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THE TEMPERATURE COEFFICIENT OF INDUCTANCE, WITH SPECIAL REFERENCE TO THE VALVE GENERATOR*

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Summary—*This paper discusses the manner in which the inductance of a coil depends on temperature, and has been written as a contribution to the problem of the temperature coefficient of frequency of a valve generator. Preliminary consideration shows that when a coil expands without change of form, it will have a temperature coefficient of inductance equal to the coefficient of linear expansion of the metal of which it is made. Such copper coils would give a negative temperature coefficient of frequency of about eight parts in* 10^6 *per degree centigrade. Since valve generators are sometimes found to have a temperature coefficient much greater than eight parts in* 10^6, *it is necessary to consider the effect of deforming the coil.*

The first type of coil examined is a concentric cable; this form is amenable to exact analysis, and the results are a valuable guide to the behavior of more common forms of coil. The temperature coefficient of "internal inductance" and of self-capacitance is calculated and found to modify the net temperature coefficient by a negligible amount; accordingly, it is argued these effects can be ignored in all forms of coil. The core is then supposed to buckle under temperature stress and this is found to make the net coefficient less than that of linear expansion. The next example is a long solenoid wound on a form which constrains one diameter, so that the cross section becomes elliptical. The effect of this deformation is found to be a second-order quantity and tends to reduce the temperature coefficient. The result for a long solenoid is considered to be an upper limit for helical coils. An approximate solution is then obtained for the elliptical deformation of a single-turn circle and the result is found to be negligible. The next example is a two- or three-turn coil in which the radial expansion is greater than the axial. It is here found that if the coil form prohibits axial expansion, the temperature coefficient of inductance would be some 40 per cent greater than that of linear expansion. The circuit as a whole, coil plus leads, is then examined, and it is argued that with probable dispositions of the coil and leads, the net coefficient cannot usually be appreciably greater than that of linear expansion.

It is suggested, in conclusion, that if a valve generator is found to have a temperature coefficient of frequency which is notably greater than half the coefficient of linear expansion of the metal of the coil, then the dominant cause is not change of inductance.

I. INTRODUCTION

THE conditions of the present day demand that the frequency of a radio transmitter shall remain constant within extremely narrow limits. For immobile stations working on an assigned wavelength, the desired constancy can be attained by the use of crystal or tuning fork control, or by the help of an elaborate drive system. But these methods of solution are impracticable for mobile stations, being

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precluded by considerations of space and cost and ability to work over a range of wavelength.

It is obvious that temperature must affect the frequency of a simple generator since it affects the geometrical dimensions of the coils and capacitances. Possibly it may affect the frequency through other agencies also. The temperature of the coils and condensers will depend both on the ambient temperature of the surroundings and on the power wasted in them. If the temperature depends appreciably on the power loss, the coils and condensers will not be at the same temperature, since they have very different heat capacities, cooling surfaces, and power losses.

If coils and condensers can be built so as to be self-compensating for temperature change, they will improve the frequency stability of a given radio transmitter. Attempts have been made to produce such coils.

The temperature coefficient of a coil must be measured by observing the relation between its temperature and the frequency of some generator of which it forms a part. A coefficient so obtained is the true temperature coefficient of that coil when associated with the particular generator, but it does not follow necessarily that the same coefficient would be obtained if it were associated with some other generator, and it does not necessarily follow that this coefficient is the temperature coefficient of inductance, tempting as it may be to assume that it is.

There are not many recorded measurements of the temperature coefficient of a coil and generator, but such as there are show that temperature coefficients of frequency of the order of 30 parts in 10^6 per degree centigrade are met with. The writer has discussed the problem with those responsible for building radio transmitters and, rightly or wrongly, has gathered the impression that temperature coefficients of 30 parts in 10^6 per degree centigrade are by no means uncommon and may be exceeded, also that the coefficient is commonly attributed to change of inductance.

When the problem of inductance change with temperature is considered from its simplest basic principles, it is astonishing that the coefficient can be as large as 30×10^6 per degree centigrade. This has led the writer to make a detailed and rather ponderous mathematical attack on the problem.

II. Temperature Coefficient of an Unsupported Metal Coil Which Expands Uniformly in all Dimensions

In the electromagnetic system of units, in which magnetic permeability is considered to be a dimensionless constant, an inductance has the dimensions of a length. Hence it follows that if a coil expands uniformly, without change of form, the temperature coefficient of inductance must be equal to the temperature coefficient of linear expansion of the metal of which it is made. A similar argument applies to the temperature coefficient of capacitance.

The tables of Kaye and Laby give the temperature coefficient of linear expansion of copper as 16.7 parts in 10^6 per degree centigrade. Thus for a copper coil, the temperature coefficient of inductance should be of the order of 17 parts in 10^6, whereas the apparent value has been observed to be twice this, or more.

A discrepancy of at least 2 to 1 is surprising, and shows that the problem must be examined in detail before attempting to design self-compensating devices.

The same tables state that the temperature coefficient of brass is 18.9, bronze 17.7, gunmetal 18.1, and phospher bronze 16.8. Hence, any alloy which is predominantly copper has a temperature coefficient near 17×10^{-6} per degree centigrade. The coefficient for aluminium is 25.5, duralumin 22.6, and aluminium bronze 17.0.

Hence, the temperature coefficient of inductance cannot be sensitive to the exact composition of the alloy. It is probable that the constraint of supports and the effect of overstrain cause a coil to deform when it is heated: Is it reasonable to assume that inductance is sufficiently sensitive to shape as to account for the temperature coefficient's being twice that of linear expansion? The writer's experimental and theoretical experience being contrary to this explanation, he has been led to attempt a detailed analysis of the changes of inductance which result from small deformations of shape, and at the same time to examine the importance of other factors which have an indirect effect on the apparent temperature coefficient. This analysis appears to show that the temperature coefficient of inductance of a copper coil is never far from 17×10^{-6} degrees centigrade, hence either the inductance changes from some cause as yet unsuspected, or the observed temperature coefficients of inductance are apparent and not real. If the first, then devices for self-compensation must be designed to act according to a law which at present is undiscovered. If the second, then devices for self-compensation must not be designed to give constant inductance but designed to change the inductance in a manner which depends on the particular generator with which the coil is associated. If the temperature coefficient of inductance is in fact about 17×10^{-6} degrees centigrade, then self-compensation for constant inductance can improve the frequency stability by only about eight parts in 10^6 degrees centigrade—an improvement which would be of comparatively little importance in the present stage of the art, save in extreme cases where the power loss heats the coil enormously, or where the range of ambient temperature is very great, as in airplanes.

III. An Analysis of the Various Factors Which Change the Inductance of a Coil

The first form of a coil to be considered is a concentric cable, because this form is amenable to exact mathematical treatment. Since this form of coil is not used in practice, it may appear to be only of academic interest to consider it. But this is not so, since the analysis shows that it is very closely related to the problem of a coil having very few turns, the form of coil used in short-wave practice, where frequency stability is a serious problem.

The other form of coil which is partly amenable to exact treatment is the closely-wound toroid, or very long solenoid. This form is also examined.

These solutions are used also to assist in reaching approximate solutions for coils consisting of a few widely spaced turns.

IV. The Concentric Cable

It is well known that the inductance per unit length of a concentric cable, solid inner conductor radius a, tubular external conductor of inner radius b, and outer radius c, is given by the expression

$$L = 2 \log_e \frac{b}{a} + \frac{1}{2} + \frac{1}{(c^2 - b^2)} \left\{ \frac{2c^4}{c^2 - b^2} \log \frac{c}{b} - \frac{3c^2 - b^2}{2} \right\}. \quad (1)$$

This expression has no dimensions; hence, since a, b, and c will all increase by the same fractional amount for a uniform change of temperature, there will be no change in the inductance per unit length. That is to say, the inductance increases with temperature only because the cable increases in length.

The expression for L consists of three distinct terms, each of which may be said to have distinct physical significance. The first term is due to the energy of the magnetic field in the tubular space. The second and third terms are due respectively to the energy of the magnetic field which is situated in the inner core and within the outer tube of thickness $(c-b)$.

As the frequency tends to infinity, the field within the conductor tends to zero and hence the inductance tends to the value

$$L' = 2 \log_e \frac{b}{a} \cdot \quad (2)$$

The inductance at any specified frequency can be evaluated in terms of complex Bessel functions, and the result will be found in Heaviside's collected papers. It differs from the limiting values expressed by (2) by an amount which will be termed the "internal inductance."

(a) *Internal inductance at high frequencies and its temperature coefficient.*

The full expression for the internal inductance is too cumbersome to deal with, but it is well known that all skin effect phenomena approach rapidly a simple asymptotic expression which is a function of a certain parameter Z, where

$$Z^2 = \frac{8\pi^2 n a^2}{\rho}$$

in which a is the radius and ρ the specific resistance of the round wire. Usually the asymptotic expression is sensibly correct if the value of Z is greater than about 4. In the wave band between 15 and 100 meters, Z for copper wires is usually greater than 20 for radii of practical interest. Using the asymptotic form of the expression, it can be shown that

$$L = 2\log_e \frac{b}{a} + \frac{1}{Z\sqrt{2}}\left(1 + \frac{a}{b}\right), \qquad (3)$$

where Z refers to the core.

The first term of (3) has no temperature coefficient but the second term has, since Z varies as $1/\sqrt{\rho}$. Let α be the temperature coefficient of linear expansion, and η that of resistance. Then

$$L_T = \left\{2\log_e \frac{b}{a} + \frac{1}{Z\sqrt{2}}\,\frac{(1+\frac{1}{2}\eta T)}{(1+\alpha T)}\left(1 + \frac{a}{b}\right)\right\}(1 + \alpha T). \qquad (4)$$

To compare the relative importance of the internal inductance, it is necessary to take a numerical example. Let $a = \frac{1}{2}$ centimeter, $b = 10$ centimeters, and $n = 10^7$ cycles ($\lambda = 30$ meters). Then it may be found that $Z = 350$.

$$\therefore L_T = \left\{6 + \frac{\left(1 + \frac{\eta}{2} - \alpha\right)T}{500}\right\}(1 + \alpha T)$$

$$\doteqdot 6\left\{1 + \left(\frac{\eta}{6000} + \alpha\right)T\right\}$$

$$= 6\left\{1 + \left(\frac{0.7}{10^6} + \frac{16.7}{10^6}\right)T\right\} \text{ for copper.}$$

Hence, the temperature increase of resistance causes the internal inductance to increase and makes the net temperature coefficient about 5 per cent greater than the coefficient of linear expansion. To the first order the effect can be ignored.

The internal inductance is not changed appreciably by curvature or by the proximity of other wires. Hence the expression which is correct for a concentric cable is approximately correct for any practical form of coil having only a few turns. The relative importance of the temperature coefficient of internal inductance becomes less as the "external inductance" increases. Hence it is likely to be of greater importance for a single turn coil or concentric cable than for a helix of few turns or a solenoid. Hence "internal inductance" may be dismissed as an inappreciable cause of temperature coefficient.

(b) *Temperature coefficient of self-capacitance of a concentric cable.*

If the concentric cable has an inductance L and a length l, it can be shown that

$$\text{apparent inductance} \doteqdot L\left(1 + \frac{4}{3}\pi^2 \frac{l^2}{\lambda^2}\right) \tag{5}$$

where the wavelength is λ and $\lambda \gg l$.

If $l = 2$ meters, $\lambda = 30$ meters, ($L = 1.205\ \mu h/cm$ if $b/a = 20$),

$$\text{apparent inductance} = L \times 1.06.$$

If such a concentric cable is the inductance of a generator whose capacitance remains constant, then, to the first order, λ^2 varies as l. Then the self-capacitance correction will have a temperature coefficient equal to that of linear expansion. For the numerical case considered this will increase the net temperature coefficient by about 5 per cent of itself.

The expression which is exact for a cable will be approximately correct for a helix having widely spaced turns. It is not clear how to estimate the effect for a long solenoid. Hence, it seems that for coils such as are used in short-wave practice, the self-capacity effect cannot increase the effective temperature coefficient by more than a small fraction of itself.

To sum up, it appears from analysis that if a copper coil expands without change of form, its temperature coefficient is unlikely to exceed 18×10^{-6} degrees centigrade; this means that the frequency of a generator should not change from this cause by more than 9 parts in 10^6 per degree centigrade.

It is now necessary to investigate the effect of deformations.

V. Change of Inductance from Deformation

(a) *Concentric cable with core slightly displaced.*

If a concentric cable were used as the inductance of a generator, the core would become hotter than the sheath since it has greater resistance and less cooling surface. Hence the core would expand more than the sheath, and would buckle if its ends were not free to expand.

The inductance can be calculated for a cable with an eccentric core. Let a be the radius of the core, b the inner radius of the sheath, and d the eccentricity; then it is well known that

$$L = 2 \log_e (\beta + \sqrt{\beta^2 - 1}) \text{ per unit length}$$

where,

$$\beta = \frac{a^2 + b^2 - d^2}{2ab}.$$

If d is small compared with $(b^2 - a^2)$, this expression reduces to

$$L = 2\left\{1 - \frac{\dfrac{d^2}{b^2}}{\log_e \dfrac{b}{a}}\right\} \log_e \frac{b}{a}. \qquad (6)$$

Take the previous numerical example in which $a = 0.5$ centimeter and $b = 10$ centimeters; then if $d = 1$ millimeter,

$$L = 2\left(1 - \frac{33}{10_e}\right) \log_e \frac{b}{a}.$$

The eccentricity could not be uniform along the length and to the first order the core would deflect into a sine curve of amplitude D. Then the change of inductance will be proportional to the mean square eccentricity and consequently will be half that given by (6). It can be shown that the relation between D and T is given by the equation.

$$\left(\frac{\pi D}{l}\right)^2 = 4\alpha T.$$

Adding the inductance due to successive arcs ds placed with eccentricity $y = D \sin \pi x / l$, it follows that

$$L_T = \left[1 +. \left\{1 - \frac{2}{\pi^2} \frac{l^2}{b^2 \log_e \dfrac{b}{a}}\right\} \alpha T\right] 2 \log_e \frac{b}{a}. \qquad (7)$$

The net coefficient will be negative so long as l is greater than, say, $4b$. If $l=10b$, the net coefficient would be about 5α and negative. In practice the net coefficient would depend enormously on the rigidity of the end fixings, but the analysis suffices to show that the deformation is almost certain to make the temperature coefficient less than that of linear expansion.

If the outer sheath is supposed to be unaffected by the temperature of the core, the inductance will decrease with rise of temperature because such will decrease the ratio b/a. If the core remains central and b constant, it can be shown that

$$L_T \doteqdot 2\left(1 - \frac{\alpha T}{\log_e \frac{b}{a}}\right)\log_e \frac{b}{a} \tag{8}$$

$$= 2\left(1 - \frac{\alpha T}{3}\right)\log_e \frac{b}{a}, \quad \text{if } \frac{b}{a} \doteqdot 20.$$

If the core is free to expand through a bush at the ground end of the sheath, the buckling of the core will be avoided and the net coefficient due to "load heating" will be negative and about 5×10^{-6} degrees centigrade. The coefficient due to ambient temperature would be that of the copper, namely 16.7×10^{-6} degrees centigrade.

(b) *Concentric cable with elliptical core and sheath.*

It is conceivable that overstrain in the metal may cause the cross section of core and sheath to deform when they are heated. To estimate the effect which such a change would have on the inductance, we shall suppose the core and sheath develop a slight ellipticity. It is well known that the inductance per unit length of a cable formed by two confocal elliptic cylinders is

$$L = 2\log\left(z + \sqrt{z^2 - 1}\right)$$

where,

$$z = \frac{a_1 a_2 - b_1 b_2}{a_1^2 - b_1^2} = \frac{\frac{a_2}{a_1}\left(1 - \frac{b_1}{a_1}\frac{b_2}{a_2}\right)}{e_1^2}$$

in which a_1, b_1, etc., have their usual significance, and where the confocal property requires that

$$a_2^2 = a_1^2 + \lambda, \quad b_2^2 = b_1^2 + \lambda.$$

Now,

$$e_1^2 = 1 - \frac{b_1^2}{a_1^2}$$

$$e_2^2 = 1 - \frac{b_2^2}{a_2^2}$$

$$= \frac{a_1^2}{a_2^2} e_1^2.$$

Let $a_1 = R_1(1+x)$ and $b_1 = R_1(1-x)$, where x is a very small fraction, and R_1 is the radius of the original circle.

Then,

$$e_1^2 = \frac{a^2 - b^2}{a^2} = \frac{4R^2 x}{(R+x)^2} \doteqdot 4x.$$

Now $a_2/a_1 \doteqdot R_0/R_i$. As before we shall assume that $R_0/R_i \doteqdot 20$. Hence it follows that $e_2 \doteqdot 1/20\ e_1$.

Hence if the core is flattened slightly, so that x is 1 per cent, we must suppose the sheath is flattened by 1/400 per cent. If $R_1 = 1/2$ centimeter and $R_0 = 10$ centimeters, this supposes that perpendicular diameters of the core differ by 0.2 millimeter and perpendicular diameters of the sheath differ by 1/100 millimeter (0.4 mil.). Thus, to the accuracy of good machining, we may consider it as an elliptic core in a circular sheath.

Let $a_1/a_2 = y$, where y is of the order of 1/20. It follows that

$$\frac{b_1}{a_1}\frac{b_2}{a_2} = \sqrt{1 - e_1^2}\sqrt{1 - y^2 e_1^2})$$

$$\doteqdot 1 - \tfrac{1}{2}(1 + y^2)e_1^2 - \tfrac{1}{8}(1 + y^2)^2 e_1^4$$

$$\therefore 1 - \frac{b_1}{a_1}\cdot\frac{b_2}{a_2} = \tfrac{1}{2}(1 + y^2)e_1^2\left\{1 + \tfrac{1}{4}(1 + y)^2 e_1^2\right\}$$

$$\therefore z = \frac{(1 + y^2)\left\{1 + \tfrac{1}{4}(1 + y)^2 e_1^2\right\}}{2y}$$

$$\doteqdot \frac{(1 + y^2)(1 + x)}{2y} \doteqdot \frac{1 + x + y^2}{2y}$$

and hence z is of the order of 10.

$$\therefore\ z+\sqrt{z^2-1} \doteqdot 2z\left(1-\frac{1}{4z^2}\right)$$
$$\doteqdot \frac{1+x}{y}$$
$$=\frac{a_2}{a_1}(1+x)$$
$$=\frac{R_0(1+xy^2)}{R_1(1+x)}(1+x).$$
$$\therefore\ L = 2\log\left\{\frac{R_0(1+xy^2)}{R_i}\right\}$$
$$\doteqdot 2\left\{1+\left(\frac{R_i}{R_0}\right)^2\frac{x}{\log\dfrac{R_0}{R_i}}\right\}\log\frac{R_0}{R_i}\cdot \qquad (9)$$

Hence, if x is 1 per cent, the inductance would be increased by about eight parts in 10^6, if $R_0/R_i=20$. Thus it would seem that small malformations of cross section are not competent to change the inductance appreciably.

Finally it would seem most unlikely that a tubular inductance can have a temperature coefficient of inductance greater than the coefficient of expansion of the metal. In practice the temperature coefficient is likely to be less than that of expansion and might even be negative.

VI. Change of Inductance Due to Deforming a Coil of Conventional Shape

We shall suppose first that a rise of temperature causes a change in the shape of the turns, but that the axial expansion is uniform along the length of the coil. We shall then suppose the shape of the turns is unchanged but that the axial expansion of the coil differs from the radial expansion.

(a) *Deformation of the turns.*

Exact solution is practicable for this deformation only when the coil is a long solenoid; an approximate solution can be obtained for a single turn coil, and the solution for a coil having a finite number of turns must be inferred by the aid of the two limiting expressions.

(1) Long Solenoid. If a coil is a long and closely wound solenoid, the inductance is proportional to the square of the number of turns per unit length and to the length and cross-sectional area of the tube. The

inductance is independent of the shape of the cross-sectional area. If a given round solenoid is flattened, the inductance will decrease by an amount proportional to the decrease of area. Let a circular tube be flattened into an ellipse of small eccentricity e, the perimeter remaining constant.

The perimeter l of an ellipse of major semiaxis a and small eccentricity e is expressed by the equation

$$l = 2\pi a\left(1 - \frac{e^2}{4} - \frac{3e^4}{64}\right).$$

Let the tube start as a circle of radius R and be flattened into an ellipse of semiaxes a and b, where $e^2 = 1 - b^2/a^2$. Then $l = 2\pi R$.

$$\begin{aligned}\text{Area} &= \pi ab \\ &= \pi a^2\sqrt{1 - e^2} \\ &= \frac{\pi R^2\sqrt{1 - e^2}}{\left(1 - \frac{e^2}{4} - \frac{3e^4}{64}\right)^2} \\ &\doteqdot \pi R^2\left(1 - \frac{3}{32}e^4\right).\end{aligned}$$

Let $b = R(1 - x)$, where x is very small.

Then it follows that $e^2 = 4x$.

$$\therefore \text{Area} = \pi R^2\left(1 - \frac{3}{2}x^2\right). \tag{10}$$

The decrease of inductance is proportional to the decrease of area, which is expressed by (10). The correction term $(3/2x^2)$ is almost certainly an upper limit and a short coil would probably decrease in inductance by a smaller amount.

Now suppose a long solenoid is constrained so that one diameter of the tube remains constant while the expansion of the wire causes the previously circular turns to become slightly elliptical. Let the fixed radius be R; this now becomes the semiminor axis of the ellipse.

First imagine the circle is allowed to expand uniformly and hence that its radius would become $R(1 + \alpha T)$; then let one diameter be compressed a fractional amount αT. Then by (10)

$$\begin{aligned}\text{area} &= \pi R^2(1 + 2\alpha T)\left(1 - \frac{3}{2}\alpha^2 T^2\right) \\ &\doteqdot \pi R^2(1 + 2\alpha T - \tfrac{1}{2}\alpha^2 T^2).\end{aligned} \tag{10a}$$

Hence such a deformation in the process of expansion causes only a second order change of inductance.

(2) Single Turn Circle. The inductance of a coil is often defined as the flux threading it per unit current; this definition draws attention to the area of the circuit. But if the circuit is only a single turn, the inductance depends mainly on the perimeter and very little on the area. That is to say, the inductance per unit length of perimeter changes very slowly with the form and area of the turn. An alternative definition of inductance is twice the energy stored in the circuit per unit of current squared. If the field strength is H at a given point of space, then the energy may be considered to be distributed through the medium at the rate of $H^2/8\pi$ per unit volume. If the circuit is a single turn, the field is intense only close to the wire, and there it tends to the value $H=2/r$, whatever be the shape of the turn. Hence the total energy is dominated by the field close to the wire and thus it is seen that L depends only slightly on the form and area of the coil. Also this can be inferred from the formal integral which expresses the external inductance of a circuit; this is

$$L = \int\int \frac{ds \cdot ds' \cos \epsilon}{r} \tag{11}$$

where ds, etc., is described by the figure (see Fig. 1) and in which the outer contour follows the center line of the wire and the inner contour is any parallel contour in the surface of the wire. Consideration of the

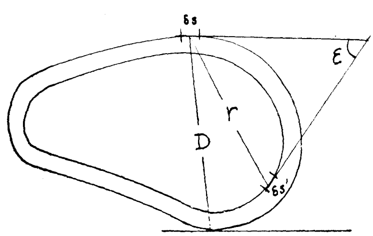

Fig. 1

figure will show that the main contribution comes from elements ds' which are near ds. For it is only such which have r small, and also in a large part of the summation $\cos \epsilon$ is very small. Hence, $ds\int(ds' \cos \epsilon/r)$ will tend to be approximately equal to $ds \log D/a$, where D is shown in the figure and a is the radius of the wire.

The integration can be performed for a circle and for any rectilinear figure. The result for a circle, a rectangle, and a regular polygon is well known. V. J. Bashenoff[1] has collected the results for a circle,

[1] Proc. I.R.E., vol. 15, p. 1013; December (1927) and *Wireless Engineer*, May, (1929).

a regular octagon, hexagon, pentagon, square, equilateral triangle, and isosceles right-angled triangle and expresses them all in the form

$$L = 2l\left\{\log\frac{2l}{r} - \theta\right\}, \tag{12}$$

where l is the total perimeter of the figure, r is the radius of the wire, and θ is a constant depending on the form of the figure.

The appropriate values of θ are shown below.

(1) Circle .... $\theta = 2.451$
(2) Regular octagon .... $\theta = 2.561$
(3) Regular hexagon .... $\theta = 2.636$
(4) Regular pentagon .... $\theta = 2.712$
(5) Square .... $\theta = 2.853$
(6) Equilateral triangle .... $\theta = 3.197$
(7) Isosceles right-angled triangle .... $\theta = 3.332$.

It is important to notice that this general formula expresses L in terms of the perimeter and it is at once obvious that the form of the circuit has comparatively little effect. This may be seen by the help of a numerical example. Suppose a wire of 1 centimeter diameter is bent into a circle of 20 centimeters diameter. The perimeter is 63 centimeters and $\log 2l/r = 5.55$. It may be found that $L = 6.188$ cm/cm. If this same wire is deformed into a square, it may be found that the inductance is decreased by only 13 per cent, though the area is decreased by 21.5 per cent. If the same wire is deformed into an octagon it may be found that the inductance is decreased by only 3.55 per cent. Hence the inductance of a single turn of given perimeter cannot be sensitive to small deformations.

Bashenoff[1] has evaluated θ for rectangles having various ratios of length a to breadth b. Some of his values are as follows:

a/b	1	1.5	2.34	4	9
θ	2.853	2.866	3.006	3.269	3.826
$l\sqrt{A}$	4	4.08	4.37	5	6.67

He shows that when $a/b > 12$, then $\theta = \log_e l^2/A$, where A is the area of the circuit, to an accuracy closer than 1 per cent.

It should be noticed that the expression for the inductance of a single turn (12) is of the same form as that for a concentric cable. Perhaps this is more obvious if Bashenoff's limiting value of θ is used, viz.,

$$\log l^2/A \doteqdot 2\log 2a - \log ab.$$

Hence,

$$L = 4a(\log 4a - \log r - 2\log 2a + \log a + \log b)$$

$$= 4a \log \frac{b}{r} \cdot \qquad (13)$$

This is the inductance of two long parallel wires separated by a distance b; it is twice the inductance of the same length of concentric cable having radii b and r. The inductance of a circular turn 20 centimeters in diameter, of a 1-centimeter rod, is 6.188 cm/cm. The inductance of a concentric cable having diameters 20 centimeters and 1 centimeter is 6 cm/cm. Hence if the given circular turn is straightened out and put inside a tube of the same diameter as the original turn, the inductance is decreased only by 3 per cent. Hence the preliminary discussion of the concentric cable was not inappropriate to the problem of a single turn circuit.

The calculation of the inductance of a slightly eccentric ellipse leads to intractable integration. But probably we can determine the result from a principle due to Bashenoff. Bashenoff suspects that all plane figures which have the same ratio of $l/\sqrt{A}$ have the same inductance. His proposition is demonstrably inexact. For example $l/\sqrt{A}$ for an isosceles triangle is 4.56, and $\theta = 3.197$; the rectangle for which $l/\sqrt{A} = 4.56$ has $\theta = 3.08$. Again an isosceles right-angled triangle has $l/\sqrt{A} = 4.82$ and $\theta = 3.33$, whereas the rectangle in which $l/\sqrt{A} = 4.82$ has $\theta = 3.19$. He quotes an example in which the measured inductance of a certain ellipse ($a/b = 2.3$) agreed within 0.1 per cent with the value calculated by using the value of θ appropriate to the particular value of $l/\sqrt{A}$ ($l/\sqrt{A} = 4.02$). It is impossible to say how exact his method of calculation is for an ellipse. However, we shall apply it to find the inductance of an ellipse of small eccentricity e. Using the previous expression for the perimeter of an ellipse, it may be found that

$$l/\sqrt{A} \doteqdot 2\sqrt{\pi}\left(1 + \frac{3}{64} e^4\right).$$

Now let $b = R(1 - x)$, then as before $e^2 = 4x$.

$$\therefore\ l/\sqrt{A} = 2\sqrt{\pi}(1 + \ x^2).$$

Using the values of θ for a circle and an octagon, we may express the value of θ for any figure in which $l/\sqrt{A}$ lies between the respective values for these two figures, by the equation

$$\theta = 2.451\left\{1 + 0.493\left(\frac{l}{\sqrt{A}} - 2\sqrt{\pi}\right)\right\}.$$

Hence for an ellipse

$$\theta = 2.451\{1 + 1.742 \times \tfrac{3}{4}x^2\} \tag{14}$$

$$\therefore \frac{L_c - L_e}{L_c} = \frac{3.2x^2}{\log\dfrac{2l}{r} - 2.451}. \tag{15}$$

Taking the same example of a rod 1 centimeter in diameter, bent into a circle of 20 centimeters diameter,

$$\frac{L_c - L_e}{L_c} = \frac{3.2}{3.1}x^2 \doteqdot x^2. \tag{16}$$

Hence, if this expression is correct, the decrease is one third less than the amount for a corresponding deformation of a long solenoid, as derived from (10). It confirms the hypothesis that (10) expressed an upper limit for the change. The fractional change is insensitive to large changes of l/r.

If the circular turn is free to expand, but one diameter is maintained at its original value R, then corresponding with (10a) the temperature coefficient will be $\alpha(1-\alpha T)$, and hence elliptical deformation during expansion will make a second order change in the temperature coefficient.

(b) *Coils in which the radial expansion differs from the axial.*

We shall now suppose that each turn remains a circle but the axial expansion differs from the radial; such a condition may involve a slight untwisting of the helix or a tension in the wire.

(1) Many Turn Solenoid of any Length. The inductance of a single turn solenoid of length b, radius R, and total turns N, is expressed by the equation

$$L = \frac{2KR^2N^2}{b}$$

where the parameter K is a function of b/R, and has been tabulated.

It may be found that K can be expressed by the equation[2]

$$K = \frac{2\pi^2 b}{0.868R + b}$$

to an accuracy of ± 2 per cent so long as $b/R > 0.4$.

[2] *See also* H. A. Wheeler, Proc. I.R.E., vol. 16, p. 1398; October, (1928), and R. R. Batcher, Proc. I.R.E., vol. 17, p. 580; March, (1929).

Hence,[3]

$$L \doteq \frac{4\pi^2 R^2 N^2}{0.868R + b}. \tag{17}$$

If the radius increases with temperature coefficient α and the axial length with temperature coefficient γ, it follows that

$$L_T = L_0\left[1 + \left\{\frac{(0.868R + 2b)\alpha - b\gamma}{(0.868R + b)}\right\}T\right]. \tag{18}$$

By arranging the ratio γ/α suitably, the temperature coefficient can be made zero, and it follows that this will occur when

$$\gamma/\alpha = 2\left(1 + 0.217\frac{R}{b}\right). \tag{19}$$

It may be found from this that the temperature coefficient is zero if the axial expansion is 3.1 times the radial for a coil in which $b/R = 0.4$, and if $\gamma = 2.22\alpha$ for a coil in which $b/R = 2$.

Coils are sometimes made by fixing separate turns to an insulating rod, which can be made of a material having a very small temperature coefficient. In such circumstances the axial expansion can be sensibly zero, and then the net temperature coefficient of inductance will be

$$2\alpha\left(\frac{1 + 0.434R/b}{1 + 0.868R/b}\right). \tag{20}$$

It follows from this expression that such a method of construction cannot cause the temperature coefficient to exceed twice that of linear expansion of the metal, and even when $b = 2R$ the coefficient is only 1.7α.

The foregoing expressions take no account of the radius of the wire and are valid only if the coil has many turns in its length. It is not strictly applicable to a helix of only a few turns. The behavior of such a coil may be investigated by calculating the inductance of a coil consisting of several coaxial turns each of radius R, of wire of radius r, placed with a distance d between the planes of successive turns.

The mutual inductance between two equal circles in parallel planes separated by a distance x is

[3] If b/R lies between 0.4 and 0.05 it may be found that

$$L \doteq \frac{4\pi^2 R^2 N^2}{0.868R + b}\left(1.025 + 0.11\frac{R}{b}\right).$$

$$M = 2l\left(\log\frac{2l}{x} - 2.451\right)$$

whereas the self-inductance of one turn is

$$L = 2l\left(\log\frac{2l}{r} - 2.451\right).$$

By means of these two expressions the self-inductance of any number of turns can be found.

Thus the self-inductance of a two-turn coil is

$$L = 4l\left(\log\frac{2l}{r} - 2.451 + \log\frac{2l}{d} - 2.451\right). \tag{21}$$

This expression shows that the proximity of the two circles increases the self-inductance per unit length of wire by an amount equal to half the self-inductance per unit length of a similar circle made of wire of radius d. When $R/r=20$, the fractional increment for various values of d/r is shown in the table below.

d/r	4	6	8	10
Increment per cent of inductance	55	41	32	25

If α is the temperature coefficient of the metal, and γ that of axial expansion, it follows that

$$4l \times L_T \doteq 4l\left\{\log\frac{2l}{r} - 2.451 + \log\frac{2l}{d} - 2.451 + (\alpha - \gamma)T\right\}$$
$$(1 + \alpha T)$$

$$\therefore\ L_T \doteq L_0\left[1 + \left\{\left(1 + \frac{1}{L_0}\right)\alpha - \frac{\gamma}{L_0}\right\}T\right]. \tag{22}$$

The temperature coefficient is zero if

$$\gamma = \alpha(1 + L_0). \tag{23}$$

It follows from this that if $R/r=20$, $\gamma=4.55\alpha$ if $d/r=4$, and $\gamma=4.25\alpha$ if $d/r=10$.

If axial expansion is prevented, then the temperature coefficient of a two-turn coil is about 22 per cent greater than it would be if uniform expansion had been permitted.

Thus, a copper coil consisting of two turns clamped to rigid terminal blocks, will have a temperature coefficient of inductance of about

20×10^{-6} degrees centigrade. Hence, this deformation is not competent to account for a large increase of temperature coefficient.

The self-inductance of a three-turn coil is

$$L = 6l\left\{\log 2\frac{l}{r} - 2.451 + 2\left(\log\frac{2l}{d} - 2.451\right) - 0.457\right\}. \qquad (24)$$

This may be interpreted by saying that the proximity of the three coils increases the self-inductance per unit length of each by an amount which is less than the self-inductance per unit length of a similar coil wound with wire of radius d.

When $R/r=20$, the fractional increment for various values of d/r is shown in the table below.

d/r	4	6	8	10
Per cent increment of inductance	100	66	50	36

If α, γ, and L_0 have their previous significance, then it can be shown that

$$L_T \doteqdot L_0\left\{1 + \left(1 + \frac{2}{L_0}\right)\alpha + \frac{2\gamma}{L_0}T\right\}. \qquad (25)$$

If a three-turn coil is clamped to rigid terminal blocks, which prevent axial expansion, then the temperature coefficient is increased in the ratio $(1+2/L_0):1$. If $R/r=20$, this ratio will be approximately equal to 1.4.

Thus if axial expansion is prohibited, the temperature coefficient will be greater than that of linear expansion by an amount which depends on the number of turns; the increment ranges from 22 per cent for two turns to 100 per cent for a very great number. This type of deformation cannot do more than double the temperature coefficient.

Self-compensation will occur in a three-turn coil, for which $R/r=20$, if $\gamma=4\alpha$ when $d/r=4$ and if $\gamma=3\alpha$ when $d/r=10$.

The process of evaluation turn by turn is laborious, and it is interesting to compare the results obtained for the two-turn and three-turn coil with those which would have been obtained from the approximate equations (19) and (20). Thus for the given two-turn coil with $d/r=4$, self-compensation occurs if $\gamma=4.55\alpha$, whereas (19) gives $\gamma=4.2\alpha$. For the three-turn coil in which $d/r=4$, self-compensation occurs when $\gamma=4\alpha$, whereas (19) gives $\gamma=3.1\alpha$. If axial expansion is prevented then the temperature coefficient is 1.22α and 1.4α when $d/r=4$, for the two-turn and three-turn coil, respectively, whereas (20) gives in these circumstances 1.19α and 1.34α. Hence it would appear that (19) and (20) give results which are substantially correct for a coil having very few turns, so long as the ratio d/r is not greater than, say, 4.

The Circuit as a Whole

So far coils have been considered in the idealized form of separate circles and no account has been taken of the leads which must be provided to connect the coil to the capacitance. Nevertheless, the temperature coefficient which is of interest is that of the complete circuit, coil plus leads. It is impossible to make a general discussion of this problem because much depends on the disposition of the leads.

Suppose the coil consists of an integral number of turns ending in rigid terminal blocks, to which run a pair of parallel leads. We have seen that the coil proper will have a temperature coefficient of the order 1.5α. Let the distance between the parallel leads be held rigid and let their length be free to expand. Then a rise of temperature will cause an increase of inductance because the length increases, and a decrease of inductance since the radius of the rods increases without a corresponding increment of their distance apart. The temperature coefficient of the leads will thus be less than α. The temperature coefficient of coil plus leads will thus be less than, say, 1.5α by an amount depending on the relative importance of the leads. The net result is a circuit in which the temperature coefficient is not much greater than α.

Now suppose that the coil and leads are disposed like the carbon filament of an incandescent lamp, the leads being fixed only at the end remote from the coil. It is possible that the result of bending the rod into a coil may be to make the radial and axial expansion of the coil greater than that of the metal. But if expansion causes the size of the coil to grow unduly, it can do so only by robbing some length from the leads. It will be noticed that the inductance of a two- or a three-turn coil has been expressed in terms of the total length of wire and a factor of the form log l/r. The logarithmic factor changes slowly with the variables, and consequently the inductance per unit length of wire is not sensitive to the size of the coil. Consequently the fractional increase in the inductance of the coil will be approximately equal to the fractional increase of the wire in it. Also the inductance per unit length of leads will be of the same order as the inductance per unit length of coil. Hence, stealing from the leads and adding to the coil will increase the inductance of the whole circuit by considerably less than the increase of inductance of the coil. With reasonably proportioned leads it seems probable that the net temperature coefficient would not differ very much from α.

No doubt it is possible to construct coils in which the leads have a very important effect on the temperature coefficient of the circuit.

For example, consider the single turn circle and parallel leads disposed as shown in Fig. 2. The radial expansion of the circle is prevented by the rollers shown diagrammatically in the figure. The increase of circumference of the circle will cause the parallel leads to approach one another. The inductance of the circle will increase in proportion to α, but the inductance of the leads will decrease because they are pushed closer together. By proportioning the leads suitably, the net temperature coefficient could be made zero or it could be made to have a large negative value.

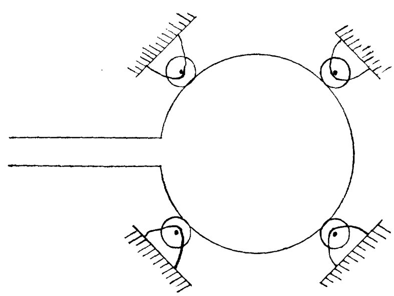

Fig. 2

The author believes that the arrangement described by Fig. 2 is the basis of the action of the self-compensated coil described by Ure, Grainger, and Cantello.[4]

VIII. Conclusion

This analysis appears to show that the construction of a coil would require to be very unusual if it is to have a positive temperature coefficient of inductance which is, say, greater than one and a half times the coefficient of linear expansion of the metal. If such large positive temperature coefficients exist in practice, the cure may lie in attention to the disposition of the leads, but this is improbable.

If a valve generator is found to have a negative temperature coefficient of frequency which is greater than the temperature coefficient of linear expansion of copper, then possibly this may not be caused by increase of inductance *per se*. It might be caused in certain circumstances by the temperature increase of resistance changing the relative phase of the grid and anode voltage, and thereby necessarily causing a change of frequency.

[4] *Jour. I.E.E.* (London), vol. 72, p. 528, (1933).