Section (1) / /

مراجعات التكامل

$$\int (f(x) + g(x) + h(x))\,dx = \int f(x)\,dx + \int g(x)\,dx + \int h(x)\,dx$$

$$\int a\,dx = ax + C \quad : \int (-7)\,dx = -7x + C$$

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + C \quad : \int (3x + \pi x^5 - 16x^{-3})\,dx \qquad n \neq -1$$

$$= 3\frac{x^2}{2} + \pi\frac{x^6}{6} - 16\frac{x^{-2}}{-2} + C$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C : \int (3x-2)^7\,dx = \frac{(3x-2)^8}{(3)(8)} + C$$

$$\int (f(x))^n \cdot f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1}\,dx : \int (x^4 - x^2)^3 \cdot (4x^3 - 2x)\,dx$$

$$= \frac{(x^4 - x^2)^4}{4} + C$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C \quad : \int \frac{-2x^{-3} - 1}{\sqrt{x^{-2} - x}}\,dx = 2\sqrt{x^{-2} - x} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C : \int \frac{2x^3 + x}{x^4 + x^2}\,dx = \frac{1}{2}\int \frac{4x^3 + 2x}{x^4 + x^2}\,dx$$

$$= \frac{1}{2}\ln|x^4 + x^2| + C$$

$$\int e^{f(x)} \cdot f'(x)dx = e^{f(x)} + C \quad : \int \frac{e^{1/x}}{x^2}dx = -\int e^{1/x} \cdot \frac{-1}{x^2}\,dx = -e^{1/x} + C$$

$$\int a^{f(x)} \cdot f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + C : \int \pi^{x^2} \cdot x\,dx = \frac{1}{2}\int \pi^{x^2} \cdot 2x\,dx$$

$$= \frac{1}{2}\,\frac{\pi^{x^2}}{\ln \pi} + C$$

$$\int \sin(f(x)) \cdot f'(x)\,dx = -\cos(f(x)) + C$$

$$\int \cos(f(x)) \cdot f'(x)\,dx = \sin(f(x)) + C$$

$$: \int e^{x} \sin(e^{x})\,dx = -\cos(e^{x}) + C$$

$$: \int \frac{\cos(\ln x)}{x}\,dx = \sin(\ln x) + C$$

$$\int \tan(f(x)) \cdot f'(x)\,dx = \int \frac{\sin(f(x))}{\cos(f(x))} \cdot f'(x)dx = -\ln|\cos(f(x))| + C$$

$$\int \cot(f(x)) \cdot f'(x)\,dx = \int \frac{\cos(f(x)) \cdot f'(x)}{\sin(f(x))}\,dx = \ln|\sin(f(x))| + C$$

$$: \int \tan(x^{-4}) \cdot x^{-5}\,dx = +\frac{1}{4}\ln|\cos(x^{-4})| + C$$

$$\int \cot(e^{x^2+1}) \cdot x e^{x^2+1} = \frac{1}{2}\ln|\sin(e^{x^2+1})| + C$$

$$\int \sec^2(f(x)) \cdot f'(x)\,dx = \tan(f(x)) + C$$

$$\int \operatorname{cosec}^2(f(x)) \cdot f'(x)\,dx = -\cot(f(x)) + C$$

$$: \int \sec^2(\sqrt{x+1})\,\frac{1}{2\sqrt{x+1}}\,dx = 2\tan(\sqrt{x+1}) + C$$

$$: \int \operatorname{cosec}^2(e^{\sqrt{x}}) \cdot e^{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\,dx = -2\cot(e^{\sqrt{x}}) + C$$

$$\int \sec(f(x)) \cdot \tan(f(x)) \cdot f'(x)\,dx = \sec(f(x)) + C$$

$$\int \csc(f(x)) \cdot \cot(f(x)) \cdot f'(x)\,dx = -\csc(f(x)) + C$$

$$: \int \frac{\sec(\sqrt[3]{x})\tan(\sqrt[3]{x})}{\sqrt[3]{x^2}}\,dx = 3\sec(\sqrt[3]{x}) + C$$

$$: \int \csc(\ln(\sin x))\cot(\ln(\sin x)) \cdot \frac{\cos x}{\sin x}\,dx$$

$$= -\csc(\ln(\sin x)) + C$$

/ /

$$\int \frac{f'(x)}{\sqrt{a^2-(f(x))^2}}\,dx = \sin^{-1}\left(\frac{f(x)}{a}\right)+C \quad \left(or = -\cos^{-1}\left(\frac{f(x)}{a}\right)+C\right)$$

$$\int \frac{f'(x)}{a^2+(f(x))^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right)+C \quad \left(or = \frac{-1}{a}\cot^{-1}\left(\frac{f(x)}{a}\right)+C\right)$$

$$\int \frac{f'(x)}{f(x)\sqrt{(f(x))^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{f(x)}{a}\right)+C \quad \left(or = \frac{-1}{a}\csc^{-1}\left(\frac{f(x)}{a}\right)+C\right)$$

: $$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right)+C$$

$$= -\cos^{-1}\left(\frac{x}{2}\right)+C$$

: $$\int \frac{\cos x}{2+\sin^2 x}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x}{\sqrt{2}}\right)+C$$

: $$\int \frac{2x}{x^2\sqrt{x^4-1}}\,dx = \frac{1}{1}\sec^{-1}(x^2)+C$$

: $$\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}\,dx = \sec^{-1}(e^x)+C$$

بعض القوانين والعلاقات الهامة

$a>0$	$a<0$
$a^0=1$	$a^0=1$
$(a>1)$ $a^{\infty}=\infty$	
$(a<1)$ $a^{\infty}=0$	$a^{\infty}=0$ $(a>-1)$
$a^{-\infty}=0$	$a^{-\infty}=0$
$\frac{a}{0}=\infty$	$\frac{a}{0}=-\infty$
$\frac{a}{\infty}=0$	$\frac{a}{\infty}=0$
$\frac{a}{-\infty}=0$	$\frac{a}{-\infty}=0$

الداله تكون زوجيه (even)

$f(-x) = f(x)$ EX: $\cos(-x) = \cos x$

الداله تكون فرديه (odd)

$f(-x) = -f(x)$ EX: $\sin(-x) = -\sin(x)$, $\tan(x)$

الداله ليست زوجيه وليست فرديه

$f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

EX : $f(x) = x^2 + x$ $\quad f(-x) = x^2 - x \neq f(x)$

$\neq -f(x)$

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1}{2}[1 - \cos 2x]$$

$$\cos^2 x = \frac{1}{2}[1 + \cos 2x]$$

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \qquad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

التكامل المحدود

$$\int_a^b e^{-x}\,dx = -e^{-x}\Big|_a^b = -[e^{-b} - e^{-a}]$$

اذا فرضنا ان $\int f(x)\,dx = g(x)$ فإن

$$\int_a^b f(x)\,dx = g(x)\Big|_a^b = g(b) - g(a)$$

نلاحظ اننا لا نضع ثابت التكامل C

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \,, \qquad a \leqslant c \leqslant b$$

if $f(x) \leqslant g(x)$ $\forall x \in [a,b]$ $\Rightarrow \int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$

if $f(x)$ is even $\Rightarrow \int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

if $f(x)$ is odd $\Rightarrow \int_{-a}^a f(x)\,dx = 0$

$$\int_0^{\pi/2} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi/2} = -\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right]$$

$$= -\left[0 - 1\right] = 1$$

$$\int_0^{-\infty} e^{x}\,dx = e^{x}\Big|_0^{-\infty} = e^{-\infty} - e^{0} = 0 - 1 = -1$$

$$\int_{-\infty}^{2} \frac{1}{x^3}\,dx = \int_{-\infty}^{2} x^{-3}\,dx = \frac{1}{-2}x^{-2}\Big|_{-\infty}^{2} = -\frac{1}{2}\left[\frac{1}{2^2} - \frac{1}{(-\infty)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{4} - \frac{1}{\infty}\right] = -\frac{1}{8}$$

$$\int_{-5}^{5} x^3\,dx = \frac{x^4}{4}\Big|_{-5}^{5} = \frac{1}{4}\left[5^4 - (-5)^4\right] = \frac{1}{4}\left[5^4 - 5^4\right] = 0$$

$$\int_{-\pi/4}^{\pi/4} \cos x\,dx = \sin x\Big|_{-\pi/4}^{\pi/4} = \sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right)$$

$$= \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) = 2\sin\left(\frac{\pi}{4}\right) = 2\cdot\frac{1}{\sqrt{2}} = \sqrt{2}$$

التكامل بالتجزئ

Integration by parts

$$d(uv) = du \cdot v + u \cdot dv \Rightarrow u\,dv = d(uv) - v\,du$$

$$\int u\,dv = uv - \int v\,du$$

عند التكامل بالتجزئ نقوم بتجزئه التكامل الى جزئين
احدهما u والاخر dv
ونختار عادة الجزء الذى يصعب تكامله على انه u

اختيار u و dv عادة يتم كالتالى

(١) $\int$ كثيرة حدود × مثلثيه dx ⇐ $dv =$ مثلثيه dx ، $u =$ كثيره حدود

اسس صحيحه موجبه

(٢) $\int$ كثيره حدود × اسيه dx ⇐ $dv =$ اسيه dx ، $u =$ كثيره حدود

(٣) $\int$ كثيره حدود × لوغارتميه dx ⇐ $dv =$ كثيره حدود dx ، $u =$ لوغارتميه
كذلك الداله اللوغارتميه مع الدوال الاخرى

(٤) $\int$ كثيره حدود × مثلثيه عكسيه dx ⇐ $dv =$ كثيره حدود dx ، $u =$ المثلثيه العكسيه
كذلك الداله المثلثيه العكسيه مع الدوال الاخرى

(٥) $\int$ مثلثيه × اسيه dx ⇐ نختار اى داله بـ u والاخرى dv

(1)

$$I = \int x^2 \sin x \, dx$$

$$u = x^2 \qquad dv = \sin x \, dx$$
$$du = 2x\,dx \qquad v = -\cos x$$

$$I = -x^2 \cdot \cos x - \int -2x \cos x \, dx$$

$$= -x^2 \cos x + 2 \int x \cos x \, dx$$

$$u = x \qquad dv = \cos x \, dx$$
$$du = dx \qquad v = \sin x$$

$$I = -x^2 \cos x + 2\left[x \sin x - \int \sin x \, dx \right]$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

حل آخر

u		dv
x^2	$+$	$\sin x$
$2x$	$-$	$-\cos x$
2	$+$	$-\sin x$
0		$\cos x$

$$I = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

/ /

(2.)

$$I = \int x^3 \ln(x)\, dx$$

$$u = \ln x \xrightarrow{+} dv = x^3 dx$$
$$du = \frac{1}{x} dx \xleftarrow{-} v = \frac{x^4}{4}$$

$$I = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \cdot \frac{1}{x}\, dx = \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \frac{x^4}{4} + C$$

$\cancel{I = \int (x^3 - 5x + 7)\, dx}$ (3) $I = \int (x^3 - 5x + 7) e^{-x}\, dx$

$\cancel{I = \int x^3 e^{-x}\, dx}$

u		dv
$x^3 - 5x + 7$	$+$	e^{-x}
$3x^2 - 5$	$-$	$-e^{-x}$
$6x$	$+$	e^{-x}
6	$-$	$-e^{-x}$
0		e^{-x}

$$I = -(x^3 - 5x + 7)e^{-x} - (3x^2 - 5)e^{-x} - 6xe^{-x} - 6e^{-x} + C$$

(4) $I = \int \tan^{-1} x\, dx$

$$u = \tan^{-1} x \xrightarrow{+} dv = dx$$
$$du = \frac{1}{1+x^2} dx \xleftarrow{-} v = x$$

$$I = x \tan^{-1} x - \int x \cdot \frac{1}{1+x^2}\, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2}\, dx = x \tan^{-1} x - \frac{1}{2} \ln |1 + x^2| + C$$

/ /

Laplace transformation

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\,dt = f(s)$$

$$\mathcal{L}\{C_1 F_1(t) + C_2 F_2(t)\} = C_1 f_1(s) + C_2 f_2(s)$$

$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$	
1	$\frac{1}{s}$	$s>0$
t^n	$\frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$\sin at$	$\frac{a}{s^2+a^2}$	$s>0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s>0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s>\lvert a\rvert$

$\Gamma(n+1) = n\Gamma(n)$ $\quad$ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(n+1) = n(n-1)(n-2)\Gamma(n-2)$

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a)$$

$$\mathcal{L}\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right)$$

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) \quad , \quad \mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

$$\mathcal{L}\{F'(t)\} = s f(s) - F(0) \quad , \quad \mathcal{L}\left\{\int_0^t F(t)\,dt\right\} = \frac{f(s)}{s}$$

$$\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$$

$$\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - sF^{(n-2)}(0) - F^{(n-1)}(0)$$

/ /

(1) $\mathcal{L}\{\sin(at)\}$

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\mathcal{L}\{\sin(at)\} = \mathcal{L}\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\} = \frac{1}{2i}\left[\mathcal{L}\{e^{iat}\} - \mathcal{L}\{e^{-iat}\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{S-ia} - \frac{1}{S+ia}\right] = \frac{1}{2i}\,\frac{S+ia-(S-ia)}{(s-ia)(S+ia)}$$

$$= \frac{1}{2i}\,\frac{2ia}{S^2+a^2} = \frac{a}{S^2+a^2}$$

(2) $\mathcal{L}\{\cosh(at)\}$

$$\cosh(at) = \frac{e^{at} + e^{-at}}{2}$$

$$\mathcal{L}\{\cosh(at)\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\left[\mathcal{L}\{e^{at}\} + \mathcal{L}\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{S-a} + \frac{1}{S+a}\right] = \frac{1}{2}\,\frac{S+a+S-a}{(S-a)(S+a)}$$

$$= \frac{1}{2}\,\frac{2S}{S^2-a^2} = \frac{S}{S^2-a^2}$$

(3) $\mathcal{L}\{e^{5t} + 2t^4 + 6\cos(3t)\}$

$$= \mathcal{L}\{e^{5t}\} + 2\mathcal{L}\{t^4\} + 6\mathcal{L}\{\cos(3t)\}$$

$$= \frac{1}{S-5} + 2\frac{4!}{S^5} + 6\frac{S}{S^2+9}$$

(4) $\mathcal{L}\{6e^{-7t} - t^{+5} + t^{3/2} - 6\sin 3t - \sinh 2t\}$

$$= 6\mathcal{L}\{e^{-7t}\} - \mathcal{L}\{t^{+5}\} + \mathcal{L}\{t^{3/2}\} - 6\mathcal{L}\{\sin 3t\} - \mathcal{L}\{\sinh 2t\}$$

$$= 6\,\frac{1}{S-(-7)} - \frac{5!}{S^6} + \frac{\Gamma(\frac{5}{2})}{S^{5/2}} - 6\,\frac{3}{S^2+9} - \frac{2}{S^2-4}$$

$$= \frac{6}{S+7} - \frac{120}{S^6} + \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{S^{5/2}} - \frac{18}{S^2+9} - \frac{2}{S^2-4}$$

(5) $\mathcal{L}\{\sin^2 4t\}$

$$\mathcal{L}\{\sin^2 4t\} = \mathcal{L}\{\tfrac{1}{2}(1-\cos 8t)\} = \tfrac{1}{2}\left[\mathcal{L}\{1\} - \mathcal{L}\{\cos 8t\}\right]$$

$$= \tfrac{1}{2}\left[\frac{1}{S} - \frac{S}{S^2+64}\right]$$