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HIGH RESOLUTION SCHEMES USING FLUX LIMITERS FOR HYPERBOLIC CONSERVATION LAWS*

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Abstract. The technique of obtaining high resolution, second order, oscillation free (TVD), explicit scalar difference schemes, by the addition of a limited antidiffusive flux to a first order scheme is explored and bounds derived for such limiters. A class of limiters is presented which includes a very compressive limiter due to Roe, and various limiters are compared both theoretically and numerically.

1. Introduction. Recently Roe [16] proposed a difference scheme which gives remarkably sharp profiles for the linear advection equation. It was the investigation of this scheme which prompted the work of this paper, which presents a unification of several independently proposed second order accurate TVD schemes, thus enabling it to be easily seen how these schemes related to each other.

Roe's sharp profile scheme, like several other recent schemes, falls into the category of flux limiters, much akin to the Flux Corrected Transport of Boris and Book [1] although differing in the respect of being essentially one-step procedures as opposed to the two-step FCT. The purpose of flux limiting/correcting is to produce a high resolution scheme without the spurious oscillations associated with the more classical second order schemes.

Some years ago Van Leer [22] derived a scheme using a flux limiter in his search for the ultimate conservative difference scheme, and more recently Roe [14] utilized flux limiting in his original monotonicity preserving second order scheme. Even more recently Chakravarthy and Osher [2] have used limiters, as has Harten [5] who also introduced the notion of TVD (Total Variation Diminishing) to characterize oscillation free schemes.

In § 2 we lay the foundation of entropy satisfying (assuring a unique solution) first order schemes to which, in § 3, we add a limited antidiffusive flux, and show the constraints this flux must satisfy to give a second order TVD scheme. A class of flux limiters is presented, which in § 4 are shown to include Roe's sharp profile limiter as well as his original limiter and a special case of the Chakravarthy and Osher limiter. Van Leer's limiter is also reformulated in the notation of § 3 and in § 5 numerical comparison of some of the limiters is given.

The schemes considered here are fully discrete, but recently Osher and Chakravarthy [13] (this issue, pp. 955–984) have also used a similar procedure to obtain a second order semi-discrete scheme from a general 3-point first order semi-discrete scheme. With the addition of artificial compression/rarefaction (ACR) they have also been able to prove entropy satisfaction for the second order semi-discrete scheme.

The schemes considered here are one-dimensional, and although they may easily be extended to two dimensions, a recent result by Goodman and LeVeque [4] shows that TVD schemes in two dimensions are at most first order accurate.

2. First order schemes. We shall consider numerical approximations to the scalar conservation law

$$\begin{aligned} &u_t+f(u)_x=0, \qquad t>0, \quad x\in\mathbb{R},\\ &u(x,0)=u_0(x). \end{aligned} \tag{2.1}$$

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In particular we shall consider numerical schemes written in conservation form

$$u^k = u_k - \lambda(h_{k+1/2} - h_{k-1/2}) \tag{2.2}$$

where h is a consistent numerical flux function

$$\begin{aligned} h_{k+1/2} &= h(u_{k+l}, \cdots, u_k, \cdots, u_{k-m}), \\ h(u, u, \cdots, u) &= f(u), \end{aligned} \tag{2.3}$$

λ is the mesh ratio

$$\lambda = \frac{\Delta t}{\Delta x}, \tag{2.4}$$

and u_k are nodal values of the piecewise constant mesh function $u_{\Delta x}(x, t)$ approximating $u(x, t)$.

Throughout we shall use the shorthand notation

$$u^k \equiv u_k^{n+1}, \qquad u_k \equiv u_k^n, \tag{2.5}$$

where k and n are the spatial and time indices respectively, whenever this is unambiguous.

Also, for clarity, we will restrict ourselves to regular grids, Δx constant, although results for irregular grids easily follow.

Recently Osher [11] defined a class of semidiscrete schemes approximating (2.1) which he names E-schemes. He showed that these schemes are at most first-order accurate but converge to the correct physical (entropy satisfying) solution of (2.1). (It is well known that weak solutions to (2.1) are nonunique and so an extra constraint is needed to select the unique physical solution. This constraint is taken to be the satisfaction of an entropy inequality, see for example [7].) These E-schemes,

$$\frac{\partial u}{\partial t} = -\frac{1}{\Delta x}(h^E_{k+1/2} - h^E_{k-1/2}), \tag{2.6}$$

may be characterized by the inequality

$$\operatorname{sgn}(u_{k+1} - u_k)[h^E_{k+1/2} - f(u)] \leqq 0 \tag{2.7}$$

for all u between u_k and u_{k+1}.

We shall consider fully discrete versions of E-schemes, in particular three point schemes:

$$\begin{aligned} u^k &= u_k - \lambda(h^E_{k+1/2} - h^E_{k-1/2}), \\ h^E_{k+1/2} &= h^E(u_{k+1}, u_k), \end{aligned} \tag{2.8}$$

where the inequality (2.7) still holds. Tadmor [21] has recently proved entropy satisfaction for these fully discrete E-schemes for a CFL like condition of $\frac{1}{2}$ and we also note that monotone schemes [6] belong to this class of schemes.

We now define, for a general three-point E-scheme (2.8), the flux differences

$$\begin{aligned} (\Delta f_{k+1/2})^+ &= -(h^E_{k+1/2} - f(u_{k+1})), \\ (\Delta f_{k+1/2})^- &= (h^E_{k+1/2} - f(u_k)) \end{aligned} \tag{2.9}$$

and note that

$$(\Delta f_{k+1/2})^+ + (\Delta f_{k+1/2})^- = \Delta f_{k+1/2}. \tag{2.10}$$

[We use the convention $\Delta_+ y_k = \Delta y_{k+1/2} = \Delta_- y_{k+1} = y_{k+1} - y_k$.]

These flux differences in turn are used to define a series of local CFL numbers:

$$\nu^{+}_{k+1/2}=\frac{\lambda(\Delta f_{k+1/2})^{+}}{\Delta u_{k+1/2}}, \tag{2.11a}$$

$$\nu^{-}_{k+1/2}=\frac{\lambda(\Delta f_{k+1/2})^{-}}{\Delta u_{k+1/2}}, \tag{2.11b}$$

$$\nu_{k+1/2}=\nu^{+}_{k+1/2}+\nu^{-}_{k+1/2}=\frac{\lambda\Delta f_{k+1/2}}{\Delta u_{k+1/2}}. \tag{2.11c}$$

Note that from the defining inequality of E-schemes, (2.7), we have

$$\nu^{+}_{k+1/2}\geqq 0,\qquad \nu^{-}_{k+1/2}\leqq 0, \tag{2.12}$$

justifying the +, − superscripts of the definitions.

It is well known (e.g. [20], [5], [6], [9]) that a crucial estimate involved in convergence proofs on difference schemes approximation (2.1) is a bound on the variation of the solution. The Total Variation, $\mathrm{TV}(u^{n+1})$, of the solution is defined by

$$\mathrm{TV}\,(u^{n+1})=\sum_{k}|u^{n+1}_{k+1}-u^{n+1}_{k}|, \tag{2.13}$$

where the shorthand (2.5) has temporarily been dropped, and an important class of difference schemes is those which are Total Variation Diminishing (TVD),

$$\mathrm{TV}\,(u^{n+1})\leqq \mathrm{TV}\,(u^{n}), \tag{2.14}$$

so-called after Harten [5].

If the general scheme (2.2) is rewritten in the form

$$u^{k}=u_k-C_{k-1/2}\Delta u_{k-1/2}+D_{k+1/2}\Delta u_{k+1/2} \tag{2.15}$$

where $C_{k-1/2}$ and $D_{k+1/2}$ are data-dependent coefficients (i.e., functions of the set $\{u_k\}$), then it is easily shown [20], [5] that sufficient conditions for the scheme to be TVD are the inequalities

$$0\leqq C_{k+1/2},\quad 0\leqq D_{k+1/2},\quad 0\leqq C_{k+1/2}+D_{k+1/2}\leqq 1. \tag{2.16}$$

From (2.9) it is seen that

$$(\Delta f_{k+1/2})^{-}+(\Delta f_{k-1/2})^{+}=h^{E}_{k+1/2}-h^{E}_{k-1/2}, \tag{2.17}$$

and therefore, using (2.11), one possibility of writing a general discrete E-scheme (2.8) in the form (2.15) is

$$u^{k}=u_k-\nu^{+}_{k-1/2}\Delta u_{k-1/2}-\nu^{-}_{k+1/2}\Delta u_{k+1/2}, \tag{2.18}$$

i.e. taking

$$C_{k+1/2}=\nu^{+}_{k+1/2},\qquad D_{k+1/2}=-\nu^{-}_{k+1/2}. \tag{2.19}$$

It is obvious from (2.12) that the first two inequalities of the set (2.16) are satisfied whilst the third inequality of the set gives the CFL-like condition

$$\nu^{+}_{k+1/2}-\nu^{-}_{k+1/2}\leqq 1 \tag{2.20}$$

for the scheme (2.8) to be TVD.

One example of an E-scheme is the Engquist–Osher scheme [3] which has numerical flux

$$h^{\mathrm{EO}}_{k+1/2}=f^{+}_{k}+f^{-}_{k+1}+f(\bar{u}) \tag{2.21}$$

where

$$f_k^+=\int_{\bar u}^{u_k}\chi(s)f'(s)\,ds,\qquad f_k^-=\int_{\bar u}^{u_k}(1-\chi(s))f'(s)\,ds \tag{2.22}$$

and

$$\chi(s)=\begin{cases}1, & f'(s)>0,\\ 0, & f'(s)\leqq 0\end{cases}$$

($\bar u$ is the sonic point of $f(u)$, $f'(\bar u)=0$).

For this scheme we have

$$\begin{aligned}(\Delta f_{k+1/2})^+&=-(f_k^++f_{k+1}^-+f(\bar u)-f_{k+1})\\ &=-(f_k^++f_{k+1}^--f_{k+1}^+-f_{k+1}^-)\\ &=f_{k+1}^+-f_k^+\\ &=\int_{u_k}^{u_{k+1}}\chi(s)f'(s)\,ds,\end{aligned} \tag{2.23a}$$

and similarly

$$(\Delta f_{k+1/2})^-=\int_{u_k}^{u_{k+1}}(1-\chi(s))f'(s)\,ds, \tag{2.23b}$$

which gives

$$\begin{aligned}\nu_{k+1/2}^+-\nu_{k+1/2}^-&=\frac{\lambda}{\Delta u_{k+1/2}}\int_{u_k}^{u_{k+1}}(2\chi(s)-1)f'(s)\,ds\\ &=\frac{\lambda}{\Delta u_{k+1/2}}\int_{u_k}^{u_{k+1}}|f'(s)|\,ds\\ &\leqq\lambda|f'|_{\max}.\end{aligned} \tag{2.24}$$

Therefore the Engquist–Osher scheme is TVD subject to a CFL condition

$$\sup_\xi(\lambda|f'(\xi)|)\leqq 1. \tag{2.25}$$

We shall assume for the remainder of this paper that the general discrete E-scheme (2.8) is TVD under a CFL condition

$$\sup_\xi(\lambda|f'(\xi)|)\leqq\mu\leqq 1. \tag{2.26}$$

It is well known that first order accurate schemes suffer from numerical diffusion, but classical higher order schemes, whilst giving higher resolution to discontinuities of the solution, exhibit spurious oscillations around such points (e.g. the Lax–Wendroff scheme [8] and Warming and Beam scheme [23]). In recent years effort has been placed into obtaining second order schemes which give high resolution whilst remaining TVD. For example, Van Leer [22], Roe [3] and Chakravarthy and Osher [2] have all proposed such high resolution schemes which incorporate some form of flux limiter.

In the next section we systematically derive a class of high resolution TVD second order schemes, which, by method of construction includes an extremely compressive limiter recently proposed by Roe [16]. Then in § 4 we investigate the schemes of the above-mentioned authors in their various formulations and compare them analytically with the class of limiters in § 3.

3. Higher resolution schemes. We now seek to derive a higher resolution TVD scheme in much the same way as the Flux Corrected Transport (FCT) of Boris and Book [1], that is, the application of a low order scheme supplemented by the addition of a "limited" (or "corrected" as in the terminology of Boris and Book) flux. This flux is a difference between the flux of a high order scheme and that of the low order scheme, which has been "limited" in such a way as to ensure the resulting scheme is TVD.

There are two main differences between the approach adopted here and that of Boris and Book [1] (and later Zalesak [24]). Firstly the FCT algorithm was essentially a two-step procedure, whereas here we adopt a single-step approach; and secondly the FCT limiter was constricted by unity whilst we allow a more generous upper limit.

For clarity of approach we first consider the linear scalar equation

$$u_t + au_x = 0, \qquad a > 0. \tag{3.1}$$

The second order Lax–Wendroff scheme [8] may be written as

$$u^k = u_k - \nu\Delta u_{k-1/2} - \Delta_-\{\tfrac{1}{2}(1-\nu)\nu\Delta u_{k+1/2}\} \tag{3.2}$$

where here

$$\nu = \frac{a\Delta t}{\Delta x}.$$

It is seen that (3.2) is in fact the result of a first order scheme

$$u^k = u_k - \nu\Delta u_{k-1/2} \tag{3.3}$$

with an additional term

$$-\Delta_-\{\tfrac{1}{2}(1-\nu)\nu\Delta u_{k+1/2}\} \tag{3.4}$$

added. That is, the numerical flux of the Lax–Wendroff scheme is that of the first order scheme (3.3) plus an additional flux

$$-\frac{1}{2\lambda}(1-\nu)\nu\Delta u_{k+1/2}. \tag{3.5}$$

We shall refer to this extra flux as an antidiffusive flux.

Since it is well known that the Lax–Wendroff scheme is not TVD, we try to remedy this by adding only a limited amount of the antidiffusive flux (3.5) to the first order scheme, i.e.

$$u^k = u_k - \nu\Delta u_{k-1/2} - \Delta_-\{\varphi_k\tfrac{1}{2}(1-\nu)\nu\Delta u_{k+1/2}\}, \tag{3.6}$$

where φ_k is some form of limiter, taken to be nonnegative so as to maintain the sign of the antidiffusive flux.

Like Roe [14], Van Leer [22] before him and more recently Chakravarthy and Osher [2] we take the limiter to be a function of consecutive gradients (in the linear case), i.e., $\varphi_k = \varphi(r_k)$ where

$$r_k = \frac{\Delta u_{k-1/2}}{\Delta u_{k+1/2}}. \tag{3.7}$$

We now seek to choose the function $\varphi(r)$ in such a way that the limited antidiffusive flux (3.5) is maximized in amplitude subject to the constraint of the resulting scheme being TVD.

Viewing the scheme (3.6), a possible choice of the coefficients $C_{k-1/2}$, $D_{k+1/2}$ in (2.15) is

$$C_{k-1/2}=\nu+\tfrac{1}{2}(1-\nu)\frac{\nu\Delta_{-}\{\varphi(r_k)\Delta u_{k+1/2}\}}{\Delta u_{k-1/2}}, \qquad D_{k+1/2}\equiv 0. \tag{3.8}$$

Reorganization of the expression for $C_{k-1/2}$ gives

$$C_{k-1/2}=\nu\{1+\tfrac{1}{2}(1-\nu)[\varphi(r_k)/r_k-\varphi(r_{k-1})]\} \tag{3.9}$$

and a bound on $C_{k-1/2}$ of

$$\nu\{1-\tfrac{1}{2}(1-\nu)\Phi\}\leqq C_{k-1/2}\leqq \nu\{1+\tfrac{1}{2}(1-\nu)\Phi\} \tag{3.10}$$

where

$$|\varphi(r_k)/r_k-\varphi(r_{k-1})|\leqq\Phi. \tag{3.11}$$

Therefore for $C_{k-1/2}$, $D_{k-1/2}$ to satisfy the TVD inequalities (2.16), which in this case reduce to

$$0\leqq C_{k-1/2}\leqq 1,$$

it is easily seen that we must have $\Phi\leqq 2$ in (3.11). If, in addition to requiring $\varphi(r)$ to be nonnegative we also insist on

$$\varphi(r)=0, \qquad r\leqq 0$$

then the bound (3.11) reduces to

$$0\leqq\left(\frac{\varphi(r)}{r},\varphi(r)\right)\leqq 2. \tag{3.12}$$

Hence for the scheme (3.6) to be TVD the limiter function $\varphi(r)$ must lie in the shaded region of Fig. 1a, which also illustrates the φ functions needed to give both the Lax–Wendroff scheme and the second order upwind scheme of Warming and Beam [23],

$$u^k=u_k-\nu\Delta u_{k-1/2}-\Delta_{-}\{\tfrac{1}{2}(1-\nu)\nu\Delta u_{k-1/2}\} \tag{3.13}$$

which is also non-TVD.

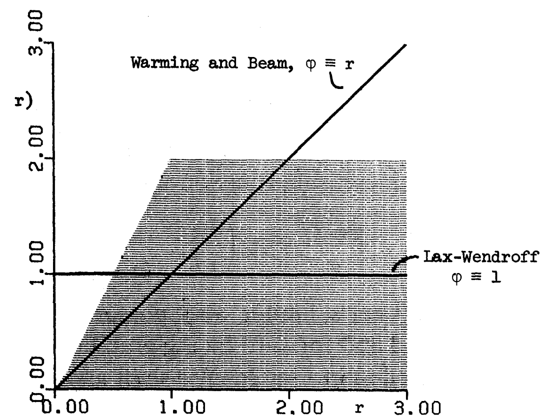
To maximize the antidiffusive flux that we add to the first order scheme, we need to maximize the limiter $\varphi(r)$ subject to the TVD constraints; so an obvious choice is

$$\varphi(r)=\min(2r,2), \qquad r>0, \tag{3.14}$$

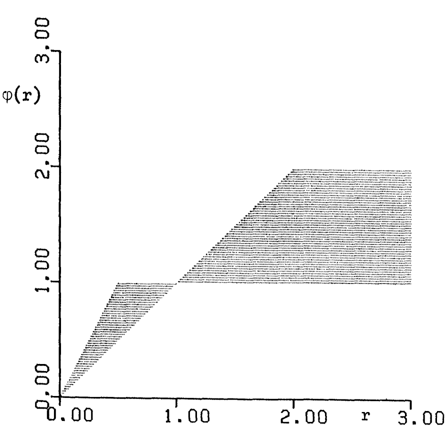
which is the upper boundary of the region in Fig. 1a. However, there is one final constraint that we impose on $\varphi(r)$ and that is that the resulting scheme (3.6) be second order accurate whenever possible. We note here that since $\varphi(r)=0$ for $r<0$, second order accuracy must be lost at extrema of the solution.

On viewing Fig. 1a it is noticed that both second order schemes depicted there pass through the point $\varphi(1)=1$, which is a general requirement for second order accuracy (as well as Lipschitz continuity of $\varphi(r)$). We also note that any second order scheme relying only on the points $(u_{k-2}, u_{k-1}, u_k, u_{k+1})$ must be a weighted average of the Lax–Wendroff scheme and the Warming and Beam upwind scheme (cf. Van Leer's [22] approach of using Fromm's scheme, the arithmetic average of these two schemes, as a starting point), i.e.

$$\varphi(r)=(1-\theta(r))\varphi^{\mathrm{LW}}(r)+\theta(r)\varphi^{\mathrm{WB}}(r)$$



(a) TVD *region.*



(b) *Second order* TVD *region.*

FIG. 1. TVD *regions.*

with

$$0 \leqq \theta(r) \leqq 1. \tag{3.15}$$

(We specify an internal average here, i.e. $0 \leqq \theta \leqq 1$, since numerical tests on external averages showed the resulting scheme to be overcompressive causing sine wave data to give square wave type solutions.)

Since $\varphi^{\mathrm{LW}}(r) \equiv 1$ and $\varphi^{\mathrm{WB}}(r) \equiv r$, this reduces to

$$\varphi(r) = 1 - \theta(r) + r\theta(r) = 1 + \theta(r)(r-1), \tag{3.16}$$

and $\varphi(r)$ is now confined to lie in the region shown in Fig. 1b. Note that the condition $\varphi(1) = 1$ is automatically imposed.

We shall later show that the upper boundary of this region is equivalent to Roe's compressive transfer function ("superbee" [16] see § 4) and that the lower boundary is equivalent to Roe's minmod transfer function [20] and is a special case of limiters used by Harten [5], and Chakravarthy and Osher [2], [12]. We shall also show that Van Leer's limiter [22] is a smooth curve lying within the region.

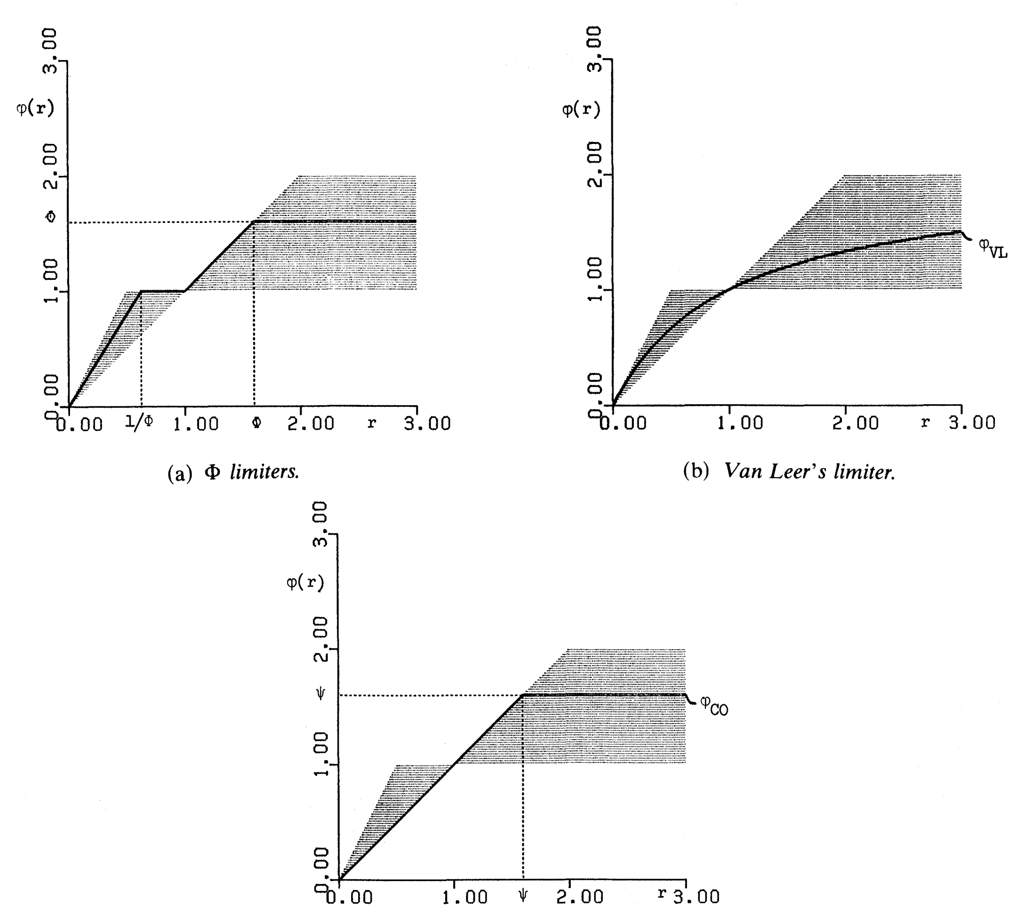
A class of flux limiters which include both extremes of the upper and lower boundaries may be defined as

$$\varphi_\Phi(r)=\max\,(0,\min\,(\Phi r,1),\min\,(r,\Phi)),\qquad 1\leqq\Phi\leqq 2 \tag{3.17}$$

which as Φ varies from 2 to 1 moves across the whole region from top to bottom. Note that $\varphi_\Phi(r)$ is a monotone increasing function and has a symmetry of

$$\frac{\varphi_\Phi(r)}{r}=\varphi_\Phi\left(\frac{1}{r}\right). \tag{3.18}$$

(We shall later see that Van Leer's limiter also possesses these properties.) This symmetry ensures that backward and forward facing gradients are treated in the same fashion. A typical $\varphi_\Phi(r)$ is sketched in Fig. 2a. We shall assume for the remainder of



(a) Φ *limiters.*

(b) *Van Leer's limiter.*

(c) *Chakravarthy–Osher limiter.*

FIG. 2. *Limiters.*

this section that the limiter function $\varphi(r)$ is a general limiter which lies within the region of Fig. 1b and now propose an extension of the above ideas to the nonlinear equation (2.1). We take the underlying first order scheme to be an E-scheme and add both limited positive and negative fluxes. That is, using the definitions of § 2, we have

$$u^k=u_k-\lambda\Delta_-h^E_{k+1/2}-\lambda\Delta_-\{\varphi(r_k^+)\alpha^+_{k+1/2}(\Delta f_{k+1/2})^+-\varphi(r^-_{k+1})\alpha^-_{k+1/2}(\Delta f_{k+1/2})^-\}, \tag{3.19}$$

where

$$\alpha^+_{k+1/2}=\tfrac{1}{2}(1-\nu^+_{k+1/2}),\qquad \alpha^-_{k+1/2}=\tfrac{1}{2}(1+\nu^-_{k+1/2}) \tag{3.20}$$

and

$$r^+_k=\frac{\alpha^+_{k-1/2}(\Delta f_{k-1/2})^+}{\alpha^+_{k+1/2}(\Delta f_{k+1/2})^+},\qquad r^-_k=\frac{\alpha^-_{k+1/2}(\Delta f_{k+1/2})^-}{\alpha^-_{k-1/2}(\Delta f_{k-1/2})^-}, \tag{3.21}$$

noting that in these definitions the superscript does not indicate the sign of the quantity.

This gives possible choices of the TVD coefficients $C_{k+1/2}$, $D_{k+1/2}$ of (2.15) as

$$C_{k+1/2}=\nu^+_{k+1/2}\{1+\alpha^+_{k+1/2}[\varphi(r^+_{k+1})/r^+_{k+1}-\varphi(r^+_k)]\}$$

and

$$D_{k+1/2}=-\nu^-_{k+1/2}\{1+\alpha^-_{k+1/2}[\varphi(r^-_k)/r^-_k-\varphi(r^-_{k+1})]\}, \tag{3.22}$$

from which, using the general bound (3.11), we see

$$C_{k+1/2}\geqq \nu^+_{k+1/2}\{1-\Phi\alpha^+_{k+1/2}\}\geqq 0\quad \text{for } \Phi\leqq 2,$$

and similarly

$$D_{k+1/2}\geqq 0\quad \text{for } \Phi\leqq 2.$$

Hence the first two inequalities of the set (2.16) are satisfied and we now investigate the third, CFL like, inequality of that set. We have

$$\begin{aligned} C_{k+1/2}+D_{k+1/2}&\leqq \nu^+_{k+1/2}\{1+\alpha^+_{k+1/2}\Phi\}-\nu^-_{k+1/2}\{1+\alpha^-_{k+1/2}\Phi\}\\ &=(\nu^+_{k+1/2}-\nu^-_{k+1/2})(1+\Phi/2)-\frac{\Phi}{2}(\nu^{+2}_{k+1/2}+\nu^{-2}_{k+1/2})\\ &\leqq(\nu^+_{k+1/2}-\nu^-_{k+1/2})(1+\Phi/2), \end{aligned} \tag{3.23}$$

and therefore, if the first order E-scheme satisfies the CFL condition (2.26), then comparison of (3.23) with (2.20) shows that the second order scheme must satisfy the CFL condition

$$\sup_{\xi}(\lambda|f'(\xi)|)\leqq\left(\frac{2}{2+\Phi}\right)\mu\leqq\frac{2}{3} \tag{3.24}$$

in order to guarantee that it is TVD.

We note that away from sonic points either $\nu^+_{k+1/2}$ or $\nu^-_{k+1/2}$ is zero and the scheme is TVD for Φ up to 2 under the original CFL condition (2.26). An alternative to the reduced CFL condition (3.24) is therefore to revert to the underlying E-scheme, i.e. $\varphi=0$, at sonic points; however, this causes the limiting function φ to be discontinuous and adds an extra complexity to implementation (since φ is then no longer a function of just r). Therefore it is not favoured by the author.

Although this paper deals only with the scalar case, we remark that the notion of flux limiters is readily extendable to systems of conservation laws where $h^E_{k+1/2}$ and $(\Delta f_{k+1/2})^{\pm}$ become vectors. The ratio $r_{k+1/2}$ must now be redefined. This is achieved using inner products with a suitable vector, e.g.

$$r^+_k=\frac{\alpha^+_{k-1/2}\langle(\Delta f_{k-1/2})^+,v\rangle}{\alpha^+_{k+1/2}\langle(\Delta f_{k+1/2})^+,v\rangle}, \tag{3.25}$$

where the vector v depends on the actual Riemann solver used as will suitable definitions of $\nu^{\pm}_{k+1/2}$. (See, for example, [2], [12], [15].)

There remains one other concern for the approximation to the nonlinear equation, and that is one of entropy satisfaction, i.e., convergence towards the correct physical solution. Although no rigorous analytical proof for a general explicit flux limiter scheme is known, we have come across no numerical evidence to suggest entropy violation and even some evidence to support entropy satisfaction. Since the underlying first order scheme is supposed entropy satisfying, it is reasonable to conjecture that if, by addition of the limited antidiffusive flux, diffusion at expansions is not decreased, then the second order scheme is entropy satisfying.

To investigate this the most convenient form of the scheme to us is that of (2.15), where $C_{k-1/2}, D_{k+1/2}$ are defined as in (2.19) for the first order scheme, and as in (3.22) for the second order scheme. These coefficients may be considered as indicating right and left moving diffusion respectively; hence if the magnitude of these coefficients as defined in (3.22) for the second order scheme is not less than the first order versions (2.19), then the diffusion is not decreased. (The coefficients are already known to be of the same sign).

Although we have not been able to show this in general, we can do so at a sonic expansion, which is where entropy violations in other schemes have been known to occur. For convex $f(u)$ a sonic expansion may be characterized by the condition

$$f'(u_{k-1/2})<0<f'(u_{k+3/2}).$$

This implies that $\nu^+_{k-1/2}=0$, $\nu^+_{k+3/2}>0$. Recalling the definition of r^+ (3.21) and ν^+ (2.11), it is seen that

$$r^+_k=0, \qquad r^+_{k+1}\neq 0.$$

Therefore the term

$$[\varphi(r^+_{k+1})/r^+_{k+1}-\varphi(r^+_k)]$$

in (3.22) is nonnegative and hence $C_{k+1/2}$ for the second order scheme is not less in magnitude than $C_{k+1/2}$ for the first order scheme (cf. (2.19)). A similar argument holds for $D_{k+1/2}$ suggesting entropy satisfaction of the second order scheme.

4. Comparison of limiters. In this section we study schemes proposed by Van Leer [22], Roe [14], [20] and [17] and Chakravarthy and Osher [2] and investigate their relationship to the framework of limiters set up in the previous section. Although others (e.g., Boris and Book [1], Zalesak [24] and Le Roux [9]) have proposed schemes involving forms of flux limiters, they do not fall into the framework considered here, not being expressible as functions only of the ratio r. For this reason we do not study them here.

Since the various schemes are presented by their authors in different formulations, we first translate the schemes into a common formulation, using notation from §§ 2 and 3 where applicable. The resulting limiters are then compared.

4.1. Van Leer. In [22] Van Leer averages nonconservative limited versions of the Lax–Wendroff and Warming and Beam schemes to give a conservative limited version of Fromm's scheme (the arithmetic average of the two schemes). The parameter he uses as a "smoothness monitor" is

$$\theta_k=\frac{\Delta u_{k+1/2}}{\Delta u_{k-1/2}}, \tag{4.1}$$

which is the reciprocal of the ratio r_k (3.7) used in the previous section. The averaged scheme is written as

$$u^k = u_k - \nu\Delta u_{k-1/2} - \frac{\nu}{4}(1-\nu)(\Delta u_{k+1/2} - \Delta u_{k-3/2}) \tag{4.2}$$
$$+\frac{\nu}{4}(1-\nu)\{S(\theta_k)(\Delta u_{k+1/2} - \Delta u_{k-1/2}) - S(\theta_{k-1})(\Delta u_{k-1/2} - \Delta u_{k-3/2})\}$$

where the function $S(\theta)$ is defined to be

$$S(\theta) = \frac{|\theta|-1}{|\theta|+1}. \tag{4.3}$$

The equation (4.2) may be manipulated to give

$$u^k = u_k - \nu\Delta u_{k-1/2} - \Delta_-\{\tfrac{1}{2}((1-S(\theta_k)) + (1+S(\theta))/\theta_k)\tfrac{1}{2}(1-\nu)\nu\Delta u_{k+1/2}\}, \tag{4.4}$$

which is in the form of (3.6) with

$$\varphi_k = \tfrac{1}{2}((1-S(\theta_k)) + (1+S(\theta_k))/\theta_k). \tag{4.5}$$

By using the relationship $\theta_k = 1/r_k$ and the definition (4.3), we get an expression for the limiter as a function of r as

$$\varphi_{\mathrm{VL}}(r) = \frac{|r|+r}{1+|r|}. \tag{4.6}$$

Note that $\varphi_{\mathrm{VL}}(r) \geqq 0$, with

$$\varphi_{\mathrm{VL}}(r) = \begin{cases} 0, & r \leqq 0, \\ \dfrac{2r}{1+r}, & r > 0, \end{cases} \tag{4.7}$$

showing it to be monotone increasing and satisfying the symmetry property

$$\varphi_{\mathrm{VL}}(r)/r = \varphi_{\mathrm{VL}}(1/r). \tag{4.8}$$

By sketching this limiter as in Fig. 2b it is seen that it lies within the second order TVD region established in § 3.

Van Leer extends his scheme to the nonlinear equation merely by substituting $\nu_{k+1/2}$ for ν; however, there is no reason why the method of extension used in § 3 should not be used.

4.2. Roe. Roe's second order scheme [14], [17], [20] is presented in "increment" formulation rather than more classical numerical flux formulation. For a given cell (x_k, x_{k+1}) an increment or fluctuation is calculated,

$$g_{k+1/2} = -\nu_{k+1/2}\Delta u_{k+1/2} \tag{4.9}$$

which is then added to the value of u at the downwind neighbour (left $\nu<0$, right $\nu>0$) of the cell to obtain a first order scheme at the next time level,

$$u^k = u_k + \tfrac{1}{2}(1+S_{k-1/2})g_{k-1/2} + \tfrac{1}{2}(1-S_{k+1/2})g_{k+1/2} \tag{4.10}$$

where $S_{k+1/2} = \operatorname{sgn}(\nu_{k+1/2})$. Next a suitably calculated flux (see below) $b_{k+1/2}$ is transferred across the cell against the direction of flow to give a second order TVD scheme:

$$u^k = u_k + \tfrac{1}{2}(1+S_{k-1/2})g_{k-1/2} + \tfrac{1}{2}(1-S_{k+1/2})g_{k+1/2} - S_{k-1/2}b_{k-1/2} + S_{k+1/2}b_{k+1/2}. \tag{4.11}$$

The complete process is illustrated in Fig. 3, and it is easily seen that the transferred flux $b_{k+1/2}$ is a form of antidiffusive flux.

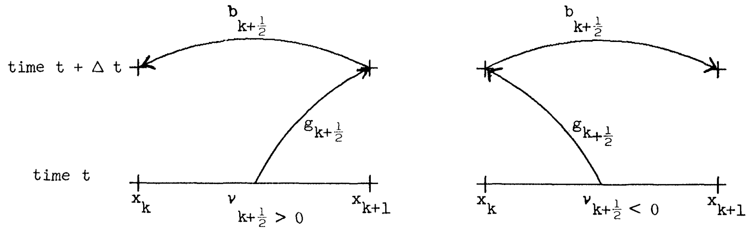


FIG. 3. *Roe's scheme.*

The transferred fluxes are defined to be a function of the quantity $\alpha_{k+1/2}g_{k+1/2}$ and its upwind neighbour, where

$$\alpha_{k+1/2}=\tfrac{1}{2}(1-|\nu_{k+1/2}|),$$

i.e.

$$b_{k+1/2}=B(\alpha_{k+1/2}g_{k+1/2},\ \alpha_{k-1/2}g_{k-1/2})\quad \text{for } \nu_{k+1/2}>0 \tag{4.12a}$$

and

$$b_{k+1/2}=B(\alpha_{k+1/2}g_{k+1/2},\ \alpha_{k+3/2}g_{k+3/2})\quad \text{for } \nu_{k+1/2}<0. \tag{4.12b}$$

The original definition of $B(x, y)$ was taken to be

$$B(x, y)=\operatorname{minmod}(x, y) \tag{4.13}$$

where

$$\operatorname{minmod}(x, y)=\begin{cases} x & \text{if } |x|\leqq|y|, \\ y & \text{if } |x|>|y|, \end{cases} \tag{4.14}$$

but more recently $B(x, y)$ is often taken as

$$B(x, y)=\begin{cases} \operatorname{minmod}(x, y), & xy>0, \\ 0, & xy\leqq 0. \end{cases} \tag{4.15}$$

To convert $B(x, y)$ to a flux limiter, we merely divide by x, to give (using (4.15))

$$\varphi_R=\begin{cases} 1 & \text{if } y/x>1, \\ y/x & \text{if } y/x<1, \\ 0 & \text{if } xy\leqq 0, \end{cases} \tag{4.16}$$

i.e.

$$\varphi_R(r)\equiv\varphi_1(r)=\max\{0, \min(r, 1)\},$$

where $\varphi_1=\varphi_\Phi$ with $\Phi=1$ and $r=y/x$. So the transfer function (4.15) of Roe is equivalent to the lower boundary of the region of Fig. 1b.

Recently Roe [16] proposed a highly compressive transfer function "superbee" defined by

$$B(x, y)=\begin{cases} \operatorname{maxmod}(x, y), & \dfrac{1}{2}\leqq\dfrac{y}{x}\leqq 2, \\ 2\operatorname{minmod}(x, y), & \dfrac{y}{x}<\dfrac{1}{2} \text{ or } \dfrac{y}{x}>2, \\ 0, & xy<0, \end{cases} \tag{4.17}$$

where maxmod has an analogous definition to minmod. On division by x this transfer function can easily be seen to be equivalent to the upper boundary of the region of Fig. 1b, i.e.

$$\varphi_{RC}(r) \equiv \varphi_2(r) = \max\{0, \min(2r, 1), \min(r, 2)\}$$

and so is the maximum possible limiter in the framework. (It was in fact the investigation of this transfer function, which produces very good results for the linear equation (see next section), which prompted the work of this paper.)

Roe's scheme, regardless of transfer function, uses the Murman upwind scheme [10] as the underlying first order scheme and has been shown to admit entropy violating shocks. However, various fixes of the scheme exist which involve replacing the Murman scheme with an E-scheme.

4.3. Chakravarthy and Osher. Chakravarthy and Osher [2] propose a limiter

$$\varphi_{CO}(r) = \max(0, \min(r, \psi)) \tag{4.18}$$

for $1 \leqq \psi \leqq 2$, and this limiter is sketched in Fig. 2c. As can be seen, their limiter lies within the second order TVD region, and for $\psi = 1$ it is equivalent to $\varphi_1(r)$, the lower boundary of the region. Note, however, that it is not symmetric in the sense of (3.18), i.e.

$$\varphi_{CO}(r)/r \neq \varphi_{CO}(1/r),$$

unless $\psi = 1$.

5. Numerical examples. We present here numerical results depicting various limiters on simple test problems. The limiters used were:

1) no limiter—i.e. first order scheme;
2) (3.17) $\Phi = 1$—φ_1 (minmod etc.);
3) (3.17) $\Phi = 2$—φ_2 (Roe's compressive limiter);
4) Van Leer—φ_{VL};
5) Chakravarthy–Osher $\psi = 2$—φ_{CO},

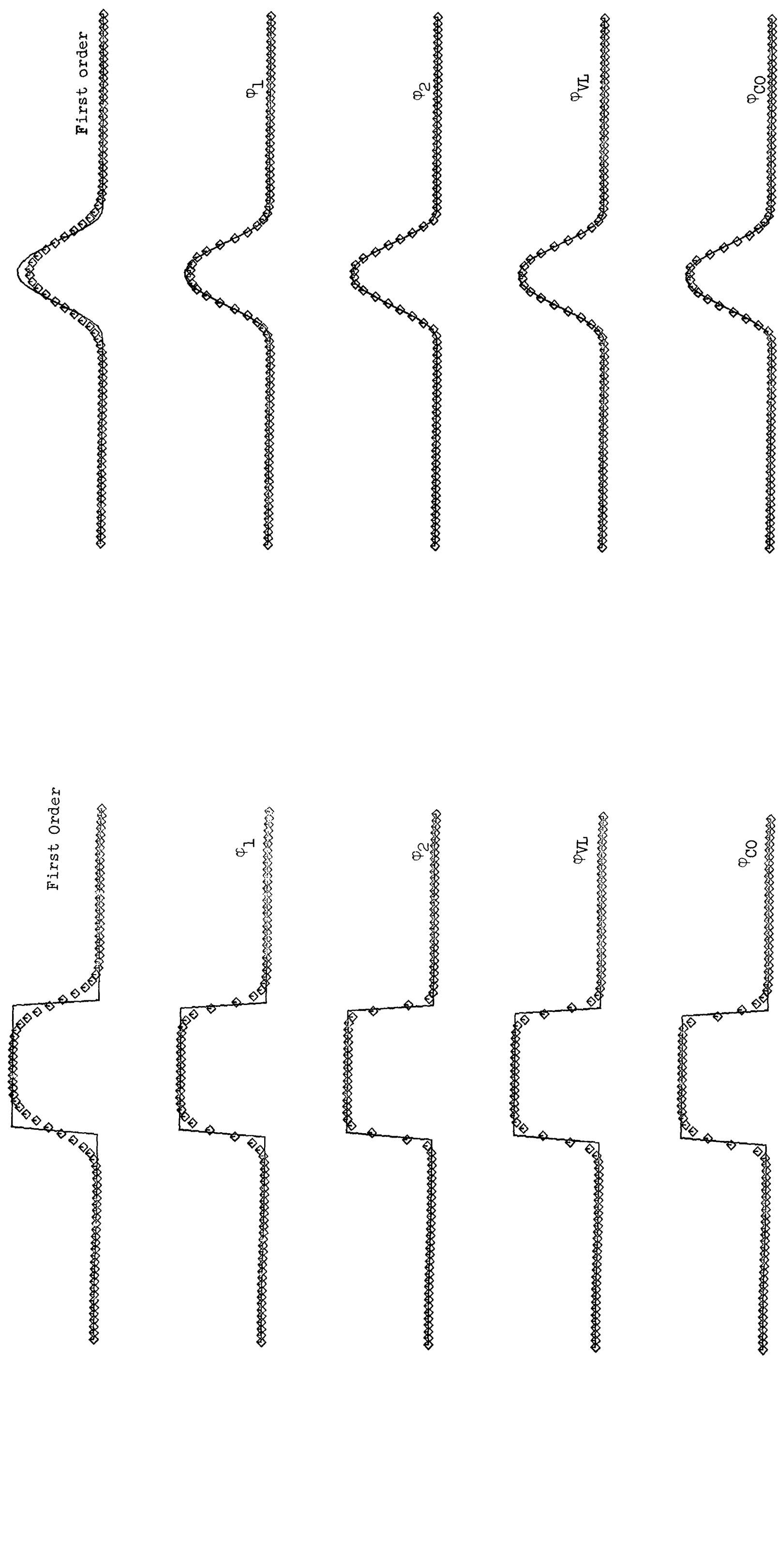
with the Engquist–Osher scheme (2.21) as the underlying first order scheme.

The first problem used was the linear advection equation (3.1). Figure 4 shows the results after 25 time steps ($\lambda = 0.5$) for both square wave and $\sin^2$ wave initial data. As can be seen all the limiter schemes show a significant improvement in resolution over the first order scheme, the solid line indicating the true solution.

For the square wave in Fig. 4a it is seen that Roe's compressive limiter (φ_2) gives remarkable resolution, followed by Van Leer (φ_{VL}) and minmod (φ_1). Although the resolution given by the Chakravarthy–Osher limiter (in the extreme case $\psi = 2$) is good, the lack of symmetry between the trailing and leading edge of the wave are clearly visible. In Fig. 4b the $\sin^2$ wave is convected well by all the second order schemes, the amplitude diminishing least by the φ_2 scheme although slight "squaring" of the top is noticeable. Again the lack of symmetry of the φ_{CO} limiter can be seen.

The second problem is the shock tube problem used by Sod [19], the schemes being extended to the system of conservation laws via Roe's approximate Riemann Solver [15]. This in effect uses scalar schemes on each characteristic field, which, in this case, consists of two nonlinear fields and a linear field. The scalar schemes applied need not be the same for each field.

Figure 5 shows the results of the limiters applied identically to all three characteristic fields, the energy plots only being shown. The leftmost discontinuity is a contact discontinuity whilst the right discontinuity is a shock. In Fig. 5b it is seen that φ_1 gives



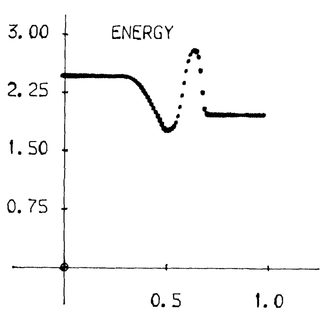
(a) *Square wave initial data.*

(b) $\sin^2$ *wave initial data.*

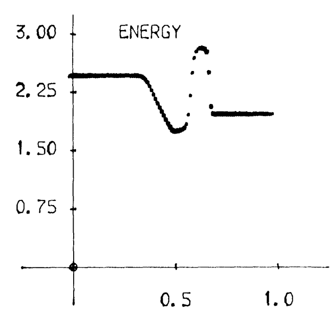
FIG. 4. *Linear advection test problem.*

a good shock resolution but the contact is not very sharp, whereas in Fig. 5c, φ_2 gives a sharp contact and shock, but due to the low CFL condition for this limiter the scheme is not TVD at the shock. Van Leer's limiter, φ_{VL}, gives better resolution of both the contact and shock compared to φ_1, as seen in Fig. 5d, but φ_{CO} gives a still sharper shock although a less sharp contact (Fig. 5e).

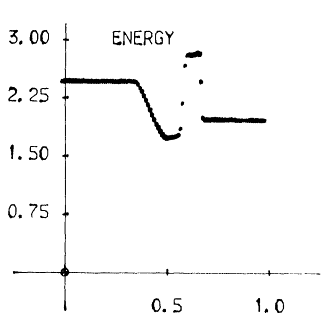
Finally, a couple of experiments using different limiters for different fields were tried, and the results can be seen in Fig. 6. For Fig. 6a φ_1 was used in the nonlinear fields and φ_2 in the linear field, thus giving a very sharp contact and shock without overshoot. By using Van Leer's limiter, φ_{VL}, in the nonlinear fields instead of φ_1, the sharpness of the shock is slightly improved still further as seen in Fig. 6b.



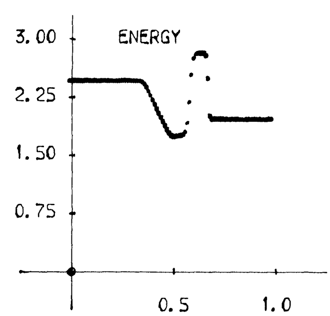
(a) *First order*



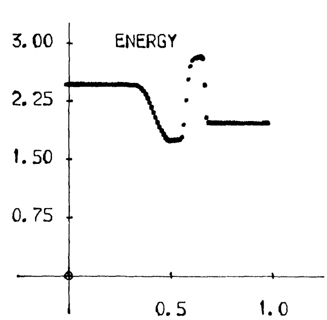
(b) φ_1.



(c) φ_2.

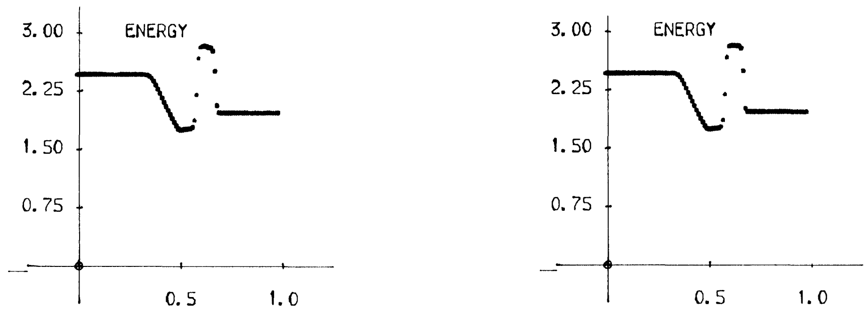


(d) φ_{VL}.



(e) φ_{CO}.

FIG. 5. *Sod's problem.*



(a) φ_1 *on nonlinear fields.* (b) φ_{VL} *on nonlinear fields.*

FIG. 6. *Sod's problem,* φ_2 *on linear field.*

6. Concluding remarks. We have investigated the derivation of high resolution second order accurate schemes by means of adding a limited antidiffusive flux to a general entropy-satisfying first order scheme. Constraints on the limiters, as functions of gradient ratios, have been obtained so that the resulting scheme is TVD, and a class of limiters proposed which satisfy these constraints. Flux limiters used by Roe and Chakravarthy and Osher have been studied and shown to be equivalent to members of the class in various cases, in particular a low diffusion limiter proposed by Roe which gives surprisingly good results in the linear case. Van Leer's flux limiter has also been investigated and shown to satisfy the TVD constraints and to exhibit results nearly as good as Roe's whilst being more reliable. It is demonstrated how good results can be obtained by using different limiters on different characteristic fields of systems of conservation laws.

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