

Algorithm Design: Nondeterministic and Randomized Algorithms

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- 1 Recap
- 2 Non-deterministic Algorithms, Generally
- 3 Non-deterministic Algorithms for Decision Problems
- 4 Randomized Algorithms
 - Random Variable
 - Example of Monte Carlo Algorithm: Primality Test
 - Las Vegas Algorithms
 - Example of Las Vegas Algorithm: k -median

Plan

- 1 Recap
- 2 Non-deterministic Algorithms, Generally
- 3 Non-deterministic Algorithms for Decision Problems
- 4 Randomized Algorithms
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 - Example of Las Vegas Algorithm: k -median

Problem solved by an algorithm

An algorithm A solves a problem P if:

- for any instance p of P , there is an initial configuration $\langle A, \sigma_p \rangle$ such that σ_p includes data structures describing p ;
- the execution starting from the initial configuration $\langle A, \sigma_p \rangle$ ends into a final configuration $\langle \cdot, \sigma' \rangle$, write $\langle A, \sigma \rangle \Rightarrow^* \langle \cdot, \sigma' \rangle$; and
- σ' includes data structures that describes $P(p)$.

Formally:

P is specified by $(pre, post)$

A solves $P \equiv (\forall \sigma)$ with $\sigma \models pre$ $(\exists \sigma')$ s.t. $\langle A, \sigma \rangle \Rightarrow^* \langle \cdot, \sigma' \rangle$ and $\sigma' \models post$

The execution time for an instance: the deterministic case

Deterministic algorithm: for any reachable configuration $\langle A_i, \sigma_i \rangle$ there is at most a successor configuration, i.e., at most a $\langle A', \sigma' \rangle$ such that $\langle A_i, \sigma_i \rangle \Rightarrow \langle A', \sigma' \rangle$.

All algorithms we considered up to now are deterministic! Later we shall consider nondeterministic algorithms as well.

Let P be a problem and A a deterministic algorithm that solves P .

For each $p \in P$ there is an unique execution path E_p .

The time for computing $P(p)$ is

$$time_d(A, p) = time_d(E_p)$$

where $d \in \{log, unif, lin\}$.

The size of an instance

The dimension of a state σ is

$$size_d(\sigma) = \sum_{x \mapsto v \in \sigma} size_d(v)$$

The dimension of a configuration is

$$size_d(\langle A, \sigma \rangle) = size_d(\sigma)$$

where $d \in \{log, unif, lin\}$.

Let P be a problem, $p \in P$, and A a deterministic algorithm that solves P .

The size of p is the the size of its intial configuration:

$$size_d(p) = size_d(\langle A, \sigma_p \rangle) (= size(\sigma_p))$$

where $d \in \{log, unif, , lin\}$.

The worst case time complexity

Let P be a problem and A a deterministic algorithm that solves P and fix $d \in \{\log, \text{unif}, \text{lin}\}$.

Group the instances p of P into equivalence classes: p and p' are in the same equivalence class iff $\text{size}(p) = \text{size}(p')$.

A natural number n can be seen as the equivalence class of instances p of size n ($\text{size}_d(p) = n$).

The **worst case time complexity**:

$$T_{A,d}(n) = \max\{\text{time}_d(A, p) \mid p \in P, \text{size}_d(p) = n\}$$

Experiments with Alk interpreter1/3

```
isPrime1(x) {  
  if (x < 2) return false;  
  for (i= 2; i <= x/2; ++i)  
    if (x % i == 0) return false;  
  return true;  
}
```


Experiments with Alk interpreter 2/3

A worst case:

```
print(isPrime1(2147483647));
```

```
$ time alki.sh -a isPrime.alk
```

```
^C
```

```
real 41m3.065s
```

```
user 40m45.849s
```

```
sys 0m15.643s
```

A more favorable case:

```
print(isPrime1(2147483647*457241)); //457241^2=209069332081
```

```
$ time alki.sh -a isPrime.alk
```

```
false
```

```
real 0m3.480s
```

```
user 0m5.921s
```

```
sys 0m0.236s
```

Experiments with Alk interpreter 3/3

A minor change could bring major improvements:

```
isPrime2(x) {  
  if (x < 2) return false;  
  for (i= 2; i*i <= x; ++i)  
    if (x % i == 0) return false;  
  return true;  
}
```

```
print(isPrime2(2147483647));
```

```
$ time alki.sh -a isPrime.alk  
true
```

```
real 0m1.659s  
user 0m3.475s  
sys 0m0.163s
```

Plan

- 1 Recap
- 2 Non-deterministic Algorithms, Generally
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 - Example of Las Vegas Algorithm: k -median

Motivație

- abstraction of the state space is useful in analysis
- non-deterministic algorithms bring an additional abstraction level, which combines state abstraction with procedural abstraction
- ignores details how some data structures are created
- useful in complexity analysis
- preliminary notion for randomized algorithms

Non-deterministic Algorithms, Intuitively

- for some configurations there are more than one way to continue the execution
- consequently, for the same input the algorithm may have many executions with different results
- **angelic version**: the algorithm "guesses" the execution that leads to the correct result
- **execution time**: the time of the execution that leads to the correct result

Extending the Language

`choose x in S ;`

- returns an element from S , arbitrarily chosen
- execution time (uniform): $O(1)$

`choose x in S s.t. B ;`

- returns an element from S that satisfies B
- equivalent to

`choose x in S ;`

`if $(\neg B(x))$ ends with failure;`

- execution time: $T(B)$

Demo with the new statements

```
choose x1 in { 1 .. 5 };
```

```
$ alki.sh -a choose.alk
```

```
x1 |-> 2
```

Note that the executed algorithm is nondeterministic.

```
$ alki.sh -a choose.alk
```

```
x1 |-> 4
```

Note that the executed algorithm is nondeterministic.

Demo with the new statements

```
odd(x) {
    return x % 2 == 1;
}
choose x1 in { 1 .. 5 } s.t. odd(x1);
choose x2 in { 1 .. 5 } s.t. odd(x2);
```

```
$ alki.sh -a choosetest.alk
```

```
x1 |-> 3
x2 |-> 1
```

Note that the executed algorithm is nondeterministic.

```
$ alki.sh -a choosetest.alk
```

```
x1 |-> 5
x2 |-> 5
```

Note that the executed algorithm is nondeterministic.

Demo with the new statements

```
odd(x) {  
    return x % 2 == 1;  
}
```

```
L = emptyList;  
for (i = 0; i < 8; i = i+2)  
    L.pushBack(i);  
choose x in L s.t. odd(x);
```

```
$ alki.sh -a failure.alk
```

Error at line 8: Choose can't find any suitable value.

Problem Solved by a Non-deterministic Algorithm

- a non-deterministic algorithm A has many executions for the same input
- so, what means that A solves a problem P ?
- we say that A solves P if $\forall x \in P$
 - \exists an execution that
 - is terminating and
 - whose final configuration includes $P(x)$

Example: N Queens Problem

Input: a chessboard $n \times n$.

Output: place n chess queens so that no two queens attack each other.

```

attacked(i, j, b) {
    attack = false;
    for (k = 0; k < i; ++k)
        if ((b[k] == j) || ((b[k]-j) == (k-i)) || ((b[k]-j) == (i-k)))
            attack = true;
    return(attack);
}

```

```

nqueens (n) {
    for (i = 0; i < n; ++i) {
        choose j in { 0 .. n-1 } s.t. ! (attacked(i, j, b));
        b[i] = j;
    }
}

```

Execution time: $O(n^2)$

Details on the blackboard.

Example: N Queens Problem, demo

```
n = 4;  
b = [-1 | i from [0..n-1]];
```

```
$ alki.sh -a nqueens.alk
```

Error at line 19: Choose can't find any suitable value. **i.e. failure**

```
$ alki.sh -a nqueens.alk  
b |-> [2, 0, 3, 1]  
i |-> 3  
n |-> 4
```

Note that the executed algorithm is nondeterministic.

Example: Subset Sum Problem (SSP)

```
/*
Input: A set  $S$  of integers,  $M$  a positive integer.
Output: A subset of  $S' \subseteq S$  s.t.  $\sum_{x \in S'} x = M$ , if any.
```

```
*/

PM = 0;
/* choose a maximal size for the subset */
choose k in {1 .. S.size()};
/* try to choose at most k-1 elements */
for(i = 0; i < k-1; ++i) {
    choose x in S s.t. PM + x <= M;
    S = S \ singletonSet(x);
    PM = PM + x;
}
/* try to choose the k-th element, if needed */
if (PM != M)
    choose x in S s.t. PM + x == M;
```

TExecution time: $O(n)$, where $n = S.size()$ (we assumed $T(S \setminus \text{singletonSet}(x)) = O(1)$).

Details on blackboard.

Example: Subset Sum Problem (SSP), demo

```

ssd.in
S |-> {1, 3, 4, 7, 9} M |-> 14
Execution:

$ alki.sh -a ssd.alk -i ssd.in
Error at line 18: Choose can't find any suitable value.  failure

$ alki.sh ssd.alk -i ssd.in
S |-> 3, 7      success
x |-> 1
i |-> 3
k |-> 4
M |-> 14
PM |-> 14

```

Note that the executed algorithm is nondeterministic.

Reduction to Deterministic Algorithms

Let \sim be an equivalence between states. For instance, $\sigma \sim \sigma'$ iff σ and σ' encode the same instance p of a problem P or both encode the answer $P(p)$.

Definition

We say that an algorithm A is **equivalent** to an algorithm B (w.r.t. \sim) iff:

- ① $\langle A, \sigma_1 \rangle \Rightarrow^* \langle \cdot, \sigma'_1 \rangle$ and $\sigma_1 \sim \sigma_2$ implies the existence of σ'_2 s.t.
 $\langle B, \sigma_2 \rangle \Rightarrow^* \langle \cdot, \sigma'_2 \rangle$ și $\sigma'_1 \sim \sigma'_2$, and
- ② reciprocally, $\langle B, \sigma_2 \rangle \Rightarrow^* \langle \cdot, \sigma'_2 \rangle$ și $\sigma_1 \sim \sigma_2$ the existence of σ'_1 s.t.
 $\langle A, \sigma_1 \rangle \Rightarrow^* \langle \cdot, \sigma'_1 \rangle$ și $\sigma'_1 \sim \sigma'_2$.

Theorem

For any non-deterministic algorithm A there is an equivalent deterministic algorithm B , which has the worst case execution time $T_B(n) = O(2^{T_A(n)})$.

Plan

- 1 Recap
- 2 Non-deterministic Algorithms, Generally
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The Approach

- there are two main steps:
 - first “guesses” a certain structure S
 - then checks if S satisfies the property requested by the question
 - if yes the execution finishes with **success**, otherwise it finishes with **failure**;
- Extending the language:
 - `success`; – signals the successful termination of an execution
 - `failure`; – signals the termination of a failing execution

Example

SAT

Instance: A set of n propositional variables and a propositional formula F in conjunctive normal form.

Question: Is F satisfiable?

```
// guess
for (i = 0; i < n; ++i) {
    choose z in {false, true};
    x[i] = z;
}
// check
if (f(x)) success;
else failure;
```

Example of SAT instance

```
n = 4;

f(x) {
    return (x[0] || x[1]) &&
           (!x[0] || x[3] || x[2]) &&
           (x[2] || !x[3]) &&
           (!x[1] || !x[2] || x[3]);
}
```

Demo

```
$ alki.sh -a sat.alk  
failure  
x |-> [false, false, false, true]  
i |-> 4  
z |-> true  
n |-> 4
```

Note that the executed algorithm is nondeterministic.

```
$ alki.sh -a sat.alk  
failure  
x |-> [true, true, true, false]  
i |-> 4  
z |-> false  
n |-> 4
```

Note that the executed algorithm is nondeterministic.

Demo

```
$ alki.sh -a sat.alk  
failure  
x |-> [false, true, true, false]  
i |-> 4  
z |-> false  
n |-> 4
```

Note that the executed algorithm is nondeterministic.

```
$ alki.sh -a sat.alk  
success  
x |-> [true, true, true, true]  
i |-> 4  
z |-> true  
n |-> 4
```

Note that the executed algorithm is nondeterministic.

Example: Subset sum Problem (SSP3)

```

/*
Instance: A set  $S$  of integers,  $M$  a positive integer.
Question: Is there a subset  $S' \subseteq S$  s.t.  $\sum_{x \in S'} x = M$ ?

*/

PM = 0;
choose k in 1 .. S.size();
for(i = 0; i < k; ++i) {
    choose x in S;
    S = S \ singletonSet(x);
    PM = PM + x;
}
if (PM == M) print("success");
else print("failure");

```

Demo (SSD3)

```
$ alki.sh -a ssd3.alk -i ssd.in
```

failure

```
S |-> 1, 3, 7
```

```
x |-> 4
```

```
i |-> 2
```

```
k |-> 2
```

```
M |-> 14
```

```
PM |-> 13
```

Note that the executed algorithm is nondeterministic.

```
$ alki.sh -a ssd3.alk -i ssd.in
```

success

```
S |-> 1, 9
```

```
x |-> 7
```

```
i |-> 3
```

```
k |-> 3
```

```
M |-> 14
```

```
PM |-> 14
```

Note that the executed algorithm is nondeterministic.

Plan

- 1 Recap
- 2 Non-deterministic Algorithms, Generally
- 3 Non-deterministic Algorithms for Decision Problems
- 4 Randomized Algorithms**
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 - Las Vegas Algorithms
 - Example of Las Vegas Algorithm: k -median

Plan

- 1 Recap
- 2 Non-deterministic Algorithms, Generally
- 3 Non-deterministic Algorithms for Decision Problems
- 4 Randomized Algorithms
 - Random Variable
 - Example of Monte Carlo Algorithm: Primality Test
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 - Example of Las Vegas Algorithm: k -median

Definition

Definition

A **random variable** is a function X defined over a set of possible outcomes Ω of a random phenomenon.

Example (only discrete variables)

1. $D2$ (two dice):

- random phenomenon: rolling two dice
- $D2$ returns the pair representing the numbers on the two dice

2. $SD2$ (the sum of two dice):

- random phenomenon: rolling two dice
- $SD2$ returns the sum of numbers on the two dice

3. CB ("chocolate bar"):

- random phenomenon: randomly choose a number i in the set $\{1, 2, \dots, n\}$, $n > 1$
- CB returns $\max(\frac{i}{n}, \frac{n-i}{n})$, $i = 1, \dots, n-1$

Probability Distribution

$$X : \Omega \rightarrow V$$

$$X(\Omega) = x_0, x_1, x_2, \dots$$

$$p_i = \text{Prob}(X = x_i) \quad (= \text{Prob}(\omega \in \Omega \mid X(\omega) = x_i))$$

SD2:

$$SD2(\Omega) = \{2, 3, 4, \dots, 12\}$$

$$\text{Prob}(SD2 = 2) = \frac{1}{36}, \text{Prob}(SD2 = 3) = \frac{2}{36}, \text{Prob}(SD2 = 4) = \frac{3}{36}, \dots$$

x_i	2	3	4	5	...
p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{4}{36}$...

Extending the Language: "Function" `random()`

`random(n)` - returns an integer $x \in \{0, 1, \dots, n - 1\}$ uniformly chosen
(i.e. with the probability $\frac{1}{n}$)

Execution time: $O(1)$

test-random.alk:

```
n = random(5);  
print(n);
```

Running test-random.alk

```
$alki.sh -a test-random.alk  
4  
n |-> 4
```

The probability for this execution is: 0.2000000000

Experiment with random()

Executing the algorithm

```
a = [0, 0, 0, 0];  
for (i = 0; i < 10000; ++i) {  
    j = random(4);  
    a[j] = a[j] + 1;  
}
```

we get the following final state:

```
$ alki.sh -a random.alk  
a |-> [2508, 2537, 2486, 2469]  
i |-> 10000  
j |-> 0
```

The probability for this execution is: $0E-10$

The probabilities experimentally computed (e.g. $\frac{2508}{10000}$) are close to the theoretical ones; in fact are approximations of them.

Random Variables as Algorithms

```
D2()  
{  
    d[0] = random(6) + 1;  
    d[1] = random(6) + 1;  
    return d;  
}
```

```
SD2()  
{  
    d1 = random(6) + 1;  
    d2 = random(6) + 1;  
    return d1 + d2;  
}
```

```
CB(n)  
{  
    i = random(n-1) + 1;  
    s1 = float(i) / float(n);  
    s2 = float(n - i) / float(n);  
    if (s1 > s2 ) return s1;  
    return s2;  
}
```

Random Variable: the Expected Value

Consider only discrete random variable X , whose values are real numbers x_1, x_2, \dots

$p_i = \Pr(X = x_i)$ - probability as X to have the value x_i

Expected Value of X : $E(X) = \sum_i x_i \cdot p_i$

Properties:

$$E(X + Y) = E(X) + E(Y)$$

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

(X și Y independente)

Expected value of CB

1 n odd:

– possible values for CB are $\left\{ \frac{k}{n} \mid k = n-1, n-2, \dots, \frac{n+1}{2} \right\}$, each of them with the probability $\frac{2}{n-1}$

$$E(CB) = \sum_{k=\frac{n+1}{2}}^{n-1} \frac{k}{n} \frac{2}{n-1} = \frac{3n-1}{4n} < \frac{3}{4}$$

2 n even:

– possible values for CB are $\left\{ \frac{k}{n} \mid k = n-1, n-2, \dots, \frac{n}{2} + 1 \right\}$, each of them with the probability $\frac{2}{n-1}$, și $\frac{1}{2}$ cu probabilitatea $\frac{1}{n-1}$

$$E(CB) = \sum_{k=\frac{n}{2}+1}^{n-1} \frac{k}{n} \frac{2}{n-1} + \frac{1}{2(n-1)} = \frac{3n-4}{4n-4} < \frac{3}{4}$$

Obs. $\frac{3n-4}{4n-4} = \frac{3(n-1)-1}{4(n-1)}$, which implies that the expected values for n and $n-1$ are the same if n is even.

Conclusion: $E(CB) < \frac{3}{4}$

Approximating the Expected Value by Successive Executions

```
sum = 0.0;
for (j = 0; j < 100; ++j)
    sum = sum + CB(31);

m = sum / float(31);
```

Values obtained using three runs: 0.7341935483870967741943,
0.7416129032258064516137, 0.7606451612903225806456.

Randomized Algorithms: Definition

There are two approaches:

① Monte Carlo Algorithms

- may produce incorrect results with some small probability, but whose execution time is deterministic
- if runned multiple times with independent random choices each time , the failure probability can be made arbitrarily small, at the cost of the running time .

② Las Vegas Algorithms

- never produce incorrect results, but whose execution time may vary from one run to another
- random choices made within the algorithm are used to establish an expected running time for the algorithm that is, essentially, independent of the input

Plan

- 1 Recap
- 2 Non-deterministic Algorithms, Generally
- 3 Non-deterministic Algorithms for Decision Problems
- 4 Randomized Algorithms
 - Random Variable
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Motivation 1/2

Naive algorithm for primality testing:

```
isPrime2(x) {  
  if (x < 2) return false;  
  for (i = 2; i*i <= x; ++i)  
    if (x % i == 0) return false;  
  return true;  
}
```

Motivation 2/2

A first execution:

```
print(isPrime2(2147483647));
```

```
$ time alki.sh -a isPrime.alk  
true
```

```
real 0m1.465s  
user 0m3.385s  
sys 0m0.155s
```

A second execution:

```
print(isPrime2(2305843009213693951));
```

```
$ time alki.sh -a isPrime.alk  
^C  
real 50m14.407s  
user 49m18.198s  
sys 0m20.615s
```

Composite numbers: problem domain 1/3

Legendre Symbol:

$$(a/p) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod{p}, \\ 1 & \text{if } a \not\equiv 0 \pmod{p} \text{ and there is } x \text{ s.t. } a \equiv x^2 \pmod{p}, \\ -1 & \text{if } a \not\equiv 0 \pmod{p} \text{ and there is NO such an } x. \end{cases}$$

where p is prime;

Jacobi Symbol :

$$(a|n) = (a/p_1)^{\alpha_1} (a/p_2)^{\alpha_2} \cdots (a/p_k)^{\alpha_k},$$

where n is a positive integer and $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ its prime factorization.

Composite numbers: problem domain 2/3

- ① If $n > 3$ and prime, the $(a|n) = (a/n)$, therefore often is used the same notation $\frac{a}{n}$.
- ② If $a \equiv b \pmod{n}$ then $(a|n) = (b|n)$.
- ③ $(a|n) = \begin{cases} 0 & \text{if } \gcd(a, n) \neq 1, \\ \pm 1 & \text{if } \gcd(a, n) = 1. \end{cases}$
- ④ $(ab|n) = (a|n)(b|n)$, so $(a^2|n) = 1 \text{ or } 0$
- ⑤ $(a|mn) = (a|m)(a|n)$, so $(a|n^2) = 1 \text{ or } 0$

Composite numbers: problem domain 3/3

- 5 If m and n are coprime, then

$$(m|n)(n|m) = (-1)^{\frac{m-1}{2} \cdot \frac{n-1}{2}} = \begin{cases} 1 & \text{dacă } n \equiv 1 \pmod{4} \text{ or } m \equiv 1 \pmod{4}, \\ -1 & \text{dacă } n \equiv m \equiv 3 \pmod{4} \end{cases}$$

6 $(-1|n) = (-1)^{\frac{n-1}{2}} = \begin{cases} 1 & \text{dacă } n \equiv 1 \pmod{4}, \\ -1 & \text{dacă } n \equiv 3 \pmod{4}, \end{cases}$

7 $(2|n) = (-1)^{\frac{n^2-1}{8}} = \begin{cases} 1 & \text{dacă } n \equiv 1, 7 \pmod{8}, \\ -1 & \text{dacă } n \equiv 3, 5 \pmod{8}. \end{cases}$

Other Properties (inferred)

$$(a \cdot 2|n) = (a|n) \cdot (2|n) = \begin{cases} -(a|n) & \text{if } n \equiv 3, 5 \pmod{8} \\ (n|a) & \text{otherwise} \end{cases}$$

$$(a|n) = \begin{cases} -(n|a) & \text{if } a \neq 0 \neq n \text{ și } n \equiv a \equiv 3 \pmod{4} \\ (n|a) & \text{otherwise} \end{cases}$$

From the problem domain to the algorithm

```
jacobi(a, n)
{
    j = 1;
    while (a != 0) {
        while (a % 2 == 0) { // a is even
            a = a / 2;
            if (n % 8 == 3 || n % 8 == 5) j = 0-j;
        }
        swap(a, n);
        if (a % 4 == 3 && n % 4 == 3) j = 0-j;
        a = a % n;
    }
    if (n == 1) return j;
    else return 0;
}
```

Solovay-Strassen Algorithm: descriptive

Input: a odd positive integer n ,

Output: "composite" if n is composite, "maybe prime" otherwise

- ① randomly choose a in $[2, n - 1]$
- ② $x = (a|n)$
- ③ if $x == 0$ or $a^{(n-1)/2} \not\equiv x \pmod{n}$ then returns "composite"
- ④ otherwise returns "maybe prime"

Solovay-Strassen Algorithm in Alk

```
isComp(n)
{
    a = random(n-3) + 2;
    if (gcd(a, n) != 1) return "composite";
    x = jacobi(a, n);
    if (x < 0) x = x + n;
    if (x != power(a, (n-1)/2, n)) return "composite";
    return "maybe prime";
}
```

Solovay-Strassen Algorithm: demo

```
m1[0] = 2147483647*457241;
```

```
m1[1] = isComp(m1[0]);
```

```
m2[0] = 2147483647;
```

```
m2[1] = isComp(m2[0]);
```

Rezultat:

```
$ time alki.sh -a compos.alk
```

```
m1 |-> [981917570237927, composite]
```

```
m2 |-> [2147483647, may be prime]
```

The probability for this execution is: 0E-10

```
real 0m0.570s
```

```
user 0m1.085s
```

```
sys 0m0.101s
```

Solovay-Strassen Algorithm as a Prime Test with a Certain Probability

Input: a positive odd integer n ,

a positive integer k representing the accuracy

Output: "composite" if n is composite, "probably prime" otherwise

```
isProbPrime(n, k) {  
    while (k > 0 && isComp(n) != "composite")  
        --k;  
    if (k == 0) return "probably prime";  
    return "composite";  
}
```

Failure probability is 2^{-k} .

(A proof can be found in Richard M. Karp. An introduction to randomized algorithms. Discrete Applied Mathematics 34 (1991) 165-201.)

Solovay-Strassen Algorithm as a Prime Test: demo

```
m4[0] = 2305843009213693951;  
m4[1] = isProbPrime(m4[0], 100);
```

Result:

```
$ time alki.sh -a compos.alk  
m4 |-> [2305843009213693951, probably prime]
```

```
real 0m1.192s  
user 0m3.043s  
sys 0m0.161s
```

The test with the naive algorithm:

```
print(isPrime2(2305843009213693951));  
$ time alki.sh -a isPrime.alk  
^C  
real 50m14.407s  
user 49m18.198s  
sys 0m20.615s
```

Solovay-Strassen Algorithm as a Prime Test: demo

```
m3[0] = 170141183460469231731687303715884105727;  
m3[1] = isProbPrime(m3[0], 100);
```

Result:

```
$ time alki.sh -a compos.alk  
m3 |-> [170141183460469231731687303715884105727, probably prime]  
  
real 0m1.851s  
user 0m4.856s  
sys 0m0.262s
```

The test with the naive algorithm: an optimistic estimation for terminating is the end of the semester ...

An Efficient Algorithm for the Power Function Modulo

From the problem domain:

$$a^n \pmod{p} = \begin{cases} 1 & \text{dacă } n = 0, \\ a \pmod{p} & \text{dacă } n = 1, \\ (a^{\frac{n}{2}} \pmod{p}) * (a^{\frac{n}{2}} \pmod{p}) \pmod{p} & \text{dacă } n \% 2 == 0, \\ (a * a^{n-1} \pmod{p}) \pmod{p} & \text{dacă } n \% 2 == 1, \end{cases}$$

to the algorithm:

```
power(a, n, p) {
    x = 1;
    while (n > 0)
        if (n % 2 == 0) {
            a = (a * a) % p ;
            n = n / 2;
        }
        else {
            x = (a * x) % p;
            n = n - 1;
        }
    return x;
}
```

Exercises

1. Find the execution time for `power(a, n, p)`.
2. Find the execution time for `isComp(n)`.
3. Find the execution time for `isProbPrime(n, k)`.

Plan

- 1 Recap
- 2 Non-deterministic Algorithms, Generally
- 3 Non-deterministic Algorithms for Decision Problems
- 4 Randomized Algorithms**
 - Random Variable
 - Example of Monte Carlo Algorithm: Primality Test
 - Las Vegas Algorithms**
 - Example of Las Vegas Algorithm: k -median

The Expected time for the Randomized Algorithms 1/2

- never produce incorrect results, but whose execution time may vary from one run to another
- random choices made within the algorithm are used to establish an expected running time for the algorithm that is, essentially, independent of the input

Notations:

$prob_{A,x}(C)$ = the probability that the algorithm A to execute C for the input x

$time_{A,x}(C)$ = the time that A to execute C for the input x (a bit different from the deterministic case)

The Expected time for the Randomized Algorithms 2/2

the expected time of A for the input x is

$$\text{exp-time}(A, x) = E[\text{time}_{A,x}] = \sum_C \text{prob}_{A,x}(C) \cdot \text{time}_{A,x}(C).$$

$\text{time}_{A,x}$ is a random variable.

the expected time of A for the worst case is

$$\text{exp-time}(A, n) = \max\{\text{exp-time}(A, x) \mid \text{size}(x) = n\}$$

If A is understood from the context, we write only $\text{exp-time}(n)$ ($\text{exp-time}(x)$) instead of $\text{exp-time}(A, n)$ (resp. $\text{exp-time}(A, x)$).

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k -median: the problem

Definition

Let S be a list with n elements from a totally ordered set. The k -median is the k -th element from the sorted list S .

Assume that S is represented by an array.

Consider the next problem:

Input an array $(a[i] \mid 0 \leq i < n)$ and a number $k \in \{0, 1, \dots, n-1\}$,
Output k -median

k -median: description of the algorithm

Select a pivot x in $a[0..n-1]$.

Partition the array a around x : the elements of the array are permuted such that $a[j] = x$ and

$$(\forall i)(i < j \implies a[i] \leq x) \wedge (i > j \implies a[i] \geq x)$$

which is equivalent to:

$$(\forall i)(i < j \implies a[i] \leq a[j]) \wedge (i > j \implies a[i] \geq a[j])$$

- ① $j = k \implies$ the problem is solved
- ② $j < k \implies$ search k in $a[j+1..n-1]$
- ③ $j > k \implies$ search k in $a[0..j-1]$

Partitioning Lomuto

```
partition(out a, p, q)
{
    pivot = a[q];
    i = p - 1;
    for (j = p; j < q; ++j)
        if (a[j] < pivot) {
            i = i + 1;
            swap(a, i, j);
        }
    swap(a, i+1, q);
    return i + 1;
}

swap(out a, i, j) {
    if (i != j) {
        temp = a[i];
        a[i] = a[j];
        a[j] = temp;
    }
}
```

Analysis of the Lomuto Partitioning

Correctness:

The for invariant:

$$i < j \wedge (\forall \ell)(p \leq \ell \leq i \implies a[\ell] \leq pivot) \wedge (i < \ell < j \implies a[\ell] > pivot)$$

After for:

the invariant and $j = q$, which implies $a[i + 1] > pivot$

After the last swap:

$$a[i + 1] = pivot \text{ și }$$

$$(\forall \ell)(p \leq \ell \leq i \implies a[\ell] \leq pivot) \wedge (i < \ell < q \implies a[\ell] > pivot).$$

The number of comparisons: $q - p$

Randomized Partitioning

The pivot is randomly chosen from $a[p..q]$:

```
randPartition(out a, p, q) {  
    if (p < q) {  
        i = p + random(q - p);  
        swap(a, i, q);  
        return partition(a, p, q);  
    }  
}
```

k -median: Las Vegas Algorithm

```
randSelectRec(out a, p, q, k)
{
    j = randPartition(a, p, q);
    if (j == k) return a[j];
    if (j < k) return randSelectRec(a, j+1, q, k);
    return randSelectRec(a, p, j-1, k);
}

randSelect(out a, k)
{
    return randSelectRec(a, 0, a.size()-1, k);
}
```

randSelect: analysis 1/3

exp-time(n, k) - the expected time to find the k -median in an array of length n

$$\text{exp-time}(n) = \max_k \text{exp-time}(n, k)$$

Since we are interested in the worst case analysis, we assume that the recursive call chooses always the longest subarray.

$$\text{Recall that } E(CB) < \frac{3}{4}$$

It follows that the expected length of the longest subarray is at most $\frac{3}{4}n$.

randSelect: analysis 2/3

Lemă

The expected length of the array after i call is at most $\left(\frac{3}{4}\right)^i n$.

Proof

L_i the random variable that returns the length of the array after i calls.

P_j the random variable that returns the fraction of the elements preserved at the level j

X_j the random variable that returns the length of the longest subarray at the level j

We have: $L_i = n \prod_{j=1}^i P_j$, $P_j = \frac{X_j}{n}$, $E(P_j) = E\left(\frac{X_j}{n}\right) = \frac{E(X_j)}{n} \leq \frac{\frac{3}{4}n}{n} = \frac{3}{4}$,

P_1, \dots, P_n are independent,

$$E(L_i) = E\left(n \prod_{j=1}^i P_j\right) = n \prod_{j=1}^i E(P_j) \leq \left(\frac{3}{4}\right)^i n$$

Now the lemma is proved.

randSelect: analysis 3/3

At the level i , the number of operations is linear, let say $\leq aX_i + b$ (recall that X_i is the length of the longest subarray).

Let $r \leq n$ be the number of recursive calls. The expected time is:

$$\begin{aligned}
 \text{exp-time}(n) &= E \left(\sum_{i=1}^r (aX_i + b) \right) \\
 &= \sum_{i=1}^r E(aX_i + b) \\
 &\leq \sum_{i=1}^n (aE(X_i) + b) \\
 &\leq \sum_{i=1}^n \left(a \left(\frac{3}{4} \right)^i n + b \right) \\
 &= an \sum_{i=1}^n \left(\frac{3}{4} \right)^i + bn \\
 &\leq 3an + bn \\
 &= O(n)
 \end{aligned}$$