Multi-criteria ordered clustering

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1 Definitions

- $\mathcal{A}$: the set of n alternatives $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ (notation: a_i or a_j, $i, j = 1, 2, \ldots, n$)
- $\mathcal{F}$: the set of m criteria $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ (notation: f_k, $k = 1, 2, \ldots, m$)
- $\mathcal{R}$: the set of l clusters $\mathcal{R} = \{r_1, r_2, \ldots, r_l\}$ (notation: r_h, $h = 1, 2, \ldots, l$)

2 Input data

- Evaluation table:

a	$f_1(\cdot)$	$f_2(\cdot)$	$\ldots$	$f_k(\cdot)$	$\ldots$	$f_m(\cdot)$
a_1	$f_1(a_1)$	$f_2(a_1)$	$\ldots$	$f_k(a_1)$	$\ldots$	$f_m(a_1)$
a_2	$f_1(a_2)$	$f_2(a_2)$	$\ldots$	$f_k(a_2)$	$\ldots$	$f_m(a_2)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_i	$f_(a_i)$	$f_2(a_i)$	$\ldots$	$f_k(a_i)$	$\ldots$	$f_m(a_i)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
a_n	$f_1(a_n)$	$f_2(a_n)$	$\ldots$	$f_k(a_n)$	$\ldots$	$f_m(a_n)$

- $\mathcal{W}$: the set of m weights for the criteria: $\mathcal{W} = \{w_1, w_2, \ldots, w_m\}$ (notation: w_k, $k = 1, 2, \ldots, m$)
- N: the number of clusters
- Big M (large value, see Note 1)

3 Decision variables

- $c_{ih} = \begin{cases} 1 & \text{if } a_i \in r_h \\ 0 & \text{otherwise} \end{cases}, \quad c_{ih} \in \{0, 1\}$

4 Equations

$$\max z = \sum_i \sum_j \sum_h \sum_l \sum_k \nu_{ijhlk} w_k - \mu_{ijhk} w_k \tag{1}$$

$$\text{s.t.} \tag{2}$$

$$\gamma_{ijk} \geq \frac{f_k(a_i) - f_k(a_j)}{M} \quad \forall i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m \qquad \text{(linearization, see Note 2)} \tag{3}$$

$$\gamma_{ijk} < \frac{f_k(a_i) - f_k(a_j)}{M} + 1 \quad \forall i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m \tag{4}$$

$$\mu_{ijhk} \geq \alpha_{ijh} + \gamma_{ijk} - 1 \tag{5}$$

$$\mu_{ijhk} \leq \frac{\alpha_{ijh} + \gamma_{ijk}}{2} \tag{6}$$

$$\alpha_{ijh} \geq c_{ih} + c_{jh} - 1 \tag{7}$$

$$\alpha_{ijh} \leq \frac{c_{ih} + c_{jh}}{2} \tag{8}$$

$$\nu_{ijhh_{+1}k} \geq \beta_{ijhh_{+1}} + \gamma_{ijk} - 1 \tag{9}$$

$$\nu_{ijhh_{+1}k} \leq \frac{\beta_{ijhh_{+1}} + \gamma_{ijk}}{2} \tag{10}$$

$$\beta_{ijhh_{+1}} \geq c_{ih} + c_{jh_{+1}} - 1 \tag{11}$$

$$\beta_{ijhh_{+1}} \leq \frac{c_{ih} + c_{jh_{+1}}}{2} \tag{12}$$

$$\sum_i c_{ih} \geq 1 \quad \forall h = 1, 2, \ldots, l \tag{13}$$

$$\sum_h c_{ih} = 1 \quad \forall i = 1, 2, \ldots, n \tag{14}$$

$$\gamma_{ijk} \in \{0, 1\} \quad \forall i, j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m \tag{15}$$

$$\alpha_{ijh} \in \{0, 1\} \tag{16}$$

$$\beta_{ijhl} \in \{0, 1\} \tag{17}$$

$$\mu_{ijhk} \in \{0, 1\} \tag{18}$$

$$\nu_{ijhlk} \in \{0, 1\} \tag{19}$$

$$c_{ih} \in \{0, 1\} \quad \forall i = 1, 2, \ldots, n;\ h = 1, 2, \ldots, l \qquad \text{(decision variables)} \tag{20}$$

5 Notes

1. Big M chosen so that $\dfrac{f_k(a_i) - f_k(a_j)}{M} \in]-1; 1[$

2. $\gamma_{ijk} = \begin{cases} 1 & \text{if } f_k(a_i) > f_k(a_j) \\ 0 & otherwise \end{cases}, \quad \gamma_{ijk} \in \{0, 1\}$

3. $\alpha_{ijh} = \begin{cases} 1 & \text{if } a_i \in r_h \text{ and } a_j \in r_h \\ 0 & otherwise \end{cases}, \quad \alpha_{ijh} \in \{0, 1\}$

4. $\mu_{ijhk} = \begin{cases} 1 & \text{if } \alpha_{ijh} = 1 \text{ and } f_k(a_i) > f_k(a_j) \\ 0 & otherwise \end{cases}, \quad \mu_{ijhk} \in \{0, 1\}$

5. $\beta_{ijhl} = \begin{cases} 1 & \text{if } a_i \in r_h \text{ and } a_j \in r_l, i \neq j, h \neq l \\ 0 & otherwise \end{cases}, \quad \beta_{ijhl} \in \{0, 1\}$

6. $\nu_{ijhlk} = \begin{cases} 1 & \text{if } \beta_{ijhl} = 1 \text{ and } f_k(a_i) > f_k(a_j) \\ 0 & otherwise \end{cases}, \quad \nu_{ijhlk} \in \{0, 1\}$