

Inverse Laplace Transforms

If $\mathcal{L}(f(t)) = F(s)$, $\mathcal{L}^{-1}(F(s)) = f(t)$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1, s > 0$$

$$\mathcal{L}^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin at, s > 0$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t, s > 0$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at, s > 0$$

$$\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n, s > 0$$

$$\mathcal{L}^{-1}\left(\frac{a}{s^2 - a^2}\right) = \sinh at, s > |a|$$

$$\mathcal{L}^{-1}\left(\frac{1}{s - a}\right) = e^{at}, s > a$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cosh at, s > |a|$$

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Inverse Laplace Transforms

$$\textcircled{Ex} \int^{\mathcal{P}^{-1}} \left(\frac{1}{s+5} \right) = e^{-5t}$$

$$\textcircled{Ex} \int^{\mathcal{P}^{-1}} \left(\frac{3}{s^2 - 9} \right)$$

$$= \sin 3t$$

$$\textcircled{Ex} \int^{\mathcal{P}^{-1}} \left(\frac{5040}{s^8} \right) = t^7$$

$$\textcircled{Ex} \quad 7! = 5040$$

$$\textcircled{Ex} \int^{\mathcal{P}^{-1}} \left(\frac{1}{s^2 + 16} \right) = \cos 4t$$

$$\int^{\mathcal{P}^{-1}} \left(\frac{12}{s^4} \right) = 2 \int^{\mathcal{P}^{-1}} \left(\frac{6}{s^4} \right)$$

$$3! = 6$$

$$= 2 t^3$$

$$\textcircled{Ex} \int^{\mathcal{P}^{-1}} \left(\frac{6}{s^2 + 36} \right)$$

$$= \sin 6t$$

Inverse Laplace Transforms

$$\textcircled{Ex} \quad \mathcal{P}^{-1} \left(\frac{1}{s+5} \right) = e^{-5t}$$

$$\textcircled{Ex} \quad \mathcal{P}^{-1} \left(\frac{5040}{2^8} \right) = t^7$$

$$\textcircled{Ex} \quad 7! = 5040$$

$$\mathcal{P}^{-1} \left(\frac{12}{2^4} \right) = 2 \mathcal{P}^{-1} \left(\frac{6}{2^4} \right)$$

$$3! = 6$$

$$= 2 t^3$$

$$\textcircled{Ex} \quad \mathcal{P}^{-1} \left(\frac{3}{s^2 - 9} \right)$$

$$= \sinh 3t$$

$$\textcircled{Ex} \quad \mathcal{P}^{-1} \left(\frac{2}{s^2 + 16} \right) = \cos 4t$$

$$\textcircled{Ex} \quad \mathcal{P}^{-1} \left(\frac{6}{s^2 + 36} \right)$$

$$= \sin 6t$$

Inverse Laplace Transforms

Linearity

$$\mathcal{L}^{-1}(a F(s) + b G(s)) = a f(t) + b g(t)$$

$$\mathcal{L}(f(t)) = F(s), \quad \mathcal{L}(g(t)) = G(s)$$

$$f(t) = \mathcal{L}^{-1}(F(s)), \quad g(t) = \mathcal{L}^{-1}(G(s))$$

(Ex:)

$$\mathcal{L}^{-1}\left(\frac{1}{s+3} + \frac{2}{s+5} + \frac{6}{s^4}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + 2 \mathcal{L}^{-1}\left(\frac{1}{s+5}\right) + \mathcal{L}^{-1}\left(\frac{6}{s^4}\right)$$

$$= e^{-3t} + 2e^{-5t} + t^3$$

Inverse Laplace Transforms

(Ex)

$$\mathcal{P}^{-1} \left(\frac{s}{4s^2 - 16} + \frac{5}{s^2 + 25} + \frac{4s}{9s^2 + 4} + \frac{1}{4s - 1} \right)$$

$$= \mathcal{P}^{-1} \left(\frac{s}{4s^2 - 16} \right) + 9 \mathcal{P}^{-1} \left(\frac{1}{s^2 + 25} \right) + 4 \mathcal{P}^{-1} \left(\frac{s}{9s^2 + 4} \right) + \mathcal{P}^{-1} \left(\frac{1}{4s - 1} \right)$$

$$= \frac{1}{4} \mathcal{P}^{-1} \left(\frac{s}{s^2 - 4} \right) + \frac{9}{5} \mathcal{P}^{-1} \left(\frac{5}{s^2 + 25} \right) + \frac{4}{9} \mathcal{P}^{-1} \left(\frac{s}{s^2 + \frac{4}{9}} \right) + \frac{1}{4} \mathcal{P}^{-1} \left(\frac{1}{s - \frac{1}{4}} \right)$$

$$= \frac{1}{4} \cdot \cosh 2t + \frac{9}{5} \cdot \sin 5t + \frac{4}{9} \cdot \cos \frac{2}{3}t + \frac{1}{4} \cdot e^{\frac{1}{4}t}$$

Inverse Laplace Transforms

$$(2) \quad \mathcal{L}^{-1} (F(s-a)) = e^{at} f(t), \quad \mathcal{L}^{-1} (F(s)) = f(t).$$

$$(Ex) \quad \mathcal{L}^{-1} \left(\frac{n!}{(s-a)^{n+1}} \right) = e^{at} \cdot t^n$$

$$\mathcal{L}^{-1} \left(\frac{n!}{s^{n+1}} \right) = t^n$$

$$(Ex) \quad \mathcal{L}^{-1} \left(\frac{s-a}{(s-a)^2 + b^2} \right) = e^{at} \cos bt$$

$$(Ex) \quad \mathcal{L}^{-1} \left(\frac{b}{(s-a)^2 - b^2} \right) = e^{at} \sinh bt = 6 \mathcal{L}^{-1} \left(\frac{2}{(s-3)^3} \right)$$

$$(Ex) \quad \mathcal{L}^{-1} \left(\frac{s-a}{(s-a)^2 - b^2} \right) = e^{at} \cosh bt = 6 \cdot e^{3t} t^2$$

Inverse Laplace Transforms

$$\textcircled{\text{Ex}} \quad \mathcal{L}^{-1} \left(\frac{3s+1}{(s+1)^4} \right) = \frac{3}{2} \mathcal{L}^{-1} \left(\frac{2}{(s+1)^3} \right) - \frac{2}{6} \mathcal{L}^{-1} \left(\frac{6}{(s+1)^4} \right)$$

$$= \mathcal{L}^{-1} \left(\frac{3s+3-2}{(s+1)^4} \right) = \frac{3}{2} \cdot e^{-t} t^2 - \frac{2}{6} \cdot e^{-t} t^3$$

$$= \mathcal{L}^{-1} \left(\frac{3(s+1)}{(s+1)^4} \right) - 2 \mathcal{L}^{-1} \left(\frac{1}{(s+1)^4} \right)$$

$$= 3 \mathcal{L}^{-1} \left(\frac{1}{(s+1)^3} \right) - 2 \mathcal{L}^{-1} \left(\frac{1}{(s+1)^4} \right)$$

Inverse Laplace Transforms

$$\textcircled{\mathcal{L}^{-1}x} \mathcal{P}^{-1} \left(\frac{3s+1}{(s+1)^4} \right)$$

$$= 3 \mathcal{P}^{-1} \left(\frac{s+1-1}{(s+1)^4} \right) + \mathcal{P}^{-1} \left(\frac{1}{(s+1)^4} \right)$$

$$= \underset{2}{3} \mathcal{P}^{-1} \left(\frac{1}{(s+1)^3} \right) - \underset{6}{3} \mathcal{P}^{-1} \left(\frac{1}{(s+1)^4} \right) + \underset{6}{1} \mathcal{P}^{-1} \left(\frac{1}{(s+1)^4} \right)$$

$$= \frac{3}{2} \cdot e^{-t} t^2 - \frac{1}{2} \cdot e^{-t} t^3 + \frac{1}{6} e^{-t} t^3$$

Transforms

$$\textcircled{3} \quad \mathcal{L}^{-1} \left(e^{-as} F(s) \right) = g(t) = \begin{cases} f(t-a) & t > a. \\ 0 & t < a \end{cases}$$

$$\textcircled{Ex 1} \quad \mathcal{L}^{-1} \left(\frac{e^{-\frac{\pi}{3}s}}{s^2 + 1} \right) = \begin{cases} \sin(t - \frac{\pi}{3}) & t > \frac{\pi}{3} \\ 0 & t < \frac{\pi}{3} \end{cases}$$

$$\mathcal{L}^{-1} \left(\frac{1}{s^2 + 1} \right) = \sin t$$

$$\textcircled{Ex 2} \quad \mathcal{L}^{-1} \left(\frac{e^{-\frac{\pi}{4}s}}{s^2 + 16} \right) = \begin{cases} \cos 4(t - \frac{\pi}{4}) & t > \frac{\pi}{4} \\ 0 & t < \frac{\pi}{4} \end{cases}$$

$$\mathcal{L}^{-1} \left(\frac{s}{s^2 + 16} \right) = \cos 4t$$

Inverse Laplace Transforms

$$\textcircled{\text{Ex}} \quad \mathcal{L}^{-1} \left(\frac{e^{-3s}}{(s-6)^4} \right) = \begin{cases} \frac{1}{6} e^{6(t-3)} (t-3)^3 & t > 3 \\ 0 & t < 3 \end{cases}$$

$$\textcircled{\text{Sol}} \quad \mathcal{L}^{-1} \left(\frac{1}{s^4} \right) = \frac{1}{6} \mathcal{L}^{-1} \left(\frac{6}{s^4} \right) = \frac{1}{6} t^3$$

$$\mathcal{L}^{-1} \left(\frac{1}{(s-6)^4} \right) = \frac{1}{6} e^{6t} t^3$$