Chapter 1 Probability / /

* Some definitions

A probability experiment is a chance process that leads to well-defined results (outcomes)

التجربه الاحتماليه هي عمليه تعتمد على الصدفة وتعطي نتائج محددة جيداً

outcome is The result of a single trial of a probability experiment

نتيجة التجربه : ناتج تجربه واحدة من التجربه الاحتماليه

Sample space (S) is The set of all possible outcome of a probability experiment

فضاء العينه S هو مجموعة كل النتائج الممكنه للتجربة العشوائيه

Example 1

consider The experiment of tossing a dice

outcomes : 1 , 2 , 3 , 4 , 5 , or 6

Sample space $S = \{1, 2, 3, 4, 5, 6\}$

Example 2

Consider The experiment of rolling two coins

outcomes : head head (HH), head tail (HT), tail head (TH), or tail tail (TT)

Sample space $S = \{HH, HT, TH, TT\}$

Event الحدث

Event is a set of outcomes of a probability experiment

Event is a subset of sample space — الحدث هو مجموعة من نواتج التجربة الاحتمالية

- لاحظ الفرق بين فضاء العينة sample space والذي به جميع النواتج والحدث Event الذي به بعض النواتج او كلها فالحدث مجموعة جزئية من فضاء العينة
- يرمز للحدث بالأحرف الأبجدية الانجليزية الكبيرة Capitale letters $A, B, C, D, \ldots$

$N(A)$ or N_A = Number of elements in A

probability of $A = P(A) = \dfrac{\text{Number of elements in } A}{\text{Number of elements in } S} \Rightarrow P(A) = \dfrac{N(A)}{N(S)}$

احتمال وقوع الحدث $A = \dfrac{\text{عدد عناصر الحدث } A}{\text{عدد عناصر فضاء العينة } S}$

تصلح هذه القاعدة عندما يكون جميع النتائج متساوية الاحتمال

Example مثال

Taking the probability experiment of tossing a dice

بأخذ التجربة الاحتمالية القاء حجر نرد

$S = \{1,2,3,4,5,6\}$, $N(S) = 6$

The event A is a set of odd numbers $\Rightarrow A = \{1,3,5\} \Rightarrow P(A) = \dfrac{N(A)}{N(S)} = \dfrac{3}{6} = \dfrac{1}{2}$

The event B is a set of numbers less than 5 $\Rightarrow B = \{1,2,3,4\} \Rightarrow P(B) = \dfrac{N(B)}{N(S)} = \dfrac{4}{6} = \dfrac{2}{3}$

The event C is a set of numbers that are divisible by 3 $\Rightarrow C = \{3,6\} \Rightarrow P(C) = \dfrac{N(C)}{N(S)} = \dfrac{2}{6} = \dfrac{1}{3}$

The event D is a set of prime numbers $\Rightarrow D = \{2,3,5\} \Rightarrow P(D) = \dfrac{N(D)}{N(S)} = \dfrac{3}{6} = \dfrac{1}{2}$

Event Relationships among events	Probability
• Event A اى حدث A	$P(A)=\frac{N(A)}{N(S)}$, $0 \leqslant P(A) \leqslant 1$
• Impossible Event $A=\phi=\{\}$ مستحيل	$P(\phi)=0$
• Sure Event $A=S$ المؤكد	$P(S)=1$
• Complement Event A^c مكملة الحدث	$P(A^c)=1-P(A)$
• union $A \cup B$ اتحاد الحدثين يعني A or B occur (at least) وقوع A أو B	$P(A \cup B)=P(A)+P(B)-P(A \cap B)$
• Intersection $A \cap B$ تقاطع الحدثين A and B occur (both) يعني وقوعهما معاً	$P(A \cap B)=P(A)+P(B)-P(A \cup B)$
• First De Morgan Rule $A^c \cup B^c=(A \cap B)^c$ عدم وقوع A أو عدم وقوع B (not both = at most)	$P(A^c \cup B^c)=P((A \cap B)^c)=1-P(A \cap B)$
• Second De Morgan Rule $A^c \cap B^c=(A \cup B)^c$ عدم وقوع A و عدم وقوع B (Neither nor)	$P(A^c \cap B^c)=P((A \cup B)^c)=1-P(A \cup B)$
• Difference between A and B $(A-B)$ هو وقوع A وعدم وقوع B A and not B	$P(A-B)=P(A)-P(A \cap B)$ $=P(A \cap B^c)$

/ /

Disjoint Events are events that never occur at the same time (Mutual exclusivity)

الاحداث المتنافية احداث لا تحدث فى نفس الوقت ويكون فيها

$$\boxed{A \cap B = \phi} \Rightarrow \boxed{P(A \cap B) = 0}$$

مثال A مجموعه الاعداد الزوجيه على حجر النرد و B مجموعه الاعداد الفردية على حجر النرد فوقوع احد الحدثين يمنع وقوع الآخر

Independent Events: occurance of one of them doesn't affect the occurrance of the other

استقلال الاحداث : وقوع حدث لا يؤثر على وقوع الآخر ويكون فيها

$$\boxed{P(A \cap B) = P(A)\, P(B)}$$

فمثلاً عند القاء عملتين فإذا كانت الاولى صورة فهذا لا يؤثر ابداً على نتيجه العملة الثانيه

$$P(TT) = P(\{T\} \cap \{T\}) = P(\{T\})\, P(\{T\}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Conditional Probability is the probability of an event given the occurrence of another event

الاحتمال المشروط : احتمال وقوع حدث بشرط وقوع حدث آخر

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

Multiplication Rule

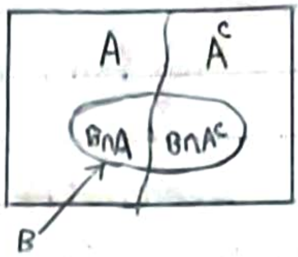
سحب على التوالي دون ارجاع

$$P(A\cap B) = P(A)\,P(B|A) = P(B)\,P(A|B)$$

$$P(A\cap B\cap C) = P(A)\,P(B|A)\,P(C|(A\cap B))$$

$$P(A_1\cap A_2\cap\ldots\cap A_n) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1\cap A_2)\cdots P(A_n|A_1\cap A_2\cap\ldots\cap A_{n-1})$$

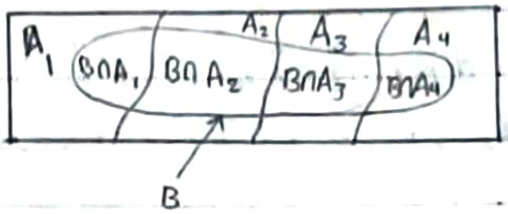
Total Probability Rule



$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

If A_1, A_2, A_3 and A_4 are disjoint events

$$P(B) = P(B\cap A_1) + P(B\cap A_2) + P(B\cap A_3) + P(B\cap A_4)$$



$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) + P(B|A_4)P(A_4)$$

Bayes' Theorem

$$P(B_i|A) = \frac{P(B_i\cap A)}{P(A)} = \frac{P(B_i)P(A|B_i)}{P(B_1)P(A|B_1) + \cdots + P(B_k)P(A|B_k)}$$

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حدثين		حدث واحد		
or = at least	$\cup$	at least	$\geq$	الشرط
and = both	$\cap$	exactly	$=$	"
not	$(\ )^c$	atmost	$\leq$	"
not both = atmost	$(\ \cap\ )^c$			
neither nor	$(\ \cup\ )^c$			

لاحظ الفرق عند التعامل مع الاحتمالات غير متساويه او متزنه (not fair)
نفرض المثالين

2 coin (not fair)

say $P(H)=\frac{1}{3}$, $P(T)=\frac{2}{3}$

$S=\{\overset{\frac{1}{3}\cdot\frac{1}{3}}{HH}, \overset{\frac{1}{3}\cdot\frac{2}{3}}{HT}, \overset{\frac{2}{3}\cdot\frac{1}{3}}{TH}, \overset{\frac{2}{3}\cdot\frac{2}{3}}{TT}\}$

$P(HH)=\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$, $P(HT)=\frac{1}{3}\times\frac{2}{3}=\frac{2}{9}$, ...

let $A=\{HH, TH\}$

$P(A)=P(HH)+P(TH)=\frac{1}{3}\times\frac{1}{3}+\frac{2}{3}\times\frac{1}{3}=\frac{3}{9}=\frac{1}{3}$

$P(A)\neq\frac{N(A)}{N(S)}=\frac{1}{2}$

2 fair coin

$P(H)=P(T)=\frac{1}{2}$

$S=\{\overset{\frac{1}{2}\cdot\frac{1}{2}}{HH}, \overset{\frac{1}{2}\cdot\frac{1}{2}}{HT}, \overset{\frac{1}{2}\cdot\frac{1}{2}}{TH}, \overset{\frac{1}{2}\cdot\frac{1}{2}}{TT}\}$

$P(HH)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$, $P(HT)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$, ...

let $A=\{HH, TH\}$

$P(A)=P(HH)+P(TH)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$

or $P(A)=\frac{N(A)}{N(S)}=\frac{2}{4}=\frac{1}{2}$

Exercise (1) Page 23 / /

1. A Event That a person attends College, and B Event That a person speaks Deutsch

(a) person doesn't speak Deutsch B^c

(b) " speaks Deutsch and does not attend College $B \cap A^C$

(c) " is either in College or speaks Deutsch $A \cup B$

2. $P(A) = 0.6$ $P(B) = 0.3$ $P(A \cap B) = 0.1$

(a) probability That A or B occurs $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.3 - 0.1 = 0.8$

(b) probability That at most one of The two events A and B occurs

$$P((A \cap B)^c) = 1 - P(A \cap B) = 1 - 0.1 = 0.9$$

(c) probability That neither A nor B occurs $P(A \cup B)^c = 1 - P(A \cup B) = 1 - 0.8 = 0.2$

3. $P(A) = 0.4$ $P(B) = 0.5$ $P(A \cap B) = 0.3$

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.3 = 0.6$

(b) $P(A \cap B^c) = P(A - B) = P(A) - P(A \cap B) = 0.4 - 0.3 = 0.1$

(c) $P(A^c \cup B^c) = P(A \cap B)^c = 1 - P(A \cap B) = 1 - 0.3 = 0.7$

4. $P(A \cup B) = 0.76$ $P(A \cup B^c) = 0.87$ Find $P(A)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (1)$$

$$P(A \cup B^c) = P(A) + P(B^c) - P(A \cap B^c) = P(A) + 1 - P(B) - (P(A) - P(A \cap B))$$

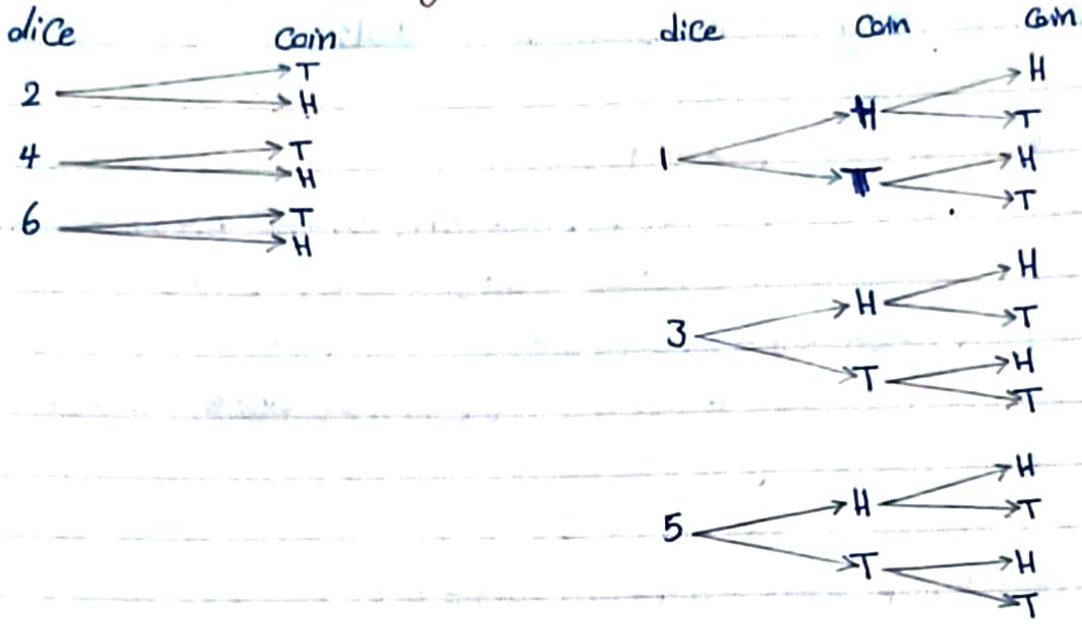
$$= \cancel{P(A)} + 1 - P(B) - \cancel{P(A)} + P(A \cap B) = 1 - P(B) + P(A \cap B) \quad (2)$$

بجمع (1) و(2)

$$P(A \cup B) + P(A \cup B^c) = P(A) + \cancel{P(B)} - \cancel{P(A \cap B)} + 1 - \cancel{P(B)} + \cancel{P(A \cap B)} = P(A) + 1$$

$$0.76 + 0.87 = P(A) + 1 \Rightarrow P(A) = 1.63 - 1 = 0.63$$

5. If the number ~~of~~ on the dice is even, the coin is ~~flipping~~ flipped once. If the number on the dice is odd, the coin is flipped twice. ~~Find~~ construct a tree diagram to show the elements of the sample space S.



$S = \{1HH, 1HT, 1TH, 1TT, 2H, 2T, 3HH, 3HT, 3TH, 3TT, 4H, 4T, 5HH, 5HT, 5TH, 5TT, 6H, 6T\}$

6. A and B are independent show That
(1) A and B^c are independent (2) A^c and B^c are independent

A and B are independent Then

$P(A \cap B) = P(A) \cdot P(B)$

(1) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)\, P(B^c)$

(2) $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B))$

$= 1 - P(A) - P(B) + P(A \cap B) = \underline{1 - P(A)} - \underline{P(B) + P(A)P(B)}$

$= (\underline{1 - P(A)}) - P(B)\,(\underline{1 - P(A)}) = (1 - P(A))(1 - P(B)) = P(A^c)\, P(B^c)$

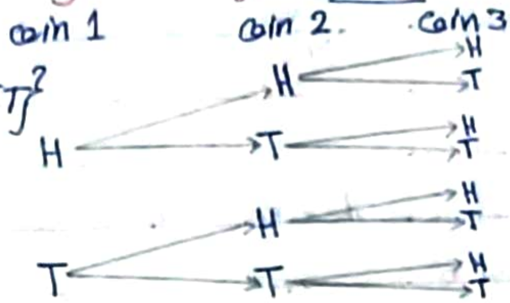
or $P(A^c \cap B^c) = P(A^c) - P(A^c \cap B) = P(A^c) - P(B \cap A^c) = P(A^c) - (P(B) - P(B \cap A))$

$= P(A^c) - P(B) + P(B)P(A) = P(A^c) - P(B)(1 - P(A)) = P(A^c) - P(B)P(A^c) = P(A^c)(1 - P(B))$

$= P(A^c)P(B^c)$

7 Toss Three fair coins

(a) probability of having at least one head, (b) probability of having exactly one head



$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(a) at least one head

$A = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$

$P(A) = \frac{7}{8}$

(b) exactly one head

$B = \{HTT, THT, TTH\}$, $P(B) = \frac{3}{8}$

8 Roll two fair dice

(a) probability that they do not show the same face

(b) probability that sum is 7 ?

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$

$(3,1), (3,2), \ldots, (3,6), (4,1), (4,2), \ldots, (4,6)$

$(5,1), (5,2), \ldots, (5,6), (6,1), (6,2), \ldots, (6,6)\}$

(a) A event does not show the same face

$A = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,4), (3,5)$

$(3,6), (4,1), (4,2), (4,3), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,6), (6,1), (6,2)$

$(6,3), (6,4), (6,5)\}$ $N(A) = 5 \times 6 = 30$

$P(A) = \frac{N(A)}{N(S)} = \frac{30}{36} = \frac{5}{6}$

(b) B event that The sum is 7

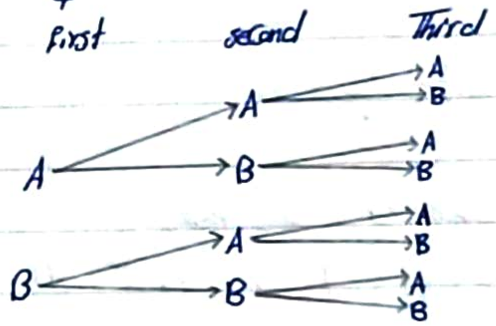
$B = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$ $N(B) = 6$

$P(B) = \frac{N(B)}{N(S)} = \frac{6}{36} = \frac{1}{6}$

$\boxed{9}$

let A be The event that a child has blue eyes $P(A) = \frac{1}{4}$ معطى

let B " " " " has not blue eyes $P(B) = \frac{3}{4}$



$S = \{\overset{\frac{1}{64}}{AAA}, \overset{\frac{3}{64}}{AAB}, \overset{\frac{3}{64}}{ABA}, \overset{\frac{9}{64}}{ABB}, \overset{\frac{3}{64}}{BAA}, \overset{\frac{9}{64}}{BAB}, \overset{\frac{9}{64}}{BBA}, \overset{\frac{27}{64}}{BBB}\}$.

$P(AAA) = P(A)P(A)P(A) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$

$P(AAB) = P(A)\,P(A)\,P(B) = \frac{1}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{3}{64}$, ---

وذلك لان الحدثين A و B مستقلين

(a) C event is to be at least one child has blue eyes $C = \{AAA, AAB, ABA, ABB, BAA, BAB, BBA\}$

D event is to be at least two children have blue eyes $D = \{AAA, AAB, ABA, BAA\}$

$$P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{P(D)}{P(C)} = \frac{P(AAA) + P(AAB) + P(ABA) + P(BAA)}{P(AAA) + P(AAB) + \cdots + P(BAB) + P(BBA)}$$

$$= \frac{\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{1}{4}\times\frac{3}{4} + \frac{1}{4}\times\frac{3}{4}\times\frac{1}{4} + \frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}}{\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{1}{4}\times\frac{3}{4} + \frac{1}{4}\times\frac{3}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{3}{4}\times\frac{3}{4} + \frac{3}{4}\times\frac{1}{4}\times\frac{1}{4} + \frac{3}{4}\times\frac{1}{4}\times\frac{3}{4} + \frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}}$$

$$\text{or} = \frac{(\frac{1}{4})^3 + 3\times(\frac{1}{4})^2 \times \frac{3}{4}}{(\frac{1}{4})^3 + 3(\frac{1}{4})^2 \times \frac{3}{4} + 3(\frac{1}{4})\times(\frac{3}{4})^2} = \frac{10}{37}$$

نلاحظ ان $D \cap C = D$

(b) E event is to be youngest child has blue eyes $E = \{AAA, ABA, BAA, BBA\}$

$$P(D|E) = \frac{P(D\cap E)}{P(E)} \qquad D\cap E = \{AAA, ABA, BAA\}$$

$$= \frac{P(AAA)+P(ABA)+P(BAA)}{P(AAA)+P(ABA)+P(BAA)+P(BBA)} = \frac{(\frac{1}{4})^3+2(\frac{1}{4})^2(\frac{3}{4})}{(\frac{1}{4})^3+2(\frac{1}{4})^2(\frac{3}{4})+(\frac{1}{4})(\frac{3}{4})^2} = \frac{7}{16}$$

[10] Factory A produces 80 % of the products with proportion of defectives 0.05

" B " 20% " " " " " 0.01

let D event be defective products

$P(A) = \frac{80}{100} = 0.8 \quad P(D|A) = 0.05, \quad P(B) = \frac{20}{100} = 0.2 \quad P(D|B) = 0.01$

(a) probability that a product picked at random comes from A and is not defective

$P(A\cap D^c) = P(A) - P(A\cap D) = P(A) - P(D|A)P(A) = 0.8 - 0.05\times 0.8 =$

(b) Probability that a product picked at random is defective

$P(D) = P(D|A)P(A) + P(D|B)P(B) = 0.05\times 0.8 + 0.01\times 0.2 =$

[11] Two dice are rolled, Given the sum is 9, what the probability that at least one dice showed 6?

$S = \{(1,1),(1,2) \dots,(1,6),(2,1),(2,2),\dots,(2,6),(3,1),(3,2),\dots,(3,6),\dots,(4,1),(4,2),\dots,(4,6)$
$(5,1),(5,2),\dots,(5,6),(6,1),(6,2),\dots,(6,6)\}$

A event that the sum is 9, B event that at least one dice showed 6

~~P(B|A)~~ $A = \{(3,6),(4,5),(5,4),(6,3)\}$ $\quad B = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

$B\cap A = \{(3,6),(6,3)\}$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{2/36}{4/36} = \frac{2}{4} = \frac{1}{2}$$

يمكن حلها بطريقة اخرى أولاً نوجد الشرط المعطى given

$A = \{(3,6), (4,5), (5,4), (6,3)\}$ ٤ عناصر

ثم نوجد العناصر التي تحقق الشرط الثاني B في A وهما $\underbrace{(3,6),(6,3)}_{\text{عنصرين}}$ ثم نوجد النسبة بين العناصر الى عدد عناصر A

$$P(B|A) = \frac{2}{4} = \frac{1}{2}$$