FLUX-VECTOR SPLITTING FOR THE EULER EQUATIONS

Bram van Leer

Institute for Computer Applications in Science and Engineering

and

Leiden State University, Leiden, The Netherlands

Abstract

The flux vector in the Euler equations of compressible flow, with the ideal-gas law inserted, is split, in the simplest possible way, in a continuously differentiable forward-flux vector and backward-flux vector. The first-order upwind scheme based on these fluxes produces steady shock profiles with at most two zones. A comparison with the split fluxes of Steger and Warming, which are not differentiable in sonic and stagnation points, shows the advantage of the present fluxes.

Research was supported under NASA Contract No. NAS1-15810 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665. Paper presented at the 8th International Conference on Numerical Methods in Fluid Dynamics, Aachen, Germany, June 28 - July 2, 1982.

Introduction

To approximate a hyperbolic system of conservation laws $w_t + \{f(w)\}_x = 0$ with so-called upwind differences, we must first establish which way the wind blows. More precisely, we must determine in which direction each of a variety of signals moves through the computational grid. For this purpose, a physical model of the interaction between computational cells is needed; at present two such models are in use.

In one model neighboring cells interact through discrete, finite-amplitude waves. The nature, propagation speed and amplitude of these waves are found by solving, exactly or approximately, Riemann's initial-value problem for the discontinuity at the cell interface. We may call this the Riemann approach (Fig. 1 (a)). The numerical technique of distinguishing between the influence of the forward- and the backward-moving waves is called flux-difference splitting; examples are the methods of Roe [1] and of Osher [2].

In the other model, the interaction of neighboring cells is accomplished through mixing of pseudo-particles that move in and out of each cell according to a given velocity distribution. We may call this the Boltzmann approach (Fig. 1 (b)). The numerical technique of distinguishing between the influence of the forward- and the backward-moving particles is called flux-vector splitting or simply flux-splitting; an example is the "beam scheme" of Prendergast [3], rediscovered by Steger and Warming [4].

Both kinds of splitting are discussed by Harten, Lax and Van Leer [5].

The present paper is restricted to flux-vector splitting for the Euler equations of compressible flow, with the ideal-gas law used as equation of state.

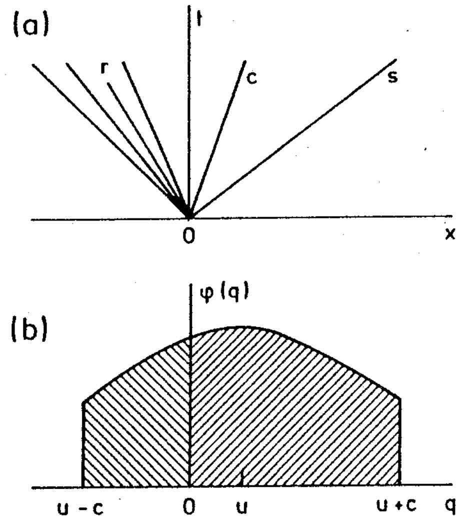

Figure 1. Two physical models for upwind differencing. (a) Space-time diagram showing waves in the solution of a Riemann problem: forward shock s and contact discontinuity c, backward rarefaction r. (b) Velocity-distribution function used in the Boltzmann model. The densities of the forward-moving and backward-moving particles are represented by the differently shaded areas.

Goal

We wish to split the flux $f(w)$ in a forward flux $f^+(w)$ and a backward flux $f^-(w)$, that is,

(1) $f(w) = f^+(w) + f^-(w)$,

(2) df^+/dw must have all eigenvalues ≥ 0,

df^-/dw must have all eigenvalues ≤ 0,

under the following restrictions:

(3) $f^{\pm}(w)$ must be continuous, with

$f^+(w) \equiv f(w)$ for Mach-numbers $M \geq 1$,

$f^-(w) \equiv f(w)$ for $M \leq -1$,

(4) the components of f^+ and f^- together must mimic the symmetry of f with respect to M (all other state quantities held constant), that is

$$f_k^+(M) = \pm f_k^-(-M) \quad \text{if} \quad f_k(M) = \pm f_k(-M),$$

(5) $df^{\pm}/dw$ must be continuous,

(6) $df^{\pm}/dw$ must have one eigenvalue vanish for $|M| < 1$,

(7) $f^{\pm}(M)$, like $f(M)$, must be a polynomial in M, and of the lowest possible degree.

Restriction (3) ensures that in supersonic regions flux-vector splitting leads to standard upwind differencing. Restrictions (4) and (5) are self-evident, although (5) was not satisfied in [4], with negative consequences for the smoothness of numerical results. Restriction (6) is crucial, greatly narrowing down the choice of functions. The degeneracy of $f^{\pm}(w)$ in subsonic cells makes it possible to build stationary shock structures with no more than two interior zones, just as does the flux-difference splitting technique in [2]. Finally, requirement (7) makes the splitting unique.

Derivation

Consider the one-dimensional Euler equations. We shall regard the full, forward, and backward fluxes as functions of density ρ, sound speed c and Mach number M. The full flux reads

$$f(\rho,c,M) = \begin{pmatrix} \rho c M \\ \rho c^2\left(M^2 + \frac{1}{\gamma}\right) \\ \rho c^3 M\left(\frac{1}{2}M^2 + \frac{1}{\gamma-1}\right) \end{pmatrix} \tag{8}$$

where γ is the specific-heat ratio.

From conditions (3) and (5) it follows that $f^+(\rho,c,M)$ as well as $\partial f^+(\rho,c,M)/\partial M$ must vanish for $M \downarrow -1$, while $f^-(\rho,c,M)$ and $\partial f^-(\rho,c,M)/\partial M$ must vanish for $M \uparrow 1$. Condition (7) then leads to the restriction that f^+ includes a factor $(M+1)^2$ and f^- a factor $(-M+1)^2$, for $|M| < 1$. Without introducing further factors depending on M we can now achieve the splitting of the mass flux:

(9) $$\rho c M \equiv \rho c\{\tfrac{1}{2}(M+1)\}^2 - \rho c\{\tfrac{1}{2}(-M+1)\}^2, \quad |M| < 1,$$

satisfying (1), (3), (4), (5) and (7). In order to split the momentum flux in agreement with (3) and (5) we need cubic polynomials in M:

(10)
$$\rho c^2(M^2 + \frac{1}{\gamma}) \equiv \rho c^2\{\tfrac{1}{2}(M+1)\}^2(\frac{\gamma-1}{\gamma}M + \frac{2}{\gamma}) + \rho c^2\{\tfrac{1}{2}(-M+1)\}^2(-\frac{\gamma-1}{\gamma}M + \frac{2}{\gamma}), \quad |M| < 1;$$

again, (1), (3), (4), (5) and (7) are satisfied.

The splitting of the energy flux can now be achieved by combining the split mass- and momentum-fluxes:

(11) $$f^{\pm}_{\text{energy}} = \frac{\gamma^2}{2(\gamma^2-1)}(f^{\pm}_{\text{momentum}})^2/f^{\pm}_{\text{mass}}, \quad |M| < 1.$$

The scale factor is needed to satisfy (3). The relation (11) between the components of $f^{\pm}$ makes these fluxes degenerate: $df^{\pm}(w)/dw$ has a zero eigenvalue for $|M| < 1$. Thus condition (6) is fulfilled. Moreover, since the vanishing eigenvalue is continuous for $M = \pm 1$, we have sufficient information about the smoothness of $f^{\pm}$ to conclude that condition (5) is fully satisfied. Testing the fulfillment of (7), (4) and (1) by (11) is trivial.

We still must determine if condition (2) is satisfied by the splitting (9), (10), (11). There is a good reason to believe this is indeed the case: the eigenvalue $\mu_1^{\pm}$ of $df^{\pm}/dw$ that is most likely to violate condition (2), i.e. to have the wrong sign, has been forced to vanish for $|M| < 1$. We find that the non-zero eigenvalues $\mu^{+}_{2,3}$ of df^{+}/dw are the roots of the quadratic equation:

$$(12)\quad (\mu^+)^2 - c\mu^+ \cdot \frac{3}{2}(1+M)[1 - \frac{\gamma-1}{12\gamma(\gamma+1)}(M-1)\ \{\gamma(M-1)^2 + 2\gamma(M-1) - 2(\gamma+3)\}]$$

$$+ c^2 \cdot \frac{1}{4}(1+M)^3[1 - \frac{M-1}{8\gamma(\gamma+1)}\ \{4\gamma(\gamma-1)(M-1) + (\gamma+1)(3-\gamma)\}] = 0,\ |M| < 1.$$

In the relevant range $1 \leqslant \gamma \leqslant 3$, the roots of (12) are positive. The negativity of $\mu^-_{2,3}$ follows by symmetry.

This completes the derivation of the split fluxes for the one-dimensional Euler equations. The formulas for the three-dimensional equations are given in Table I (full equations) and Table II (constant enthalpy assumed).

Stability

The stability analysis for the first-order upwind scheme based on the above split fluxes is complicated by the fact that df^+/dw and df^-/dw commute neither with each other nor with df/dw, for $|M| < 1$. This leads to a reduction of the CFL limit; in the worst case, $M = 0$, we find for the shortest waves:

?

$$(13)\qquad \frac{\Delta t}{\Delta t}c \leqslant 2\gamma/(\gamma+3).$$

A practical local stability criterion is

$$(14)\qquad \frac{\Delta t}{\Delta t}(|u|+c) \leqslant \{2\gamma + |M|(3-\gamma)\}/(\gamma+3),\ |M| < 1.$$

Steady discontinuities

The degeneracy of $f^\pm$ for $|M| < 1$ makes it possible to build numerical shock structures that satisfy the steady upwind-difference equations using only two interior zones.

Table I. Flux-splitting for the Euler equations. $M = u/c$.

conserved quantity	x-flux f	forward x-flux f^+, $\|M\| < 1$
mass	ρu	$\rho c\{ \tfrac{1}{2}(M+1)\}^2$
x-momentum	$\rho(u^2+c^2/\gamma)$	$f^+_{mass}\cdot\{(\gamma-1)u+2c\}/\gamma$
y-momentum	ρuv	$f^+_{mass}\cdot v$
z-momentum	ρuw	$f^+_{mass}\cdot w$
total energy	$\rho u\{\tfrac{1}{2}(u^2+v^2+w^2) + c^2/(\gamma-1)\}$	$f^+_{mass}\cdot[\{(\gamma-1)u+2c\}^2/\{2(\gamma^2-1)\} + \tfrac{1}{2}(v^2+w^2)]$

Table II. Flux-splitting for the isenthalpic Euler equations.

$H \equiv$ enthalpy, $c_* = \surd[\{2(\gamma-1)/(\gamma+1)\}\{H - \tfrac{1}{2}(v^2+w^2)\}]$, $M_* = u/c_*$.

conserved quantity	x-flux f	forward x-flux f^+, $\|M_*\| < 1$
mass	ρu	$\rho c_*\{ \tfrac{1}{2}(M_*+1)\}^2$
x-momentum	$\rho\{(\gamma+1)/(2\gamma)\}(u^2+c_*^2)$	$f^+_{mass}\cdot\{(\gamma+1)/\gamma\}c_*$
y-momentum	ρuv	$f^+_{mass}\cdot v$
z-momentum	ρuw	$f^+_{mass}\cdot w$

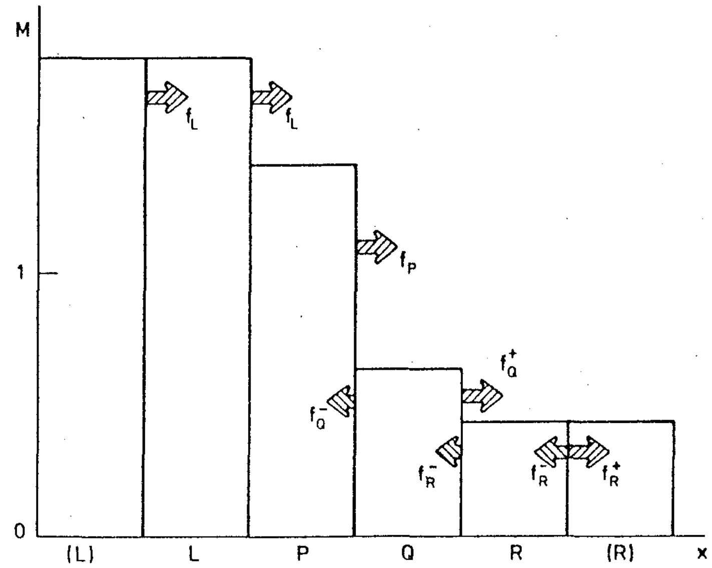

Figure 2. Steady shock profile from a flux-split upwind scheme.

Consider the one-dimensional Euler equations. We denote the supersonic pre-shock state by L, the subsonic post-shock state by R and the interior states by P and Q, as in Fig. 2. To require stationarity means to require constancy of net flux:

(15)
$$\begin{aligned} f_R &= f_L \\ &= f_L^+ + f_P^- \\ &= f_P^+ + f_Q^- \\ &= f_Q^+ + f_R^-. \end{aligned}$$

Assume that the first equality, i.e., the jump condition across the full shock structure, holds. The second equality is automatically satisfied if zone P, like L, is supersonic, with $f_P^- = 0$, $f_L^+ = f_L$. The fourth equality boils down to

$$f_R^+ = f_Q^+, \tag{16}$$

which, if zone Q is subsonic, implies only 2 independent equations. The third equality stands for 3 equations, so that we end up with 5 equations for 6 unknowns, the components of w_P and w_Q. The solutions form a one-parameter family of steady shock profiles, the parameter relating to the sub-grid shock position. There exists one profile with only one interior zone that is precisely sonic. In constrast, Godunov's and Roe's schemes yield steady shocks with one or no interior zone.

Another scheme that yields steady shock structures with two interior states is Osher's flux-difference scheme [2]. For a scalar conservation law it boils down to a flux-splitting method.

Unlike Osher's scheme, the present flux-vector splitting can not preserve a stationary contact discontinuity; in fact, no flux-vector splitting scheme can (see [5]). This is readily understood from the underlying physical model: cell-interactions are achieved through mixing, a diffusive process. Present research is aimed at the development of a computationally simple "collision term" that could prevent the diffusion across a steady contact discontinuity.

A comparison

For one-dimensional isothermal flow ($\gamma=1$, constant c) the flux-vector splitting of Steger and Warming, alias the beam scheme, can be derived from the assumption that there are two kinds of gas particles, equally abundant but moving at different rates: the beams. Per unit volume, $\frac{1}{2}\rho$ moves with velocity $u-c$ and $\frac{1}{2}\rho$ with velocity $u+c$, yielding the correct average flow speed u. For $|M| < 1$ there is a forward and a backward beam, with associated fluxes satisfying (1), (3) and (4):

$$f^{\pm}_{B/SW} = \begin{pmatrix} \frac{1}{2}\rho(u\pm c) \\ \frac{1}{2}\rho(u\pm c)^2 \end{pmatrix}, \quad |u| < c. \tag{17}$$

In Fig. 3 (a) the dependence of the components of $f_{B/SW}$ on M is shown. The momentum flux is continuously differentiable at $M = \pm 1$; the mass flux is not. The splitting given by Eqs. (9) and (10) for $\gamma = 1$ reads

$$f^{\pm}_{VL} = \begin{pmatrix} \pm \frac{1}{4}\rho(u\pm c)^2/c \\ \frac{1}{2}\rho(u\pm c)^2 \end{pmatrix}, \quad |u| < c, \tag{18}$$

with the same momentum-flux splitting but improved mass-flux splitting. The dependence on M is shown in Fig. 3 (b). The Figs. 4 (a) and 4 (b) are graphs

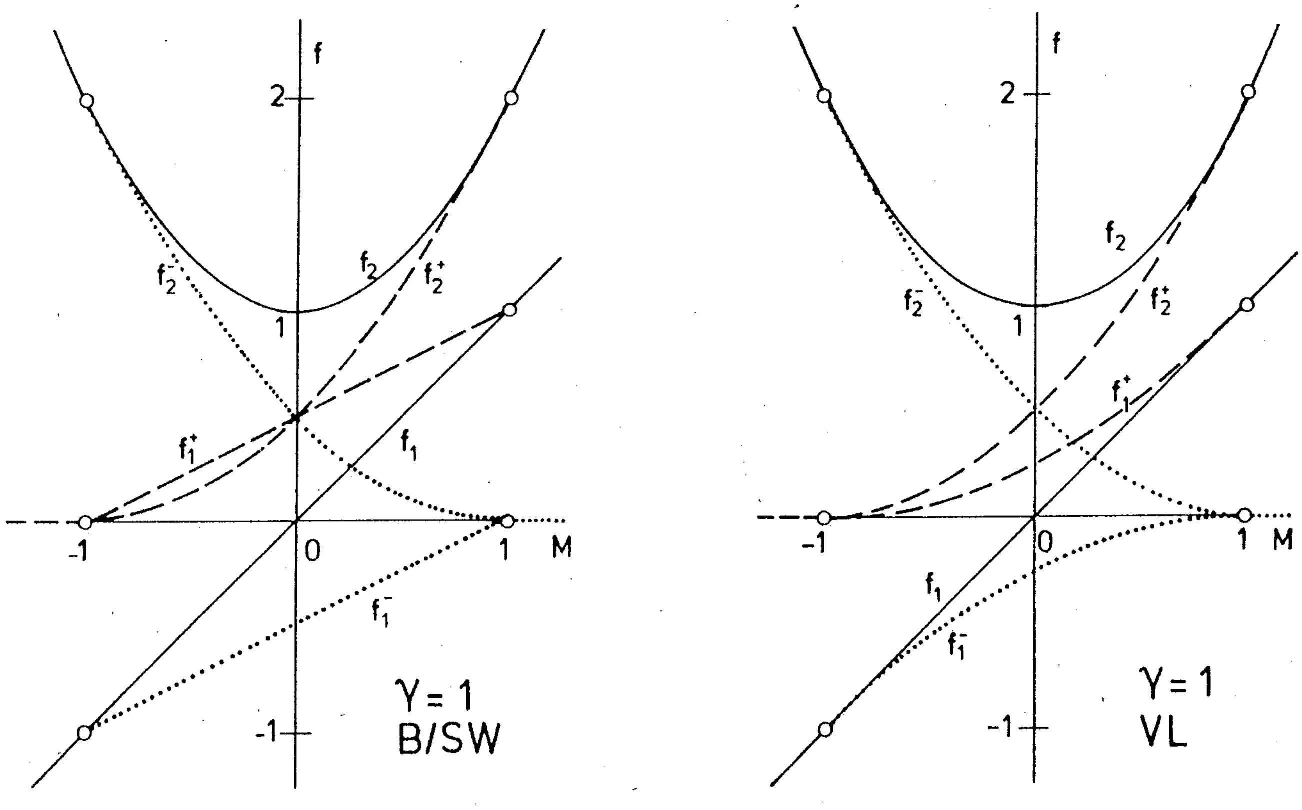

Figure 3. Flux-splitting for the isothermal Euler equations (γ=1). Plotted against Mach number are mass fluxes (subscript 1) and momentum fluxes (subscript 2), normalized by ρc and ρc^2, respectively. (a) Beam scheme/ Steger and Warming; (b) Van Leer.

of the eigenvalues of $df^{\pm}_{B/SW}/dw$ and $df^{\pm}_{VL}/dw$. Their values are given by

$$
(19) \qquad
\begin{aligned}
(\mu^{+}_{1,2})_{B/SW} &= \tfrac{1}{4}\, c\{3+2M \mp \surd(5+4M)\}, \\
(\mu^{+}_{1})_{VL} &= 0, \qquad\qquad\qquad |M| < 1, \\
(\mu^{+}_{2})_{VL} &= \tfrac{1}{4}\, c(5-M)(1+M),
\end{aligned}
$$

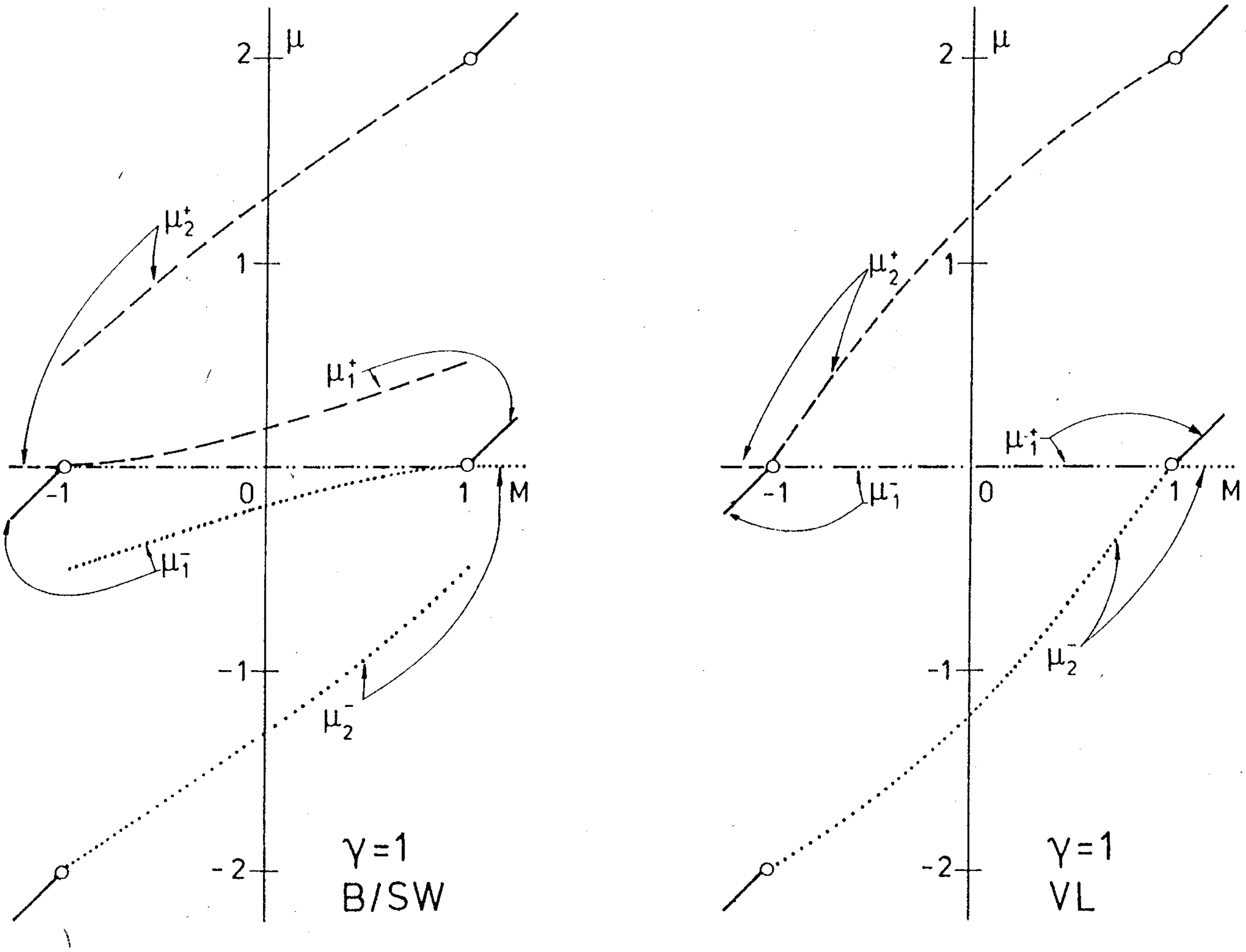

Figure 4. Eigenvalues of split-flux derivatives (γ=1). Plotted against Mach number are the eigenvalues of df^{+}/dw and df^{-}/dw, normalized by c. (a) Beam scheme/Steger and Warming; (b) Van Leer.

and the symmetry condition (4). The eigenvalues shown in Fig. 4 (a) while satisfying (2), have a discontinuity at $M = +1$ or -1. In a stationary numerical solution this causes a discontinuity in the gradient at the sonic point, as seen in Fig. 5, curve B/SW. Correcting the mass-flux splitting performs a small miracle, as seen from curve VL. The gradient-discontinuity disappears, and the numerical diffusion in the subsonic region is substantially reduced. This reduction corresponds to the reduction of the eigenvalues $\mu_1^{\pm}$ to zero.

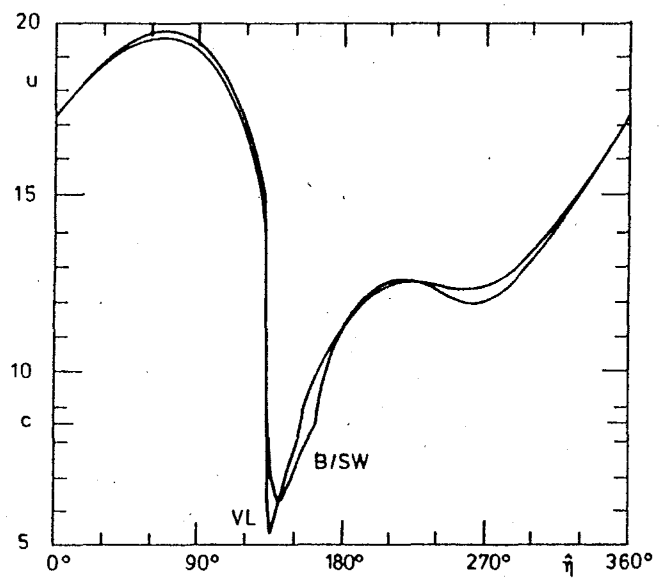

Figure 5. Numerical solutions of the periodic, nozzle-type, cosmic flow problem ($\gamma=1$) from [7] by flux-split upwind schemes, on a 128-zone grid. Plotted is velocity against coordinate angle. B/SW ≡ steady solution by Beam scheme/Steger and Warming; VL ≡ steady solution by Van Leer.

Conclusions

For the full or isenthalpic Euler equations combined with the ideal-gas law, the flux-vector splitting presented here is, by a great margin, the simplest means to implement upwind differencing. For a polytropic gas law, with $\gamma > 1$, closed formulas have not yet been derived.

The scheme produces steady shock profiles with two interior zones. It has been conjectured by Engquist and Osher [10] that, among implicit versions of upwind methods, those with a two-zone steady-shock representation may give faster convergence to a steady solution than those with a one-zone representation. This has not yet been demonstrated in practice.

A disadvantage in using any flux-vector splitting is that it leads to numerical diffusion of a contact discontinuity at rest. This diffusion can be removed; present research is aimed at achieving this with minimal computational effort. Numerical solutions obtained with first- and second-order schemes including the above split fluxes can be found in Refs. [6], [7] (one-dimensional) and [8], [9] (two-dimensional).

References

1. P. L. Roe, J. Computational Phys. 43 (1981), 357.
2. S. Osher, "Numerical solution of singular perturbation problems and hyperbolic systems of conservation laws", North-Holland Mathematical Studies 47 (1981), 179.
3. R. H. Sanders and K. H. Prendergast, Astrophys. J. 188 (1974), 489.
4. J. L. Steger and R. F. Warming, J. Computational Phys. 40 (1981), 263.
5. A. Harten, P. D. Lax and B. van Leer, "Upstream differencing and Godunov-type schemes," ICASE Report No. 82-5; to appear in SIAM Review.
6. J. L. Steger, "A preliminary study of relaxation methods for the inviscid conservative gasdynamics equations using flux-vector splitting," Report No. 80-4, August 1980, Flow Simulations, Inc.
7. G. D. van Albada, B. van Leer and W. W. Roberts, Jr., "A comparative study of computational methods in cosmic gas dynamics," Astron. Astrophys. 108 (1982), 76.
8. G. D. van Albada and W. W. Roberts, Jr., Ap. J. 246 (1981), 740.
9. M. D. Salas, "Recent developments in transonic flow over a circular cylinder," NASA Technical Memorandum 83282 (April, 1982).
10. B. Engquist and S. Osher, Math. Comp. 34 (1980), 45.