Propositional Logic. Questions and exercises for testing your understanding of propositional logic.

The following list of questions and exercises was desgined to test your understanding of the concepts in the first 7 lectures, on Propositional Logic. Some questions have a fixed answer, while others are open-ended.

You may use the questions to test your degree of understanding of the concepts as follows: for each question, you should know the answer and be able to argue it convincingly and confidently.

If you do not know the answer to a question, there is no point in asking your colleagues or the instructor – it will not help you to learn the answer. Instead, you should just study the lectures again.

- The Syntax of Propositional Logic
  - Any propositional formula has at least one logical connective.
  - Any propositional variable is a propositional formula.
  - Any propositional formula has an even number of brackets.
  - Any propositional formula in $PL_{\wedge,\vee}$ has an odd number of symbols.
  - If the abstract syntax tree of a formula has an even number of nodes, then the formula contains at least one negation.
  - There are propositional formulas with an abstract syntax tree having 0 nodes.
  - The number of subformulae of a formula is equal to the number of nodes in its abstract syntax tree.
  - The height of the abstract syntax tree of a formula is smaller than or equal to the number of connectives in the formula.
  - The number of nodes in the abstract syntax tree of a formula can be equal to the height of the formula.
  - The main connective of the formula $(\neg p \vee q)$ is the negation.
  - The main connective of the formula $((\neg p \wedge q) \vee q)$ is the conjunction.
  - The main connective of the formula $\neg((\neg p \wedge q) \vee q)$ is the negation.
  - The following words are formulae in $PL_{\neg,\wedge,\vee}$: $(\neg\mathtt{p})$, $(\neg\neg\mathtt{p})$, $\mathtt{p}+\mathtt{q}$, $\mathtt{p}\cdot\mathtt{q}$, $(\mathtt{p}\wedge\mathtt{q})$, $\neg(\neg\mathtt{p})$, $\neg\neg\mathtt{q}$, $(\neg\mathtt{p})\vee\mathtt{q}$, $(\neg\mathtt{p}\vee(\mathtt{q}))$, $(\neg\neg\mathtt{p}\vee\mathtt{q})$, $\neg(\neg\mathtt{p}\vee\mathtt{q})$, $(\neg\neg\mathtt{p})\vee\mathtt{q}$, $(\neg\mathtt{p}\vee\neg\mathtt{q})$, $\mathtt{q}$, $(\mathtt{q})$, $\neg\neg\neg$, $\vee(\mathtt{p},\mathtt{q})$.
- The Semantics of Propositional Logic
  - The negation of any tautology is a tautology.
  - The negation of any contingency (satisfiable but invalid formula) is a contingency.
  - The negation of a formula is a contradiction.
  - If φ is unsatisfiable, then $\varphi \vee \psi$ is unsatisfiable.
  - If φ is a tautology, then $\neg\varphi \vee \psi$ is unsatisfiable.
  - If φ_1 and φ_2 are tautologies, then $\varphi_1 \wedge \varphi_2$ is a tautology.
  - If φ_1 and φ_2 are tautologies, then $\neg\varphi_1 \wedge \varphi_2$ is satisfiable.
  - Any tautology is satisfiable.
  - Any satisfiable formula is also a tautology.
  - The negation of any contradiction is a tautology.
  - The negation of a satisfiable formula is unsatisfiable.
  - There are no invalid formula that are also satisfiable.

- Any propositional variable, seen as a formula, is satisfiable.
- Any tautology is a semantical consequence of the empty set of formulae.
- Any tautology is a logical consequence of any other satisfiable formula.
- Any satisfiable formula is a semantic consequence of any tautology.
- Any formula is a logical consequence of any unsatisfiable formula.
- The disjunction of two satisfiable formulae is also satisfiable.
- The conjunction of any two tautologies is a tautology.
- If a formula is a logical consequence of any tautology, then the formula is itself a tautology.
- No formula is a semantical consequence of a contradiction.
- An unsatisfiable formula is equivalent only to other unsatisfiable formulae.
- If $\varphi_1 \equiv \varphi_2$ and φ_1 is a semantical consequence of a set of formulae Γ, then $\Gamma \models \varphi_2$.
- A contingency can be equivalent to a valid formula.
- If $\varphi_1 \equiv \varphi_2$, then $\varphi_1 \models \varphi_2$.
- If $\varphi_1 \models \varphi_2$ and $\varphi_1 \models \varphi_2$ and $\varphi_2 \models \varphi_1$, then φ_1 and φ_2 are contradictions.
- If φ_1 is a tautology, then $\varphi_1 \to \varphi_2$ is a tautology.
- If φ_1 is satisfiable, then $\varphi_1 \to \varphi_2$ is a tautology.
- If φ_2 is a tautology, then $\varphi_1 \to \varphi_2$ is a tautology.
- If φ_2 is satisfiable, then $\varphi_1 \to \varphi_2$ is a tautology.
- If $\varphi_1 \models \varphi_2$ and $\neg\varphi_3 \models \neg\varphi_2$, then $\varphi_1 \models \varphi_2$.
- A formula is valid iff it is a semantical consequence of the empty set of formulae.
- If $\models \varphi_1 \to \varphi_2$, then $\models \neg\varphi_2 \to \neg\varphi_1$.
- If $\varphi_1 \models \neg\varphi_2$ and $\varphi_2 \models \neg\varphi_1$, then $\varphi_1 \equiv \varphi_2$.
- If $\Gamma \models \varphi_1$ and $\Gamma \models \neg\varphi_1$, then $\Gamma \models \bot$.
- If $\varphi_1 \models \neg\varphi_1$, then φ_1 is a contradiction.
- If $\neg\varphi_1 \models \varphi_1$, then φ_1 is a contradiction.
- There is a satisfiable formula φ such that $\varphi \models \bot$.
- If φ is a tautology and $\Gamma \cup \{\varphi\} \models \varphi'$, then $\Gamma \models \varphi'$.
- If $\Gamma \cup \{\varphi\} \models \varphi'$ and $\Gamma \models \varphi'$, then φ is a tautology.

- Syntax and Semantics
  - Any propositional formula is equivalent to a formula that contains at most one negation along any path from a leaf towards the root of its abstract syntax tree.
  - If $\varphi_1 \equiv \varphi_2$, then $\varphi_1 = \varphi_2$.
  - Any unsatisfiable formula has at least 6 symbols.
  - Any tautology has at least 6 symbols.
  - Any contradiction is the negation of a tautology.
  - Any contradiction is equivalent to the negation of a tautology.
  - Any formula in $PL_{\neg,\wedge,\vee}$ that does not contain the symbol $\neg$ is valid.
  - Any formula in $PL_{\wedge,\vee}$ is satisfiable.
  - The logic $PL_{\neg,\bot}$ is as expressive as $PL_{\wedge,\vee}$.
  - Any contradiction in the logic $PL_{\neg,\wedge,\vee}$ has at least one $\wedge$ connective.

- Any contradiction in the logic $PL_{\neg,\wedge,\vee}$ has at least one $\neg$ connective.
- Any contradiction in the logic $PL_{\neg,\wedge,\vee}$ has at least one $\vee$ connective.
- The logic $PL_{\wedge,\rightarrow}$ is as expressive as the logic $PL_{\vee,\neg}$.
- Consider the binary connective $\mathtt{X}$ with the following semantics: $\hat{\tau}\Big(\varphi_1 \mathtt{X} \varphi_2\Big) = \overline{\hat{\tau}\Big(\varphi_1\Big) \cdot \hat{\tau}\Big(\varphi_2\Big)}$. The logic $PL_{\mathtt{X}}$ is as expressive as $PL_{\neg,\wedge,\vee}$.
- Consider the binary connective $\mathtt{O}$ with the following semantics: $\hat{\tau}\Big(\varphi_1 \mathtt{O} \varphi_2\Big) = \overline{\hat{\tau}\Big(\varphi_1\Big) + \hat{\tau}\Big(\varphi_2\Big)}$. The logic $PL_{\mathtt{O}}$ is equally expressive to $PL_{\neg,\wedge,\vee}$.
- The logic $PL_{\neg,\wedge,\vee}$ is as expressive as $PL_{\leftrightarrow,\perp}$.

• Natural Deduction
  - Show that $\neg\neg e$ and LEM are derivable one from the other (you may also use the other 13 inference rules).
  - Show that the rule LEM is equivalent (it can be derived from the others, and the others can be derived from it) to any of the following rules:

$$\textsc{Peirce}\ \frac{}{((\varphi_1 \rightarrow \varphi_2) \rightarrow \varphi_1) \rightarrow \varphi_1} \qquad \text{DM1}\ \frac{}{\neg(\neg\varphi_1 \wedge \neg\varphi_2) \rightarrow (\varphi_1 \vee \varphi_2)}$$

$$\text{IO}\ \frac{}{(\varphi_1 \rightarrow \varphi_2) \rightarrow (\neg\varphi_1 \vee \varphi_2)}$$

  - There exists a derivation by natural deduction of the sequent $\neg\neg(p \vee \neg p)$ that does not use the rule $\neg\neg e$.
  - Any derivation of the sequent $\vdash p \vee q$ uses at least once the rule $\neg\neg e$.
  - The sequent $\vdash p \wedge \neg p$ is derivable.
  - If $\varphi_1 \vdash \varphi_2$ and $\varphi_2 \vdash \varphi_1$ are derivable sequents, then $\varphi_1 \equiv \varphi_2$.
  - If $\Gamma \vdash \varphi$ and $\varphi \equiv \varphi'$, then $\Gamma \vdash \varphi'$.
  - If $\Gamma \models \varphi$ and the sequent $\Gamma \vdash \neg\varphi$ is valid, then φ is a contradiction.
  - If $\Gamma \vdash \varphi_1$ and $\models \varphi_2 \rightarrow \varphi_2$, then $\Gamma \vdash \varphi_2$.
  - If $\varphi_1 \vdash \perp$, then φ_1 is unsatisfiable.
  - If $\varphi_1 \in PL_{\wedge,\vee,\neg}$ and $\varphi_2 \in PL_{\rightarrow,\perp}$ so that $\varphi_1 \equiv \varphi_2$, then it does not necessarily follow that $\varphi_1 \vdash \varphi_2$.
  - If $\varphi_1 \in PL_{\wedge,\vee,\neg}$ and $\varphi_2 \in PL_{\rightarrow,\perp}$ so that $\varphi_1 \equiv \varphi_2$, then it does not necessarily follow that $\varphi_1 \vdash \varphi_2$.
  - The notions of valid sequent and valid formula are similar.
  - Show that the following sequents are valid:
    * $\{((\mathtt{p} \wedge \mathtt{q}) \wedge \mathtt{r})\} \vdash (\mathtt{q} \wedge \mathtt{r})$;
    * $\{((\mathtt{p} \wedge \mathtt{q}) \wedge \mathtt{r}), \mathtt{r}'\} \vdash (\mathtt{r}' \wedge \mathtt{q})$;
    * $\{((\mathtt{p} \wedge \mathtt{q}) \wedge \mathtt{r})\} \vdash (\mathtt{r} \wedge (\mathtt{q} \wedge \mathtt{p}))$;
    * $\{((\mathtt{p} \wedge \mathtt{q}) \rightarrow \mathtt{r}), \mathtt{p}, \mathtt{q}\} \vdash \mathtt{r}$;
    * $\{(\mathtt{p} \rightarrow \mathtt{r}), \mathtt{p}, \mathtt{q}\} \vdash (\mathtt{q} \wedge \mathtt{r})$;
    * $\{((\mathtt{p} \wedge \mathtt{q}) \rightarrow \mathtt{r}), \mathtt{p}, \mathtt{q}\} \vdash \mathtt{r}$;
    * $\{((\mathtt{p} \wedge \mathtt{q}) \rightarrow \mathtt{r})\} \vdash (\mathtt{p} \rightarrow (\mathtt{q} \rightarrow \mathtt{r}))$;
    * $\{(\mathtt{p} \rightarrow (\mathtt{q} \rightarrow \mathtt{r}))\} \vdash ((\mathtt{p} \wedge \mathtt{q}) \rightarrow \mathtt{r})$;

* $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash (\mathtt{r} \vee \mathtt{p})$;
* $\{(\mathtt{p} \vee \mathtt{q}), (\mathtt{p} \to \mathtt{r}), (\mathtt{q} \to \mathtt{r})\} \vdash \mathtt{r}$;
* $\{(\mathtt{p} \to \mathtt{r}), (\mathtt{q} \to \mathtt{r})\} \vdash ((\mathtt{p} \vee \mathtt{q}) \to \mathtt{r})$;
* $\{(\mathtt{p} \vee \mathtt{q})\} \vdash \neg(\neg\mathtt{p} \wedge \neg\mathtt{q})$;
* $\{(\mathtt{p} \wedge \mathtt{q})\} \vdash \neg(\neg\mathtt{p} \vee \neg\mathtt{q})$;
* $\{(\neg\mathtt{p} \vee \neg\mathtt{q})\} \vdash \neg(\mathtt{p} \wedge \mathtt{q})$;
* $\{(\neg\mathtt{p} \wedge \neg\mathtt{q})\} \vdash \neg(\mathtt{p} \vee \mathtt{q})$;
* $\{\neg(\mathtt{p} \vee \mathtt{q})\} \vdash (\neg\mathtt{p} \wedge \neg\mathtt{q})$;
* $\{\neg(\mathtt{p} \wedge \mathtt{q})\} \vdash (\neg\mathtt{p} \vee \neg\mathtt{q})$;
* $\{\neg(\neg\mathtt{p} \vee \neg\mathtt{q})\} \vdash (\mathtt{p} \wedge \mathtt{q})$;
* $\{\neg(\neg\mathtt{p} \wedge \neg\mathtt{q})\} \vdash (\mathtt{p} \vee \mathtt{q})$.

- Normal Forms
  - Any formula in CNF is satisfiable.
  - Any formula in DNF is satisfiable.
  - Any formula in CNF is valid.
  - Any formula in DNF is valid.
  - If a formula is in CNF, it is trivial to say whether is is valid or not.
  - If a formula is in DNF, it is trivial to say whether it is satisfiable or not.
  - A formula is in NNF (negation normal form) if its only subformulae of the form $\neg\varphi$ are literals, meaning $\varphi \in A$. Find an algorithm to find a NNF for any given formula.
  - Disjunction is associative.
  - Conjunction is associative.
  - Disjunction is commutative.
  - Conjunction is commutative.
  - Implication is associative.
  - Double implication is associative.
  - Implication is commutative.
  - Double implication is commutative.
  - Disjunctions/conjunctions/implications/double implications distribute over disjunctions/conjunctions/implicat implications.

- Resolution
  - If $\varphi_1, \dots, \varphi_n, \varphi$ are clauses and $\varphi_1, \dots, \varphi_n \models \varphi$, then φ can be obtained by resolution from $\varphi_1, \dots, \varphi_n$.
  - If $\varphi_1, \dots, \varphi_n, \varphi$ are clauses and φ can be obtained by resolution from $\varphi_1, \dots, \varphi_n$, then φ is a contradiction.
  - If $\varphi_1, \dots, \varphi_n, \varphi$ are clauses and φ can be obtained by resolution from $\varphi_1, \dots, \varphi_n$, then $\{\varphi_1, \dots, \varphi_n\}$ is inconsistent.
  - If $\varphi_1, \dots, \varphi_n, \varphi$ are literals and $\varphi_1, \dots, \varphi_n \models \varphi$, then φ can be obtained by resolution from $\varphi_1, \dots, \varphi_n$.
  - If φ is a formula in DNF, then φ^c is in CNF.
  - If φ is a formula in DNF, then φ is unsatisfiable if the empty clause can be obtained by resolution from the clauses of the formula φ^c.

- If $\models \varphi_1 \rightarrow \mathtt{p}$, then $\mathtt{p}$ can be obtained by resolution starting from the CNF of the formula φ_1.
- The CNF of a formula can be exponentially larger than the formula itself.
- However, the DNF of a formula is not much larger than the formula itself.
- The CNF of a valid formula contains just clauses that are tautologies.
- If the hypotheses of the binary resolution rule are tautologies, then its conclusion is also a tautology.
- If the hypotheses of the binary resolution rule are satisfiable, then its conclusion is also satisfiable.
- In the context of resolution, the formulae $p \vee q$ and $p \wedge q$ are both denoted by $\{p, q\}$.
- What CNF formulae are represented by the following sets of sets of literals: $\{\}, \emptyset, \{\{\}\}, \{\emptyset\}, \{\{p, q\}\}, \{\{p\}, \{q\}\}$?
- Prove, using resolution, that the following formulae are valid:
  * $p \rightarrow p$;
  * $(p \wedge q) \rightarrow (p \vee q)$;
  * $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$.
- Prove, using resolution, the semantical consequence corresponding to the sequents in the section dedicated to natural deduction.