Logică pentru Informatică - Săptămâna 11
Deducția naturală
Exerciții pentru Seminar

1 Regulile deducției naturale

$$\wedge i\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \varphi'}{\Gamma \vdash (\varphi \wedge \varphi'),} \qquad \wedge e_1\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi,} \qquad \wedge e_2\ \frac{\Gamma \vdash (\varphi \wedge \varphi')}{\Gamma \vdash \varphi',}$$

$$\rightarrow e\ \frac{\Gamma \vdash (\varphi \rightarrow \varphi') \qquad \Gamma \vdash \varphi}{\Gamma \vdash \varphi',} \qquad \rightarrow i\ \frac{\Gamma, \varphi \vdash \varphi'}{\Gamma \vdash (\varphi \rightarrow \varphi'),} \qquad \vee i_1\ \frac{\Gamma \vdash \varphi_1}{\Gamma \vdash (\varphi_1 \vee \varphi_2),}$$

$$\vee i_2\ \frac{\Gamma \vdash \varphi_2}{\Gamma \vdash (\varphi_1 \vee \varphi_2),} \qquad \vee e\ \frac{\Gamma \vdash (\varphi_1 \vee \varphi_2) \qquad \Gamma, \varphi_1 \vdash \varphi' \qquad \Gamma, \varphi_2 \vdash \varphi'}{\Gamma \vdash \varphi',}$$

$$\neg e\ \frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \neg\varphi}{\Gamma \vdash \bot,} \qquad \neg i\ \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi,} \qquad \bot e\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi,}$$

$$\text{IPOTEZĂ}\ \frac{}{\Gamma \vdash \varphi}\ \varphi \in \Gamma, \qquad \text{EXTINDERE}\ \frac{\Gamma \vdash \varphi}{\Gamma, \varphi' \vdash \varphi,} \qquad \neg\neg e\ \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi.} \qquad \forall e\ \frac{\Gamma \vdash (\forall x.\varphi)}{\Gamma \vdash \varphi[x \mapsto t]}$$

$$\exists e\ \frac{\Gamma \vdash (\exists x.\varphi) \qquad \Gamma \cup \{\varphi[x \mapsto x_0]\} \vdash \psi}{\Gamma \vdash \psi}\ x_0 \notin vars(\Gamma, \varphi, \psi)$$

$$\forall i\ \frac{\Gamma \vdash \varphi[x \mapsto x_0]}{\Gamma \vdash (\forall x.\varphi)}\ x_0 \notin vars(\Gamma, \varphi) \qquad \exists i\ \frac{\Gamma \vdash \varphi[x \mapsto t]}{\Gamma \vdash (\exists x.\varphi)}$$

În exercițiile de mai jos vom lucra cu o signatură $\Sigma = (\{P, Q\}, \{a, b, f, g\})$, unde predicatele P și Q au aritate 1, simbolurile funcționale f și g au aritate 1, iar simbolurile a și b sunt constante (aritate 0).

2 Exerciții rezolvate

1. Arătați că secvența $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(b)}$ este validă.

Rezolvare:

1. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(a)}$; (IPOTEZĂ)
2. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \neg\mathtt{P(a)}$; (IPOTEZĂ)

3. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \bot$; $(\neg e,\ 1,\ 2)$

4. $\{\mathtt{P(a)}, \neg\mathtt{P(a)}\} \vdash \mathtt{P(b)}$. $(\bot e,\ 3)$

2. Arătați că secvența $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{q} \vee \mathtt{p})$ este validă.

Rezolvare:

1. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{P(a)}\} \vdash \mathtt{P(a)}$; (Ipoteză)
2. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{P(a)}\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$; $(\vee i_2,\ 1)$
3. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{Q(a)}\} \vdash \mathtt{Q(a)}$; (Ipoteză)
4. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)}), \mathtt{Q(a)}\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$; $(\vee i_1,\ 1)$
5. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{P(a)} \vee \mathtt{Q(a)})$; (Ipoteză)
6. $\{(\mathtt{P(a)} \vee \mathtt{Q(a)})\} \vdash (\mathtt{Q(a)} \vee \mathtt{P(a)})$. $(\vee e,\ 5,\ 2,\ 4)$

3. Arătați că secvența $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ este validă.

Rezolvare:

1. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)})$ (Ipoteză)
2. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \mathtt{P(a)}$ (Ipoteză)
3. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash (\mathtt{P(a)} \to \mathtt{Q(a)})$ $(\forall e, 1, \mathtt{a})$
4. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \mathtt{Q(a)}$ $(\to e, 3, 2)$
5. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(a)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ $(\exists i, 4)$

4. Arătați că secvența $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ este validă.

Rezolvare:

1. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}\} \vdash \exists \mathtt{x}.\mathtt{P(x)}$ (Ipoteză)
2. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash \mathtt{P(x_0)}$ (Ipoteză)
3. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash \forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)})$ (Ipoteză)
4. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash (\mathtt{P(x_0)} \to \mathtt{Q(x_0)})$ $(\forall e, 3, \mathtt{x_0})$
5. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash \mathtt{Q(x_0)}$ $(\to e, 4, 2)$
6. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}, \mathtt{P(x_0)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ $(\exists i, 5)$
7. $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \exists \mathtt{x}.\mathtt{P(x)}\} \vdash \exists \mathtt{x}.\mathtt{Q(x)}$ $(\exists e, 1, 6)$

5. Arătați că secvența $\{\forall \mathtt{x}.(\mathtt{P(x)} \to \mathtt{Q(x)}), \mathtt{P(x)}\} \vdash \forall \mathtt{x}.\mathtt{Q(x)}$ este validă

Rezolvare:

1. $\{\forall x.(P(x) \to Q(x)), \forall x.P(x)\} \vdash \forall x.(P(x) \to Q(x))$ (IPOTEZĂ)
2. $\{\forall x.(P(x) \to Q(x)), \forall x.P(x)\} \vdash \forall x.P(x)$ (IPOTEZĂ)
3. $\{\forall x.(P(x) \to Q(x)), \forall x.P(x)\} \vdash (P(x_0) \to Q(x_0))$ $(\forall e, 1, x_0)$
4. $\{\forall x.(P(x) \to Q(x)), \forall x.P(x)\} \vdash P(x_0)$ $(\forall e, 2, x_0)$
5. $\{\forall x.(P(x) \to Q(x)), \forall x.P(x)\} \vdash Q(x_0)$ $(\to e, 3, 4)$
6. $\{\forall x.(P(x) \to Q(x)), \forall x.P(x)\} \vdash \forall x.Q(x)$ $(\forall i, 5)$

3 Exerciții propuse

Secvențele de mai jos sunt valide?

1. $\{((P(a) \land Q(a)) \land \forall x.P(x))\} \vdash (Q(a) \land \forall x.P(x))$;
2. $\{((P(a) \land Q(a)) \land \forall x.P(x)), \forall x.Q(x)\} \vdash (\forall x.Q(x) \land Q(a))$;
3. $\{((P(a) \land Q(a)) \land \forall x.P(x))\} \vdash (\forall x.P(x) \land (Q(a) \land P(a)))$;
4. $\{((P(a) \land Q(a)) \to \forall x.P(x)), P(a), Q(a)\} \vdash \forall x.P(x)$;
5. $\{(P(a) \to \forall x.P(x)), P(a), Q(a)\} \vdash (Q(a) \land \forall x.P(x))$;
6. $\{(P(a) \to P(b)), (Q(a) \to P(b))\} \vdash ((P(a) \lor Q(a)) \to P(b))$;
7. $\{\neg(P(a) \land Q(a))\} \vdash (\neg P(a) \lor \neg Q(a))$;
8. $\{\neg(\neg P(a) \lor \neg Q(a))\} \vdash (P(a) \land Q(a))$;
9. $\{\neg(\neg P(a) \land \neg Q(a))\} \vdash (P(a) \lor Q(a))$;
10. $\{\forall x.(P(x) \land Q(x))\} \vdash \forall x.P(x)$;
11. $\{\forall x.Q(x), P(a)\} \vdash P(a) \land Q(a)$;
12. $\{\forall x.P(x), \forall x.Q(x))\} \vdash \forall x.(P(x) \land Q(x))$;
13. $\{\exists x.\exists y.P(x, y)\} \vdash \exists y.\exists x.P(x, y)$;
14. $\{\exists x.\forall y.P(x, y)\} \vdash \forall y.\exists x.P(x, y)$; Dar invers: $\{\forall y.\exists x.P(x, y)\} \vdash \exists x.\forall y.P(x, y)$?
15. $\{\neg(\exists x.P(x))\} \vdash \forall x.\neg P(x)$;
16. $\{\forall x.\neg P(x)\} \vdash \neg(\exists x.P(x))$;