ME663 - Computational Fluid Dynamics
Assignment 2

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1 Methods

```
(*Initial setup*)
U = {
     U1,
     U2,
     U3
};

F = {
     U[[2]],
     U[[2]]^2/U[[1]] +(\[Gamma]-1)*(U[[3]]-U[[2]]^2/(2*U[[1]])),
     U[[2]]/U[[1]] * (U[[3]]+(\[Gamma]-1)*(U[[3]]-U[[2]]^2/(2*U[[1]])))
};
```

Listing 1: Mathematica notebook used for symbolic analysis of the discretized schemes.

2 (Q1) - Question #1

Derive the Jacobian matrix $A = \frac{\partial F}{\partial U}$ given:

$$U = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}, \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho E + p)u \end{bmatrix} \tag{1}$$

where ρ is the density, u is the velocity, E is the total energy, and p is the pressure.

2.1 Solution

The Jacobian matrix A is defined as:

$$A = \frac{\partial F}{\partial U} = \begin{bmatrix} \frac{\partial F_1}{\partial U_1} & \frac{\partial F_1}{\partial U_2} & \frac{\partial F_1}{\partial U_3} \\ \frac{\partial F_2}{\partial U_1} & \frac{\partial F_2}{\partial U_2} & \frac{\partial F_2}{\partial U_3} \\ \frac{\partial F_3}{\partial U_1} & \frac{\partial F_3}{\partial U_2} & \frac{\partial F_3}{\partial U_3} \end{bmatrix} \tag{2}$$

where F_i is the i-th component of the vector F and U_i is the i-th component of the vector U.
Before proceeding with the derivation of the Jacobian matrix, it's useful to express the components of the vector F in terms of the components of the vector U:

$$F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho E + p)u \end{bmatrix} = \begin{bmatrix} U_2 \\ \frac{U_2^2}{U_1} + (\gamma - 1)\left(U_3 - \frac{U_2^2}{2U_1}\right) \\ \frac{U_2}{U_1}\left(U_3 + (\gamma - 1)\left(U_3 - \frac{U_2^2}{2U_1}\right)\right) \end{bmatrix} \tag{3}$$

Now, we can proceed with the derivation of the Jacobian matrix A based on its definition in Equation 2:

$$A = \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma - 1 \\ (\gamma-1)u^3 - \gamma u E & -\frac{3}{2}(\gamma-1)u^2 + \gamma E & \gamma u \end{bmatrix} \tag{4}$$

References