ME621 - Advanced Finite Element Methods
Assignment 1

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1 Requests

A system of two aluminum bars of the same material is shown in the following figure. The system is subjected to two external loads, P_x and P_y, at joint B. A and C are connected to pinned supports.

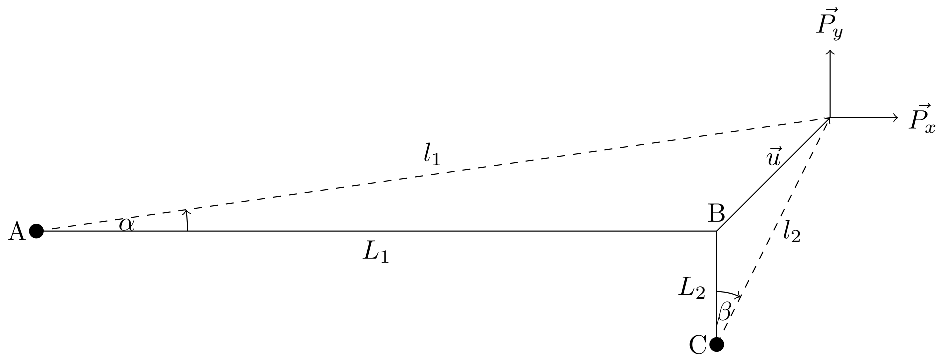

Figure 1: Problem representation

The problem asks to:

- Obtain the external loads P_x and P_y as a function of horizontal and vertical displacements at point B (namely u and v).
- Determine the displacements in both x and y directions for 1000 load increments of +5N for both P_x and P_y (from zero).
- Find the displacement of point B after the final increment.

Write a MATLAB code with a convergence error of 10^-5 to numerically solve the problem. Use a combination of (a) Euler and N-R, and (b) Euler and modified N-R. Also plot the resultant force versus the resultant displacement.
Use the Green strain measure:

$$\epsilon = \frac{l^2 - L^2}{2L^2} \tag{1}$$

Parameter	**Value**	**Unit**
$E_1 = E_2 = E$	70	GPa
L_1	3	m
L_2	0.5	m
$A_1 = A_2 = A$	0.0001	m^2

Table 1: Parameters of the system

2 Methodology

We will start by writing the equilibrium equations for the system. We will further work on the equilibrium equations to obtain the force-displacement relationship. Finally, we will solve the system of equations $\vec{P} = f(\vec{u})$ to obtain the displacement of point B for a given load.
Notice that we will likely obtain a system of non-linear equations, which we will solve using the both Euler and Newton-Raphson methods. We will also perform a linearization of the system to successively compare the results obtained with the linearized and the non-linearized system.
Even if the problem asked to produce a MATLAB code, we will first set up the system of equations in Mathematica, and then we will translate the code to MATLAB. Mathematica is a more powerful tool for symbolic computation, and it will allow us to obtain the system of equations that can be easily updated in case of necessity (for example a different order of the Taylor series).

3 Solution

3.1 Equilibrium equations

We can start writing the equilibrium equations for the system. Since we have no data about the mass of the bars, we will assume negligible mass of the bars. This assumption will allow us to neglect the effect of gravity on the system and avoid the introduction of the inertia terms in the equilibrium equations.

$$\vec{F_{ext}} + \vec{F_{int}} = \vec{0} \tag{2}$$

$$\begin{cases} \vec{F_{ext}} = \vec{P_x} + \vec{P_y} \\ \vec{F_{int}} = \vec{F_{int,x}} + \vec{F_{int,y}} \end{cases} \tag{3}$$

By decomposing the equations in the x and y directions, we obtain:

$$\begin{cases} P_x = F_{int,x} = |F_1| * \cos(\alpha) + |F_2| * \sin(\beta) \\ P_y = F_{int,y} = |F_1| * \sin(\alpha) + |F_2| * \cos(\beta) \end{cases} \tag{4}$$

3.2 Force displacement relationship

So far we have obtained the equilibrium equations for the system. We can now proceed to obtain the force displacement relationship, which will allow us to solve the system of equations $\vec{P} = f(\vec{u})$. To do so, we try to express everything on the right-hand side of the equilibrium equations in terms of the displacements u and v. From simple trigonometrical considerations, we can obtain the following relationships:

$$\cos(\alpha) = \frac{\mathrm{L}_1 + u}{l_1} \tag{5}$$

$$\sin(\alpha) = \frac{v}{l_1} \tag{6}$$

$$\cos(\beta) = \frac{\mathrm{L}_2 + u}{l_2} \tag{7}$$

$$\sin(\beta) = \frac{v}{l_2} \tag{8}$$

We can now proceed working on the forces, knowing that the internal forces are linked to the strains by the following relationship:

$$\begin{Bmatrix} \vec{F_{int,x}} \\ \vec{F_{int,y}} \end{Bmatrix} = \begin{Bmatrix} \mathrm{A}_1 * \mathrm{E}_1 * \epsilon_1 \\ \mathrm{A}_2 * \mathrm{E}_2 * \epsilon_2 \end{Bmatrix} \tag{9}$$

And the strains are linked to the displacements by the following relationship:

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \end{Bmatrix} = \begin{Bmatrix} \frac{l_1^2 - \mathrm{L}_1^2}{2\mathrm{L}_1^2} \\ \frac{l_2^2 - \mathrm{L}_2^2}{2\mathrm{L}_2^2} \end{Bmatrix} \tag{10}$$

Finally, we can give the following definition to the real length of the bars:

$$\begin{Bmatrix} l_1 \\ l_2 \end{Bmatrix} = \begin{Bmatrix} \sqrt{(\mathrm{L}_1 + u)^2 + v^2} \\ \sqrt{u^2 + (\mathrm{L}_2 + v)^2} \end{Bmatrix} \tag{11}$$

We can now substitute the equations above in the equilibrium equations 4 to obtain the force displacement relationship:

$$\begin{Bmatrix} \vec{P_x} \\ \vec{P_y} \end{Bmatrix} = \begin{Bmatrix} \frac{\mathrm{A}_1\mathrm{E}_1(\mathrm{L}_1+u)\left(-\mathrm{L}_1^2+(\mathrm{L}_1+u)^2+v^2\right)}{2\mathrm{L}_1^2\sqrt{(\mathrm{L}_1+u)^2+v^2}} + \frac{\mathrm{A}_2\mathrm{E}_2 v\left(-\mathrm{L}_2^2+u^2+(\mathrm{L}_2+v)^2\right)}{2\mathrm{L}_2^2\sqrt{u^2+(\mathrm{L}_2+v)^2}} \\ \frac{\mathrm{A}_1\mathrm{E}_1 v\left(-\mathrm{L}_1^2+(\mathrm{L}_1+u)^2+v^2\right)}{2\mathrm{L}_1^2\sqrt{(\mathrm{L}_1+u)^2+v^2}} + \frac{\mathrm{A}_2\mathrm{E}_2(\mathrm{L}_2+u)\left(-\mathrm{L}_2^2+u^2+(\mathrm{L}_2+v)^2\right)}{2\mathrm{L}_2^2\sqrt{u^2+(\mathrm{L}_2+v)^2}} \end{Bmatrix} \tag{12}$$

Even if quite long, the equation 12 are nothing more than a function $\vec{P} = f(\vec{u})$ which can be solved numerically for a given value of $\vec{P}*$ to find the corresponding value of $\vec{u}*$.

3.3 Linearization

Before proceeding with the numerical solution of the system, we can try to linearize the system of equations to obtain a linear system of equations. To do so, we can use a Taylor series expansion of the force-displacement relationship 12 around the point $\vec{u} = \vec{0}$. A general Taylor series expansion of a function $f(x, y)$ around the point (x_0, y_0) is given by:

$$\begin{aligned} f(x,y) = & f(x_0,y_0)+ \\ & + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)+ \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\cdot(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\cdot(y-y_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\cdot(x-x_0)\cdot(y-y_0)+ \\ & + \dots \end{aligned} \tag{13}$$

We can now try to apply the (1° order) Taylor series expansion to the force-displacement relationship 12 to obtain the following system of linear equations:
We can now try to apply the Taylor series expansion to the force-displacement relationship 12.

3.3.1 Taylor series of order 1

By applying the Taylor series expansion of order 1, we obtain the following system of linear equations:

$$\left\{ \begin{matrix} \widehat{\vec{P}_x} \\ \widehat{\vec{P}_y} \end{matrix} \right\} = \left\{ \begin{matrix} \frac{\mathrm{A}_1 \mathrm{E}_1 u}{\mathrm{L}_1} \\ \frac{\mathrm{A}_2 \mathrm{E}_2 v}{\mathrm{L}_2} \end{matrix} \right\} \tag{14}$$

3.3.2 Taylor series of order 2

By applying the Taylor series expansion of order 2, we obtain the following system of quadratic equations:

$$\left\{ \begin{matrix} \widehat{\widehat{\vec{P}_x}} \\ \widehat{\widehat{\vec{P}_y}} \end{matrix} \right\} = \left\{ \begin{matrix} \frac{\mathrm{A}_1 \mathrm{E}_1 u}{\mathrm{L}_1} + \frac{\mathrm{A}_2 \mathrm{E}_2 v^2}{\mathrm{L}_2^2} + \frac{\mathrm{A}_1 \mathrm{E}_1 \left(u^2+v^2\right)}{2\mathrm{L}_1^2} \\ \frac{\mathrm{A}_2 \mathrm{E}_2 v}{\mathrm{L}_2} + \frac{\mathrm{A}_1 \mathrm{E}_1 uv}{\mathrm{L}_1^2} + \frac{\mathrm{A}_2 \mathrm{E}_2 \left(u^2+2uv-v^2\right)}{2\mathrm{L}_2^2} \end{matrix} \right\} \tag{15}$$

3.3.3 Taylor series of order 3

By applying the Taylor series expansion of order 3, we obtain the following system of cubic equations:

$$\left\{ \begin{matrix} \widehat{\widehat{\widehat{\vec{P}_x}}} \\ \widehat{\widehat{\widehat{\vec{P}_y}}} \end{matrix} \right\} = \left\{ \begin{matrix} \frac{\mathrm{A}_1 \mathrm{E}_1 u}{\mathrm{L}_1} + \frac{\mathrm{A}_2 \mathrm{E}_2 v^2}{\mathrm{L}_2^2} - \frac{\mathrm{A}_1 \mathrm{E}_1 uv^2}{2\mathrm{L}_1^3} - \frac{\mathrm{A}_2 \mathrm{E}_2 v\left(-u^2+v^2\right)}{2\mathrm{L}_2^3} + \frac{\mathrm{A}_1 \mathrm{E}_1 \left(u^2+v^2\right)}{2\mathrm{L}_1^2} \\ \frac{\mathrm{A}_2 \mathrm{E}_2 v}{\mathrm{L}_2} + \frac{\mathrm{A}_1 \mathrm{E}_1 uv}{\mathrm{L}_1^2} + \frac{\mathrm{A}_2 \mathrm{E}_2 \left(u^2+2uv-v^2\right)}{2\mathrm{L}_2^2} + \frac{\mathrm{A}_1 \mathrm{E}_1 v\left(-u^2+v^2\right)}{2\mathrm{L}_1^3} + \frac{\mathrm{A}_2 \mathrm{E}_2 \left(u^3-2u^2v-uv^2+v^3\right)}{2\mathrm{L}_2^3} \end{matrix} \right\} \tag{16}$$

3.3.4 Comparison

As it's easy to understand, the system of equations obtained with the Taylor series expansion of order 2 (since it has a higher order) is more accurate than the system of equations obtained with the Taylor series expansion of order 1. We can now compare the results obtained with the two systems of equations to understand the error introduced by the linearization. In the following figures, we can see the error introduced by the linearization with respect to the original force-displacement relationship 12.

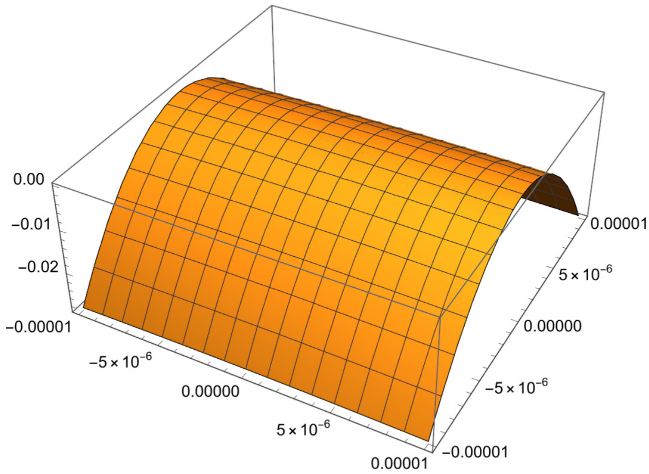

Figure 2: Error analysis for a Taylor series of order 1

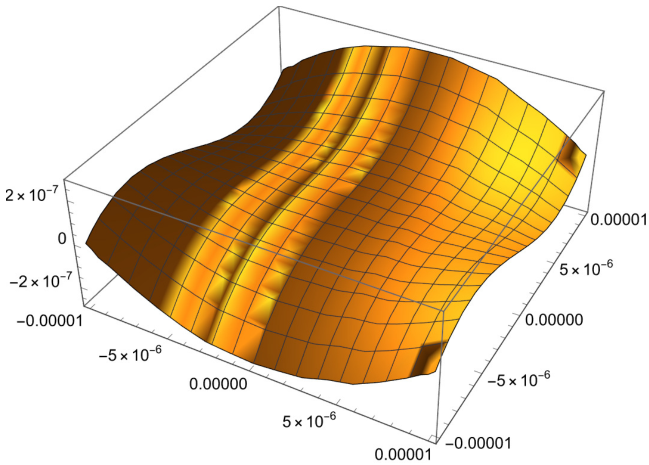

Figure 3: Error analysis for a Taylor series of order 2

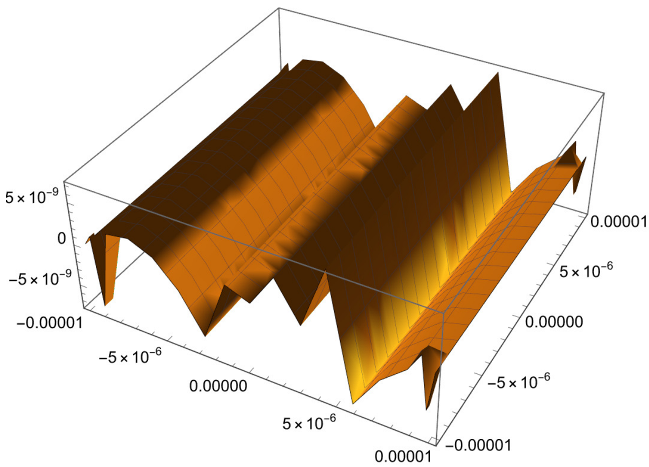

Figure 4: Error analysis for a Taylor series of order 3

As we can see, around the point $\vec{u} = \vec{0}$, the error is negligible for both the approximations. However, as we move away from the point $\vec{u} = \vec{0}$, the error introduced by the linearization increases.
In particular, we can see that the error introduced by the linearization decreases as we increase the order of the Taylor series expansion.

3.4 Numerical solution

We can now proceed to solve the system of equations $\vec{P} = f(\vec{u})$ numerically. To do so, we will use two different combined methods:

- Euler and Newton-Raphson
- Euler and modified Newton-Raphson

3.5 Results

Results here.

A Mathematica code

Here is the Mathematica code used to set up the system of equations and perform some symbolic computations.

```
Clear["Global`*"];
$Assumptions=Element[{L1,L2,A1,A2,E1,E2,u,v},Reals]&&L1>0&&L2>0&&A1>0&&A2>0&&E1>0&&E2>0;

(*Beam lenght*)
l1[u_,v_] := Sqrt[(L1+u)^2+v^2];
l2[u_,v_] := Sqrt[u^2+(L2+v)^2];
l ={l1,l2};

(*Green strain*)
eps1[u_,v_] :=(l[[1]][u,v]^2-L1^2)/(2*L1^2);
eps2[u_,v_] :=(l[[2]][u,v]^2-L2^2)/(2*L2^2);
eps = {eps1, eps2};

(*F_{int}*)
Fi1[u_,v_] :=E1*A1*eps[[1]][u,v];
Fi2[u_,v_] :=E2*A2*eps[[2]][u,v];
Fi = {Fi1,Fi2};

(*Trigonometry relations*)
cosAlpha[u_,v_] = (L1+u)/l[[1]][u,v];
sinAlpha[u_,v_] = v/l[[1]][u,v];
cosBeta[u_,v_] = (L2+u)/l[[2]][u,v];
sinBeta[u_,v_] = v/l[[2]][u,v];

(*Equilibrium of the system*)
Px[u_,v_] = Fi[[1]][u,v]*cosAlpha[u,v]+Fi[[2]][u,v]*sinBeta[u,v];
Py[u_,v_] = Fi[[1]][u,v]*sinAlpha[u,v]+Fi[[2]][u,v]*cosBeta[u,v];
P[u_,v_]={Px[u,v],Py[u,v]}; // FullSimplify

(*Setting values for the problem datas*)
problemDatas = {
E1-> 70*10^9,A1->10^-3,L1->3,
E2->70*10^9,A2->10^-3,L2->0.5
};
Pevaluation[u_,v_]= P[u,v]/. problemDatas; // FullSimplify

(*Defining the taylor expansion series*)
Ptaylor[x_, y_, taylorOrder_] := Normal @ Series[P[u, v] /. Thread[{u, v} -> {0, 0} + t {x, y}], {t, 0, taylorOrder}] /. t -> 1; // FullSimplify
Ptaylorevaluation[x_, y_, taylorOrder_] := Normal @ Series[Pevaluation[u, v] /. Thread[{u, v} -> {0, 0} + t {x, y}], {t, 0, taylorOrder}] /. t -> 1; // FullSimplify
```

Listing 1: Mathematica notebook used for symbolic analysis of the equations.