Ch ② Random Variables and Probability Distributions

Ex Coin two times

$S = \{HH, HT, TH, TT\}$

X := عدد الصور ⟶ عدد الكتابات

X : 2 , 0 , 0 , -2

R.V. المتغير العشوائي

X	-2	0	2
$P(X)$	$1/4$	$2/4$	$1/4$

Prob Distr

* $P(X) \geq 0$
* $\sum_{\forall x} P(X) = 1$

R.V is a real Valued function of Sample space element, always denoted y Capital Letters

X, Y, Z, ...

Probabity Distribu → Discrete

is a function of R.V

Discrete

* $P(X) \geq 0$
* $\sum P(X) = 1$

Continous

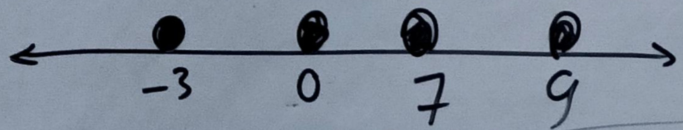
* $F(X) \geq 0$
* $\int_{\forall x} f(x)\,dx = 1$

Discrete R.V

$X = -2, 0, 2$

$X = 0, 1, 2, \ldots$

$X = -3, 0, 7, 9$



نعرف علية داله كتلة احتمالية

P.M.F $P(x)$

① $P(x) \geq 0$

② $\sum_{\forall x} P(x) = 1$

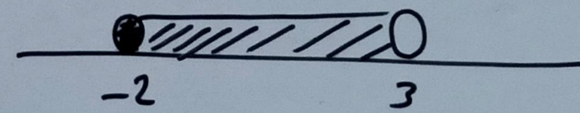
Continuous R.V

$5 < X \leq 13$

$X > 0$

$a \leq X \leq b$

$-2 \leq X < 3$



نعرف عليه داله كثافة احتمالية

P.D.F $f(x)$

① $f(x) \geq 0$

② $\int_{\forall x} f(x)\, dx = 1$

Discrete R.V

Continous R.V

التوقع الرياضي

Expectation $\quad E(x) = \mu \quad$ Expected value $\quad$ Mean $\quad$ avarge

$$E(x) = \sum_{\forall x} x\,P(x)$$

$$E(x) = \int_{\forall x} x\,f(x)\,dx$$

$$E(x^2) = \sum_{\forall x} x^2\,P(x)$$

$$E(x^2) = \int_{\forall x} x^2\,f(x)\,dx$$

التباين

Variance $\quad V(x) = \sigma^2$

$$V(x) = E(x^2) - (E(x))^2$$

$$V(x) = E(x^2) - (E(x))^2$$

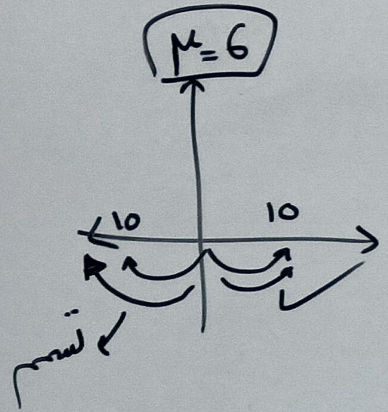
Discrete (R.V) | Continous R.V

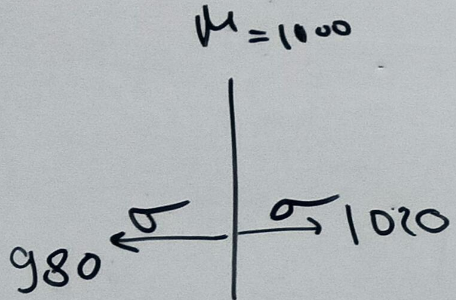
الانحراف المعياري

$\boxed{\text{Standard Deviation } \sigma}$

Discrete R.V

$$\boxed{\sigma = \sqrt{V(x)}}$$





$$\boxed{\sigma^2} = V(x)$$

التباين

Continous R.V

$$\boxed{\sigma = \sqrt{V(x)}}$$

Ex① for P.P Given in the table

X	1	2	3	4
$P(X)$	1/8	3/8	K	1/8

1. Find K
2. " $E(X)$, $E(X^2)$
3. Find $V(X)$, σ
4. find The following Probabilities
   a. $P(2 \leq X < 4)$
   b. $P(X > 3)$
   c. $P(X < 1)$
   d. $P(X > 1 \mid X \leq 3)$
   e. $P(X > 4)$

Sol ① $\frac{1}{8} + \frac{3}{8} + + \frac{1}{8} = 1$

$K = 3/8$

$\therefore$

X	1	2	3	4
$P(X)$	1/8	3/8	3/8	1/8
$X P(X)$	1/8	6/8	9/8	4/8
	1/8	12/8	27/8	16/8

② (i) $E(X) = \sum_{x=1}^{4} X\,P(X) = \frac{20}{8} = \frac{5}{2}$

(ii) $E(X^2) = \sum_{x=1}^{4} X^2\,P(X) = \frac{56}{8}$

③ $V(X) = E(X^2) - (E(X))^2$

$= 56/8 - \left(\frac{5}{2}\right)^2 = \frac{3}{4}$

$\sigma = \sqrt{V(X)} = \sqrt{3/4} = 0.86$

Ex(1) for P.P. Given in the table

X	1	2	3	4
P(x)	1/8	3/8	K	1/8

(1.) Find K

(2) " $E(x)$, $E(x^2)$

(3) Find $V(x)$, σ

(4) find The following Probabilities

(a) $P(2 \le X < 4)$

(b) $P(X > 3)$

(c) $P(X < 1)$

(d) $P(X > 1 \mid X \le 3)$

(e) $P(X > 4)$

(4) (a) $P(2 \le X < 4)$

$= P(2) + P(3)$

$= 3/8 + 3/8 = \frac{6}{8} = \frac{3}{4}$

(b) $P(X > 3) = P(4) = 1/8$

(c) $P(X < 1) = P(\phi) = 0$

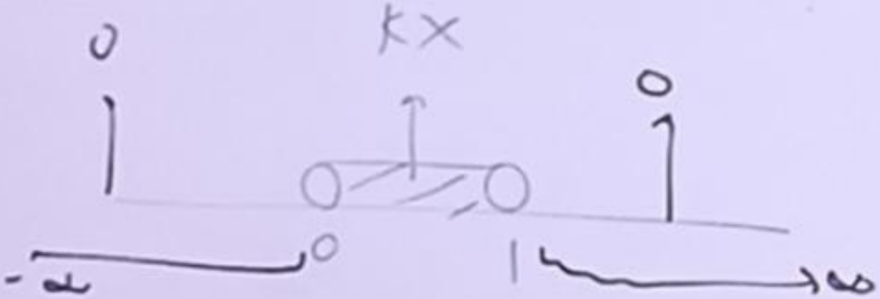
(d) $P[\overset{A}{X > 1} \mid \overset{B}{X \le 3}]$

$= \frac{P(\underset{2,3,4}{X>1} \cap \underset{1,2,3}{X \le 3})}{P(X \le 3)} = \frac{P(2)+P(3)}{P(1)+P(2)+P(3)}$

$= \frac{3/8 + 3/8}{1/8 + 3/8 + 3/8} = \frac{6}{7}$

(e) $P(X > 4) = P(\phi) = 0$

__Ex 1__ Let $f(x) = \begin{cases} kx, & 0<x<1, \\ 0, & o.w. \end{cases}$



Ex② Let $f(x) = \begin{cases} KX &, 0 < X < 1 \\ 0 &, \text{ow} \end{cases}$

is P.D.F

Find

1. K
2. $E(X)$, $E(X^2)$
3. $V(X)$, σ
4. find The following probabilities

ⓐ $P(0 < X < 0.25)$

ⓑ $P(0.25 < X < 0.5)$

ⓒ $P(X = 0.7)$

ⓓ $P(X > 0.3 \mid X < 0.8)$

Sol $\boxed{1}$ Since $f(x)$ is P.D.F.

$$\therefore \int_0^1 f(x)\, dx = 1$$

$$\int_0^1 K X\, dX = 1$$

$$K \int_0^1 X\, dX = 1$$

ألكاسيو

$$K\left(\tfrac{1}{2}\right) = 1$$

$$\boxed{K = 2}$$

$$K \left.\frac{X^2}{2}\right|_0^1 = 1$$

$$K\left(\frac{1^2}{2} - \frac{0^2}{2}\right) = 1$$

$$K\left(\tfrac{1}{2}\right) = 1$$

$$K = 2$$

Ex ② Let $f(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & o.w \end{cases}$

is P.D.F

Find

① K

② $E(x)$, $E(x^2)$

③ $V(x)$, σ

④ Find The following probabilities

ⓐ $P(0 < X < 0.25)$

ⓑ $P(0.25 < X < 0.5)$

ⓒ $P(X = 0.7)$

ⓓ $P(X > 0.3 \mid X < 0.8)$

Sol $\boxed{2}$ (i) $E(X) = \int_0^1 x f(x)\, dx$

أهم حاجة

$= \int_0^1 x(2x)\, dx = 2\int_0^1 x^2 dx$

$= 2 \frac{x^3}{3} \Big|_0^1 = \frac{2}{3}[1^3 - 0^3] = \frac{2}{3}$

$\boxed{ii}$ $E(x^2) = \int_0^1 x^2 f(x) = \int_0^1 x^2 (2x)\, dx$

$= 2 \frac{x^4}{4} \Big|_0^1 = 1/2$

$\boxed{3}$ $V(X) = E(x^2) - (E(x))^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2$

$\sigma = \sqrt{V(x)} = \sqrt{\frac{1}{2} - \left(\frac{2}{3}\right)^2}$

Ex② Let $f(x) = \begin{cases} 2x & , 0<x<1 \\ 0 & , ow \end{cases}$

is P.D.F

Find

1. K
2. $E(X),\ E(X^2)$
3. $V(X),\ \sigma$
4. Find The following probabilities

ⓐ $P(0 < X \leq 0.25)$

ⓑ $P(0.25 < X \leq 0.5)$

ⓒ $P(X = 0.7)$

ⓓ $P(X > 0.6)$ $\quad \int_{0.6}^{1} f(x)\,dx$

ⓔ $P(X > 0.3 \mid X < 0.8)$

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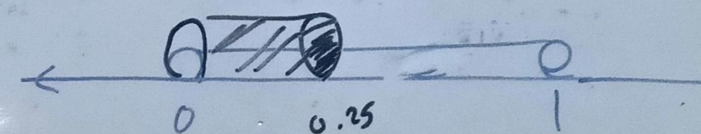
④ ⓐ $P(0 < X \leq 0.25)$

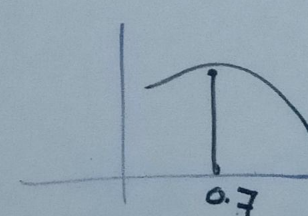


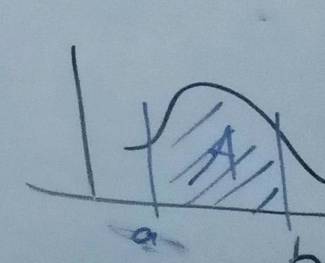
$$= \int_0^{0.25} f(x)\,dx = \int_0^{0.25} (2x)\,dx$$

$$= (0.25)^2$$

ⓑ $P(0.25 < X < 0.5) = \int_{0.25}^{0.5} (2x)\,dx$

ⓒ $P(X = \overset{\text{Constant}}{0.7}) = \text{Zero}$





Ex② Let $f(x) = \begin{cases} 2x , & 0<x<1 \\ 0 , & ow \end{cases}$

is P.D.F

Find

1. K
2. $E(x)$, $E(x^2)$
3. $V(x)$, σ
4. Find The following probabilities

a) $P(0 < X \leq 0.25)$

b) $P(0.25 < X \leq 0.5)$

c) $P(X = 0.7)$

d) $P(X > 0.6)$ $\quad \int_{0.6}^{1} f(x)\,dx$

e) $P(X > 0.3 \mid X < 0.8)$

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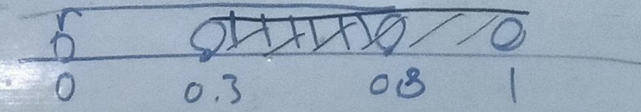
e) $P(X > 0.3 \mid X < 0.8)$



$$\frac{P(X>0.3 \cap X<0.8)}{P(X<0.8)}$$

$$= \frac{P(0.3<X<0.8)}{P(X<0.8)}$$

$$= \frac{\int_{0.3}^{0.8} (2x)\,dx}{\int_{0}^{0.8} (2x)\,dx} =$$

Discrete (R.V) | Continous (R.V)

دالة التوزيع التراكمية Commulative Probabity function

$$F_X(x) = \sum_{t=-\infty}^{x} P(t) \quad \Bigg| \quad F_X(x) = \int_{-\infty}^{x} f(x)\, dx$$

الفائدة منها هو حساب الاحتمالات

Contiuous

$$P(X \leq a) = F_X(a)$$

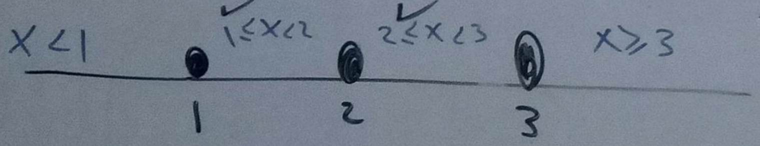
$$P(a < X < b) = F_X(b) - F(a)$$

Ex① let $p(x) = \begin{cases} \left(\frac{8}{7}\right)\left(\frac{1}{2}\right)^x, & x = 1, 2, 3 \\ 0, & ow \end{cases}$

find the Cummulative function

Sol

X	1	2	3
$p(x)$	$4/7$	$2/7$	$1/7$



X	$x<1$	$1\le x<2$	$2\le x<3$	$x\ge 3$
$F_x(x)$	0	$4/7$	$6/7$	1

Ques 1

Find $P(X \le 2)$

$= F_X(2)$

$= \frac{6}{7}$

$P(X \le 2) = P(1) + P(2)$

$= \frac{4}{7} + \frac{2}{7}$

$= \frac{6}{7}$

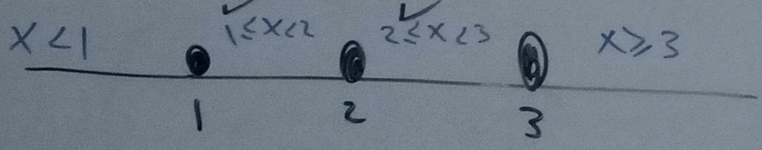
Ex ① let $P(x) = \begin{cases} \left(\frac{8}{7}\right)\left(\frac{1}{2}\right)^x, & x = 1, 2, 3 \\ 0, & \text{ow} \end{cases}$

Find the Cummulative function

Sol

X	1	2	3
P(x)	4/7	2/7	1/7

يصب



بعض

X	$x<1$	$1 \le x < 2$	$2 \le x < 3$	$x \ge 3$
$F_x(x)$	0	4/7	6/7	1

X	1	2	3
P(x)	4/7	2/7	1/7

Ex ②

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & ow \end{cases}$$

Find $F_X(x)$

Sol

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} e^{-t}\,dt$$

رمزية

$$F_X(x) = \frac{e^{-t}}{-1}\Big|_{0}^{x}$$

$$= -\left[e^{-x} - e^{0}\right]$$

$$F_X(x) = \begin{cases} 1 - e^{-x}, & x > 0 \\ 0, & ow \end{cases}$$

Find (a) $P(X \le 8)$

(b) $P(1 < X < 6)$

Sol

(a) $P(X \le 8) = F_X(8) = 1 - e^{-8}$

(b) $P(1 < X < 6) = F(6) - F(1)$

$$= (1 - e^{-6}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-6}$$

Moments Generating Function

داله توليد العزوم

Disecrete	Continous
$M_x(t) = E(e^{tx})$	$M_x(t) = E(e^{tx})$
$= \sum_{\forall x} e^{tx} P(x)$	$= \int_{\forall x} e^{tx} f(x)\, dx$

$E(x), E(x^2), E(x^3)$ لتوليد العزوم

العزم الاول

$$E(x) = \frac{d\, M_x(t)}{dt}\Big]_{t=0}$$

العزم الثاني

$$E(x^2) = \frac{d^2 M_x(t)}{dt^2}\Big]_{t=0}$$

$\vdots$ So, on and on

Ex① <u>Find</u> Moment Generating fun

of

X	-2	-1	0	1	2
$p(x)$	$1/8$	$2/8$	$2/8$	$2/8$	$1/8$

Then use it to find

① $E(X)$

② $E(X^2)$

③ $V(X)$

$\frac{4}{8} + \frac{2}{8} + 0 + \frac{2}{8} + \frac{4}{8}$

Sol

$$M_X(t) = E(e^{tX}) = \sum_{x=-2}^{2} e^{tx}\, p(x)$$

$$= e^{-2t}\frac{1}{8} + e^{-t}\left(\frac{2}{8}\right) + e^{0t}\left(\frac{2}{8}\right) + e^{t}\left(\frac{2}{8}\right) + e^{2t}\frac{1}{8}$$

$$M_X(t) = \frac{1}{8}\left[e^{-2t} + 2e^{-t} + 2 + 2e^{t} + e^{2t}\right]$$

$*$

① $E(X) = \left.\frac{d M_X(t)}{dt}\right|_{t=0}$

$$= \frac{1}{8}\left[-2e^{-2t} - 2e^{-t} + 2e^{t} + 2e^{2t}\right]\Big|_{t=0}$$

$$= \frac{1}{8}\left[-2 - 2 + 2 + 2\right] = 0$$

② $E(X^2) = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0}$

$$= \frac{1}{8}\left[4e^{-2t} + 2e^{-t} + 2e^{t} + 4e^{2t}\right]\Big|_{t=0}$$

$$= \frac{1}{8}\left[4 + 2 + 2 + 4\right]$$

$$= 3/2$$

③ $V(X) = \frac{3}{2} - 0^2 = 3/2$

Ex② Find $M_X(t)$ of

$$f(x) = \begin{cases} e^{-x} & , \boxed{x>0} \\ 0 & , ow \end{cases}$$

then find ① $E(x)$ ② $E(x^2)$, ③ $V(x)$

Sol

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} e^{-x} dx = \int_0^\infty e^{tx-x} dx$$

$$= \int_\infty^0 e^{\boxed{-(1-t)}x} dx = \frac{e^{-(1-t)x}}{-(1-t)} \Big|_0^\infty$$

$$= \frac{e^{-(1-t)\infty}}{(1-t)} - \frac{e^{-(1-t)0}}{-(1-t)}$$

$$M_X(t) = \frac{1}{1-t} = (1-t)^{-1}$$

① $E(x) = \dfrac{d\,M_X(t)}{dt} \Big\}_{t=0}$

$= (-1)(1-t)^{-2}(-1) \Big|_{t=0}$

$= 1$

② $E(x^2) = \dfrac{d^2 M_X(t)}{dt^2} \Big\}_{t=0}$

$= (-2)(1-t)^{-3}(-1) \Big|_{t=0}$

$= 2$

③ $V(x) = 2 - 1^2 = 1$

Dis—

$x = 1, 2$

$P(x)$ - P.M.F ① $P(x) \geq 0$ ② $\sum P(x) = 1$

Con

$a < x < b$

$f(x)$ - P.D.F ① $f(x) \geq 0$ ② $\int f(x)\,dx = 1$

$\sum x\,P(x)$ — $E(x)$ — $\int x\,f(x)\,dx$

$\sum x^2 P(x)$ — $E(x^2)$ — $\int x^2 f(x)\,dx$

$V(x) = E(x^2) - (E(x))^2$

$\sigma = \sqrt{V(x)}$

Commutative

$$F_x(x) = \sum_{-\infty}^{x} P(t) \qquad F_x(x) = \int_{-\infty}^{x} f(t)\,dt$$

Moments Generating fn

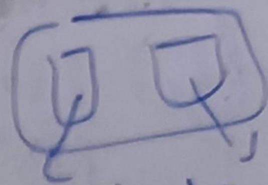
$$M_x(t) = E(e^{tx}) = \sum e^{tx} P(x) \qquad M_x(t) = E(e^{tx}) = \int e^{tx} F(x)\,dx$$

Some ... الورقة

(1)

$p = 0.96$

$q = 0.04$ ($1-p$)



indep

$p = 1 - q$

$n = 2$

سليمة
معيبة
تكون غير المعيب

Bin $\left(\overset{}{n}, \overset{\text{نجاح}}{p}, \overset{\text{الفشل}}{q} \right)$

$$P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x} \quad , \quad x = 0, 1, 2$$

(i) $P(X = 1) = \boxed{{}^{2}C_{1}}\,(0.96)^{1}(0.04)^{2-1} =$

$\left(\boxed{2}\ \boxed{\text{shift}}\ \boxed{\div}\ \boxed{1} \right)$

(ii) $P(X = 0) = {}^{2}C_{0}\,(0.96)^{0}(0.04)^{2-0} =$

② $P(A) = 0.3$ $P(B) = 0.4$ $P(A \cap B) = 0.1$

ⓐ $\underline{P(A \cup B')} = P(A) + \underline{P(B^c)} - P(A \cap B^c)$

$= 0.3 + (1 - 0.4) - [P(A) - P(A \cap B)]$

$= 0.3 + (1 - 0.4) - [0.3 - 0.1] =$

② $P(A - B^c) = P(A \cap B) = 0.1$

③ $P(A | B^c) = \dfrac{P(A \cap B^c)}{P(B^c)} = \dfrac{P(A) - P(A \cap B)}{1 - P(B)} = \dfrac{0.3 - 0.1}{1 - 0.4}$

$$\boxed{P(A \cap B^c) = P(A - B) = P(A) - P(A \cap B)}$$

$$f(x) = \begin{cases} C(x^2+1), & x = 1,2,3,4 \\ 0, & \text{ow} \end{cases}$$

X	1	2	3	4
$f(x)$	$2C$	$5C$	$10C$	$17C$

fm C

① $34C = 1 \Rightarrow C = \frac{1}{34}$

$34C = 1$

$C = \frac{1}{34}$

② $P(0 < X < 3) = P(1) + P(2) = 7\left(\frac{1}{34}\right)$

$P(X > 2) = P(3) + P(4) = 10\left(\frac{1}{34}\right) + 17\left(\frac{1}{34}\right) =$

③ $E(X) = \frac{110}{34}$

X	1	2	3	4	
$F(x)$	$\frac{2}{34}$	$\frac{5}{34}$	$\frac{10}{34}$	$\frac{17}{34}$	
	$\frac{2}{34}$	$\frac{10}{34}$	$\frac{30}{34}$	$\frac{68}{34}$	Σ

4 $f(x) = \begin{cases} C\sqrt{x} & 0 \le x \le 2 \\ 0 & ow \end{cases}$

$$E(x) = \int_0^2 x\, f(x)\, dx$$

1 $\int_0^2 C x^{\frac{1}{2}}\, dx = 1$

$C \left. \frac{x^{3/2}}{3/2} \right|_0^2 = 1$

$\frac{2C}{3} \left[2^{3/2} - 0^{3/2} \right] = 1$

$\frac{2C}{3} (2\sqrt{2}) = 1$

$C = \frac{3}{4\sqrt{2}}$

$$= \int_0^2 \left(x \frac{3}{4\sqrt{2}} \sqrt{x} \right) dx$$

$$= \frac{3}{4\sqrt{2}} \int_0^2 x^{3/2}\, dx = \frac{3}{4\sqrt{2}} \left. \frac{x^{5/2}}{5/2} \right|_0^2$$

$$= \frac{6}{20\sqrt{2}} \left[2^{5/2} \right] *$$

$$E(x^2) = \int_0^2 x^2 f(x)\, dx = \frac{3}{4\sqrt{2}} \int_0^2 x^{5/2}\, dx$$

$$= \frac{3}{4\sqrt{2}} \left. \frac{x^{7/2}}{7/2} \right|_0^2 = \frac{6}{28\sqrt{2}} \left[2^{7/2} \right] *$$

$$V(x) = E(x^2) - (E(x))^2$$