# Inverse Laplace Transforms

If $\mathcal{L}$ ... $\mathcal{L}^{-1}(F(s)) = f(t)$

$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$ $\qquad$ $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

$\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$ $\qquad$ $\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

$\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$

# Inverse Laplace Transforms

① Linearity $\mathcal{L}^{-1}(aF(s)+bG(s)) = a\,\mathcal{L}^{-1}(F(s)) + b\,\mathcal{L}^{-1}(G(s))$

Ex $\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\sin at$

Ex $\mathcal{L}^{-1}\left(\frac{3}{s-5}\right) = 3\,\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) = 3e^{5t}$

② First Shifting $\mathcal{L}^{-1}(F(s-a)) = e^{at} f(t)$

Ex $\mathcal{L}^{-1}\left(\frac{b}{(s-a)^2-b^2}\right) = e^{at}\sinh bt$

# Inverse Laplace Transforms

3 Second Shifting $\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

Ex: $\mathcal{L}^{-1}\left(\dfrac{e^{-\frac{\pi}{3}s}}{s^2+4}\right)$.

$\mathcal{L}^{-1}\left(\dfrac{1}{s^2+4}\right) = \dfrac{1}{2}\mathcal{L}^{-1}\left(\dfrac{2}{s^2+4}\right)$

$= \dfrac{1}{2}\sin 2t$

$y = \begin{cases} \dfrac{1}{2}\sin 2\left(t-\dfrac{\pi}{3}\right) & t > \dfrac{\pi}{3} \\ 0 & t < \dfrac{\pi}{3} \end{cases}$

# Inverse Laplace Transforms

4 Change of scale

$$\mathcal{L}^{-1}\left(\frac{1}{a}F\left(\frac{s}{a}\right)\right) = f(at)$$

$$\mathcal{L}^{-1}\left(F(s/a)\right) = a\, f(at).$$

Ex

$$\mathcal{L}^{-1}\left(\frac{b}{\left(\frac{s}{a}\right)^2 + b^2}\right) = a \cdot \sin abt$$

$$\mathcal{L}^{-1}\left(\frac{b}{s^2 + b^2}\right) = \sin bt = f(t)$$

$$f(at) = \sin abt$$

# Inverse Laplace Transforms

## 4 Change of scale

Ex $\mathcal{L}^{-1}\left(\dfrac{s}{\left(\frac{s}{7}\right)^2+9}\right) = 7\,\mathcal{L}^{-1}\left(\dfrac{s/7}{\left(\frac{s}{7}\right)^2+9}\right)$

$\mathcal{L}^{-1}\left(\dfrac{s}{s^2+9}\right) = \cos 3t = f(t)$

$7\,\mathcal{L}^{-1}\left(\dfrac{s/7}{\left(s/7\right)^2+9}\right) = 7 \times 7 \cos 3 \times 7 t$

$= 49 \cos 21t.$

# Inverse LaPlace Transforms

5) $$\mathcal{L}^{-1}\left(\frac{\boxed{F(s)}}{s}\right) = \int_0^t f(u)\,du$$

Ex) $$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{s^2+a^2}}{s}\right) \qquad \frac{1}{s^2+a^2} = F(s)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\cdot \sin at = f(t)$$

$$= \frac{1}{a}\int_0^t \sin au\,du = -\frac{1}{a^2}\cos au\Big|_0^t = -\frac{1}{a^2}(\cos at - 1)$$

# Inverse Laplace Transforms

5) $\mathcal{L}^{-1}\left(\frac{\boxed{F(s)}}{s}\right) = \int_0^t f(u)\,du \quad \ldots F(s)$

Ex) $\mathcal{L}^{-1}\left(\frac{1}{s^3+s}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{s^2+1}}{s}\right)$

$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = f(t)$

Ex) $\mathcal{L}^{-1}\left(\frac{1}{s^4+s^2}\right)$

$= \int_0^t \sin u\,du = -\cos u\Big|_0^t = -(\cos t - 1)$

# Inverse Laplace Transforms

6 $\mathcal{L}^{-1}\left((-1)^n \frac{d^n}{ds^n} F(s)\right) = t^n f(t)$

$$\boxed{\mathcal{L}^{-1}\left(\frac{d}{ds} F(s)\right) = -t\, f(t).}$$

Ex $\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)^2}\right) = \frac{1}{4} t \sin 2t.$

$$\frac{d}{ds}\frac{1}{s^2+4} = \frac{-1}{(s^2+4)^2}\cdot 2s = \frac{-2s}{(s^2+4)^2}$$

$$\mathcal{L}^{-1}\left(\frac{d}{ds}\frac{1}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{-2s}{(s^2+4)^2}\right) = -2\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)^2}\right)$$

$$\| $$

$$-t\ \frac{1}{2}\sin 2t = -2\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)^2}\right)$$

# Inverse Laplace Transforms

(7) $\mathcal{L}^{-1}\left(\int_s^\infty F(u)\,du\right) = \dfrac{f(t)}{t}$

Ex $\mathcal{L}^{-1}\left(\int_s^\infty \dfrac{du}{u^2+u}\right) = \dfrac{1-e^{-t}}{t}$

$\mathcal{L}^{-1}\left(\dfrac{1}{S^2+S}\right) = \mathcal{L}^{-1}\left(\dfrac{\frac{1}{S+1}}{S}\right) = \int_0^t e^{-u}\,du = -e^{-u}\Big|_0^t = -(e^{-t}-1)$

$= 1-e^{-t}$

$\mathcal{L}^{-1}\left(\dfrac{1}{S+1}\right) = e^{-t}$

# Inverse Laplace Transforms

Ex: $\mathcal{L}^{-1}\left( \int_s^{\infty} \left( \frac{2u}{u^2+a^2} - \frac{2}{u} \right) du \right).$

S.P.

$$\mathcal{L}^{-1}\left( \frac{2s}{s^2+a^2} - \frac{2}{s} \right) = 2\mathcal{L}^{-1}\left( \frac{s}{s^2+a^2} \right) - 2\mathcal{L}^{-1}\left( \frac{1}{s} \right)$$

$$= 2\cos at - 2 = f(t)$$

$$= \frac{2\cos at - 2}{t}$$

$$\frac{x}{(x^2-a^2)} = \frac{x}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$\frac{x}{(x^2-a^2)(x+a)}$$

$$= \frac{x}{(x-a)(x+a)^2} = \frac{A}{x-a} + \frac{B}{x+a} + \frac{C}{(x+a)^2}$$