

Exercice 1

1) Pour \Rightarrow_R :

$$\begin{aligned}
 \Gamma, A \Rightarrow B, \Delta &\equiv (A \wedge (\neg r)) \Rightarrow (B \vee (V\Delta)) \\
 &\equiv \neg(A \wedge (\neg r)) \vee B \vee (V\Delta) \\
 &\equiv \neg A \vee \neg(\neg r) \vee B \vee (V\Delta) \\
 &\equiv \neg(\neg r) \vee (A \Rightarrow B) \vee (V\Delta) \\
 &\equiv (\neg r) \Rightarrow (A \Rightarrow B) \vee (V\Delta) \\
 &\equiv \Gamma \vdash A \Rightarrow B, \Delta
 \end{aligned}$$

Pour \Rightarrow_L :

$$\begin{aligned}
 A \Rightarrow B, \Gamma \vdash \Delta &\equiv ((A \Rightarrow B) \wedge (\neg r)) \Rightarrow (V\Delta) \\
 &\equiv \neg((A \Rightarrow B) \wedge (\neg r)) \vee (V\Delta) \\
 &\equiv \neg(A \Rightarrow B) \vee \neg(\neg r) \vee (V\Delta) \\
 &\equiv \neg(\neg A \vee B) \vee \neg(\neg r) \vee (V\Delta) \\
 &\equiv (A \wedge \neg B) \vee \neg(\neg r) \vee (V\Delta) \\
 &\equiv (A \vee (\neg \neg r) \vee (V\Delta)) \wedge (\neg B \vee \neg(\neg r) \vee (V\Delta)) \\
 &\equiv ((\neg r) \Rightarrow (A \vee (V\Delta))) \wedge ((B \wedge (\neg r)) \Rightarrow (V\Delta)) \\
 &\equiv (\Gamma \vdash A, \Delta) \wedge (B, \Gamma \vdash \Delta)
 \end{aligned}$$

3)

$$\begin{array}{c}
 \frac{\frac{\frac{}{A \Rightarrow \neg B, A \vdash A} \wedge_x}{A \Rightarrow \neg B \vdash A, \neg A} \neg_r}{A \Rightarrow B, A \Rightarrow \neg B \vdash \neg A} \Rightarrow_L \\
 \frac{\frac{\frac{\frac{}{A, B \vdash A} \wedge_x}{B \vdash A, \neg A} \neg_r}{B \vdash A, \neg A} \neg_r \quad \frac{\frac{\frac{}{B \vdash \neg A, B} \wedge_x}{\neg B, B \vdash \neg A} \neg_r}{\neg B, B \vdash \neg A} \neg_r}{A \Rightarrow \neg B, B \vdash \neg A} \Rightarrow_L \\
 \frac{}{A \Rightarrow B, A \Rightarrow \neg B \vdash \neg A} \Rightarrow_L
 \end{array}$$

Exercise 2

1)

$$\begin{aligned} (A, r \vdash \Delta) \vee (r \vdash A, \Delta) &\equiv ((A \wedge (\lambda r)) \Rightarrow (V_\Delta)) \vee ((\lambda r) \Rightarrow (A \vee (V_\Delta))) \\ &\equiv \neg A \vee \neg (\lambda r) \vee (V_\Delta) \vee \neg (\lambda r) \vee A \vee (V_\Delta) \\ &\equiv \neg A \vee A \vee \neg (\lambda r) \vee (V_\Delta) \\ &\equiv (\lambda r) \Rightarrow (V_\Delta) \\ &\equiv r \vdash \Delta \end{aligned}$$

Donc pour toute affectation v , v satisfait les deux hypothèses de Cutssi $\Leftrightarrow v$ satisfait la conclusion de Cut.

Donc si les deux hypothèses sont prouvables dans LK , alors la conclusion l'est aussi.

Exercise 3

1)

$$\begin{array}{c}
 \frac{}{A_1} \quad \frac{}{A_2} \\
 \frac{A \vdash A, B, C \quad B, A \vdash B, C}{\Rightarrow_L} \quad \frac{}{A_x} \\
 \frac{A \Rightarrow B, A \vdash B, C \quad C, A \Rightarrow B, A \vdash C}{\Rightarrow_L} \\
 \frac{A \Rightarrow B, A \vdash A, C \quad B \Rightarrow C, A \Rightarrow B, A \vdash C}{\Rightarrow_L} \\
 \frac{A \Rightarrow (B \Rightarrow C), A \Rightarrow B, A \vdash C}{\Rightarrow_R} \\
 \frac{A \Rightarrow (B \Rightarrow C), A \Rightarrow B \vdash A \Rightarrow C}{\Rightarrow_R} \\
 \frac{A \Rightarrow (B \Rightarrow C) \vdash (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}{\Rightarrow_R} \\
 \frac{}{\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))} \Rightarrow_R
 \end{array}$$

$$\frac{\Gamma, A \vdash B, \Delta \Rightarrow_R \quad \Gamma \vdash A \Rightarrow B, \Delta}{\Rightarrow_R}$$

$$\frac{\Gamma \vdash A, \Delta, B, \Gamma \vdash \Delta \Rightarrow_L \quad A \Rightarrow B, \Gamma \vdash \Delta}{\Rightarrow_L}$$

2)

$$\frac{\frac{\frac{\frac{}{B, A \vdash B, A}}{B \vdash A, A \Rightarrow B} \Rightarrow_R}{\vdash A \Rightarrow B, B \Rightarrow A} \Rightarrow_R}{\vdash (A \Rightarrow B) \vee (B \Rightarrow A)} \vee_R$$

3)

$$\frac{\frac{\frac{}{A \vdash A}}{A \vdash A} A_x}{\vdash A, A} \rightarrow_R \quad \frac{}{\vdash A, A} \rightarrow_L \quad \frac{}{\vdash A, A} \Rightarrow_R$$

4)

$$\begin{array}{r} \text{---} \quad \text{A} \\ \vdash \text{A} \quad \text{A} \\ \text{---} \quad \neg \text{A} \\ \vdash \text{A}, \neg \text{A} \\ \text{---} \quad \vee \text{A} \\ \vdash \text{A} \vee \text{A} \\ \text{---} \quad \neg \text{A} \\ \neg (\text{A} \vee \neg \text{A}) \vdash \\ \text{---} \quad \neg \text{A} \\ \vdash \neg \neg (\text{A} \vee \neg \text{A}) \end{array}$$