RUNSER Laure

DM de mathématiques n°1

02/11/2020

n° 21955060
L2 info, groupe 3

Exercice 1

1) sin et exp admettent toutes les deux un DL à l'ordre 4 en 0.

$$\sin(x) = x - \frac{x^3}{6} + o_0(x^4)$$

$$\exp(u) = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + o_0(u^4)$$

avec $u = x - \frac{x^3}{6}$ on a donc

$$\exp(\sin(x)) = 1 + x - \frac{x^3}{6} + o_0(x_4) + \frac{\left(x - \frac{x^3}{6} + o_0(x^4)\right)^2}{2} + \frac{\left(x - \frac{x^3}{6} + o_0(x^4)\right)^3}{6} + \frac{\left(x - \frac{x^3}{6} + o_0(x_4)\right)^4}{24} + o_0(x^4)$$

$$\exp(\sin(x)) = 1 + x - \cancel{\frac{x^3}{6}} + \frac{x^2}{2} - \frac{2}{2} \cdot x \cdot \frac{x^3}{6} + \cancel{\frac{x^3}{6}} + \frac{x^4}{24} + o_0(x^4)$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{6} + \frac{x^4}{24} + o_0(x^4)$$

$$= 1 + x + \frac{x^2}{2} - \frac{3x^4}{24} + o_0(x^4)$$

$$\boxed{\exp(\sin(x)) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + o_0(x^4)}$$

2) $\sqrt{1 + \sqrt{1+x}} = \left(1 + (1+x)^{1/2}\right)^{1/2} = f(x)$

$(1+x)^{1/2}$ admet un DL à l'ordre 2 quand $x \to 0$

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + o_0(x^2) \qquad (*)$$

donc $\left(1 + \sqrt{1+x}\right)^{1/2} = \left(1 + 1 + \frac{x}{2} - \frac{x^2}{8} + o_0(x^2)\right)^{1/2}$

$$= \left(2 + \frac{x}{2} - \frac{x^2}{8} + o_0(x^2)\right)^{1/2}$$

$$= \left(2\left(1 + \frac{x}{4} - \frac{x^2}{16} + o_0(x^2)\right)\right)^{1/2}$$

$$= \sqrt{2}\left(1 + \underbrace{\frac{x}{4} - \frac{x^2}{16} + o_0(x^2)}_{\to 0 \text{ quand } x \to 0}\right)^{1/2}$$

donc en réutilisant $(*)$ avec u à la place de x on trouve

$$f(x) = \sqrt{2}\left(1 + \frac{1}{2}\left(\frac{x}{4} - \frac{x^2}{16} + o_0(x^2)\right) - \frac{1}{8}\left(\frac{x}{4} - \frac{x^2}{16} + o_0(x^2)\right)^2 + o_0(x^2)\right)$$

$$= \sqrt{2}\left(1 + \frac{x}{8} - \frac{x^2}{32} + o_0(x^2) - \frac{1}{8}\left(\frac{x^2}{16} - 2\,\frac{x}{4} \cdot \frac{x^2}{16} + o_0(x^2)\right) + o_0(x^2)\right)$$

$$= \sqrt{2}\left(1 + \frac{x}{8} - \frac{x^2}{16.2} - \frac{x^2}{16.8} + o_0(x^2)\right)$$

$$= \sqrt{2}\left(1 + \frac{x}{8} - \frac{(8x^2 + 2x^2)}{16.2.8} + o_0(x^2)\right)$$

$$= \sqrt{2} + \frac{x}{4\sqrt{2}} - \frac{\sqrt{2} \times 10x^2}{16.2.8} + o_0(x^2)$$

donc $\boxed{f(x) = \sqrt{2} + \frac{x}{4\sqrt{2}} - \frac{5x^2}{64\sqrt{2}} + o_0(x^2)}$

3) $f(x) = \dfrac{\cos(x)}{1+\tan(x)} = \cos(x) \times \dfrac{1}{1+\tan(x)}$

or :

$\cos(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + o_0(x^4)$

$\tan(x) = x + \dfrac{x^3}{3} + o_0(x^4)$

donc $f(x) = \left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + o_0(x^4)\right) \times \overbrace{\dfrac{1}{1+\underbrace{\left(x + \dfrac{x^3}{3} + o_0(x^4)\right)}_{\to 0 \text{ qd } x \to 0}}}^{=g(x)}$

or $\dfrac{1}{1+u} = 1 - u + u^2 - u^3 + u^4 + o_0(u^4)$ qd $u \to 0$

donc $g(x) = 1 - x - \dfrac{x^3}{3} + \left(x + \dfrac{x^3}{3}\right)^2 - x^3 + x^4 + o_0(x^4)$ on simplifie car les autres termes sont d'un ordre > 4.

$g(x) = 1 - x - \dfrac{x^3}{3} + x^2 + \dfrac{2x^4}{3} - x^3 + x^4 + o_0(x^4)$

$= 1 - x + x^2 - \dfrac{4x^3}{3} + \dfrac{5x^4}{3} + o_0(x^4)$

donc $f(x) = \left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + o_0(x^4)\right) \cdot \left(1 - x + x^2 - \dfrac{4x^3}{3} + \dfrac{5x^4}{3} + o_0(x^4)\right)$

$f(x) = 1 - x + x^2 - \dfrac{4x^3}{3} + \dfrac{5x^4}{3} + o_0(x^4) - \dfrac{x^2}{2} + \dfrac{x^3}{2} - \dfrac{x^4}{2} + o_0(x^4) + \dfrac{x^4}{24} + o_0(x^4)$

$\boxed{f(x) = 1 - x + \dfrac{x^2}{2} - \dfrac{5x^3}{6} + \dfrac{29}{24}x^4 + o_0(x^4)}$

4) $f(x) = \dfrac{1}{1+e^x}$

or $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + o_0(x^4) \underset{x \to 0}{\to} 0$

et $\dfrac{1}{1+u} = 1 - u + u^2 - u^3 + u^4 + o_0(x^4)$

donc $f(x) = \dfrac{1}{1 + 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + o_0(x^4)} = \dfrac{1}{2 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + o_0(x^4)}$

$f(x) = \dfrac{1}{2} \times \dfrac{1}{1 + \underbrace{\dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{12} + \dfrac{x^4}{48} + o_0(x^4)}_{\to 0 \text{ qd } x \to 0}}$

on pose donc $u = \dfrac{x}{2} + \dfrac{x^2}{4} + \dfrac{x^3}{12} + \dfrac{x^4}{48} + o_0(x^4)$

donc $f(x) = \dfrac{1}{2}\left(1 - u + u^2 - u^3 + u^4 + o_0(u^4)\right) = \dfrac{1}{2}\left(1 - \left(\dfrac{x}{2} + \cancel{\dfrac{x^2}{4}} + \dfrac{x^3}{12} + \cancel{\dfrac{x^4}{48}} + o_0(x^4)\right) + \left(\cancel{\dfrac{x^2}{4}} + \dfrac{x^3}{4} + \cancel{\dfrac{7x^4}{48}} + o_0(x^4)\right)\right.$

$\left. - \left(\dfrac{x^3}{8} + \cancel{\dfrac{3x^4}{16}} + o_0(x^4)\right) + \left(\cancel{\dfrac{x^4}{16}} + o_0(x^4)\right) + o_0(x^4)\right)$

donc $\boxed{f(x) = \dfrac{1}{2}\left(1 - \dfrac{x}{2} + \dfrac{x^3}{24} + o_0(x^4)\right)}$

6) $f(x) = (1+x)^{\frac{1}{1+x}} = \exp\left(\frac{1}{1+x}\ln(1+x)\right)$

or $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o_0(x^3)$

et $\frac{1}{1+x} = 1 - x + x^2 - x^3 + o_0(x^3)$

donc $f(x) = \exp\left(\left(1-x+x^2-x^3+o_0(x^2)\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + o_0(x^3)\right)\right)$

$f(x) = \exp\left(x - \frac{x^2}{2} + \frac{x^3}{3} - x^2 + \frac{x^3}{2} + x^3 + o_0(x^3)\right)$

$f(x) = \exp\Big(\underbrace{x - \frac{3x^2}{2} + \frac{11x^3}{6} + o_0(x^3)}_{\to 0 \text{ qd } x \to 0}\Big)$

or $\exp(u) = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + o_0(u^3)$

donc $f(x) = 1 + x - \frac{3x^2}{2} + \frac{11x^3}{6} + \frac{1}{2}\left(x^2 - \frac{6x^3}{2}\right) + \frac{1}{6}x^3 + o_0(x^3)$

$$\boxed{f(x) = 1 + x - x^2 + \frac{x^3}{2} + o_0(x^3)}$$

Exercice 2

1) $f(x) = \dfrac{\sin(x) - x\cos(x)}{\sin(x) - x}$

or $\sin(x) = x - \dfrac{x^3}{6} + o_0(x^3)$

et $\cos(x) = 1 - \dfrac{x^2}{2} + o_0(x^3)$

donc $f(x) = \left(x - \dfrac{x^3}{6} + o_0(x^3) - x\left(1 - \dfrac{x^2}{2} + o_0(x^3)\right)\right) \times \dfrac{1}{x - \dfrac{x^3}{6} - x + o_0(x^3)}$

$f(x) = \left(x - \dfrac{x^3}{6} - x + \dfrac{x^3}{2} + o_0(x^3)\right) \times \dfrac{1}{\dfrac{x^3}{6} + o_0(x^3)}$

$f(x) = \dfrac{-\dfrac{2x^3}{6} + o_0(x^3)}{\dfrac{x^3}{6} + o_0(x^3)}$

donc $\boxed{\lim\limits_{x \to 0} f(x) = -2}$

2) $f(x) = \left(\dfrac{\sin(x)}{x}\right)^{1/x} = \exp\left(\dfrac{1}{x}\ln\left(\dfrac{\sin(x)}{x}\right)\right)$

or $\sin(x) = x - \dfrac{x^3}{6} + o_0(x^3)$ donc $f(x) = \exp\left(\dfrac{1}{x}\ln\left(\dfrac{x - \dfrac{x^3}{6} + o_0(x^3)}{x}\right)\right)$

$f(x) = \exp\left(\dfrac{1}{x}\ln\left(1 - \underbrace{\dfrac{x^2}{6} + o_0(x^3)}_{\to 0 \text{ qd } x \to 0}\right)\right)$

or $\ln(1+u) = u + o_0(u)$

donc $f(x) = \exp\left(\dfrac{1}{x} \times \dfrac{-x^2}{6} + o_0(x^2)\right)$

$f(x) = \exp\left(\dfrac{-x}{6} + o_0(x^2)\right)$

donc $\boxed{\lim\limits_{x \to 0} f(x) = e^0 = 1}$

3) $f(x) = \dfrac{\sin(x) - x}{\tan(x) - x}$

or $\sin(x) = x - \dfrac{x^3}{6} + o_0(x^3)$

$\tan(x) = x + \dfrac{x^3}{3} + o_0(x^3)$

donc $f(x) = \dfrac{\dfrac{-x^3}{6} + o_0(x^3)}{\dfrac{x^3}{3} + o_0(x^3)} = \dfrac{-\dfrac{1}{2} \times \dfrac{x^3}{3} + o_0(x^3)}{\dfrac{x^3}{3} + o_0(x^3)}$

donc $\boxed{\lim\limits_{x \to 0} f(x) = \dfrac{-1}{2}}$

4) $f(x) = \dfrac{\cos(x) - \exp(x^2)}{x\tan(x) - x^2}$

or $\cos(x) = 1 - \dfrac{x^2}{2} + o_0(x^3)$

$\exp(x) = 1 + x + o_0(x)$

donc $\exp(x^2) = 1 + x^2 + o_0(x^3)$

$\tan(x) = x + \dfrac{x^3}{3} + o_0(x^3)$

donc $f(x) = \dfrac{\cancel{1} - \dfrac{x^2}{2} + o_0(x^3) - (\cancel{1} + x^2 + o_0(x^3))}{x \times \left(\cancel{x} + \dfrac{x^3}{3} + o_0(x^3)\right) - \cancel{x^2}}$

$f(x) = \dfrac{-\dfrac{3}{2}x^2 + o_0(x^3)}{\dfrac{x^4}{3} + o_0(x^3)}$

$f(x) = \dfrac{-9}{2} \times \dfrac{x^2 + o_0(x^3)}{x^4 + o_0(x^3)} = \dfrac{-9}{2} \times \dfrac{1}{x^2} + o_0(x^3)$

donc $\boxed{\lim\limits_{x \to 0^+} f(x) = -\infty}$

5) $f(x) = \dfrac{\ln(x)}{(x-1)^2} - \dfrac{1}{x-1}$

changement de variable : $t = x - 1$ alors $x \to 1 \Rightarrow t \to 0$

$x = t + 1$

$f(t) = \dfrac{\ln(t+1)}{t^2} - \dfrac{1}{t}$

or $\ln(1+t) = t - \dfrac{t^2}{2} + o_0(t^2)$

donc $f(t) = \dfrac{\cancel{t}}{t^2} - \dfrac{1}{2}\left(\dfrac{t^2}{t^2}\right) - \cancel{\dfrac{1}{t}} + o_0(t^2) = -\dfrac{1}{2} + o_0(t^2) \xrightarrow[\substack{t \to 0 \\ x \to 1}]{} -\dfrac{1}{2}$

donc $\boxed{\lim\limits_{x \to 1} f(x) = -\dfrac{1}{2}}$

Exercice 3

1) Soit $f(x) = \exp(x)$ $\quad \forall n \in \mathbb{Z}, n \geqslant 0 \quad f^{(n)}(x) = \exp(x) = f(x)$

On applique la formule de Taylor-Lagrange entre 0 et 1,

sachant que $\forall n \in \mathbb{Z}, n \geqslant 0 \quad (1-0)^n = 1^n = 1$

On a :

$$f(1) = f(0) + f(0) + \frac{f(0)}{2!} + \dots + \frac{f(0)}{n!} + \frac{f(\varepsilon)}{(n+1)!}$$

soit

$$e = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{f(\varepsilon)}{(n+1)!}$$

On pose $\theta_n = f(\varepsilon)$ et on retrouve donc bien la formule

$$\boxed{e = 1 + \sum_{i=1}^{n} \left(\frac{1}{n!}\right) + \frac{\theta_n}{(n+1)!}}$$

2) Montrons que e est irrationel.

D'après 1), $e = \sum_{k=0}^{n} \left(\frac{1}{k!}\right) + r_n$ où $r_n = \frac{\theta_n}{(n+1)!}$

D'après la méthode de construction de e utilisée dans 1), on a

$$r_n = \frac{1}{(n+1)!} + r_{n+1}, \quad \forall n \in \mathbb{Z}, n > 0$$

Montrons que $0 \leqslant r_n \leqslant \frac{e}{n!}$

- $\forall k \in \mathbb{R}, k! > 0$ donc $\forall n, a \in \mathbb{R} \; (n+a)! \geqslant 0$

  donc $\forall n, a \in \mathbb{R} \quad r_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{\theta_n}{(n+a)!} \geqslant 0$

- $e = \sum_{k=0}^{n} \left(\frac{1}{k!}\right) + r_n$ donc $r_n = e - \sum_{k=0}^{n} \left(\frac{1}{k!}\right)$