Inverse Laplace Transforms

If $\mathcal{L}(f(t)) = F(s)$, $\quad \mathcal{L}^{-1}(F(s)) = f(t)$

$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1, \; s > 0$ $\qquad \mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at, \; s > 0$

$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t, \; s > 0$ $\qquad \mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at, \; s > 0$

$\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n, \; s > 0$ $\qquad \mathcal{L}^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at, \; s > |a|$

$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}, \; s > a$ $\qquad \mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at \quad s > |a|$

Inverse Laplace Transforms

Ex $\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = e^{-5t}$

Ex $\mathcal{L}^{-1}\left(\frac{5040}{s^8}\right) = t^7$

Ex $7! = 5040$

$\mathcal{L}^{-1}\left(\frac{12}{s^4}\right) = 2\,\mathcal{L}^{-1}\left(\frac{6}{s^4}\right)$

$3! = 6$ $\quad = 2.\,t^3.$

Ex $\mathcal{L}^{-1}\left(\frac{3}{s^2-9}\right)$

$= \sinh 3t$

Ex $\mathcal{L}^{-1}\left(\frac{s}{s^2+16}\right) = \cos 4t$

Ex $\mathcal{L}^{-1}\left(\frac{6}{s^2+36}\right)$

$= \sin 6t$

Inverse Laplace Transforms

Linearity

$$\mathcal{L}^{-1}\left(aF(s)+bG(s)\right)=a\,f(t)+b\,g(t)$$

$$\mathcal{L}(f(t))=F(s),\quad \mathcal{L}(g(t))=G(s)$$

$$f(t)=\mathcal{L}^{-1}(F(s)),\quad g(t)=\mathcal{L}^{-1}(G(s)).$$

Ex: $\mathcal{L}^{-1}\left(\frac{1}{s+3}+\frac{2}{s+5}+\frac{6}{s^4}\right)$

$$=\mathcal{L}^{-1}\left(\frac{1}{s+3}\right)+2\,\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+\mathcal{L}^{-1}\left(\frac{6}{s^4}\right)$$

$$=e^{-3t}+2e^{-5t}+t^3$$

Inverse Laplace Transforms

Ex $\mathcal{L}^{-1}\left(\frac{s}{4s^2-16}+\frac{9}{s^2+25}+\frac{4s}{9s^2+4}+\frac{1}{4s-1}\right)$

$=\mathcal{L}^{-1}\left(\frac{s}{4s^2-16}\right)+9\,\mathcal{L}^{-1}\left(\frac{1}{s^2+25}\right)+4\,\mathcal{L}^{-1}\left(\frac{s}{9s^2+4}\right)+\mathcal{L}^{-1}\left(\frac{1}{4s-1}\right)$

$=\frac{1}{4}\mathcal{L}^{-1}\left(\frac{s}{s^2-4}\right)+\frac{9}{5}\mathcal{L}^{-1}\left(\frac{5}{s^2+25}\right)+\frac{4}{9}\mathcal{L}^{-1}\left(\frac{s}{s^2+\frac{4}{9}}\right)$

$+\frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-\frac{1}{4}}\right)$

$=\frac{1}{4}\cdot\cosh 2t+\frac{9}{5}\cdot\sin 5t+\frac{4}{9}\cdot\cos\frac{2}{3}t+\frac{1}{4}\cdot e^{\frac{1}{4}t}$

Inverse Laplace Transforms

(2) $\mathcal{L}^{-1}\left(F(s-a)\right) = e^{at} f(t)$, $\mathcal{L}^{-1}\left(F(s)\right) = f(t)$.

Ex $\mathcal{L}^{-1}\left(\frac{n!}{(S-a)^{n+1}}\right) = e^{at} \cdot t^{n}$

$\mathcal{L}^{-1}\left(\frac{n!}{S^{n+1}}\right) = t^{n}$

Ex $\mathcal{L}^{-1}\left(\frac{b}{(S-a)^2-b^2}\right) = e^{at}\sinh bt$

Ex $\mathcal{L}^{-1}\left(\frac{S-a}{(S-a)^2-b^2}\right) = e^{at}\cosh bt$

Ex $\mathcal{L}^{-1}\left(\frac{S-a}{(S-a)^2+b^2}\right) = e^{at}\cos bt$

Ex $\mathcal{L}^{-1}\left(\frac{12}{(S-3)^3}\right)$

$= 6\,\mathcal{L}^{-1}\left(\frac{2}{(S-3)^3}\right)$

$= 6 \cdot e^{3t}\, t^{2}$

Inverse Laplace Transforms

Ex) $\mathcal{L}^{-1}\left(\dfrac{3S+1}{(S+1)^4}\right)$

$= \mathcal{L}^{-1}\left(\dfrac{3S+3-2}{(S+1)^4}\right)$

$= \mathcal{L}^{-1}\left(\dfrac{3(S+1)}{(S+1)^4}\right) - 2\,\mathcal{L}^{-1}\left(\dfrac{1}{(S+1)^4}\right)$

$= 3\,\mathcal{L}^{-1}\left(\dfrac{1}{(S+1)^3}\right) - 2\,\mathcal{L}^{-1}\left(\dfrac{1}{(S+1)^4}\right)$

$= 3/2\,\mathcal{L}^{-1}\left(\dfrac{2}{(S+1)^3}\right) - \dfrac{2}{6}\,\mathcal{L}^{-1}\left(\dfrac{6}{(S+1)^4}\right)$

$= \dfrac{3}{2}\cdot e^{-t}\,t^2 - \dfrac{2}{6}\cdot e^{-t}\,t^3$

Inverse Laplace Transforms

Ex $\mathcal{L}^{-1}\left(\dfrac{3S+1}{(S+1)^4}\right)$

$$=3\,\mathcal{L}^{-1}\left(\frac{S+1-1}{(S+1)^4}\right)+\mathcal{L}^{-1}\left(\frac{1}{(S+1)^4}\right)$$

$$=\frac{3}{2}\,\mathcal{L}^{-1}\left(\frac{2}{(S+1)^3}\right)-\frac{3}{6}\,\mathcal{L}^{-1}\left(\frac{6}{(S+1)^4}\right)+\frac{1}{6}\,\mathcal{L}^{-1}\left(\frac{6}{(S+1)^4}\right)$$

$$=\frac{3}{2}\cdot e^{-t}\,t^2-\frac{1}{2}\cdot e^{-t}\,t^3+\frac{1}{6}\,e^{-t}\,t^3.$$

Inverse Laplace Transforms

(3) $\mathcal{L}^{-1}\left(e^{-as} F(s)\right) = g(t) = \begin{cases} f(t-a) & t > a. \\ 0 & t < a \end{cases}$

Ex: $\mathcal{L}^{-1}\left(\dfrac{e^{-\frac{\pi}{3}s}}{s^2+1}\right) = \begin{cases} \sin(t - \frac{\pi}{3}). & t > \frac{\pi}{3} \\ 0 & t < \frac{\pi}{3} \end{cases}$

$\mathcal{L}^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$

Ex) $\mathcal{L}^{-1}\left(\dfrac{e^{-\frac{\pi}{4}s}\, s}{s^2+16}\right) = \begin{cases} \cos 4(t - \frac{\pi}{4}) & t > \frac{\pi}{4} \\ 0, & t < \frac{\pi}{4} \end{cases}$

$\mathcal{L}^{-1}\left(\dfrac{s}{s^2+16}\right) = \cos 4t$

Inverse Laplace Transforms

Ex: $$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{(s-6)^4}\right) = \begin{cases} \frac{1}{6} e^{6(t-3)} (t-3)^3 & t > 3 \\ 0, & t < 3 \end{cases}$$

Sol:

$$\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{6}\mathcal{L}^{-1}\left(\frac{6}{s^4}\right) = \frac{1}{6} t^3$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-6)^4}\right) = \frac{1}{6} e^{6t} t^3.$$