Vector Spaces

Part I

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Outline

Definitions, examples, and basic properties

Definition 1

Let $(F,+,-,0,\circ,',e)$ be a field. A vector space over F is an algebraic system $(V,\oplus,\ominus,\mathbf{0},\cdot)$ which consists of a commutative group $(V,\oplus,\ominus,\mathbf{0})$ and a function $\cdot : F \times V \to V$ such that:

1. $\alpha \cdot (x \oplus y) = \alpha \cdot x \oplus \alpha \cdot y$, for any $\alpha \in F$ and $x, y \in V$;
2. $(\alpha + \beta) \cdot x = \alpha \cdot x \oplus \beta \cdot x$, for any $\alpha, \beta \in F$ and $x \in V$;
3. $(\alpha \circ \beta) \cdot x = \alpha \cdot (\beta \cdot x)$, for any $\alpha, \beta \in F$ and $x \in V$;
4. $e \cdot x = x$, for any $x \in V$.

The elements of V are called vectors, the elements of F are called scalars, and F is called the field of scalars of V. The operation $\oplus$ is called the vector addition and the operation $\cdot$ is called the scalar multiplication.

Remark 2

To simplify the notation, we will denote the operations of F by $(F, +, -, 0, \cdot, ', 1)$ and the operations of V by $(V, +, -, 0, \cdot)$. Moreover, the symbol of the operation $\cdot$ will be mostly omitted. Therefore, the axioms of V can be rewritten as follows:

1. $\alpha(x+y) = \alpha x + \alpha y$, *for any* $\alpha \in F$ *and* $x, y \in V$;
2. $(\alpha+\beta)x = \alpha x + \beta x$, *for any* $\alpha, \beta \in F$ *and* $x \in V$;
3. $(\alpha\beta)x = \alpha(\beta x)$, *for any* $\alpha, \beta \in F$ *and* $x \in V$;
4. $1x = x$, *for any* $x \in V$.

Vector subtraction is defined by $x - y = x + (-y)$, for any $x, y \in V$.

The vector space which consists of the only element 0 is called the trivial vector space (it is unique up to an isomorphism).

Examples of vector spaces

Example 3

1. Let F be a field and $n \geq 1$. F^n, the set of all n-tuples over F, usually called *n-dimensional vectors over F*, can be organized as a vector space over F. To this, define vector addition by
$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$$
and scalar multiplication by
$$b(a_1, \ldots, a_n) = (ba_1, \ldots, ba_n),$$
for any $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in F^n$ and $b \in F$.
With these operations, F^n is a vector space over F. If we identify F^1 with F, then F can be viewed as a vector space over itself.

2. The set of all $m \times n$ matrices over F, denoted ${}^mF^n$, can be organized as a vector space over F. Vector addition is matrix addition, and scalar multiplication is the usual multiplication with scalars.

Examples of vector spaces

Example 4

1. $\mathbb{Q}^n$, $\mathbb{R}^n$, and $\mathbb{C}^n$ are vector spaces over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. .
2. $\mathbb{C}$ can be viewed as a vector space over $\mathbb{R}$, and both $\mathbb{C}$ and $\mathbb{R}$ can be viewed as vector spaces over $\mathbb{Q}$.
3. The set of all functions from $\mathbb{R}$ to $\mathbb{R}$, together with the addition $f + g$ and scalar multiplication αf ($(\alpha f)(x) = \alpha f(x)$, for any x), form a vector space over $\mathbb{R}$.

Some basic properties

Prove the following properties!

Proposition 5

Let V be a vector space over a field F. Then, for any $x, y \in V$ and $\alpha, \beta \in F$, the following properties hold

1. $0x = 0$;
2. $(-1)x = -x$;
3. $(-\alpha)x = \alpha(-x) = -\alpha x$;
4. $\alpha 0 = 0$;
5. *If* $\alpha x = 0$, *then* $\alpha = 0$ *or* $x = 0$;
6. *If* $\alpha x = \alpha y$, *then* $\alpha = 0$ *or* $x = y$;
7. *If* $\alpha x = \beta x$, *then* $\alpha = \beta$ *or* $x = 0$.

Vector subspace

Definition 6

Let V and U be vector spaces over a field F. We say that U is a subspace of V, denoted $U \leq V$, if $U \subseteq V$ and the restriction of V's operations to U coincide with U's operations.

Example 7

1. If V is a vector space over F, then $\{0\}$ and V are subspaces of V.
2. Let F be a field and $n \geq 1$. The set U of all vectors of F^n whose first coordinate is 0 is a subspace of F^n. When $n \geq 2$, this subspace can be identified with F^{n-1}.

Sums and direct sums

Sums of vector subspaces

Given V a vector space over a field F and $U_1, U_2 \subseteq V$, define

$$U_1 + U_2 = \{u_1 + u_2 \mid u_1 \in U_1,\ u_2 \in U_2\}$$

Prove the following property!

Proposition 8

If V is a vector space over a field F and $U_1, U_2 \leq V$, then $U_1 + U_2 \leq V$.

Definition 9

If a vector space V can be written as $V = U_1 + U_2$, where $U_1, U_2 \leq V$, then V is called the sum of U_1 and U_2. Moreover, U_2 (U_1) is called a complement of U_1 (U_2) in V.

Direct sums of vector subspaces

Definition 10

Let V be a vector space over a field F and $U_1, U_2 \leq V$. V is called the direct sum of U_1 and U_2 if $V = U_1 + U_2$ and each vector $x \in V$ has a unique expression as a sum $x = x_1 + x_2$ of vectors $x_1 \in U_1$ and $x_2 \in U_2$.

Denote $V = U_1 \oplus U_2$ whenever V is the direct sum of U_1 and U_2. In this case, U_2 (U_1) is called a direct complement of U_1 (U_2) in V.

Theorem 11

Given V a vector space over a field F and $U_1, U_2 \leq V$, $V = U_1 \oplus U_2$ if and only if $V = U_1 + U_2$ and $U_1 \cap U_2 = \{0\}$.

Proof.

See textbook [2], page 364. □

In some research papers and textbooks, the concept of "complement" is used only for the direct sum!

Linear combinations

Linear combination of vectors

Let V be a vector space over a field F and $B \subseteq V$ non-empty. A linear combination of vectors in B is an expression (sum)

$$\alpha_1 x_1 + \ldots + \alpha_k x_k,$$

where $k \geq 1$, $x_1, \ldots, x_k \in B$, and $\alpha_1, \ldots, \alpha_k \in F$.

In this definition, B may be infinite!

The set of all linear combinations of vectors in B forms a subspace of V, called the subspace generated by B and denoted $\langle B \rangle_V$ (or $\langle B \rangle$, when V is clear from the context). Therefore,

$$\langle B \rangle_V = \{\alpha_1 x_1 + \cdots + \alpha_k x_k | k \geq 1 \wedge (\forall 1 \leq i \leq k)(\alpha_i \in F \wedge x_i \in B)\}.$$

When B is finite, $B = \{x_1, \ldots, x_k\}$ for some $k \geq 1$, a sum as that above will simply be called "linear combination of $x_1, \ldots, x_k$", and the subspace generated by B will also be denoted by $\langle x_1, \ldots, x_k \rangle_V$.

Linearly independent vectors

Definition 12

Let V be a vector space over a field F and $B \subseteq V$ non-empty. B is called linearly independent if for any $k \geq 1$, distinct vectors $x_1, \ldots, x_k \in B$, and $\alpha_1, \ldots, \alpha_k \in F$, the following property holds:

$$\sum \alpha_i x_i = 0 \;\Rightarrow\; (\forall 1 \leq i \leq k)(\alpha_i = 0)$$

If B is not linearly independent, then it is called linearly dependent. That is, in such a case, there are $k \geq 1$, $x_1, \ldots, x_k \in B$ distinct vectors, and $\alpha_1, \ldots, \alpha_k \in F$ not all 0, such that $\sum \alpha_i x_i = 0$.

When B is finite, we can consider only linear combinations with all vectors in B without restricting the generality in the above definitions!

Remarks on linear independence of vectors

Remark 13

Let V be a vector space over a field F.

1. $x \in V$ *is linearly independent if and only if* $x \neq 0$.
2. *If a "sequence of vectors"* $x_1, \ldots, x_k \in V$ *is linearly independent, then* $x_i \neq 0$*, for any* i*, and* $x_i \neq x_j$*, for any* $i \neq j$.

In everything that follows, when we write $x_1, \ldots, x_n \in V$, we mean that these vectors are different than 0 and pairwise distinct!

Proposition 14

Let V be a vector space over a field F. $x_1, \ldots, x_k$ from V are linearly dependent if and only if there exists $1 \leq i \leq k$ such that x_i is a linear combination of the other vectors.

Proof.

See textbook [2], page 356. □

Basis and dimension

Basis

Definition 15

Let V be a non-trivial vector space over a field F. A finite subset $B \subseteq V$ is called a basis of V if it is linearly independent and generates V (each element in V is a linear combination of vectors in B).

Remark 16

- *If $x_1, \ldots, x_k$ form a basis for V, then $x_i \neq x_j$, for any $i \neq j$. Therefore, $\{x_1, \ldots, x_k\}$ has exactly k vectors.*
- *We have considered only finite basis. There are approaches for infinite basis too.*

Examples of bases

Example 17

1. Let F be a field and $n \geq 1$. The vector space F^n can be generated by

$$\begin{array}{rcl} \mathbf{e_1} & = & (1, 0, 0, \ldots, 0, 0) \\ \mathbf{e_2} & = & (0, 1, 0, \ldots, 0, 0) \\ & \ldots & \\ \mathbf{e_n} & = & (0, 0, 0, \ldots, 0, 1). \end{array}$$

2. Let F be a field and $m, n \geq 1$. The vector space ${}^mF^n$ can be generated by E_{ij}, where

$$E_{ij}(u, v) = \begin{cases} 1, & \text{if } u = i \text{ and } v = j \\ 0, & \text{otherwise,} \end{cases}$$

for any $i, u \in \{1, \ldots, m\}$ and $v, j \in \{1, \ldots, n\}$.

Basis and coordinate

Theorem 18

Let V be a vector space over a field F. $B = \{x_1, \ldots, x_k\} \subseteq V$ is a basis of V if and only if any $x \in V$ can be uniquely written as a linear combination of vectors in B.

Proof.

See textbook [2], page 358. $\square$

Given a basis $B = \{x_1, \ldots, x_n\}$ of V, each element $x \in V$ can be uniquely written as

$$x = \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

with $\alpha_1, \ldots, \alpha_n \in F$. We will say that α_i in the i-th coordinate of x, and $(\alpha_1, \ldots, \alpha_n)$ is the coordinate of x, in the basis B.

Pictorial view of bases

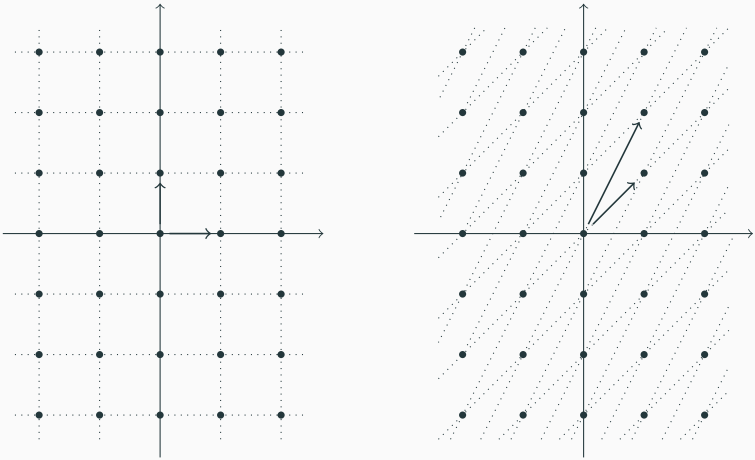

Figure 1: Generic examples of bases

Exchange lemma

Lemma 19 (Steinitz Exchange Lemma)

Let V be a vector space over a field F. If V is generated by $A = \{x_1, \ldots, x_k\}$ and $B = \{y_1, \ldots, y_m\} \subseteq V$ is a subset of linearly independent vectors, then $m \leq k$. Moreover, some vectors from A can be replaced by vectors from B so that the set thus obtained contains exactly k vectors, still generates V, and includes B.

Proof.

See textbook [2], page 359. $\square$

Directly from Steinitz's exchange lemma we obtain the following result.

Corollary 20

If A and B are finite linearly independent sets that generate a vector space V, then $|A| = |B|$.

Bases from subspace bases

Clearly, if V is a vector space finitely generated by a subset A, then A must include a basis for V (see textbook [2], page 358).

Then, Steinitz's exchange lemma leads to the following result.

Corollary 21

If A is a subspace of a vector space V finitely generated, then any basis of A can be extended to a basis of V.

Dimension of a vector space

Definition 22

Let V be a vector space over a field F. V is called finite dimensional if there exists a finite basis B for V. In this case, $|B|$ is called the dimension of V, denoted $dim(V)$. If V is not finite dimensional then it is called infinite dimensional.

Example 23

1. Given $n \geq 1$, $dim(F^n) = n$;
2. Given $n, m \geq 1$, $dim({}^mF^n) = mn$;
3. $F^{\mathbb{N}}$ is infinite dimensional.

We accept that the trivial vector space is finite dimensional.

Basis and direct sum of vector spaces

Prove the following properties!

Theorem 24

Let V be a vector space over a field F and B a basis.

- *For any partition $B = B_1 \cup B_2$ ($B_1, B_2 \neq \emptyset$, $B_1 \cap B_2 = \emptyset$) of B we have $V = \langle B_1 \rangle \oplus \langle B_2 \rangle$.*
- *For any decomposition $V = U_1 \oplus U_2$, any basis B_1 of U_1 and any basis B_2 of U_2, $B = B_1 \cup B_2$ is a basis of V and $B_1 \cap B_2 = \emptyset$.*

Proof.

Hint: any linear combination of vectors in B, $\sum_{v_i \in B} \alpha_i v_i$, can be decomposed in two linear combinations

$$\sum_{v_i \in B} \alpha_i v_i = \sum_{v_i \in B_1} \alpha_i v_i + \sum_{v_i \in B_2} \alpha_i v_i$$

□

Dimension and (direct) sum of vector spaces

Theorem 25

Let V be a finitely generated vector space over a field F and $U_1, U_2 \leq V$. Then,

$$dim(U_1) + dim(U_2) = dim(U_1 + U_2) + dim(U_1 \cap U_2)$$

In particular, if $V = U_1 \oplus U_2$, then

$$dim(U_1) + dim(U_2) = dim(V)$$

Proof.

Start with a basis for $U_1 \cap U_2$, extend it to a basis for U_1, then to a basis for U_2, and finally prove that their union is a basis for $U_1 + U_2$. $\square$

Remark 26

Using Zorn's Lemma [1], one can show that any non-trivial vector space has a basis. The theorem above can then be proven in the general case.

Reading and exercise guide

Reading and exercise guide

It is highly recommended that you do all the exercises marked in red from the slides.

Course readings:

1. Pages 351-368 from textbook [2].

References

[1] Ferucio Laurențiu Țiplea. *Introduction to Set Theory*. "Alexandru Ioan Cuza" University Publishing House, Iași, Romania, 1998.

[2] Ferucio Laurențiu Țiplea. *Algebraic Foundations of Computer Science*. "Alexandru Ioan Cuza" University Publishing House, Iași, Romania, second edition, 2021.