiv$ is a *congruence.*

Example 2.1. *Let $\varphi = \mathtt{p}$ and $\varphi' = \neg\neg\mathtt{p}$.*

Let $\varphi_1 = (\mathtt{p} \vee \mathtt{q})$ and $\varphi_2 = (\neg\neg\mathtt{p} \vee \mathtt{q})$.

By the replacement theorem, we get tgat $\varphi_1 \equiv \varphi_2$. In other words, by replacing a subformula with an equivalent one, the new formula is equivalent to the starter formula

3 Literal

Definition 3.1 (Literal)**.** *A formula φ is called a* literal *if there exists a propositional variable $a \in A$ such that*

$$\varphi = a \qquad \text{or} \qquad \varphi = \neg a.$$

Example 3.1. *The formulae $\mathtt{p}, \mathtt{q}, \neg\mathtt{p}, \neg\mathtt{p}', \mathtt{q}_1$ are literals, but the formulae $(\mathtt{p} \vee \mathtt{q}), \neg\neg\mathtt{p}, \neg\neg\neg\mathtt{q}_1, (\neg\mathtt{p} \wedge \mathtt{q})$ are not literals.*

4 Clauses

Definition 4.1. *A formula φ is called a* clause *if there exist n literals $\varphi_1, \ldots, \varphi_n$ such that*

$$\varphi = \varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_n.$$

In other words, a clause is a *disjunction of literals.*

Example 4.1. *The following formulae are clauses:*

1. $\mathtt{p} \vee \mathtt{q} \vee \mathtt{r}$;
2. $\mathtt{p} \vee \neg\mathtt{q} \vee \neg\mathtt{r}$;
3. $\neg\mathtt{p} \vee \neg\mathtt{q} \vee \neg\mathtt{r}$;
4. $\neg\mathtt{p}_1 \vee \mathtt{p}_1 \vee \mathtt{p}_2 \vee \neg\mathtt{q} \vee \neg\mathtt{r}$;
5. $\neg\mathtt{p}_1 \vee \mathtt{p}_1$;

6. $\neg \mathtt{p}_1$ *($n = 1$).*

7. $\mathtt{p}$ *($n = 1$).*

The following formulae are not clauses:

1. $\mathtt{p} \wedge \mathtt{r}$; *(conjunction instead of disjunction)*
2. $\mathtt{p} \vee \neg\neg \mathtt{q} \vee \neg \mathtt{r}$; *(not a disjunction of literals)*
3. $\neg\neg \mathtt{p} \vee \mathtt{p} \wedge \neg \mathtt{p}_1$. *(we have $\neg\neg$, so no literals; we also have $\wedge$)*

Remark 4.1. *For $n = 0$, we get the* empty clause*, denoted by $\square$. We consider that $\square \in PL$ is un unsatisfiable formula (i.e., the empty clause is a formula equivalent to $\bot$).*

5 Conjunctive Normal Form

Definition 5.1 (CNF)**.** *A formula φ* is in CNF *if there exist n clauses $\varphi_1, \ldots, \varphi_n$ such that*

$$\varphi = \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n.$$

In other words, a formula in CNF is a conjunction of disjunctions of literals. Or, a formula in CNF is a conjunction of clauses. CNF is conveniently also an abbreviation of clausal normal form, which means the same thing as conjunctive normal form.

Example 5.1. *The following formulae are in CNF:*

1. $(\neg \mathtt{p} \vee \mathtt{q}) \wedge (\mathtt{r} \vee \neg \mathtt{p} \vee \mathtt{r}') \wedge (\neg \mathtt{p} \vee \neg \mathtt{r})$;
2. $\neg \mathtt{p} \wedge (\mathtt{r} \vee \neg \mathtt{p} \vee \mathtt{r}') \wedge \neg \mathtt{r}$;
3. $\neg \mathtt{p} \wedge \mathtt{r} \wedge \neg \mathtt{r}$;
4. $\neg \mathtt{p}$;
5. $\mathtt{p}$.

The following formulae are not in CNF:

1. $\neg(\neg \mathtt{p} \vee \mathtt{q}) \wedge (\mathtt{r} \vee \neg \mathtt{p} \vee \mathtt{r}') \wedge (\neg \mathtt{p} \vee \neg \mathtt{r})$; *(the first clause is negated)*
2. $\neg \mathtt{p} \wedge \neg(\mathtt{r} \vee \neg \mathtt{p} \vee \mathtt{r}')$; *(second clause is negated)*
3. $\neg \mathtt{p} \vee (\mathtt{r} \wedge \neg \mathtt{p} \wedge \mathtt{r}')$; *(the main connective is the disjunction, instead of a conjunction)*
4. $\neg\neg \mathtt{p}$; *(we have $\neg\neg$, so no literals)*
5. $\mathtt{p} \vee (\mathtt{q} \wedge \mathtt{r})$. *(disjunction of conjunctions, not a conjunction of disjunctions (of literals))*

6 Bringing a Formula into CNF

Theorem 6.1 (Theorem for Bringing a Formula into CNF)**.** *For any formula $\varphi \in PL$, there exists a formula $\varphi' \in PL$ that is in CNF such that $\varphi \equiv \varphi'$.*

Proof Sketch. By repeatedly applying the replacement theorem using the following equivalences:

1. $(\varphi_1 \leftrightarrow \varphi_2) \equiv ((\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1))$;
2. $(\varphi_1 \rightarrow \varphi_2) \equiv (\neg\varphi_1 \vee \varphi_2)$;
3. $(\varphi_1 \vee (\varphi_2 \wedge \varphi_3)) \equiv ((\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \varphi_3))$;
4. $((\varphi_2 \wedge \varphi_3) \vee \varphi_1) \equiv ((\varphi_2 \vee \varphi_1) \wedge (\varphi_3 \vee \varphi_1))$;
5. $(\varphi_1 \vee (\varphi_2 \vee \varphi_3)) \equiv ((\varphi_1 \vee \varphi_2) \vee \varphi_3)$;
6. $(\varphi_1 \wedge (\varphi_2 \wedge \varphi_3)) \equiv ((\varphi_1 \wedge \varphi_2) \wedge \varphi_3)$;
7. $\neg(\varphi_1 \vee \varphi_2) \equiv (\neg\varphi_1 \wedge \neg\varphi_2)$;
8. $\neg(\varphi_1 \wedge \varphi_2) \equiv (\neg\varphi_1 \vee \neg\varphi_2)$;
9. $\neg\neg\varphi \equiv \varphi$.

The first two equivalences ensure the fact that all double implications and all implications disappear from the formula.

The third and the fourth equivalence ensure that in the tree of the formula, all disjunctions "go under" the conjunctions.

Equivalences 5 and 6 ensure the associativity of the chains of disjunctions and conjunctions, respectively (it allows us to write $\varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \ldots \vee \varphi_n$ instead of $((((\varphi_1 \vee \varphi_2) \vee \varphi_3) \vee \ldots) \vee \varphi_n)$.

Equivalences 7 and 8 ensure that the negations end up "under" the conjunctions and disjunctions in the tree of the formula.

The last equivalence ensures that there are no double negations "one under the other" in the abstract syntax tree.

Applying the equivalences above eventually stops (they cannot be applied ad infinitum – why?).

The result will be a formula in which all conjunctions are "above" all disjunctions, which are "above" all possible negations, meaning a formula in CNF. $\square$

Example 6.1. *Let us bring the formula* $((\neg p \to \neg q) \leftrightarrow (q \to p))$ *to CNF*

$$
\begin{aligned}
& ((\neg p \to \neg q) \leftrightarrow (q \to p)) \\
\stackrel{1}{\equiv}\ & \Big((\neg p \to \neg q) \to (q \to p)\Big) \wedge \Big((q \to p) \to (\neg p \to \neg q)\Big) \\
\stackrel{2}{\equiv}\ & \Big((\neg\neg p \vee \neg q) \to (\neg q \vee p)\Big) \wedge \Big((\neg q \vee p) \to (\neg\neg p \vee \neg q)\Big) \\
\stackrel{2}{\equiv}\ & \Big(\neg(\neg\neg p \vee \neg q) \vee (\neg q \vee p)\Big) \wedge \Big(\neg(\neg q \vee p) \vee (\neg\neg p \vee \neg q)\Big) \\
\stackrel{7}{\equiv}\ & \Big((\neg\neg\neg p \wedge \neg\neg q) \vee (\neg q \vee p)\Big) \wedge \Big((\neg\neg q \wedge \neg p) \vee (\neg\neg p \vee \neg q)\Big) \\
\stackrel{9}{\equiv}\ & \Big((\neg p \wedge q) \vee (\neg q \vee p)\Big) \wedge \Big((q \wedge \neg p) \vee (p \vee \neg q)\Big) \\
\stackrel{4}{\equiv}\ & \Big(\big(\neg p \vee (\neg q \vee p)\big) \wedge \big(q \vee (\neg q \vee p)\big)\Big) \wedge \Big(\big(q \vee (p \vee \neg q)\big) \wedge \big(\neg p \vee (p \vee \neg q)\big)\Big) \\
\stackrel{5}{\equiv}\ & \Big(\big((\neg p \vee \neg q) \vee p\big) \wedge \big((q \vee \neg q) \vee p\big)\Big) \wedge \Big(\big((q \vee p) \vee \neg q\big) \wedge \big((\neg p \vee p) \vee \neg q\big)\Big) \\
=\ & \big((\neg p \vee \neg q \vee p) \wedge (q \vee \neg q \vee p)\big) \wedge \big((q \vee p \vee \neg q) \wedge (\neg p \vee p \vee \neg q)\big) \\
\stackrel{6}{\equiv}\ & \Big(\big((\neg p \vee \neg q \vee p) \wedge (q \vee \neg q \vee p)\big) \wedge (q \vee p \vee \neg q)\Big) \wedge (\neg p \vee p \vee \neg q) \\
=\ & (\neg p \vee \neg q \vee p) \wedge (q \vee \neg q \vee p) \wedge (q \vee p \vee \neg q) \wedge (\neg p \vee p \vee \neg q).
\end{aligned}
$$

7 Disjunctive Normal Form

Definition 7.1 (DNF)**.** *A formula is in DNF if it is a disjunction of conjunctions of literals.*

Example 7.1. *The following formulae are in DNF:*

1. $(\neg \mathtt{p} \wedge \mathtt{q}) \vee (\mathtt{r} \wedge \neg \mathtt{p} \wedge \mathtt{r}') \vee (\neg \mathtt{p} \wedge \neg \mathtt{r})$;
2. $\neg \mathtt{p} \vee (\mathtt{r} \wedge \neg \mathtt{p} \wedge \mathtt{r}') \vee \neg \mathtt{r}$;
3. $\neg \mathtt{p} \vee \mathtt{r} \vee \neg \mathtt{r}$;
4. $\neg \mathtt{p}$;
5. $\mathtt{p}$.

The following formulae are not in DNF:

1. $\neg(\neg \mathtt{p} \wedge \mathtt{q}) \vee (\mathtt{r} \wedge \neg \mathtt{p} \wedge \mathtt{r}') \vee (\neg \mathtt{p} \wedge \neg \mathtt{r})$;
2. $\neg \mathtt{p} \vee \neg(\mathtt{r} \wedge \neg \mathtt{p} \wedge \mathtt{r}')$;
3. $\neg \mathtt{p} \wedge (\mathtt{r} \vee \neg \mathtt{p} \vee \mathtt{r}')$;
4. $\neg\neg \mathtt{p}$;
5. $\mathtt{p} \wedge (\mathtt{q} \vee \mathtt{r})$.

Exercise 7.1. *State and sketch the proof of a constructive theorem to bring a formula into DNF.*

8 The Link between DNF and CNF

Definition 8.1. *The complement of a formula* $\varphi \in PL_{\neg,\vee,\wedge}$ *is denoted by* φ^c *and is defined as follows:*

1. $a^c = \neg a$, *for any* $a \in A$;
2. $(\neg\varphi)^c = \varphi$, *for any* $\varphi \in PL$;
3. $(\varphi_1 \vee \varphi_2)^c = (\varphi_1^c \wedge \varphi_2^c)$, *for any* $\varphi_1, \varphi_2 \in PL$;
4. $(\varphi_1 \wedge \varphi_2)^c = (\varphi_1^c \vee \varphi_2^c)$, *for any* $\varphi_1, \varphi_2 \in PL$.

Example 8.1.

1. $\neg\mathtt{p}^c = \mathtt{p}$ *(pay attention! the complement of* $\neg\mathtt{p}$ *is not* $\neg\neg\mathtt{p}$*);*
2. $\neg(\neg\mathtt{p} \wedge \mathtt{q}) \vee (\mathtt{r} \wedge \neg\mathtt{p} \wedge \mathtt{r}') \vee (\neg\mathtt{p} \wedge \neg\mathtt{r})^c = (\neg\mathtt{p} \vee \mathtt{q}) \wedge (\mathtt{r} \vee \neg\mathtt{p} \vee \mathtt{r}') \wedge (\neg\mathtt{p} \vee \neg\mathtt{r})$;
3. $(\neg\mathtt{p} \vee \mathtt{q}) \wedge (\mathtt{r} \vee \neg\mathtt{p} \vee \mathtt{r}') \wedge (\neg\mathtt{p} \vee \neg\mathtt{r})^c = \neg(\neg\mathtt{p} \wedge \mathtt{q}) \vee (\mathtt{r} \wedge \neg\mathtt{p} \wedge \mathtt{r}') \vee (\neg\mathtt{p} \wedge \neg\mathtt{r})$.

Theorem 8.1 (The Complement is Equivalent to the Negation)**.** *For any formula* $\varphi \in PL_{\neg,\wedge,\vee}$, *we have that*

$$\varphi^c \equiv \neg\varphi.$$

Exercise: prove the theorem above by structural induction.

Example 8.2. $(\mathtt{p} \wedge (\mathtt{q} \vee \neg\mathtt{r}))^c = (\neg\mathtt{p} \vee (\neg\mathtt{q} \wedge \mathtt{r})) \equiv \neg(\mathtt{p} \wedge (\mathtt{q} \vee \neg\mathtt{r}))$.

Theorem 8.2 (The Link between CNFs and DNFs)**.** *Let* φ_1 *be a formula in CNF and* φ_2 *a formula in DNF.*

Then φ_1^c *is in DNF and* φ_2^c *is in CNF.*