Laplace Transforms

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)\, e^{-st}\, dt, \quad s \in \mathbb{R},\ t > 0$$

$$\mathcal{L}(1) = \frac{1}{s}, \quad s > 0 \qquad \mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}, \quad s > 0$$

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0 \qquad \mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}, \quad s > 0$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad s > 0 \qquad \mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}, \quad s > |a|$$

$$\mathcal{L}(e^{at}) = \frac{1}{s - a}, \quad s > a \qquad \mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2}, \quad s > |a|$$

Laplace Transforms

1. **Linearity**

$$\mathcal{L}(a f(t) + b g(t)) = a \mathcal{L}(f(t)) + b \mathcal{L}(g(t))$$

2. **First shifting property**

If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at} f(t)) = F(s-a)$

Ex $\mathcal{L}(e^{at} t^n)$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad s > 0$$

$$\mathcal{L}(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}, \quad s > a$$

Ex $\mathcal{L}(e^{at} \sinh bt)$

$$\mathcal{L}(\sinh bt) = \frac{b}{s^2 - b^2}, \quad s > |b|$$

$$\mathcal{L}(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$$

$s - a > |b|$

Laplace Transforms

Ex) $\mathcal{L}(e^{at}\cos bt)$

$\mathcal{L}(\cos bt) = \dfrac{s}{s^2+b^2}, \; s>0$

$\mathcal{L}(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}, \; s>a$

$\mathcal{L}(\cos 5t) = \dfrac{s}{s^2+25}, \; s>$

$\mathcal{L}(\sin 5t) = \dfrac{5}{s^2+25}, \; s>0$

Ex) $\mathcal{L}(e^{-3t}\, 2\cos 5t - 3\sin 5t)$

$= 2\,\mathcal{L}(e^{-3t}\cos 5t) - 3\,\mathcal{L}(\sin 5t)$

$= 2 \cdot \dfrac{s+3}{(s+3)^2+25} - 3 \cdot \dfrac{5}{(s+3)^2+25}$

$s > -3$

Laplace Transforms

Ex) $\mathcal{L}\left(e^{-t}\left(3\sinh 2t-5\cosh 2t\right)\right)$

Sol
$$=3\,\mathcal{L}\left(e^{-t}\sinh 2t\right)-5\,\mathcal{L}\left(e^{-t}\cosh 2t\right)$$

$$\mathcal{L}(\sinh 2t)=\frac{2}{s^2-4},\quad s>2$$

$$\mathcal{L}(\cosh 2t)=\frac{s}{s^2-4},\quad s>2$$

$$=3\cdot\frac{2}{(s+1)^2-4}-5\cdot\frac{s+1}{(s+1)^2-4},\quad s>-1$$

Laplace Transforms

Ex) If $\mathcal{L}(f(t)) = F(s)$, find $\mathcal{L}(\sinh at\, f(t))$

$$= \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2} f(t) \right)$$

$$= \frac{1}{2}\left[\mathcal{L}(e^{at} f(t)) - \mathcal{L}(e^{-at} f(t)) \right]$$

$$= \frac{1}{2}\left[F(s-a) - F(s+a) \right]$$

Laplace Transforms

Second shifting property

If $\mathcal{L}(f(t)) = F(s)$, $\mathcal{L}(g(t))$: $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

$$\boxed{\mathcal{L}(g(t)) = e^{-as} F(s)}$$

Ex) Find $\mathcal{L}(g(t))$: $g(t) = \begin{cases} \cos(t - \frac{\pi}{2}), & t > \frac{\pi}{2} \\ 0, & t < \frac{\pi}{2}. \end{cases}$

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1}, \quad s > 0$$

$$\mathcal{L}(g(t)) = e^{-\frac{\pi}{2}s} \frac{s}{s^2+1}$$

Laplace Transforms

Second shifting property

Ex Find $\mathcal{L}(g(t))$: $g(t) = \begin{cases} (t-2)^3 & , t > 2 \\ 0 & , t < 2 \end{cases}$

$$\mathcal{L}(t^3) = \frac{3!}{s^4}, s > 0$$

$$= \frac{6}{s^4}$$

$$\mathcal{L}(g(t)) = e^{-2s} \cdot \frac{6}{s^4}, s > 0$$

Laplace Transforms

Change of Scale

If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$

Ex: If $\mathcal{L}\left(\frac{\sin t}{t}\right) = \tan^{-1}\frac{1}{s}$, find $\mathcal{L}\left(\frac{\sin at}{at}\right)$

$$= \frac{1}{a} \tan^{-1}\frac{1}{\frac{s}{a}} = \frac{1}{a} \tan^{-1}\frac{a}{s}$$

Laplace Transforms

Change of Scale

If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$

Ex) If $\mathcal{L}\left(\frac{\sin t}{t}\right) = \tan^{-1}\frac{1}{s}$, find $\mathcal{L}\left(\frac{\sin at}{t}\right)$

$$\mathcal{L}\left(\frac{\sin at}{at}\right) = \frac{1}{a}\tan^{-1}\frac{a}{s}$$

a) $\mathcal{L}\left(\frac{\sin at}{at}\right) = \tan^{-1}\frac{a}{s}$

$\mathcal{L}\left(\frac{\sin at}{t}\right) = \tan^{-1}\frac{a}{s}$

Laplace Transforms

Laplace Transform of derivatives

If $\mathcal{L}(f(t)) = F(s)$, then

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

Ex: Use $\mathcal{L}(f'(t))$ to compute $\mathcal{L}(t^{n+1})$

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$f(t) = t^{n+1}$, $f(0) = 0$, $f'(t) = (n+1)t^n$

$$\mathcal{L}((n+1)t^n) = s\mathcal{L}(f(t))$$

$$(n+1)\mathcal{L}(t^n) = s\mathcal{L}(f(t))$$

$$(n+1)\frac{n!}{s^{n+1}} = s\mathcal{L}(f(t))$$

$$\mathcal{L}(f(t)) = \frac{(n+1)!}{s^{n+2}}$$

Laplace Transforms

Laplace Transform of derivatives

Ex: $\mathcal{L}(t \sin bt)$ Use $\mathcal{L}(f''(t))$.

Sol: $f(t) = t \sin bt$ $\qquad f(0) = 0$

$f'(t) = bt \cos bt + \sin bt \cdot 1$ $\qquad f'(0) = 0$

$f''(t) = -b^2 t \sin bt + b\cos bt + b\cos bt$

$\mathcal{L}(f''(t)) = -b^2 \mathcal{L}(t \sin bt) + 2b\,\mathcal{L}(\cos bt)$ $\qquad \rightarrow \mathcal{L}(t \sin t) = \dfrac{2bs}{(s^2+b^2)^2}$

$\parallel$

$s^2 \mathcal{L}(t \sin bt)$

$(s^2+b^2)\,\mathcal{L}(t \sin bt) = 2b \cdot \dfrac{s}{s^2+b^2}$

Laplace Transforms

Laplace Transform of Integral

If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{F(s)}{s}$

Ex) If $\mathcal{L}\left(\frac{\sin t}{t}\right) = \tan^{-1}\frac{1}{s}$

Find $\mathcal{L}\left(\int_0^t \frac{\sin u}{u}\,du\right)$.

$= \frac{1}{s}\tan^{-1}\frac{1}{s}$

$-$

$$\boxed{\mathcal{L}\{f(t)\}} = \int_0^\infty e^{-st} f(t)\,dt = \boxed{F(s)}$$

1) $\mathcal{L}\{1\} = \frac{1}{s}$

2) $\mathcal{L}\{t^b\} = \frac{\Gamma(b+1)}{s^{b+1}}$

3) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

4) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

5) $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

6) $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

7) $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$

8) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

1) First Shifting Theorem $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

2) Second Shifting If $\boxed{\mathcal{L}\{f(t)\} = F(s)}$

$$\Rightarrow \mathcal{L}\{g(t)\} = e^{-as} F(s)$$

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

3) Change of scale property:

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

4) Laplace Transforms of Derivative

a) $L(f') = sL(f) - f(0)$

b) $L(f'') = s^2 L(f) - s f(0) - f'(0)$

c) $L(f''') = s^3 L(f) - s^2 f(0) - s f'(0) - f''(0)$