Vector Spaces

Part II

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Linear maps

Linear map

Definition 1

Let U and V be vector spaces over a field F. A linear map from U to V is a mapping $f : U \to V$ such that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y),$$

for all $\alpha, \beta \in F$ and $x, y \in U$.

Some authors use the term linear transformation or homomorphism, which means the same as linear map!

Example 2

Let U and V be vector spaces over a field F.

1. Zero linear map: $0 : U \to V$ given by $0(u) = 0$, for all $u \in U$.
2. Identity linear map: $I : U \to U$ given by $I(u) = u$, for all $u \in U$.
3. $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = (2x - 4y + 3z, 6x)$, for any $(x, y, z) \in \mathbb{R}^3$, is a linear map.

Isomorphism of vector spaces

Definition 3

Two vector spaces U and V over a field F are called isomorphic, denoted $U \cong V$, if there exists a bijective linear map from U to V.

Proposition 4

If U and V are two vector spaces of the same (finite) dimension over a field F, then U and V are isomorphic.

Proof.

Let $n = dim(U) = dim(V)$. Consider $f : F^n \to U$ given by

$$f(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} \alpha_i x,$$

where $\{x_1, \ldots, x_n\}$ is a basis of U.

f is a bijective linear map. Therefore, F^n is isomorphic to U. In a similar way, F^n is isomorphic to V. $\square$

The vector space of linear maps

Given two vector spaces U and V over a field F, denote by $Hom_F(U,V)$ or $\mathcal{L}_F(U,V)$ the set of all linear maps from U to V.

Define the addition of linear maps $f,g \in Hom_F(U,V)$ in the standard way, namely

$$(f+g)(u) = f(u) + g(u),$$

and the multiplication by scalars in F by

$$(\alpha f)(u) = \alpha f(u),$$

for any $u \in U$ and $\alpha \in F$.

Prove the following result!

Proposition 5

For any two vector spaces U and V over the same field F, $Hom_F(U,V)$ with the two operations as defined above is a vector space over F.

The matrix of a linear map

Let U and V be vector spaces over a field F, with bases $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$, respectively. Assume that the two bases are totally ordered by the index of its vectors.

Any $f \in Hom_F(U, V)$ is completely and uniquely specified by a matrix

$$A_f = \begin{array}{c} \\ f(x_1) \\ \vdots \\ f(x_m) \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & \cdots & y_n \end{array} \\ \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \end{array}$$

where $f(x_i) = a_{i,1}y_1 + \cdots + a_{i,n}y_n$, for all i.

A_f is called the matrix of the linear map f.

A_f depends on the bases of U and V, as well as the vectors' order in the two bases!

Evaluating linear maps by their matrices

Let U and V be vector spaces over a field F, with bases $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$, respectively.

Assume that $f \in Hom_F(U, V)$ is given by the matrix A_f. Then, for any $x = \alpha_1 x_1 + \cdots + \alpha_m x_m \in U$,

$$\begin{array}{rcl}
[f(x)] & = & [f(\alpha_1 x_1 + \cdots + \alpha_m x_m)] \\
& = & \alpha_1 [f(x_1)] + \cdots + \alpha_m [f(x_m)] \\
& = & \begin{pmatrix} \alpha_1 & \cdots & \alpha_m \end{pmatrix} \begin{pmatrix} [f(x_1)] \\ \vdots \\ [f(x_m)] \end{pmatrix} \\
& = & [x] A_f
\end{array}$$

where $[z]$ stands for the coordinate of z in the corresponding basis of U or V. For instance, $[x] = (\alpha_1, \ldots, \alpha_m)$, $[f(x_1)] = (a_{1,1}, \ldots, a_{1,n})$, and so on.

Evaluating linear maps by their matrices

When the vector identifies itself with its coordinate, that is $[z] = z$, we obtain $f(x) = xA_f$. The next example illustrate this.

Example 6

Let F be a field and $f \in Hom_F(F^2, F^3)$ given by

$$f(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$$

Considering the standard bases of F^2 and F^3, we have $f(1, 0) = (1, 2, 7)$ and $f(0, 1) = (3, 5, 9)$. So,

$$A_f = \begin{pmatrix} 1 & 2 & 7 \\ 3 & 5 & 9 \end{pmatrix}$$

Then, $f(3, 2) = \begin{pmatrix} 3 & 2 \end{pmatrix} A_f = (9, 16, 39)$.

Uniquely identifying a linear map with a matrix leads to the following result.

Proposition 7

Let U and V be vector spaces over a field F with $dim(U) = m$ and $dim(V) = n$. Then, $Hom_F(U, V)$ is isomorphic to ${}^mF^n$.

Proof.

See textbook [1], pages 362-363. □

Some authors represent vectors in column format and not row as we have adopted. In the linear map matrix A_f, the vectors $f(x_i)$ are then represented by columns of the matrix. The calculations then change via the matrix transpose operation!

The null space and the range of a linear map

Definition 8

Let $f \in Hom_F(U, V)$.

1. The null space of f, denoted $\mathcal{N}(f)$, is the subset of U consisting of all vectors u such that $f(u) = 0$.
2. The range of f, denoted $\mathcal{R}(f)$, is the subset of V consisting of all vectors $v \in V$ such that $f(u) = v$ for some $u \in U$.

Prove the following properties!

Proposition 9

Let $f \in Hom_F(U, V)$. Then, $\mathcal{N}(f) \leq U$ and $\mathcal{R}(f) \leq V$.

Fundamental theorem of linear maps

Theorem 10 (Fundamental theorem of linear maps)

Let $f \in Hom_F(U, V)$. Then,

$$dim(U) = dim(\mathcal{N}(f)) + dim(\mathcal{R}(f)).$$

Proof.

Let $n = dim(U)$ and $k = dim(\mathcal{N}(f))$. Given a basis B_1 of $\mathcal{N}(f)$, extend it to a basis B of U.

Prove that $f(B_2)$, where $B_2 = B \setminus B_1$, is a basis of $\mathcal{R}(f)$. $\square$

Our assumption in this course was that all vector spaces we work with have finite bases. The fundamental theorem of linear maps remains valid even if V is infinite-dimensional (but U finite-dimensional)!

Inner product and orthogonality

Inner product

The inner product, also called the scalar product, of two vectors in a vector space is a binary operation that allows defining the length (magnitude) of a vector and the angle between two vectors.

The definition of an inner product on a vector space depends a lot on its field of scalars:

- Whether or not it is endowed with an involutory automorphism to capture the conjugation aspects that appear, for example, in $\mathbb{C}$;
- Whether or not it is endowed with a total order to capture the aspect of positive semi-definiteness;
- Whether or not it allows an extension to positive definiteness.

The dot product in $\mathbb{R}^n$

The dot product of two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{i=1}^{n} x_i y_i$$

$\langle x, y \rangle$ is offten written as $x \cdot y$, which justifies the name of "dot product"!

The dot product allows computing the length of a vector and the angle between two vectors in $\mathbb{R}^2$:

- $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + x_2^2}$;
- If θ is the angle between x and y, then

$$\theta = \arccos\left(\frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}\right)$$

The vectors x and y are perpendicular, also called orthogonal, if $\langle x, y \rangle = 0$.

Properties of the dot product in $\mathbb{R}^n$

The dot product on vectors in $\mathbb{R}^n$ has many useful properties:

1. Symmetry: $\langle x, y \rangle = \langle y, x \rangle$, for all $x, y \in \mathbb{R}^n$;
2. Linearity: $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for all $x, y, z \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Combining with the symmetry we get bi-linearity;
3. Non-degeneracy: if $\langle x, y \rangle = 0$ for all y, then $x = 0$;
4. Positive semi-definite: $\langle x, x \rangle \geq 0$, for all $x \in \mathbb{R}^n$;
5. Positive definite: add "$\langle x, x \rangle = 0$ iff $x = 0$" to the positive semi-definiteness.

Clearly, positive definiteness implies non-degeneracy.

Generalizing the dot product

The generalization of the dot product to vector spaces over arbitrary fields should be made with great care. A few facts are in order:

- For $\mathbb{C}^n$, the dot product is defined in a different way:

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$$

  and so the properties above need to be adapted ($\bar{z}$ stands for the complex conjugate);

- In finite fields, such as $\mathbb{Z}_p$ with p prime, the requirement

$$\langle x, x \rangle = 0 \text{ iff } x = 0$$

  does not usually hold. For instance, if $V = \mathbb{Z}_5^2$,

$$\langle (1,2), (1,2) \rangle = 1 + 4 \equiv 0 \bmod 5$$

- On arbitrary fields, the condition $\langle x, x \rangle \geq 0$ does not make any sense (unless we define a total order relation).

Inner product

Definition 11

Let V be a vector space over a field F. An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to F$ such that:

1. $\langle x, y \rangle = \langle y, x \rangle$;
2. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,

for any $x, y, z \in V$ and $\alpha, \beta \in F$.

As we can see, the definition borrows only the properties of symmetry and linearity in the first argument from the dot product. This is not the most general definition, but it satisfies our needs.

Orthogonality

Definition 12

Let V be a vector space over a field F. The vectors $x, y \in V$ are called orthogonal, denoted $x \perp y$, if $\langle x, y \rangle = 0$.

Let $x \in V$ and $X, Y \subseteq V$. We define:

1. $x \perp Y$ if $x \perp y$, for all $y \in Y$;
2. $X \perp Y$ if $x \perp Y$, for all $x \in X$;
3. $X^{\perp} = \{z \in V \mid z \perp X\}$ – this is the orthogonal complement of X.

Remark 13

- *The orthogonal complement of a set of vectors is always a subspace of V;*
- $X \subseteq (X^{\perp})^{\perp}$*, for all* $X \subseteq V$*. In* $\mathbb{R}$ *and* $\mathbb{C}$ *we can prove the equality when* X *is a sub-space.*

Definition 14

An orthogonal basis of a vector space V over a field F is any basis B such that $x \perp y$, for any $x, y \in B$ with $x \neq y$.

Theorem 15 (Gram-Schmidt)

For any finite dimensional vector space there exists an orthogonal basis.

Proof.

See textbook [1], page 367. $\square$

Theorem 16

Let V be an n-dimensional vector space over a field F. Then, for any subspace U of V, $dim(U) + dim(U^{\perp}) = n$.

Proof.

See textbook [1], page 368. $\square$

Reading and exercise guide

Reading and exercise guide

It is highly recommended that you do all the exercises marked in red from the slides.

Course readings:

1. Pages 351-368 from textbook [1].

References

[1] Ferucio Laurențiu Țiplea. *Algebraic Foundations of Computer Science*. "Alexandru Ioan Cuza" University Publishing House, Iași, Romania, second edition, 2021.