Maths 101: Brief answers to relevant questions

WHAT IS A LINEAR FORM?

A linear functional or linear form is a linear map from a vector space to its field of scalars. In general, if *V* is a vector space over a field *k*, then a linear functional *f* is a function from *V* to *k* that is linear:

$$f(\vec{v}+\vec{w}) = f(\vec{v}) + f(\vec{w}) \text{ for all } \vec{v}, \vec{w} \in V$$
$$f(a\vec{v}) = af(\vec{v}) \text{ for all } \vec{v} \in V, a \in k.$$

WHAT IS AN AFFINE FUNCTION?

Let $(V,+,\cdot)$ be a linear space. A function $f : V \to R$ is called an affine functional if there exist a linear functional $f0 \in V*$ and a constant $c \in R$ such that $f(v) = f0(v) + c$, $\forall v \in V$.

WHAT IS V*?

The linear space $L(V;R)$ of all linear forms is called the dual of V and is denoted $V*$.

WHAT IS A HYPERPLANE?

Let $(V,+,\cdot)$ be a linear space. A linear subspace $W \subseteq V$ is called a (vector) hyperplane if there exists $f \in V* \setminus \{0_{V^*}\}$ such that $\ker f = W$. But this is completely irrelevant.

WHAT IS A BILINEAR FORM?

Let $(V,+,\cdot)$ and $(W,+,\cdot)$ two linear spaces. A function $g : V \times W \to R$ is called a bilinear form (bilinear map/mapping) on $V \times W$ if the following conditions are fulfilled:
1. $g(\alpha u+ \beta v,w) = \alpha g(u,w) + \beta g(v,w)$, $\forall \alpha, \beta \in R$, $\forall u,v \in V$, $\forall w \in W$;
2. $g(v,\lambda w + \mu z) = \lambda g(v,w) + \mu g(v,z)$, $\forall \lambda,\mu \in R$, $\forall v \in V$, $\forall w,z \in W$.

In the case $W = V$, a bilinear form on $V \times V$ is also called bilinear form (functional, map/mapping) on V.

WHAT IS A SYMMETRIC BILINEAR FORM?

A bilinear form $g : V \times V \to R$ is called symmetric if $g(u,v) = g(v,u), \forall u,v \in V$, respectively antisymmetric if $g(u,v) = -g(v,u), \forall u,v \in V$.

WHAT IS THE DIMENSION THEOREM?

$\text{rank}g + \dim(\ker g) = \dim V$.

WHAT IS SYLVESTER'S LAW OF INERTIA AND WHAT DOES IT MEAN?

Well, first you need to know that…
Let $(V,+,\cdot)$ be a finite-dimensional linear space and $g : V \times V \to R$ a symmetric bilinear form. If $\{b1,...,bn\}$ is a basis of V which is g-orthogonal, then rankg is precisely the number of elements among $g(b1,b1),g(b2,b2),...,g(bn,bn)$ which are non-zero.
Then…
Let $(V,+,\cdot)$ be a finite-dimensional linear space and $g : V \times V \to R$ a symmetric bilinear form. Then there exist $p,q,r \in R$ such that for every g-orthogonal basis $\{b1,...,bn\}$ of V, p, q and r represent the number of positive, negative, respectively null elements among $g(b1,b1),g(b2,b2),...,g(bn,bn)$.
Therefore…

The numbers p and q are called the positive, respectively the negative index of inertia.
The triple (p,q,r) is called the signature of g.
Of course, p + q + r = n (n = dimV); moreover, rankg = p + q.
This will help you later on.

WHAT IS A QUADRATIC FORM?

Let (V,+,·) be a linear space and g : V ×V → R a symmetric bilinear form. The function h : V → R, defined by *h(v) := g(v,v)*, v ∈ V is called the quadratic form (functional) associated to g.
In other words, you just have to find h by computing g(v,v) – then you would only have one variable, as you can see.

WHAT IS THE QUADRATIC POLYNOMIAL?

You probably don't need it. But here it is:
Let Ag B,B = $(aij)_{1\le i,j\le n}$ be the matrix of g with respect to B. If x1,...,xn ∈ R are the coefficients of a vector v ∈ V with respect to B, then

$$h(\mathbf{v}) = h(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j.$$

The right-hand side of this relation is a homogeneous polynomial of degree 2 and is called *the quadratic polynomial* associated to the quadratic form h and the basis B.

HOW DO I GET TO THE REDUCED FORM?

Well… there are several methods. Gauss, Jacobi, Eigenvalues. However, only Gauss method can be used for any quadratic form and so it's better to keep this one in mind.

HOW DOES GAUSS METHOD WORK?

The idea is to get to something like *h(x1b1 +···+ xnbn) = ω1x^2 1 + ω2x^2 2 +···+ ωnx^2 n*. You need to create squares all the way. Again, this should *always* work.

HOW TO CREATE SQUARES?

There are a few steps you should keep in mind. So.. if the general expression looks like this

$h(x) = \sum_{i,j=1}^{n} a_{ij}\, x_i x_j$ then the canonic form should look like this $h(x) = \sum_{i,j=1}^{n} b_{ii}\, x_i^2$.

Now, first thing you should look for: Is there any $a_{ij} \neq 0$?
1. YES? → $a_{ii} \neq 0$ OR $a_{jj} \neq 0$ (one of them).
    THEN h(x)=[(*all the terms containing x_i*)^2 – *rest*].
    So you have to create a square using any $a_{ii} \neq 0$ you find, then separate the terms that don't appear in you square. After this, you go back to step 1 and do the same thing with the rest.

2. NO? → $a_{ii} = a_{jj} = 0$ → example: $h(x)=x_2^2-x_1x_3$
    THEN what you have to change the coordinates (basis).
    $x_i=y_i+y_j$
    $x_j=y_i-y_j$
    and any other $x_k=y_k$ for any k≠i,j.

for the example above: rewrite expression → $x_1=y_1-y_3$ $x_3=y_1+y_3$ and $x_2=y_2$
So everything becomes $h(x)=y_2^2-2(y_1^2-y_3^2) = -2y_1^2+y_2^2+2y_3^2$
So p=2 and q=1.

WHAT SHOULD I KNOW ABOUT LIMITS?

Besides the abstract theory, the most important thing is… well… computing it. And so, if you have a function whose limit you need to find, here is what you have to take into account:

1. There are more types of limits for a function in R^n. *Iterate, directional, partial & global.* You probably need all of them.
2. First thing you should check is the Domain. Where does your function have a "problem"? That's where you have to look.
3. A limit may simply not exist.

HOW DO I COMPUTE THE ITERATE LIMIT?

For a function in R^2, there are two iterate limits. We will denote them L_{12} and L_{21}.

$L_{12}= \lim\limits_{x\to x_0} (\lim\limits_{y\to y_0} f(x,y))$

$L_{21}= \lim\limits_{y\to y_0} (\lim\limits_{x\to x_0} f(x,y))$

You should pay attention here. There might be some cases in which either L_{12} or L_{21} doesn't exist, but the global limit still does.

However, if L_{12} and L_{21} DO exist BUT they have different values ($L_{12} \neq L_{21}$), then you can be sure the global limit doesn't exist.

WHAT ABOUT THE DIRECTIONAL LIMIT?

Here all you have to do is replace x by x_0 and add a direction $u=(u_1,u_2)$.

And so $x=x_0+t*u$.

The limit becomes $\lim\limits_{x\to x_0} f(x,u)$ which is the same as $\lim\limits_{t\to 0} f(x_0+t*u)$ = $\lim\limits_{t\to 0} f(tu_1,tu_2)$

AND THE PARTIAL LIMIT?

The partial limit is, as its name says, partial. You should compute $\lim\limits_{x\to x_0} f(x,y)$ and respectively $\lim\limits_{y\to y_0} f(x,y)$.

SO WHAT'S THE GLOBAL LIMIT THING THEN?

If it exists, then it is generally equal to all the other limits (if the conditions are satisfied). However, the general purpose of an exercise is to prove that this global limit doesn't actually exist. For that we shall use the characterization with sequences.

HOW DOES IT WORK?

If you manage to find two different sequences (let's call them xn^1 and xn^2) having the same limit (so xn^1 and $xn^2 \to x_0$) BUT $f(xn^1) \to A1$ and $f(xn^2) \to A2$, where $A1 \neq A2$, then the limit doesn't exist.

WHAT SHOULD I KNOW ABOUT DIFFERENTIABILITY?

Gâteau differentiability is pretty useless. But a function f is Gâteau differentiable in x_0 if f is derivable in x_0 along each direction.

We'll assume you already know how to obtain a derivative. And a 2nd degree one.

The derivative of f along a direction u can be written as:

$$f'(\mathbf{x}_0;\mathbf{u}) := \lim_{t\to 0} \frac{1}{t}\left(f(\mathbf{x}_0+t\mathbf{u})-f(\mathbf{x}_0)\right) \in \mathbb{R}^m$$

The partial derivative is denoted by:

$$\frac{\partial f}{\partial x_k}(\mathbf{x}_0) := f'(\mathbf{x}_0; \mathbf{e}_k).$$

Where e_k=(0,0,0….1,0,….0), 1 being the k^{th} element.

The matrix in M_{nm} associated to Df (x0) (with respect to the canonical bases R^n and R^m) is called the Jacobian matrix of f in x0 and is denoted Jf(x0). The Jacobian looks like this and contains all the partial derivatives of the function:

$$J_f(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_1}(\mathbf{x}_0) & \dots & \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) \\ \frac{\partial f_1}{\partial x_2}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}_0) & \dots & \frac{\partial f_m}{\partial x_2}(\mathbf{x}_0) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_n}(\mathbf{x}_0) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}.$$

In the case m = 1, the Jacobian matrix of f in x0 is also called the gradient of f and is denoted by ∇f (x0).

$$\nabla f(\mathbf{x}_0) = \left[\frac{\partial f}{\partial x_1}(\mathbf{x}_0) \quad \dots \quad \frac{\partial f}{\partial x_n}(\mathbf{x}_0)\right]^{\mathrm{T}}.$$

WHAT'S THE FRECHET THING, ANYWAY?

For x0 ∈ D, we say that f is Frechet differentiable in x0 if there exists a linear operator T ∈L(R^n, R^m) such that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|}\left(f(\mathbf{x}) - f(\mathbf{x}_0) - T(\mathbf{x}-\mathbf{x}_0)\right) = \mathbf{0}_{\mathbb{R}^m}.$$

T is called the Frechet differential of f in x0 and is denoted by df (x0).

SO HOW DO I ACTUALLY CALCULATE F'S DIFFERENTIAL?

Here's a formula you should use.

$$\mathrm{d}f(\mathbf{x}_0)(\mathbf{u}) = f'(\mathbf{x}_0; \mathbf{u}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0) u_i = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\mathrm{d}x_i(\mathbf{u}),$$
$$\forall \mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n,$$

You don't care about u unless it is given. And so what you generally need becomes.

$$\mathrm{d}f(\mathbf{x}_0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\mathrm{d}x_i.$$

Oh… and then there's Schwartz Theorem. Here it goes:

Let $D \subseteq \mathbb{R}^n$ be an open set, $\mathbf{x}_0 \in D$, $f : D \to \mathbb{R}^m$ a function and $i, j \in \{1, \dots n\}$ with $i \neq j$. If the mixt partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ exist on a neighbourhood of $\mathbf{x}_0$ and they are continuous in $\mathbf{x}_0$, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0).$$

And the Taylor series:

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \mathrm{d}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2!}\mathrm{d}^2 f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \dots$$
$$\dots + \frac{1}{p!}\mathrm{d}^p f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \frac{1}{(p+1)!}\mathrm{d}^p f(\xi)(\mathbf{x} - \mathbf{x}_0),$$

You don't care about this

HOW TO CALCULATE $d^2f(x_0)$

$$d^2f(x_0) = \frac{\partial^2 f}{\partial x^2}(x0)dx^2 + \frac{\partial^2 f}{\partial y^2}(x0)dy^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x0)dxdy$$

WHAT TO DO FOR THE CRITICAL POINTS?

You know it – x_0 is a critical point if $f'(x_0)=0$. Here it works just the same, except there are more variables, so you just have to create a system.

HOW DO I KNOW IF MY CRITICAL POINT IS EITHER MAXIMUM, MINIMUM OR SADDLE?

Here's where Taylor's formula is useful. If you substract f(x0), it will result in something really close to $d^2f(x_0)$ (really close because there's that thing you don't care about which kinda gives you an error, but it's insignificant). And so you use $d^2f(x_0)$, since it's equal to f(x)-f(x0) (and that's basically the translation for an extreme point). You should check if $d^2f(x_0)$ is more or less than 0 by using the Gauss method ($d^2f(x_0)$ gives you a quadratic form after all). However, if the signature contains both positive and negative terms, then it's undefined – you've got a saddle point there.

WHAT TO DO WITH A RESTRICTED EXTREMA?

A restriction is just another function. You should use Lagrange here, but I'll show you an example. Let f(x,y,z)=x-2y+2z. You've got some restriction such as $x^2+y^2+z^2-9=0$ and you should find the extreme points.

As I said, a restriction is just another function. Let's call it g. $g(x,y,z)= x^2+y^2+z^2-9$.

Combining our two functions, we get a 3rd function, L (from Lagrange), which looks like this:

L(x,y,z)=f + $\sum_{i=1}^{p} \lambda_i g_i$

Or, shorter, L(x,y,z)= f + λg

Which is, in our case, equal to $x-2y+2z+\lambda(x^2+y^2+z^2-9)$

Now all you have to do is find the extrema for this new function, by computing a system that looks like this:

$$\begin{cases} dL(x^0)=0 \\ g(x,y,z)=0 \end{cases}$$

Then you just do the normal calculus, including that scary lambda.

WHAT IS THERE TO BE KNOWN ABOUT INTEGRABILITY?

We'll assume you know the basics about how to integrate a function. And the formulae.
Now, there are some *special* functions requiring some more theorems so that you can solve them.
Functions such that:

1. Rational functions (I'll just copy-paste this one for you)

$$f(x) = G(x) + \frac{H(x)}{Q(x)} = G(x) + \sum\nolimits_1 \frac{A_{k,m}}{(x-x_k)^m} + \sum\nolimits_2 \frac{B_{k,m}x + C_{k,m}}{(x^2+p_kx+q_k)^m},$$

where:

- G is a polynomial function (equal to 0 when $\deg P < \deg Q$),
- H still a polynomial function with $\deg H < \deg Q$,
- $\sum_1$ is a finite sum with respect to all real roots x_k of Q and
- $\sum_2$ is a finite sum with respect to all complex roots of Q (with $p_k, q_k \in \mathbb{R}$ such that $p_k^2 - 4q_k < 0$).

And Gauss-Ostrogradski:

If Q has multiple roots, computing the antiderivative of $\frac{P(x)}{Q(x)}$ can be also done by *Gauss-Ostrogradski method*, based on the formula

$$(*) \qquad \int \frac{P(x)}{Q(x)}dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)}dx, \; x \in I,$$

where

- $Q_1 \in \mathbb{R}[X]$ is the greatest common divisor of Q and Q',
- $Q_2 = \frac{Q}{Q_1}$ and
- P_1, P_2 are polynomials having the degree one unit smaller than $\deg Q_1$, respectively $\deg Q_2$.

2. Trigonometric functions

- For *trigonometric integrals* of the form

$$\int E(\sin x, \cos x)dx, \; x \in I = (-\pi, \pi),$$

where E is a rational function of two variabiles: substitution $\operatorname{tg}\frac{x}{2} = t$.

Short reminder:

Since $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $x = 2\operatorname{arctg} t$, $dx = \frac{2dt}{1+t^2}$: transformation into a rational function in the new variable t.

ALSO,

There are some cases in which the computations can be simplified, by avoiding the standard substitution $\operatorname{tg}\frac{x}{2}=t$:

i) if $E(-\sin x, \cos x) = -E(\sin x, \cos x)$, *i.e.* E is odd in $\sin x$, then the substitution $\cos x = t$ is recommended;

ii) if $E(\sin x, -\cos x) = -E(\sin x, \cos x)$, *i.e.* E is odd in $\cos x$, then the substitution $\sin x = t$ is recommended;

iii) if $E(-\sin x, -\cos x) = E(\sin x, \cos x)$, *i.e.* E is even in $\sin x$ and $\cos x$, then the substitution $\operatorname{tg} x = t$ is recommended.

3. Irrational integrals – there are Euler's substitutions you should know

We still apply the substitution method for computing the so-called *irrational integrals*, in order to reduce them to integrals of rational functions.
We use the *Euler substitutions* for integrals of the form

$$\int E\left(x, \sqrt{ax^2+bx+c}\right) dx,\ \ x \in I,$$

with $a, b, c \in \mathbb{R}$ and E a rational function of two variabiles. The change of variable is done according to each of the following case:

i) $\sqrt{ax^2+bx+c} = \pm x\sqrt{a} \pm t$, when $a > 0$;

ii) $\sqrt{ax^2+bx+c} = \pm tx \pm \sqrt{c}$, when $c > 0$;

iii) $\sqrt{ax^2+bx+c} = t(x-x_0)$, when $b^2-4ac>0$, where x_0 is a real root of the equation $ax^2+bx+c=0$.

ALSO,

For irrational integrals of the form

$$\int E\left(x, \left(\frac{ax+b}{cx+d}\right)^{p_1/q_1}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{p_k/q_k}\right) dx,\ \ x \in I,$$

where E is a rational function of $k+1$ ($k \in \mathbb{N}^*$) real variables, $a, b, c, d \in \mathbb{R}$, $a^2+b^2+c^2+d^2 \neq 0$, $cx+d \neq 0$, $\forall x \in I$, $\dfrac{ax+b}{cx+d} > 0$, $\forall x \in I$, $p_i \in \mathbb{Z}$, $q_i \in \mathbb{N}^*$, $\forall i = \overline{1,k}$, we use the substitution $\dfrac{ax+b}{cx+d} = t^{q_0}$, where q_0 is the least common multiple of $q_1, q_2, \ldots, q_k$.

4. Bynomial integrals – this is there Chebyshev substitutions are needed:

Chebyshev substitutions are used for the calculus of *bynomial integrals*, having the form

$$\int x^p (ax^q + b)^r \, dx, \ x \in I,$$

where $a \in \mathbb{R}^*$, $b \in \mathbb{R}$ and $p, q, r \in \mathbb{Q}$. The computation of such integrals is reduced to that of the antiderivatives of irrational functions only in the following three cases:

i) $r \in \mathbb{Z}$: the substitution $x = t^m$, with m being the least common multiple of p and q;

ii) $\frac{p+1}{q} \in \mathbb{Z}$: the substitution $ax^q + b = t^\ell$, where ℓ is the denominator of r.

iii) $\frac{p+1}{q} + r \in \mathbb{Z}$: the substitution $a + bx^{-q} = t^\ell$, ℓ being the denominator r.

Computing integrals of the form

$$\int E\left(a^{r_1 x}, a^{r_2 x}, \ldots, a^{r_n x}\right) \, dx,$$

where $a \in \mathbb{R}_+^* \setminus \{1\}$, $r_1, r_2, \ldots, r_n \in \mathbb{Q}$ and E is a rational functions of n $(n \in \mathbb{N}^*)$ real variables can be done by the substitution $a^x = t^\nu$, where $t > 0$ and ν is the least common multiple of the denominators of $r_1, r_2, \ldots, r_n$.

These should be enough so you can compute any possible integral for now ☺

WHAT DO I HAVE TO DO WITH MULTIPLE INTEGRALS?

Let the fun begin.

So a Jordan measure is basically what you would call an area or a volume, but translated in R^n. You don't need it, most likely. But what you ACTUALLY need to know is how to compute a multiple integral. This is the main idea:

$$\iint_D f(x,y)dxdy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy \right) dx$$

Well... how to do that? I'll just leave this pic since my eyes are bleeding now.

Watch below.

THE DOUBLE INTEGRAL ON COMPACT SETS

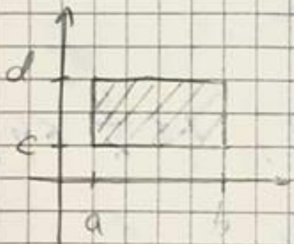
$f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$

$J = \iint_\Delta f(x,y)\, dx\, dy$

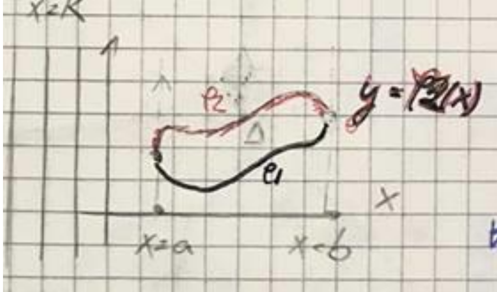
1. $\Delta = \underset{x}{[a,b]} \times \underset{y}{[c,d]}$, f int basic $/_{ox,oy}$

$\Rightarrow J = \int_a^b \Big(\overbrace{\int_c^d f}^{E(x)}\Big) dx\, dy$

$= \int_c^d \Big(\underbrace{\int_a^b f(x,y)\, dx}_{E(y)}\Big) dy$

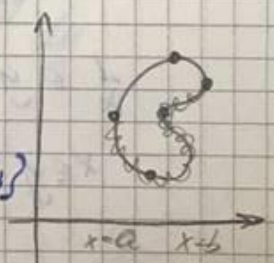


2. Δ = simple (basic) $/_{oy}$ $\begin{cases} a \le x \le b \\ \varphi_1(x) \le y \le \varphi_2(x) \quad \varphi_1, \varphi_2 \text{ cont.} \\ y = \varphi_1(x) \end{cases}$



not basic

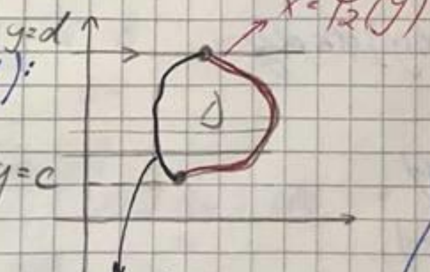
$\forall \,//oy \cap Fr_{G_f} = \{A_1, A_2\}$



$\Rightarrow J = \int_a^b \Big(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy\Big) dx$

3. (symmetry):

Δ = basic $/_{ox}$

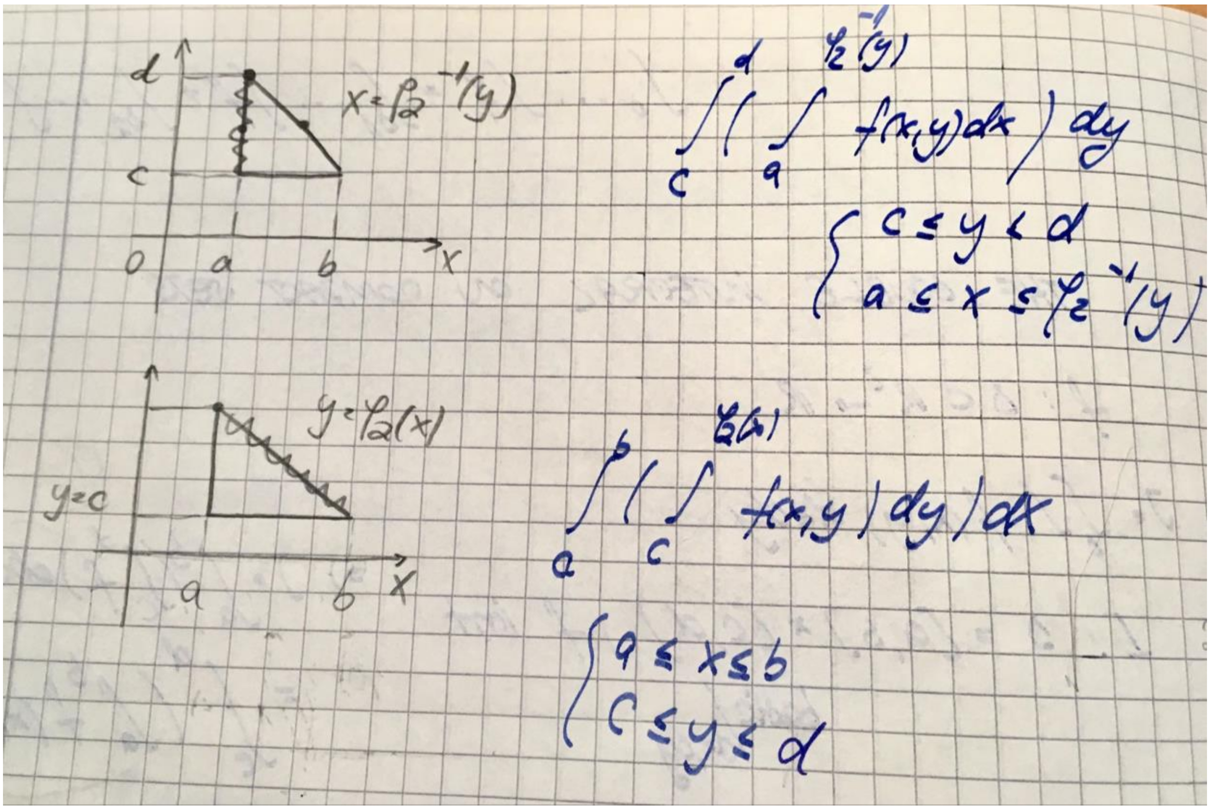


$\forall \,//ox \cap Fr_{G_f} = \{B_1, B_2\}$

$\begin{cases} c \le y \le b \\ \psi_1(y) \le x \le \psi_2(y) \\ \psi_1, \psi_2 \text{ cont.} \end{cases}$

$J = \int_c^d \Big(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\Big) dy$

Here’s the basic thing explained a little

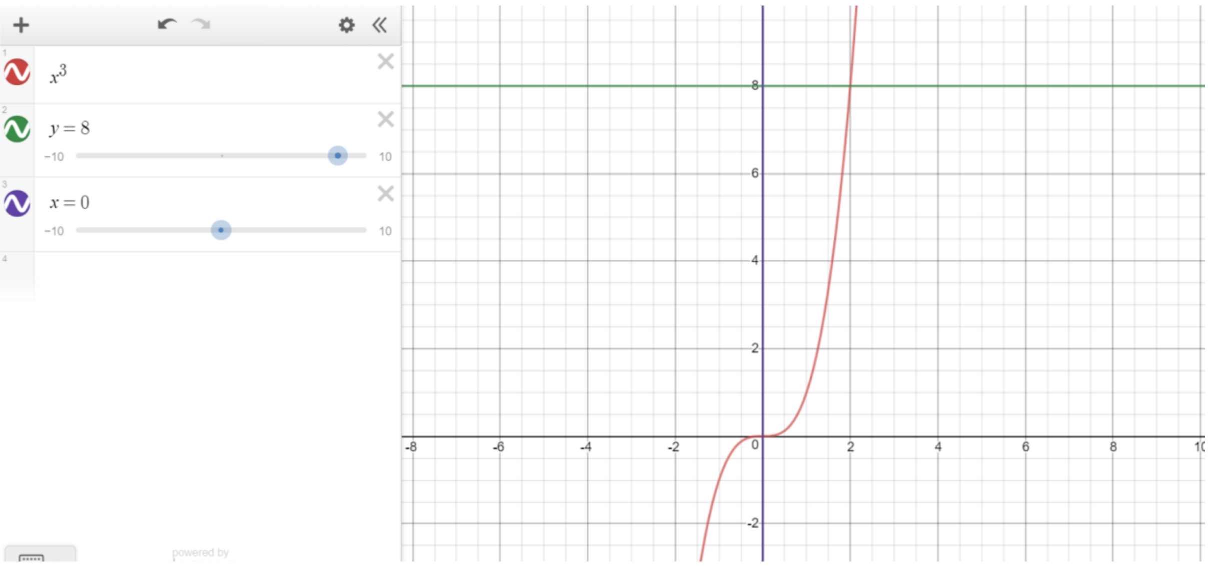


1. Best case scenario

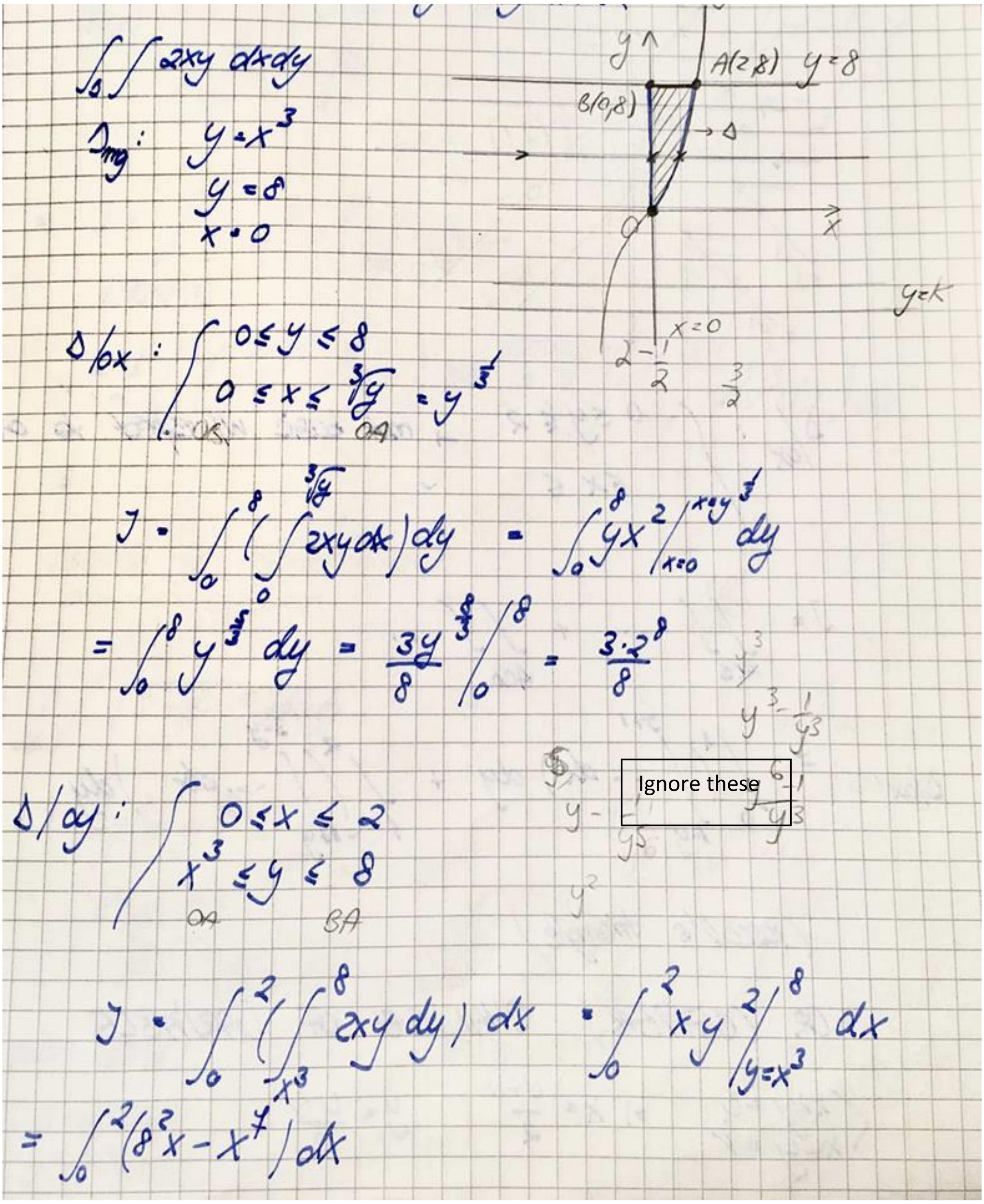
You’ve got some integral that generally looks like this:

$$\iint_D f(x,y)\,dxdy$$

Let’s take an example.



Our Domain is simple with respect to both OX and OY (this means that our inf and sup are continuous). Lucky us. And now, because I'm really tired, I'll just insert this pic containing what it should look like.



2. Worst case scenario – you'll have to split your domain or just find another solution (change of coordinates, variables etc.)