# ME621 - Advanced Finite Element Methods
# Assignment 4

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# 1 Requests

The goal of this assignment is to write an elastic-plastic stress update algorithm for simple shear deformation using MATLAB. It is important to note that this algorithm is independent of the finite element analysis and is strictly on the implementation of computational plasticity. This implementation is similar to writing a custom user-defined material subroutine (UMAT) for commercial FE software, with the exception that you must obtain the kinematic variables across time.

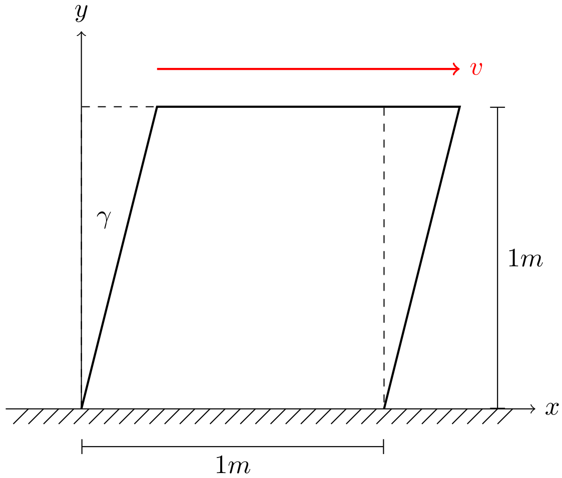


Figure 1: Problem representation in its initial (dashed lines) and final (solid lines) configuration.

Consider a square-shape aluminum plate under simple shear deformation. The deformation gradient for simple shear is:

$$F = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \tag{1}$$

- Obtain infinitesimal strain in the simple shear problem when the shear deformation is linearly increased from zero to $\gamma = 1$. Use constant time increments.
- Implement von-Mises plasticity according to the radial return algorithm described in the lecture notes. Use the following elastic constitutive equation in modeling of the elastic-plastic deformation. Do not consider kinematic hardening.

$$\dot{\sigma} = C : \dot{\varepsilon} \tag{2}$$

Plot the shear stress as a function of shear strain, $\gamma$.
Assume the material has a Young's modulus $E = 70$GPa and a Poisson's ratio of $\nu = 0.3$. During plastic deformation, the material hardens according to the following isotropic hardening equation

$$\bar{\sigma} = \sigma_{Y0} + K(\bar{\varepsilon}^p)^n \tag{3}$$

where $\sigma_{Y0} = 200$MPa, $K = 325$MPa, and $n = 0.125$.

- Use the following elastic constitutive equation instead of the one at the second request and compare the results

$$\sigma^{oT} = C : D \tag{4}$$

Is there any significant difference between the results? Why?
$\sigma^{oT}$ is the Truesdell rate of Cauchy stress and $D$ is the rate-of-deformation tensor.

## 2 Methodology

To solve the given problem, we will implement the radial return algorithm on top of the code developed for the previous assignment.
In particular, inside our Updated Lagrangian Formulation, we will check if the current configuration is in the elastic or plastic regime. If the material is in the elastic regime, we will proceed with the standard strain/stress update algorithm. On the other hand, if the material is in the plastic regime, we will apply the radial return algorithm to compute the elastic and plastic components of the strain tensor and update the stress tensors accordingly.
In the plastic region, we will neglect any hardening due to kinematic and will assume the material to follow the isotropic hardening model:

$$\bar{\sigma} = \bar{\sigma}(\bar{\varepsilon_p}, \dot{\bar{\varepsilon_p}}) \approx \bar{\sigma}(\bar{\varepsilon_p}) \tag{5}$$

## 3 Solution

The majority of the code for this assignment is based on the one developed for the previous one.
In the following, we limit our self to describe how the main iterative algorithm for plasticity region works.

### 3.1 Radial Return Algorithm

The radial return algorithm is a method used to determine the plastic strain increment in a material subjected to a stress state that is outside the yield surface.
The base assumption is that the stress state of a material must always be contained within the yield surface (or alternatively, its effective stress must be less or equal to the yield strength). From here, two cases can be distinguished:

- Effective stress is less than the yield strength: the material is in the elastic regime and the plastic components of the strain are zero.
- Effective stress is greater than the yield strength: the material is in the plastic regime and the plastic components of the strain must be determined using the radial return algorithm in order to satisfy the yield condition.

#### 3.1.1 First steps of the algorithm & material considerations

Before diving into the core of the radial return algorithm, we need to introduce some preliminary concepts and considerations.
At first, we need to evaluate the trial stress tensor $\sigma_{trial} = C : \varepsilon$, where $C$ is the stiffness tensor and $\varepsilon$ is the strain tensor (supposed at this point to be purely elastic $\varepsilon = \varepsilon_e$).
Then we check whether the trial stress tensor is inside the yield surface or not. In general, different criteria can be chosen to define the yield surface (or yield strength), but the most common is the von Mises yield criterion. Based on the von Mises criteria, the yield strength can be computed as:

$$S_y = \sqrt{\frac{1}{2}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)]} \tag{6}$$

Moreover, the von Mises yield criterion can be expressed starting from the deviatoric stress tensor $S$, which represent the shape deformation of the material (volume deformation is represented by the hydrostatic stress tensor $P$). Given a generic stress tensor $\sigma$, the deviatoric stress tensor $S$ can be computed as:

$$S = \sigma - P = \sigma - \frac{1}{3}\text{trace}(\sigma)I \tag{7}$$

The yield surface of the current stress state, based on the von Mises criterion, can then be expressed as:

$$f(\sigma) = \sqrt{\frac{3}{2}S : S} \tag{8}$$

where : is the double contraction product between two tensors (i.e. $A : B = \sum_i \sum_j A_{ij} \times B_{ij}$). Of course, given that the deviatoric stress tensor is a second order tensor, the result of the double contraction product $S : S$ is a scalar.
To check whether the trial stress tensor is inside the yield surface, we can define a $\Phi$ function as:

$$\Phi(\sigma) = f(\sigma) - \bar{\sigma}(\bar{\varepsilon_p}) \tag{9}$$

where $\bar{\sigma}(\bar{\varepsilon_p})$ is the hardening function that describe the behavior of the material in the plastic regime. In particular we know that, in general, in the plastic regime the material undergoes hardening, which means that the yield strength increases with the plastic strain. There exist many model of the hardening function, but the most common are:

- Linear hardening: $\bar{\sigma}(\bar{\varepsilon_p}) = S_{y0} + H\bar{\varepsilon_p}$
- Exponential hardening: $\bar{\sigma}(\bar{\varepsilon_p}) = S_{y0} + (S_{yf} - S_{y0})(1 - e^{-k\bar{\varepsilon_p}})$
- Power hardening: $\bar{\sigma}(\bar{\varepsilon_p}) = S_{y0} + H\bar{\varepsilon_p}^n$

Notice that $\bar{\sigma}(\bar{\varepsilon_p})$ is known as the flow stress (or flow rule considering the function it self), while the hardening variable $\bar{\varepsilon_p}$ is known as the effective plastic strain and can be computed starting from the plastic strain tensor $\varepsilon_p$ as:

$$\bar{\varepsilon_p} = \sqrt{\frac{2}{3}\varepsilon_p : \varepsilon_p} \tag{10}$$

If $\Phi(\sigma, \bar{\varepsilon_p}) \leq 0$, the material is in the elastic regime and $\varepsilon_p = 0$ & $\sigma = \sigma_{trial}$, being the stress tensor still inside the yield surface.
If, increasing the stress state, $\Phi(\sigma, \bar{\varepsilon_p}) \geq 0$, the material is in the plastic regime and the radial return algorithm must be applied to compute the elastic and plastic components of the strain tensor and update the stress tensor considering only the elastic component.

### 3.1.2 Radial return core algorithm

In the latter case of the two above, we need to evaluate $\varepsilon_p$ using an iterative algorithm, given that a change of $\varepsilon_p$ will change both the stress tensor ($\sigma = C : (\varepsilon - \varepsilon_p)$) and the yield surface (Equation 8).
The exact value of $\varepsilon_p$, correspond to the zero of the $\Phi$ function.
The core block of the radial return algorithm is essentially similar to a Newton-Raphson algorithm, where the unknown is the plastic strain $\varepsilon_p$ and the function to minimize is $\Phi(\sigma, \bar{\varepsilon_p})$.
In particular, by using a while loop, we can iterate the following steps:

1. Compute $G$ (tangent modulus) and $\frac{\partial\bar{\sigma}}{\partial\bar{\varepsilon_p}}$, evaluated at the current $\bar{\varepsilon_p}$.
2. Compute the direction of the plastic strain increment $n = \frac{\partial f}{\partial\sigma}$
3. Compute $\lambda = \frac{\Phi(\sigma,\bar{\varepsilon_p})}{n:C:n+\frac{\partial\bar{\sigma}}{\partial\bar{\varepsilon_p}}}$
4. Update $\sigma = \sigma - \lambda C : n$
5. Update $\bar{\varepsilon_p} = \bar{\varepsilon_p} + \lambda$
6. Check if $\Phi(\sigma, \bar{\varepsilon_p}) \leq \delta$, if not repeat from step 1.

The algorithm will converge to the exact value of $\varepsilon_p$ that satisfy the yield condition, and the stress tensor will be correctly updated considering only the elastic component of the strain tensor.

### 3.1.3 Adaptation to the problem at hand

Given the hardening law $\bar{\sigma}(\bar{\varepsilon_p}) = 200MPa + 325MPa \times \bar{\varepsilon_p}^{0.125}$, we can compute its derivative with respect to $\bar{\varepsilon_p}$ as:

$$\frac{\partial\bar{\sigma}}{\partial\bar{\varepsilon_p}} = 325MPa \times 0.125 \times \bar{\varepsilon_p}^{0.125-1} \tag{11}$$

Moreover, we are going to adopt the von Mises criteria. Because of that, quantities such as $\lambda$ and $n$ can be further specified.

$$n = \frac{\partial f}{\partial\sigma} = \frac{3}{2}\frac{S}{f} = \sqrt{\frac{3}{2}}\frac{S}{\sqrt{S : S}} \tag{12}$$

$$\lambda = \frac{\Phi(\sigma, \bar{\varepsilon_p})}{n : C : n + \frac{\partial\bar{\sigma}}{\partial\bar{\varepsilon_p}}} = \frac{\Phi(\sigma, \bar{\varepsilon_p})}{3G + \frac{\partial\bar{\sigma}}{\partial\bar{\varepsilon_p}}} \tag{13}$$

### 3.2 Flow Charts

To better understand the flow used in the solution of the problem, we leave in the Appendix section (Section A) a couple of flow charts that represent the main steps involved in the solution of the problem.

# 4 Results

The numerical results for the problem described in Section 1 are reported in the following.

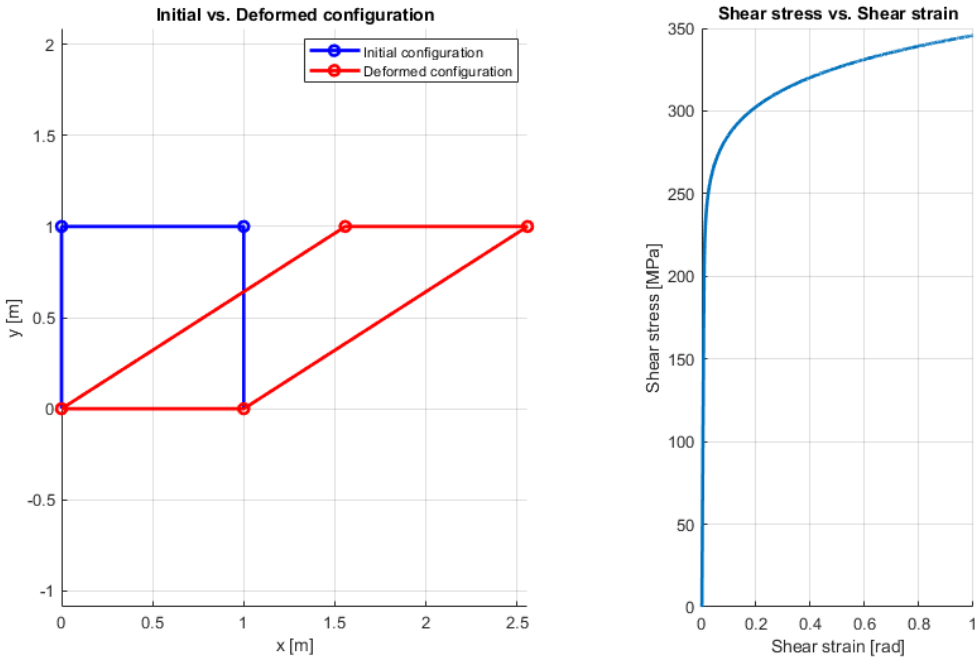


Figure 2: Shear stress $\sigma_{12}$ as function of $\gamma$.

In Figure 2 the curve $\sigma_{12}(\gamma)$ is plotted.
As we can see, in the first part of the curve the material behaves linearly. Here the material is in the elastic region, and the slope of the curve is constant.
After this first linear region, the material starts to behave non-linearly. This is due to the fact that the material has reached the yield strength, and the material starts to deform plastically. The slope of the curve is not constant anymore, and the material starts to harden.

### 4.1 Different elastic constitutive equations

In the requests of the problem (Section 1), we were asked to compare the following two different elastic constitutive equations:

$$\dot{\sigma} = C : \dot{\varepsilon} \tag{14}$$
$$\sigma^{oT} = C : D \tag{15}$$

In our code we implemented the possibility to chose between the two constitutive equations for the elastic predictor step. However, we weren't sure how to proceed with the implementation of the Trusdell objective rate inside the Radial Return algorithm.

### 4.2 Loading and unloading cycles

In order to better understand the behavior of the material after the yield strength is reached, we can try to simulate a loading and unloading cycle.
In particular, considering the same problem as before, we can simulate the following loading and unloading cycle:

| Step | Loading direction | Stop condition |
|---|---|---|
| 1 | Positive | $\gamma = 0.03$ |
| 2 | Negative | $\sigma_{12} = 0$ |
| 3 | Positive | $\gamma = 0.05$ |
| 4 | Negative | $\gamma = -0.01$ |
| 5 | Positive | $\gamma = 0.1$ |

Table 1: Loading and unloading cycle stages and relative conditions.

From the conditions above, the shear stress-strain curve in Figure 3 is obtained.

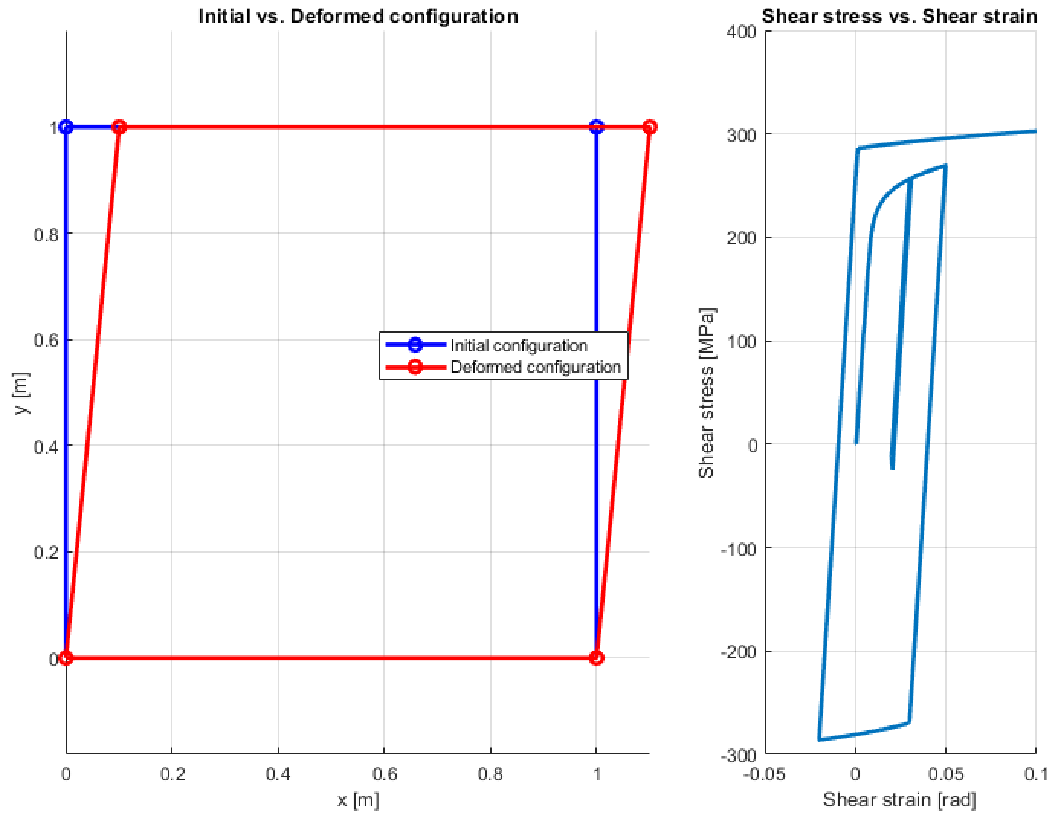


Figure 3: Shear stress $\sigma_{12}$ as function of $\gamma$ for the loading and unloading cycle.

It's easy to see how only the elastic deformation is reversible and can be recovered after the unloading stage. Instead, the plastic deformation is not recovered even after a complete unloading of the material.
This behavior can be explained by the model used to describe the material, where in the elastic region molecular structures are deformed elastically and the binding between atoms act as springs. On the other hand, in the plastic region, the material undergoes permanent deformation due to the breaking of atomic bonds and the formation of new bonds in different positions (plane slip).
Among the files attached to this report, a video is provided to show the relationship between the deformed configuration of the structure and the stress-strain curve.

# A Flow Charts

The following flow charts mimics the structure of the MATLAB code used to solve the problem.
The code is structured in three nested loops: the main loop (**Explicit Time Integration Algorithm**), the inner loop (**Stress Update Algorithm**) and the innermost loop (**Radial Return Algorithm**).
The Explicit Time Integration Algorithm is the main loop that iterates until the convergence criterion is met.
Inside the main loop, the Stress Update Algorithm is called to update the stress state of each element (at each integration point).
Inside the Stress Update Algorithm, the Radial Return Algorithm is called to update the stress state of each element (at each integration point) in case of plastic behavior.

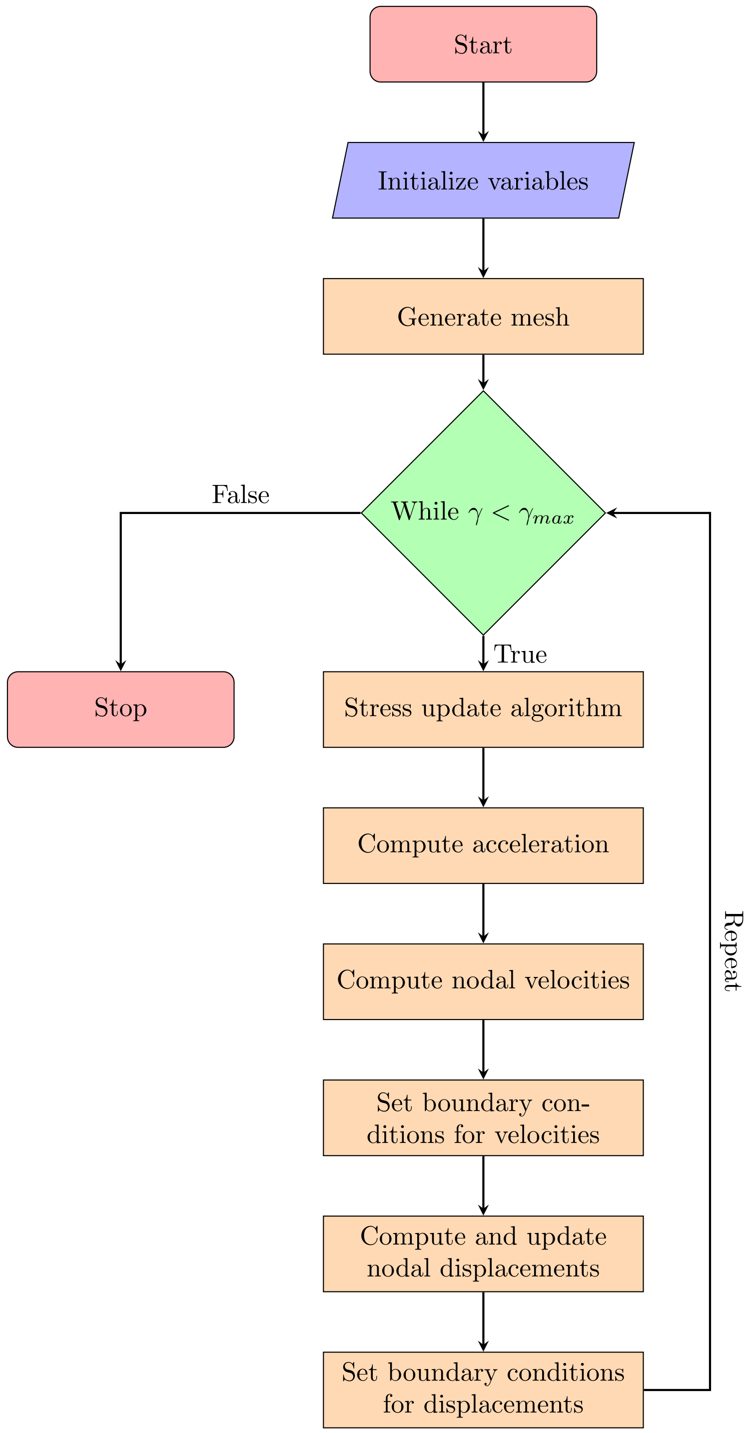


Figure 4: Flowchart for the **Explicit Time Integration Algorithm**. The convergence criterion $\gamma < \gamma_{max}$ is relative to the given problem, while the rest of the algorithm is general.

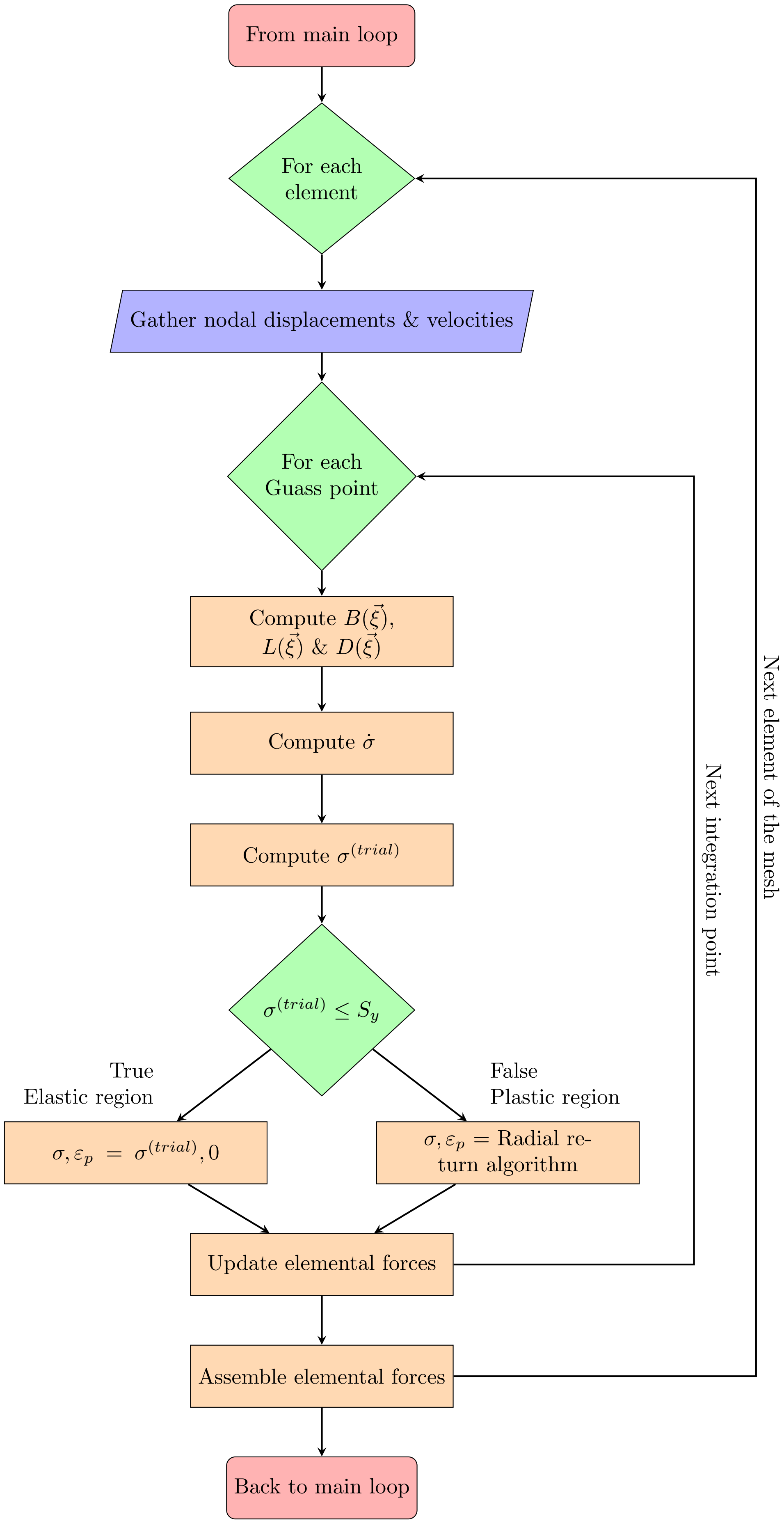


Figure 5: Flowchart for the **Stress Update Algorithm**. The decision blocks in this case represent for loops over the elements and the integration points.

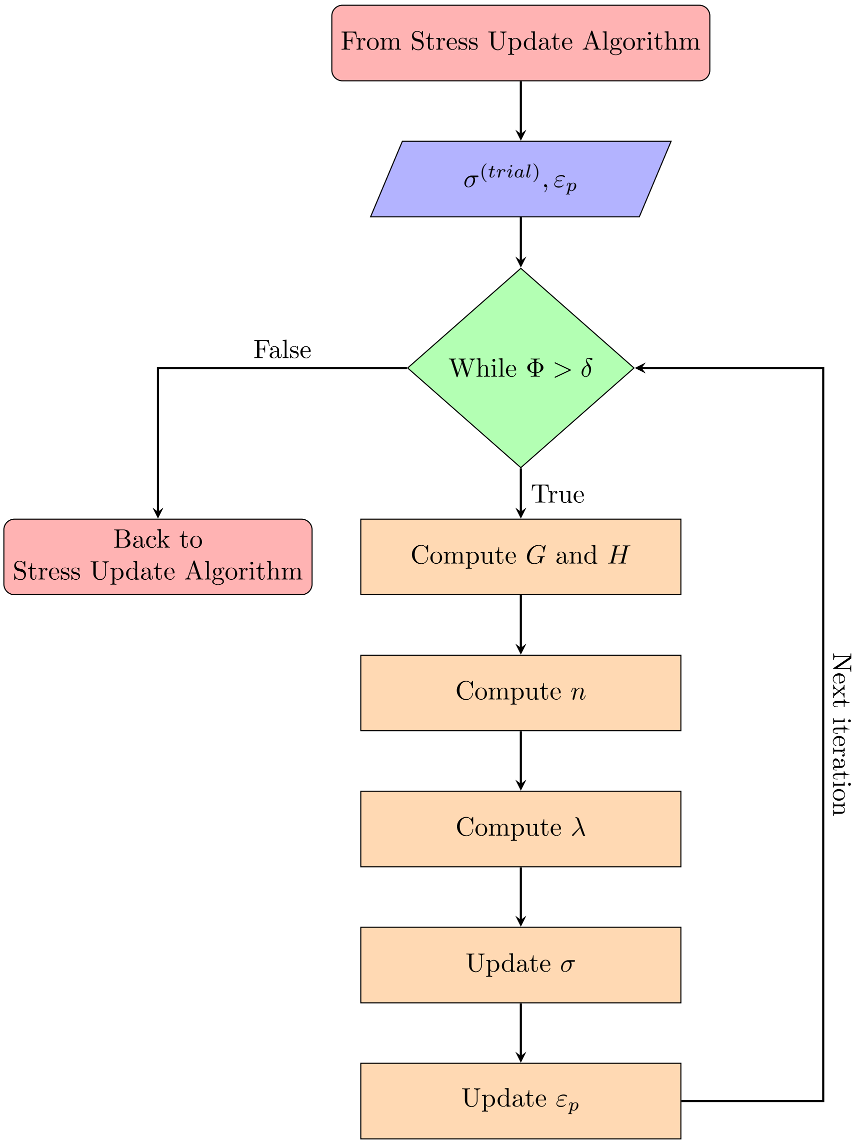


Figure 6: Flowchart for the **Radial Return Algorithm**.