ME621 - Advanced Finite Element Methods
Assignment 1

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1 Requests

A system of two aluminum bars of the same material is shown in the following figure. The system is subjected to two external loads, P_x and P_y, at joint B. A and C are connected to pinned supports.

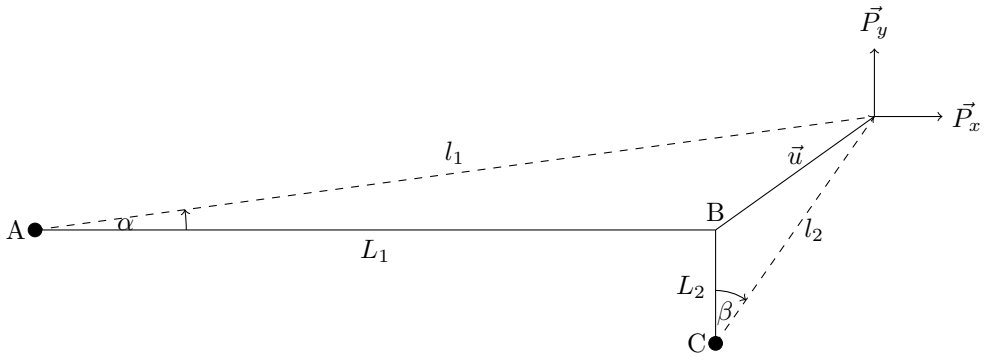


Figure 1: Problem representation

The problem asks to:

- Obtain the external loads P_x and P_y as a function of horizontal and vertical displacements at point B (namely u and v).
- Determine the displacements in both x and y directions for 1000 load increments of +5N for both P_x and P_y (from zero).
- Find the displacement of point B after the final increment.

Write a `MATLAB` code with a convergence error of 10^-5 to numerically solve the problem. Use a combination of (a) Euler and N-R, and (b) Euler and modified N-R. Also plot the resultant force versus the resultant displacement. Use the Green strain measure:

$$E_i = \frac{l_i^2 - L^2}{2L^2} \tag{1}$$

From now on, we will refer to the Green strain measure as $\epsilon_{1,2}$ to differentiate it from the Young's modulus $E_{1,2}$.

Parameter	**Value**	**Unit**
$E_1 = E_2 = E$	70	GPa
L_1	3	m
L_2	0.5	m
$A_1 = A_2 = A$	0.0001	m^2

Table 1: Parameters of the system

2 Methodology

We will start by writing the equilibrium equations for the system. We will further work on the **equilibrium equations to obtain the force-displacement relationship**. Finally, we will solve the system of equations $\vec{P} = f(\vec{u})$ to obtain the displacement of point B for a given load.
Notice that we will likely obtain a system of non-linear equations, which **we will solve using the both Euler and Newton-Raphson methods**. We will also perform a linearization of the system to successively compare the results obtained with the linearized and the non-linearized system.
Even if the problem asked to produce a `MATLAB` code, we will first set up the system of equations in `Mathematica`, and then we will translate the code to `MATLAB`. `Mathematica` is a more powerful tool for symbolic computation, and it will allow us to obtain the system of equations that can be easily updated in case of necessity (for example a different order of the Taylor series).

3 Solution

3.1 Equilibrium equations

We can start writing the equilibrium equations for the system. Since we have no data about the mass of the bars, we will assume negligible mass of the bars. This assumption will allow us to neglect the effect of gravity on the system and avoid the introduction of the inertia terms in the equilibrium equations.

$$\vec{F_{ext}} + \vec{F_{int}} = \vec{0} \tag{2}$$

$$\begin{cases} \vec{F_{ext}} = \vec{P_x} + \vec{P_y} \\ \vec{F_{int}} = \vec{F_1} + \vec{F_2} \end{cases} \tag{3}$$

By decomposing the equations in the x and y directions, we obtain:

$$\begin{cases} P_x = F_{int,x} = |F_1|\cos(\alpha) + |F_2|\sin(\beta) \\ P_y = F_{int,y} = |F_1|\sin(\alpha) + |F_2|\cos(\beta) \end{cases} \tag{4}$$

3.2 Force displacement relationship

So far we have obtained the equilibrium equations for the system. We can now proceed to obtain the force displacement relationship, which will allow us to solve the system of equations $\vec{P} = f(\vec{u})$. To do so, we try to express everything on the right-hand side of the equilibrium equations in terms of the displacements u and v. From simple trigonometrical considerations, we can obtain the following relationships:

$$\cos(\alpha) = \frac{\mathrm{L}_1 + u}{l_1} \tag{5}$$

$$\sin(\alpha) = \frac{v}{l_1} \tag{6}$$

$$\cos(\beta) = \frac{\mathrm{L}_2 + v}{l_2} \tag{7}$$

$$\sin(\beta) = \frac{u}{l_2} \tag{8}$$

We can now proceed working on the forces, knowing that the internal forces are linked to the strains by the following relationship:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} \mathrm{A}_1\mathrm{E}_1\epsilon_1 \\ \mathrm{A}_2\mathrm{E}_2\epsilon_2 \end{Bmatrix} \tag{9}$$

And the strains are linked to the displacements by the following relationship:

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \end{Bmatrix} = \begin{Bmatrix} \frac{l_1^2 - \mathrm{L}_1^2}{2\mathrm{L}_1^2} \\ \frac{l_2^2 - \mathrm{L}_2^2}{2\mathrm{L}_2^2} \end{Bmatrix} \tag{10}$$

Finally, we can give the following definition to the real length of the bars:

$$\begin{Bmatrix} l_1 \\ l_2 \end{Bmatrix} = \begin{Bmatrix} \sqrt{(\mathrm{L}_1 + u)^2 + v^2} \\ \sqrt{u^2 + (\mathrm{L}_2 + v)^2} \end{Bmatrix} \tag{11}$$

We can now substitute the equations above in the equilibrium equations 4 to obtain the force displacement relationship:

$$\begin{Bmatrix} P_x \\ P_y \end{Bmatrix} = \begin{Bmatrix} \frac{\mathrm{A}_1\mathrm{E}_2(\mathrm{L}_1+u)\left(-\mathrm{L}_1^2+(\mathrm{L}_1+u)^2+v^2\right)}{2\mathrm{L}_1^2\sqrt{(\mathrm{L}_1+u)^2+v^2}} + \frac{\mathrm{A}_2\mathrm{E}_2 u\left(-\mathrm{L}_2^2+(\mathrm{L}_2+v)^2+u^2\right)}{2\mathrm{L}_2^2\sqrt{(\mathrm{L}_2+v)^2+u^2}} \\ \frac{\mathrm{A}_1\mathrm{E}_2 v\left(-\mathrm{L}_1^2+(\mathrm{L}_1+u)^2+v^2\right)}{2\mathrm{L}_1^2\sqrt{(\mathrm{L}_1+u)^2+v^2}} + \frac{\mathrm{A}_2\mathrm{E}_2(\mathrm{L}_2+v)\left(-\mathrm{L}_2^2+(\mathrm{L}_2+v)^2+u^2\right)}{2\mathrm{L}_2^2\sqrt{(\mathrm{L}_2+v)^2+u^2}} \end{Bmatrix} \tag{12}$$

Even if quite long, the equation 12 are nothing more than a function $\vec{P} = f(\vec{u})$ which can be solved numerically for a given value of $\vec{P^*}$ to find the corresponding value of $\vec{u^*}$.

3.3 Linearization of P = f(u)

Before proceeding with the numerical solution of the system, we can try to linearize the system of equations to obtain a linear system of equations. To do so, we can use a Taylor series expansion of the force-displacement relationship 12 around the equilibrium position $\vec{u} = \vec{0}$. A general Taylor series expansion of a function $f(x, y)$ around the point (x_0, y_0) is given by:

$$\begin{aligned} f(x,y) = {} & f(x_0,y_0) + \\ & + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0) + \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\cdot(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\cdot(y-y_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\cdot(x-x_0)\cdot(y-y_0) + \\ & + \dots \end{aligned} \tag{13}$$

3.3.1 Taylor series of order 1

By applying the Taylor series expansion of order 1, we obtain the following system of linear equations:

$$\begin{Bmatrix} \widehat{P_x} \\ \widehat{P_y} \end{Bmatrix} = \begin{Bmatrix} \frac{\mathrm{A_1 E_2} u}{\mathrm{L_1}} \\ \frac{\mathrm{A_2 E_2} v}{\mathrm{L_2}} \end{Bmatrix} \tag{14}$$

3.3.2 Taylor series of order 2

By applying the Taylor series expansion of order 2, we obtain the following system of quadratic equations:

$$\begin{Bmatrix} \widehat{\widehat{P_x}} \\ \widehat{\widehat{P_y}} \end{Bmatrix} = \begin{Bmatrix} \frac{\mathrm{A_1 E_2}\left(u^2+v^2\right)}{2\mathrm{L_1^2}} + \frac{\mathrm{A_1 E_2} u}{\mathrm{L_1}} + \frac{\mathrm{A_2 E_2} uv}{\mathrm{L_2^2}} \\ \frac{\mathrm{A_1 E_2} uv}{\mathrm{L_1^2}} + \frac{\mathrm{A_2 E_2}\left(u^2+v^2\right)}{2\mathrm{L_2^2}} + \frac{\mathrm{A_2 E_2} v}{\mathrm{L_2}} \end{Bmatrix} \tag{15}$$

3.3.3 Taylor series of order 3

By applying the Taylor series expansion of order 3, we obtain the following system of cubic equations:

$$\begin{Bmatrix} \widehat{\widehat{\widehat{P_x}}} \\ \widehat{\widehat{\widehat{P_y}}} \end{Bmatrix} = \begin{Bmatrix} -\frac{\mathrm{A_1 E_2} uv^2}{2\mathrm{L_1^3}} + \frac{\mathrm{A_1 E_2}\left(u^2+v^2\right)}{2\mathrm{L_1^2}} + \frac{\mathrm{A_1 E_2} u}{\mathrm{L_1}} + \frac{\mathrm{A_2 E_2} u\left(u^2-v^2\right)}{2\mathrm{L_2^3}} + \frac{\mathrm{A_2 E_2} uv}{\mathrm{L_2^2}} \\ \frac{\mathrm{A_1 E_2} v\left(v^2-u^2\right)}{2\mathrm{L_1^3}} + \frac{\mathrm{A_1 E_2} uv}{\mathrm{L_1^2}} - \frac{\mathrm{A_2 E_2} u^2 v}{2\mathrm{L_2^3}} + \frac{\mathrm{A_2 E_2}\left(u^2+v^2\right)}{2\mathrm{L_2^2}} + \frac{\mathrm{A_2 E_2} v}{\mathrm{L_2}} \end{Bmatrix} \tag{16}$$

3.4 Approximation of P = f(u)

Under the hypothesis of small displacements, we can approximate the force-displacement relationship 12 to obtain a simpler system of equations.

3.4.1 Approximation (acceptable/soft)

Working on the trigonometric functions, we can approximate the cosine and sine functions as follows:

$$lim_{\alpha \to 0} \cos(\alpha) = 1 \tag{17}$$
$$lim_{\alpha \to 0} \sin(\alpha) = \alpha \tag{18}$$
$$lim_{\beta \to 0} \cos(\beta) = 1 \tag{19}$$
$$lim_{\beta \to 0} \sin(\beta) = \beta \tag{20}$$

By substituting in the force-displacement relationship 12, we obtain the following approximated system of equations:

$$\begin{Bmatrix} P_{x,approx,soft} \\ P_{y,approx,soft} \end{Bmatrix} = \begin{Bmatrix} \frac{\mathrm{A_1 E_1}\left(-\mathrm{L_1^2}+(\mathrm{L_1}+u)^2+v^2\right)}{2\mathrm{L_1^2}} + \frac{\mathrm{A_2 E_2} v\left(-\mathrm{L_2^2}+(\mathrm{L_2}+v)^2+u^2\right)}{2\mathrm{L_2^2}} \\ \frac{\mathrm{A_1 E_1} v\left(-\mathrm{L_1^2}+(\mathrm{L_1}+u)^2+v^2\right)}{2\mathrm{L_1^2}} + \frac{\mathrm{A_2 E_2}\left(-\mathrm{L_2^2}+(\mathrm{L_2}+v)^2+u^2\right)}{2\mathrm{L_2^2}} \end{Bmatrix} \tag{21}$$

3.4.2 Approximation (bad/hard)

To see how a bad approximation could bring to a shifted or wrong solution, we will assume that:

$$\cos(\alpha) = \frac{\mathrm{L}_1 + u}{l_1} \approx 1 \tag{22}$$

$$\sin(\alpha) = \frac{v}{l_1} \approx 0 \tag{23}$$

$$\cos(\beta) = \frac{\mathrm{L}_2 + u}{l_2} \approx 1 \tag{24}$$

$$\sin(\beta) = \frac{v}{l_2} \approx 0 \tag{25}$$

By substituting in the force-displacement relationship 12, we obtain the following approximated system of equations:

$$\begin{Bmatrix} P_{x,approx} \\ P_{y,approx} \end{Bmatrix} = \begin{Bmatrix} \frac{\mathrm{A}_1\mathrm{E}_2\left(-\mathrm{L}_1^2+(\mathrm{L}_1+u)^2+v^2\right)}{2\mathrm{L}_1^2} \\ \frac{\mathrm{A}_2\mathrm{E}_2\left(-\mathrm{L}_2^2+(\mathrm{L}_2+v)^2+u^2\right)}{2\mathrm{L}_2^2} \end{Bmatrix} \tag{26}$$

3.5 Comparison for Px

Now, it can be interesting to compare the results obtained with the different linearization/approximation described in the previous sections. For simplicity, we will perform this type of error analysis only for the force $P_x = f(\vec{u})$, but conceptually it would be the same for the force $P_y = f(\vec{u})$.
Since the following doesn't want to be a formal error analysis, we will not use any formal error measure, but we will simply plot the difference between the approximated / linearized solutions and the exact one, that is:

$$z(u,v) = \mathrm{Error}[P_x(u,v)] = P_{x,approximated/linearized}(u,v) - P_{x,exact}(u,v) \tag{27}$$

In this way, the error analysis can be performed graphically, by plotting a 3D surface of the error in the space (u, v, z).
Moreover, since all our linearized / approximated solutions were obtained in the neighborhood of the point $\vec{u} = \vec{0}$ (because of the hypothesis of small displacements), we will limit our plot to a domain of $(u, v) \in [-0.01, 0.01] \times [-0.01, 0.01]$ [1].

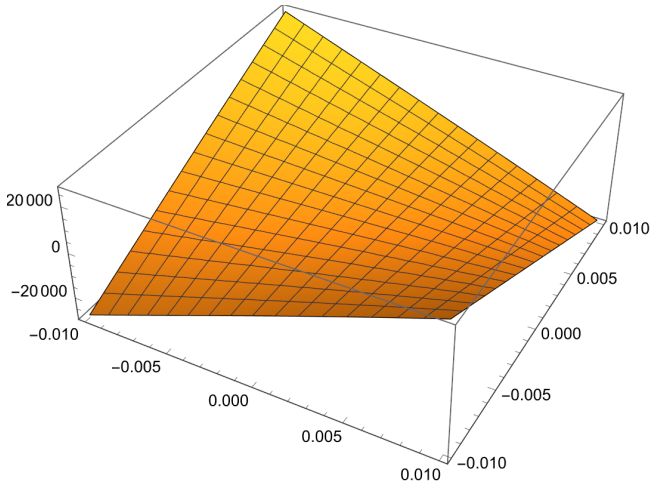


Figure 2: Error analysis for a Taylor series of order 1

[1] At this point the choice of the domain dimension has been done per intuition. A post analysis with the numerical result at hand, would show that this domain is far too large with respect to the solutions effective domain (Factor = 10^2).

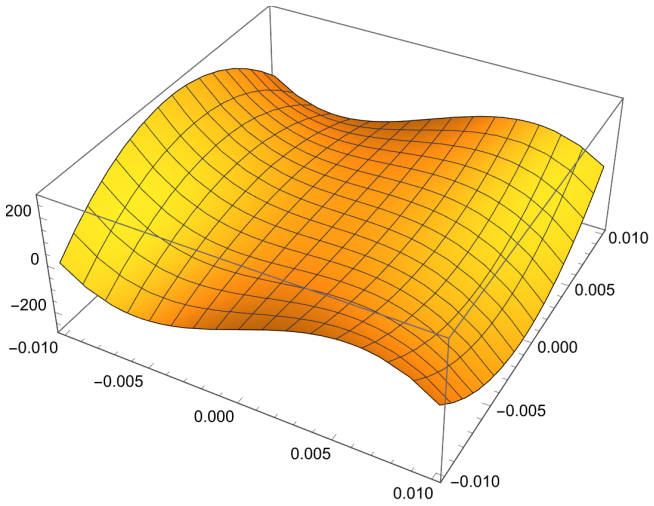


Figure 3: Error analysis for a Taylor series of order 2

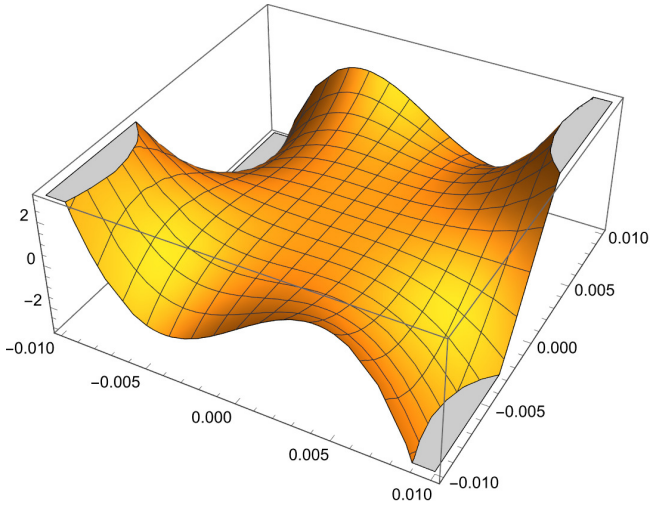


Figure 4: Error analysis for a Taylor series of order 3

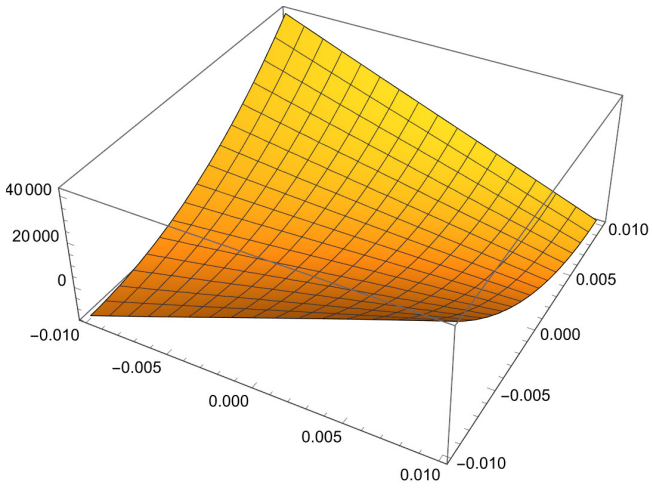


Figure 5: Error analysis for approximated (soft) solution

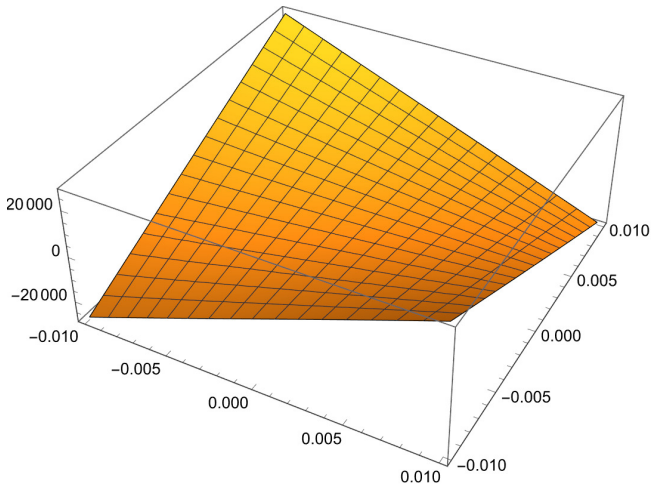


Figure 6: Error analysis for approximated (hard) solution

As we can see, in the neighborhood of point $\vec{u} = \vec{0}$, the error is negligible for every linearization and/or approximations.
However, as we move away from the point $\vec{u} = \vec{0}$, the error increases. In particular, we can see that the error introduced by the linearization decreases as we increase the order of the Taylor series expansion.

4 Numerical solution

We can now proceed to solve the system of equations $\vec{P} = f(\vec{u})$ numerically. To do so, we will use two different combined methods, in the form of Predictor-Corrector, that are:

- Euler and Newton-Raphson
- Euler and modified Newton-Raphson

Just as a review of the methods, we report in the following the algorithms for the tree methods applied to a 2D problem (easily extendable to 3D).
Notice that with the symbol $[K_t]$ we are referring to the tangent stiffness matrix, that by definition is:

$$[K_t] = \frac{\partial \vec{P_{ext}}}{\partial \vec{u}} = \begin{bmatrix} \frac{\partial P_{ext,x}}{\partial u} & \frac{\partial P_{ext,x}}{\partial v} \\ \frac{\partial P_{ext,y}}{\partial u} & \frac{\partial P_{ext,y}}{\partial v} \end{bmatrix} \tag{28}$$

Instead, with the symbol $\vec{R}$ we are referring to the residual vector, that by definition is:

$$\vec{R} = \vec{F_{int}} - \vec{P_{ext}} \tag{29}$$

Since we will work in a discretized domain, we will use the subscript n to refer to the current iteration, and the subscript $n+1$ to refer to the next iteration. So $[K_{t,n}]$ and $\vec{R_n}$ will simply be the tangent stiffness matrix and the residual vector evaluated at the current iteration with the associated values of $\vec{u}$.

4.1 Euler

Used as predictor, the Euler method is defined as:

$$\vec{u}_{n+1} = \vec{u}_n + [K_{t,n}]^{-1} * \vec{P_{ext,n}} \tag{30}$$

4.2 Newton-Raphson

Used as corrector, the Newton-Raphson method is defined as:

$$\vec{u}_{n+1} = \vec{u}_n - [K_{t,n}]^{-1} * \vec{R_n} \tag{31}$$

4.3 Modified Newton-Raphson

Similarly to the Newton-Raphson method, the modified Newton-Raphson method is defined as:

$$\vec{u}_{n+1} = \vec{u}_n - [K_{t,0}]^{-1} * \vec{R_n} \tag{32}$$

Where $[K_{t,0}]$ is the tangent stiffness matrix evaluated at the first iteration.

5 Implementation

For each of the previous models (12,14, 15, 16, 21, 26) we can then compute the tangent stiffness matrix and the internal force vector as function of $\vec{u}$. By applying the definition 28 and 29 we are able to basically compute for each model two functions $f_{K_t}(\vec{u})$ and $f_R(\vec{u})$ that will return the tangent stiffness matrix and the residual vector evaluated at the current iteration with the associated values of $\vec{u}$.
As an example, we report the functions $f_{K_t}(\vec{u})$ and $f_R(\vec{u})$ for the model 14:

$$f_{K_t}(\vec{u}) = \begin{bmatrix} \frac{\partial P_{ext,x}}{\partial u} & \frac{\partial P_{ext,x}}{\partial v} \\ \frac{\partial P_{ext,y}}{\partial u} & \frac{\partial P_{ext,y}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial u}\left(\frac{\mathrm{A_1E_1}}{L_1}(u-u_0)\right) & \frac{\partial}{\partial v}\left(\frac{\mathrm{A_1E_1}}{L_1}(u-u_0)\right) \\ \frac{\partial}{\partial u}\left(\frac{\mathrm{A_2E_2}}{L_2}(v-v_0)\right) & \frac{\partial}{\partial v}\left(\frac{\mathrm{A_2E_2}}{L_2}(v-v_0)\right) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{A_1E_1}}{\mathrm{L_1}} & 0 \\ 0 & \frac{\mathrm{A_2E_2}}{\mathrm{L_2}} \end{bmatrix} \tag{33}$$

$$f_R(\vec{u}) = \vec{F_{int}} - \vec{P_{ext}} \tag{34}$$

$$\vec{F_{int}} = \begin{bmatrix} \frac{\mathrm{A_1E_1}}{\mathrm{L_1}}(u-u_0) \\ \frac{\mathrm{A_2E_2}}{\mathrm{L_2}}(v-v_0) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{A_1E_1}}{\mathrm{L_1}}u \\ \frac{\mathrm{A_2E_2}}{\mathrm{L_2}}v \end{bmatrix} \tag{35}$$

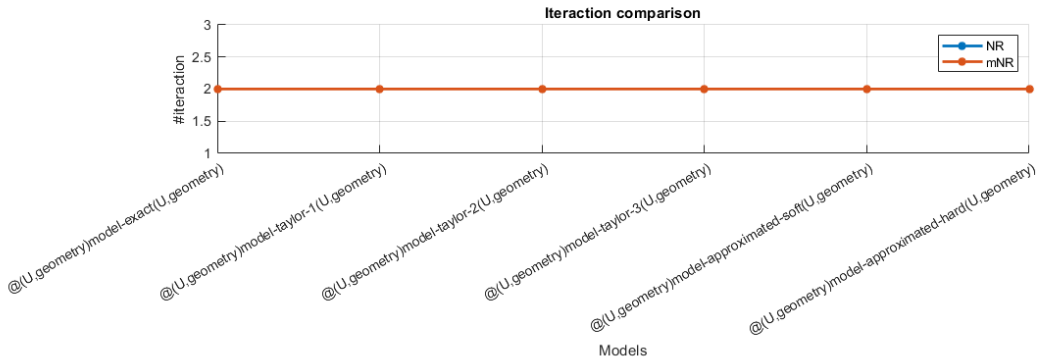
When it comes to `MATLAB` code, the functions $f_{K_t}(\vec{u})$ and $f_{F_{int}}(\vec{u})$ are implemented as follows:

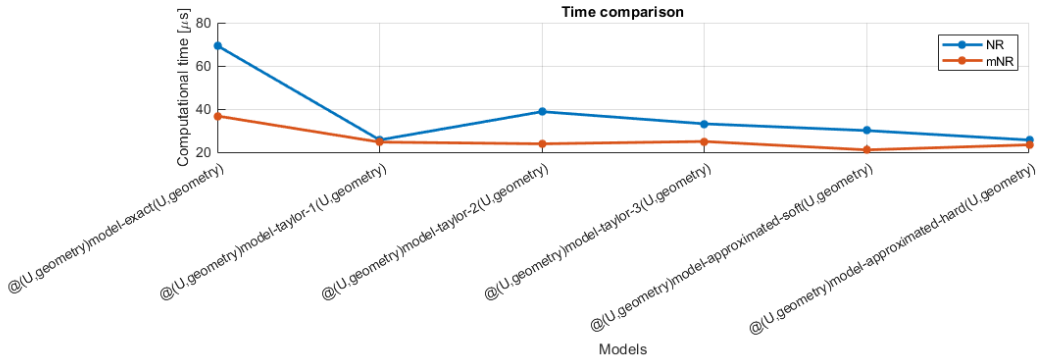
```
function [Kt, Fint] = model_taylor_1(U, geometry)

[L1, A1, E1, L2, A2, E2] = decompose_geometry(geometry); % From a struct to variables
[u, v] = decompose_u(U); % From a vector to variables

Kt = [
    [A1.*E1.*L1.^(-1) 0];
    [0 A2.*E2.*L2.^(-1)]
    ];

Fint = [
    A1.*E1.*L1.^(-1).*u;
    A2.*E2.*L2.^(-1).*v
    ];

end
```

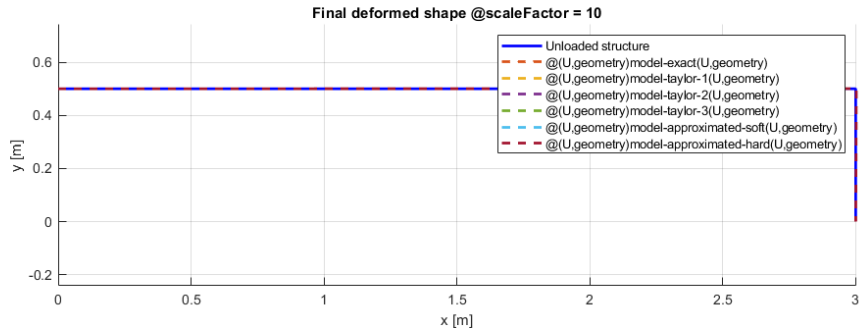
6 Results

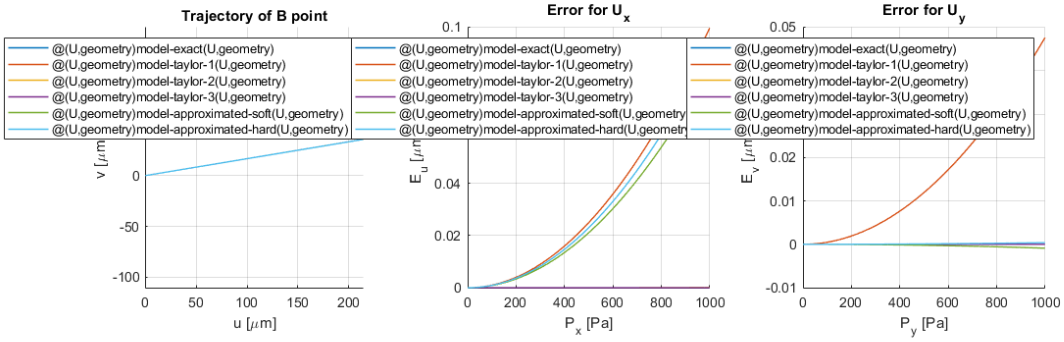
The numerical solution results for the problem described in 1 are reported in the following table and figures.

"Model name"	"Corrector"	"Time (s)"	"Iterations"









A Mathematica code

Here follows the `Mathematica` notebook used for symbolic analysis of the equations.

```
In[234]:= Clear["Global`*"];
plots := True;
$Assumptions=Element[{L1, L2, A1, A2, E1, E2, u, v}, Reals]&&L1>0&&L2>0&&A1>0&&A2>0&&E1>0&&E2>0;

(*Setting values for the problem datas*)
```

```
problemDatas = {
E1-> 70*10^9,A1->10^-3,L1->3,
E2->70*10^9,A2->10^-3,L2->0.5
};

domainPlotLimit:=0.01;

In[239]:= (*Beam lenght*)
l[u_,v_] ={
{Sqrt[(L1+u)^2+v^2]},
{Sqrt[u^2+(L2+v)^2]}
};

(*Green strain*)
eps[u_,v_] = {
{(l[u,v][[1, 1]]^2-L1^2)/(2*L1^2)},
{(l[u,v][[2, 1]]^2-L2^2)/(2*L2^2)}
};

(*F_{int}*)
Fint[u_,v_]={
{E1*A1*eps[u,v][[1,1]]},
{E2*A2*eps[u,v][[2,1]]}
};

(*Trigonometry relations*)
cosAlpha[u_,v_]=(L1+u)/l[u,v][[1, 1]];
sinAlpha[u_,v_]=v/l[u,v][[1, 1]];
cosBeta[u_,v_]=(L2+v)/l[u,v][[2, 1]];
sinBeta[u_,v_]=u/l[u,v][[2, 1]];

(*Equilibrium of the system*)
Pexact[u_,v_]={
{Fint[u,v][[1, 1]]*cosAlpha[u,v]+Fint[u,v][[2, 1]]*sinBeta[u,v]},
{Fint[u,v][[1, 1]]*sinAlpha[u,v]+Fint[u,v][[2, 1]]*cosBeta[u,v]}
};

(*Defining the taylor expansion series*)
Ptaylor[x_, y_, taylorOrder_] := Normal @ Series[Pexact[u, v] /. Thread[{u, v} -> {0, 0} + t
    {x, y}], {t, 0, taylorOrder}] /. t -> 1;

(*Approximated models*)
PapproximatedSoft[u_,v_]={
{Fint[u,v][[1, 1]]*1+Fint[u,v][[2, 1]]*v},
{Fint[u,v][[1, 1]]*v+Fint[u,v][[2, 1]]*1}
};

PapproximatedHard[u_,v_]={
{Fint[u,v][[1, 1]]*1+Fint[u,v][[2, 1]]*0},
{Fint[u,v][[1, 1]]*0+Fint[u,v][[2, 1]]*1}
};
```

Listing 1: Mathematica notebook used for symbolic analysis of the equations.