Logic(s) for Computer Science - Week 11
Natural Deduction in First-Order Logic

1 Introduction

In the previous lecture, we discussed some notions regarding the *semantics* of first order logic:

1. the value of a term in an assignment;
2. the truth value of a formula in a structure and an assignment;
3. the satisfiability;
4. validity;
5. equivalence;
6. semantical consequence.

We saw, for instance, that for establishing that two formulae are equivalent, we need to reason semantically (that is, a reasoning that uses the semantic notions of truth value, structure, etc.).

In this lecture, we present natural deduction for first order logic. We present the notion of substitution, we recall some notions that are specific for natural deductions (that we previously discussed in the case of propositional logic) and we present the extended deductive system with the corresponding properties.

2 Substitutions

Definition 2.1. *A* substitution *is a function $\sigma : \mathcal{X} \to \mathcal{T}$, with the property that $\sigma(x) \neq x$ for a finite number of variables $x \in \mathcal{X}$.*

Definition 2.2. *If $\sigma : \mathcal{X} \to \mathcal{T}$ is a substitution, the set $dom(\sigma) = \{x \in \mathcal{X} \mid \sigma(x) \neq x\}$ is the* domain of the substitution *σ.*

Remark 2.1. *By definition, the domain of a substitution is a finite set.*

Definition 2.3. *If $\sigma : \mathcal{X} \to \mathcal{T}$ is a substitution, then the unique* extension *of the substitution σ to the set of terms is the function $\sigma^{\sharp} : \mathcal{T} \to \mathcal{T}$, recursively defined as follows:*

1. $\sigma^\sharp(x) = \sigma(x)$, *for any* $x \in \mathcal{X}$;

2. $\sigma^\sharp(c) = c$, *for any constant symbol* $c \in \mathcal{F}_0$;

3. $\sigma^\sharp(f(t_1, \ldots, t_n)) = f(\sigma^\sharp(t_1), \ldots, \sigma^\sharp(t_n))$, *for any functional symbol* $f \in \mathcal{F}_n$ *of arity* $n \in \mathbb{N}*$ *and any terms* $t_1, \ldots, t_n \in \mathcal{T}$.

The substitutions are noted by $\sigma, \tau, \sigma_0, \tau_1, \sigma'$, etc.

Remark 2.2. *If* $t \in \mathcal{T}$ *is a term, then* $\sigma^\sharp(t) \in \mathcal{T}$ *is the term obtained from* t *by* applying *to the substitution* σ *or the term obtained by* applying the substitution σ on the term t.

Practically, in order to obtain $\sigma^\sharp(t)$ from t, all occurences of a variable x in t are replaced simultaneously with the corresponding term $\sigma(x)$.

Example 2.1. *Let consider the substitution* $\sigma_1 : \mathcal{X} \to \mathcal{T}$ *defined as follows:*

1. $\sigma_1(x_1) = x_2$;

2. $\sigma_1(x_2) = f(x_3, x_4)$;

3. $\sigma_1(x) = x$ *for any* $x \in \mathcal{X} \setminus \{x_1, x_2\}$.

Let consider the term $t = f(f(x_1, x_2), f(x_3, e))$. *We have that:*

$$\begin{aligned} \sigma_1^\sharp\Big(t\Big) &= \sigma_1^\sharp\Big(f(f(x_1,x_2), f(x_3,e))\Big) \\ &= f(\sigma_1^\sharp\Big(f(x_1,x_2)\Big), \sigma_1^\sharp\Big(f(x_3,e)\Big)) \\ &= f(f(\sigma_1^\sharp\Big(x_1\Big), \sigma_1^\sharp\Big(x_2\Big)), f(\sigma_1^\sharp\Big(x_3\Big), \sigma_1^\sharp\Big(e\Big))) \\ &= f(f(\sigma_1\Big(x_1\Big), \sigma_1\Big(x_2\Big)), f(\sigma_1\Big(x_3\Big), e)) \\ &= f(f(x_2, f(x_3,x_4)), f(x_3,e)). \end{aligned}$$

Note that by applying a substitution over a term, we replace (in the same time) all occurrences of the variables from the domain of the substitution with the corresponding terms.

Notation 2.1. *If* $dom(\sigma) = \{x_1, \ldots, x_n\}$, *then the substitution* σ *cam be also written as:*

$$\sigma = \{x_1 \mapsto \sigma(x_1), \ldots, x_n \mapsto \sigma(x_n)\}.$$

Attention, it is not a set, but only some notation for substitutions.

Example 2.2. *For the substitution from the previous example, we have*

$$\sigma_1 = \{x_1 \mapsto x_2, x_2 \mapsto f(x_3, x_4)\}.$$

Definition 2.4. *If $\sigma : \mathcal{X} \to \mathcal{T}$ is a substitution and $V \subseteq \mathcal{X}$ is a subset of variables, then* the restriction of the substitution σ to the set V is another substitution $\sigma|_V : \mathcal{X} \to \mathcal{T}$*, defined as follows:*

1. $\sigma|_V(x) = \sigma(x)$ *for any* $x \in V$*;*

2. $\sigma|_V(x) = x$ *for any* $x \in \mathcal{X} \setminus V$*.*

Example 2.3. $\sigma_1|_{\{x_1\}} = \{x_1 \mapsto x_2\}$.

In other words, by restricting a substitution to a set of variables, we remove some the other variables from the domain of the substitution.

Definition 2.5. *for any substitution $\sigma : \mathcal{X} \to \mathcal{T}$,* the extention *of σ to the set of formulae is the function $\sigma^\flat :$ `LP1` $\to$ `LP1`, defined as:*

1. $\sigma^\flat(P(t_1, \ldots, t_n)) = P(\sigma^\sharp(t_1), \ldots, \sigma^\sharp(t_n))$*;*

2. $\sigma^\flat(\neg\varphi) = \neg\sigma^\flat(\varphi)$*;*

3. $\sigma^\flat(\varphi_1 \wedge \varphi_2) = \sigma^\flat(\varphi_1) \wedge \sigma^\flat(\varphi_2)$*;*

4. $\sigma^\flat(\varphi_1 \vee \varphi_2) = \sigma^\flat(\varphi_1) \vee \sigma^\flat(\varphi_2)$*;*

5. $\sigma^\flat(\varphi_1 \rightarrow \varphi_2) = \sigma^\flat(\varphi_1) \rightarrow \sigma^\flat(\varphi_2)$*;*

6. $\sigma^\flat(\varphi_1 \leftrightarrow \varphi_2) = \sigma^\flat(\varphi_1) \leftrightarrow \sigma^\flat(\varphi_2)$*;*

7. $\sigma^\flat(\forall x.\varphi) = \forall x.(\rho^\flat(\varphi))$*, where* $\rho = \sigma|_{dom(\sigma)\setminus\{x\}}$*;*

8. $\sigma^\flat(\exists x.\varphi) = \exists x.(\rho^\flat(\varphi))$*, where* $\rho = \sigma|_{dom(\sigma)\setminus\{x\}}$*.*

In other words, to obtain the formula $\sigma^\flat(\varphi)$ from φ, every free occurrence of the variable x from the formula φ is replaced by the term $\sigma(x)$.

Example 2.4.

$$
\begin{array}{l}
\sigma_1^\flat\Big(\big(\forall x_2.P(x_1, x_2)\big) \wedge P(x_2, x_2)\Big) \equiv \\
\sigma_1^\flat\Big(\big(\forall x_2.P(x_1, x_2)\big)\Big) \wedge \sigma_1^\flat\Big(P(x_2, x_2)\Big) \equiv \\
\big(\forall x_2.\sigma_1|^\flat_{\{x_1\}}\Big(P(x_1, x_2)\Big)\big) \wedge P(\sigma_1^\sharp\Big(x_2\Big), \sigma_1^\sharp\Big(x_2\Big)) \equiv \\
\big(\forall x_2.P(\sigma_1|^\sharp_{\{x_1\}}\Big(x_1\Big), \sigma_1|^\sharp_{\{x_1\}}\Big(x_2\Big))\big) \wedge P(\sigma_1\Big(x_2\Big), \sigma_1\Big(x_2\Big)) \equiv \\
\big(\forall x_2.P(\sigma_1|_{\{x_1\}}\Big(x_1\Big), \sigma_1|_{\{x_1\}}\Big(x_2\Big))\big) \wedge P(f(x_3, x_4), f(x_3, x_4)) \equiv \\
\big(\forall x_2.P(\sigma_1\Big(x_1\Big), x_2)\big) \wedge P(f(x_3, x_4), f(x_3, x_4)) \equiv \\
\big(\forall x_2.P(x_2, x_2)\big) \wedge P(f(x_3, x_4), f(x_3, x_4)).
\end{array}
$$

Remark 2.3. *Attention: the bound occurrences of variables are NOT replaced when applying the substitution!*

Notation 2.2. *According to Notation 2.1, for the substitutions with a finite domain, we also use the notation* $\{x_1 \mapsto \sigma(x_1), \ldots, x_n \mapsto \sigma(x_n)\}$. *We will also use substitutions without associating a name to them, since they are simple having the form :* $\{x \mapsto t\}$. *In order to express the fact that we apply this substitution to a formula, according to our notation, we should write* $\{x \mapsto t\}(\varphi)$. *However, in the literature other notations are preferred that we will also use. One variant is to write* $\varphi[t/x]$. *Another one is* $\varphi[x \mapsto t]$. *In this document we will use the last notation.*

3 Sequences

Definition 3.1 (Sequence)**.** *A* sequence *is a pair formed by a set of formulae* $\{\varphi_1, \ldots, \varphi_n\} \subseteq \texttt{LP1}$ *and a formula* $\varphi \in \texttt{LP1}$*, denoted by:*

$$\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi.$$

sometimes we read the notation $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$ as φ *is a syntactic consequence from* $\{\varphi_1, \ldots, \varphi_n\}$. Usually, we will note with $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ the set of hypothesis and we will write the sequence as $\Gamma \vdash \varphi$.

Remark 3.1. *We recall that the usual notation in the literature allows us to write* $\varphi_1, \ldots, \varphi_n \vdash \varphi$ *(without curly brackets) instead of* $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$. *However, we have to remember that on the left side of the symbol* $\vdash$ *is all the time a set. The notation without brackets allows us to write* $\varphi_1, \ldots, \varphi_n, \psi \vdash \varphi$ *instead of* $\{\varphi_1, \ldots, \varphi_n\} \cup \{\psi\} \vdash \varphi$.

Example 3.1. *In many examples from this material, we work with the signature* $\Sigma = (\{P, Q\}, \{a, b, f, g\})$*, where the predicates* P *and* Q *have arity 1, the functional symbols* f *and* g *have arity 1, and the symbols* a *and* b *are constants (arity 0).*

Example 3.2. *Let consider the signature* Σ *from Example 3.1. The following are some examples of sequences:*

1. $\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash \mathtt{P(a)} \land \mathtt{Q(a)}$;
2. $\{\forall\mathtt{x}.\mathtt{Q(x)}, \mathtt{P(a)}\} \vdash \mathtt{P(a)} \land \mathtt{Q(a)}$;
3. $\{\exists\mathtt{x}.\mathtt{Q(x)}\} \vdash \mathtt{Q(a)}$.

Later we will see that the first two sequences from above are valid,*and the last one is not valid.*

4 Inference rules

Definition 4.1. *An* inference rule *is a tuple formed from:*

1. *a set of sequences* $S_1, \ldots, S_n$*, called* hypothesis *of the rule;*

2. *a sequence S called* conclusion *of the rule;*

3. *a* condition *for the applicability of the rule;*

4. *a name.*

An inference rule is noted as follows:

$$\textsc{name}\ \frac{S_1 \qquad \ldots \qquad S_n}{S}\ \textit{condition.}$$

Remark 4.1. *The inference rules that have $n = 0$ hypothesis, are called axioms. Also, the applicability conditions may be absent.*

Example 4.1. *The following are some inference rules from propositional logic:*

$$\wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,} \qquad \wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi'.}$$

As in the case of propositional logic, all three inference rules from above are correct*. None of them has a applicability condition. The following is an example of inference rule with $n = 0$ hypothesis, but with one condition.*

$$\textsc{Hypothesis}\ \frac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma.$$

Below we have an example of incorrect inference rule (in a way that we will clarify later, but that can be already perceived).

$$\textsc{incorrect rule}\ \frac{\Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi').}$$

Remark 4.2. *The hypothesis of the inference rule, as the conclusion, are in fact* patterns *for sequences and not the sequences themselves. These patterns can be instantiated, meaning that an inference rule (presented above) has several* instances *obtained by replacing mathematical variables $\varphi, \varphi', \Gamma$ with concrete formulae. For instance, here is a new instance for the rule $\wedge i$ from above:*

$$\wedge i\ \frac{\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash \mathtt{P(a)} \qquad \{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash \mathtt{Q(a)}}{\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash (\mathtt{P(a)} \wedge \mathtt{Q(a)});}$$

$$\wedge i\ \frac{\{\mathtt{P(a)}, \mathtt{Q(a)}, \mathtt{Q(b)}\} \vdash (\mathtt{P(a)} \wedge \mathtt{Q(a)}) \qquad \{\mathtt{P(a)}, \mathtt{Q(a)}, \mathtt{Q(b)}\} \vdash \mathtt{P(a)}}{\{\mathtt{P(a)}, \mathtt{Q(a)}, \mathtt{Q(b)}\} \vdash ((\mathtt{P(a)} \wedge \mathtt{Q(a)}) \wedge \mathtt{P(a)}).}$$

We first replaced the mathematical variable Γ with the set of formulae $\{\texttt{P(a)}, \texttt{Q(a)}\}$, the mathematical variable φ with the formula $\texttt{P(a)}$ and the mathematical variable φ' with the formula $\texttt{Q(a)}$. Exercise: establish with what was replaced each mathematical variable from the second instance.

Here is an example of rule that is not an instance of a rule $\wedge i$ (exercise: explain why not):

$$? \ \frac{\{\texttt{P(a)}, \texttt{Q(a)}\} \vdash \texttt{P(a)} \qquad \{\texttt{P(a)}, \texttt{Q(a)}\} \vdash \texttt{Q(a)}}{\{\texttt{P(a)}, \texttt{Q(a)}\} \vdash (\texttt{P(a)} \wedge \texttt{P(a)});}$$

5 Deductive system

Definition 5.1. *A* deductive system *is a set of inference rules.*

Example 5.1. *Let consider the deductive system D_1, formed from the following four inference rules:*

$$\textsc{Hypothesis}\ \frac{}{\Gamma \vdash \varphi,}\ \varphi \in \Gamma \qquad \wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,}$$

$$\wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi'.}$$

6 Formal proof

Definition 6.1 (Formal proof)**.** *A* formal proof *in a deductive system is a list of sequences*

1. S_1
2. S_2

...

n. S_n,

with the property that each sequence S_i is justified by a inference rule of the deductive system applied on the previous sequences ($S_1, \ldots, S_{i-1}$), meaning that S_i is the conclusion of an instance of an inference rule from the deductive rule, rule that uses as hypothesis sequences chosen from $S_1, \ldots, S_{i-1}$. In addition, if the inference rule has a condition, this condition has to be true. Note also that any prefix of a formal proof is also a proof.

Example 6.1. *Here is an example of formal proof in the deductive system D_1 from above:*

1. $\{\texttt{P(a)}, \texttt{Q(a)}\} \vdash \texttt{P(a)}$; *(*Hypothesis*)*

2. $\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash \mathtt{Q(a)}$; *(*HYPOTHESIS*)*

3. $\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash (\mathtt{P(a)} \wedge \mathtt{Q(a)})$; *(*$\wedge i$, 1, 2*)*

4. $\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash (\mathtt{Q(a)} \wedge (\mathtt{P(a)} \wedge \mathtt{Q(a)}))$. *(*$\wedge i$, 2, 3*)*

As in the case of propositional logic, each line has the name of the applied inference rule and the lines where the needed hypothesis are found (in the same order used to present the deductive system).

Remark 6.1. *The definition of the formal proof in the first order logic is the same as in the case of propositional logic. However, we will see later that in order to apply the new inference rules, associated to the quantifiers, we will use new annotations for the lines of the formal proof.*

Definition 6.2 (Valid sequence)**.** *A sequence* $\Gamma \vdash \varphi$ *is* valid in a deductive system D *if there is a formal proof* $S_1, \ldots, S_n$ *in* D *such that* $S_n = \Gamma \vdash \varphi$.

Example 6.2. *The sequence* $\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash (\mathtt{P(a)} \wedge \mathtt{Q(a)})$ *is valid in the deductive system* D_1 *from above because is the last sequence from the following formal proof:*

1. $\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash \mathtt{P(a)}$; *(*HYPOTHESIS*)*

2. $\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash \mathtt{Q(a)}$; *(*HYPOTHESIS*)*

3. $\{\mathtt{P(a)}, \mathtt{Q(a)}\} \vdash (\mathtt{P(a)} \wedge \mathtt{Q(a)})$. *(*$\wedge i$, 1, 2*)*

Remark 6.2. *Attention! Do not mix the notions of* valid sequence in a deductive system *and the notion of* valid formula*.*

7 Natural deduction

Natural deduction is a deductive system for the first order logic. In other words, the deductive system for first order logic includes all the rules of natural deduction from propositional logic. In addition, for first order logic we have new rules for the introduction and elimination of quantifiers. In this section we will present each inference rule from natural deduction of first order logic.

7.1 Rules for conjunctions

We already saw the inference rules for the introduction and elimination for the "and" connector:

$$\wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,} \qquad \wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi'.}$$

This deductive system is called *natural* because the inference rules mimic the reasoning of humans, based in essence on an intuitive semantics for the notion of truth:

1. The rule for the introduction of the connector $\wedge$ indicates that we can prove a conjunction $\varphi \wedge \varphi'$ from the set of hypothesis Γ if we already know that each part of the conjunction, φ and respectively φ', are consequences of the hypotheses from Γ.

   In other words, in order to prove a conjunction from a set of hypotheses, is enough to establish individually that each part of the conjunction is a consequence of the hypothesis.

2. For the $\wedge$ connector, we have two rules for elimination. First elimination rule for the $\wedge$ connector says that if we already established that some conjunction $(\varphi \wedge \varphi')$ is the consequence of a set Γ of hypotheses, then the left side of the conjunction, φ, is a consequence of the set Γ.

   The second rule is symmetric with respect to the first and says that we can conclude that the right side of the conjunction is the consequence of a set of formulae if the conjunction is is the consequence of this set of formulae.

Here is an example of formal proof that uses the inference rules for the connector $\wedge$:

1. $\{(\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash (\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a}))$; (HYPOTHESIS)
2. $\{(\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \forall \mathtt{x}.\mathtt{P}(\mathtt{x})$; (HYPOTHESIS)
3. $\{(\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \mathtt{P}(\mathtt{a})$; ($\wedge e_1$, 1)
4. $\{(\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})), \forall \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash (\mathtt{P}(\mathtt{a}) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x}))$. ($\wedge i$, 3, 2)

Exercises:

1. $\{((\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x}))\} \vdash (\mathtt{Q}(\mathtt{a}) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x}))$;
2. $\{((\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x})), \forall \mathtt{x}.\mathtt{Q}(\mathtt{x})\} \vdash (\forall \mathtt{x}.\mathtt{Q}(\mathtt{x}) \wedge \mathtt{Q}(\mathtt{a}))$;
3. $\{((\mathtt{P}(\mathtt{a}) \wedge \mathtt{Q}(\mathtt{a})) \wedge \forall \mathtt{x}.\mathtt{P}(\mathtt{x}))\} \vdash (\forall \mathtt{x}.\mathtt{P}(\mathtt{x}) \wedge (\mathtt{Q}(\mathtt{a}) \wedge \mathtt{P}(\mathtt{a})))$.

7.2 Rules for implication

The rule for the elimination of the implication, also called *modus ponens* in latin, is one of the most important rules of inference that we apply.

$$\to e\ \frac{\Gamma \vdash (\varphi \to \varphi') \qquad \Gamma \vdash \varphi}{\Gamma \vdash \varphi'}$$

The rule shows that, supposing that we proved $\varphi \to \varphi'$ (from Γ) and in addition we proved that φ (also from Γ), then we can prove φ' (from Γ).

Here is an example of formal proof that uses the rule for the elimination of the implication:

1. $\{(\mathtt{P(a)} \to \forall\mathtt{x.P(x)}), (\mathtt{P(a)} \land \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \land \mathtt{Q(a)})$; (HYPOTHESIS)
2. $\{(\mathtt{P(a)} \to \forall\mathtt{x.P(x)}), (\mathtt{P(a)} \land \mathtt{Q(a)})\} \vdash \mathtt{P(a)}$; ($\land e_1$, 1)
3. $\{(\mathtt{P(a)} \to \forall\mathtt{x.P(x)}), (\mathtt{P(a)} \land \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \to \forall\mathtt{x.P(x)})$; (HYPOTHESIS)
4. $\{(\mathtt{P(a)} \to \forall\mathtt{x.P(x)}), (\mathtt{P(a)} \land \mathtt{Q(a)})\} \vdash \forall\mathtt{x.P(x)}$. ($\to e$, 3, 1)

This proof shows that the sequence $\{(\mathtt{P(a)} \to \forall\mathtt{x.P(x)}), (\mathtt{P(a)} \land \mathtt{Q(a)})\} \vdash \forall\mathtt{x.P(x)}$ is valid, meaning that the formula $\forall\mathtt{x.P(x)}$ is a consequence of the set of formulae $\{(\mathtt{P(a)} \to \forall\mathtt{x.P(x)}), (\mathtt{P(a)} \land \mathtt{Q(a)})\}$. Observe the order in which the lines 3 and 1 appear in the explanation for line 4: they follow the same order, fixed by the inference rule.

Exercise 7.1. *Prove that the following sequences are valid:*

1. $\{((\mathtt{P(a)} \land \mathtt{Q(a)}) \to \forall\mathtt{x.P(x)}), \mathtt{P(a)}, \mathtt{Q(a)}\} \vdash \forall\mathtt{x.P(x)}$;
2. $\{(\mathtt{P(a)} \to \forall\mathtt{x.P(x)}), \mathtt{P(a)}, \mathtt{Q(a)}\} \vdash (\mathtt{Q(a)} \land \forall\mathtt{x.P(x)})$.

The rule for introducing the implication is subtle. In order to prove that an implication $(\varphi \to \varphi')$ follows from Γ, *we suppose* φ (besides Γ) and prove φ'. In other words, in the hypothesis for the rule, we add the formula φ to the formulae from Γ. The rule may be written in two equivalent ways, that differ only by the fact that the first rule uses the convention referring to the curly brackets around premises from the notations of sequences, while in the second one the brackets appear explicitly:

$$\to i\ \frac{\Gamma, \varphi \vdash \varphi'}{\Gamma \vdash (\varphi \to \varphi'),} \qquad \to i\ \frac{\Gamma \cup \{\varphi\} \vdash \varphi'}{\Gamma \vdash (\varphi \to \varphi').}$$

It is important to observe and understand for rule of introduction of implication is that the set of premises changes. In the conclusion we have that the formula $(\varphi \to \varphi')$ follows from Γ, while in the hypothesis we have to prove that φ' follows from the premises $\Gamma \cup \{\varphi\}$. In other words, intuitively, in order to prove an implication $(\varphi \to \varphi')$, we suppose the antecedent φ and prove φ'.

Example 7.1. *Let's prove that the sequence* $\{\} \vdash (\mathtt{P(a)} \to \mathtt{P(a)})$ *is valid:*

1. $\{\mathtt{P(a)}\} \vdash \mathtt{P(a)}$; *(*HYPOTHESIS*)*
2. $\{\} \vdash (\mathtt{P(a)} \to \mathtt{P(a)})$. *(*$\to i$, 1*)*

Example 7.2. *Let's prove that the sequence* $\{(\mathtt{P(a)} \to \mathtt{Q(a)}), (\mathtt{Q(a)} \to \mathtt{P(b)})\} \vdash (\mathtt{P(a)} \to \mathtt{P(b)})$ *is valid:*

1. $\{(\mathtt{P(a)} \to \mathtt{Q(a)}), (\mathtt{Q(a)} \to \mathtt{P(b)}), \mathtt{P(a)}\} \vdash (\mathtt{P(a)} \to \mathtt{Q(a)})$; *(*HYPOTHESIS*)*
2. $\{(\mathtt{P(a)} \to \mathtt{Q(a)}), (\mathtt{Q(a)} \to \mathtt{P(b)}), \mathtt{P(a)}\} \vdash \mathtt{P(a)}$; *(*HYPOTHESIS*)*

3. $\{(\mathtt{P(a)} \to \mathtt{Q(a)}), (\mathtt{Q(a)} \to \mathtt{P(b)}), \mathtt{P(a)}\} \vdash \mathtt{Q(a)}$; *($\to e$, 1, 2)*
4. $\{(\mathtt{P(a)} \to \mathtt{Q(a)}), (\mathtt{Q(a)} \to \mathtt{P(b)}), \mathtt{P(a)}\} \vdash (\mathtt{Q(a)} \to \mathtt{P(b)})$; *(Hypothesis)*
5. $\{(\mathtt{P(a)} \to \mathtt{Q(a)}), (\mathtt{Q(a)} \to \mathtt{P(b)}), \mathtt{P(a)}\} \vdash \mathtt{P(b)}$; *($\to e$, 4, 3)*
6. $\{(\mathtt{P(a)} \to \mathtt{Q(a)}), (\mathtt{Q(a)} \to \mathtt{P(b)})\} \vdash (\mathtt{P(a)} \to \mathtt{P(b)})$. *($\to i$, 5)*

Exercise 7.2. *Prove that the following sequences are valid:*

1. $\{((\mathtt{P(a)} \wedge \mathtt{Q(a)}) \to \mathtt{P(b)}), \mathtt{P(a)}, \mathtt{Q(a)}\} \vdash \mathtt{P(b)}$;
2. $\{((\mathtt{P(a)} \wedge \mathtt{Q(a)}) \to \mathtt{P(b)})\} \vdash (\mathtt{P(a)} \to (\mathtt{Q(a)} \to \mathtt{P(b)}))$;
3. $\{(\mathtt{P(a)} \to (\mathtt{Q(a)} \to \mathtt{P(b)}))\} \vdash ((\mathtt{P(a)} \wedge \mathtt{Q(a)}) \to \mathtt{P(b)})$.

7.3 Rules for disjunction

The connector $\vee$ has two introduction rules:

$$\vee i_1 \; \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2),} \qquad\qquad \vee i_2 \; \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2).}$$

The first rule shows that if we know φ_1 (from Γ), then we also know $(\varphi_1 \vee \varphi_2)$ (from Γ), no matter what φ_2 is. The second rule of elimination is symmetric, for the right side of the disjunction.

Example 7.3. *Let's prove that* $\{(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \vee \mathtt{Q(a)})$ *is valid:*

1. $\{(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \wedge \mathtt{Q(a)})$; *(Hypothesis)*
2. $\{(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash \mathtt{P(a)}$; *($\wedge e_1$, 1)*
3. $\{(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \vee \mathtt{Q(a)})$. *($\vee i_1$, 2)*

Another formal proof of the sequence is the following:

1. $\{(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \wedge \mathtt{Q(a)})$; *(Hypothesis)*
2. $\{(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash \mathtt{Q(a)}$; *($\wedge e_2$, 1)*
3. $\{(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \vee \mathtt{Q(a)})$. *($\vee i_2$, 2)*

Exercise 7.3. *Prove that* $\{(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash (\mathtt{P(b)} \vee \mathtt{P(a)})$ *is valid.*

The proof for the elimination of disjunction is a little mor complicated, being another rule in which the set of premises of the sequences changes from hypotheses to conclusion:

$$\vee e \; \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \qquad \Gamma, \varphi_1 \vdash \varphi' \qquad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi'}$$

The first hypothesis of the rule, $\Gamma \vdash (\varphi_1 \vee \varphi_2)$, is easy to understand: in order to "remove" a disjunction, we need a disjunction between the hypotheses (disjunction that we want to "eliminate"). The last two hypothesis of the elimination rule of disjunction has to be understood intuitively as follows. From the first hypothesis we know $(\varphi_1 \vee \varphi_2)$ (from Γ); in other words, at least one of the formulae φ_1 and respectively φ_2 follows from Γ. The hypotheses 2 and 3 indicates that, no matter which of the formulae φ_1 or φ_2 holds, in any case φ' holds. That is, if we suppose φ_1 (besides Γ), φ' holds, and if we suppose φ_2 (besides Γ), φ' still holds. And therefore the conclusion indicates that φ' holds no matter which one of the formulae φ_1 and respectively φ_2 would hold.

Example 7.4. *Let us prove that the sequence* $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$ *is valid:*

1. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{P(a)}\} \vdash \mathtt{P(a)}$; *(*Hypothesis*)*
2. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{P(a)}\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$; *(*$\vee i_2$, 1*)*
3. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{Q(a)}\} \vdash \mathtt{Q(a)}$; *(*Hypothesis*)*
4. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{Q(a)}\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$; *(*$\vee i_1$, 1*)*
5. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \vee \mathtt{Q(a)})$; *(*Hypothesis*)*
6. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$. *(*$\vee e$, 5, 2, 4*)*

Note the way in which the set of premises changes from a sequence to the other in the formal proof, following the inference rules.

Exercise 7.4. *Prove that the sequence* $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), (\mathtt{P(a)} \rightarrow \mathtt{P(b)}), (\mathtt{Q(a)} \rightarrow \mathtt{P(b)})\} \vdash \mathtt{P(b)}$ *is valid.*

Exercise 7.5. *Prove that the following sequence is valid:*

$$\{(\mathtt{P(a)} \rightarrow \mathtt{P(b)}), (\mathtt{Q(a)} \rightarrow \mathtt{P(b)})\} \vdash ((\mathtt{P(a)} \vee \mathtt{Q(a)}) \rightarrow \mathtt{P(b)}).$$

7.4 Proofs for negation

The rules for the introduction end elimination of the negation are presented together with a rule for the elimination of $\perp$:

$$\neg i\ \frac{\Gamma, \varphi \vdash \perp}{\Gamma \vdash \neg\varphi} \qquad \neg e\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \perp} \qquad \perp e\ \frac{\Gamma \vdash \perp}{\Gamma \vdash \varphi}$$

Let us recall that $\perp$ is a logical connector of arity 0. In other words, the connector $\perp$ does not change in the formula, The semantics of the formula $\perp$ is such that it is false in any structure and any assignment. In any words, $\perp$ is a contradiction.

The first rule , for the introduction of negation, is easy to explayn intuitively: how can we prove that a formula of the form $\neg\varphi$ follows from Γ? We suppose, in addition besides Γ, that we have φ and prove that from Γ and φ follows a contradiction ($\Gamma, \varphi \vdash \bot$). In this way, we prove that $\neg\varphi$ follows from Γ.

The second rule, for the elimination of negation, indicates that if a formula φ, as well as its negation, $\neg\varphi$, follow from the same set of premises Γ, then, from Γ also follow a contradiction, $\bot$. A set Γ from which follow a contradiction is called *inconsistent*.

The third rule indicates that, if Γ is an inconsistent set of formulae, then any formula φ follow from Γ.

There is no rule for the introduction of $\bot$ (or, the rule for the elimination of negation can be considered also as being the rule for the introduction of $\bot$).

Example 7.5. *Let us prove that the sequence* $\{\mathtt{P(a)}\} \vdash \neg\neg\mathtt{P(a)}$ *is valid:*

1. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(a)}$; (Hypothesis)
2. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \neg\mathtt{P(a)}$; (Hypothesis)
3. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \bot$; ($\neg e$, 1, 2)
4. $\{\mathtt{P(a)}\} \vdash \neg\neg\mathtt{P(a)}$. ($\neg i$, 3)

Example 7.6. *Let us prove that the sequence* $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(b)}$ *is valid:*

1. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(a)}$; (Hypothesis)
2. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \neg\mathtt{P(a)}$; (Hypothesis)
3. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \bot$; ($\neg e$, 1, 2)
4. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(b)}$. ($\bot e$, 3)

Exercise 7.6. *Prove that the following sequences are valid::*

1. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash \neg(\neg\mathtt{P(a)} \wedge \neg\mathtt{Q(a)})$;
2. $\{(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash \neg(\neg\mathtt{P(a)} \vee \neg\mathtt{Q(a)})$;
3. $\{(\neg\mathtt{P(a)} \vee \neg\mathtt{Q(a)})\} \vdash \neg(\mathtt{P(a)} \wedge \mathtt{Q(a)})$;
4. $\{(\neg\mathtt{P(a)} \wedge \neg\mathtt{Q(a)})\} \vdash \neg(\mathtt{P(a)} \vee \mathtt{Q(a)})$;
5. $\{\neg(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\neg\mathtt{P(a)} \wedge \neg\mathtt{Q(a)})$.

7.4.1 Elimination of double negation

In the case of propositional logic, we also saw the following rule for the elimination of the double negation:

$$\neg\neg e\ \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}$$

Example 7.7. *Let us prove that the sequence* $\{(\neg\mathtt{P(a)} \rightarrow \mathtt{Q(a)}), \neg\mathtt{Q(a)}\} \vdash \mathtt{P(a)}$ *is valid:*

1. $\{(\neg\mathtt{P(a)} \rightarrow \mathtt{Q(a)}), \neg\mathtt{Q(a)}, \neg\mathtt{P(a)}\} \vdash \neg\mathtt{P(a)}$; (Hypothesis)
2. $\{(\neg\mathtt{P(a)} \rightarrow \mathtt{Q(a)}), \neg\mathtt{Q(a)}, \neg\mathtt{P(a)}\} \vdash (\neg\mathtt{P(a)} \rightarrow \mathtt{Q(a)})$; (Hypothesis)
3. $\{(\neg\mathtt{P(a)} \rightarrow \mathtt{Q(a)}), \neg\mathtt{Q(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{Q(a)}$; ($\rightarrow e$, 2, 1)
4. $\{(\neg\mathtt{P(a)} \rightarrow \mathtt{Q(a)}), \neg\mathtt{Q(a)}, \neg\mathtt{P(a)}\} \vdash \neg\mathtt{Q(a)}$; (Hypothesis)
5. $\{(\neg\mathtt{P(a)} \rightarrow \mathtt{Q(a)}), \neg\mathtt{Q(a)}, \neg\mathtt{P(a)}\} \vdash \bot$; ($\neg i$, 4, 3)
6. $\{(\neg\mathtt{P(a)} \rightarrow \mathtt{Q(a)}), \neg\mathtt{Q(a)}\} \vdash \neg\neg\mathtt{P(a)}$; ($\neg i$, 5)
7. $\{(\neg\mathtt{P(a)} \rightarrow \mathtt{Q(a)}), \neg\mathtt{Q(a)}\} \vdash \mathtt{P(a)}$. ($\neg\neg e$, 6)

Example 7.8. *Let us prove that the sequence* $\{\} \vdash (P(a) \vee \neg P(a))$ *is valid:*

1. $\{\neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)}), \mathtt{P(a)}\} \vdash \neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)})$; (Hypothesis)
2. $\{\neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)}), \mathtt{P(a)}\} \vdash \mathtt{P(a)}$; (Hypothesis)
3. $\{\neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)}), \mathtt{P(a)}\} \vdash (\mathtt{P(a)} \vee \neg\mathtt{P(a)})$; ($\vee i_1$, 2)
4. $\{\neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)}), \mathtt{P(a)}\} \vdash \bot$; ($\neg e$, 1, 3)
5. $\{\neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)})\} \vdash \neg\mathtt{P(a)}$; ($\neg i$, 4)
6. $\{\neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)})\} \vdash (\mathtt{P(a)} \vee \neg\mathtt{P(a)})$; ($\vee i_2$, 5)
7. $\{\neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)})\} \vdash \neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)})$; (Hypothesis)
8. $\{\neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)})\} \vdash \bot$; ($\neg e$, 7, 6)
9. $\{\} \vdash \neg\neg(\mathtt{P(a)} \vee \neg\mathtt{P(a)})$; ($\neg i$, 8)
10. $\{\} \vdash (\mathtt{P(a)} \vee \neg\mathtt{P(a)})$. ($\neg\neg e$, 9)

Exercise 7.7. *Prove that the following sequences are valid::*

1. $\{\neg(\mathtt{P(a)} \wedge \mathtt{Q(a)})\} \vdash (\neg\mathtt{P(a)} \vee \neg\mathtt{Q(a)})$;
2. $\{\neg(\neg\mathtt{P(a)} \vee \neg\mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \wedge \mathtt{Q(a)})$;
3. $\{\neg(\neg\mathtt{P(a)} \wedge \neg\mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \vee \mathtt{Q(a)})$.

7.5 Rules for quantifiers

7.5.1 Elimination of universal quantifier

The rule for the elimination of the universal quantifier is:

$$\forall e \ \frac{\Gamma \vdash \forall x.\varphi}{\Gamma \vdash \varphi[x \mapsto t]}$$

The elimination rule for the universal quantifier is quite simple: if we know that $\forall x.\varphi$ is a syntactic consequence from Γ, then we can instantiate the bound variable x with any term t.

Exercise 7.8. *Question: does the previous rule make sense if x does not appear in φ? For instance, from $\Gamma \vdash \forall \mathtt{x}.\mathtt{P}(\mathtt{a})$ can we deduce $\Gamma \vdash \mathtt{P}(\mathtt{a})[\mathtt{x} \mapsto \mathtt{b}]$?*

Example 7.9. *Let us go back to an example previously discussed in which we have the two affirmations:* Any human is mortal *and* Socrate is a human*. Can we conclude that* Socrate is mortal*? In order to answer to the question, we have to prove the sequence: $\{\forall \mathtt{x}.(\mathit{Human}(\mathtt{x}) \to \mathit{Mortal}(\mathtt{x})), \mathit{Human}(\mathtt{s})\} \vdash \mathit{Mortal}(\mathtt{s})$, where* Human *and* Mortal *are predicates of arity 1 and s is a constant (functional symbol of arity 0) associated to the name Socrates. Here is the formal proof for the sequence:*

1. $\{\forall \mathtt{x}.(\mathit{Human}(\mathtt{x}) \to \mathit{Mortal}(\mathtt{x})), \mathit{Human}(\mathtt{s})\} \vdash \forall \mathtt{x}.(\mathit{Human}(\mathtt{x}) \to \mathit{Mortal}(\mathtt{x}))$(Hypothesis)
2. $\{\forall \mathtt{x}.(\mathit{Human}(\mathtt{x}) \to \mathit{Mortal}(\mathtt{x})), \mathit{Human}(\mathtt{s})\} \vdash (\mathit{Human}(\mathtt{s}) \to \mathit{Mortal}(\mathtt{s}))(\forall e, 1, s)$
3. $\{\forall \mathtt{x}.(\mathit{Human}(\mathtt{x}) \to \mathit{Mortal}(\mathtt{x})), \mathit{Human}(\mathtt{s})\} \vdash \mathit{Human}(\mathtt{s})$ (Hypothesis)
4. $\{\forall \mathtt{x}.(\mathit{Human}(\mathtt{x}) \to \mathit{Mortal}(\mathtt{x})), \mathit{Human}(\mathtt{s})\} \vdash \mathit{Mortal}(\mathtt{s})$ $(\to e, 2, 3)$

Note that at step 2 of the proof, we used the rule $\forall e$ which instantiates in the formula $\forall \mathtt{x}.(\mathit{Human}(\mathtt{x}) \to \mathit{Mortal}(\mathtt{x}))$ the bound variable $\mathtt{x}$ with $\mathtt{s}$: $(\mathit{Human}(\mathtt{s}) \to \mathit{Mortal}(\mathtt{s}))$. In natural language, this is similar with deducing by reasoning that If Socrates is a human, then he is mortal *from* Any human is mortal.

7.5.2 Introduction of existential quantifier

There is a duality of rules for the introduction and the elimination of quantifiers in the sense that the rule for introducing the existential quantifier from below can be seen as a dual rule for the elimination of universal quantifier:

$$\exists i \ \frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash \exists x.\varphi}$$

The rule indicates that we can deduce $\exists x.\varphi$ when $\varphi[x \mapsto t]$ is a semantical consequence from Γ. Informally, if there is a concrete x — namely t — such that $\varphi[x \mapsto t]$ is true, we conclude that $\exists \mathtt{x}.\varphi$ is true.

Example 7.10. *Let us prove that the sequence* $\{\mathtt{P(a)}\} \vdash \exists\mathtt{x.P(x)}$ *is valid:*

1. $\{\mathtt{P(a)}\} \vdash \mathtt{P(a)}$ (HYPOTHESIS)
2. $\{\mathtt{P(a)}\} \vdash \exists\mathtt{x.P(x)}$ $(\exists i, 1)$

Note that in this case φ *is* $\mathtt{P(x)}$ *and* $\varphi[\mathtt{x} \mapsto \mathtt{a}]$ *is* $\mathtt{P(x)}[\mathtt{x} \mapsto \mathtt{a}] = \mathtt{P(a)}$.

Example 7.11. *Let us prove that the sequence* $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \mathtt{P(a)}\} \vdash \exists\mathtt{x.Q(x)}$ *is valid:*

1. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \mathtt{P(a)}\} \vdash \forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}$ (HYPOTHESIS)
2. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \mathtt{P(a)}\} \vdash \mathtt{P(a)}$ (HYPOTHESIS)
3. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \mathtt{P(a)}\} \vdash (\mathtt{P(a)} \rightarrow \mathtt{Q(a)})$ $(\forall e, 1, \mathtt{a})$
4. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \mathtt{P(a)}\} \vdash \mathtt{Q(a)}$ $(\rightarrow e, 3, 2)$
5. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \mathtt{P(a)}\} \vdash \exists\mathtt{x.Q(x)}$ $(\exists i, 4)$

7.5.3 Introduction of universal quantifier.

The rule for the introduction of the universal quantifier is:

$$\forall i \ \frac{\Gamma \vdash \varphi[x \mapsto x_0]}{\Gamma \vdash \forall x.\varphi} \ x_0 \notin \mathit{vars}(\Gamma, \varphi)$$

The rule from above says that we can conclude $\Gamma \vdash \forall x.\varphi$ if we first prove that $\varphi[x \mapsto x_0]$ is a syntactical consequence from Γ, where x_0 is a *new* variable: it does not appear in other formulae and we make no assumption over it.

Example 7.12. *Let us prove that the sequence* $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \forall\mathtt{x.P(x)}\} \vdash \forall\mathtt{x.Q(x)}$ *is valid:*

1. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \forall\mathtt{x.P(x)}\} \vdash \forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}$ (HYPOTHESIS)
2. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \forall\mathtt{x.P(x)}\} \vdash \forall\mathtt{x.P(x)}$ (HYPOTHESIS)
3. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \forall\mathtt{x.P(x)}\} \vdash (\mathtt{P(x_0)} \rightarrow \mathtt{Q(x_0)})$ $(\forall e, 1, \mathtt{x_0})$
4. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \forall\mathtt{x.P(x)}\} \vdash \mathtt{P(x_0)}$ $(\forall e, 2, \mathtt{x_0})$
5. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \forall\mathtt{x.P(x)}\} \vdash \mathtt{Q(x_0)}$ $(\rightarrow e, 3, 4)$
6. $\{\forall\mathtt{x.(P(x)} \rightarrow \mathtt{Q(x))}, \forall\mathtt{x.P(x)}\} \vdash \forall\mathtt{x.Q(x)}$ $(\forall i, 5)$

Note that for the sequences 3, 4 and 5, we use the variable x_0 *about which we make no assumption. Therefore, intuitively,* $Q(x_0)$ *holds for any* x_0.

Exercise 7.9. *Prove that the following sequences are valid:*

1. $\{\forall\mathtt{x.(P(x)} \land \mathtt{Q(x))}\} \vdash \forall\mathtt{x.P(x)}$;
2. $\{\forall\mathtt{x.Q(x)}, \mathtt{P(a)}\} \vdash \mathtt{P(a)} \land \mathtt{Q(a)}$;
3. $\{\forall\mathtt{x.P(x)}, \forall\mathtt{x.Q(x))}\} \vdash \forall\mathtt{x.(P(x)} \land \mathtt{Q(x))}$.

7.5.4 Elimination of existential quantifier

The rule for the elimination of the existential quantifier is the following:

$$\exists e \; \frac{\Gamma \vdash \exists x.\varphi \qquad \Gamma \cup \{\varphi[x \mapsto x_0]\} \vdash \psi}{\Gamma \vdash \psi} \; x_0 \notin \mathit{vars}(\Gamma, \varphi)$$

The first hypothesis of the rule is $\Gamma \vdash \exists x.\varphi$, which, intuitively, ensures us that there is at least one term (they can be several) that can replace x such that φ is a syntactical consequence from Γ. However, we don't know which are this terms (in the case they are several). We only know that there is at least one and we call it x_0. In order to prove the conclusion, that ψ is a syntactical consequence from Γ, we have to analyze several cases for x_0. This is summarized by the second hypothesis of the rule, where we have to prove that ψ is a syntactical consequence from $\Gamma \cup \{\varphi[x \mapsto x_0]\}$.

Example 7.13. *Let us prove that the sequence* $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \exists \mathtt{x}.\mathtt{Q}(\mathtt{x})$ *is valid:*

1. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \exists \mathtt{x}.\mathtt{P}(\mathtt{x})$ (Hypothesis)
2. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x_0})\} \vdash \mathtt{P}(\mathtt{x_0})$ (Hypothesis)
3. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x_0})\} \vdash \forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x}))$ (Hypothesis)
4. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x_0})\} \vdash (\mathtt{P}(\mathtt{x_0}) \to \mathtt{Q}(\mathtt{x_0}))$ $(\forall e, 3, \mathtt{x_0})$
5. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x_0})\} \vdash \mathtt{Q}(\mathtt{x_0})$ $(\to e, 4, 2)$
6. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x}), \mathtt{P}(\mathtt{x_0})\} \vdash \exists \mathtt{x}.\mathtt{Q}(\mathtt{x})$ $(\exists i, 5)$
7. $\{\forall \mathtt{x}.(\mathtt{P}(\mathtt{x}) \to \mathtt{Q}(\mathtt{x})), \exists \mathtt{x}.\mathtt{P}(\mathtt{x})\} \vdash \exists \mathtt{x}.\mathtt{Q}(\mathtt{x})$ $(\exists e, 1, 6)$

Note that in order to prove the 7th sequence, we used sequences 1 and 6. The former one was proved by steps 2,3,4 and 5, where we also used as a hypothesis the formula $\mathtt{P}(\mathtt{x_0})(= \mathtt{P}(\mathtt{x})[\mathtt{x} \mapsto \mathtt{x_0}])$.

7.6 Other rules

Another useful rule, that does not necessary corresponds to some operator, is the extension rule, that was also presented in the case of natural deduction for propositional logic:

$$\textsc{Extension} \; \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi}$$

This rule indicates the fact that, if φ is a consequence from a set of formulas Γ, then φ is also a consequence of $\Gamma \cup \{\varphi'\}$ (for any φ'). In other words, we can extend the set of premises of a valid sequence and we get another valid sequence.

Example 7.14. *Let us prove that the sequence* $\{\mathtt{P(a)}, \neg\mathtt{Q(a)}, \mathtt{P(f(a))}, (\mathtt{P(b)} \wedge \mathtt{Q(b)})\} \vdash \neg\neg\mathtt{P(a)}$ *is valid:*

1. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(a)}$; (HYPOTHESIS)
2. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \neg\mathtt{P(a)}$; (HYPOTHESIS)
3. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \bot$; ($\neg e$, 1, 2)
4. $\{\mathtt{P(a)}\} \vdash \neg\neg\mathtt{P(a)}$; ($\neg i$, 3)
5. $\{\mathtt{P(a)}, \neg\mathtt{Q(a)}\} \vdash \neg\neg\mathtt{P(a)}$; (EXTENSION, 4)
6. $\{\mathtt{P(a)}, \neg\mathtt{Q(a)}, \mathtt{P(f(a))}\} \vdash \neg\neg\mathtt{P(a)}$; (EXTENSION, 5)
7. $\{\mathtt{P(a)}, \neg\mathtt{Q(a)}, \mathtt{P(f(a))}, (\mathtt{P(b)} \wedge \mathtt{Q(b)})\} \vdash \neg\neg\mathtt{P(a)}$. (EXTENSION, 6)

8 Natural deduction system

The natural deduction for first order logic is the deductive system formed by all rules presented in the previous section. Here is the sum up of all rules:

$$\wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,} \qquad \wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi',}$$

$$\rightarrow e\ \frac{\Gamma \vdash (\varphi \rightarrow \varphi') \qquad \Gamma \vdash \varphi}{\Gamma \vdash \varphi',} \qquad \rightarrow i\ \frac{\Gamma, \varphi \vdash \varphi'}{\Gamma \vdash (\varphi \rightarrow \varphi'),} \qquad \vee i_1\ \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2),}$$

$$\vee i_2\ \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2),} \qquad \vee e\ \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \qquad \Gamma, \varphi_1 \vdash \varphi' \qquad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi',}$$

$$\neg e\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \bot,} \qquad \neg i\ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi,} \qquad \bot e\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi,}$$

$$\text{HYPOTHESIS}\ \frac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma, \qquad \text{EXTENSION}\ \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi,} \qquad \neg\neg e\ \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi.}$$

$$\forall e\ \frac{\Gamma \vdash \forall x.\varphi}{\Gamma \vdash \varphi[x \mapsto t]} \qquad \exists e\ \frac{\Gamma \vdash \exists x.\varphi \qquad \Gamma \cup \{\varphi[x \mapsto x_0]\} \vdash \psi}{\Gamma \vdash \psi}\ x_0 \notin \mathit{vars}(\Gamma, \varphi)$$

$$\forall i\ \frac{\Gamma \vdash \varphi[x \mapsto x_0]}{\Gamma \vdash \forall x.\varphi}\ x_0 \notin \mathit{vars}(\Gamma, \varphi) \qquad \exists i\ \frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash \exists x.\varphi}$$

Of course that we can also use in a proof derivated rules (also presented in the case of propositional logic).

9 Soundness and Completeness of Natural Deduction for Forst Order Logic

Theorem 9.1 (Soundness for natural deduction)**.** *For any set Γ of formulae and any formula φ, if the sequence $\Gamma \vdash \varphi$ is valid, then $\Gamma \models \varphi$.*

Exercise: to prove in seminar.

Theorem 9.2 (Completeness for natural deduction)**.** *For any set Γ of formulae and any formula φ, if $\Gamma \models \varphi$ then the sequence $\Gamma \vdash \varphi$ is valid.*

This proof exceeds the level of the lecture.

Remark 9.1. *Note that, using the two soundness and completeness theorems, the relation $\vdash$ coincides with $\models$, even if they have different meanings.*