Inverse Laplace Transforms

If $\mathcal{L}(f(t)) = F(s)$, $\mathcal{L}^{-1}(F(s)) = f(t)$

$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$ $\quad$ $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$ $\quad$ $\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

$\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$

Inverse Laplace Transforms

1 Linearity $\mathcal{L}^{-1}(aF(s) + bG(s)) = a\mathcal{L}^{-1}(F(s)) + b\mathcal{L}^{-1}(G(s))$

Ex $\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\sin at$

x $\mathcal{L}^{-1}\left(\frac{3}{s-5}\right) = 3\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) = 3e^{5t}$

irst Shifting $\mathcal{L}^{-1}(F(s-a)) = e^{at} f(t)$

$\mathcal{L}^{-1}\left(\frac{b}{(s-a)^2-b^2}\right) = e^{at}\sinh bt$

Inverse Laplace Transforms

(3) Second shifting $\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = g(t) = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$

Ex $\mathcal{L}^{-1}\left(\dfrac{e^{-\frac{\pi}{3}s}}{s^2+4}\right)$

$\mathcal{L}^{-1}\left(\dfrac{1}{s^2+4}\right) = \dfrac{1}{2}\mathcal{L}^{-1}\left(\dfrac{2}{s^2+4}\right)$

$= \dfrac{1}{2}\sin 2t$

$= \begin{cases} \frac{1}{2}\sin 2(t-\frac{\pi}{3}) & t>\frac{\pi}{3} \\ 0 & t<\frac{\pi}{3} \end{cases}$

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④ Change of scale

$$\mathcal{L}^{-1}\left(\frac{1}{a}F\left(\frac{s}{a}\right)\right) = f(at)$$

$$\mathcal{L}^{-1}\left(F(s/a)\right) = a\,f(at).$$

Ex

$$\mathcal{L}^{-1}\left(\frac{b}{\left(\frac{s}{a}\right)^2 + b^2}\right) = a \sin abt$$

$$\mathcal{L}^{-1}\left(\frac{b}{s^2+b^2}\right) = \sin bt = f(t)$$

$$f(at) = \sin abt$$

Inverse Laplace Transforms

④ Change of scale

Ex

$$\mathcal{L}^{-1}\left(\frac{s}{\left(\frac{s}{7}\right)^2 + 9}\right) = 7\,\mathcal{L}^{-1}\left(\frac{s/7}{\left(\frac{s}{7}\right)^2 + 9}\right)$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t = f(t)$$

$$7\,\mathcal{L}^{-1}\left(\frac{s/7}{\left(\frac{s}{7}\right)^2+9}\right) = 7 \times 7 \cos 3\times 7t$$

$$= 49 \cos 21t.$$

Inverse Laplace Transforms

5) $\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(u)\,du$ $\quad = F(s)$

Ex) $\mathcal{L}^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{s^2+a^2}}{s}\right)$

$\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\cdot \sin at = f(t)$

$= \frac{1}{a}\int_0^t \sin au\,du = -\frac{1}{a^2}\cos au\Big|_0^t = -\frac{1}{a^2}\left[\cos at - 1\right]$

Inverse Laplace Transforms

(6) $$\mathcal{L}^{-1}\left((-1)^n \frac{d^n}{ds^n} F(s)\right) = t^n f(t)$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{d}{ds} F(s)\right) = -t\, f(t)}$$

Ex $$\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)^2}\right) = \frac{1}{4}\, t \sin 2t.$$

$$\frac{d}{ds}\,\frac{1}{s^2+4} = \frac{-1}{(s^2+4)^2}\cdot 2s = \frac{-2s}{(s^2+4)^2}$$

$$\mathcal{L}^{-1}\left(\frac{d}{ds}\,\frac{1}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{-2s}{(s^2+4)^2}\right) = -2\,\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)^2}\right)$$

$$\shortparallel$$

$$-t\,\frac{1}{2}\sin 2t = -2\,\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)^2}\right)$$

Inverse Laplace Transforms

(7) $\mathcal{L}^{-1}\left(\int_s^\infty F(u)\,du\right) = \dfrac{f(t)}{t}$

Ex $\mathcal{L}^{-1}\left(\int_s^\infty \dfrac{du}{u^2+u}\right) = \dfrac{1-e^{-t}}{t}$

$\mathcal{L}^{-1}\left(\dfrac{1}{s^2+s}\right) = \mathcal{L}^{-1}\left(\dfrac{\frac{1}{s+1}}{s}\right) = \int_0^t e^{-u}\,du = -e^{-u}\Big|_0^t = -(e^{-t}-1)$

$= 1-e^{-t}$

$\mathcal{L}^{-1}\left(\dfrac{1}{s+1}\right) = e^{-t}$

Inverse Laplace Transforms

Ex $\mathcal{L}^{-1}\left(\int_s^\infty \left(\dfrac{2u}{u^2+a^2} - \dfrac{2}{u}\right) du\right)$

S.P.

$\mathcal{L}^{-1}\left(\dfrac{2s}{s^2+a^2} - \dfrac{2}{s}\right) = 2\mathcal{L}^{-1}\left(\dfrac{s}{s^2+a^2}\right) - 2\mathcal{L}^{-1}\left(\dfrac{1}{s}\right)$

$= 2\cos at - 2 = f(t)$

$= \dfrac{2\cos at - 2}{t}$

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Ex $\mathcal{L}^{-1}\left(\int_s^\infty \frac{du}{u^2+u}\right) = \frac{1-e^{-t}}{t}$

$\mathcal{L}^{-1}\left(\frac{1}{s^2+s}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{s+1}}{s}\right) = \int_0^t e^{-u}\,du = -e^{-u}\Big|_0^t = -(e^{-t}-1)$
$= 1-e^{-t}$

$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$

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Ex $\mathcal{L}^{-1}\left(\int_s^\infty \left(\frac{2u}{u^2+a^2} - \frac{2}{u}\right)du\right)$

S.P.

$\mathcal{L}^{-1}\left(\frac{2s}{s^2+a^2} - \frac{2}{s}\right) = 2\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s}\right)$

$= 2\cos at - 2 = f(t)$

$= \frac{2\cos at - 2}{t}$

$\frac{x}{(x^2-a^2)} = \frac{x}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$

$\frac{x}{(x^2-a^2)(x+a)}$

$= \frac{x}{(x-a)(x+a)^2} = \frac{A}{x-a} + \frac{B}{x+a} + \frac{C}{(x+a)^2}$