قسم الرياضيات

Mathematics III

For

Faculty of Computers & Information

Second Year Students

Prepared By

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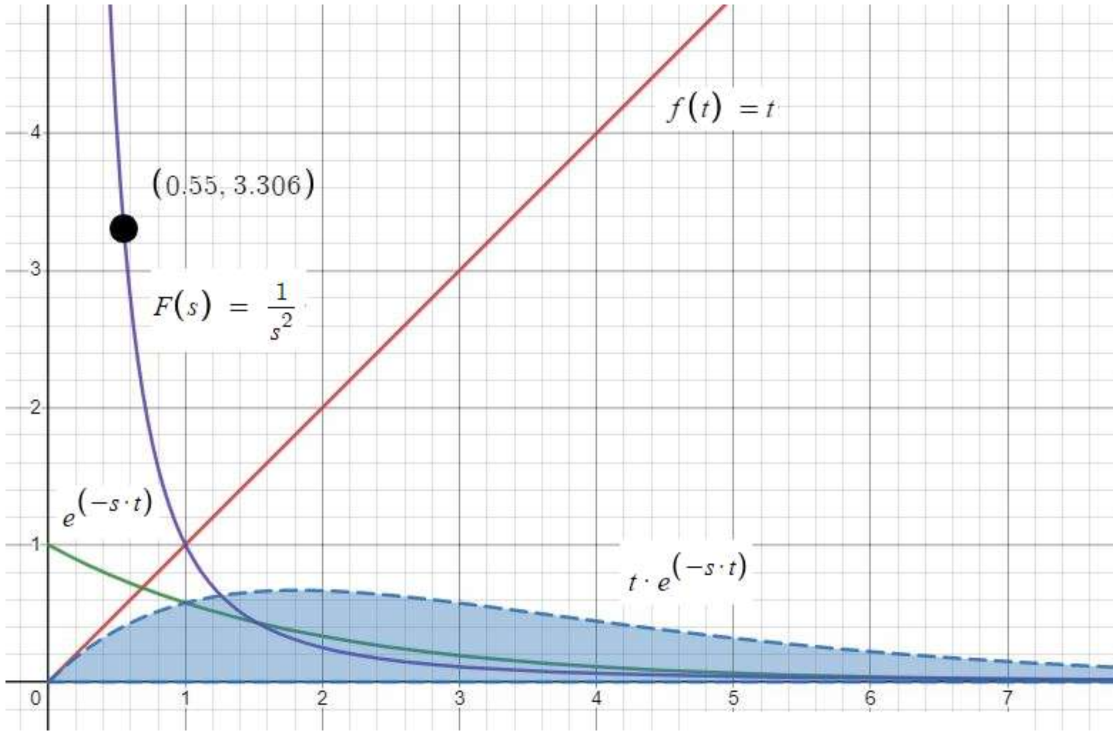
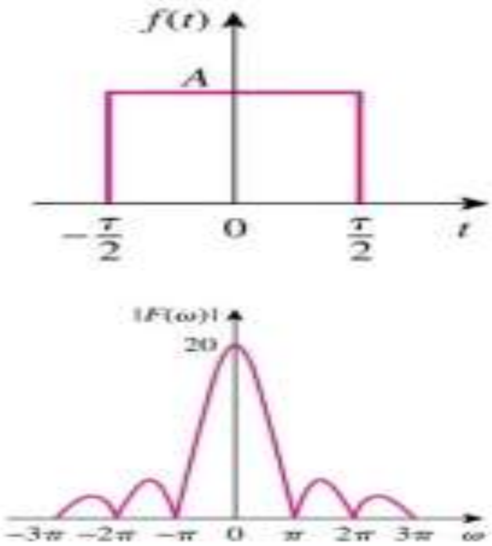
2024-2025


(0.55, 3.306)
$F(s) = \frac{1}{s^2}$
$f(t) = t$
$e^{(-s \cdot t)}$
$t \cdot e^{(-s \cdot t)}$


$f(t)$
A
$-\frac{\tau}{2}$
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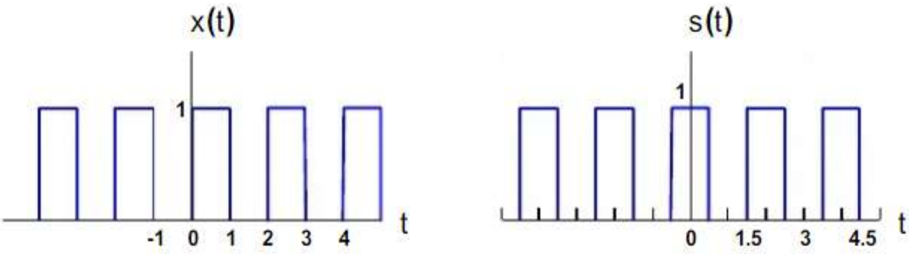
$|F(\omega)|$
20
-3π -2π $-\pi$ 0 π 2π 3π ω


x(t)
1
-1 0 1 2 3 4
t
s(t)
1
0 1.5 3 4.5
t

(i)

Lectures

In

Laplace and Fourier Transforms

SYMBOLS

Greek Alphabets

Α	α	Alpha	Ι	ι	Iota	Ρ	ρ	Rho
Β	β	Beta	Κ	κ	Kappa	Σ	σ	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	Τ	τ	Tau
D	δ	Delta	Μ	μ	Mu	Υ	υ	Upsilon
Ε	ε	Epsilon	Ν	ν	Nu	Φ	φ	Phi
Ζ	ζ	Zeta	Ξ	ξ	Xi	Χ	χ	Chi
Η	η	Eta	Ο	ο	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π	Pi	Ω	ϖ	Omega
	∃	there exists		∀	for all			

Preface

This work has been primarily written for the student of mathematics who is in the second year or the early part of the third year of an undergraduate course. It will also be very useful for students of engineering and physical sciences for whom.

Laplace and Fourier transforms continue to be extremely useful tools. The work demands no more than an elementary knowledge of calculus and linear algebra of the type found in many first-year mathematics modules for applied subjects. For mathematics majors and specialists, it is not the mathematics that will be challenging but the applications to the real world.

These transforms are used in anger to solve real problems, as well as spending rather more years within mathematics where accuracy and logic are of primary importance.

Also, these transforms play an important role in the analysis of all kinds of physical phenomena. As a link between the various applications of these transforms, the authors use the theory of signals and systems, as well as the theory of ordinary and partial differential equations.

This work has been prepared with the following salient features:

(*1*) The language of the book is simple and easy to understand.

(*2*) Each topic has been presented in a systematic, simple, lucid, and exhaustive manner.

(*3*) A large number of important solved examples properly selected from the previous university question papers have been provided to enable the students to have a clear grasp of the subject and to equip them for attempting problems in the university examination without any difficulty.

(*4*) Apart from providing a large number of examples, different types of questions in ample quantity have been provided for thorough practice for the students.

(*5*) A large number of ‘notes’ and ‘remarks’ have been added for better understanding of the subject.

A serious effort has been made to keep the book free from mistakes and errors. In fact, no pains have been spared to make the book interesting and useful. Suggestions and comments for further improvement of the book will be welcomed.

Hamdy

CONTENTS

Chapter *Pages*

Laplace Transforms

1

1.1. INTRODUCTION

The Laplace transform* of a suitably defined function f of a real variable t is a related function F of a real variable s. The use of Laplace transforms provide a powerful method of solving differential and integral equations. The Laplace transform method also has the advantage that it solves initial value problems directly without first finding a general solution. The ready tables of Laplace transforms has reduced the problem of solving differential equations to merely algebraic manipulation.

1.2. LAPLACE TRANSFORM OF A FUNCTION

Let f be a real valued function of the real variable t, defined for $t \geq 0$. Let s be a real variable and consider the function F of s defined by

$$\mathrm{F}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

for all values of s for which this integral exists (that has some finite value). The function F(s) defined by the integral $\int_0^\infty e^{-st} f(t)\, dt$ is called the **Laplace transform** of the function f and is denoted by L(f).

$$\therefore \qquad \mathbf{F(s) = L(f) = \int_0^\infty e^{-st}\, f(t)\, dt}.$$

Remarks 1. The original function f depends on t and its Laplace transform (F) depends on s.

2. Original functions are denoted by *lower case letters* and their Laplace transforms by the same letters in *capitals.*

WORKING STEPS TO FIND LAPLACE TRANSFORMS OF f(t), t ≥ 0.

Step I. Write $\mathrm{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$.

Step II. Write this improper integral as $\lim\limits_{\mathrm{T} \to \infty} \int_0^{\mathrm{T}} e^{-st} f(t)\, dt$ and simplify $e^{-st} f(t)$.

Step III. Evaluate $\int_0^{\mathrm{T}} e^{-st} f(t)\, dt$.

Step IV. Find limit of $\int_0^{\mathrm{T}} e^{-st} f(t)\, dt$ as $\mathrm{T} \to \infty$. This gives the value of L(f).

*__Pierre Simon Marquis De Laplace__ (1749—1827) French mathematician, made contribution to special functions, probability theory, potential theory and astronomy.

1.3. LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

In this section, we shall find the Laplace transforms of some elementary functions.

Theorem. Prove that for t ≥ 0,

1. $\mathbf{L(1) = \frac{1}{s}, s > 0}$

2. $\mathbf{L(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, a > -1, s > 0}$

3. $\mathbf{L(t^n) = \frac{n!}{s^{n+1}}, n = 0, 1, 2, \ldots\ldots; s > 0}$

4. $\mathbf{L(e^{at}) = \frac{1}{s-a}, \ s > a}$

5. $\mathbf{L(\sinh at) = \frac{a}{s^2 - a^2}, \ s > |a|}$

6. $\mathbf{L(\cosh at) = \frac{s}{s^2 - a^2}, \ s > |a|}$

7. $\mathbf{L(\sin at) = \frac{a}{s^2 + a^2}, \ s > 0}$

8. $\mathbf{L(\cos at) = \frac{s}{s^2 + a^2}, \ s > 0}$.

Proof. By definition, the improper integeral $\int_k^\infty g(t)\,dt$ is equal to $\lim_{T\to\infty} \int_k^T g(t)\,dt$.

1. By definition, $L(1) = \int_0^\infty e^{-st}.1\,dt$

$$\therefore \qquad L(1) = \lim_{T\to\infty} \int_0^T e^{-st}\,dt = \lim_{T\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^T$$

$$= \frac{1}{s} \lim_{T\to\infty} \left[-\frac{1}{e^{st}}\right]_0^T = \frac{1}{s} \lim_{T\to\infty} \left[-\frac{1}{e^{sT}} + 1\right]$$

$$= \frac{1}{s}[0+1] = \frac{1}{s} \text{ for } s > 0.$$

$$\therefore \qquad \mathbf{L(1) = \frac{1}{s}, s > 0}.$$

2. By definition, $L(t^a) = \int_0^\infty e^{-st} t^a\,dt$.

Let $st = u$. $\therefore$ $s\,dt = du$.

$$\therefore \qquad L(t^a) = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a\,du$$

$$= \frac{\Gamma(a+1)}{s^{a+1}}, \text{ provided } a > -1$$

$$\therefore \qquad \mathbf{L(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, a > -1, \ s > 0}. \qquad (\because \ t \geq 0, u \geq 0, u = st \Rightarrow s \geq 0)$$

3. By definition, $L(t^n) = \int_0^\infty e^{-st} t^n\,dt$.

Let $st = u$. $\therefore$ $s\,dt = du$

$\therefore \quad L(t^n) = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n \, du$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \text{ , provided } n + 1 > 0.$$

Let $\quad n = 0, 1, 2, \ldots\ldots \quad \therefore \quad \Gamma(n + 1) = n\,!$

$\therefore \quad \mathbf{L(t^n) = \frac{n\,!}{s^{n+1}}, n = 0, 1, 2, \ldots\ldots; s > 0.} \qquad (\because \quad t \geq 0, u \geq 0, u = st \Rightarrow s \geq 0)$

4. By definition, $\quad L(e^{at}) = \int_0^\infty e^{-st} e^{at} \, dt$

$$= \lim_{T \to \infty} \int_0^T e^{(a-s)t} \, dt = \lim_{T \to \infty} \left[\frac{e^{(a-s)t}}{a-s}\right]_0^T$$

$$= \frac{1}{a-s} \lim_{T \to \infty} \left[\frac{1}{e^{(s-a)T}} - 1\right] = \frac{1}{a-s} [0 - 1] \text{ for } s > a \; = \frac{1}{s-a}$$

$\therefore \quad \mathbf{L(e^{at}) = \frac{1}{s-a}, \; s > a.}$

5. $\quad L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2}\right) dt$

$$= \frac{1}{2} \int_0^\infty \left(e^{-(s-a)t} - e^{-(s+a)t}\right) dt \; = \frac{1}{2} \lim_{T \to \infty} \int_0^T \left(e^{-(s-a)t} - e^{-(s+a)t}\right) dt$$

$$= \frac{1}{2} \lim_{T \to \infty} \left[\frac{e^{-(s-a)t}}{a-s} + \frac{e^{-(s+a)t}}{s+a}\right]_0^T$$

$$= \frac{1}{2} \lim_{T \to \infty} \left[\frac{1}{a-s}\left(\frac{1}{e^{(s-a)T}} - 1\right) + \frac{1}{s+a}\left(\frac{1}{e^{(s+a)T}} - 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{a-s}(0-1) + \frac{1}{s+a}(0-1)\right] \text{ provided } s > a, s > -a$$

$$= \frac{1}{2}\left[\frac{2a}{s^2 - a^2}\right] = \frac{a}{s^2 - a^2} \;, s > |\,a\,|.$$

$\therefore \quad \mathbf{L(\sinh at) = \frac{a}{s^2 - a^2} \;, s > |\,a\,|.}$

7. $\quad L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$

$\therefore \quad L(\sin at) = \lim_{T \to \infty} \int_0^T e^{-st} \sin at \, dt$

Let $\quad I = \int e^{-st} \sin at \, dt$

$\therefore \quad I = e^{-st} . \frac{-\cos at}{a} - \int -se^{-st} . \frac{-\cos at}{a} \, dt$

$$= -\frac{e^{-st}\cos at}{a} - \frac{s}{a}\int e^{-st}\cos at\,dt$$

$$= -\frac{e^{-st}\cos at}{a} - \frac{s}{a}\left[e^{-st}\frac{\sin at}{a} - \int -se^{-st}\,.\,\frac{\sin at}{a}\,dt\right]$$

$$\therefore \qquad I = -\frac{e^{-st}}{a^2}[a\cos at + s\sin at] - \frac{s^2}{a^2}I$$

$$\therefore \qquad I = -\frac{e^{-st}}{s^2+a^2}[a\cos at + s\sin at]$$

$$\therefore \qquad L(\sin at) = \lim_{T\to\infty}\frac{-e^{-st}}{s^2+a^2}\Big[a\cos at + s\sin at\Big]_0^T$$

$$= \frac{-1}{s^2+a^2}\lim_{T\to\infty}\left[\frac{a\cos aT + s\sin aT}{e^{sT}} - \frac{a+0}{1}\right]$$

$$= \frac{-1}{s^2+a^2}[0-a] \text{ , provided } s > 0$$

$$= \frac{a}{s^2+a^2} \text{ , } s > 0.$$

$$\therefore \qquad \mathbf{L(\sin at) = \frac{a}{s^2+a^2} \text{ , } s > 0.}$$

Remark. The proofs of the parts 6 and 8 are left for readers as exercises.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Laplace transform of the following function :*

$$f(t) = \begin{cases} t+1, & 0 \le t \le 2 \\ 3, & t > 2. \end{cases}$$

Sol.

$$Lf(t) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st}(t+1)\,dt + \int_2^\infty e^{-st}\,.\,3\,dt$$

$$= \left[(t+1)\,.\,\frac{e^{-st}}{-s}\Bigg|_0^2 - \int_0^2 1\,.\,\frac{e^{-st}}{-s}\,dt\right] + 3\,.\,\frac{e^{-st}}{-s}\Bigg|_2^\infty$$

$$= -\frac{3}{s}e^{-2s} + \frac{1}{s} - \frac{e^{-st}}{s^2}\Bigg|_0^2 - \frac{3}{s}\left[\lim_{T\to\infty} e^{-sT} - e^{-2s}\right]$$

$$= -\frac{3e^{-2s}}{s} + \frac{1}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} - \frac{3}{s}(0 - e^{-2s})$$

$$= -\frac{3e^{-2s}}{s} + \frac{1}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} + \frac{3e^{-2s}}{s} = \mathbf{\frac{1}{s^2}[s - e^{-2s} + 1]}\,.$$

Example 2. *Find the Laplace transform of the following function :*

$$f(t)=\begin{cases}0, & 0\le t<1\\ t, & 1<t<2\\ 1, & t>2.\end{cases}$$

Sol. $$\mathrm{L}(f(t))=\int_0^\infty e^{-st}\,f(t)\,dt$$

$$=\int_0^1 e^{-st}\,f(t)\,dt+\int_1^2 e^{-st}\,f(t)\,dt+\int_2^\infty e^{-st}\,f(t)\,dt$$

$$=\int_0^1 e^{-st}\,.\,0\,dt+\int_1^2 e^{-st}\,t\,dt+\int_2^\infty e^{-st}\,.\,1\,dt$$

$$=\int_0^1 0\,dt+\left[t\,.\,\frac{e^{-st}}{-s}\bigg|_1^2-\int_1^2 1\,.\,\frac{e^{-st}}{-s}\,dt\right]+\lim_{T\to\infty}\left[\frac{e^{-st}}{-s}\right]_2^T$$

$$=0-\frac{2}{se^{2s}}+\frac{1}{se^{s}}+\frac{e^{-st}}{-s^2}\bigg|_1^2-\frac{1}{s}\lim_{T\to\infty}\left[\frac{1}{e^{sT}}-\frac{1}{e^{2s}}\right]$$

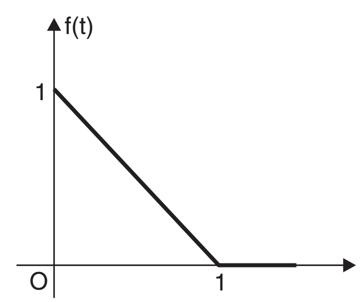
$$=-\frac{2}{se^{2s}}+\frac{1}{se^{s}}-\frac{1}{s^2}\left[\frac{1}{e^{2s}}-\frac{1}{e^{s}}\right]-\frac{1}{s}[0-1],\quad s>0$$

$$=-\frac{2}{se^{2s}}+\frac{1}{se^{s}}-\frac{1}{s^2e^{2s}}+\frac{1}{s^2e^{s}}+\frac{1}{se^{2s}},\quad s>0$$

$$=\frac{1}{s^2}[-se^{-2s}+se^{-s}-e^{-2s}+e^{-s}],\quad s>0$$

$$=\boldsymbol{\frac{1}{s^2}(e^{-s}-e^{-2s})(s+1),\quad s>0.}$$

Example 3. *Find the Laplace transform of the function shown in the graph.*



Sol. From the graph, we have

$$f(t)=\begin{cases}t\tan 135^\circ+1 & 0\le t\le 1\\ 0 & t\ge 1\end{cases}$$

or $$f(t)=\begin{cases}-t+1 & 0\le t\le 1\\ 0 & t\ge 1.\end{cases}$$

$$\therefore\quad \mathrm{L}(f(t))=\int_0^\infty e^{-st}\,f(t)\,dt$$

$$=\int_0^1 e^{-st}\,f(t)\,dt+\int_1^\infty e^{-st}\,f(t)\,dt=\int_0^1 e^{-st}\,(-t+1)\,dt+\int_1^\infty e^{-st}\,.\,0\,dt$$

$$=\int_0^1 (1-t)\,e^{-st}\,dt+\int_0^\infty 0\,dt=(1-t)\,.\,\frac{e^{-st}}{-s}\bigg|_0^1-\int_0^1 -1\,.\,\frac{e^{-st}}{-s}\,dt+0$$

$$=0+\frac{e^0}{s}-\frac{e^{-st}}{-s^2}\bigg|_0^1=\frac{1}{s}+\frac{1}{s^2}\left[\frac{1}{e^s}-\frac{1}{e^0}\right]=\boldsymbol{\frac{1}{s}+\frac{1}{s^2}\left[\frac{1}{e^s}-1\right],\quad s\neq 0.}$$

Example 4. *Show that* $\int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Sol. Let $\quad I = \int_0^\infty \cos x^2\, dx$.

$$t = x^2 \Rightarrow dt = 2x\, dx \Rightarrow dx = \frac{dt}{2\sqrt{t}}$$

$x \to \infty \Rightarrow t \to \infty$, $x = 0 \Rightarrow t = (0)^2 = 0$

$$\therefore \quad I = \int_0^\infty \frac{\cos t\, dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty \frac{e^{it} + e^{-it}}{2\sqrt{t}}\, dt$$

$$= \frac{1}{4}\left[\int_0^\infty e^{it} \,.\, t^{-1/2}\, dt + \int_0^\infty e^{-it} \,.\, t^{-1/2}\, dt\right]$$

$$= \frac{1}{4}\left[L(t^{-1/2})\,\big|_{s=-i} + L(t^{-1/2})\,\big|_{s=i}\right]$$

$$= \frac{1}{4}\left[\frac{\Gamma(1/2)}{s^{1/2}}\bigg|_{s=-i} + \frac{\Gamma(1/2)}{s^{1/2}}\bigg|_{s=i}\right]$$

$$= \frac{1}{4}\left[\frac{\sqrt{\pi}}{\sqrt{-i}} + \frac{\sqrt{\pi}}{\sqrt{i}}\right] = \frac{\sqrt{\pi}}{4}\left[\frac{1}{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i} + \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}\right] \qquad \text{(Note this step)}$$

$$= \frac{\sqrt{2\pi}}{4}\left[\frac{1}{1-i} + \frac{1}{1+i}\right] = \frac{\sqrt{2\pi}}{4}\left[\frac{2}{2}\right] = \mathbf{\frac{1}{2}\sqrt{\frac{\pi}{2}}}.$$

TEST YOUR KNOWLEDGE

1. By using standard formulae, write the Laplace transform of the following functions :

(*i*) $t^{3/2}$	(*ii*) $t^{5/3}$	(*iii*) t^7
(*iv*) e^{3t}	(*v*) e^{-10t}	(*vi*) sinh $5t$
(*vii*) cosh $7t$	(*viii*) sin $4t$	(*ix*) cos $6t$.

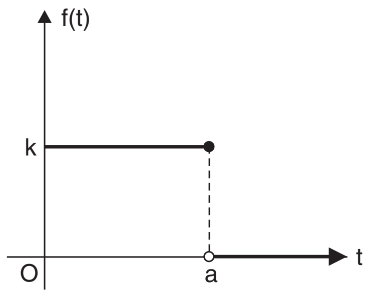
2. Find the Laplace transform of the function $f(t) = t$, $t \geq 0$.
3. Find the Laplace transform of the function $f(t) = 4t + 3$, $t \geq 0$.
4. Find the Laplace transform of the function $f(t) = \cosh at$, $t \geq 0$.
5. Find the Laplace transform of the function $f(t) = \cos at$, $t \geq 0$.
6. Find the Laplace transform of the following functions :

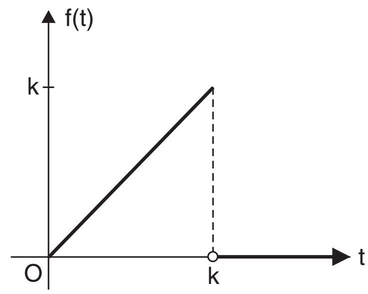
(*i*) $f(t) = \begin{cases} 4, & 0 \leq t < 3 \\ 2, & t > 3 \end{cases}$ (*ii*) $f(t) = \begin{cases} t, & 0 \leq t < 2 \\ 3, & t > 2 \end{cases}$.

7. Find the Laplace transform of the following function :

$$f(t) = \begin{cases} t & 0 \leq t \leq 1/2 \\ t-1 & 1/2 < t \leq 1 \\ 0 & t > 1 \end{cases}.$$

8. Find the Laplace transform of the functions whose graphs are given :

(*i*) 

(*ii*) 

9. Show that :

(*i*) $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$ (*ii*) $\int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Answers

1. (*i*) $\frac{3\sqrt{\pi}}{4s^{5/2}}, s > 0$ (*ii*) $\frac{10\,\Gamma(2/3)}{9s^{8/3}}, s > 0$ (*iii*) $\frac{5040}{s^8}, s > 0$

(*iv*) $\frac{1}{s-3}, s > 3$ (*v*) $\frac{1}{s+10}, s > -10$ (*vi*) $\frac{s}{s^2-25}, s > 5$

(*vii*) $\frac{s}{s^2-49}, s > 7$ (*viii*) $\frac{4}{s^2+16}, s > 0$ (*ix*) $\frac{s}{s^2+36}, s > 0$

2. $\frac{1}{s^2},\ s > 0$ **3.** $\frac{3s+4}{s^2},\ s > 0$ **4.** $\frac{s}{s^2-a^2},\ s > |a|$

5. $\frac{s}{s^2+a^2},\ s > 0$ **6.** (*i*) $\frac{2}{s}(2-e^{-3s}),\ s > 0$ (*ii*) $\frac{1}{s^2}+\frac{e^{-2s}}{s}-\frac{e^{-2s}}{s^2},\ s > 0$

7. $\frac{1}{s^2}\left[1-se^{-s/2}-e^{-s}\right],\ s > 0$ **8.** (*i*) $\frac{k}{s}(1-e^{-as}),\ s > 0$

(*ii*) $\frac{1}{s^2}(1-e^{-ks})-\frac{k}{s}e^{-ks},\ s > 0$.

1.4. LINEARITY OF THE LAPLACE TRANSFORM

Theorem. *If f(t) and g(t) be any functions of t for t ≥ 0 whose Laplace transforms exists and a and b be any constants, then*

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}.$$

Proof. We have $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ and $L\{g(t)\} = \int_0^\infty e^{-st} g(t)\, dt$.

Now $$L\{af(t) + bg(t)\} = \int_0^\infty e^{-st} [af(t) + bg(t)]\, dt$$

$$= a\int_a^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt$$

$$= aL\{f(t)\} + bL\{g(t)\}$$

$\therefore$ $$\mathbf{L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find Laplace transform of the function cosh at, $t \geq 0$ using linearity property.*

Sol. We have $L(\cosh at) = L\left(\dfrac{e^{at}+e^{-at}}{2}\right)$

$$= L\left(\frac{1}{2}e^{at}+\frac{1}{2}e^{-at}\right) = \frac{1}{2}L(e^{at})+\frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\cdot\frac{1}{s-a}+\frac{1}{2}\cdot\frac{1}{s-(-a)},\ s>a, s>-a$$

$$= \frac{s}{s^2-a^2},\ s>|a| \qquad \left(\because\ L(e^{at})=\frac{1}{s-a}, s>a\right)$$

$\therefore$ $$\mathbf{L(\cosh at) = \frac{s}{s^2-a^2},\ s>|a|.}$$

Example 2. *Find the Laplace transform of the following functions of t, $t \geq 0$:*

(*i*) $6t + 9$ (*ii*) $2t^2 - t + 5$ (*iii*) $\sin^2 t$

(*iv*) $\sin 2t \sin 3t$ (*v*) $\cos^3 t$ (*vi*) $e^{4t} + e^{2t} + t^3 + \sin^2 t$.

Sol. (*i*) $L(6t + 9) = 6L(t) + L(9)$ (By linearity)

$$= 6\left(\frac{1!}{s^{1+1}}\right)+\frac{1}{s},\ s>0$$

$$= \mathbf{\frac{6+9s}{s^2},\ s>0}$$

(*ii*) $L(2t^2 - t + 5) = 2L(t^2) - L(t) + L(5)$

$$= 2\cdot\left(\frac{2!}{s^3}\right)-\frac{1!}{s^2}+\frac{5}{s},\ s>0$$

$$= \mathbf{\frac{4-s+5s^2}{s^3},\ s>0.}$$

(*iii*) $L(\sin^2 t) = L\left(\dfrac{1-\cos 2t}{2}\right) = L\left(\dfrac{1}{2}-\dfrac{1}{2}\cos 2t\right)$

$$= \frac{1}{2}L(1)-\frac{1}{2}L(\cos 2t) = \frac{1}{2}\left(\frac{1}{s}\right)-\frac{1}{2}\left(\frac{s}{s^2+4}\right),\ s>0$$

$$= \mathbf{\frac{2}{s(s^2+4)},\ s>0.}$$

(*iv*) $L(\sin 2t \sin 3t) = L\left(\dfrac{1}{2}(2\sin 2t\sin 3t)\right)$

$$= L\left(\frac{1}{2}\cos t-\frac{1}{2}\cos 5t\right) = \frac{1}{2}L(\cos t)-\frac{1}{2}L(\cos 5t)$$

$$= \frac{1}{2} \cdot \left(\frac{s}{s^2+1^2}\right) - \frac{1}{2} \cdot \left(\frac{s}{s^2+5^2}\right) s > |1|, s > |5|$$

$$= \frac{\mathbf{12s}}{\mathbf{(s^2+1)\,(s^2+25)}}\ \ \mathbf{, s > 5\,.}$$

(*v*) We have cos 3θ = 4 cos³ θ – 3 cos θ.

$$\Rightarrow \qquad \cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$$

$$\therefore \qquad \cos^3 t = \frac{1}{4} \cos 3t + \frac{3}{4} \cos t\,.$$

$$\therefore \qquad \mathrm{L}(\cos^3 t) = \frac{1}{4} \mathrm{L}(\cos 3t) + \frac{3}{4} \mathrm{L}(\cos t) \qquad \text{(By linearity)}$$

$$= \frac{1}{4}\left(\frac{s}{s^2+3^2}\right) + \frac{3}{4}\left(\frac{s}{s^2+1^2}\right) , s > 0$$

$$= \frac{s}{4}\left[\frac{1}{s^2+9} + \frac{3}{s^2+1}\right], s > 0$$

$$= \frac{\mathbf{(s^2+7)\,s}}{\mathbf{(s^2+9)\,(s^2+1)}}\ \ \mathbf{, s > 0.}$$

(*vi*) $\mathrm{L}(e^{4t} + e^{2t} + t^3 + \sin^2 t)$

$$= \mathrm{L}(e^{4t}) + \mathrm{L}(e^{2t}) + \mathrm{L}(t^3) + \mathrm{L}\left(\frac{1-\cos 2t}{2}\right) \qquad \text{(By linearity)}$$

$$= \mathrm{L}(e^{4t}) + \mathrm{L}(e^{2t}) + \mathrm{L}(t^3) + \frac{1}{2}\mathrm{L}(1) - \frac{1}{2}\mathrm{L}(\cos 2t)$$

$$= \frac{1}{s-4} + \frac{1}{s-2} + \frac{3\,!}{s^{3+1}} + \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+2^2}\right),\ s > 4, s > 2, s > 0$$

$$\mathbf{= \frac{1}{s-4} + \frac{1}{s-2} + \frac{6}{s^4} + \frac{1}{2s} - \frac{s}{2(s^2+4)},\ \ s > 4.}$$

Example 3. *Find the Laplace transform of the function sin (ωt + δ), t ≥ 0.*

Sol. L(sin (ωt + δ)) = L(sin ω*t* cos δ + cos ω*t* sin δ)

$$= \cos \delta\, \mathrm{L}(\sin \omega t) + \sin \delta\, \mathrm{L}(\cos \omega t) \qquad \text{(By linearity)}$$

$$= \cos \delta \left(\frac{\omega}{s^2+\omega^2}\right) + \sin \delta \left(\frac{s}{s^2+\omega^2}\right),\ s > 0$$

$$\mathbf{= \frac{1}{s^2+\omega^2}\,[\omega \cos \delta + s \sin \delta],\ \ s > 0.}$$

WORKING STEPS FOR SOLVING PROBLEMS

Step I. Simplify the given functions and write it as a linear combination of functions with known Laplace transforms.

Step II. Apply linearity theorem.

Step III. Simplify the terms as far as possible.

TEST YOUR KNOWLEDGE

1. Find the Laplace transform of the following functions of t, $t \geq 0$:

(*i*) $5t + 9$ (*ii*) $t^2 + 8t - 15$

(*iii*) $t^3 + 4$ (*iv*) $t^{10} + t^4 + 2$.

2. Find the Laplace transform of the following functions of t, $t \geq 0$:

(*i*) $e^{5t} + t^2$ (*ii*) $e^{-4t} + e^{2t}$

(*iii*) $e^{3t} + t^3$ (*iv*) $e^{-5t} + t^2 + 6t$.

3. Find the Laplace transform of the following functions of t, $t \geq 0$:

(*i*) $\sinh 4t + \cosh 4t$ (*ii*) $t - \sinh 2t$

(*iii*) $\sin \pi t + e^{6t}$ (*iv*) $\cos (at + b)$.

4. Find the Laplace transform of the following functions of t, $t \geq 0$:

(*i*) $\sin^2 2t$ (*ii*) $\sin^2 3t$

(*iii*) $\cos^2 4t$ (*iv*) $7 \cos^2 t$.

5. Find the Laplace transform of the following functions of t, $t \geq 0$:

(*i*) $\sin^3 t$ (*ii*) $\sin^3 2t$

(*iii*) $\cos^3 2t + t^2$ (*iv*) $\cos^3 4t + t$.

6. Find the Laplace transform of the following functions of t, $t \geq 0$:

(*i*) $\sin 3t \cos 2t$ (*ii*) $\cos 5t \sin 2t$

(*iii*) $\cos 4t \cos t$ (*iv*) $\sin 3t \sin 7t$.

7. Find the Laplace transform of the following functions of t :

(*i*) $(\sin t - \cos t)^2, t \geq 0$ (*ii*) $1 + \sqrt{t} + \dfrac{3}{\sqrt{t}}, t > 0$

(*iii*) $\sin at \sin bt, t \geq 0$ (*iv*) $\cosh at - \cos at, t \geq 0$.

Answers

1. (*i*) $\dfrac{5}{s^2} + \dfrac{9}{s}, s > 0$ (*ii*) $\dfrac{2}{s^3} + \dfrac{8}{s^2} - \dfrac{15}{s}, s > 0$

(*iii*) $\dfrac{6}{s^4} + \dfrac{4}{s}, s > 0$ (*iv*) $\dfrac{10\,!}{s^{11}} + \dfrac{24}{s^5} + \dfrac{2}{s}, s > 0$

2. (*i*) $\dfrac{1}{s-5} + \dfrac{2}{s^3}, s > 5$ (*ii*) $\dfrac{2\,(s+1)}{(s+4)\,(s-2)}, s > 2$

(*iii*) $\dfrac{1}{s-3} + \dfrac{6}{s^4}, s > 3$ (*iv*) $\dfrac{1}{s+5} + \dfrac{2}{s^3} + \dfrac{6}{s^2}, s > 0$

3. (*i*) $\dfrac{1}{s-4}$, $s > 4$ (*ii*) $\dfrac{4+s^2}{s^2(4-s^2)}$, $s > 2$

(*iii*) $\dfrac{\pi}{s^2+\pi^2}+\dfrac{1}{s-6}$, $s > 6$ (*iv*) $\dfrac{s\cos b - a\sin b}{s^2+a^2}$, $s > 0$

4. (*i*) $\dfrac{8}{s(s^2+16)}$, $s > 0$ (*ii*) $\dfrac{18}{s(s^2+36)}$, $s > 0$

(*iii*) $\dfrac{s^2+32}{s(s^2+64)}$, $s > 0$ (*iv*) $\dfrac{7(s^2+2)}{s(s^2+4)}$, $s > 0$

5. (*i*) $\dfrac{6}{(s^2+1)(s^2+9)}$, $s > 0$ (*ii*) $\dfrac{48}{(s^2+4)(s^2+36)}$, $s > 0$

(*iii*) $\dfrac{s(s^2+28)}{(s^2+4)(s^2+36)}+\dfrac{2}{s^3}$, $s > 0$ (*iv*) $\dfrac{s(s^2+112)}{(s^2+16)(s^2+144)}+\dfrac{1}{s^2}$, $s > 0$

6. (*i*) $\dfrac{3s^2+15}{(s^2+1)(s^2+25)}$, $s > 0$ (*ii*) $\dfrac{2s^2-42}{(s^2+9)(s^2+49)}$, $s > 0$

(*iii*) $\dfrac{s(s^2+17)}{(s^2+9)(s^2+25)}$, $s > 0$ (*iv*) $\dfrac{42s}{(s^2+16)(s^2+100)}$, $s > 0$

7. (*i*) $\dfrac{s^2-2s+4}{s(s^2+4)}$, $s > 0$ (*ii*) $\dfrac{1}{s}+\dfrac{\sqrt{\pi}}{2s^{3/2}}+\dfrac{3\sqrt{\pi}}{s^{1/2}}$, $s > 0$

(*iii*) $\dfrac{2abs}{(s^2+(a-b)^2)(s^2+(a+b)^2)}$, $s > 0$ (*iv*) $\dfrac{2a^2s}{s^4-a^4}$, $s > |a|$.

1.5. SHIFTING THEOREMS

There are two shifting theorems. In the first theorem the variable s of the function $L(f(t))$ is replaced by $s - a$ and in the second theorem the variable t of the function $f(t)$ is replaced by $t - a$.

1.6. FIRST SHIFTING THEOREM

If f(t) be a function of t for t ≥ 0 whose Laplace transform F(s) exists for s > k then for any constant 'a' the function e^{at} f(t) has the Laplace transform F(s – a) for s > k + a.

Proof. We have $F(s) = L(f) = \int_0^\infty e^{-st} f(t)\,dt, s > k$.

Replacing s by $s - a$, we get $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt, s - a > k$.

$$\Rightarrow \quad F(s-a) = \int_0^\infty e^{-st}\,(e^{at} f(t))\,dt, s > k + a$$

$\therefore$ Laplace transform of the function $e^{at} f(t)$ is $F(s - a)$ for $s > k + a$.

Remark. The result of above theorem can be easily remembered as follows :

If L(f(t)) = F(s), s > k then for any a, L(e^{at} f(t)) = F(s – a), s > k + a.

Theorem. Using first shifting theorem. Show that for $t \geq 0$:

1. $\mathbf{L(e^{at}) = \dfrac{1}{s-a},\ s > a}$

2. $\mathbf{L(e^{at}\ t^{b}) = \dfrac{\Gamma(b+1)}{(s-a)^{b+1}},\ b > -1,\ s > a}$

3. $\mathbf{L(e^{at}\ t^{n}) = \dfrac{n!}{(s-a)^{n+1}},\ n = 0, 1, 2, \ldots\ldots, s > a}$

4. $\mathbf{L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2 - b^2},\ s > |b| + a}$

5. $\mathbf{L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2},\ s > |b| + a}$

6. $\mathbf{L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2 + b^2},\ s > a}$

7. $\mathbf{L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2 + b^2},\ s > a}$.

Proof. 1. We have $\mathrm{L}(1) = \dfrac{1}{s},\ s > 0$.

$\therefore$ By **first shifting theorem,**

$$\mathrm{L}(e^{at}.1) = \frac{1}{s-a},\ s > a$$

$\therefore$ $$\mathbf{L(e^{at}) = \frac{1}{s-a},\ s > a}.$$

2. We have $\mathrm{L}(t^b) = \dfrac{\Gamma(b+1)}{s^{b+1}}, b > -1,\ s > 0$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at}\ t^{b}) = \frac{\Gamma(b+1)}{(s-a)^{b+1}},\ b > -1,\ s > a}.$$

3. We have $\mathrm{L}(t^n) = \dfrac{n!}{s^{n+1}},\ n = 0, 1, 2, \ldots\ldots; s > 0$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at}\ t^{n}) = \frac{n!}{(s-a)^{n+1}},\ n = 0, 1, 2, \ldots\ldots; s > a}.$$

4. We have $\mathrm{L}(\sinh bt) = \dfrac{b}{s^2 - b^2},\ s > |b|$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at}\sinh bt) = \frac{b}{(s-a)^2 - b^2},\ s > |b| + a}.$$

5. We have $L(\cosh bt) = \dfrac{s}{s^2 - b^2}$, $s > |b|$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}, \quad s > |b| + a.}$$

6. We have $L(\sin bt) = \dfrac{b}{s^2 + b^2}$, $s > 0$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}, \quad s > a.}$$

7. We have $L(\cos bt) = \dfrac{s}{s^2 + b^2}$, $s > 0$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}, \quad s > a.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Laplace transform of the following functions* :

(*i*) $t^3 e^{-3t}$, $t \geq 0$ (*ii*) e^{-3t} $(2 \cos 5t - 3 \sin 5t)$, $t \geq 0$

(*iii*) $e^{4t} \sin 2t \cos t$, $t \geq 0$ (*iv*) $\sinh t \cos t$, $t \geq 0$

(*v*) e^{-t} $(3 \sinh 2t - 5 \cosh 2t)$.

Sol. (*i*) We have $L(t^3) = \dfrac{3!}{s^{3+1}} = \dfrac{6}{s^4}$, $s > 0$.

$\therefore$ By **first shifting theorem,**

$$L(t^3 e^{-3t}) = \frac{6}{(s-(-3))^4}, s > 0 + (-3) = \mathbf{\frac{6}{(s+3)^4}, \quad s > -3.}$$

(*ii*) We have $L(2 \cos 5t - 3 \sin 5t)$

$$= 2L(\cos 5t) - 3L(\sin 5t) \qquad \text{(By linearity)}$$

$$= 2\left(\frac{s}{s^2 + 5^2}\right) - 3\left(\frac{5}{s^2 + 5^2}\right), \ s > 0 = \frac{2s - 15}{s^2 + 25}, \ s > 0.$$

$\therefore$ By **first shifting theorem,**

$L(e^{-3t} (2 \cos 5t - 3 \sin 5t))$

$$= \frac{2(s-(-3)) - 15}{(s-(-3))^2 + 25}, \ s > 0 + (-3) = \mathbf{\frac{2s - 9}{s^2 + 6s + 34}, \quad s > -3.}$$

(*iii*) $L(\sin 2t \cos t) = L\left(\dfrac{1}{2}(\sin 3t + \sin t)\right)$

$$= \frac{1}{2}\,\text{L}(\sin 3t) + \frac{1}{2}\,\text{L}(\sin t) \;= \frac{1}{2}\left(\frac{3}{s^2+3^2}\right) + \frac{1}{2}\left(\frac{1}{s^2+1^2}\right), s > 0$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right], s > 0.$$

$\therefore$ By **first shifting theorem,**

$$\text{L}(e^{4t} \sin 2t \cos t) = \frac{1}{2}\left[\frac{3}{(s-4)^2+9} + \frac{1}{(s-4)^2+1}\right], s > 0 + 4$$

$$= \mathbf{\frac{1}{2}\left[\frac{3}{s^2 - 8s + 25} + \frac{1}{s^2 - 8s + 17}\right], s > 4.}$$

(iv) $\sinh t \cos t = \dfrac{e^t - e^{-t}}{2} \cos t \;= e^t \left(\dfrac{\cos t}{2}\right) - e^{-t}\left(\dfrac{\cos t}{2}\right).$ $\left[\because \;\; \sinh at = \dfrac{e^{at} - e^{-at}}{2}\right]$

Now $\quad \text{L}\left(\dfrac{\cos t}{2}\right) = \dfrac{1}{2}\,\text{L}\,(\cos t) = \dfrac{1}{2}\left(\dfrac{s}{s^2+1^2}\right) = \dfrac{s}{2\,(s^2+1)}, \; s > 0.$

$$\therefore \quad \text{L}(\sinh t \cos t) = \text{L}\left(e^t \cdot \frac{\cos t}{2} - e^{-t} \cdot \frac{\cos t}{2}\right)$$

$$= \text{L}\left(e^t \cdot \frac{\cos t}{2}\right) - \text{L}\left(e^{-t} \cdot \frac{\cos t}{2}\right) \qquad \text{(By linearity)}$$

$$= \frac{s-1}{2((s-1)^2+1)} - \frac{s-(-1)}{2((s-(-1))^2+1)}, \; s > 0+1, s > 0 + (-1)$$

$$= \frac{s-1}{2(s^2-2s+2)} - \frac{s+1}{2(s^2+2s+2)}, \; s > 1$$

$$= \mathbf{\frac{s^2-2}{s^4+4}, \; s > 1.}$$

(v) We have L(3 sinh $2t$ – 5 cosh $2t$)

$$= 3\text{L}(\sinh 2t) - 5\text{L}(\cosh 2t) \;= 3 \cdot \frac{2}{s^2-4} - 5 \cdot \frac{s}{s^2-4}, \; s > |2|$$

$$= \frac{6-5s}{s^2-4}, \; s > 2.$$

By **first shifting theorem,**

$$\text{L}(e^{-t}\,(3 \sinh 2t - 5 \cosh 2t)) = \frac{6-5\,(s+1)}{(s+1)^2-4}, s > 2 + (-1) \;\; = \mathbf{\frac{1-5s}{s^2+2s-3}, s > 1.}$$

Example 2. *If the Laplace transform of the function f(t) of t for t ≥ 0 is F(s), then show that*

$$L[(\sinh at)\, f(t)] = \frac{1}{2}\,[F(s-a) - F(s+a)].$$

Hence evaluate L(sinh 2t sin 3t).

Sol. $L[(\sinh at)\, f(t)] = L\left[\dfrac{e^{at} - e^{-at}}{2} f(t)\right]$ $\left(\because \sinh at = \dfrac{e^{at} - e^{-at}}{2}\right)$

$$= L\left[\frac{1}{2} e^{at} f(t) - \frac{1}{2} e^{-at} f(t)\right]$$

$$= \frac{1}{2} L(e^{at} f(t)) - \frac{1}{2} L(e^{-at} f(t)) \qquad \text{(By linearity)}$$

$$= \frac{1}{2} F(s-a) - \frac{1}{2} F(s-(-a)) \qquad \text{(By first shifting theorem)}$$

$$= \mathbf{\frac{1}{2}\,[F(s-a) - F(s+a)].}$$

Also, $L(\sin 3t) = \dfrac{3}{s^2+3^2} = \dfrac{3}{s^2+9}$, $s > 0$.

$$\therefore \quad L(\sinh 2t \sin 3t) = \frac{1}{2}\left[\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9}\right], \; s > 0+2, \; s > 0+(-2)$$

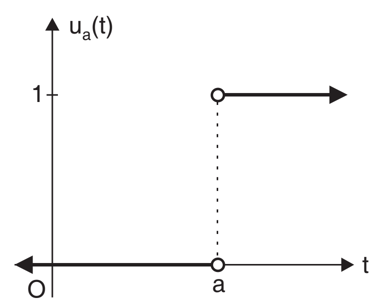
$$= \frac{3}{2}\left[\frac{1}{s^2-4s+13} - \frac{1}{s^2+4s+13}\right], \; s > 2$$

$$= \mathbf{\frac{12s}{s^4+10s^2+169}, \; s > 2.}$$

1.7. UNIT STEP FUNCTION

Let $a \geq 0$. The function of t taking value 0 if $t < a$ and 1 if $t > a$, is called a **unit step function** and is denoted by $u_a(t)$.

$$\therefore \qquad u_a(t) = \begin{cases} 0 & \text{if} \;\; t < a \\ 1 & \text{if} \;\; t > a. \end{cases}$$



1.8. SECOND SHIFTING THEOREM

If f(t) be a function of t for t ≥ 0 whose Laplace transform F(s) exists then for any constant a (≥ 0), the function f(t – a) $u_a(t)$ has the Laplace transform e^{-as} F(s).

Proof. $t < a \;\Rightarrow\; f(t-a)\, u_a(t) = f(t-a) \,.\, 0 = 0$

and $t > a \;\Rightarrow\; f(t-a)\, u_a(t) = f(t-a) \,.\, 1 = f(t-a)$

$$\therefore \qquad f(t-a)\, u_a(t) = \begin{cases} 0 & \text{if} \;\; t < a \\ f(t-a) & \text{if} \;\; t > a \end{cases}$$

Also, $$\mathrm{F}(s) = \mathrm{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \qquad \text{...(1)}$$

Now $$\mathrm{L}(f(t-a)\, u_a(t)) = \int_0^\infty e^{-st} f(t-a)\, u_a(t)\, dt$$

$$= \int_0^a e^{-st} f(t-a)\, u_a(t)\, dt + \int_a^\infty e^{-st} f(t-a)\, u_a(t)\, dt$$

$$= \int_0^a e^{-st} . 0\, dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

$$= 0 + \int_0^\infty e^{-s(a+u)} f(u)\, du, \text{where } u = t - a$$

$$= e^{-sa} \int_0^\infty e^{-su} f(u)\, du = e^{-sa} \int_0^\infty e^{-st} f(t)\, dt$$

(By replacing u by t)

$$= e^{-as}\ \mathrm{F}(s) \qquad \text{(By using (1))}$$

∴ **$\mathbf{L(f(t-a)\ u_a(t)) = e^{-as}\ F(s)}$, where $\mathbf{F(s) = L(f(t))}$.**

Remark. The result of above theorem can be easily remembered as follows :

If **$\mathbf{L(f(t)) = F(s)}$, then for any $\mathbf{a \geq 0}$, $\mathbf{L(f(t-a)\ u_a(t)) = e^{-as}\ F(s)}$.**

Example 3. *Find the Laplace transform of the function $t^2\, u_1(t)$.*

Sol. Let $g(t) = t^2\, u_1(t)$.

∴ $g(t) = ((t-1)+1)^2\, u_1(t) = f(t-1)\, u_1(t)$, where $f(t) = (t+1)^2$.

Now $\mathrm{L}(f(t)) = \mathrm{L}((t+1)^2) = \mathrm{L}(t^2 + 2t + 1)$

$$= \mathrm{L}(t^2) + 2\mathrm{L}(t) + \mathrm{L}(1) = \frac{2!}{s^3} + 2\left(\frac{1!}{s^2}\right) + \frac{1}{s},\ (s > 0) = \frac{2 + 2s + s^2}{s^3},\ s > 0.$$

∴ By **second shifting theorem,**

$$\mathrm{L}(g(t)) = \mathrm{L}(f(t-1))\, u_1(t)) = e^{-1.s}\ \mathrm{L}(f(t))$$

$$= \mathbf{e^{-s} . \frac{2 + 2s + s^2}{s^3},\ s > 0.}$$

Example 4. *Find the Laplace transform of the following function of t :*

$$g(t) = \begin{cases} 0, & 0 < t < \pi/2 \\ \sin t, & t > \pi/2. \end{cases}$$

Sol. Given function $g(t)$ can be written as

$$g(t) = \begin{cases} 0, & 0 < t < \pi/2 \\ \cos\left(t - \dfrac{\pi}{2}\right), & t > \dfrac{\pi}{2}. \end{cases}$$

or $$g(t) = \cos\left(t - \frac{\pi}{2}\right) u_{\pi/2}\,(t)$$

∴ $$g(t) = f\left(t - \frac{\pi}{2}\right) u_{\pi/2}\,(t),\ \text{ where } f(t) = \cos t$$

Now $\quad L(f(t)) = L(\cos t) = \dfrac{s}{s^2+1^2} = \dfrac{s}{s^2+1}, s > 0$

$\therefore$ By **second shifting theorem,**

$$L(g(t)) = L\left(f\left(t - \frac{\pi}{2}\right)u_{\pi/2}(t)\right)$$

$$= e^{-(\pi/2)s} L(f(t)) = \mathbf{\frac{e^{-\pi s/2}\, s}{s^2+1}, s > 0.}$$

Example 5. *Find the Laplace transform of the function* $g(t) = \begin{cases} 0, & 0 < t < 2 \\ t, & t > 2 \end{cases}$ *by using (i) definition (ii) second shifting theorem. Verify that the results are same :*

Sol. We have $g(t) = \begin{cases} 0, & 0 < t < 2 \\ t, & t > 2. \end{cases}$

(*i*) $L(g) = \displaystyle\int_0^\infty e^{-st}\, g(t)\, dt$

$$= \int_0^2 e^{-st}\, g(t)\, dt + \int_2^\infty e^{-st}\, g(t)\, dt = \int_0^2 e^{-st}\,.\,0\, dt + \int_2^\infty e^{-st}\, t\, dt$$

$$= 0 + t\,.\,\frac{e^{-st}}{-s}\Bigg|_2^\infty - \int_2^\infty 1\,.\,\frac{e^{-st}}{-s}\, dt = -\frac{1}{s}\left[\lim_{T\to\infty}\frac{T}{e^{sT}} - \frac{2}{e^{2s}}\right] + \frac{e^{-st}}{-s^2}\Bigg|_2^\infty$$

$$= -\frac{1}{s}\left[\lim_{T\to\infty}\frac{1}{se^{sT}} - \frac{2}{e^{2s}}\right] - \frac{1}{s^2}\left[\lim_{T\to\infty}\frac{1}{e^{sT}} - \frac{1}{e^{2s}}\right]$$

(Using L'Hospital's Rule)

$$= -\frac{1}{s}\left[0 - \frac{2}{e^{2s}}\right] - \frac{1}{s^2}\left[0 - \frac{1}{e^{2s}}\right] = \frac{2}{se^{2s}} + \frac{1}{s^2e^{2s}} = \mathbf{\frac{2s+1}{s^2\, e^{2s}},\ s > 0.}$$

(*ii*) $\quad g(t) = \begin{cases} 0, & 0 < t < 2 \\ (t-2)+2, & t > 2 \end{cases}.$

$\therefore \quad g(t) = f(t-2)\, u_2(t)$, where $f(t) = t + 2$

Now $\quad L(f(t)) = L(t + 2) = L(t) + 2L(1)$

$$= \frac{1!}{s^{1+1}} + 2\left(\frac{1}{s}\right), (s > 0) = \frac{1+2s}{s^2}, s > 0$$

By **second shifting theorem,**

$$L(g(t)) = L(f(t-2)\, u_2(t)) = e^{-2s}\, L(f(t))$$

$$= e^{-2s}\,.\,\frac{1+2s}{s^2}, (s > 0) = \mathbf{\frac{2s+1}{s^2\, e^{2s}},\ s > 0.}$$

1.9. CHANGE OF SCALE PROPERTY

Theorem. *If f(t) be a function of t for t ≥ 0 whose Laplace transform F(s) exists, then for any positive constant 'a', the function f(at) has the Laplace transform* $\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$.

Proof. We have $\quad F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$.

Now $\quad L(f(at)) = \int_0^\infty e^{-st} f(at)\, dt = \int_0^\infty e^{-sz/a} f(z) \frac{dz}{a}$, where $z = at$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)z} f(z)\, dz = \frac{1}{a} \int_0^\infty e^{-(s/a)t} f(t)\, dt$$

(By replacing variable z by t)

$$= \frac{1}{a} F\left(\frac{s}{a}\right).$$

$\therefore \quad \mathbf{L(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right).}$

Remark. The above theorem can be easily remembered as follows :

If $\quad$ **L(f(t)) = F(s), then for any a (> 0),** $\mathbf{L(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right).}$

Example 6. *Find $L(f(\lambda t))$ where λ is any positive constant and $f(t)$ is a function of t for $t \ge 0$:*

(*i*) *$f(t) = t^n$, $n \in \mathbf{N}$* $\quad$ (*ii*) *$f(t) = e^t$* $\quad$ (*iii*) *$f(t) = \sinh t$*

(*iv*) *$f(t) = \cosh t$* $\quad$ (*v*) *$f(t) = \sin t$* $\quad$ (*vi*) *$f(t) = \cos t$.*

Sol. (*i*) $f(t) = t^n$, $n \in \mathbf{N}$

$\therefore \quad L(f(t)) = L(t^n) = \frac{n!}{s^{n+1}}, s > 0$

By **change of scale property,** $L(f(\lambda t)) = \frac{1}{\lambda} \cdot \frac{n!}{(s/\lambda)^{n+1}}, \frac{s}{\lambda} > 0$

$$= \mathbf{\frac{\lambda^n\, n!}{s^{n+1}}, s > 0.} \qquad (\because \lambda > 0)$$

(*ii*) $f(t) = e^t$

$\therefore \quad L(f(t)) = L(e^t) = \frac{1}{s-1}, \quad s > 1$

By **change of scale property,** $L(f(\lambda t)) = \frac{1}{\lambda} \cdot \frac{1}{(s/\lambda) - 1}, \quad \frac{s}{\lambda} > 1$

$$= \mathbf{\frac{1}{s - \lambda}, s > \lambda.}$$

(*iii*) $f(t) = \sinh t$

$\therefore \quad L(f(t)) = L(\sinh t) = \frac{1}{s^2 - (1)^2}, (s > |1|) = \frac{1}{s^2 - 1}, s > 1$

By **change of scale property,** $L(f(\lambda t)) = \frac{1}{\lambda} \cdot \frac{1}{(s/\lambda)^2 - 1}, \quad \frac{s}{\lambda} > 1$

$$= \mathbf{\frac{\lambda}{s^2 - \lambda^2}, s > \lambda.}$$

(*iv*) $f(t) = \cosh t$

$$\therefore \quad L(f(t)) = L(\cosh t) = \frac{s}{s^2 - 1}, \; s > 1$$

By **change of scale property,** $L(f(\lambda t)) = \frac{1}{\lambda} \cdot \frac{s/\lambda}{(s/\lambda)^2 - 1}, \quad \frac{s}{\lambda} > 1$

$$= \frac{\mathbf{s}}{\mathbf{s^2} - \lambda^2}, \; \mathbf{s} > \lambda.$$

(*v*) $f(t) = \sin t$

$$\therefore \quad L(f(t)) = L(\sin t) = \frac{1}{s^2 + 1}, \; s > 0$$

By **change of scalar property,** $L(f(\lambda t)) = \frac{1}{\lambda} \cdot \frac{1}{(s/\lambda)^2 + 1}, \quad \frac{s}{\lambda} > 0$

$$= \frac{\lambda}{\mathbf{s^2} + \lambda^2}, \; \mathbf{s} > 0.$$

(*vi*) $f(t) = \cos t$

$$\therefore \quad L(f(t)) = L(\cos t) = \frac{s}{s^2 + 1}, \; s > 0$$

By **change of scalar property,** $L(f(\lambda t)) = \frac{1}{\lambda} \cdot \frac{s/\lambda}{(s/\lambda)^2 + 1}, \quad \frac{s}{\lambda} > 0$

$$= \frac{\mathbf{s}}{\mathbf{s^2} + \lambda^2}, \; \mathbf{s} > 0.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L(f(t)) = F(s)$, $s > k$ then for any a, $L(e^{at} f(t)) = F(s - a)$, $s > k + a$.

Rule II. For $a \geq 0$, the unit step function $u(t - a)$ is defined as $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a. \end{cases}$

Rule III. If $L(f(t)) = F(s)$, then for any $a \geq 0$, $L(f(t - a)\, u(t - a)) = e^{-as}\, L(f(t))$.

Rule IV. If $L(f(t)) = F(s)$, then for any $a (> 0)$, $L(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$.

TEST YOUR KNOWLEDGE

1. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $e^t t^k$, $k > -1$ (*ii*) $e^{2t} t^4$ (*iii*) $e^t \sinh t$

(*iv*) $e^{3t} \cosh 3t$ (*v*) $e^{2t} \sin t$ (*vi*) $e^{2t} \cos 2t$.

2. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $e^{-4t}\, t^{3/2}$ (*ii*) $e^{-6t}\, t^7$ (*iii*) $e^{-2t} \sinh 6t$

(*iv*) $e^{-5t} \cosh 3t$ (*v*) $e^{-5t} \sin 8t$ (*vi*) $e^{-3t} \cos 4t$.

3. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $e^{3t} \sin^2 t$ (*ii*) $(t + 2)^2 e^t$

(*iii*) $\sinh 3t \cos^2 t$ (*iv*) $\cosh at \sin at$. (*v*) $e^{-2t} \sin t \cos 3t$.

4. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $e^{-t} \sin^2 t$ (*ii*) $5e^{2t} \sinh 2t$

(*iii*) $e^{-at} \sinh bt$ (*iv*) $\cosh 4t \sin 6t$.

5. If the Laplace transform of the function $f(t)$ of t for $t \geq 0$ is $F(s)$, then show that

$$L[(\cosh at)\, f(t)] = \frac{1}{2}[F(s-a) + F(s+a)].$$

Hence evaluate $L(\cosh 3t \cos 2t)$.

6. Show that $L\{(1+te^{-t})^3\} = \dfrac{1}{s} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4},\ s > 0.$

7. Find the Laplace transform of the following functions :

(*i*) $(t - 1)\, u_1(t)$ (*ii*) $(5 \cos t)\, u_\pi(t)$.

8. Find the Laplace transform of the function : $g(t) = \begin{cases} 0, & 0 < t < \pi/2 \\ \cos t, & t > \pi/2. \end{cases}$

9. Find the Laplace transform of the function : $g(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & 0 < t < 2\pi/3. \end{cases}$

10. Find the Laplace transform of the function $g(t) = \begin{cases} 0, & 0 < t < 5 \\ t, & t > 5 \end{cases}$ by using (*i*) definition (*ii*) second shifting theorem. Verify that the results are same.

Answers

1. (*i*) $\dfrac{\Gamma(k+1)}{(s-1)^{k+1}},\ s > 1$ (*ii*) $\dfrac{24}{(s-2)^5},\ s > 2$ (*iii*) $\dfrac{1}{s^2 - 2s},\ s > 2$

(*iv*) $\dfrac{s}{s^2 - 6s},\ s > 6$ (*v*) $\dfrac{1}{s^2 - 4s + 5},\ s > 2$ (*vi*) $\dfrac{s-2}{s^2 - 4s + 8},\ s > 2$

2. (*i*) $\dfrac{3\sqrt{\pi}}{4\,(s+4)^{5/2}},\ s > -4$ (*ii*) $\dfrac{5040}{(s+6)^8},\ s > -6$ (*iii*) $\dfrac{6}{s^2 + 4s - 32},\ s > 4$

(*iv*) $\dfrac{s+5}{s^2 + 10s + 16},\ s > -2$ (*v*) $\dfrac{8}{s^2 + 10s + 89},\ s > -5$ (*vi*) $\dfrac{s+3}{s^2 + 6s + 25},\ s > -3$

3. (*i*) $\dfrac{2}{(s-3)(s^2 - 6s + 13)},\ s > 3$ (*ii*) $\dfrac{2}{(s-1)^3}[(2s^2 - 2s + 1)],\ s > 1$

(*iii*) $\dfrac{1}{2}\left[\dfrac{s^2 - 6s + 11}{(s-3)(s^2 - 6s + 13)} + \dfrac{s^2 + 6s + 11}{(s+3)(s^2 + 6s + 13)}\right],\ s > 3$ (*iv*) $\dfrac{a(s^2 + 2a^2)}{s^4 + 4a^4},\ s > |a|$

(*v*) $\dfrac{2}{s^2 + 4s + 20} - \dfrac{1}{s^2 + 4s + s},\ s > -2$

4. (i) $\dfrac{2}{(s+1)(s^2+2s+5)}$, $s > -1$ (ii) $\dfrac{10}{s^2-4s}$, $s > 4$

(iii) $\dfrac{b}{s^2+2as+a^2-b^2}$, $s > |b| - a$ (iv) $\dfrac{6(s^2+52)}{s^4+40s^2+2704}$, $s > 4$

5. $\dfrac{s^3-5s}{s^4-10s^2+169}$, $s > 3$ **7.** (i) $\dfrac{e^{-s}}{s^2}$, $s > 0$ (ii) $\dfrac{-5se^{-\pi s}}{s^2+1}$, $s > 0$

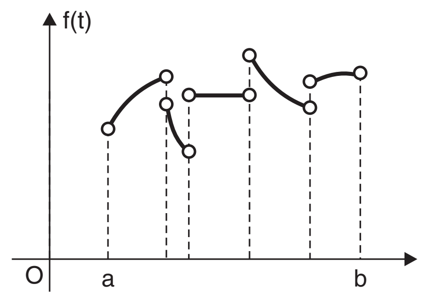
8. $-\dfrac{e^{-\pi s/2}}{s^2+1}$, $s > 0$ **9.** $\dfrac{e^{-2\pi s/3}s}{s^2+1}$, $s > 0$ **10.** $\dfrac{(5s+1)e^{-5s}}{s^2}$, $s > 0$

Hint

8. Use $\cos t = -\sin\left(t - \dfrac{\pi}{2}\right)$.

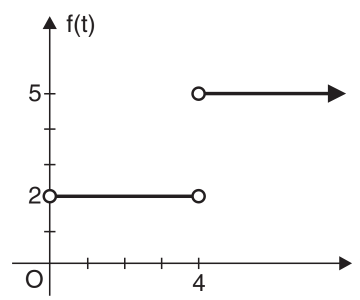
1.10. PIECEWISE CONTINUOUS FUNCTION

A function $f(t)$ is called **piecewise continuous (**or **sectionally continuous)** on a finite interval $[a, b]$ if this interval can be divided into a finite number of sub-intervals such that (i) $f(t)$ is continuous in the interior of each of these sub-intervals and (ii) $f(t)$ approaches a finite limit as t approaches either endpoint of each of the sub-interval from its interior.



Thus a piecewise continuous function has finitely many jumps as discontinuities. Clearly every continuous function is a piecewise continuous function. Also, if $f(t)$ is a piecewise continuous function on $[a, b]$, then $f(x)$ is integrable on $[a, b]$.

For example, let $f(t) = \begin{cases} 2, & 0 < t < 4 \\ 5, & t > 4. \end{cases}$



The function $f(x)$ is piecewise continuous on every finite interval $[0, b]$, where b is a positive real number.

1.11. EXISTENCE THEOREM FOR LAPLACE TRANSFORMS

The Laplace transform of any arbitrary function may not exist. For the existence of the Laplace transform of a function, the function is expected to fulfil certain conditions. In the ensuing theorem, we shall establish the criterion for the existence of Laplace transform of a function.

Theorem. *Let f(t) be a real function such that*

(i) *f(t) is piece wise continuous on every finite interval in the range* $t \geq 0$ *and*

(ii) *there exists constants k and M such that* $|f(t)| \leq Me^{kt}$ *for* $t \geq 0$, *then prove that the Laplace transform of f(t) exists for all* $s > k$.

Proof. Since $f(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$, $f(t)$ is piece wise continuous on the finite interval [0, T] for T > 0.

Also e^{-st} is continuous on the finite interval [0, T] for T > 0

$\therefore$ $e^{-st} f(t)$ is piecewise continuous on [0, T] for T > 0.

$\therefore$ $e^{-st} f(t)$ is integrable on [0, T] for T > 0.

$\therefore$ $\int_0^T e^{-st} f(t)\, dt$ exists for all T > 0.

Now, $$\int_0^\infty e^{-st} f(t)\, dt = \lim_{T \to \infty} \int_0^T e^{-st} f(t)\, dt$$

and $$\left| \int_0^\infty e^{-st} f(t)\, dt \right| \leq \int_0^\infty \left| e^{-st} f(t) \right| dt = \int_0^\infty e^{-st} |f(t)|\, dt$$

$$\leq \int_0^\infty e^{-st} \,.\, Me^{kt}\, dt = M \int_0^\infty e^{(k-s)t}\, dt = M \lim_{T \to \infty} \left[\frac{e^{(k-s)t}}{k-s} \right]_0^T$$

$$= \frac{M}{k-s} \lim_{T \to \infty} \left[\frac{1}{e^{(s-k)T}} - 1 \right] = \frac{M}{k-s} [0-1] \quad \text{for } s > k$$

$$= \frac{M}{s-k} \quad \text{for } s > k\,.$$

$$\therefore \quad \int_0^\infty e^{-st} f(t)\, dt \leq \frac{M}{s-k} \quad \text{for } s > k$$

$$\therefore \quad L(f) = \int_0^\infty e^{-st} f(t)\, dt \text{ exists for } s > k.$$

1.12. LAPLACE TRANSFORMS OF DERIVATIVES

Theorem 1. *(Laplace transform of the first derivative). Let f(t) be a real function such that*

(*i*) *f(t) is continuous for all $t \geq 0$*

(*ii*) *there exists constants k and M such that $|f(t)| \leq Me^{kt}$ for $t \geq 0$*

(*iii*) *$f'(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$,*

then the Laplace transform of the derivative $f'(t)$ exists and

$$L(f') = sL(f) - f(0) \quad \text{for } s > k.$$

Proof. By definition

$L(f') = \lim_{T \to \infty} \int_0^T e^{-st} f'(t)\, dt$, provided this limit has some finite value.

Let T be any positive number.

Using (*iii*), $f'(t)$ is piecewise continuous on [0, T].

$\therefore$ $f'(t)$ has at most a finite number of discontinuities, say $t_1, t_2, \ldots\ldots, t_n$, where

$$0 < t_1 < t_2 < \ldots\ldots < t_n \leq T$$

$$\therefore \quad \int_0^T e^{-st} f'(t)\, dt = \int_0^{t_1} e^{-st} f'(t)\, dt + \ldots\ldots + \int_{t_n}^T e^{-st} f'(t)\, dt$$

The integrand of each of the integrals on the right is continuous.

$\therefore$ By using integration by parts, we have

$$\int_0^T e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]_0^{t_1} + s\int_0^{t_1} e^{-st} f(t)\,dt + \ldots\ldots + \left[e^{-st} f(t)\right]_{t_n}^{T} + s\int_{t_n}^{T} e^{-st} f(t)\,dt$$

$$= \left[e^{-st_1} f(t_1) - f(0)\right] + s\int_0^{t_1} e^{-st} f(t)\,dt + \ldots\ldots$$

$$+ \left[e^{-sT} f(T) - e^{-st_n} f(t_n)\right] + s\int_{t_n}^{T} e^{-st} f(t)\,dt$$

$$= e^{-sT} f(T) - f(0) + s\left[\int_0^{t_1} e^{-st} f(t)\,dt + \ldots\ldots + \int_{t_n}^{T} e^{-st} f(t)\,dt\right]$$

$$\therefore \quad \int_0^T e^{-st} f'(t)\,dt = e^{-sT} f(T) - f(0) + s\int_0^T e^{-st} f(t)\,dt$$

($\because$ f is continuous for $t \geq 0$)

$$\therefore \quad \lim_{T\to\infty} \int_0^T e^{-st} f'(t)\,dt = \lim_{T\to\infty} e^{-sT} f(T) - f(0) + \lim_{T\to\infty} s\int_0^T e^{-st} f(t)\,dt \qquad \ldots(1)$$

Now using (*ii*), $\quad e^{-sT} f(T) \leq e^{-sT}\,|f(T)| \leq e^{-sT}\,.\,Me^{kT} = \dfrac{M}{e^{(s-k)T}}$

$$\therefore \quad \lim_{T\to\infty} e^{-sT} f(T) = 0 \text{ for } s > k$$

Also $\quad \displaystyle\lim_{T\to\infty} s\int_0^T e^{-st} f(t)\,dt = sL(f)$

$\therefore$ (1) $\Rightarrow$ $\displaystyle\lim_{T\to\infty} \int_0^T e^{-st} f'(t)\,dt = 0 - f(0) + sL(f)$.

$\therefore$ **L(f′) = sL(f) – f(0) for s > k**.

ILLUSTRATIVE EXAMPLES

Example 1. *Using* $L(t^n) = \dfrac{n!}{s^{n+1}}$ *, find the value of* $L(t^{n+1})$.

Sol. Let $\quad f(t) = t^{n+1}$.

$\therefore \quad f'(t) = (n+1)\,t^n$.

We have $\quad L(f') = sL(f) - f(0)$.

$\therefore \quad L((n+1)\,t^n) = sL(t^{n+1}) - (0)^{n+1} \Rightarrow (n+1)\,L(t^n) = sL(t^{n+1})$

$$\therefore \quad L(t^{n+1}) = \frac{n+1}{s} L(t^n) = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \boldsymbol{\frac{(n+1)!}{s^{n+2}}}.$$

Example 2. *Find the Laplace transform of the function* $\sin^2 at \cos at$, $t \geq 0$.

Sol. Let $\quad f(t) = \sin^3 at$.

$\therefore \quad f'(t) = 3\sin^2 at\,.\,a\cos at = 3a\cos at\sin^2 at$

We have $\quad L(f') = sL(f) - f(0).$

$\therefore \quad L(3a \cos at \sin^2 at) = sL(\sin^3 at) - \sin^3 0$

$\Rightarrow \quad 3a\, L(\cos at \sin^2 at) = sL(\sin^3 at)$ ...(1)

Now, $\quad \sin 3x = 3 \sin x - 4 \sin^3 x \quad \Rightarrow \quad \sin^3 x = \dfrac{3}{4}\sin x - \dfrac{1}{4} \sin 3x$

$$\therefore \quad \sin^3 at = \frac{3}{4}\sin at - \frac{1}{4}\sin 3at$$

$$\therefore \quad L(\sin^3 at) = \frac{3}{4} L(\sin at) - \frac{1}{4} L(\sin 3at)$$

$$= \frac{3}{4}\left(\frac{a}{s^2+a^2}\right) - \frac{1}{4}\left(\frac{3a}{s^2+(3a)^2}\right)$$

$$= \frac{3a}{4}\left(\frac{1}{s^2+a^2} - \frac{1}{s^2+9a^2}\right) = \frac{6a^3}{(s^2+a^2)(s^2+9a^2)}$$

$$\therefore \quad (1) \quad \Rightarrow \quad 3a\, L(\cos at \sin^2 at) = s \,.\, \frac{6a^3}{(s^2+a^2)(s^2+9a^2)} - 0$$

$$\therefore \quad L(\cos at \sin^2 at) = \frac{1}{3a}\cdot\frac{6a^3 s}{(s^2+a^2)(s^2+9a^2)} = \boldsymbol{\frac{2a^2 s}{(s^2+a^2)(s^2+9a^2)}}.$$

Alternative method

$$\sin^2 at \cos at = \frac{1}{2}.\sin at \,.\, 2 \sin at \cos at$$

$$= \frac{1}{2}\sin at \sin 2at = \frac{1}{4}.\, 2 \sin 2at \sin at$$

$$= \frac{1}{4}(\cos at - \cos 3at).$$

$$\therefore \quad L(\sin^2 at \cos at) = L\left(\frac{1}{4}(\cos at - \cos 3at)\right)$$

$$= \frac{1}{4}[L(\cos at) - L(\cos 3at)] = \frac{1}{4}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+9a^2}\right]$$

$$= \frac{s}{4}\left[\frac{8a^2}{(s^2+a^2)(s^2+9a^2)}\right] = \boldsymbol{\frac{2a^2 s}{(s^2+a^2)(s^2+9a^2)}}.$$

Theorem 2. *(Laplace transform of the derivative of any order n). Let f(t) be a real function such that*

(*i*) *$f(t), f'(t), f''(t), \ldots\ldots, f^{(n-1)}(t)$ are continuous for all $t \geq 0$*

(*ii*) *there exists constants k and M such that*

$| f(t) | \leq Me^{kt}$ *for* $t \geq 0$ *and* $\quad | f^{(i)}(t) | \leq Me^{kt}$ *for* $t \geq 0$ *and* $i = 1, 2, \ldots\ldots, n-1$

(*iii*) *$f^{(n)}(t)$ is piece wise continuous on every finite interval in the range $t \geq 0$ then the Laplace transform of $f^{(n)}(t)$ exists and*

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots\ldots - f^{(n-1)}(0) \quad \text{for } s > k.$$

Proof. Using (*i*), $f'(t)$ is piece wise continuous on every finite interval in the range $t \geq 0$.

$\therefore$ By **theorem 1,** $L(f') = s\,L(f) - f(0),\ s > k$

By applying **theorem 1** to the function $f'(t)$, we get

$$L(f'') = s\,L(f') - f'(0).$$

$$\therefore \quad L(f'') = s\,[s\,L(f) - f(0)] - f'(0)$$

or
$$L(f'') = s^2\,L(f) - s\,f(0) - f'(0)$$

Let the result be true for $f^{(m)}(t)$, the mth derivative of $f(t)$.

$$\therefore \quad L(f^{(m)}) = s^m\,L(f) - s^{m-1}\,f(0) - \ldots\ldots f^{(m-1)}(0) \qquad \ldots(1)$$

Now (*i*) $f^{(m)}(t)$ is continuous for all $t \geq 0$.

(*ii*) $|\,f^{(m)}(t)\,| \leq M\,e^{kt}$ for $t \geq 0$.

(*iii*) By given condition (*i*), $f^{(m+1)}(t)$ is piece wise continuous on every finite interval in the range $t \geq 0$.

$\therefore$ By **theorem 1**, the Laplace transform of the derivative $f^{(m+1)}(t)$ exists and

$$L(f^{(m+1)}) = s\,L(f^{(m)}) - f^{(m)}(0). \qquad \ldots(2)$$

Using (1), we get

$$L(f^{(m+1)}) = s\,[s^m\,L(f) - s^{m-1}\,f(0) - \ldots\ldots - f^{(m-1)}(0)] - f^{(m)}(0)$$

or
$$L(f^{(m+1)}) = s^{m+1}\,L(f) - s^m\,f(0) - s^{m-1}\,f(0) - \ldots\ldots - f^{m}(0)$$

$\therefore$ By P.M.I., we have

$$\mathbf{L(f^{(n)}) = s^n\,L(f) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \ldots\ldots - f^{(n-1)}(0) \quad for\ s > k.}$$

In particular

$n = 1 \Rightarrow$ $\mathbf{L(f') = sL(f) - f(0)}$

$n = 2 \Rightarrow$ $\mathbf{L(f'') = s^2L(f) - sf(0) - f'(0)}$

$n = 3 \Rightarrow$ $\mathbf{L(f''') = s^3L(f) - s^2f(0) - sf'(0) - f''(0).}$

Example 3. *Given* $L(t^3) = \dfrac{6}{s^4}$, *find the value of* $L(t^6)$.

Sol. Let $f(t) = t^6$.

$\therefore$ $f'(t) = 6t^5,\ f''(t) = 30t^4$ and $f'''(t) = 120t^3$.

We have $L(f''') = s^3L(f) - s^2\,f(0) - sf'(0) - f''(0)$.

$$\therefore \quad L(120t^3) = s^3L(t^6) - s^2\,.\,(0)^6 - s\,.\,6(0)^5 - 30(0)^4$$

$$\Rightarrow \quad 120L(t^3) = s^3L(t^6) - 0 - 0 - 0$$

$$\Rightarrow \quad 120\left(\frac{6}{s^4}\right) = s^3L(t^6) \qquad \text{(Given)}$$

$$\Rightarrow \quad L(t^6) = \frac{1}{s^3}\left(\frac{720}{s^4}\right) = \mathbf{\frac{720}{s^7}}.$$

Example 4. *Find the Laplace transform of the function cos at, t ≥ 0.*

Sol. Let $f(t) = \cos at,\ t \geq 0$.

$\therefore$ $f'(t) = -\,a\sin at$ and $f''(t) = -\,a^2\cos at$.

We have $L(f'') = s^2\,L(f) - sf(0) - f'(0)$.

$\therefore \quad L(-a^2 \cos at) = s^2 L(\cos at) - s \cos 0 - (-a \sin 0)$

$\Rightarrow \quad -a^2 L(\cos at) = s^2 L(\cos at) - s - 0$

$\Rightarrow \quad (s^2 + a^2) L(\cos at) = s$

$$\therefore \quad L(\cos at) = \mathbf{\frac{s}{s^2 + a^2}}.$$

Example 5. *Find the Laplace transform of the function t sin bt.*

Sol. Let $f(t) = t \sin bt$.

$\therefore \quad f'(t) = t(b \cos bt) + 1 \,.\, \sin bt = bt \cos bt + \sin bt$

and $\quad f''(t) = b[t(-b \sin bt) + 1 \,.\, \cos bt] + b \cos bt = -b^2 t \sin bt + 2b \cos bt$

We have $\quad L(f'') = s^2 L(f) - sf(0) - f'(0)$.

$\therefore \quad L(-b^2 t \sin bt + 2b \cos bt) = s^2 L(t \sin bt) - s(0 \,.\, \sin 0) - (b \,.\, 0 \,.\, \cos 0 + \sin 0)$

$\Rightarrow \quad -b^2 L(t \sin bt) + 2b L(\cos bt) = s^2 L(t \sin bt) - 0 - (0 + 0)$

$$\Rightarrow \quad (s^2 + b^2) L(t \sin bt) = 2b L(\cos bt) = 2b \left(\frac{s}{s^2 + b^2}\right)$$

$$\Rightarrow \quad L(t \sin bt) = \frac{1}{s^2 + b^2}\left(\frac{2bs}{s^2 + b^2}\right) = \mathbf{\frac{2bs}{(s^2 + b^2)^2}}.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. $L(f') = sL(f) - f(0)$.

Rule II. $L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots\ldots - f^{(n-1)}(0)$.

Rule III. (*i*) $L(f'') = s^2 L(f) - sf(0) - f'(0)$

(*ii*) $L(f''') = s^3 L(f) - s^2 f(0) - sf'(0) - f''(0)$.

TEST YOUR KNOWLEDGE

1. Using $L(1) = \frac{1}{s}$, find $L(t)$.

2. (*i*) Using $L(\sinh at) = \frac{a}{s^2 - a^2}$, find $L(\cosh at)$.

(*ii*) Using $L(\cosh at) = \frac{s}{s^2 - a^2}$, find $L(\sinh at)$.

3. (*i*) Using $L(\sin bt) = \frac{b}{s^2 + b^2}$, find $L(\cos bt)$.

(*ii*) Using $L(\cos bt) = \frac{s}{s^2 + b^2}$, find $L(\sin bt)$.

4. Find the Laplace transform of the function $\sin bt \cos^2 bt$, $t \geq 0$.

5. Using $L(1) = \frac{1}{s}$, find $L(t^2)$.

6. Given $L(t^2) = \frac{2}{s^3}$, find the value of $L(t^5)$.

7. Find the Laplace transform of the following functions by using the result of Laplace transform of second derivative of a function :

(*i*) e^{at} (*ii*) sinh at (*iii*) cosh at (*iv*) sin at.

8. Find the Laplace transform of the function t cos at.

Answers

1. $\dfrac{1}{s^2}$ **2.** (*i*) $\dfrac{s}{s^2 - a^2}$ (*ii*) $\dfrac{a}{s^2 - a^2}$ **3.** (*i*) $\dfrac{s}{s^2 + b^2}$ (*ii*) $\dfrac{b}{s^2 + b^2}$

4. $\dfrac{b\,(s^2 + 3b^2)}{(s^2 + b^2)\,(s^2 + 9b^2)}$ **5.** $\dfrac{2}{s^3}$ **6.** $\dfrac{120}{s^6}$ **7.** (*i*) $\dfrac{1}{s - a}$

(*ii*) $\dfrac{a}{s^2 - a^2}$ (*iii*) $\dfrac{s}{s^2 - a^2}$ (*iv*) $\dfrac{a}{s^2 + a^2}$ **8.** $\dfrac{s^2 - a^2}{(s^2 + a^2)^2}$.

Hint

8. Let $f(t) = t \cos at$.
Use $L(f'') = s^2 L(f) - sf(0) - f'(0)$.

1.13. LAPLACE TRANSFORMS OF INTEGRALS

Theorem. *Let f(t) be a real function such that*

(*i*) *f(t) is piece wise continuous on every finite interval in the range* $t \geq 0$

(*ii*) *there exists constants k and M such that* $|f(t)| \leq Me^{kt}$ *for* $t \geq 0$, *then the Laplace transform of the integral* $\int_0^t f(T)\,dT$ *exists and*

$$L\left(\int_0^t f(T)\,dT\right) = \frac{1}{s}\,L(f(t)) \text{ for } s > k.$$

Proof. If k in (*ii*) is negative, then $-k > 0$ and

$$(ii) \Rightarrow |f(t)| \leq Me^{kt} = \frac{M}{e^{-kt}} \leq Me^{-kt}. \qquad (\because \; e^{-kt} > 1)$$

$\therefore$ We can assume that k is positive.

Let $$g(t) = \int_0^t f(T)\,dT, \quad t \geq 0$$

$\therefore$ $g(t)$ is continuous for all $t \geq 0$.

Now $$|g(t)| = \left| \int_0^t f(T)\,dT \right| \leq \int_0^t |f(T)|\,dT \leq \int_0^t |Me^{kT}|\,dT = M\int_0^t e^{kT}\,dT$$

$$= \frac{M}{k}(e^{kt} - 1) \leq \frac{M}{k}e^{kt}. \qquad (\because \; k > 0 \Rightarrow e^{kt} > 1)$$

$$\therefore \quad |g(t)| \leq \frac{M}{k}e^{kt} \quad \text{for } t \geq 0$$

Also $g'(t) = f(t)$ except for points at which $f(t)$ is discontinuous.

$\therefore$ $g'(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$.

$\therefore$ The Laplace transform of $g'(t)$ exists and

$$L(g') = sL(g) - g(0) \quad \text{for } s > k$$

$$\Rightarrow \qquad L(g') = sL(g) \quad \text{for } s > k \qquad \left(\because \ g(0) = \int_0^0 f(T)\, dT = 0\right)$$

$$\Rightarrow \qquad L(g) = \frac{1}{s} L(g')$$

$$\Rightarrow \qquad \mathbf{L\left(\int_0^t f(T)\, dT\right) = \frac{1}{s} L(f(t)) \quad \text{for } s > k.}$$

WORKING RULE FOR SOLVING PROBLEMS

Rule $\quad L\left(\int_0^t f(T)\, dT\right) = \frac{1}{s} L(f(t))$.

ILLUSTRATIVE EXAMPLES

Example 1. *Using* $L(1) = \frac{1}{s}$, *find* $L(t^2)$.

Sol. Let $f(t) = 1$. $\quad \therefore \quad L(f) = L(1) = \frac{1}{s}$ (Given)

We have $\quad L\left(\int_0^t f(T)\, dT\right) = \frac{1}{s} L(f)$.

$$\therefore \qquad L\left(\int_0^t 1\, dT\right) = \frac{1}{s}\left(\frac{1}{s}\right)$$

$$\Rightarrow \qquad L\left(T\Big|_0^t\right) = \frac{1}{s^2} \ \Rightarrow \ L(t) = \frac{1}{s^2}.$$

Let $\quad g(t) = t. \quad \therefore \quad L(g) = L(t) = \frac{1}{s^2}$

We have $\quad L\left(\int_0^t g(T)\, dT\right) = \frac{1}{s} L(g)$

$$\therefore \qquad L\left(\int_0^t T\, dT\right) = \frac{1}{s} \cdot \frac{1}{s^2}$$

$$\Rightarrow \qquad L\left(\frac{T^2}{2}\Bigg|_0^t\right) = \frac{1}{s^3} \ \Rightarrow \ L\left(\frac{t^2}{2}\right) = \frac{1}{s^3}$$

$$\Rightarrow \qquad \frac{1}{2} L(t^2) = \frac{1}{s^3} \ \Rightarrow \ L(t^2) = \mathbf{\frac{2}{s^3}}.$$

Example 2. *Using* $L\,(\sin bt) = \dfrac{b}{s^2 + b^2}$ *, find L (cos bt).*

Sol. Let $f(t) = \sin bt$. $\therefore$ $L(f) = L\,(\sin bt) = \dfrac{b}{s^2 + b^2}$. (Given)

We have $$L\left(\int_0^t f(T)\,dT\right) = \frac{1}{s}\,L(f).$$

$$\therefore \quad L\left(\int_0^t \sin bT\,dT\right) = \frac{1}{s}\cdot\frac{b}{s^2+b^2} \quad \Rightarrow \quad L\left(-\left.\frac{\cos bT}{b}\right|_0^t\right) = \frac{b}{s\,(s^2+b^2)}$$

$$\Rightarrow \quad L\left(\frac{1}{b} - \frac{\cos bT}{b}\right) = \frac{b}{s\,(s^2+b^2)} \quad \Rightarrow \quad \frac{1}{b}\,L(1) - \frac{1}{b}\,L\,(\cos bt) = \frac{b}{s\,(s^2+b^2)}$$

$$\Rightarrow \quad \frac{1}{b}\,L(\cos bt) = \frac{1}{b}\cdot\frac{1}{s} - \frac{b}{s\,(s^2+b^2)} = \frac{s}{b\,(s^2+b^2)}$$

$$\therefore \quad L(\cos bt) = \frac{\mathbf{s}}{\mathbf{s^2 + b^2}}.$$

Example 3. *Using* $L(1) = \dfrac{1}{s}$*, find the Laplace transform of the function cos bt.*

Sol. Let $f(t) = \cos bt$.

We have $$L\left(\int_0^t f(T)\,dT\right) = \frac{1}{s}\,L(f).$$

$$\Rightarrow \quad L\left(\int_0^t \cos bT\,dT\right) = \frac{1}{s}\,L(\cos bt) \quad \Rightarrow \quad L\left(\left.\frac{\sin bT}{b}\right|_0^t\right) = \frac{1}{s}\,L(\cos bt)$$

$$\Rightarrow \quad L\left(\frac{\sin bt}{b} - 0\right) = \frac{1}{s}\,L(\cos bt) \quad \Rightarrow \quad L(\sin bt) = \frac{b}{s}\,L(\cos bt) \qquad \text{...(1)}$$

Let $g(t) = \sin bt$.

We have $$L\left(\int_0^t g(T)\,dT\right) = \frac{1}{s}\,L(g).$$

$$\Rightarrow \quad L\left(\int_0^t \sin bT\,dT\right) = \frac{1}{s}\,L(\sin bt) = \frac{1}{s}\cdot\left(\frac{b}{s}\,L(\cos bt)\right) \qquad \text{(By using (1))}$$

$$\Rightarrow \quad L\left(-\left.\frac{\cos bT}{b}\right|_0^t\right) = \frac{b}{s^2}\,L(\cos bt) \quad \Rightarrow \quad L\left(-\frac{\cos bt}{b} + \frac{1}{b}\right) = \frac{b}{s^2}\,L(\cos bt)$$

$$\Rightarrow \quad -\frac{1}{b}\,L(\cos bt) + \frac{1}{b}\,L(1) = \frac{b}{s^2}\,L(\cos bt) \quad \Rightarrow \quad \frac{1}{b}\cdot\frac{1}{s} = \left(\frac{b}{s^2} + \frac{1}{b}\right)L(\cos bt)$$

$$\Rightarrow \quad \frac{1}{bs} = \frac{b^2 + s^2}{bs^2}\,L(\cos bt) \quad \Rightarrow \quad L(\cos bt) = \frac{\mathbf{s}}{\mathbf{s^2 + b^2}}.$$

TEST YOUR KNOWLEDGE

1. Using $L(t) = \frac{1}{s^2}$, find the values of $L(t^2)$ and $L(t^3)$.

2. Using $L(\cos at) = \frac{s}{s^2 + a^2}$, find $L(3 \sin 8t)$.

3. Using $L(1) = \frac{1}{s}$ and $L(\sinh at) = \frac{a}{s^2 - a^2}$, find $L(4 \cosh 6t)$.

4. Using $L(\cosh at) = \frac{s}{s^2 - a^2}$, find $L(2 \sinh pt)$.

5. Using $L(1) = \frac{1}{s}$, find the Laplace transform of the function $\frac{1}{5} \sin \pi t$.

6. Using $L(1) = \frac{1}{s}$, find the Laplace transform of the function $b \cosh pt$.

7. Using $L(1) = \frac{1}{s}$, find the Laplace transform of the function $b \sinh at$.

8. Evaluate $L\left(\int_0^t e^{-x} \cos x \, dx\right)$.

Answers

1. $\frac{2}{s^3}, \frac{6}{s^4}$ **2.** $\frac{24}{s^2 + 64}$ **3.** $\frac{4s}{s^2 - 36}$ **4.** $\frac{2p}{s^2 - p^2}$

5. $\frac{\pi}{5(s^2 + \pi^2)}$ **6.** $\frac{bs}{s^2 - p^2}$ **7.** $\frac{ab}{s^2 - a^2}$ **8.** $\frac{s+1}{s(s^2 + 2s + 2)}$.

1.14. DIFFERENTIATION OF LAPLACE TRANSFORMS

Theorem. *Let f(t) be a real function such that*

(*i*) *f(t) is piecewise continuous on every finite interval in the range* $t \geq 0$

(*ii*) *there exists constants k and M such that* $|f(t)| \leq Me^{kt}$ *for* $t \geq 0$, *then*

$$F'(s) = -L(t\, f(t)), \text{ where } F(s) = L(f(t)).$$

Note. The proof of this theorem is beyond the scope of this book.

Remark. The result of the above theorem can be easily remembered as follows :

If $L(f(t)) = F(s)$, then $L(t\, f(t)) = -F'(s)$.

Corollary. We have $L(t\, f(t)) = -F'(s)$.

Applying this result successively, we get

$$L(t^2 f(t)) = (-1)^2 F''(s)$$

$$L(t^3 f(t)) = (-1)^3 F'''(s)$$

....

$$\therefore \quad \mathbf{L(t^n f(t)) = (-1)^n F^{(n)}(s).}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Laplace transform of the following functions of t for t ≥ 0 :*

(*i*) $t\, e^{at}$ (*ii*) $t^2 \cosh \pi t$ (*iii*) $t^2 \sin 3t$.

Sol. (*i*) We have $\quad L(e^{at}) = \dfrac{1}{s-a}$

∴ By **differentiability of Laplace transform,**

$$L(t \,.\, e^{at}) = -\frac{d}{ds}\left(\frac{1}{s-a}\right) = -(-(s-a)^{-2}) = \mathbf{\frac{1}{(s-a)^2}}.$$

(*ii*) We have $\quad L(\cosh \pi t) = \dfrac{s}{s^2 - \pi^2}$

∴ By **differentiability of Laplace transform,**

$$L(t \,.\, \cosh \pi t) = -\frac{d}{ds}\left(\frac{s}{s^2-\pi^2}\right) = -\left[\frac{(s^2-\pi^2)\,.\,1 - s\,.\,2s}{(s^2-\pi^2)^2}\right] = \frac{s^2+\pi^2}{(s^2-\pi^2)^2}$$

Again by using the same formula,

$$L(t \,.\, t \cosh \pi t) = -\frac{d}{ds}\left(\frac{s^2+\pi^2}{(s^2-\pi^2)^2}\right)$$

$$= -\left[\frac{(s^2-\pi^2)^2\,.\,2s - (s^2+\pi^2)\,.\,2(s^2-\pi^2)\,.\,2s}{(s^2-\pi^2)^4}\right] = \frac{2s(s^2+3\pi^2)}{(s^2-\pi^2)^3}.$$

$$\therefore \qquad L(t^2 \cosh \pi t) = \mathbf{\frac{2s(s^2+3\pi^2)}{(s^2-\pi^2)^3}}.$$

Alternative method

$$L(t^2 \cosh \pi t) = L\left(t^2 \,.\, \frac{e^{\pi t} + e^{-\pi t}}{2}\right) = \frac{1}{2}\, L(e^{\pi t}\, t^2 + e^{-\pi t}\, t^2)$$

$$= \frac{1}{2}\,[L(e^{\pi t}\, t^2) + L(e^{-\pi t}\, t^2)] = \frac{1}{2}\left[\frac{2\,!}{(s-\pi)^3} + \frac{2\,!}{(s-(-\pi))^3}\right]$$

(Using $L(t^2) = \dfrac{2\,!}{s^3}$ and first shifting theorem)

$$= \frac{1}{(s-\pi)^3} + \frac{1}{(s+\pi)^3} = \frac{(s+\pi)^3 + (s-\pi)^3}{(s^2-\pi^2)^3}$$

$$= \frac{2s^3 + 6s\pi^2}{(s^2-\pi^2)^3} = \frac{2s(s^2+3\pi^2)}{(s^2-\pi^2)^3}.$$

(*iii*) We have $\quad L(\sin 3t) = \dfrac{3}{s^2+(3)^2} = \dfrac{3}{s^2+9}$

By **differentiability of Laplace transform,**

$$L(t \cdot \sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -3(-(s^2+9)^{-2} \cdot 2s) = \frac{6s}{(s^2+9)^2}$$

Again by using the same formula,

$$L(t\,.\,t\sin 3t) = -\frac{d}{ds}\left(\frac{6s}{(s^2+9)^2}\right)$$

$$= -6\left[\frac{(s^2+9)^2\,.\,1 - s\,.\,2\,(s^2+9)\,.\,2s}{(s^2+9)^4}\right] = \frac{18(s^2-3)}{(s^2+9)^3}$$

$\therefore$ $$L(t^2\sin 3t) = \mathbf{\frac{18(s^2-3)}{(s^2+9)^3}}.$$

Example 2. *Find the Laplace transform of the function $3te^{5t}\sin 7t$ for $t \geq 0$.*

Sol. We have $L(\sin 7t) = \dfrac{7}{s^2+49}$.

By **first shifting theorem,**

$$L(e^{5t}\sin 7t) = \frac{7}{(s-5)^2+49} = \frac{7}{s^2-10s+74}.$$

By **differentiability of Laplace transform,**

$$L(te^{5t}\sin 7t) = -\frac{d}{ds}\left(\frac{7}{s^2-10s+74}\right) = \frac{7\,.\,(2s-10)}{(s^2-10s+74)^2} = \frac{14\,(s-5)}{(s^2-10s+74)^2}$$

By **linearity of Laplace transform,**

$$L(3te^{5t}\sin 7t) = 3L(te^{5t}\sin 7t) = \mathbf{\frac{42\,(s-5)}{(s^2-10s+74)^2}}.$$

Example 3. *Prove that* $L\left[\dfrac{1}{2a^3}(\sin at - at\cos at)\right] = \dfrac{1}{(s^2+a^2)^2}$.

Sol. We have $L(\sin at) = \dfrac{a}{s^2+a^2}$. ...(1)

Also $$L(\cos at) = \frac{s}{s^2+a^2}$$

By **differentiability of Laplace transform,**

$$L\,(t\cos at) = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right).$$

$$= -\left[\frac{(s^2+a^2)\,.\,1 - s\,.\,2s}{(s^2+a^2)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

Now $$L\left[\frac{1}{2a^3}(\sin at - at\cos at)\right]$$

$$= \frac{1}{2a^3}\,[L(\sin at) - aL(t\cos at)]$$

$$= \frac{1}{2a^3}\left[\frac{a}{s^2+a^2} - a\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right)\right]$$

$$= \frac{1}{2a^3}\left[\frac{as^2+a^3-as^2+a^3}{(s^2+a^2)^2}\right] = \mathbf{\frac{1}{(s^2+a^2)^2}}.$$

Example 4. *Evaluate* $\int_0^\infty te^{-2t} \sin t \, dt$.

Sol. We have $\qquad L(\sin t) = \dfrac{1}{s^2 + 1^2} = \dfrac{1}{s^2 + 1}$.

$$\therefore \qquad L(t \sin t) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -(-(s^2+1)^2 \,.\, 2s) = \frac{2s}{(s^2+1)^2}$$

$$\therefore \qquad \int_0^\infty e^{-st} \,.\, t \sin t \, dt = \frac{2s}{(s^2+1)^2}$$

$$\therefore \qquad \int_0^\infty e^{-2t} \, t \sin t \, dt = \left.\frac{2s}{(s^2+1)^2}\right|_{s=2} = \frac{2(2)}{((2)^2+1)^2} = \mathbf{\frac{4}{25}}.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L(f(t)) = F(s)$, then $L(t\, f(t)) = -F'(s)$.

Rule II. If $L(f(t)) = F(s)$, then

$L(t^2 f(t)) = (-1)^2 F''(s)$

$L(t^3 f(t)) = (-1)^3 F'''(s)$

....

$L(t^n f(t)) = (-1)^n F^{(n)}(s)$.

TEST YOUR KNOWLEDGE

1. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $t\, e^{5t}$ (*ii*) $4t \cos 3t$ (*iii*) $6t \sin 5t$

(*iv*) $2t \sinh 2t$ (*v*) $3t \cosh 4t$.

2. Using $L(t^2) = \dfrac{2}{s^3}$, find the Laplace transform of t^3 and t^4.

3. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $2t^2 e^{3t}$ (*ii*) $t^2 \cos 4t$ (*iii*) $t^2 \sin at$

(*iv*) $6t^2 \sinh 3t$ (*v*) $5t^2 \cosh 7t$.

4. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $t^3 e^{2t}$ (*ii*) $3t^3 \cos \pi t$ (*iii*) $4t^3 \sin 4t$

(*iv*) $5t^3 \sinh 4t$ (*v*) $6t^3 \cosh (t/2)$.

5. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $t\, e^{2t} \cos 4t$ (*ii*) $t\, e^{-t} \sin 3t$

(*iii*) $t\, e^{3t} \sinh 3t$ (*iv*) $3t\, e^{2t} \cosh 7t$ (*v*) $te^{-2t} \sin 3t$

6. Prove that $L\left(\dfrac{1}{2a}(\sin at + at \cos at)\right) = \dfrac{s^2}{(s^2+a^2)^2}$.

7. Evaluate $\int_0^\infty t^3 e^{-t} \sin t \, dt$.

Answers

1. (*i*) $\frac{1}{(s-5)^2}$ (*ii*) $\frac{4(s^2-9)}{(s^2+9)^2}$ (*iii*) $\frac{60s}{(s^2+25)^2}$

(*iv*) $\frac{8s}{(s^2-4)^2}$ (*v*) $\frac{3(s^2+16)}{(s^2-16)^2}$ **2.** $\frac{6}{s^4}, \frac{24}{s^5}$

3. (*i*) $\frac{4}{(s-3)^3}$ (*ii*) $\frac{2s(s^2-48)}{(s^2+16)^3}$ (*iii*) $\frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

(*iv*) $\frac{108(s^2+3)}{(s^2-9)^3}$ (*v*) $\frac{10s(s^2+147)}{(s^2-49)^3}$ **4.** (*i*) $\frac{6}{(s-2)^4}$

(*ii*) $\frac{18(\pi^4-6\pi^2s^2-s^4)}{(s^2+\pi^2)^4}$ (*iii*) $\frac{384s(s^2-16)}{(s^2+16)^4}$ (*iv*) $\frac{480s(s^2+16)}{(s^2-16)^4}$

(*v*) $\frac{576(16s^4+24s^2+1)}{(4s^2-1)^4}$ **5.** (*i*) $\frac{s^2-4s-12}{(s^2-4s+20)^2}$ (*ii*) $\frac{6(s+1)}{(s^2+2s+10)^2}$

(*iii*) $\frac{6(s-3)}{(s^2-6s)^2}$ (*iv*) $\frac{3(s^2-4s+53)}{(s^2-4s-45)^2}$ (*v*) $\frac{6(s+2)}{(s^2+4s+13)^2}$

7. 0.

1.15. INTEGRATION OF LAPLACE TRANSFORMS

Theorem. *Let f(t) be a real function such that*

(*i*) *f(t) is piecewise continuous on every finite interval in the range* $t \geq 0$

(*ii*) *there exists constants k and M such that* $|f(t)| \leq Me^{kt}$ *for* $t \geq 0$

(*iii*) $\lim_{t \to 0^+} \frac{f(t)}{t}$ *exists finitely, then*

$$\int_s^\infty F(S)\, dS = L\left(\frac{f(t)}{t}\right), s > k \text{ where } F(s) = L(f(t)).$$

Note. The proof of this theorem is beyond the scope of this book.

Remark. The result of the above theorem can be easily remembered as follows :

If $\quad \mathbf{L(f(t)) = F(s), \text{ then } L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(S)\, dS}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Laplace transform of the function* $\frac{2}{t}(1-\cos at)$, *for* $t \geq 0$.

Sol. We have L(2(1 – cos at)) = 2L(1 – cos at) = 2[L(1) – L(cos at)]

$$= 2\left[\frac{1}{s} - \frac{s}{s^2 + a^2}\right] = \left(\frac{2}{s} - \frac{2s}{s^2 + a^2}\right)^{*}.$$

By **integration of Laplace transform,**

$$\mathrm{L}\left(\frac{2(1-\cos at)}{t}\right) = \int_s^\infty \left[\frac{2}{\mathrm{S}} - \frac{2\mathrm{S}}{\mathrm{S}^2 + a^2}\right] d\mathrm{S}$$

$$= \left[2 \log \mathrm{S} - \log (\mathrm{S}^2 + a^2)\right]_s^\infty = \log \frac{\mathrm{S}^2}{\mathrm{S}^2 + a^2}\Bigg|_s^\infty$$

$$= \lim_{\mathrm{S}\to\infty}\left(\log \frac{\mathrm{S}^2}{\mathrm{S}^2 + a^2}\right) - \log \frac{s^2}{s^2 + a^2}$$

$$= \log\left(\lim_{\mathrm{S}\to\infty} \frac{1}{1 + \dfrac{a^2}{\mathrm{S}^2}}\right) - \log \frac{s^2}{s^2 + a^2}$$

$$= \log\left(\frac{1}{1+0}\right) - \log \frac{s^2}{s^2 + a^2} = 0 + \log \frac{s^2}{s^2 + a^2} = \log \frac{s^2 + a^2}{s^2}.$$

$$\therefore \qquad \mathrm{L}\left(\frac{2}{t}(1-\cos at)\right) = \mathbf{log}\ \frac{\mathbf{s^2 + a^2}}{\mathbf{s^2}}.$$

Example 2. *Find the Laplace transformation of the function* $\dfrac{2\,(e^t - \cos t)}{t}$.

Sol. We have $\mathrm{L}(2(e^t - \cos t)) = 2\mathrm{L}(e^t - \cos t)$

$$= 2\left[\frac{1}{s-1} - \frac{s}{s^2 + 1}\right] = \frac{2}{s-1} - \frac{2s}{s^2+1}.$$

By **integration of Laplace transform,**

$$\mathrm{L}\left(\frac{2\,(e^t - \cos t)}{t}\right) = \int_s^\infty \left(\frac{2}{\mathrm{S}-1} - \frac{2\mathrm{S}}{\mathrm{S}^2+1}\right) d\mathrm{S}$$

$$= \left[2 \log (\mathrm{S}-1) - \log (\mathrm{S}^2 + 1)\right]\Bigg|_s^\infty = \log \frac{(\mathrm{S}-1)^2}{\mathrm{S}^2+1}\Bigg|_s^\infty$$

$$= \lim_{\mathrm{S}\to\infty} \log \frac{(\mathrm{S}-1)^2}{\mathrm{S}^2+1} - \log \frac{(s-1)^2}{s^2+1}$$

$$= \log \lim_{\mathrm{S}\to\infty} \frac{(1-1/\mathrm{S})^2}{1+1/\mathrm{S}^2} - \log \frac{(s-1)^2}{s^2+1}$$

***Why this step.** We have not simplified $\dfrac{2}{s} - \dfrac{2s}{s^2 + a^2}$ because we shall be integrating it in the next step.

$$= \log \frac{(1-0)^2}{1+0} - \log \frac{(s-1)^2}{s^2+1} = 0 + \log \frac{s^2+1}{(s-1)^2}$$

$$\therefore \quad L\left(\frac{2\,(e^t - \cos t)}{t}\right) = \mathbf{log}\ \frac{\mathbf{s^2+1}}{\mathbf{(s-1)^2}}\ .$$

Example 3. *Find the Laplace transform of the function* $\dfrac{\sin at}{t}$.

Sol. We have $L\,(\sin at) = \dfrac{a}{s^2+a^2} = F(s)$ (say)

By **integration of Laplace transform,**

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty F(S)\,dS\,.$$

$$\therefore \qquad L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{S^2+a^2}\,dS = a\,.\,\frac{1}{a}\tan^{-1}\frac{S}{a}\bigg|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}\ .$$

$$\therefore \qquad L\left(\frac{\sin at}{t}\right) = \mathbf{cot^{-1}\,\frac{s}{a}}\,.$$

Example 4. *Show that :*

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\,dt = \log 3.$$

Sol. $\qquad L(e^{-t} - e^{-3t}) = L(e^{-t}) - L(e^{-3t}) = \dfrac{1}{s+1} - \dfrac{1}{s+3}$

$\therefore$ *By integration of Laplace transform,*

$$L\left(\frac{e^{-t} - e^{-3t}}{t}\right) = \int_s^\infty \left(\frac{1}{S+1} - \frac{1}{S+3}\right) dS$$

$$= \Big[\log(S+1) - \log(S+3)\Big]_s^\infty = \log \frac{S+1}{S+3}\bigg|_s^\infty$$

$$= \lim_{S\to\infty}\left(\log\frac{S+1}{S+3}\right) - \log\frac{s+1}{s+3} = \log\left(\lim_{S\to\infty}\frac{1+\frac{1}{S}}{1+\frac{3}{S}}\right) - \log\frac{s+1}{s+3}$$

$$= \log\left(\frac{1+0}{1+0}\right) + \log\frac{s+3}{s+1} = \log\frac{s+3}{s+1}$$

$$\therefore \qquad L\left(\frac{e^{-t} - e^{-3t}}{t}\right) = \log\frac{s+3}{s+1}$$

$$\therefore \int_0^\infty e^{-st} \cdot \frac{e^{-t} - e^{-3t}}{t}\, dt = \log \frac{s+3}{s+1}$$

In particular, let $s = 0$,

$$\therefore \qquad \int_0^\infty e^0 \cdot \frac{e^{-t} - e^{-3t}}{t}\, dt = \log \frac{0+3}{0+1}$$

$$\Rightarrow \qquad \int_0^\infty \frac{\mathbf{e^{-t} - e^{-3t}}}{\mathbf{t}}\, \mathbf{dt} = \mathbf{\log 3.}$$

WORKING RULE FOR SOLVING PROBLEMS

Rule If $L(f(t)) = F(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(S)\, dS$.

TEST YOUR KNOWLEDGE

1. Find the Laplace transform of the following functions of t for $t > 0$:

(*i*) $\frac{\sin t}{t}$ (*ii*) $\frac{1 - \cos 2t}{t}$.

2. Find the Laplace transform of the following functions of t for $t > 0$:

(*i*) $\frac{1 - e^t}{t}$ (*ii*) $\frac{e^{-at} - e^{-bt}}{t}$.

3. Find the Laplace transform of the following functions of t for $t > 0$:

(*i*) $\frac{e^{at} - \cos bt}{t}$ (*ii*) $\frac{\cos 2t - \cos 3t}{t}$.

4. Find the Laplace transform of the following functions of t for $t > 0$:

(*i*) $\frac{\cos \alpha t - \cos \beta t}{t}$ (*ii*) $\frac{e^{-t} \sin t}{t}$.

5. Show that the Laplace transform of the function $\frac{\cos at}{t}$ does not exist.

6. Show that :

(*i*) $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$ (*ii*) $\int_0^\infty \frac{e^{-t} \sin t}{t}\, dt = \frac{\pi}{4}$.

(*iii*) $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt = \log \frac{2}{3}$.

7. Show that :

(*i*) $L\left(\int_0^t \frac{\sin x}{x}\, dx\right) = \frac{1}{s} \cot^{-1} s$ (*ii*) $L\left(\int_0^t \frac{1 - e^{-2x}}{x}\, dx\right) = \frac{1}{s} \log\left(1 + \frac{2}{s}\right)$.

(*iii*) $L\left(\int_0^t \frac{e^x \sin x}{x}\, dx\right) = \frac{\cot^{-1}(s-1)}{s}$.

Answers

1. (*i*) $\cot^{-1} s$ (*ii*) $\frac{1}{2}\log\frac{s^2+4}{s^2}$ **2.** (*i*) $\log\frac{s-1}{s}$ (*ii*) $\log\frac{s+b}{s+a}$

3. (*i*) $\frac{1}{2}\log\frac{s^2+b^2}{(s-a)^2}$ (*ii*) $\frac{1}{2}\log\frac{s^2+9}{s^2+4}$ **4.** (*i*) $\frac{1}{2}\log\frac{s^2+\beta^2}{s^2+\alpha^2}$ (*ii*) $\cot^{-1}(s+1)$

Hints

6. We have $$L\left(\frac{\sin t}{t}\right) = \cot^{-1} s$$

$$\therefore \quad \int_0^\infty \frac{e^{-st}\sin t}{t}\,dt = \cot^{-1} s$$

For part (*i*), take $s = 0$.

For part (*ii*), take $s = 1$.

7. (*i*) $L(\sin t) = \frac{1}{s^2+1} \quad \Rightarrow \quad L\left(\frac{\sin t}{t}\right) = \cot^{-1} s.$

Now using Laplace transform of integral of $\frac{\sin t}{t}$, we have

$$L\left(\int_0^t \frac{\sin T}{T}\,dT\right) = \frac{1}{s}\,.\,\cot^{-1} s.$$

Inverse Laplace Transforms

2

2.1. INTRODUCTION

We have already noted that the main purpose of studying Laplace transforms is for solving various types of differential equations. During the process of solving a differential equation, we shall also require to find a function when its Laplace transform is known. This is the reverse process of finding the Laplace transform of a function. In the present chapter, we shall learn to find the function whose Laplace transform is known.

2.2. INVERSE LAPLACE TRANSFORM OF A FUNCTION

Let f be a real valued function of the real variable t, defined for $t \geq 0$. Let the Laplace transform F(s) of $f(t)$ exists. Therefore the infinite integral $\int_0^\infty e^{-st} f(t)\, dt$ exists and equals F(s). The function $f(t)$ is called the **inverse Laplace transform** of the function F(s) and we write $L^{-1}(F(s)) = f(t)$.

In other words, $L^{-1}(F(s))$ is that function whose Laplace transform is the function F(s).

For example,

(i) $L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}$, because $L(e^{at}) = \dfrac{1}{s-a}$

(ii) $L^{-1}\left(\dfrac{s}{s^2+9}\right) = \cos 3t$, because $L(\cos 3t) = \dfrac{s}{s^2+9}$

2.3. EXISTENCE AND UNIQUENESS OF INVERSE LAPLACE TRANSFORM

A given function F(s) of s may or may not have its inverse Laplace transform. So far as the uniqueness of inverse Laplace transforms, we have the following result :

If $f_1(t)$ and $f_2(t)$ be two continuous functions for $t \geq 0$ having the same Laplace transform $F(s)$ i.e. $L^{-1}(F(s)) = f_1(t)$ and $L^{-1}(F(s)) = f_2(t)$, then $f_1(t) = f_2(t) \quad \forall \ t \geq 0$.

We accept this result without proof.

Illustration : The functions $f_1(t) = 1$

and
$$f_2(t) = \begin{cases} 1, & 0 \leq t < 4 \\ 5, & t = 4 \\ 1, & t > 4 \end{cases}$$

have the same Laplace transform $\frac{1}{s}$. Here the above result is not applicable because the function $f_1(t)$ is continuous for $t \geq 0$ but the function $f_2(t)$ is not continuous for $t \geq 0$.

2.4. ELEMENTARY INVERSE LAPLACE TRANSFORM FORMULAE

In this section, we shall find some elementary inverse Laplace transform formulae.

1. We have $$\mathrm{L}(1) = \frac{1}{s}, \; s > 0$$

∴ By definition, $$\mathbf{L^{-1}\left(\frac{1}{s}\right) = 1, \; s > 0}.$$

2. We have $$\mathrm{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, \; a > -1, \; s > 0.$$

∴ By definition, $$\mathbf{L^{-1}\left(\frac{\Gamma(a+1)}{s^{a+1}}\right) = t^a, a > -1, \; s > 0}.$$

3. We have $$\mathrm{L}(t^n) = \frac{n!}{s^{n+1}}, \; n = 0, 1, 2, \ldots\ldots; s > 0$$

∴ By definition, $$\mathbf{L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n, \; n = 0, 1, 2, \ldots\ldots; s > 0}.$$

4. We have $$\mathrm{L}(e^{at}) = \frac{1}{s-a}, \; s > a.$$

∴ By definition, $$\mathbf{L^{-1}\left(\frac{1}{s-a}\right) = e^{at}, \; s > a}.$$

5. We have $$\mathrm{L}(\sinh at) = \frac{a}{s^2 - a^2}, \; s > |a|$$

∴ By definition, $$\mathbf{L^{-1}\left(\frac{a}{s^2 - a^2}\right) = \sinh at, \; s > |a|}.$$

6. We have $$\mathrm{L}(\cosh at) = \frac{s}{s^2 - a^2}, \; s > |a|$$

∴ By definition, $$\mathbf{L^{-1}\left(\frac{s}{s^2 - a^2}\right) = L(\cosh at), \; s > |a|}.$$

7. We have $$\mathrm{L}(\sin at) = \frac{a}{s^2 + a^2}, \; s > 0$$

∴ By definition, $$\mathbf{L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin at, \; s > 0}.$$

8. We have $$\mathrm{L}(\cos at) = \frac{s}{s^2 + a^2}, \; s > 0.$$

∴ By definition, $$\mathbf{L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at, \; s > 0.}$$

Remarks 1. $L^{-1}\left(\dfrac{\Gamma(a+1)}{s^{a+1}}\right)=t^a \Rightarrow L^{-1}\left(\dfrac{1}{s^{a+1}}\right)=\dfrac{t^a}{\Gamma(a+1)}$; $a>-1,\ s>0$.

$\therefore$ $$\mathbf{L^{-1}\left(\dfrac{1}{s^a}\right)=\dfrac{t^{a-1}}{\Gamma(a)},\ a>0,\ s>0.}$$

2. $L^{-1}\left(\dfrac{n!}{s^{n+1}}\right)=t^n \Rightarrow L^{-1}\left(\dfrac{1}{s^{n+1}}\right)=\dfrac{t^n}{n!}$, $n=0,1,2,\ldots\ldots;\ s>0$

$\therefore$ $$\mathbf{L^{-1}\left(\dfrac{1}{s^n}\right)=\dfrac{t^{n-1}}{(n-1)!},\quad n\in N, s>0.}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L(f(t)) = F(s)$, then $L^{-1}(F(s)) = f(t)$.

Rule II. (*i*) $L^{-1}\left(\dfrac{1}{s}\right)=1$ (*ii*) $L^{-1}\left(\dfrac{\Gamma(a+1)}{s^{a+1}}\right)=t^a$, $a>-1$

(*iii*) $L^{-1}\left(\dfrac{n!}{s^{n+1}}\right)=t^n$ (*iv*) $L^{-1}\left(\dfrac{1}{s-a}\right)=e^{at}$

(*v*) $L^{-1}\left(\dfrac{a}{s^2-a^2}\right)=\sinh at$ (*vi*) $L^{-1}\left(\dfrac{s}{s^2-a^2}\right)=\cosh at$

(*vii*) $L^{-1}\left(\dfrac{a}{s^2+a^2}\right)=\sin at$ (*viii*) $L^{-1}\left(\dfrac{s}{s^2+a^2}\right)=\cos at$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the values of :*

(*i*) $L^{-1}\left(\dfrac{\Gamma(5/2)}{s^{5/2}}\right)$ (*ii*) $L^{-1}\left(\dfrac{5040}{s^8}\right)$

(*iii*) $L^{-1}\left(\dfrac{1}{s+5}\right)$ (*iv*) $L^{-1}\left(\dfrac{3}{s^2-9}\right)$

(*v*) $L^{-1}\left(\dfrac{s}{s^2+16}\right)$ (*vi*) $L^{-1}\left(\dfrac{6}{s^2+36}\right)$.

Sol. (*i*) We have $L^{-1}\left(\dfrac{\Gamma(a+1)}{s^{a+1}}\right)=t^a$.

$\therefore$ $$L^{-1}\left(\dfrac{\Gamma(5/2)}{s^{5/2}}\right)=t^{\frac{5}{2}-1}=\mathbf{t^{3/2}}.$$

(*ii*) We have $L^{-1}\left(\dfrac{n!}{s^{n+1}}\right)=t^n$.

$\therefore$ $$L^{-1}\left(\dfrac{5040}{s^8}\right)=L^{-1}\left(\dfrac{7!}{s^{7+1}}\right)=\mathbf{t^7}.$$

(*iii*) We have $\quad L^{-1}\left(\dfrac{1}{s-a}\right)=e^{at}$.

$\therefore \quad L^{-1}\left(\dfrac{1}{s+5}\right)=L^{-1}\left(\dfrac{1}{s-(-5)}\right)=\mathbf{e^{-5t}}.$

(*iv*) We have $\quad L^{-1}\left(\dfrac{a}{s^2-a^2}\right)=\sinh at.$

$\therefore \quad L^{-1}\left(\dfrac{3}{s^2-9}\right)=L^{-1}\left(\dfrac{3}{s^2-3^2}\right)=\mathbf{\sinh 3t.}$

(*v*) We have $\quad L^{-1}\left(\dfrac{s}{s^2+a^2}\right)=\cos at$

$\therefore \quad L^{-1}\left(\dfrac{s}{s^2+16}\right)=L^{-1}\left(\dfrac{s}{s^2+4^2}\right)=\mathbf{\cos 4t.}$

(*vi*) We have $\quad L^{-1}\left(\dfrac{a}{s^2+a^2}\right)=\sin at$

$\therefore \quad L^{-1}\left(\dfrac{6}{s^2+36}\right)=L^{-1}\left(\dfrac{6}{s^2+6^2}\right)=\mathbf{\sin 6t.}$

2.5. LINEARITY OF THE INVERSE LAPLACE TRANSFORM

Theorem. *If f(t) and g(t) be any functions of t for t ≥ 0 such that L(f(t)) = F(s) and L(g(t)) = G(s) and a and b be any constants, then*

$$L^{-1}(aF(s) + bG(s)) = aL^{-1}(F(s)) + bL^{-1}(G(s)).$$

Proof. We have $L(f(t)) = F(s)$ and $L(g(t)) = G(s)$.

$\therefore \quad aF(s) + bG(s) = aL(f(t)) + bL(g(t))$

$\quad = L(af(t) + bg(t))$

$\therefore \quad L^{-1}(aF(s) + bG(s)) = af(t) + bg(t)$

or $\quad \mathbf{L^{-1}(aF(s) + bG(s)) = aL^{-1}(F(s)) + bL^{-1}(G(s))}.$

Example 2. *Find the value of* $L^{-1}\left(\dfrac{1}{s+3}+\dfrac{2}{s+5}+\dfrac{6}{s^4}\right)$.

Sol. $\quad L^{-1}\left(\dfrac{1}{s+3}+\dfrac{2}{s+5}+\dfrac{6}{s^4}\right)$

$$=L^{-1}\left(\frac{1}{s+3}\right)+2L^{-1}\left(\frac{1}{s+5}\right)+L^{-1}\left(\frac{6}{s^4}\right) \qquad \text{(Using linearity)}$$

$$= L^{-1}\left(\frac{1}{s-(-3)}\right) + 2L^{-1}\left(\frac{1}{s-(-5)}\right) + L^{-1}\left(\frac{3!}{s^{3+1}}\right)$$

$$= \mathbf{e^{-3t} + 2e^{-5t} + t^3.}$$

Example 3. *Find the value of* $L^{-1}\left(\frac{s}{4s^2-16} + \frac{9}{s^2+25} + \frac{4s}{9s^2+4} + \frac{1}{4s-1}\right)$.

Sol. $L^{-1}\left(\frac{s}{4s^2-16} + \frac{9}{s^2+25} + \frac{4s}{9s^2+4} + \frac{1}{4s-1}\right)$

$$= L^{-1}\left(\frac{s}{4s^2-16}\right) + L^{-1}\left(\frac{9}{s^2+25}\right) + L^{-1}\left(\frac{4s}{9s^2+4}\right) + L^{-1}\left(\frac{1}{4s-1}\right)$$

$$= \frac{1}{4} L^{-1}\left(\frac{s}{s^2-2^2}\right) + \frac{9}{5} L^{-1}\left(\frac{5}{s^2+5^2}\right) + \frac{4}{9} L^{-1}\left(\frac{s}{s^2+\left(\frac{2}{3}\right)^2}\right) + \frac{1}{4} L^{-1}\left(\frac{1}{s-\frac{1}{4}}\right)$$

$$= \mathbf{\frac{1}{4}\cosh 2t + \frac{9}{5}\sin 5t + \frac{4}{9}\cos\frac{2}{3}t + \frac{1}{4}e^{\frac{1}{4}t}}.$$

Example 4. *Find the inverse Laplace transform of the following functions :*

(*i*) $\frac{4s-8}{9-s^2}$ (*ii*) $\frac{2s-5}{4s^2+25}$

Sol. (*i*) $\frac{4s-8}{9-s^2} = -4\left(\frac{s-2}{s^2-9}\right) = -4 \cdot \frac{s}{s^2-9} + \frac{8}{3} \cdot \frac{3}{s^2-9}$

$\therefore$ $L^{-1}\left(\frac{4s-8}{9-s^2}\right) = L^{-1}\left(-4 \cdot \frac{s}{s^2-9} + \frac{8}{3} \cdot \frac{3}{s^2-9}\right)$

$$= -4L^{-1}\left(\frac{s}{s^2-3^2}\right) + \frac{8}{3} L^{-1}\left(\frac{3}{s^2-3^2}\right)$$

$$= \mathbf{-4\cosh 3t + \frac{8}{3}\sinh 3t.}$$

(*ii*) $\frac{2s-5}{4s^2+25} = \frac{2}{4}\left(\frac{s-\frac{5}{2}}{s^2+\frac{25}{4}}\right) = \frac{1}{2} \cdot \frac{s}{s^2+\frac{25}{4}} - \frac{5}{4} \cdot \frac{1}{s^2+\frac{25}{4}}$

$\therefore$ $L^{-1}\left(\frac{2s-5}{4s^2+25}\right) = L^{-1}\left(\frac{1}{2} \cdot \frac{s}{s^2+(5/2)^2} - \frac{1}{2} \cdot \frac{5/2}{s^2+(5/2)^2}\right)$

$$= \frac{1}{2} L^{-1}\left(\frac{s}{s^2+(5/2)^2}\right) - \frac{1}{2} L^{-1}\left(\frac{5/2}{s^2+(5/2)^2}\right)$$

$$= \mathbf{\frac{1}{2}\cos\frac{5}{2}t - \frac{1}{2}\sin\frac{5}{2}t.}$$

TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions :

1. (*i*) $\dfrac{720}{s^7}$ (*ii*) $\dfrac{40320}{s^9}$ (*iii*) $\dfrac{\Gamma(3/2)}{s^{3/2}}$ (*iv*) $\dfrac{1}{s-7}$

2. (*i*) $\dfrac{s}{s^2+25}$ (*ii*) $\dfrac{s}{s^2-49}$ (*iii*) $\dfrac{8}{s^2-64}$ (*iv*) $\dfrac{4}{s^2+16}$

3. (*i*) $\dfrac{60+6s^2+s^4}{s^7}$ (*ii*) $\dfrac{s-4}{s^2-16}$ (*iii*) $\dfrac{0.1s+0.9}{s^2+3.24}$ (*iv*) $\dfrac{s}{a^2s^2+b^2c^2}$

4. (*i*) $\dfrac{6}{s-4}+\dfrac{3}{s+5}+\dfrac{9}{s-7}$ (*ii*) $\dfrac{2}{2s+1}-\dfrac{4}{5s+7}+\dfrac{11}{1+5s}$

5. (*i*) $\dfrac{4}{s^2+1}-\dfrac{9s}{s^2+5}+\dfrac{6}{4s^2+1}+\dfrac{8s}{s^2+25}$ (*ii*) $\dfrac{2}{(s-1)(s+1)}+\dfrac{6s}{s^2-25}-\dfrac{4s}{s^2-9}+\dfrac{1}{6s^2-1}$

6. (*i*) $\dfrac{3}{s^2-1}-\dfrac{4}{s^2+9}+\dfrac{s}{s^2+16}-\dfrac{5}{s^2-16}$ (*ii*) $\dfrac{5}{3s^2+27}+\dfrac{6}{5s^2-9}-\dfrac{5}{36s^2+1}+\dfrac{9}{5+s^2}$.

Answers

1. (*i*) t^6 (*ii*) t^8 (*iii*) $\sqrt{t}$ (*iv*) e^{7t}

2. (*i*) $\cos 5t$ (*ii*) $\cosh 7t$ (*iii*) $\sinh 8t$ (*iv*) $\sin 4t$

3. (*i*) $\dfrac{1}{12}t^6+\dfrac{1}{4}t^4+\dfrac{1}{2}t^2$ (*ii*) e^{-4t}

(*iii*) $\dfrac{1}{10}\cos 1.8\,t+\dfrac{1}{2}\sin 1.8\,t$ (*iv*) $\dfrac{1}{a^2}\cos\dfrac{bc}{a}t$

4. (*i*) $6e^{4t}+3e^{-5t}+9e^{7t}$ (*ii*) $e^{-\frac{1}{2}t}-\dfrac{4}{5}e^{-\frac{7}{5}t}+\dfrac{11}{5}e^{-\frac{1}{5}t}$

5. (*i*) $4\sin t-9\cos\sqrt{5}\,t+3\sin\dfrac{t}{2}+8\cos 5t$ (*ii*) $2\sinh t+6\cosh 5t-4\cosh 3t+\dfrac{1}{\sqrt{6}}\sinh\dfrac{t}{\sqrt{6}}$

6. (*i*) $3\sinh t-\dfrac{4}{3}\sin 3t+\cos 4t-\dfrac{5}{4}\sinh 4t$ (*ii*) $\dfrac{5}{9}\sin 3t+\dfrac{2}{\sqrt{5}}\sinh\dfrac{3}{\sqrt{5}}t-\dfrac{5}{6}\sin\dfrac{t}{6}+\dfrac{9}{\sqrt{5}}\sin\sqrt{5}t.$

2.6. VALUE OF $L^{-1}(F(s-a))$ IN TERMS OF $L^{-1}(F(s))$

By the **first shifting theorem** of Laplace transforms,

if $\quad L(f(t)) = F(s)$, then $L(e^{at} f(t)) = F(s-a)$.

$\therefore$ If $\quad \mathbf{L^{-1}(F(s)) = f(t)}$ **then** $\mathbf{L^{-1}(F(s-a)) = e^{at} f(t)}$.

Equivalently, $\mathbf{L^{-1}(F(s-a)) = e^{at} L^{-1}(F(s))}$.

Remark. In the formula, $L^{-1}(F(s-a)) = e^{at} L^{-1}(F(s))$, if we replace a by $-a$, the formula takes the form,

$$\mathbf{L^{-1}(F(s+a)) = e^{-at} L^{-1}(F(s))}.$$

For example, $\quad L^{-1}\left(\dfrac{s-2}{(s-2)^2+5^2}\right) = e^{2t}\,L^{-1}\left(\dfrac{s}{s^2+5^2}\right) = e^{2t}\cos 5t.$

Theorem. Using the formula $L^{-1}(F(s-a)) = e^{at} L^{-1}(F(s))$ prove that :

1. $\mathbf{L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}}$

2. $\mathbf{L^{-1}\left(\dfrac{\Gamma(b+1)}{(s-a)^{b+1}}\right) = e^{at}\, t^{b}\ ,\ b > -1}$

3. $\mathbf{L^{-1}\left(\dfrac{n\,!}{(s-a)^{n+1}}\right) = e^{at}\, t^{n}\ ,\ n = 0, 1, 2, \ldots\ldots}$

4. $\mathbf{L^{-1}\left(\dfrac{b}{(s-a)^2 - b^2}\right) = e^{at} \sinh bt}$

5. $\mathbf{L^{-1}\left(\dfrac{s-a}{(s-a)^2 - b^2}\right) = e^{at} \cosh bt}$

6. $\mathbf{L^{-1}\left(\dfrac{b}{(s-a)^2 + b^2}\right) = e^{at} \sin bt}$

7. $\mathbf{L^{-1}\left(\dfrac{s-a}{(s-a)^2 + b^2}\right) = e^{at} \cos bt.}$

Proof. 1. We have $\quad L^{-1}\left(\dfrac{1}{s}\right) = 1$.

$$\therefore \qquad L^{-1}\left(\frac{1}{s-a}\right) = e^{at} \,.\, L^{-1}\left(\frac{1}{s}\right) = e^{at}\,.1$$

$$\therefore \qquad \mathbf{L^{-1}\left(\frac{1}{s-a}\right) = e^{at}}.$$

2. We have $\quad L^{-1}\left(\dfrac{\Gamma(b+1)}{s^{b+1}}\right) = t^{b}\ , b > -1$.

$$\therefore \qquad L^{-1}\left(\frac{\Gamma(b+1)}{(s-a)^{b+1}}\right) = e^{at}\, L^{-1}\left(\frac{\Gamma(b+1)}{s^{b+1}}\right) = e^{at}\, t^{b}$$

$$\therefore \qquad \mathbf{L^{-1}\left(\frac{\Gamma(b+1)}{(s-a)^{b+1}}\right) = e^{at}\, t^{b}\ ,\ b > -1}.$$

3. We have $\quad L^{-1}\left(\dfrac{n\,!}{s^{n+1}}\right) = t^{n}\ , n = 0, 1, 2, \ldots\ldots$

$$\therefore \qquad L^{-1}\left(\frac{n\,!}{(s-a)^{n+1}}\right) = e^{at}\, L^{-1}\left(\frac{n\,!}{s^{n+1}}\right) = e^{at}\, t^{n}$$

$$\therefore \qquad \mathbf{L^{-1}\left(\frac{n\,!}{(s-a)^{n+1}}\right) = e^{at}\, t^{n}\ ,\ n = 0, 1, 2, \ldots\ldots}$$

4. We have $\quad L^{-1}\left(\dfrac{b}{s^2 - b^2}\right) = \sinh bt$.

$\therefore \qquad L^{-1}\left(\dfrac{b}{(s-a)^2 - b^2}\right) = e^{at}\, L^{-1}\left(\dfrac{b}{s^2 - b^2}\right) = e^{at} \sinh bt$

$\therefore \qquad \mathbf{L^{-1}\left(\dfrac{b}{(s-a)^2 - b^2}\right) = e^{at} \sinh bt}$.

5. We have $\quad L^{-1}\left(\dfrac{s}{s^2 - b^2}\right) = \cosh bt$.

$\therefore \qquad L^{-1}\left(\dfrac{s-a}{(s-a)^2 - b^2}\right) = e^{at}\, L^{-1}\left(\dfrac{s}{s^2 - b^2}\right) = e^{at} \cosh bt$

$\therefore \qquad \mathbf{L^{-1}\left(\dfrac{s-a}{(s-a)^2 - b^2}\right) = e^{at} \cosh bt}$.

6. We have $\quad L^{-1}\left(\dfrac{b}{s^2 + b^2}\right) = \sin bt$.

$\therefore \qquad L^{-1}\left(\dfrac{b}{(s-a)^2 + b^2}\right) = e^{at}\, L^{-1}\left(\dfrac{b}{s^2 + b^2}\right) = e^{at} \sin bt$

$\therefore \qquad \mathbf{L^{-1}\left(\dfrac{b}{(s-a)^2 + b^2}\right) = e^{at} \sin bt}$.

7. We have $\quad L^{-1}\left(\dfrac{s}{s^2 + b^2}\right) = \cos bt$.

$\therefore \qquad L^{-1}\left(\dfrac{s-a}{(s-a)^2 + b^2}\right) = e^{at}\, L^{-1}\left(\dfrac{s}{s^2 + b^2}\right) = e^{at} \cos bt$

$\therefore \qquad \mathbf{L^{-1}\left(\dfrac{s-a}{(s-a)^2 + b^2}\right) = e^{at} \cos bt.}$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse Laplace transform of the function* $\dfrac{12}{(s-4)^3}$.

Sol.

$$L^{-1}\left(\frac{12}{(s-4)^3}\right) = 6L^{-1}\left(\frac{2}{(s-4)^3}\right)$$

$$= 6e^{4t}\, L^{-1}\left(\frac{2}{s^3}\right) = 6e^{4t}\, L^{-1}\left(\frac{2!}{s^{2+1}}\right)$$

$(\because \;\; L^{-1}(F(s-a)) = e^{at}\, L^{-1}(F(s)))$

$$\mathbf{= 6e^{4t}\, t^2.}$$

Example 2. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{3s+1}{(s+1)^4}$ (*ii*) $\displaystyle\sum_{k=1}^{5} \frac{\lambda_k}{s+k^2}$.

Sol. (*i*) $L^{-1}\left(\dfrac{3s+1}{(s+1)^4}\right) = L^{-1}\left(\dfrac{3(s+1)-2}{(s+1)^4}\right)$

$$= e^{-t} L^{-1}\left(\frac{3s-2}{s^4}\right) = e^{-t} L^{-1}\left(\frac{3}{s^3} - \frac{2}{s^4}\right)$$

$(\because\ L^{-1}(F(s-a)) = e^{at} L^{-1}(F(s)))$

$$= e^{-t}\left[\frac{3}{2} L^{-1}\left(\frac{2!}{s^3}\right) - \frac{2}{6} L^{-1}\left(\frac{3!}{s^4}\right)\right] = \mathbf{e^{-t}\left[\frac{3}{2} t^2 - \frac{1}{3} t^3\right]}.$$

(*ii*) $L^{-1}\left(\displaystyle\sum_{k=1}^{5} \frac{\lambda_k}{s+k^2}\right) = \displaystyle\sum_{k=1}^{5} \lambda_k L^{-1}\left(\frac{1}{s+k^2}\right) = \mathbf{\displaystyle\sum_{k=1}^{5} \lambda_k e^{-k^2 t}}$.

Example 3. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{s+3}{(s+3)^2+4}$ (*ii*) $\dfrac{3s}{s^2+2s-8}$.

Sol. (*i*) $L\left(\dfrac{s+3}{(s+3)^2+4}\right) = e^{-3t} L^{-1}\left(\dfrac{s}{s^2+2^2}\right) = \mathbf{e^{-3t} \cos 2t}$.

$(\because\ L^{-1}(F(s-a)) = e^{at} L^{-1}(F(s)))$

(*ii*) $$\frac{3s}{s^2+2s-8} = \frac{3s}{(s+1)^2-9} = \frac{3s+3-3}{(s+1)^2-9} = \frac{3(s+1)}{(s+1)^2-9} - \frac{3}{(s+1)^2-9}$$

$\therefore$ $$L^{-1}\left(\frac{3s}{s^2+2s-8}\right) = L^{-1}\left(\frac{3(s+1)}{(s+1)^2-9} - \frac{3}{(s+1)^2-9}\right)$$

$$= 3L^{-1}\left(\frac{s+1}{(s+1)^2-9}\right) - L^{-1}\left(\frac{3}{(s+1)^2-9}\right)$$

$$= 3e^{-1.t} L^{-1}\left(\frac{s}{s^2-3^2}\right) - e^{-1.t} L^{-1}\left(\frac{3}{s^2-3^2}\right)$$

$$= 3e^{-t} \cosh 3t - e^{-t} \sinh 3t$$

$$= \mathbf{e^{-t}(3 \cosh 3t - \sinh 3t)}.$$

Example 4. *Find the inverse Laplace transform of the function*

$$\frac{1}{s^2+4s+13} - \frac{s+4}{s^2+8s+97} + \frac{s+2}{s^2-4s+29}.$$

Sol. $$\frac{1}{s^2+4s+13} - \frac{s+4}{s^2+8s+97} + \frac{s+2}{s^2-4s+29}$$

$$= \frac{1}{(s+2)^2+9} - \frac{s+4}{(s+4)^2+81} + \frac{s+2}{(s-2)^2+25}$$

$$= \frac{1}{(s+2)^2+9} - \frac{s+4}{(s+4)^2+81} + \frac{s-2}{(s-2)^2+25} + \frac{4}{(s-2)^2+25}$$

$$\therefore \quad L^{-1}\left(\frac{1}{s^2+4s+13} - \frac{s+4}{s^2+8s+97} + \frac{s+2}{s^2-4s+29}\right)$$

$$= L^{-1}\left(\frac{1}{(s+2)^2+9}\right) - L^{-1}\left(\frac{s+4}{(s+4)^2+81}\right) + L^{-1}\left(\frac{s-2}{(s-2)^2+25}\right) + L^{-1}\left(\frac{4}{(s-2)^2+25}\right)$$

$$= e^{-2t} L^{-1}\left(\frac{1}{s^2+9}\right) - e^{-4t} L^{-1}\left(\frac{s}{s^2+81}\right) + e^{2t} L^{-1}\left(\frac{s}{s^2+25}\right) + e^{2t} L^{-1}\left(\frac{4}{s^2+25}\right)$$

$$= \frac{1}{3} e^{-2t} L^{-1}\left(\frac{3}{s^2+3^2}\right) - e^{-4t} L^{-1}\left(\frac{s}{s^2+9^2}\right) + e^{2t} L^{-1}\left(\frac{s}{s^2+5^2}\right) + \frac{4}{5} e^{2t} L^{-1}\left(\frac{5}{s^2+5^2}\right)$$

$$= \mathbf{\frac{1}{3} e^{-2t} \sin 3t - e^{-4t} \cos 9t + e^{2t} \cos 5t + \frac{4}{5} e^{2t} \sin 5t.}$$

2.7. VALUE OF L^{-1} (e^{-as} F(s)) IN TERMS OF L^{-1} (F(s))

By the **second shifting theorem** of Laplace transforms,

if $\qquad L(f(t)) = F(s)$ then for any $a \geq 0$, $L(f(t-a)\, u_a(t)) = e^{-as} F(s)$.

$\therefore$ **If $L^{-1}(F(s)) = f(t)$ then $L^{-1}(e^{-as} F(s)) = f(t-a)\, u_a(t)$ for any $a \geq 0$.**

Remark. It may be recalled that for $a \geq 0$, $u_a(t)$ is the *unit step function* defined as :

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a. \end{cases}$$

$\therefore$ The function $f(t-a)\, u_a(t)$ can also be written as

$$f(t-a)\, u_a(t) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a. \end{cases}$$

Example 5. *Find the inverse Laplace transform of the function* $\dfrac{2s+1}{s^2 e^{2s}}$.

Sol. $\qquad \dfrac{2s+1}{s^2 e^{2s}} = e^{-2s} \cdot \dfrac{2s+1}{s^2} = e^{-2s} F(s)$, where $F(s) = \dfrac{2s+1}{s^2}$.

$$L^{-1}(F(s)) = L^{-1}\left(\frac{2s+1}{s^2}\right) = L^{-1}\left(\frac{2}{s} + \frac{1}{s^2}\right) = 2L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^2}\right)$$

$= 2(1) + t = t + 2 = f(t)$, say.

$$\therefore \quad L^{-1}\left(\frac{2s+1}{s^2 e^{2s}}\right) = L^{-1}(e^{-2s} F(s))$$

$$= f(t-2)\, u_2(t) = ((t-2)+2)\, u_2(t) = t \cdot u_2(t) = \begin{cases} 0 & \text{if } t < 2 \\ t & \text{if } t > 2 \end{cases}.$$

$$\therefore \quad L^{-1}\left(\frac{2s+1}{s^2 e^{2s}}\right) = g(t), \text{ where } g(t) = \begin{cases} 0 & \text{if } t < 2 \\ t & \text{if } t > 2 \end{cases}.$$

Example 6. *Find the inverse Laplace transform of the function* $\dfrac{e^{-\pi s/2}\, s}{s^2+1}$.

Sol. $$\frac{e^{-\pi s/2} s}{s^2+1} = e^{-(\pi/2)s} \cdot \frac{s}{s^2+1} = e^{-(\pi/2)s}\, F(s), \text{ where } F(s) = \frac{s}{s^2+1}.$$

$$L^{-1}(F(s) = L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t = f(t), \text{ say.}$$

$$\therefore \quad L^{-1}\left(\frac{e^{-\pi s/2}\, s}{s^2+1}\right) = L^{-1}\,(e^{-(\pi/2)s}\, F(s))$$

$$= f\left(t-\frac{\pi}{2}\right) u_{\pi/2}(t) = \cos\left(t-\frac{\pi}{2}\right) u_{\pi/2}\,(t)$$

$$= \sin t \,.\, u_{\pi/2}\,(t) = \begin{cases} \mathbf{0} & \textbf{if } \mathbf{t < \pi/2} \\ \mathbf{\sin t} & \textbf{if } \mathbf{t > \pi/2} \end{cases}.$$

Example 7. *Find the inverse Laplace transform of the function* $\dfrac{3\,(1-e^{-\pi s})}{s^2+9}$.

Sol. $$L^{-1}\left(\frac{3(1-e^{-\pi s})}{s^2+9}\right) = L^{-1}\left(\frac{3}{s^2+9} - \frac{3e^{-\pi s}}{s^2+9}\right) = L^{-1}\left(\frac{3}{s^2+9}\right) - L^{-1}\left(\frac{3e^{-\pi s}}{s^2+9}\right)$$

$$\therefore \quad L^{-1}\left(\frac{3(1-e^{-\pi s})}{s^2+9}\right) = \sin 3t - L^{-1}\left(\frac{3e^{-\pi s}}{s^2+9}\right) \qquad ...(1)$$

$$\frac{3e^{-\pi s}}{s^2+9} = e^{-\pi s} \cdot \frac{3}{s^2+9} = e^{-\pi s}\, F(s), \text{ where } F(s) = \frac{3}{s^2+9}.$$

$$L^{-1}\,(F(s)) = L^{-1}\left(\frac{3}{s^2+9}\right) = \sin 3t = f(t), \text{ say.}$$

$$\therefore \quad L^{-1}\left(\frac{3e^{-\pi s}}{s^2+9}\right) = L^{-1}(e^{-\pi s}\, F(s)) = f(t-\pi)\, u_\pi(t) = \sin 3(t-\pi)\, u_\pi(t)$$

$$= -\sin(3\pi - 3t)\, u_\pi(t) = -\sin 3t \,.\, u_\pi(t).$$

$$\therefore \quad (1) \Rightarrow \quad L^{-1}\left(\frac{3(1-e^{-\pi s})}{s^2+9}\right) = \sin 3t - (-\sin 3t \,.\, u_\pi(t))$$

$$= \mathbf{(1 + u_\pi(t))\, \sin 3t.}$$

2.8. VALUE OF L^{-1} F(s/a)) IN TERMS OF L^{-1} (F(s))

By the **change of scale property,**

if $$L(f(t)) = F(s) \text{ then for any } a > 0,\ L\,(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right) \quad \textit{i.e.} \quad L(af(at)) = F\left(\frac{s}{a}\right).$$

$$\therefore \quad \textbf{If } \mathbf{L^{-1}\,(F(s)) = f(t)} \textbf{ then } \mathbf{L^{-1}\left(F\left(\frac{s}{a}\right)\right) = a\, f(at)} \textbf{ for any } \mathbf{a > 0.}$$

Example 8. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{a}{\left(\dfrac{s}{\lambda}\right)^2 + a^2}$ (*ii*) $\dfrac{s}{\left(\dfrac{s}{7}\right)^2 + 9}$

Sol. (*i*) Let $\quad F(s) = \dfrac{a}{s^2 + a^2}$.

$\therefore \quad L^{-1}(F(s)) = \sin at = f(t)$, say

$$\therefore \quad L^{-1}\left(\frac{a}{\left(\dfrac{s}{\lambda}\right)^2 + a^2}\right) = L^{-1}\left(F\left(\frac{s}{\lambda}\right)\right) = \lambda f(\lambda t) = \boldsymbol{\lambda \sin \lambda at.}$$

Alternative method

$$L^{-1}\left(\frac{a}{\left(\dfrac{s}{\lambda}\right)^2 + a^2}\right) = L^{-1}\left(\frac{\lambda^2 a}{s^2 + a^2\lambda^2}\right) = \lambda L^{-1}\left(\frac{\lambda a}{s^2 + (\lambda a)^2}\right) = \boldsymbol{\lambda \sin \lambda at.}$$

(*ii*) Let $\quad F(s) = \dfrac{s}{s^2 + 9}$.

$\therefore \quad L^{-1}(F(s)) = \cos 3t = f(t)$, say

$$\therefore \quad L^{-1}\left(\frac{s}{\left(\dfrac{s}{7}\right)^2 + 9}\right) = 7L^{-1}\left(\frac{s/7}{(s/7)^2 + 9}\right) = 7(7\, f(7t))$$

$$= 49 \cos(3(7t)) = \mathbf{49 \cos 21t.}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L^{-1}(F(s)) = f(t)$, then $L^{-1}(F(s-a)) = e^{at} f(t)$.

Rule II. If $L^{-1}(F(s)) = f(t)$, then

$L^{-1}(e^{-as} F(s)) = f(t-a)\, u_a(t),\ a \geq 0$.

Rule III. If $L^{-1}(F(s)) = f(t)$, then

$L^{-1}\left(F\left(\dfrac{s}{a}\right)\right) = a\, f(at),\ a > 0$.

TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions :

1. (*i*) $\dfrac{1}{(s-4)^5}$ (*ii*) $\dfrac{4}{(s+2)^7}$

2. (*i*) $\dfrac{1}{s^2+6s+13}$ (*ii*) $\dfrac{s+3}{s^2-4s+13}$

3. (*i*) $\dfrac{s}{\left(s+\frac{1}{2}\right)^2+1}$ (*ii*) $\dfrac{4}{s^2+6s+18}$

4. (*i*) $\dfrac{5}{s^2-4s-3}$ (*ii*) $\dfrac{9}{s^2+s+\frac{1}{2}}$

5. (*i*) $\dfrac{s+2}{s^2+4s+7}$ (*ii*) $\dfrac{s+10}{s^2+8s+20}$

6. (*i*) $\dfrac{1}{(s-4)^5}+\dfrac{5}{s^2-4s+29}$ (*ii*) $\dfrac{1}{(s-3)^4}+\dfrac{s+3}{s^2+6s+45}$

7. (*i*) $\dfrac{1}{2s-5}+\dfrac{5s-2}{3s^2+s+8}+\dfrac{s+1}{s^2+2s+2}$

(*ii*) $\dfrac{1}{s^2-6s+10}+\dfrac{1}{s^2+8s+16}+\dfrac{3s-2}{s^2-4s+20}+\dfrac{3s+7}{s^2-2s-3}$

8. (*i*) $\dfrac{e^{-s}}{s^2}$ (*ii*) $\dfrac{e^{-3s}}{s^3}$

9. (*i*) $\dfrac{(5s+1)\,e^{-5s}}{s^2}$ (*ii*) $\dfrac{2+2s+s^2}{s^3\,e^s}$

10. (*i*) $-\dfrac{e^{-\pi s/2}}{s^2+1}$ (*ii*) $-\dfrac{5se^{-\pi s}}{s^2+1}$

11. (*i*) $\dfrac{se^{-2s}}{s^2+\pi^2}$ (*ii*) $\dfrac{e^{-2\pi s/3}}{s^2+1}$

12. (*i*) $\dfrac{e^{4-3s}}{(s+4)^{5/2}}$ (*ii*) $\dfrac{(s+1)\,e^{-\pi s}}{s^2+s+1}$.

Answers

1. (*i*) $\dfrac{1}{24}t^4 e^{4t}$ (*ii*) $\dfrac{1}{180}t^6 e^{-2t}$

2. (*i*) $\dfrac{1}{2}e^{-3t}\sin 2t$ (*ii*) $e^{2t}\left(\cos 3t+\dfrac{5}{3}\sin 3t\right)$

3. (*i*) $e^{-t/2}\left(\cos t-\dfrac{1}{2}\sin t\right)$ (*ii*) $\dfrac{4}{3}e^{-3t}\sin 3t$

4. (*i*) $\dfrac{5}{\sqrt{7}}e^{2t}\sinh\sqrt{7}t$ (*ii*) $18e^{-t/2}\sin\dfrac{t}{2}$

5. (*i*) $e^{-2t}\cos\sqrt{3}t$ (*ii*) $e^{-4t}\,[\cos 2t+3\sin 2t]$

6. (*i*) $\frac{1}{24} e^{4t} t^4 + e^{2t} \sin 5t$ (*ii*) $\frac{1}{6} e^{3t} t^3 + e^{-3t} \cos 6t$

7. (*i*) $\frac{1}{2} e^{5t/2} + e^{-\frac{t}{6}} \left[\frac{5}{3} \cos \frac{\sqrt{95}}{6} t - \frac{25}{3\sqrt{95}} \sin \frac{\sqrt{95}}{6} t \right] + e^{-t} \cos t$

(*ii*) $e^{3t} \sin t + te^{-4t} + 3e^{2t} \cos 4t + e^{2t} \sin 4t + 3e^t \cosh 2t + 5e^t \sinh 2t$

8. (*i*) $(t - 1)\, u_1(t)$ (*ii*) $\frac{1}{2} (t - 3)^2\, u_3(t)$

9. (*i*) $t \,.\, u_5(t)$ (*ii*) $t^2 u_1(t)$

10. (*i*) $\cos t \,.\, u_{\pi/2}(t)$ (*ii*) $5 \cos t \,.\, u_\pi(t)$

11. (*i*) $\cos \pi t \,.\, u_2(t)$ (*ii*) $-\cos\left(\frac{\pi}{6} - t\right) . \,u_{2\pi/3}(t)$

12. (*i*) $\frac{4}{3\sqrt{\pi}} e^{16-4t} (t - 3)^{3/2}\, u_3(t)$

(*ii*) $e^{-\frac{1}{2}(t-\pi)} \left[\cos \frac{\sqrt{3}}{2} (t - \pi) + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} (t - \pi) \right] u_\pi(t).$

2.9. VALUE OF L^{-1} (F(s)/s) IN TERMS OF L^{-1} (F(s))

By the property of **Laplace transforms of integrals,**

if $$L(f(t)) = F(s), \text{ then } L\left(\int_0^t f(T)\, dT \right) = \frac{1}{s} F(s).$$

$\therefore$ If $\mathbf{L^{-1} (F(s)) = f(t), \text{ then } L^{-1} \left(\frac{1}{s} F(s) \right) = \int_0^t f(T)\, dT}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse Laplace transform of the function* $\frac{1}{s(s^2 + a^2)}$.

Sol. $$\frac{1}{s(s^2 + a^2)} = \frac{1}{s}\left(\frac{1}{s^2 + a^2} \right) = \frac{1}{s} F(s), \text{ where } F(s) = \frac{1}{s^2 + a^2}.$$

$$L^{-1} (F(s)) = L^{-1} \left(\frac{1}{s^2 + a^2} \right) = \frac{1}{a} L^{-1} \left(\frac{a}{s^2 + a^2} \right) = \frac{1}{a} \sin at = f(t), \text{ say.}$$

$\therefore$ By the **property of Laplace transform of integrals,**

$$L^{-1} \left(\frac{1}{s} F(s) \right) = \int_0^t f(T)\, dT = \int_0^t \frac{1}{a} \sin aT\, dT = \frac{1}{a} \left(-\frac{\cos aT}{a} \right) \Bigg|_0^t$$

$$= -\frac{1}{a^2} (\cos at - \cos 0) = \frac{1 - \cos at}{a^2}$$

$$\therefore \quad L^{-1} \left(\frac{1}{s(s^2 + a^2)} \right) = \mathbf{\frac{1 - \cos at}{a^2}}.$$

Example 2. *Find the inverse Laplace transform of the function* $\frac{1}{s^5+s^3}$.

Sol. $$\frac{1}{s^5+s^3} = \frac{1}{s^3(s^2+1)} = \frac{1}{s^3}\left(\frac{1}{s^2+1}\right) = \frac{1}{s^3}\,\mathrm{F}(s), \text{ where } \mathrm{F}(s) = \frac{1}{s^2+1}.$$

$$\mathrm{L}^{-1}(\mathrm{F}(s)) = \mathrm{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t = f(t), \text{ say}$$

∴ By the **property of Laplace transform of integrals,**

$$\mathrm{L}^{-1}\left(\frac{1}{s}\,\mathrm{F}(s)\right) = \int_0^t \mathrm{F}(\mathrm{T})\,d\mathrm{T} = \int_0^t \sin \mathrm{T}\,d\mathrm{T} = -\cos \mathrm{T}\Big|_0^t = -\cos t + \cos 0 = 1 - \cos t.$$

Again by using the same formula,

$$\mathrm{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s}\,\mathrm{F}(s)\right) = \int_0^t (1-\cos \mathrm{T})\,d\mathrm{T} = (\mathrm{T} - \sin \mathrm{T})\Big|_0^t = t - \sin t.$$

∴ $$\mathrm{L}^{-1}\left(\frac{1}{s^2}\,\mathrm{F}(s)\right) = t - \sin t.$$

Again by using the same formula,

$$\mathrm{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2}\,\mathrm{F}(s)\right) = \int_0^t (\mathrm{T} - \sin \mathrm{T})\,d\mathrm{T} = \left(\frac{\mathrm{T}^2}{2} + \cos \mathrm{T}\right)\Bigg|_0^t = \frac{t^2}{2} + \cos t - 1$$

∴ $$\mathrm{L}^{-1}\left(\frac{1}{s^3}\,\mathrm{F}(s)\right) = \frac{t^2}{2} + \cos t - 1$$

or $$\mathrm{L}^{-1}\left(\frac{1}{s^5+s^3}\right) = \boldsymbol{\frac{t^2}{2} + \cos t - 1.}$$

Remark. The above problem can also be solved by the "method of partial fractions".

2.10. VALUE OF L^{-1} (F′(s)) IN TERMS OF L^{-1}(F(s))

By the property of **differentiation of Laplace transforms**

if $$\mathrm{L}(f(t)) = \mathrm{F}(s), \text{ then } \mathrm{L}(t\,f(t)) = -\,\mathrm{F}'(s).$$

∴ If $\mathrm{L}^{-1}(\mathrm{F}(s)) = f(t)$, then $\mathrm{F}'(s) = \mathrm{L}(-\,t\,f(t))$.

∴ **If** $\mathbf{L^{-1}(F(s)) = f(t)}$**, then** $\mathbf{L^{-1}(F'(s)) = -\,t\,f(t)}$**.**

Example 3. *Find the inverse Laplace transform of the function* $\frac{s}{(s^2+4)^2}$.

Sol. $$\int \frac{s}{(s^2+4)^2}\,ds = \frac{1}{2}\int \frac{2s}{(s^2+4)^2}\,ds = \frac{1}{2}\cdot\frac{(s^2+4)^{-1}}{-1} = -\frac{1}{2(s^2+4)} = \mathrm{F}(s), \quad \text{say.}$$

∴ $$\mathrm{F}'(s) = \frac{s}{(s^2+4)^2}$$

Also $$\mathrm{L}^{-1}(\mathrm{F}(s)) = \mathrm{L}^{-1}\left(-\frac{1}{2(s^2+4)}\right) = -\frac{1}{4}\,\mathrm{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = -\frac{1}{4}\sin 2t = f(t), \quad \text{say.}$$

$\therefore$ By the **property of differentiation of Laplace transforms,** we have

$$L^{-1}(F'(s)) = -t\, f(t).$$

$$\therefore \qquad L^{-1}\left(\frac{s}{(s^2+4)^2}\right) = -t\left(-\frac{1}{4}\sin 2t\right) = \frac{1}{4}\, t \sin 2t$$

$$\therefore \qquad L^{-1}\left(\frac{s}{(s^2+4)^2}\right) = \frac{\mathbf{1}}{\mathbf{4}}\,\mathbf{t \sin 2t.}$$

Example 4. *Find the inverse Laplace transform of the function* $\dfrac{2s+6}{(s^2+6s+10)^2}$.

Sol. $$L^{-1}\left[\frac{2s+6}{(s^2+6s+10)^2}\right] = L^{-1}\left[\frac{2(s+3)}{((s+3)^2+1)^2}\right] = e^{-3t}\, L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] \qquad \text{...(1)}$$

Now, we require the value of $L^{-1}\left[\dfrac{2s}{(s^2+1)^2}\right]$.

$$\int \frac{2s}{(s^2+1)^2}\, ds = \frac{(s^2+1)^{-1}}{-1} = -\frac{1}{s^2+1} = F(s), \quad \text{say.}$$

$$\therefore \qquad F'(s) = \frac{2s}{(s^2+1)^2}$$

Also $$L^{-1}(F(s)) = L^{-1}\left(-\frac{1}{s^2+1}\right) = -\sin t = f(t), \quad \text{say.}$$

$\therefore$ By the **property of differentiation of Laplace transforms,** we have

$$L^{-1}(F'(s)) = -t\, f(t).$$

$$\Rightarrow \qquad L^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = -t\,(-\sin t) \quad \Rightarrow \quad L^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = t \sin t.$$

$$\therefore \quad (1) \quad \Rightarrow \quad L^{-1}\left(\frac{2s+6}{(s^2+6s+10)^2}\right) = \mathbf{e^{-3t}\, t \sin t\,.}$$

Example 5. *Find the inverse Laplace transform of the function* $\dfrac{1}{(s^2+a^2)^2}$.

Sol. Let $$F(s) = \frac{1}{s^2+a^2}.$$

$$\therefore \qquad L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\, L^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\sin at = f(t), \text{say.}$$

Using, $L^{-1}(F'(s)) = -t\, f(t)$, we get

$$L^{-1}\left(\left(\frac{1}{s^2+a^2}\right)'\right) = -t \cdot \frac{1}{a}\sin at\,.$$

$$\Rightarrow \qquad L^{-1}\left(\frac{-2s}{(s^2+a^2)^2}\right) = -\frac{1}{a}\,t\sin at$$

$$\Rightarrow \qquad L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a}\,t\sin at = g(t), \text{ say}.$$

Using formula for **Laplace transform of integrals,** we get

$$L^{-1}\left(\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right) = \int_0^t g(T)\,dT\,.$$

$$\Rightarrow \qquad L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \int_0^t \frac{1}{2a}\,T\sin aT\,dT = \frac{1}{2a}\left[-\frac{T\cos aT}{a}+\frac{\sin aT}{a^2}\right]_0^t$$

$$= \frac{1}{2a^3}\,[-\,at\cos at + \sin at]$$

$$\therefore \qquad L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \mathbf{\frac{1}{2a^3}\,[\sin at - at\cos at]}\,.$$

2.11. VALUE OF $L^{-1}\left(\int_s^\infty F(s)\,dS\right)$ IN TERMS OF L^{-1} (F(s))

By the property of **integration of Laplace transforms**

if $\qquad L(f(t)) = F(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(S)\,dS.$

$\therefore$ **If** $\qquad \mathbf{L^{-1}(F(s)) = f(t)}$**, then** $\mathbf{L^{-1}\left(\int_s^\infty F(S)\,dS\right) = \frac{f(t)}{t}}$**.**

Example 6. *Find the inverse Laplace transform of the function log* $\left(1+\frac{a^2}{s^2}\right)$.

Sol. $\frac{d}{ds}\left(\log\left(1+\frac{a^2}{s^2}\right)\right) = \frac{d}{ds}\left(\log(s^2+a^2) - 2\log s\right) = \frac{2s}{s^2+a^2} - \frac{2}{s} = F(s)$, say.

$$\therefore \qquad \int F(s)\,ds = \log\left(1+\frac{a^2}{s^2}\right)$$

Also, $\qquad L^{-1}(F(s)) = L^{-1}\left(\frac{2s}{s^2+a^2}-\frac{2}{s}\right) = 2L^{-1}\left(\frac{s}{s^2+a^2}\right) - 2L^{-1}\left(\frac{1}{s}\right)$

$$= 2\cos at - 2(1) = 2\cos at - 2 = f(t), \text{ say}.$$

$\therefore$ By the **property of integration of Laplace transforms**, we have

$$L^{-1}\left(\int_s^\infty F(S)\,dS\right) = \frac{f(t)}{t}\,.$$

$$\therefore \qquad L^{-1}\left(\log\left(1+\frac{a^2}{S^2}\right)\Bigg|_s^\infty\right)=\frac{2\cos at-2}{t}$$

$$\therefore \qquad L^{-1}\left(0-\log\left(1+\frac{a^2}{s^2}\right)\right)=\frac{2\cos at-2}{t}$$

$$\therefore \qquad L^{-1}\left(\log\left(1+\frac{a^2}{s^2}\right)\right)=\mathbf{\frac{2-2\cos at}{t}}.$$

Example 7. *Find the inverse Laplace transform of the function log* $\dfrac{s^2+1}{(s-1)^2}$.

Sol. $\dfrac{d}{ds}\left(\log\dfrac{s^2+1}{(s-1)^2}\right)=\dfrac{d}{ds}(\log(s^2+1)-2\log(s-1))=\dfrac{2s}{s^2+1}-\dfrac{2}{s-1}=F(s)$, say.

$$\therefore \qquad \int F(s)\,ds=\log\frac{s^2+1}{(s-1)^2}$$

Also $\qquad L^{-1}(F(s))=L^{-1}\left(\dfrac{2s}{s^2+1}-\dfrac{2}{s-1}\right)=2L^{-1}\left(\dfrac{s}{s^2+1}\right)-2L^{-1}\left(\dfrac{1}{s-1}\right)$

$$=2\cos t-2e^t=f(t),\quad \text{say.}$$

By the **property of integration of Laplace transforms**, we have

$$L^{-1}\left(\int_s^\infty F(S)\,dS\right)=\frac{f(t)}{t}.$$

$$\therefore \qquad L^{-1}\left(\left(\log\frac{S^2+1}{(S-1)^2}\right)\Bigg|_s^\infty\right)=\frac{2\cos t-2e^t}{t}$$

$$\therefore \qquad L^{-1}\left(0-\log\frac{s^2+1}{(s-1)^2}\right)=-\frac{2e^t-2\cos t}{t}$$

$$\therefore \qquad L^{-1}\left(\log\frac{s^2+1}{(s-1)^2}\right)=\mathbf{\frac{2e^t-2\cos t}{t}}.$$

Example 8. *Find the inverse Laplace transform of the function* $\cot^{-1}(s+1)$.

Sol. $\dfrac{d}{ds}(\cot^{-1}(s+1))=\dfrac{-1}{(s+1)^2+1}\,.\,1=-\dfrac{1}{(s+1)^2+1}=F(s)$, say.

$$\therefore \qquad \int F(s)\,ds=\cot^{-1}(s+1)$$

Also, $$L^{-1}(F(s)) = L^{-1}\left(-\frac{1}{(s+1)^2+1}\right) = -L^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$= -e^{-t} L^{-1}\frac{1}{s^2+1} = -e^{-t}\sin t = f(t), \quad \text{say.}$$

$\therefore$ By the **property of integration of Laplace transforms**, we have

$$L^{-1}\left(\int_s^\infty F(S)\, dS\right) = \frac{f(t)}{t}.$$

$$\therefore \quad L^{-1}\left(\cot^{-1}(S+1)\Big|_s^\infty\right) = -\frac{e^{-t}\sin t}{t}$$

$$\Rightarrow \quad L^{-1}(0 - \cot^{-1}(s+1)) = -\frac{e^{-t}\sin t}{t}$$

$$\therefore \quad L^{-1}(\cot^{-1}(s+1)) = \mathbf{\frac{e^{-t}\sin t}{t}}.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L^{-1}(F(s)) = f(t)$, then $L^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t f(T)\, dT$.

Rule II. If $L^{-1}(F(s)) = f(t)$, then $L^{-1}(F'(s)) = -t\, f(t)$.

Rule III. If $L^{-1}(F(s)) = f(t)$, then $L^{-1}\left(\int_s^\infty F(S) dS\right) = \frac{f(t)}{t}$.

TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions :

1. (*i*) $\frac{1}{s^2+4s}$ (*ii*) $\frac{1}{s(s^2+16)}$

2. (*i*) $\frac{1}{s^3-s}$ (*ii*) $\frac{4}{s^3-2s^2}$

3. (*i*) $\frac{1}{s(s+a)^3}$ (*ii*) $\frac{1}{s(s^2+2s+2)}$

4. (*i*) $\frac{7}{s^4-s^2}$ (*ii*) $\frac{\pi^5}{s^4(s^2+\pi^2)}$

5. (*i*) $\frac{s}{(s^2+a^2)^2}$ (*ii*) $\frac{s}{(s^2-9)^2}$

6. (*i*) $\frac{s^2-\pi^2}{(s^2+\pi^2)^2}$ (*ii*) $\frac{s+1}{(s^2+2s+2)^2}$

7. (*i*) $\log\frac{s+a}{s+b}$ (*ii*) $\log\frac{s+1}{s-1}$

8. (*i*) $\log\left(1-\frac{a^2}{s^2}\right)$ (*ii*) $\log\frac{1+s}{s}$

9. (*i*) $\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$ (*ii*) $\log\frac{s^2+1}{s(s+1)}$

10. (*i*) $\log\frac{(s+1)^2}{(s+2)(s+3)}$ (*ii*) $\log\frac{s^2+1}{(s-1)^2}$

11. (*i*) $\cot^{-1}\frac{s}{\pi}$ (*ii*) $\tan^{-1}\frac{2}{s}$.

Answers

1. (*i*) $\frac{1}{4} - \frac{e^{-4t}}{4}$ (*ii*) $\frac{1}{16} - \frac{\cos 4t}{16}$ **2.** (*i*) $\cosh t - 1$ (*ii*) $e^{2t} - 2t - 1$

3. (*i*) $\frac{1}{a^3} - \frac{e^{-at}}{a^3}\left(\frac{a^2}{2}t^2 + at + 1\right)$ (*ii*) $\frac{1}{2}\left[1 - (\sin t + \cos t)\, e^{-t}\right]$

4. (*i*) $7(\sinh t - t)$ (*ii*) $\sin \pi t + \frac{\pi^3}{6}t^3 - \pi t$

5. (*i*) $\frac{1}{2a} t \sin at$ (*ii*) $\frac{1}{6} t \sinh 3t$ **6.** (*i*) $t \cos \pi t$ (*ii*) $\frac{1}{2} e^{-t}\, t \sin t$

7. (*i*) $\frac{e^{-bt} - e^{-at}}{t}$ (*ii*) $\frac{e^{t} - e^{-t}}{t}$ **8.** (*i*) $\frac{2(1 - \cos at)}{t}$ (*ii*) $\frac{1 - e^{-t}}{t}$

9. (*i*) $\frac{\cos at - \cos bt}{t}$ (*ii*) $\frac{1 + e^{-t} - 2\cos t}{t}$

10. (*i*) $\frac{e^{-2t} + e^{-3t} - 2e^{-t}}{t}$ (*ii*) $\frac{2(e^{t} - \cos t)}{t}$

11. (*i*) $\frac{\sin \pi t}{t}$ (*ii*) $\frac{\sin 2t}{t}$.

Hint

6. (*i*) Use $\left(\frac{s}{s^2 + \pi^2}\right)' = -\frac{s^2 - \pi^2}{(s^2 + \pi^2)^2}$.

2.12. CONVOLUTION THEOREM

The convolution theorem is used to find the inverse Laplace transform of the product of two functions with known inverse Laplace transforms of the factors of the product. Let F(s) and G(s) be two functions with known inverse Laplace transforms $f(t)$ and $g(t)$ respectively. The convolution theorem would help us to find the inverse Laplace transform of the product F(s) G(s). In this theorem, we shall prove that the inverse Laplace transformation of the product F(s) G(s) is given by the function $\int_0^t f(\mathrm{T})\, g(t - T)\, d\mathrm{T}$ of t. This integral is called the **convolution** of the functions f and g and it is denoted by $f * g$.

$$\therefore \quad (f * g)(t) = \int_0^t f(\mathrm{T})\, g(t - T)\, d\mathrm{T}.$$

Let $z = t - \mathrm{T}$. $\therefore$ $dz = -\,d\mathrm{T}$

$$\therefore \quad (f * g)(t) = \int_t^0 f(t - z)\, g(z)\,.\, - dz = -\int_t^0 g(z)\, f(t - z)\, dz$$

$$= \int_0^t g(z)\, f(t - z)\, dz = \int_0^t g(\mathrm{T})\, f(t - \mathrm{T})\, d\mathrm{T} \qquad \text{(Replacing } z \text{ by T)}$$

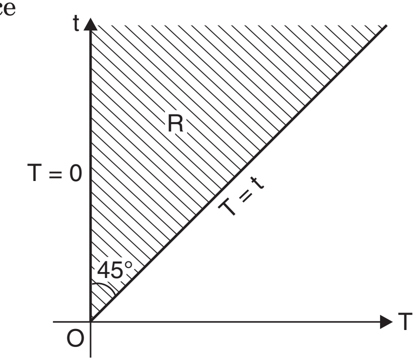
$$= (g * f)(t)$$

$$\therefore \quad \mathbf{f * g = g * f}$$

Statement. *Let f(t) and g(t) be functions such that their Laplace transforms exist. Prove that*

$$L(f * g) = L(f)\, L(g).$$

Proof. It can be proved mathematically that the Laplace transform of the convolution of f and g (*i.e.*, $f * g$) exists.



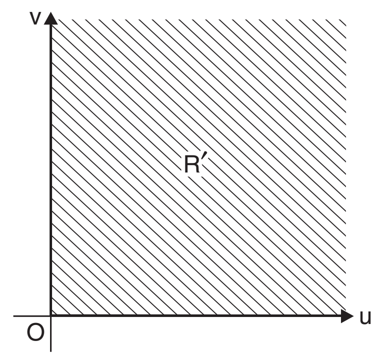
By definition, $\mathrm{L}(f * g) = \int_0^\infty e^{-st}\,(f * g)(t)\,dt$

$$\therefore \qquad \mathrm{L}(f * g) = \int_0^\infty e^{-st}\left[\int_0^t f(\mathrm{T})\, g(t - T)\, d\mathrm{T}\right] dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(\mathrm{T})\, g(t - T)\, d\mathrm{T}\, dt$$

$$= \iint_{\mathrm{R}} e^{-st} f(\mathrm{T})\, g(t - T)\, d\mathrm{T}\, dt,$$

where R is the 45° wedge bounded by the lines T = 0 and T = t in the Tt-plane.



Let $\qquad u = \mathrm{T}$ and $v = t - \mathrm{T}$.

$\therefore \qquad \mathrm{T} = u$ and $t = u + v$

$$\therefore \qquad \mathrm{J} = \begin{vmatrix} \dfrac{\partial \mathrm{T}}{\partial u} & \dfrac{\partial \mathrm{T}}{\partial v} \\ \dfrac{\partial t}{\partial u} & \dfrac{\partial t}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

New variables transform the region R into the region R′ in the uv-plane defined by $u \ge 0$, $v \ge 0$.

$$\therefore \qquad \mathrm{L}\,(f * g) = \iint_{\mathrm{R}'} e^{-s(u+v)} f(u)\, g(v)\, |J|\, du\, dv$$

$$= \int_0^\infty \int_0^\infty e^{-su}\, e^{-sv} f(u)\, g(v)\, du\, dv \qquad (\because\ |\mathrm{J}| = |1| = 1)$$

$$= \left(\int_0^\infty e^{-su} f(u)\, du\right)\left(\int_0^\infty e^{-sv} g(v)\, dv\right)$$

$$= \mathrm{L}(f)\, \mathrm{L}(g).$$

$\therefore \qquad$ **L(f ∗ g) = L(f) L(g).**

Corollary 1. Let $\mathrm{L}(f) = \mathrm{F}(s)$ and $\mathrm{L}(g) = \mathrm{G}(s)$.

$\therefore$ The result of the convolution theorem can be written as

$$\mathrm{L}(f * g) = \mathrm{F}(s)\, \mathrm{G}(s).$$

$$\Rightarrow \qquad \mathrm{L}^{-1}\,(\mathrm{F}(s)\, \mathrm{G}(s)) = f * g$$

$$\Rightarrow \qquad \mathrm{L}^{-1}\,(\mathrm{F}(s)\, \mathrm{G}(s)) = \int_0^t f(\mathrm{T})\, g(t - T)\, d\mathrm{T}.$$

$\therefore$ **If** $\qquad \mathbf{L^{-1}\,(F(s)) = f(t)}$ **and** $\mathbf{L^{-1}\,(G(s)) = g(t)}$,

then $\qquad \mathbf{L^{-1}\,(F(s)\, G(s)) = \int_0^t f(T)\, g(t - T)\, dT.}$

This formula is used to find the inverse Laplace transform of the product of two functions.

Corollary 2. Since $f * g = g * f$, we can also write $L^{-1}(F(s)\,G(s)) = \int_0^t g(T)\,f(t-T)\,dT.$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse Laplace transform of the function* $\dfrac{1}{(s+3)(s-2)}$.

Sol.
$$\frac{1}{(s+3)(s-2)} = \left(\frac{1}{s+3}\right)\left(\frac{1}{s-2}\right) = F(s)\,G(s),$$

where
$$F(s) = \frac{1}{s+3} \quad \text{and} \quad G(s) = \frac{1}{s-2}$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s+3}\right) = e^{-3t} = f(t), \quad \text{say}$$

and
$$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s-2}\right) = e^{2t} = g(t), \quad \text{say}.$$

By **convolution theorem**, we have

$$L^{-1}(F(s)\,G(s)) = f * g = \int_0^t f(T)\,g(t-T)\,dT.$$

$$\therefore \quad L^{-1}\left(\frac{1}{(s+3)(s-2)}\right) = \int_0^t e^{-3T}\,e^{2(t-T)}\,dT$$

$$= \int_0^t e^{2t-5T}\,dT = \left.\frac{e^{2t-5T}}{-5}\right|_0^t = -\frac{1}{5}[e^{-3t} - e^{2t}] = \mathbf{\frac{e^{2t}}{5} - \frac{e^{-3t}}{5}}.$$

Example 2. *Use convolution theorem to evaluate* $L^{-1}\left(\dfrac{1}{s^2(s^2+a^2)}\right)$.

Sol.
$$\frac{1}{s^2(s^2+a^2)} = \left(\frac{1}{s^2}\right)\left(\frac{1}{s^2+a^2}\right) = F(s)\,G(s), \text{ where}$$

$$F(s) = \frac{1}{s^2} \quad \text{and} \quad G(s) = \frac{1}{s^2+a^2}.$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2}\right) = t = f(t), \text{ say}$$

and
$$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\sin at = g(t), \text{ say}$$

By **convolution theorem,** we have

$$L^{-1}(F(s)\,G(s)) = f * g = g * f = \int_0^t g(T)\,f(t-T)\,dT$$

$$\therefore \quad L^{-1}\left(\frac{1}{s^2(s^2+a^2)}\right) = \int_0^t \left(\frac{1}{a}\sin a\mathrm{T}\right)(t-\mathrm{T})\,d\mathrm{T}$$

$$= \frac{t}{a}\int_0^t \sin a\mathrm{T}\,d\mathrm{T} - \frac{1}{a}\int_0^t \mathrm{T}\sin a\mathrm{T}\,d\mathrm{T}$$

$$= \frac{t}{a}\left(-\frac{\cos a\mathrm{T}}{a}\right)\Bigg|_0^t - \frac{1}{a}\left[\mathrm{T}\left(-\frac{\cos a\mathrm{T}}{\mathrm{a}}\right)\Bigg|_0^t - \int_0^t 1.\left(-\frac{\cos a\mathrm{T}}{a}\right)d\mathrm{T}\right]$$

$$= -\frac{t}{a^2}(\cos at - 1) - \frac{1}{a}\left[-\frac{t\cos at}{a} + 0 + \frac{1}{a}.\frac{\sin a\mathrm{T}}{a}\Bigg|_0^t\right]$$

$$= \frac{t}{a^2}(1-\cos at) + \frac{t}{a^2}\cos at - \frac{1}{a^3}\sin at - 0$$

$$= \frac{t}{a^2} - \frac{1}{a^3}\sin at = \frac{\mathbf{1}}{\mathbf{a^3}}\ \mathbf{(at - \sin at).}$$

Example 3. *Find the inverse Laplace transform of the function* $\dfrac{s}{(s^2+\pi^2)^2}$.

Sol.
$$\frac{s}{(s^2+\pi^2)^2} = \left(\frac{s}{s^2+\pi^2}\right)\left(\frac{1}{s^2+\pi^2}\right) = \mathrm{F}(s)\,\mathrm{G}(s),$$

where
$$\mathrm{F}(s) = \frac{s}{s^2+\pi^2} \quad \text{and} \quad \mathrm{G}(s) = \frac{1}{s^2+\pi^2}.$$

$$L^{-1}(\mathrm{F}(s)) = L^{-1}\left(\frac{s}{s^2+\pi^2}\right) = \cos\pi t = f(t), \quad \text{say}$$

and
$$L^{-1}(\mathrm{G}(s)) = L^{-1}\left(\frac{1}{s^2+\pi^2}\right) = \frac{1}{\pi}L^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) = \frac{1}{\pi}\sin\pi t = g(t), \quad \text{say.}$$

By **convolution theorem,** we have

$$L^{-1}(\mathrm{F}(s)\,\mathrm{G}(s)) = f * g = \int_0^t f(\mathrm{T})\,g(t-T)\,d\mathrm{T}.$$

$$\therefore\ L^{-1}\left(\frac{s}{(s^2+\pi^2)^2}\right) = \int_0^t \cos\pi\mathrm{T}\,.\,\frac{1}{\pi}\sin\pi(t-\mathrm{T})\,d\mathrm{T}$$

$$= \frac{1}{2\pi}\int_0^t [2\sin\pi(t-\mathrm{T})\cos\pi\mathrm{T}\,d\mathrm{T}$$

$$= \frac{1}{2\pi}\int_0^t [\sin\pi t + \sin\pi(t-2\mathrm{T})]\,d\mathrm{T} = \frac{1}{2\pi}\left[\mathrm{T}\sin\pi t - \frac{\cos\pi(t-2\mathrm{T})}{-2\pi}\right]_0^t$$

$$= \frac{1}{2\pi}\left[t\sin\pi t + \frac{1}{2\pi}\cos\pi t - 0 - \frac{1}{2\pi}\cos\pi t\right] = \frac{\mathbf{1}}{\mathbf{2\pi}}\ \mathbf{t\sin\pi t.}$$

Example 4. *Find the inverse Laplace transform of the function* $\dfrac{s+3}{(s^2+6s+13)^2}$.

Sol. $$\frac{s+3}{(s^2+6s+13)^2} = \left(\frac{s+3}{s^2+6s+13}\right)\left(\frac{1}{s^2+6s+13}\right) = \text{F}(s)\ \text{G}(s),$$

where $$\text{F}(s) = \frac{s+3}{s^2+6s+13} \quad \text{and} \quad \text{G}(s) = \frac{1}{s^2+6s+13}.$$

$$\text{L}^{-1}\ (\text{F}(s)) = \text{L}^{-1}\left(\frac{s+3}{s^2+6s+13}\right) = \text{L}^{-1}\left(\frac{s+3}{(s+3)^2+4}\right) = e^{-3t}\ \text{L}^{-1}\left(\frac{s}{s^2+4}\right)$$
$$= e^{-3t} \cos 2t = f(t), \text{ say}$$

and $$\text{L}^{-1}\ (\text{G}(s)) = \text{L}^{-1}\left(\frac{1}{s^2+6s+13}\right) = \text{L}^{-1}\left(\frac{1}{(s+3)^2+4}\right) = e^{-3t}\ \text{L}^{-1}\left(\frac{1}{s^2+4}\right)$$
$$= e^{-3t}\ \frac{\sin 2t}{2} = g(t), \text{ say.}$$

By **convolution theorem,** we have

$$\text{L}^{-1}\ (\text{F}(s)\ \text{G}(s)) = f * g = \int_0^t f(\text{T})\ g(t-\text{T})\ dt.$$

$$\therefore \quad \text{L}^{-1}\left(\frac{s+3}{(s^2+6s+13)^2}\right) = \int_0^t e^{-3\text{T}} \cos 2\text{T} \cdot \frac{1}{2} e^{-3(t-\text{T})} \sin 2(t-\text{T})\ d\text{T}.$$
$$= \frac{1}{2}\int_0^t e^{-3t} \cos 2\text{T} \sin 2(t-\text{T})\ d\text{T}$$
$$= \frac{e^{-3t}}{4}\int_0^t 2 \sin 2(t-\text{T}) \cos 2\text{T}\ d\text{T}$$
$$= \frac{e^{-3t}}{4}\int_0^t (\sin 2t + \sin (2t-4\text{T}))\ d\text{T}$$
$$= \frac{e^{-3t}}{4}\left[(\sin 2t)\ \text{T} - \frac{\cos (2t-4\text{T})}{-4}\right]\Bigg|_0^t$$
$$= \frac{e^{-3t}}{4}\left[t \sin 2t + \frac{1}{4}\cos 2t - 0 - \frac{1}{4}\cos 2t\right] = \mathbf{\frac{1}{4}\ te^{-3t} \sin 2t.}$$

Example 5. *Find the inverse Laplace transform of the function* $\dfrac{s}{(s^2+a^2)^3}$.

Sol. $$\frac{s}{(s^2+a^2)^3} = \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+a^2}\right)\left(\frac{s}{s^2+a^2}\right) = \text{F}(s)\ \text{F}(s)\ \text{G}(s),$$

where $\text{F}(s) = \dfrac{1}{s^2+a^2}$ and $\text{G}(s) = \dfrac{s}{s^2+a^2}$.

$$\text{L}^{-1}\ (\text{F}(s)) = \text{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\ \text{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a} \sin at = f(t), \text{ say}$$

and $$\text{L}^{-1}\ (\text{G}(s)) = \text{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at = g(t), \text{ say.}$$

By **convolution theorem,** we have

$$L^{-1}(F(s)\,F(s)) = f * f = \int_0^t f(T)\, f(t-T)\, dT.$$

$$\therefore \quad L^{-1}(F(s)\,F(s)) = \int_0^t \frac{1}{a}\sin aT \cdot \frac{1}{a}\sin a(t-T)\, dT$$

$$= \frac{1}{2a^2}\int_0^t 2\sin aT \sin(at - aT)\, dT$$

$$= \frac{1}{2a^2}\int_0^t [\cos(2aT - at) - \cos at]\, dT$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2aT - at)}{2a} - T(\cos at)\right]\Bigg|_0^t$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at - \frac{\sin(-at)}{2a} + 0\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right] = \frac{1}{2a^3}\,[\sin at - at\cos at] = h(t), \text{ say.}$$

Again, by **convolution theorem,** we have

$$L^{-1}(F(s)\,F(s)\,.\,G(s)) = h * g = \int_0^t h(T)\, g(t-T)\, dT.$$

$$\therefore \quad L^{-1}\left(\frac{s}{(s^2+a^2)^3}\right) = \int_0^t \frac{1}{2a^3}\,(\sin aT - aT\cos aT)\cos a(t-T)\, dT$$

$$= \frac{1}{2a^3}\int_0^t (\sin aT\cos(at - aT) - aT\cos aT\cos(at - aT))\, dT$$

$$= \frac{1}{4a^3}\int_0^t (2\sin aT\cos(at - aT) - aT\,.\,2\cos aT\cos(at - aT))\, dT$$

$$= \frac{1}{4a^3}\int_0^t (\sin at + \sin(2aT - at) - aT(\cos at + \cos(2aT - at)))\, dT$$

$$= \frac{1}{4a^3}\Bigg[\sin at\int_0^t dT + \int_0^t \sin(2aT - at)\, dT - a\cos at\int_0^t T dt$$
$$- a\int_0^t T\cos(2aT - at)\, dT\Bigg]$$

$$= \frac{1}{4a^3}\Bigg[(\sin at)T\Big|_0^t - \frac{\cos(2aT - at)}{2a}\Big|_0^t - (a\cos at)\frac{T^2}{2}\Big|_0^t$$
$$- a\left(T\,.\,\frac{\sin(2aT - at)}{2a}\Big|_0^t - \int_0^t 1\,.\,\frac{\sin(2aT - at)}{2a}\, dT\right)\Bigg]$$

$$= \frac{1}{4a^3}\left[t \sin at - 0 - \frac{1}{2a}(\cos at - \cos at) - \frac{at^2 \cos at}{2} - \frac{1}{2}(t \sin at - 0) + \frac{1}{2}\left(\frac{-\cos(2aT - at)}{2a}\right)\Bigg|_0^t\right]$$

$$= \frac{1}{4a^3}\left[t \sin at - \frac{1}{2}at^2 \cos at - \frac{1}{2}t \sin at - \frac{1}{4a}(\cos at - \cos at)\right]$$

$$= \frac{1}{4a^3}\left[\frac{1}{2}t \sin at - \frac{1}{2}at^2 \cos at\right]. = \frac{\mathbf{t}}{\mathbf{8a^3}}\,\textbf{[sin at – at cos at].}$$

WORKING STEPS FOR SOLVING PROBLEMS

Step I. Write the given function as the product of two functions say F(s) and G(s).

Step II. Find the inverse Laplace transforms of the functions F(s) and G(s) and call these functions as $f(t)$ and $g(t)$ respectively.

Step III. Write the result of convolution theorem as

$$L^{-1}(F(s)\,G(s)) = \int_0^t f(T)\, g(t - T)\, dT.$$

Step IV. Substitute the values of f(T) and $g(t$ – T) and simplify the integral on the right side.

TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions by using convolution theorem :

1. (i) $\dfrac{1}{(s-2)(s+4)}$ (ii) $\dfrac{4}{(s-3)(s+7)}$
2. (i) $\dfrac{1}{s^2(s-1)}$ (ii) $\dfrac{1}{s(s^2+1)}$
3. (i) $\dfrac{1}{s^2(s+1)^2}$ (ii) $\dfrac{1}{s^3(s^2+1)}$
4. (i) $\dfrac{1}{(s+1)(s^2+1)}$ (ii) $\dfrac{1}{(s^2+1)^2}$
5. (i) $\dfrac{1}{s^2(s^2-a^2)}$ (ii) $\dfrac{a^2}{s(s+a)^2}$
6. (i) $\dfrac{1}{s(s+1)^3}$ (ii) $\dfrac{4}{s^3+s^2+s+1}$
7. (i) $\dfrac{s^2}{(s^2+4)^2}$ (ii) $\dfrac{s^2}{(s^2-9)^2}$
8. (i) $\dfrac{1}{s^3-a^3}$ (ii) $\dfrac{s+2}{(s^2+4s+5)^2}$
9. (i) $\dfrac{s}{(s^2+2s)^2}$ (ii) $\dfrac{1}{s(s+2)^3}$.

Answers

1. (*i*) $\frac{1}{6}\,[e^{2t} - e^{-4t}]$ (*ii*) $\frac{2}{5}\,[e^{3t} - e^{-7t}]$

2. (*i*) $e^t - t - 1$ (*ii*) $1 - \cos t$

3. (*i*) $(t + 2)\, e^{-t} + t - 2$ (*ii*) $\cos t + \frac{t^2}{2} - 1$

4. (*i*) $\frac{1}{2}\,[e^{-t} + \sin t - \cos t]$ (*ii*) $\frac{1}{2}\,(\sin t - t \cos t)$

5. (*i*) $\frac{1}{a^3}\,[\sinh at - at]$ (*ii*) $1 - e^{-at}\,(1 + at)$

6. (*i*) $-\frac{1}{e^t}\left[\frac{t^2}{2} + t + 1\right] + 1$ (*ii*) $2\,[e^{-t} + \sin t - \cos t]$

7. (*i*) $\frac{1}{2}\left[t \cos 2t + \frac{1}{2} \sin 2t\right]$ (*ii*) $\frac{1}{2}\left[t \cosh 3t + \frac{1}{3} \sinh 3t\right]$

8. (*i*) $\frac{1}{3a^2}\left[e^{at} - e^{-\frac{at}{2}}\left(\cos \frac{\sqrt{3}}{2}\, at + \sqrt{3} \sin \frac{\sqrt{3}}{2}\, at\right)\right]$ (*ii*) $\frac{1}{2}\, t\, e^{-2t} \sin t$

9. (*i*) $-\frac{1}{4}\,[2te^{-2t} + e^{-2t} - 1]$ (*ii*) $-\frac{1}{8e^{2t}}(2t^2 + 2t + 1) + \frac{1}{8}$.

2.13. INVERSE LAPLACE TRANSFORMS BY THE METHOD OF PARTIAL FRACTIONS

Let $\dfrac{a_0 s^m + a_1 s^{m-1} + \ldots\ldots + a_m}{b_0 s^n + b_1 s^{n-1} + \ldots\ldots + b_n}$ be a proper rational algebraic function of s with $m < n$. The denominator of this quotient can be factorised into linear and quadratic factors. The given rational function can be expressed as the sum of partial fractions as per the rules given below :

(*i*) If $as + b$ is any linear non-repeated factor in the denominator, then there corresponds a partial fraction of the form $\dfrac{\text{A}}{as + b}$.

(*ii*) If $as + b$ is any linear factor repeated r ($\in$ **N**) times in the denominator, then there corresponds partial fractions of the form $\dfrac{\text{A}}{as + b}, \dfrac{\text{B}}{(as + b)^2}, \dfrac{\text{C}}{(as + b)^3}, \ldots\ldots r$ terms.

(*iii*) If $as^2 + bs + c$ is any irreducible quadratic factor in the denominator, then there corresponds partial fraction of the form $\dfrac{\text{A}s + \text{B}}{as^2 + bs + c}$.

(*iv*) If $as^2 + bs + c$ is any irreducible factor repeated r ($\in$ **N**) times in the denominator, then there corresponds partial fractions of the form $\dfrac{\text{A}s + \text{B}}{as^2 + bs + c}, \dfrac{\text{C}s + \text{D}}{(as^2 + bs + c)^2}$, ... r terms.

The quantities A, B, C, D, ... are all constants independent of s.

The constants A, B, C, D, ... occurring in the numerators of the partial fractions are determined by simplifying the sum of partial fractions and then giving various values to s, to obtain equations involving unknown constants or by comparing the coefficients of like powers of s.

$\therefore$ The given proper fraction can be expressed as the sum of its partial fractions. Thus, by using linearity of inverse Laplace transform and elementary inverse Laplace transform formulae, the inverse Laplace transform of the given proper fraction is found.

If the given fraction is not proper, then division of numerator by denominator is carried first.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse Laplace transform of the following functions* :

(*i*) $\dfrac{s^2+s-2}{s(s+3)(s-2)}$ (*ii*) $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$.

Sol. (*i*) $\dfrac{s^2+s-2}{s(s+3)(s-2)}$ is a proper fraction.

Let
$$\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-2}$$

Multiplying both sides by $s(s + 3)(s - 2)$, we get

$$s^2 + s - 2 = A(s + 3)(s - 2) + Bs(s - 2) + Cs(s + 3) \qquad ...(1)$$

$s = 0$ in (1) $\Rightarrow$ $-2 = A(3)(-2) + 0 + 0$ $\Rightarrow$ $A = -2/-6 = 1/3$

$s = -3$ in (1) $\Rightarrow$ $9 - 3 - 2 = 0 + B(-3)(-5) + 0$ $\Rightarrow$ $B = 4/15$

$s = 2$ in (1) $\Rightarrow$ $4 + 2 - 2 = 0 + 0 + C(2)(5)$ $\Rightarrow$ $C = 4/10 = 2/5$

$$\therefore \quad \frac{s^2+s-2}{s(s+3)(s-2)}=\frac{1/3}{s}+\frac{4/15}{s+3}+\frac{2/5}{s-2}=\frac{1}{3}\left(\frac{1}{s}\right)+\frac{4}{15}\left(\frac{1}{s+3}\right)+\frac{2}{5}\left(\frac{1}{s-2}\right)$$

$$\therefore \quad L^{-1}\left(\frac{s^2+s-2}{s(s+3)(s-2)}\right)=\frac{1}{3}L^{-1}\left(\frac{1}{s}\right)+\frac{4}{15}L^{-1}\left(\frac{1}{s+3}\right)+\frac{2}{5}L^{-1}\left(\frac{1}{s-2}\right)$$

$$=\frac{1}{3}(1)+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}=\mathbf{\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}}.$$

Note. Since the factors in the denominator are linear and non-repeated, the short-cut method can also be used.

(ii) $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)} = \dfrac{s^2-10s+13}{(s-7)(s-2)(s-3)}$ is a proper fraction.

The factors in the denominator are linear and non-repeated.

$\therefore$ By short-cut method,

$$\frac{s^2-10s+13}{(s-7)(s^2-5s+6)} = \frac{7^2+10(7)+13}{(s-7)(5)(4)} + \frac{2^2-10(2)+13}{(-5)(s-2)(-1)} + \frac{3^2-10(3)+13}{(-4)(1)(s-3)}$$

$$= \frac{-8}{20(s-7)} + \frac{-3}{5(s-2)} + \frac{-8}{-4(s-3)}$$

$$= -\frac{2}{5(s-7)} - \frac{3}{5(s-2)} + 2\,.\,\frac{1}{s-3}$$

$$\therefore \quad L^{-1}\left(\frac{s^2-10s+13}{(s-7)(s^2-5s+6)}\right) = L^{-1}\left(-\frac{2}{5(s-7)} - \frac{3}{5(s-2)} + 2\,.\,\frac{1}{s-3}\right)$$

$$= -\frac{2}{5}L^{-1}\left(\frac{1}{s-7}\right) - \frac{3}{5}L^{-1}\left(\frac{1}{s-2}\right) + 2L^{-1}\left(\frac{1}{s-3}\right)$$

$$= \mathbf{-\frac{2}{5}e^{7t} - \frac{3}{5}e^{2t} + 2e^{3t}.}$$

Example 2. *Find the inverse Laplace transform of the function* $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ *by using (i) method of partial fractions (ii) convolution theorem.*

Sol. (i) $$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{z}{(z+a^2)(z+b^2)}, \text{ where } z = s^2$$

$$= \frac{-a^2}{(z+a^2)(-a^2+b^2)} + \frac{-b^2}{(-b^2+a^2)(z+b^2)}$$

$$= \frac{a^2}{a^2-b^2}\cdot\frac{1}{z+a^2} - \frac{b^2}{a^2-b^2}\cdot\frac{1}{z+b^2}$$

$$= \frac{a}{a^2-b^2}\cdot\frac{a}{s^2+a^2} - \frac{b}{a^2-b^2}\cdot\frac{b}{s^2+b^2}.$$

$$\therefore \quad L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right) = \frac{a}{a^2-b^2}L^{-1}\left(\frac{a}{s^2+a^2}\right) - \frac{b}{a^2-b^2}L^{-1}\left(\frac{b}{s^2+b^2}\right)$$

$$= \frac{a}{a^2 - b^2} \sin at - \frac{b}{a^2 - b^2} \sin bt$$

$$= \frac{\mathbf{a \sin at - b \sin bt}}{\mathbf{a^2 - b^2}}.$$

(*ii*) $$\frac{s^2}{(s^2 + a^2)(s^2 + b^2)} = \left(\frac{s}{s^2 + a^2}\right)\left(\frac{s}{s^2 + b^2}\right) = \text{F}(s)\ \text{G}(s),$$

where $\text{F}(s) = \dfrac{s}{s^2 + a^2}$ and $\text{G}(s) = \dfrac{s}{s^2 + b^2}$.

Also $$\text{L}^{-1}(\text{F}(s)) = \text{L}^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at = f(t), \text{ say}$$

and $$\text{L}^{-1}(\text{G}(s)) = \text{L}^{-1}\left(\frac{s}{s^2 + b^2}\right) = \cos bt = g(t), \text{ say.}$$

By **convolution theorem,** we have

$$\text{L}^{-1}(\text{F}(s)\ \text{G}(s)) = f * g = \int_0^t f(\text{T})\, g(t - \text{T})\, d\text{T}.$$

$$\therefore \quad \text{L}^{-1}\left(\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}\right) = \int_0^t \cos a\text{T} \cos b(t - \text{T})\, d\text{T}$$

$$= \frac{1}{2}\int_0^t \left[\cos(a\text{T} + bt - b\text{T}) + \cos(a\text{T} - bt + b\text{T})\right] d\text{T}$$

$$= \frac{1}{2}\left[\frac{\sin((a - b)\text{T} + bt)}{a - b} + \frac{\sin((a + b)\text{T} - bt)}{a + b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{a - b} + \frac{\sin at}{a + b} - \frac{\sin bt}{a - b} + \frac{\sin bt}{a + b}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{a^2 - b^2} \sin at - \frac{2b}{a^2 - b^2} \sin bt\right]$$

$$= \frac{\mathbf{a \sin at - b \sin bt}}{\mathbf{a^2 - b^2}}.$$

Example 3. *Find the inverse Laplace transform of the function* $\dfrac{1}{(s - 1)^5 (s + 2)}$ *by using (i) method of partial fractions (ii) convolution theorem.*

Sol. (*i*) $$\frac{1}{(s - 1)^5 (s + 2)} = \frac{1}{z^5 (z + 3)}, \text{ where } \quad z = s - 1.$$

We divide 1 by $z + 3$ as follows :

$$
\begin{array}{r|l}
 & \dfrac{1}{3}-\dfrac{z}{9}+\dfrac{z^2}{27}-\dfrac{z^3}{81}+\dfrac{z^4}{243} \\
\hline
3+z & 1 \\
 & 1+\dfrac{z}{3} \\
\cline{2-2}
 & -\dfrac{z}{3} \\
 & -\dfrac{z}{3}-\dfrac{z^2}{9} \\
\cline{2-2}
 & \dfrac{z^2}{9} \\
 & \dfrac{z^2}{9}+\dfrac{z^3}{27} \\
\cline{2-2}
 & -\dfrac{z^3}{27} \\
 & -\dfrac{z^3}{27}-\dfrac{z^4}{81} \\
\cline{2-2}
 & \dfrac{z^4}{81} \\
 & \dfrac{z^4}{81}+\dfrac{z^5}{243} \\
\cline{2-2}
 & -\dfrac{z^5}{243} \\
\cline{2-2}
\end{array}
$$

$$\therefore \qquad \frac{1}{3+z}=\frac{1}{3}-\frac{z}{9}+\frac{z^2}{27}-\frac{z^3}{81}+\frac{z^4}{243}-\frac{z^5/243}{3+z}$$

$$\therefore \qquad \frac{1}{z^5(z+3)}=\frac{1}{z^5}\left(\frac{1}{3+z}\right)=\frac{1}{3z^5}-\frac{1}{9z^4}+\frac{1}{27z^3}-\frac{1}{81z^2}+\frac{1}{243z}-\frac{1}{243(3+z)}$$

$$\therefore \qquad \frac{1}{(s-1)^5\,(s+2)}=\frac{1}{3(s-1)^5}-\frac{1}{9(s-1)^4}+\frac{1}{27(s-1)^3}$$

$$-\frac{1}{81(s-1)^2}+\frac{1}{243(s-1)}-\frac{1}{243(s+2)}$$

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{1}{(s-1)^5(s+2)}\right)=\frac{1}{3}\mathrm{L}^{-1}\left(\frac{1}{(s-1)^5}\right)-\frac{1}{9}\mathrm{L}^{-1}\left(\frac{1}{(s-1)^4}\right)$$

$$+\frac{1}{27}\mathrm{L}^{-1}\left(\frac{1}{(s-1)^3}\right)-\frac{1}{81}\mathrm{L}^{-1}\left(\frac{1}{(s-1)^2}\right)+\frac{1}{243}\mathrm{L}^{-1}\left(\frac{1}{s-1}\right)-\frac{1}{243}\mathrm{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{3} e^t \, L^{-1}\left(\frac{1}{s^5}\right) - \frac{1}{9} e^t \, L^{-1}\left(\frac{1}{s^4}\right) + \frac{1}{27} e^t \, L^{-1}\left(\frac{1}{s^3}\right)$$

$$- \frac{1}{81} e^t \, L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{243} e^t \, L^{-1}\left(\frac{1}{s}\right) - \frac{1}{243} e^{-2t} \, L^{-1}\left(\frac{1}{s}\right)$$

$$= \frac{e^t}{3} \cdot \frac{1}{24} L^{-1}\left(\frac{4!}{s^5}\right) - \frac{e^t}{9} \cdot \frac{1}{6} L^{-1}\left(\frac{3!}{s^4}\right) - \frac{e^t}{27} \cdot \frac{1}{2} L^{-1}\left(\frac{2!}{s^3}\right) - \frac{e^t}{81} \cdot t + \frac{e^t}{243} \cdot 1 - \frac{e^{-2t}}{243} \cdot 1$$

$$= \mathbf{\frac{e^t}{72} t^4 - \frac{e^t}{54} t^3 - \frac{e^t}{54} t^2 - \frac{e^t}{81} t + \frac{e^t}{243} - \frac{e^{-2t}}{243}}.$$

(*ii*) $$\frac{1}{(s-1)^5 (s+2)} = \left(\frac{1}{(s-1)^5}\right)\left(\frac{1}{s+2}\right) = F(s)\, G(s),$$

where $$F(s) = \frac{1}{(s-1)^5} \quad \text{and} \quad G(s) = \frac{1}{s+2}.$$

Also $$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{(s-1)^5}\right) = e^{1.t} \, L^{-1}\left(\frac{1}{s^5}\right)$$

$$= e^t \cdot \frac{1}{24} L^{-1}\left(\frac{4!}{s^5}\right) = \frac{e^t}{24} t^4 = f(t), \text{ say}$$

and $$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s+2}\right) = e^{-2t} = g(t), \quad \text{say}.$$

By **convolution theorem,** we have

$$L^{-1}(F(s)\, G(s)) = f * g = \int_0^t f(T)\, g(t-T)\, dT.$$

$$\therefore \quad L^{-1}\left(\frac{1}{(s-1)^5 (s+2)}\right) = \int_0^t \frac{e^T T^4}{24} \cdot e^{-2(t-T)}\, dT = \frac{e^{-2t}}{24} \int_0^t T^4 e^{3T}\, dT.$$

Now $$\int_0^t T^4 e^{3T}\, dT = \frac{T^4 e^{3T}}{3}\Bigg|_0^t - \int_0^t 4T^3 \cdot \frac{e^{3T}}{3}\, dT$$

$$= \frac{t^4 e^{3t}}{3} - \frac{4}{3}\left[\frac{T^3 e^{3T}}{3}\Bigg|_0^t - \int_0^t 3T^2 \cdot \frac{e^{3T}}{3}\, dT\right]$$

$$= \frac{t^4 e^{3t}}{3} - \frac{4}{9} t^3 e^{3t} + \frac{4}{3}\left[\frac{T^2 e^{3T}}{3}\Bigg|_0^t - \int_0^t 2T \cdot \frac{e^{3T}}{3}\, dT\right]$$

$$= \frac{t^4 e^{3t}}{3} - \frac{4}{9} t^3 e^{3t} + \frac{4}{9} t^2 e^{3t} - \frac{8}{9}\left[\frac{Te^{3T}}{3}\Bigg|_0^t - \int_0^t 1 \cdot \frac{e^{3T}}{3}\, dT\right]$$

$$= \frac{t^4 e^{3t}}{3} - \frac{4}{9} t^3 e^{3t} + \frac{4}{9} t^2 e^{3t} - \frac{8}{27} te^{3t} + \frac{8}{81}(e^{3t} - 1)$$

$$\therefore \quad L^{-1}\left(\frac{1}{(s-1)^5 (s+2)}\right) = \frac{e^{-2t}}{24}\left[\frac{t^4 e^{3t}}{3} - \frac{4}{9} t^3 e^{3t} + \frac{4}{9} t^2 e^{3t} - \frac{8}{27} te^{3t} + \frac{8}{81} e^{3t} - \frac{8}{81}\right]$$

$$= \mathbf{\frac{t^4 e^t}{72} - \frac{t^3 e^t}{54} + \frac{t^2 e^t}{54} - \frac{te^t}{81} + \frac{e^t}{243} - \frac{e^{-2t}}{243}}.$$

Example 4. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{4s+5}{(s-1)^2 (s+2)}$ (*ii*) $\dfrac{1+2s}{(s+2)^2 (s-1)^2}$.

Sol. (*i*) $\dfrac{4s+5}{(s-1)^2 (s+2)}$ is a proper fraction.

Let $\dfrac{4s+5}{(s-1)^2 (s+2)} = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{s+2}$

Multiplying both sides by $(s-1)^2 (s+2)$, we get

$$4s + 5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2. \qquad ...(1)$$

$s = 1$ in (1) $\Rightarrow$ $9 = 0 + 3B + 0$ $\Rightarrow$ $B = 3$

$s = -2$ in (1) $\Rightarrow$ $-3 = 0 + 0 + 9C$ $\Rightarrow$ $C = -1/3$

Let $s = 0$. $\therefore$ $5 = -2A + 2B + C$

$\Rightarrow$ $2A = 2(3) - (1/3) - 5 = 2/3$ $\Rightarrow$ $A = 1/3$.

$$\therefore \quad \frac{4s+5}{(s-1)^2 (s+2)} = \frac{1/3}{s-1} + \frac{3}{(s-1)^2} + \frac{-1/3}{s+2} = \frac{1}{3}\left(\frac{1}{s-1}\right) + 3\left(\frac{1}{(s-1)^2}\right) - \frac{1}{3}\left(\frac{1}{s+2}\right)$$

$$\therefore \quad L^{-1}\left(\frac{4s+5}{(s-1)^2 (s+2)}\right) = \frac{1}{3} L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{(s-1)^2}\right) - \frac{1}{3} L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{3} e^{1.t} + 3e^{1.t} L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{3} e^{-2t}$$

$$= \frac{1}{3} e^t + 3e^t t^1 - \frac{1}{3} e^{-2t} = \mathbf{\frac{1}{3}((9t+1)\, e^t - e^{-2t})}.$$

(*ii*) $\dfrac{1+2s}{(s+2)^2 (s-1)^2}$ is a proper fraction.

Let $\dfrac{1+2s}{(s+2)^2 (s-1)^2} = \dfrac{A}{s+2} + \dfrac{B}{(s+2)^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2}$

Multiplying both sides by $(s+2)^2 (s-1)^2$, we get

$$1 + 2s = A(s+2)(s-1)^2 + B(s-1)^2 + C(s+2)^2 (s-1) + D(s+2)^2. \qquad ...(1)$$

$s = 1$ in (1) $\Rightarrow$ $3 = 0 + 0 + 0 + 9D$ $\Rightarrow$ $D = 1/3$

$s = -2$ in (1) $\Rightarrow$ $-3 = 0 + 9B + 0 + 0$ $\Rightarrow$ $B = -1/3$

Comparing coefficients of s^3 and s^2 in (1), we get

$$0 = A + C \qquad ...(2) \qquad \text{and} \qquad 0 = B + 3C + D \qquad ...(3)$$

$$(3) \Rightarrow 3C = -B - D = \frac{1}{3} - \frac{1}{3} = 0 \Rightarrow C = 0$$

$\therefore$ (2) $\Rightarrow$ $A = -C = 0$.

$$\therefore \quad \frac{1+2s}{(s+2)^2\,(s-1)^2} = \frac{0}{s+2} + \frac{-1/3}{(s+2)^2} + \frac{0}{s-1} + \frac{1/3}{(s-1)^2} = -\frac{1}{3}\left(\frac{1}{(s+2)^2}\right) + \frac{1}{3}\left(\frac{1}{(s-1)^2}\right)$$

$$\therefore \quad L^{-1}\left(\frac{1+2s}{(s+2)^2\,(s-1)^2}\right) = -\frac{1}{3}L^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{1}{3}L^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$= -\frac{1}{3}e^{-2t}\,L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{3}e^{1.t}\,L^{-1}\left(\frac{1}{s^2}\right)$$

$$= -\frac{1}{3}e^{-2t}\,.\,t + \frac{1}{3}e^{t}\,.\,t = \mathbf{\frac{t}{3}(e^t - e^{-2t}).}$$

Example 5. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{3s+1}{(s+1)\,(s^2+1)}$ (*ii*) $\dfrac{2s^3+2s^2+4s+1}{(s^2+1)\,(s^2+s+1)}$.

Sol. (*i*) $\dfrac{3s+1}{(s+1)(s^2+1)}$ is a proper fraction.

Let $$\frac{3s+1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}.$$

Multiplying both sides by $(s + 1)(s^2 + 1)$, we get

$$3s + 1 = A(s^2 + 1) + (Bs + C)\,(s + 1). \qquad ...(1)$$

$s = -1$ in (1) $\Rightarrow$ $-2 = 2A + 0$ $\Rightarrow$ $A = -1$

Comparing coefficients of s^2 and s in (1), we get

$$0 = A + B \qquad ...(2) \qquad \text{and} \qquad 3 = B + C \qquad ...(3)$$

(2) $\Rightarrow$ $B = -A = -(-1) = 1$

(3) $\Rightarrow$ $C = 3 - B = 3 - 1 = 2$.

$$\therefore \quad \frac{3s+1}{(s+1)(s^2+1)} = \frac{-1}{s+1} + \frac{1\,.\,s+2}{s^2+1} = -\frac{1}{s+1} + \frac{s}{s^2+1} + \frac{2}{s^2+1}$$

$$\therefore \quad L^{-1}\left(\frac{3s+1}{(s+1)\,(s^2+1)}\right) = -L^{-1}\left(\frac{1}{s+1}\right) + L^{-1}\left(\frac{s}{s^2+1}\right) + 2L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \mathbf{-e^{t} + \cos t + 2 \sin t.}$$

(*ii*) Let $$F(s) = \frac{2s^3 + 2s^2 + 4s + 1}{(s^2 + 1)(s^2 + s + 1)}.$$

$\therefore$ F(s) is a proper fraction.

Let $$\frac{2s^3 + 2s^2 + 4s + 1}{(s^2 + 1)(s^2 + s + 1)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + s + 1}.$$

Multiplying both sides by $(s^2 + 1)(s^2 + s + 1)$, we get

$$2s^3 + 2s^2 + 4s + 1 = (As + B)(s^2 + s + 1) + (Cs + D)(s^2 + 1).$$

Comparing coefficients of s^3, s^2, s and constant terms, we get

$$2 = A + C \quad \text{...(1)}$$

$$2 = A + B + D \quad \text{...(2)}$$

$$4 = A + B + C \quad \text{...(3)}$$

$$1 = B + D \quad \text{...(4)}$$

(2) – (4) $\Rightarrow$ A = 1

(1) $\Rightarrow$ C = 2 – A = 2 – 1 = 1

(3) $\Rightarrow$ B = 4 – A – C = 4 – 1 – 1 = 2

(4) $\Rightarrow$ D = 1 – B = 1 – 2 = – 1.

$$\therefore \quad F(s) = \frac{1.s + 2}{s^2 + 1} + \frac{1.s - 1}{s^2 + s + 1} = \frac{s}{s^2 + 1} + 2\left(\frac{1}{s^2 + 1}\right) + \frac{s - 1}{s^2 + s + 1}$$

$$\therefore \quad L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2 + 1}\right) + 2L^{-1}\left(\frac{1}{s^2 + 1}\right) + L^{-1}\left(\frac{s - 1}{s^2 + s + 1}\right)$$

$$= \cos t + 2\sin t + L^{-1}\left(\frac{\left(s + \frac{1}{2}\right) - \frac{3}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}\right)$$

$$= \cos t + 2\sin t + e^{-\frac{1}{2}t}\, L^{-1}\left(\frac{s - \frac{3}{2}}{s^2 + \frac{3}{4}}\right)$$

$$= \cos t + 2\sin t + e^{-\frac{1}{2}t}\, L^{-1}\left(\frac{s}{s^2 + \frac{3}{4}} - \sqrt{3}\,\frac{\sqrt{3}/2}{s^2 + \frac{3}{4}}\right)$$

$$= \mathbf{\cos t + 2\sin t + e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right)}.$$

Example 6. *Find the inverse Laplace transform of the function* $\dfrac{s}{s^4+4a^4}$.

Sol. $$\frac{s}{s^4+4a^4}=\frac{s}{(s^2+2a^2)^2-4a^2s^2}=\frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$$

$\therefore$ Let $$\frac{s}{s^4+4a^4}=\frac{As+B}{s^2+2as+2a^2}+\frac{Cs+D}{s^2-2as+2a^2}.$$

Multiplying both sides by $s^4 + 4a^4$, we get

$$s = (As + B)(s^2 - 2as + 2a^2) + (Cs + D)(s^2 + 2as + 2a^2).$$

Comparing coefficients of s^3, s^2, s and constant terms, we get

$$0 = A + C \qquad ...(1)$$

$$0 = -2aA + B + 2aC + D \qquad ...(2)$$

$$1 = 2a^2A - 2aB + 2a^2C + 2aD \qquad ...(3)$$

$$0 = 2a^2B + 2a^2 D \qquad ...(4)$$

$$(4) \Rightarrow B + D = 0 \qquad ...(5)$$

$$(2) - (5) \Rightarrow -2a(A - C) = 0 \Rightarrow A = C \qquad ...(6)$$

$\therefore$ (1) and (6) $\Rightarrow$ A = 0, C = 0

$$(3) \Rightarrow 1 = -2a(B - D) \Rightarrow B - D = -1/2a \qquad ...(7)$$

Solving (5) and (7), we get $B = -1/4a$, $D = 1/4a$.

$$\therefore \quad \frac{s}{s^4+4a^4}=\frac{0.s-1/4a}{s^2+2as+2a^2}+\frac{0.s+1/4a}{s^2-2as+2a^2}$$

$$=-\frac{1}{4a}\left(\frac{1}{s^2+2as+a^2}\right)+\frac{1}{4a}\left(\frac{1}{s^2-2as+2a^2}\right)$$

$$=-\frac{1}{4a}\left(\frac{1}{(s+a)^2+a^2}\right)+\frac{1}{4a}\left(\frac{1}{(s-a)^2+a^2}\right)$$

$$\therefore L^{-1}\left(\frac{s}{s^4+4a^4}\right)=-\frac{1}{4a}L^{-1}\left(\frac{1}{(s+a)^2+a^2}\right)+\frac{1}{4a}L^{-1}\left(\frac{1}{(s-a)^2+a^2}\right)$$

$$=-\frac{1}{4a}e^{-at}L^{-1}\left(\frac{1}{s^2+a^2}\right)+\frac{1}{4a}e^{at}L^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$=\frac{1}{4a}(e^{at}-e^{-at}).\frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right)=\frac{1}{2a^2}\left(\frac{e^{at}-e^{-at}}{2}\right).\sin at$$

$$=\mathbf{\frac{1}{2a^2}\sinh at\sin at.}$$

TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions by the method of partial fractions (Q No 1–12) :

1. (*i*) $\dfrac{1}{(s-1)(s-2)}$ (*ii*) $\dfrac{1}{(s+a)(s+b)}$

2. (*i*) $\dfrac{3s+7}{s^2-2s-3}$ (*ii*) $\dfrac{3s-11}{s^2-7s+12}$

3. (*i*) $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$ (*ii*) $\dfrac{s^2-6}{s^3+4s^2+3s}$

4. (*i*) $\dfrac{1}{s^2(s^2+1)}$ (*ii*) $\dfrac{1}{s^2(s^2+1)(s^2+9)}$

5. (*i*) $\dfrac{s^2-2s+3}{(s-1)^2(s+1)}$ (*ii*) $\dfrac{1}{(s+2)^2(s-2)}$

6. (*i*) $\dfrac{5s-2}{s^2(s+2)(s-1)}$ (*ii*) $\dfrac{1}{s^2(s-a)^2}$

7. (*i*) $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}$ (*ii*) $\dfrac{3s^3-3s^2-40s+36}{(s^2-4)^2}$

8. (*i*) $\dfrac{1}{(s+1)(s^2+1)}$ (*ii*) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

9. (*i*) $\dfrac{3s+1}{(s-1)(s^2+1)}$ (*ii*) $\dfrac{1}{s^3(s^2+1)}$

10. (*i*) $\dfrac{s}{s^4+s^2+1}$ (*ii*) $\dfrac{1}{s^3-a^3}$

11. (*i*) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$ (*ii*) $\dfrac{s^3}{s^4-a^4}$

12. (*i*) $\dfrac{s}{(s+1)^2(s^2+1)}$ (*ii*) $\dfrac{a(s^2-2a^2)}{s^4+4a^4}$.

Answers

1. (*i*) $e^{2t}-e^t$ (*ii*) $\dfrac{e^{-at}-e^{-bt}}{b-a}$

2. (*i*) $4e^{3t}-e^{-t}$ (*ii*) $e^{4t}+2e^{3t}$

3. (*i*) $2+e^t-e^{-2t}$ (*ii*) $-2+\dfrac{5}{2}e^{-t}+\dfrac{1}{2}e^{-3t}$

4. (*i*) $t-\sin t$ (*ii*) $\dfrac{1}{9}t-\dfrac{1}{8}\sin t+\dfrac{1}{216}\sin 3t$

5. (*i*) $\left(t-\dfrac{1}{2}\right)e^t+\dfrac{3}{2}e^{-t}$ (*ii*) $\dfrac{1}{16}(e^{2t}-(4t+1)e^{-2t})$

6. (*i*) $t + e^t + e^{-2t} - 2$ (*ii*) $\frac{1}{a^3}[(2 + at) - (2 - at)e^{at}]$

7. (*i*) $e^{2t}\left[\frac{1}{3} + 4t - \frac{7}{2}t^2\right] - \frac{1}{3}e^{-t}$ (*ii*) $-10t \sinh 2t + 3e^{-2t} + 3te^{2t}$

8. (*i*) $\frac{1}{2}[e^{-t} - \cos t + \sin t]$ (*ii*) $e^t - e^{-t}\left(\cos 2t - \frac{3}{2}\sin 2t\right)$

9. (*i*) $2e^t - 2\cos t + \sin t$ (*ii*) $\frac{1}{2}t^2 + \cos t - 1$

10. (*i*) $\frac{2}{\sqrt{3}} \sin \frac{t\sqrt{3}}{2} \sinh \frac{t}{2}$

(*ii*) $\frac{1}{3a^2}\left[e^{at} - e^{-at/2}\left(\cos\left(\frac{\sqrt{3}}{2}at\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}at\right)\right)\right]$

11. (*i*) $\frac{1}{5}(1 + e^{-t})\sin t + \frac{3}{5}(1 - e^{-t})\cos t$ (*ii*) $\frac{1}{2}[\cos at + \cosh at]$

12. (*i*) $\frac{1}{2}[\sin t - te^{-t}]$ (*ii*) $\cos at \sinh at$.

Hints

4. (*i*) $\frac{1}{s^2(s^2 + 1)} = \frac{1}{z(z + 1)}$, where $z = s^2$

$$= \frac{1}{z(1)} + \frac{1}{(-1)(z + 1)} = \frac{1}{z} - \frac{1}{z + 1} = \frac{1}{s^2} - \frac{1}{s^2 + 1}.$$

10. (*i*) $s^4 + s^2 + 1 = (s^4 + 2s^2 + 1) - s^2$.

2.14. SOLUTION OF DIFFERENTIAL EQUATIONS BY USING LAPLACE TRANSFORMATION

We know that a differential equation with initial conditions is solved by first finding the general solution of the given equation and then finding the values of the arbitrary constants by using the given initial conditions. By using Laplace transformations, we find the required solution without first finding the general solution of the given equation.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following equations :*

(*i*) $\frac{d^2y}{dt^2} + y = 6\cos 2t$, *where* $y'(0) = 1$, $y(0) = 3$.

(*ii*) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$, *where* $y = \frac{dy}{dt} = 0$, *when* $t = 0$.

(*iii*) $(D^2 + m^2)\, y = a \cos nt$, *where* $y = 0 = \frac{dy}{dt}$ *when* $t = 0$.

(*iv*) $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$, *where* $y = 1$, $\frac{dy}{dt} = 2$, $\frac{d^2y}{dt^2} = 2$ at $t = 0$.

Sol. (*i*) We have

$$\frac{d^2y}{dt^2} + y = 6 \cos 2t.$$

$$\Rightarrow \qquad y'' + y = 6 \cos 2t$$

Taking Laplace Transform, we have

$$L(y'') + L(y) = 6L(\cos 2t)$$

$$\Rightarrow \qquad [s^2 L(y) - s \,.\, y(0) - y'(0)] + L(y) = 6 \, \frac{s}{s^2 + (2)^2}$$

$$\Rightarrow \qquad s^2 \,\overline{y} - s \,.\, 3 - 1 + \overline{y} = \frac{6s}{s^2 + 4} \qquad \text{(Putting } \overline{y} = L(y))$$

$$\Rightarrow \qquad (s^2 + 1)\, \overline{y} = \frac{6s}{s^2 + 4} + 3s + 1$$

$$\Rightarrow \qquad \overline{y} = \frac{6s}{(s^2 + 1)(s^2 + 4)} + \frac{3s}{s^2 + 1} + \frac{1}{s^2 + 1}$$

Taking inverse Laplace transform, we have

$$y = 6L^{-1}\left(\frac{s}{(s^2 + 1)(s^2 + 4)}\right) + 3L^{-1}\left(\frac{s}{s^2 + 1}\right) + L^{-1}\left(\frac{1}{s^2 + 1}\right) \qquad ...(1)$$

$$\frac{s}{(s^2 + 1)(s^2 + 4)} = \frac{s}{s^2 + 1} \cdot \frac{1}{s^2 + 4} = F(s)\, G(s),$$

where $F(s) = \dfrac{s}{s^2 + 1}$ and $G(s) = \dfrac{1}{s^2 + 4}$.

$$L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2 + 1}\right) = \cos t = f(t), \text{ say}$$

and $L^{-1}(G(s)) = L^{-1}\left(\dfrac{1}{s^2 + 4}\right) = \dfrac{1}{2} \sin 2t = g(t)$, say

By **convolution theorem,**

$$L^{-1}(F(s)\, G(s)) = f * g = \int_0^t f(T)\, g(t - T)\, dT$$

$$= \int_0^t (\cos T)\left(\frac{1}{2} \sin 2(t - T)\right) dT = \frac{1}{4}\int_0^t 2 \sin\, (2t - 2T) \cos T \, dT$$

$$= \frac{1}{4}\int_0^t (\sin\,(2t - T) + \sin\,(2t - 3T))\, dT$$

$$= \frac{1}{4}\left[-\frac{\cos\,(2t - T)}{-1} - \frac{\cos\,(2t - 3T)}{-3}\right]_0^t$$

$$= \frac{1}{4}\left[\cos t - \cos 2t + \frac{\cos t}{3} - \frac{\cos 2t}{3}\right]$$

$$= \frac{1}{4}\left[\frac{4}{3}\cos t - \frac{4}{3}\cos 2t\right] = \frac{1}{3}\, [\cos t - \cos 2t]$$

Also, $\quad L^{-1}\left(\dfrac{s}{s^2+1}\right) = \cos t \quad$ and $\quad L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$

$\therefore \quad (1) \Rightarrow \qquad y = 6 \cdot \dfrac{1}{3}(\cos t - \cos 2t) + 3\cos t + \sin t$

or $\qquad y = \mathbf{5\cos t + \sin t - 2\cos 2t.}$

(*ii*) We have

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t.$$

$\Rightarrow \qquad y'' + 2y' - 3y = \sin t$

Taking Laplace transform, we have

$$L(y'') + 2L(y') - 3L(y) = L(\sin t)$$

$$\Rightarrow \quad [s^2L(y) - s \cdot y(0) - y'(0)] + 2[sL(y) - y(0)] - 3L(y) = \frac{1}{s^2+1}$$

$$\Rightarrow \quad s^2\bar{y} - s.0 - 0 + 2[s\bar{y} - 0] - 3\bar{y} = \frac{1}{s^2+1}$$

$$\Rightarrow \quad (s^2 + 2s - 3)\,\bar{y} = \frac{1}{s^2+1}$$

$$\Rightarrow \quad \bar{y} = \frac{1}{(s^2+1)(s^2+2s-3)} = \frac{1}{(s^2+1)(s+3)(s-1)}$$

Taking inverse Laplace transform, we have

$$y = L^{-1}\left(\frac{1}{(s^2+1)(s+3)(s-1)}\right)$$

Let $\quad \dfrac{1}{(s^2+1)(s+3)(s-1)} = \dfrac{As+B}{s^2+1} + \dfrac{C}{s+3} + \dfrac{D}{s-1}$

Multiplying by $(s^2 + 1)(s + 3)(s - 1)$, we get

$$1 = (As + B)(s + 3)(s - 1) + C(s^2 + 1)(s - 1) + D(s^2 + 1)(s + 3) \qquad ...(1)$$

$s = -3 \Rightarrow \quad 1 = 0 + c(10)(-4) + 0 \qquad \Rightarrow \quad C = -\dfrac{1}{40}$

$s = 1 \quad \Rightarrow \quad 1 = 0 + 0 + D(2)(4) \qquad \Rightarrow \quad D = \dfrac{1}{8}$

Comparing the coefficients of s^3 and constant terms in (1), we get

$$0 = A + C + D \qquad ...(2)$$

$$1 = -3B - C + 3D \qquad ...(3)$$

$(2) \quad \Rightarrow \qquad A = -C - D = \dfrac{1}{40} - \dfrac{1}{8} = -\dfrac{1}{10}$

$(3) \quad \Rightarrow \qquad 3B = -1 - C + 3D = -1 + \dfrac{1}{40} + \dfrac{3}{8} = -\dfrac{3}{5} \quad \therefore \quad B = -\dfrac{1}{5}$

$$\frac{1}{(s^2+1)(s+3)(s-1)} = \frac{-\frac{1}{10}s - \frac{1}{5}}{s^2+1} + \frac{-\frac{1}{40}}{s+3} + \frac{\frac{1}{8}}{s-1}$$

$$= -\frac{1}{10}\cdot\frac{s}{s^2+1} - \frac{1}{5}\cdot\frac{1}{s^2+1} - \frac{1}{40}\cdot\frac{1}{s+3} + \frac{1}{8}\cdot\frac{1}{s-1}$$

$$\therefore \quad y = -\frac{1}{10}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{5}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{40}L^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{8}L^{-1}\left(\frac{1}{s-1}\right)$$

$$= \mathbf{-\frac{1}{10}\cos t - \frac{1}{5}\sin t - \frac{1}{40}e^{-3t} + \frac{1}{8}e^{t}.}$$

(*iii*) We have $(D^2 + m^2)\, y = a \cos nt.$

$\Rightarrow$ $y'' + m^2 y = a \cos nt$

Taking Laplace transform, we have

$$L(y'') + m^2 L(y) = a\, L(\cos nt)$$

$$\Rightarrow \quad [s^2 L(y) - s\,.\,y(0) - y'(0)] + m^2 L(y) = a\,.\,\frac{s}{s^2+n^2}$$

$$\Rightarrow \quad s^2\bar{y} - s.0 - 0 + m^2\bar{y} = \frac{as}{s^2+n^2}$$

$$\Rightarrow \quad (s^2+m^2)\,\bar{y} = \frac{as}{s^2+n^2}$$

$$\Rightarrow \quad \bar{y} = \frac{as}{(s^2+m^2)(s^2+n^2)}$$

$$\therefore \quad y = aL^{-1}\left(\frac{s}{(s^2+m^2)(s^2+n^2)}\right) \qquad ...(1)$$

$$\frac{s}{(s^2+m^2)(s^2+n^2)} = \frac{s}{s^2+m^2}\cdot\frac{1}{s^2+n^2} = F(s)\,G(s), \quad \text{where } F(s) = \frac{s}{s^2+m^2}$$

and
$$G(s) = \frac{1}{s^2+n^2}.$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2+m^2}\right) = \cos mt = f(t), \text{ say}$$

and
$$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s^2+n^2}\right) = \frac{1}{n}\sin nt = g(t), \text{ say}$$

By **convolution theorem,**

$$L^{-1}(F(s)\,G(s)) = f * g = \int_0^t f(T)\, g(t-T)\, dT$$

$$= \int_0^t \cos mT\,.\,\frac{1}{n}\sin n(t-T)\, dT$$

$$= \frac{1}{2n}\int_0^t 2\sin(nt - nT)\cos mT\, dT$$

$$= \frac{1}{2n}\int_0^t [\sin(nt + (m-n)T) + \sin(nt - (m+n)T)]\, dT$$

$$= \frac{1}{2n}\left[-\frac{\cos(nt+(m-n)\,T)}{m-n} - \frac{\cos(nt-(m+n)\,T)}{-(m+n)}\right]_0^t$$

$$= \frac{1}{2n}\left[-\frac{\cos mt}{m-n} + \frac{\cos nt}{m-n} + \frac{\cos mt}{m+n} - \frac{\cos nt}{m+n}\right]$$

$$= \frac{1}{2n}\left[\left(\frac{1}{m+n} - \frac{1}{m-n}\right)\cos mt + \left(\frac{1}{m-n} - \frac{1}{m+n}\right)\cos nt\right]$$

$$= \frac{1}{2n}\left[-\frac{2n\cos mt}{m^2-n^2} + \frac{2n\cos nt}{m^2-n^2}\right] = \frac{1}{m^2-n^2}\ (\cos nt - \cos mt)$$

$\therefore$ (1) $\Rightarrow$ $$y = \frac{\mathbf{a}}{\mathbf{m^2 - n^2}}\ \mathbf{(\cos nt - \cos mt).}$$

(*iv*) We have

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0.$$

$\Rightarrow$ $$y''' + 2y'' - y' - 2y = 0$$

Taking Laplace transform, we have

$$L(y''') + 2L(y'') - L(y') - 2L(y) = 0$$

$\Rightarrow$ $$[s^3L(y) - s^2 \,.\, y(0) - s \,.\, y'(0) - y''(0)] + 2[s^2L(y) - s \,.\, y(0) - y'(0)] - [sL(y) - y(0)] - 2L(y) = 0$$

$\Rightarrow$ $$(s^3\bar{y} - s^2.1 - s.2 - 2) + 2(s^2\bar{y} - s.1 - 2) - (s\bar{y} - 1) - 2\bar{y} = 0$$

$\Rightarrow$ $$s^3\bar{y} - s^2 - 2s - 2 + 2s^2\bar{y} - 2s - 4 - s\bar{y} + 1 - 2\bar{y} = 0$$

$\Rightarrow$ $$(s^3 + 2s^2 - s - 2)\ \bar{y} = s^2 + 4s + 5$$

$$\bar{y} = \frac{s^2+4s+5}{s^3+2s^2-s-2} = \frac{s^2+4s+5}{s^2(s+2)-(s+2)}$$

$\therefore$ $$\bar{y} = \frac{s^2+4s+5}{(s-1)\,(s+1)\,(s+2)}$$

$$= \frac{1+4+5}{(s-1)\,(2)\,(3)} + \frac{1-4+5}{(-2)\,(s+1)\,(1)} + \frac{4-8+5}{(-3)\,(-1)\,(s+2)}$$

$$= \frac{5}{3}\cdot\frac{1}{s-1} - \frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{s+2}$$

$\therefore$ $$y = \frac{5}{3}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \mathbf{\frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}.}$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{d^2y}{dt^2} + y = t$, where $y(0) = 1, y'(0) = -2$.

2. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y = e^{-t}$, where $y(0) = y'(0) = 1$.

3. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} = 2$, where $x(0) = 3, x'(0) = 1$.

4. $y'' - 3y' + 2y = 4t + e^{3t}$, where $y(0) = 1, y'(0) = -1$.

5. $\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - y = t^2e^t$, where $y(0) = 1, y'(0) = 0, y''(0) = -2$.

6. $(D^4 + 2D^2 + 1)\, y = 0$, where $y(0) = 0, y'(0) = 1, y''(0) = 2, y'''(0) = -3$.

7. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = t^2 + 2t$, where $y(0) = 4, y'(0) = -2$.

Answers

1. $y = t - 3\sin t + \cos t$

2. $y = -\dfrac{3}{4}e^{-3t} + \dfrac{7}{4}e^{-t} + \dfrac{1}{2}te^{-t}$

3. $x = 2t + e^{-t} + 2$

4. $y = 2t + 3 + \dfrac{1}{2}e^{3t} - \dfrac{1}{2}e^{t} - 2e^{2t}$

5. $y = e^t\left[\dfrac{t^5}{60} - \dfrac{t^2}{2} - t + 1\right]$

6. $y = t(\sin t + \cos t)$

7. $y = \dfrac{1}{3}t^3 + 2e^{-t} + 2$.

Solution of Integral Equations Using Laplace Transformation

3

3.1. INTRODUCTION

Integral equations occur very frequently in the fields of mechanics and mathematical physics. The development of the theory of integral equations began with the works of Italian mathematician **V. Volterra** (1896) and the Swedish mathematician **I. Fredholm** (1900).

3.2. DEFINITION OF INTEGRAL EQUATION

An equation involving an unknown function under the integral and perhaps also outside it is called an **integral equation.**

For example the equation $y(t) = f(t) + \int_a^b y(u)\, k(t, u)\, du$, where $f(t)$, $k(t, u)$ are known functions and the function $y(t)$ is to be determined, is an integral equation.

If the integral involved in an integral equation is of the form $\int_0^t y(u)\, k(t-u)\, du$, then the integral equation is said to be an **integral equation of convolution type.**

3.3. METHOD OF SOLVING INTEGRAL EQUATION OF CONVOLUTION TYPE

Let

$$y(t) = f(t) + \int_0^t y(u)\, k(t-u)\, du \qquad \text{...(1)}$$

be an integral equation of convolution type.

The integral on the right side is the convolution of the functions $y(t)$ and $k(t)$.

$\therefore$ (1) can be written as $y(t) = f(t) + (y(t) * k(t))$

Taking the Laplace transform on both sides, we get

$$L(y(t)) = L(f(t)) + L(y(t) * k(t)).$$

$$\Rightarrow \quad L(y(t)) = L(f(t)) + L(y(t)).L(k(t)) \qquad \text{(By convolution theorem)}$$

$$\Rightarrow \quad Y(s) = F(s) + Y(s)\, K(s),$$

where $L(y(t)) = Y(s)$, $L(f(t)) = F(s)$ and $L(k(t)) = K(s)$.

$$\Rightarrow \quad (1 - K(s))\, Y(s) = F(s) \quad \Rightarrow \quad Y(s) = \frac{F(s)}{1 - K(s)}$$

$\Rightarrow \quad L^{-1}(Y(s)) = L^{-1}\left(\dfrac{F(s)}{1-K(s)}\right) \quad \therefore \quad \mathbf{y(t) = L^{-1}\left(\dfrac{L(f(t))}{1-L(k(t))}\right)}.$

This gives the solution of the given integral equation.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following integral equation by using Laplace transforms :*

$$y(t) = 1 + \int_0^t y(u) \sin(t-u)\, du.$$

Sol. We have $\quad y(t) = 1 + \int_0^t y(u) \sin(t-u)\, du$.

$\Rightarrow \quad y(t) = 1 + (y(t) * \sin t)$

Taking the Laplace transform, we get

$$L(y(t)) = L(1) + L(y(t) * \sin t)$$

$\Rightarrow \quad L(y(t)) = \dfrac{1}{s} + L(y(t)) \,.\, L(\sin t)$ (By convolution theorem)

$\Rightarrow \quad Y(s) = \dfrac{1}{s} + Y(s) \,.\, \dfrac{1}{s^2+1}$, where $Y(s) = L(y(t))$

$\Rightarrow \quad \left(1 - \dfrac{1}{s^2+1}\right) Y(s) = \dfrac{1}{s} \quad \Rightarrow \quad Y(s) = \dfrac{s^2+1}{s^3} = \dfrac{1}{s} + \dfrac{1}{s^3}$

$\therefore \quad y(t) = L^{-1}(Y(s)) = L^{-1}\left(\dfrac{1}{s}\right) + L^{-1}\left(\dfrac{1}{s^3}\right) = 1 + \dfrac{1}{2} L^{-1}\left(\dfrac{2!}{s^{2+1}}\right) = \mathbf{1 + \dfrac{1}{2} t^2}.$

Example 2. *Solve the following integral equation by using Laplace transforms :*

$$y(t) = te^t - 2e^t \int_0^t y(u)\, e^{-u}\, du.$$

Sol. We have $\quad y(t) = te^t - 2e^t \int_0^t y(u)\, e^{-u}\, du$.

$\Rightarrow \quad y(t) = te^t - 2 \int_0^t y(u)\, e^{t-u}\, du$

$\Rightarrow \quad y(t) = te^t - 2\,(y(t) * e^t)$

Taking the Laplace transform, we get

$$L(y(t)) = L(te^t) - 2L(y(t) * e^t).$$

$\Rightarrow \quad L(y(t)) = L(te^t) - 2(L(y(t)) \,.\, L(e^t))$ (By convolution theorem)

$\Rightarrow \quad Y(s) = -\dfrac{d}{ds}\left(\dfrac{1}{s-1}\right) - 2Y(s)\left(\dfrac{1}{s-1}\right) \quad \Rightarrow \quad Y(s) = \dfrac{1}{(s-1)^2} - 2Y(s)\left(\dfrac{1}{s-1}\right)$

$\Rightarrow \quad \left(1 + \dfrac{2}{s-1}\right) Y(s) = \dfrac{1}{(s-1)^2} \quad \Rightarrow \quad Y(s) = \dfrac{1}{(s-1)(s+1)} = \dfrac{1}{s^2-1}$

$\therefore \quad y(t) = L^{-1}(Y(s)) = L^{-1}\left(\dfrac{1}{s^2-1}\right) = \mathbf{\sinh t}.$

Example 3. *Solve the following integral equation by using Laplace transforms :*

$$y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)\,du.$$

Sol. We have $\quad y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)\,du.$

$\Rightarrow \quad y(t) = e^{-t} - 2\,(y(t) * \cos t)$

Taking the Laplace transform, we get $\quad L(y(t)) = L(e^{-t}) - 2L(y(t) * \cos t)$

$\Rightarrow \quad L(y(t)) = L(e^{-t}) - 2(L(y(t))\,.\,L(\cos t))$ (By convolution theorem)

$$\Rightarrow \quad Y(s) = \frac{1}{s+1} - 2Y(s)\,.\,\frac{s}{s^2+1^2} \quad \Rightarrow \quad \left(1+\frac{2s}{s^2+1}\right)Y(s) = \frac{1}{s+1}$$

$$\Rightarrow \quad Y(s) = \frac{s^2+1}{(s+1)^3}$$

$$\therefore \quad y(t) = L^{-1}(Y(s)) = L^{-1}\left(\frac{s^2+1}{(s+1)^3}\right) = e^{-t}L^{-1}\left(\frac{(s-1)^2+1}{s^3}\right)$$

$$= e^{-t}L^{-1}\left(\frac{s^2-2s+2}{s^3}\right) = e^{-t}\left[L^{-1}\left(\frac{1}{s}\right) - 2L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{2\,!}{s^3}\right)\right]$$

$$= e^{-t}[1-2t+t^2] = \mathbf{e^{-t}\,(1-t)^2}.$$

Example 4. *Solve the following integral equation by using Laplace transforms :*

$$y(t) = 1 - \sinh t + \int_0^t (1+u)\,y(t-u)\,du.$$

Sol. We have $\quad y(t) = 1 - \sinh t + \int_0^t (1+u)\,y(t-u)\,du.$

$\Rightarrow \quad y(t) = 1 - \sinh t + ((1+t) * y(t))$

Taking the Laplace transform, we get

$$L(y(t)) = L(1) - L(\sinh t) + L((1+t) * y(t)).$$

$\Rightarrow \quad L(y(t)) = \dfrac{1}{s} - \dfrac{1}{s^2-1} + L(1+t)\,.\,L(y(t))$ (By convolution theorem)

$$\Rightarrow \quad Y(s) = \frac{1}{s} - \frac{1}{s^2-1} + \left(\frac{1}{s}+\frac{1}{s^2}\right)Y(s), \text{ where } Y(s) = L(y(t))$$

$$\Rightarrow \quad \left(1-\frac{1}{s}-\frac{1}{s^2}\right)Y(s) = \frac{s^2-1-s}{s(s^2-1)}$$

$$\Rightarrow \quad Y(s) = \frac{s^2(s^2-1-s)}{(s^2-s-1)\,s(s^2-1)} = \frac{s}{s^2-1}$$

$$\therefore \quad y(t) = L^{-1}(Y(s)) = L^{-1}\left(\frac{s}{s^2-1}\right) = \textbf{cosh t}.$$

Example 5. *Solve the following integral equation using Laplace transforms :*

$$\int_0^t \frac{y(u)}{\sqrt{t-u}}\,du = 1 + 2t - t^2.$$

Sol. We have $\displaystyle\int_0^t y(u)\,(t-u)^{-1/2}\,du = 1 + 2t - t^2$.

$\Rightarrow$ $$y(t) * t^{-1/2} = 1 + 2t - t^2$$

Taking the Laplace transform, we get

$$\text{L}\,(y(t) * t^{-1/2}) = \text{L}\,(1 + 2t - t^2)$$

$$\text{L}(y(t))\,.\,\text{L}(t^{-1/2}) = \text{L}(1) + 2\text{L}(t) - \text{L}(t^2) \qquad \text{(By convolution theorem)}$$

$\Rightarrow$ $$\text{Y}(s)\,.\,\frac{\Gamma\,(1/2)}{\sqrt{s}} = \frac{1}{s} + \frac{2}{s^2} - \frac{2\,!}{s^3}, \text{ where } \text{Y}(s) = \text{L}(y(t))$$

$\Rightarrow$ $$\text{Y}(s)\,\frac{\sqrt{\pi}}{\sqrt{s}} = \frac{s^2 + 2s - 2}{s^3} \quad \Rightarrow \quad \text{Y}(s) = \frac{1}{\sqrt{\pi}}\left[\frac{1}{s^{1/2}} + \frac{2}{s^{3/2}} - \frac{2}{s^{5/2}}\right]$$

$\therefore$ $$y(t) = \text{L}^{-1}\,(\text{Y}(s)) = \frac{1}{\sqrt{\pi}}\left[\text{L}^{-1}\left(\frac{1}{s^{1/2}}\right) + 2\text{L}^{-1}\left(\frac{1}{s^{3/2}}\right) - 2\text{L}^{-1}\left(\frac{1}{s^{5/2}}\right)\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\Gamma\,(1/2)} + 2\,.\,\frac{t^{1/2}}{\Gamma\,(3/2)} - 2\,.\,\frac{t^{3/2}}{\Gamma\,(5/2)}\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{t^{-1/2}}{\sqrt{\pi}} + \frac{2t^{1/2}}{\frac{1}{2}\,.\,\sqrt{\pi}} - 2\,.\,\frac{t^{3/2}}{\frac{3}{2}\,.\,\frac{1}{2}\,.\,\sqrt{\pi}}\right]$$

$\therefore$ $$y(t) = \frac{\mathbf{1}}{\pi}\left[\mathbf{t^{-1/2}} + \mathbf{4t^{1/2}} - \frac{\mathbf{8}}{\mathbf{3}}\,\mathbf{t^{3/2}}\right].$$

3.4. INTEGRO-DIFFERENTIAL EQUATION

An equation involving an unknown function under the integral and perhaps also outside it and the derivatives of the unknown function is called an **integro-differential equation**.

The method of solving an integro-differential equation is same as that of solving an integral equation.

Example 6. *Solve the following integro-differential equation using Laplace transforms :*

$$y'(t) = t + \int_0^t (\cos u)\,y(t-u)\,du,\ y(0) = 4.$$

Sol. We have $\displaystyle y'(t) = t + \int_0^t (\cos u)\,y(t-u)\,du,\ y(0) = 4$.

$\Rightarrow$ $$y'(t) = t + (\cos t * y(t))$$

Taking the Laplace transform, we get

$$\text{L}(y'(t)) = \text{L}(t) + \text{L}(\cos t * y(t)).$$

$$sL(y(t)) - y(0)^* = \frac{1}{s^2} + L(\cos t) \, . \, L(y(t)) \qquad \text{(By convolution theorem)}$$

$$\Rightarrow \qquad sY(s) - 4 = \frac{1}{s^2} + \frac{s}{s^2+1} \, . \, Y(s) \text{, where } Y(s) = L(y(t))$$

$$\Rightarrow \quad \left(s - \frac{s}{s^2+1}\right) Y(s) = \frac{1}{s^2} + 4 \quad \Rightarrow \quad Y(s) = \frac{s^2+1}{s^3}\left(\frac{1}{s^2} + 4\right)$$

$$\Rightarrow \qquad Y(s) = \frac{1}{s^3} + \frac{1}{s^5} + \frac{4}{s} + \frac{4}{s^3} = \frac{5}{s^3} + \frac{1}{s^5} + \frac{4}{s}$$

$$\therefore \qquad y(t) = L^{-1}(Y(s)) = 5L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s^5}\right) + 4L^{-1}\left(\frac{1}{s}\right)$$

$$= 5\left(\frac{t^2}{2\,!}\right) + \frac{t^4}{4\,!} + 4(1) = \mathbf{\frac{5}{2}\, t^2 + \frac{1}{24}\, t^4 + 4.}$$

Example 7. *Solve the following integro-differential equation by using Laplace transforms :*

$$y'(t) + 4y(t) + 5\int_0^t y(u)\, du = e^{-t},\ y(0) = 0.$$

Sol. We have $\quad y'(t) + 4y(t) + 5\int_0^t y(u)\, du = e^{-t},\ y(0) = 0.$

$$\Rightarrow \qquad y'(t) + 4y(t) + 5(y(t) * 1) = e^{-t}$$

Taking the Laplace transform, we get

$$L(y'(t) + 4L(y(t)) + 5L(y(t) * 1) = L(e^{-t}).$$

$$\Rightarrow \quad (sL(y(t)) - y(0)) + 4L(y(t)) + 5(L(y(t)) \, . \, L(1)) = \frac{1}{s-(-1)} \qquad \text{(By convolution theorem)}$$

$$\Rightarrow \qquad sY(s) - 0 + 4Y(s) + 5Y(s) \, . \, \frac{1}{s} = \frac{1}{s+1}.$$

$$\Rightarrow \qquad \left(s + 4 + \frac{5}{s}\right) Y(s) = \frac{1}{s+1} \quad \Rightarrow \quad Y(s) = \frac{s}{(s+1)(s^2+4s+5)}$$

Using partial fractions, we have

$$Y(s) = -\frac{1}{2(s+1)} + \frac{s+5}{2(s^2+4s+5)} = -\frac{1}{2(s+1)} + \frac{(s+2)+3}{2((s+2)^2+1)}$$

$$\therefore \qquad y(t) = L^{-1}(Y(s)) = -\frac{1}{2} L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2} L^{-1}\left(\frac{(s+2)+3}{(s+2)^2+1}\right)$$

$$= -\frac{1}{2} e^{-t} + \frac{1}{2} e^{-2t} \, L^{-1}\left(\frac{s+3}{s^2+1}\right)$$

***Why this step.** Using L(f') = sL(f') – f(0).

$$= -\frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t}\left(L^{-1}\left(\frac{s}{s^2+1}\right) + 3L^{-1}\left(\frac{1}{s^2+1}\right)\right)$$

$\therefore$
$$y(t) = -\frac{1}{2}\mathbf{e^{-t}} + \frac{1}{2}\mathbf{e^{-2t}}\ (\mathbf{\cos t + 3\sin t}).$$

WORKING STEPS FOR SOLVING PROBLEMS

Step I. Take the Laplace transform of the given equation.

Step II. Substitute the values of Laplace transforms.

Step III. Simplify the equation to get the value of Laplace transform of the unknown function.

Step IV. Find the unknown functions by taking the inverse Laplace transform.

TEST YOUR KNOWLEDGE

Solve the following integral equations by using Laplace transforms :

1. $y(t) = t + \int_0^t y(u)\sin(t-u)\,du$
2. $y(t) = 1 - \int_0^t y(u)(t-u)\,du$
3. $y(t) = t + 2\int_0^t y(u)\cos(t-u)\,du$
4. $y(t) = 1 + 2\int_0^t y(u)\cos(t-u)\,du$
5. $y(t) = \sin t + 2\int_0^t y(u)\cos(t-u)\,du$
6. $y(t) = 3t^2 + \int_0^t y(u)\sin(t-u)\,du$
7. $y(t) = t^2 + \int_0^t y(u)\sin(t-u)\,du$
8. $y(t) = t + \frac{1}{6}\int_0^t y(u)(t-u)^3\,du$
9. $\int_0^t \frac{y(u)}{\sqrt{t-u}}\,du = 1$
10. $\int_0^t \frac{y(u)}{\sqrt{t-u}}\,du = 1 + t + t^2$
11. $\int_0^t y(u)\,y(t-u)\,du = 2y(t) + t - 2$
12. $y'(t) = 3\int_0^t y(u)\cos 2(t-u)\,du + 2,\ y(0) = 1$
13. $y'(t) + 3y(t) + 2\int_0^t y(u)\,du = t,\ y(0) = 1$
14. $y'(t) = \int_0^t y(u)\cos(t-u)\,du,\ y(0) = 1$
15. $y'(t) = t + \int_0^t (\cos u)\,y(t-u)\,du, y(0) = 1.$

Answers

1. $y(t) = t + \frac{t^3}{6}$
2. $y = \cos t$
3. $y(t) = 2e^t(t-1) + 2 + t$
4. $y(t) = 1 + 2te^t$
5. $y = te^t$
6. $y(t) = 3t^2 + \frac{t^4}{4}$
7. $y(t) = t^2 + \frac{t^4}{12}$
8. $y(t) = \frac{1}{2}(\sin t + \sinh t)$

9. $y(t) = \dfrac{1}{\pi\sqrt{t}}$ **10.** $y(t) = \dfrac{1}{\pi}\left[\dfrac{1}{\sqrt{t}} + 2\sqrt{t} + \dfrac{8}{3}t^{3/2}\right]$ **11.** $y(t) = 1$

12. $y(t) = 4 + 8t - 3\cos t - 6\sin t$ **13.** $y(t) = \dfrac{1}{2} - 2e^{-t} + \dfrac{5}{2}e^{-2t}$ **14.** $y(t) = 1 + \dfrac{t^2}{2}$

15. $y(t) = 1 + t^2 + \dfrac{t^4}{24}$.

Hint

11. If $L(y(t)) = Y(s)$, then $(Y(s))^2 - 2Y(s) - \dfrac{1-2s}{s^2} = 0$.

$\therefore$ $Y(s) = 2 - \dfrac{1}{s}$ or $\dfrac{1}{s}$

Rejecting $Y(s) = 2 - \dfrac{1}{s}$, we have $Y(s) = \dfrac{1}{s}$.

Solution of Systems of Differential Equations Using the Laplace Transformation

4

4.1. INTRODUCTION

In the present chapter, we shall study the method of solving a system of differential equations using Laplace transformations of functions. Our knowledge of finding Laplace transforms and inverse Laplace transforms would help us to solve the systems of differential equations. In the method of Laplace transforms, we would be able to find the required particular solution of the given system of differential equations with known initial conditions, without the necessity of first finding the general solution and then evaluating the arbitrary constants by using the given conditions.

4.2. METHOD OF SOLVING SYSTEM OF DIFFERENTIAL EQUATIONS

Let t be the independent variable and x, y the dependent variables of a given system of differential equations. We shall denote the Laplace transforms of the functions $x(t)$ and $y(t)$ by $\bar{x}(s)$ and $\bar{y}(s)$ respectively. Thus,

$$L(x(t)) = \bar{x}(s) \quad \text{and} \quad L(y(t)) = \bar{y}(s).$$

We take the Laplace transform of each differential equation of the given system. These equations are simplified to get the values of the functions $\bar{x}(s)$ and $\bar{y}(s)$. The values of the functions $x(t)$ and $y(t)$ are found by taking the inverse Laplace transforms of the functions $\bar{x}(s)$ and $\bar{y}(s)$ respectively.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following system of differential equations by the method of Laplace transforms :*

$$x' + y = 2\cos t,\ x + y' = 0,\ x(0) = 0,\ y(0) = 1.$$

Sol. Given system is $\quad x' + y = 2\cos t \qquad$ ...(1)

$$x + y' = 0. \qquad ...(2)$$

Taking the Laplace transform, we get

$$L(x') + L(y) = L(2\cos t) \qquad ...(3)$$

and $\quad L(x) + L(y') = L(0). \qquad$ ...(4)

(3) $\Rightarrow$ $$[sL(x) - x(0)] + \bar{y} = \frac{2s}{s^2+1}$$

$\Rightarrow$ $$s\bar{x} + \bar{y} = \frac{2s}{s^2+1} \qquad (\because\ x(0) = 0) \quad ...(5)$$

(4) $\Rightarrow$ $$\bar{x} + [sL(y) - y(0)] = 0 \ \Rightarrow\ \bar{x} + s\bar{y} - 1 = 0 \qquad (\because\ y(0) = 1)$$

$\Rightarrow$ $$\bar{x} + s\bar{y} = 1 \quad ...(6)$$

(5) – s × (6) $\Rightarrow$ $$\bar{y} - s^2\bar{y} = \frac{2s}{s^2+1} - s = \frac{s-s^3}{s^2+1}$$

$\Rightarrow$ $$\bar{y} = \frac{s}{s^2+1} \ \Rightarrow\ y(t) = L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$$

(6) $\Rightarrow$ $$\bar{x} = 1 - s\bar{y} = 1 - s\left(\frac{s}{s^2+1}\right) = \frac{1}{s^2+1}$$

$\Rightarrow$ $$x = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$$

$\therefore$ $$x = \mathbf{\sin t},\ y = \mathbf{\cos t}.$$

Example 2. *Solve the following system of differential equations by the method of Laplace transforms :*

$$x' + 5x - 2y = t,\ y' + 2x + y = 0\ ;\ x(0) = 0,\ y(0) = 0.$$

Sol. Given system is $x' + 5x - 2y = t$ ...(1)

$$y' + 2x + y = 0. \quad ...(2)$$

Taking the Laplace transform, we get

$$L(x') + 5L(x) - 2L(y) = L(t) \quad ...(3)$$

and $$L(y') + 2L(x) + L(y) = L(0). \quad ...(4)$$

(3) $\Rightarrow$ $$[sL(x) - x(0)] + 5\bar{x} - 2\bar{y} = \frac{1}{s^2}$$

$\Rightarrow$ $$s\bar{x} - 0 + 5\bar{x} - 2\bar{y} = \frac{1}{s^2} \ \Rightarrow\ (s+5)\,\bar{x} - 2\bar{y} - \frac{1}{s^2} = 0 \quad ...(5)$$

(4) $\Rightarrow$ $$[sL(y) - y(0)] + 2\bar{x} + \bar{y} = 0$$

$\Rightarrow$ $$s\bar{y} - 0 + 2\bar{x} + \bar{y} = 0 \ \Rightarrow\ 2\bar{x} + (s+1)\,\bar{y} + 0 = 0 \quad ...(6)$$

Solving (5) and (6), we get

$$\frac{\bar{x}}{0 + \frac{s+1}{s^2}} = \frac{\bar{y}}{-\frac{2}{s^2} - 0} = \frac{1}{(s+5)\,(s+1) + 4}$$

$\therefore$ $$\bar{x} = \frac{s+1}{s^2\,(s^2+6s+9)} = \frac{s+1}{s^2\,(s+3)^2}$$

Also $$\bar{y} = -\frac{2}{s^2\,(s^2+6s+9)} = -\frac{2}{s^2\,(s+3)^2}\,.$$

Using partial fractions,

$$\bar{x} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}.$$

$$\therefore \qquad x = L^{-1}(\bar{x}) = \frac{1}{27} L^{-1}\left(\frac{1}{s}\right) + \frac{1}{9} L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{27} L^{-1}\left(\frac{1}{s+3}\right) - \frac{2}{9} L^{-1}\left(\frac{1}{(s+3)^2}\right)$$

$$= \frac{1}{27}(1) + \frac{1}{9}.(t) - \frac{1}{27}(e^{-3t}) - \frac{2}{9} e^{-3t} L^{-1}\left(\frac{1}{s^2}\right)$$

$$\therefore \qquad x = \mathbf{\frac{1}{27} + \frac{t}{9} - \frac{1}{27} e^{-3t} - \frac{2}{9} e^{-3t} t.}$$

Using partial fractions,

$$\bar{y} = \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}.$$

$$\therefore \qquad y = L^{-1}(\bar{y}) = -\frac{4}{27} L^{-1}\left(\frac{1}{s}\right) - \frac{2}{9} L^{-1}(s^{-2}) - \frac{4}{27} L^{-1}\left(\frac{1}{s+3}\right) - \frac{2}{9} L^{-1}\left(\frac{1}{(s+3)^2}\right)$$

$$= -\frac{4}{27}(1) - \frac{2}{9}(t) - \frac{4}{27}(e^{-3t}) - \frac{2}{9} e^{-3t} L^{-1}\left(\frac{1}{s^2}\right)$$

$$\therefore \qquad y = \mathbf{-\frac{4}{27} - \frac{2}{9} t - \frac{4}{27} e^{-3t} - \frac{2}{9} e^{-3t} t.}$$

Example 3. *Solve the following system of differential equations by the method of Laplace transforms :*

$$x' + y = 3e^{2t},\ y' + x = 0,$$
$$x(0) = 2,\ y(0) = 0.$$

Sol. Given system is $x' + y = 3e^{2t}$ ...(1)

$y' + x = 0.$ ...(2)

Taking the Laplace transform, we get

$L(x') + L(y) = 3L(e^{2t})$ ...(3)

and $L(y') + L(x) = 0.$ ...(4)

(3) $\Rightarrow$ $[sL(x) - x(0)] + \bar{y} = 3.\dfrac{1}{s-2}$ $\Rightarrow$ $s\bar{x} - 2 + \bar{y} = \dfrac{3}{s-2}$

$\Rightarrow$ $s\bar{x} + \bar{y} = 2 + \dfrac{3}{s-2}$ ...(5)

(4) $\Rightarrow$ $[sL(y) - y(0)] + \bar{x} = 0 \Rightarrow s\bar{y} - 0 + \bar{x} = 0$

$\Rightarrow$ $\bar{x} + s\bar{y} = 0$ ...(6)

(5) – (6) $\times s$ $\Rightarrow$ $\bar{y} - s^2\bar{y} = 2 + \dfrac{3}{s-2} = \dfrac{2s-1}{s-2}$ $\Rightarrow$ $\bar{y} = \dfrac{2s-1}{(s-2)(1-s)(1+s)}$

(6) $\Rightarrow$ $\bar{x} = -s\bar{y} = \dfrac{-s(2s-1)}{(s-2)(1-s)(1+s)} = \dfrac{-s(2s-1)}{(s-2)(1-s)(1+s)}$

Using partial fractions,

$$\bar{x} = \frac{(-2)(3)}{(s-2)(-1)(3)} + \frac{(-1)(1)}{(-1)(1-s)(2)} + \frac{(1)(-3)}{(-3)(2)(1+s)}$$

$$= \frac{2}{s-2} - \frac{1}{2(s-1)} + \frac{1}{2(s+1)}.$$

$$\therefore \qquad x = L^{-1}(\bar{x}) = 2L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right)$$

$$= 2e^{2t} - \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}.$$

Using partial fractions,

$$\bar{y} = \frac{3}{(s-2)(-3)} + \frac{1}{(-1)(1-s)(2)} + \frac{-3}{(-3)(2)(1+s)}$$

$$= -\frac{1}{s-2} + \frac{1}{2(s-1)} + \frac{1}{2(s+1)}.$$

$$\therefore \qquad y = L^{-1}(\bar{y}) = -L^{-1}\left(\frac{1}{s-2}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right)$$

$$= -e^{2t} + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}.$$

$$\therefore \qquad x = \mathbf{2e^{2t}} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{t}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{-t}} \quad \text{and} \quad y = -\mathbf{e^{2t}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{t}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{-t}}.$$

Example 4. *Solve the following system of differential equations by the method of Laplace transforms :*

$$(D-2)x - (D+1)y = 6e^{3t}$$
$$(2D-3)x + (D-3)y = 6e^{3t},$$
$$x(0) = 3,\ y(0) = 0.$$

Sol. Given system is $\quad Dx - Dy - 2x - y = 6e^{3t}$ ...(1)

$$2Dx + Dy - 3x - 3y = 6e^{3t}. \qquad \text{...(2)}$$

Taking the Laplace transform, we get

$$L(Dx) - L(Dy) - 2L(x) - L(y) = 6L(e^{3t}) \qquad \text{...(3)}$$

and

$$2L(Dx) + L(Dy) - 3L(x) - 3L(y) = 6L(e^{3t}). \qquad \text{...(4)}$$

$$(3) \quad \Rightarrow \quad [sL(x) - x(0)] - [sL(y) - y(0)] - 2\bar{x} - \bar{y} = \frac{6}{s-3}$$

$$\Rightarrow \qquad s\bar{x} - 3 - s\bar{y} + 0 - 2\bar{x} - \bar{y} = \frac{6}{s-3} \quad \Rightarrow \quad (s-2)\bar{x} - (s+1)\bar{y} = 3 + \frac{6}{s-3} \qquad \text{...(5)}$$

$$(4) \quad \Rightarrow \quad 2[sL(x) - x(0)] + [sL(y) - y(0)] - 3\bar{x} - 3\bar{y} = \frac{6}{s-3}$$

$$\Rightarrow \qquad 2(s\bar{x} - 3) + s\bar{y} - 0 - 3\bar{x} - 3\bar{y} = \frac{6}{s-3} \quad \Rightarrow \quad (2s-3)\bar{x} + (s-3)\bar{y} = 6 + \frac{6}{s-3} \qquad \text{...(6)}$$

$(5) \times (s - 3) + (6) \times (s + 1)$

$$\Rightarrow \quad [(s-3)(s-2)+(2s-3)(s+1)]\,\bar{x} = 3(s-3)+6+6(s+1)+\frac{6(s+1)}{s-3}$$

$$\Rightarrow \quad 3(s-1)^2\,\bar{x} = \frac{3(3s^2-6s-1)}{s-3} \quad \Rightarrow \quad \bar{x} = \frac{3s^2-6s-1}{(s-1)^2\,(s-3)}.$$

Using partial fractions,

$$\bar{x} = \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{2}{s-3}.$$

$$\therefore \quad x = L^{-1}\left(\frac{1}{s-1}\right) + 2\left(L^{-1}\left(\frac{1}{(s-1)^2}\right)\right) + 2L^{-1}\left(\frac{1}{s-3}\right)$$

$$= e^t + 2e^t\,L^{-1}\left(\frac{1}{s^2}\right) + 2e^{3t} \;= e^t + 2e^t\,t + 2e^{3t}.$$

$(5) \times (2s - 3) - (6) \times (s - 2)$

$$\Rightarrow \quad [-(2s-3)(s+1)-(s-2)(s-3)]\,\bar{y} \;= 3(2s-3)+\frac{6(2s-3)}{s-3}-6(s-2)-\frac{6(s-2)}{s-3}$$

$$\Rightarrow \quad -3(s-1)^2\bar{y} = 3+\frac{6s-6}{s-3}$$

$$\Rightarrow \quad \bar{y} = -\frac{1}{(s-1)^2} - \frac{2}{(s-1)(s-3)}$$

$$= -\frac{1}{(s-1)^2} - 2\left[\frac{1}{(s-1)(-2)} + \frac{1}{2(s-3)}\right]$$

$$= -\frac{1}{(s-1)^2} + \frac{1}{s-1} - \frac{1}{s-3}.$$

$$\therefore \quad y = -L^{-1}\left(\frac{1}{(s-1)^2}\right) + L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-3}\right)$$

$$= -e^tL^{-1}\left(\frac{1}{s^2}\right) + e^t - e^{3t} = -e^t t + e^t - e^{3t}.$$

$$\therefore \quad \mathbf{x = e^t + 2te^t + 2e^{3t}} \text{ and } \mathbf{y = -te^t + e^t - e^{3t}}.$$

Example 5. *Solve the following system of differential equations by the method of Laplace transforms :*

$$x'' = x + 3y,\; y'' = 4x - 4e^t,$$
$$x(0) = 2,\; x'(0) = 3,\; y(0) = 1,\; y'(0) = 2.$$

Sol. Given system is $x'' = x + 3y$ ...(1)

$y'' = 4x - 4e^t.$ ...(2)

Taking the Laplace transform, we get

$$L(x'') = L(x) + 3L(y) \quad ...(3)$$

and $$L(y'') = 4L(x) - 4L(e^t). \quad ...(4)$$

(3) $\Rightarrow$ $$[s^2L(x) - sx(0) - x'(0)] = \bar{x} + 3\bar{y}$$

$\Rightarrow$ $$s^2\bar{x} - s.2 - 3 = \bar{x} + 3\bar{y}$$

$\Rightarrow$ $$(s^2 - 1)\bar{x} - 3\bar{y} = 2s + 3 \quad ...(5)$$

(4) $\Rightarrow$ $$[s^2L(y) - sy(0) - y'(0)] = 4\bar{x} - 4 \cdot \frac{1}{s-1}$$

$\Rightarrow$ $$s^2\bar{y} - s\,.1 - 2 = 4\bar{x} - \frac{4}{s-1}$$

$\Rightarrow$ $$-4\bar{x} + s^2\bar{y} = s + 2 - \frac{4}{s-1} \quad ...(6)$$

(5) × 4 + (6) × $(s^2 - 1)$ $\Rightarrow$ $$-12\bar{y} + s^2(s^2 - 1)\bar{y} = 8s + 12 + (s^2 - 1)(s + 2) - 4(s + 1)$$

$\Rightarrow$ $$(s^2 - 4)(s^3 + 3)\bar{y} = 4(s + 2) + (s^2 - 1)(s + 2)$$

$\Rightarrow$ $$\bar{y} = \frac{(s+2)(s^2+3)}{(s^2-4)(s^2+3)} = \frac{1}{s-2}.$$

$\therefore$ $$y = L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$$

(5) $\Rightarrow$ $$(s^2 - 1)\bar{x} = 3\left(\frac{1}{s-2}\right) + 2s + 3 = \frac{(s+1)(2s-3)}{s-2}$$

$\therefore$ $$\bar{x} = \frac{2s-3}{(s-1)(s-2)} = \frac{(-1)}{(s-1)(-1)} + \frac{1}{1.(s-2)} = \frac{1}{s-1} + \frac{1}{s-2}.$$

$\therefore$ $$x = L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{1}{s-2}\right) = e^t + e^{2t}.$$

$\therefore$ $$x = \mathbf{e^t + e^{2t}} \text{ and } y = \mathbf{e^{2t}}.$$

Example 6. *Solve the following system of differential equations by the method of Laplace transforms :*

$$(D^2 + 2)x - Dy = 1,\ Dx + (D^2 + 2)y = 0,$$
$$x(0) = 0,\ x'(0) = 0,\ y(0) = 0,\ y'(0) = 0.$$

Sol. Given system is $$D^2x + 2x - Dy = 1 \quad ...(1)$$
$$Dx + D^2y + 2y = 0. \quad ...(2)$$

Taking the Laplace transform, we get

$$L(x'') + 2L(x) - L(y') = L(1) \quad ...(3)$$

and $$L(x') + L(y'') + 2L(y) = L(0). \quad ...(4)$$

(3) $\Rightarrow$ $$[s^2L(x) - sx(0) - x'(0)] + 2\bar{x} - [sL(y) - y(0)] = L(1)$$

$\Rightarrow$ $$s^2\bar{x} - s\,.\,0 - 0 + 2\bar{x} - s\bar{y} + 0 = \frac{1}{s}$$

$\Rightarrow$ $$(s^2 + 2)\bar{x} - s\bar{y} - \frac{1}{s} = 0 \quad ...(5)$$

(4) $\Rightarrow$ $[s\text{L}(x) - x(0)] + [s^2\text{L}(y) - sy(0) - y'(0)] + 2\bar{y} = 0$

$\Rightarrow$ $s\bar{x} - 0 + s^2\bar{y} - s \,.\, 0 - 0 + 2\bar{y} = 0$

$\Rightarrow$ $$s\bar{x} + (s^2 + 2)\bar{y} + 0 = 0 \qquad \text{...(6)}$$

Solving (5) and (6), we get

$$\frac{\bar{x}}{0 + \dfrac{s^2+2}{s}} = \frac{\bar{y}}{-1-0} = \frac{1}{(s^2+2)^2 + s^2}$$

$\therefore$ $$\bar{x} = \frac{s^2+2}{s}\left(\frac{1}{s^4 + 5s^2 + 4}\right) = \frac{s^2+2}{s(s^2+1)(s^2+4)}$$

and $$\bar{y} = \frac{-1}{(s^2+1)(s^2+4)}.$$

Using partial fractions, we get

$$\bar{x} = \frac{1}{2s} - \frac{s}{3(s^2+1)} - \frac{s}{6(s^2+4)}$$

and $$\bar{y} = -\frac{1}{3(s^2+1)} + \frac{1}{3(s^2+4)}.$$

$\therefore$ $$x = \frac{1}{2}\text{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{3}\text{L}^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{6}\text{L}^{-1}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{1}{2}\,.\,1 - \frac{1}{3}\,.\cos t - \frac{1}{6}\cos 2t$$

and $$y = -\frac{1}{3}\text{L}^{-1}\left(\frac{1}{s^2+1}\right) + \frac{1}{3}\text{L}^{-1}\left(\frac{1}{s^2+4}\right) = -\frac{1}{3}\sin t + \frac{1}{3}\,.\,\frac{1}{2}\sin 2t.$$

$\therefore$ $$x = \mathbf{\frac{1}{2} - \frac{1}{3}\cos t - \frac{1}{6}\cos 2t} \;\text{ and }\; y = \mathbf{-\frac{1}{3}\sin t + \frac{1}{6}\sin 2t.}$$

Example 7. *Solve the following system of differential equations by the method of Laplace transforms :*

$$\frac{dx}{dt} + \frac{dy}{dt} = t, \quad \frac{d^2x}{dt^2} - y = e^{-t} \textit{ with } x(0) = 3,\ x'(0) = -2,\ y(0) = 0.$$

Sol. Given system is $$\frac{dx}{dt} + \frac{dy}{dt} = t \qquad \text{...(1)}$$

$$\frac{d^2x}{dt^2} - y = e^{-t}. \qquad \text{...(2)}$$

Taking the Laplace transform, we get

$$\text{L}\left(\frac{dx}{dt}\right) + \text{L}\left(\frac{dy}{dt}\right) = \text{L}(t) \qquad \text{...(3)}$$

and $$\text{L}\left(\frac{d^2x}{dt^2}\right) - \text{L}(y) = \text{L}(e^{-t}). \qquad \text{...(4)}$$

$$(3) \quad \Rightarrow \quad [s\mathrm{L}(x) - x(0)] + [s\mathrm{L}(y) - y(0)] = \frac{1}{s^2}$$

$$\Rightarrow \quad [s\bar{x} - 3] + [s\bar{y} - 0] = \frac{1}{s^2}$$

$$\Rightarrow \quad s\bar{x} + s\bar{y} = 3 + \frac{1}{s^2}$$

$$\Rightarrow \quad \bar{x} + \bar{y} = \frac{3}{s} + \frac{1}{s^3} \qquad ...(5)$$

$$(4) \quad \Rightarrow \quad [s^2\mathrm{L}(x) - sx(0) - x'(0)] - \bar{y} = \frac{1}{s+1}$$

$$\Rightarrow \quad [s^2\bar{x} - s(3) - (-2)] - \bar{y} = \frac{1}{s+1}$$

$$\Rightarrow \quad s^2\bar{x} - \bar{y} = \frac{1}{s+1} + 3s - 2 \qquad ...(6)$$

$$(5) + (6) \quad \Rightarrow \quad (1+s^2)\bar{x} = \frac{3}{s} + \frac{1}{s^3} + \frac{1}{s+1} + 3s - 2$$

$$\therefore \quad \bar{x} = \frac{3}{s(s^2+1)} + \frac{1}{s^3(s^2+1)} + \frac{1}{(s+1)(s^2+1)} + \frac{3s}{s^2+1} - \frac{2}{s^2+1}.$$

We have $\quad \mathrm{L}^{-1} \dfrac{1}{s^2+1} = \sin t.$

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{1}{s} \cdot \frac{1}{s^2+1}\right) = \int_0^t \sin \mathrm{T}\, d\mathrm{T} = -\cos \mathrm{T}\Big|_0^t = 1 - \cos t$$

$$\mathrm{L}^{-1}\left(\frac{1}{s} \cdot \frac{1}{s(s^2+1)}\right) = \int_0^t (1 - \cos \mathrm{T})\, d\mathrm{T} = (T - \sin \mathrm{T})\Big|_0^t = t - \sin t$$

$$\mathrm{L}^{-1}\left(\frac{1}{s} \cdot \frac{1}{s^2(s^2+1)}\right) = \int_0^t (\mathrm{T} - \sin \mathrm{T})\, d\mathrm{T} = \left(\frac{\mathrm{T}^2}{2} + \cos \mathrm{T}\right)\Bigg|_0^t = \frac{t^2}{2} + \cos t - 1.$$

Using partial fractions,

$$\frac{1}{(s+1)(s^2+1)} = \frac{1}{2(s+1)} + \frac{1}{2}\left(\frac{1-s}{s^2+1}\right)$$

$$= \frac{1}{2(s+1)} + \frac{1}{2(s^2+1)} - \frac{s}{2(s^2+1)}$$

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right) = \frac{1}{2}e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t.$$

$$\therefore \quad x = 3(1 - \cos t) + \left(\frac{t^2}{2} + \cos t - 1\right) + \frac{1}{2}(e^{-t} + \sin t - \cos t) + 3\cos t - 2\sin t$$

or $$x = \mathbf{2} + \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{cos\ t} + \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{t^2} + \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{e^{-t}} - \frac{\mathbf{3}}{\mathbf{2}}\,\mathbf{sin\ t}$$

(5) $\Rightarrow$ $$\bar{y} = \frac{3}{s} + \frac{1}{s^3} - \bar{x}$$

$\therefore$ $$y = 3L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^3}\right) - L^{-1}(\bar{x})$$

$$= 3(1) + \frac{1}{2}t^2 - 2 - \frac{1}{2}\cos t - \frac{1}{2}t^2 - \frac{1}{2}e^{-t} + \frac{3}{2}\sin t$$

or $$y = \mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{cos\ t} - \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{e^{-t}} + \frac{\mathbf{3}}{\mathbf{2}}\,\mathbf{sin\ t.}$$

WORKING STEPS FOR SOLVING PROBLEMS

Step I. Take the Laplace transform of the given equations.

Step II. Substitute the values of Laplace transforms.

Step III. Simplify the equations to get the values of $\bar{x}$ and $\bar{y}$.

Step IV. Find the inverse Laplace transforms of $\bar{x}$ and $\bar{y}$ to get the values of x and y.

TEST YOUR KNOWLEDGE

Solve the following systems of differential equations by the method of Laplace transforms :

1. $x' + x - 4y = 0$, $y' = 3x - 2y$,
$x(0) = 3$, $y(0) = 4$.

2. $x' + x - y = 0$, $y' + x + y = 0$,
$x(0) = 1$, $y(0) = 0$.

3. $3x' + y' + 2x = 1$, $x' + 4y' + 3y = 0$,
$x(0) = 0$, $y(0) = 0$.

4. $(D - 2)x + 3y = 0$, $2x + (D - 1)y = 0$,
$x(0) = 8$, $y(0) = 3$.

5. $x' = 6x + 9y$, $y' = x + 6y$,
$x(0) = 3$, $y(0) = 3$.

6. $x' = 5x + y$, $y' = x + 5y$,
$x(0) = -3$, $y(0) = 7$.

7. $x' = -y$, $y' = x$,
$x(0) = 1$, $y(0) = 0$.

8. $x' = 2x - 4y$, $y' = x - 3y$,
$x(0) = 3$, $y(0) = 0$.

9. $x' = 2x + 4y$, $y' = x + 2y$,
$x(0) = -4$, $y(0) = -4$.

10. $x' = -2x + 3y$, $y' = 4x - y$,
$x(0) = 4$, $y(0) = 3$.

11. $x' + 2x - 4y = 0$, $y' - 2x = t$,
$x(0) = 0$, $y(0) = 3$.

12. $2x' + y' - x - y = e^{-t}$, $x' + y' + 2x + y = e^{t}$,
$x(0) = 2$, $y(0) = 1$.

13. $x'' - 3x' + y' + 2x - y = 0$,
$x' + y' - 2x + y = 0$,
$x(0) = 0$, $y(0) = -1$, $x'(0) = 0$.

14. $x'' = -5x + 2y$, $y'' = 2x - 2y$,
$x(0) = 3$, $x'(0) = 0$, $y(0) = 1$, $y'(0) = 0$.

15. $x'' + 3x - 2y = 0$, $x'' + y'' - 3x + 5y = 0$,
$x(0) = 0$, $x'(0) = 3$, $y(0) = 0$, $y'(0) = 2$.

16. $x'' - x + 5y' = t$, $-2x' + y'' - 4y = -2$
$x(0) = 0$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = 0$.

Answers

1. $x = 4e^{2t} - e^{-5t},\ y = 3e^{2t} + e^{-5t}$

2. $x = e^{-t} \cos t,\ y = -e^{-t} \sin t$

3. $x = \dfrac{1}{10}(5 - 2e^{-t} - 3e^{-6t/11}),\ y = \dfrac{1}{5}(e^{-t} - e^{-6t/11})$

4. $x = 5e^{-t} + 3e^{4t},\ y = 5e^{-t} - 2e^{4t}$

5. $x = -3e^{3t} + 6e^{9t},\ y = e^{3t} + 2e^{9t}$

6. $x = -5e^{4t} + 2e^{6t},\ y = 5e^{4t} + 2e^{6t}$

7. $x = \cos t,\ y = \sin t$

8. $x = 4e^{t} - e^{-2t},\ y = e^{t} - e^{-2t}$

9. $x = 2 - 6e^{4t},\ y = -1 - 3e^{4t}$

10. $x = 3e^{2t} + e^{-5t},\ y = 4e^{2t} - e^{-5t}$

11. $x = \dfrac{13}{6}e^{2t} - \dfrac{49}{24}e^{-4t} - \dfrac{1}{2}t - \dfrac{1}{8},\ y = \dfrac{13}{6}e^{2t} + \dfrac{49}{48}e^{-4t} - \dfrac{1}{4}t - \dfrac{3}{16}$

12. $x = 8 \sin t + 2 \cos t,$

$y = -13 \sin t + \cos t + \dfrac{e^{t}}{2} - \dfrac{e^{-t}}{2}$

13. $x = 2e^{t} - e^{2t} - 1,$

$y = e^{t} - 2$

14. $x = \cos t + 2 \cos \sqrt{6}\, t,\ y = 2 \cos t - \cos \sqrt{6}\, t$

15. $x = \dfrac{11}{4} \sin t + \dfrac{1}{12} \sin 3t,\ y = \dfrac{11}{4} \sin t - \dfrac{1}{4} \sin 3t$

16. $x = -t + 5 \sin t - 2 \sin 2t,\ y = 1 - 2 \cos t + \cos 2t.$

Fourier Transforms

5

5.1. INTRODUCTION

Fourier* transforms play an important part in the theory of many branches of science. The use of Fourier transforms is indispensible in solving the problems in the field of mathematics, physics and engineering. Many types of integrals and differential equations can be solved by using Fourier transforms.

5.2. FOURIER'S INTEGRAL THEOREM

Statement. *Let f be a real valued function of the real variable x such that f(x) and f′(x) are piecewise continuous in every finite interval and the integral of |f(x)| from – ∞ to ∞ exists. Then f(x) can be expressed as*

$$f(x) = \int_0^\infty (A(s)\cos sx + B(s)\sin sx)\, ds, \text{ where}$$

$$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos sv\, dv \quad \text{and} \quad B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin sv\, dv.$$

At a point where f(x) is discontinuous the value of the integral on the right side equals the average of left hand limit and right hand limit of f(x) at that point.

Note. The proof of this theorem is beyond the scope of this book.

The integral $\int_0^\infty (A(s)\cos sx + B(s)\sin sx)\, ds$ is called the **Fourier integral expression** of the function $f(x)$.

Theorem. *Prove that the Fourier integral expression of a function f(x) is same as the integral* $\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\, e^{is(x-v)}\, dv\right) ds$.

Proof. The Fourier integral expression of $f(x)$ is :

$$f(x) = \int_0^\infty (A(s)\cos sx + B(s)\sin sx)\, ds, \text{ where}$$

$$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos sv\, dv \quad \text{and} \quad B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin sv\, dv$$

* **Baron Baptiste Joseph Fourier** (1768–1830) French physicist and mathematician lived and taught in Paris. Fourier was appointed as Perfect of Isère by Nepoleon in 1802. He never married.

$\therefore \quad f(x) = \int_0^{\infty}\left(\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos sv\, dv\right)\cos sx + \left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin sv\, dv\right)\sin sx\right) ds$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(v)\,[\cos sv\cos sx + \sin sv \sin sx]\, dv\right) ds$$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(v)\cos (sv - sx)\, dv\right) ds$$

$\therefore \quad f(x) = \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(v)\cos s(x - v)\, dv\right) ds \quad$ ...(1)

The function $f(v)\cos s(x - v)$ is an even function of s.

$\therefore$ The function $\int_{-\infty}^{\infty} f(v)\cos s(x - v)\, dv$ is also an even function of s.

($\because$ The integration is not w.r.t. s)

$\therefore$ (1) $\Rightarrow$ $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\cos s(x - v)\, dv\right) ds \quad$ ...(2)

$\left(\because \text{ If } \phi(x) \text{ is an even function of } x \text{ then } \int_{-\infty}^{\infty}\phi(x)\, dx = 2\int_0^{\infty}\phi(x)\, dx\right)$

The function $f(v)\sin s(x - v)$ is an odd function of s.

$\therefore$ The function $\int_{-\infty}^{\infty} f(v)\sin s(x - v)\, dv$ is also an odd function of s.

$\therefore \quad \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\sin s(x - v)\, dv\right) ds = 0$

$\Rightarrow \quad 0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\sin s(x - v)\, dv\right) ds \quad$ ...(3)

Adding (2) and 'i' times (3), we get

$$f(x) + i \,.\, 0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\,(\cos s(x - v) + i\sin s(x - v))\, dv\right) ds$$

$\Rightarrow \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\, e^{is(x - v)}\, dv\right) ds$*.

$\therefore$ The result holds.

5.3. FOURIER TRANSFORM AND ITS INVERSE

Let f be a real valued function of the real variable x such that $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval and the integral of $|f(x)|$ exists from $-\infty$ to ∞.

The integral $\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$ is called the **Fourier transform** of the function $f(x)$ and we write this as $F(f(x))$ or as $\bar{f}(s)$.

***Why this step.** We have used the **Euler formula :**

$$e^{ix} = \cos x + i \sin x.$$

Thus, $$\mathbf{\bar{f}(s) = F(\mathit{f}(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx}.$$

By Fourier integral expression of $f(x)$, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\, e^{is(x-v)}\, dv\right) ds.$$

$$\Rightarrow \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\, e^{-isv}\, dv\right) e^{isx}\, ds$$

$$\Rightarrow \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds.$$

The function $\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds$ *i.e.*, $f(x)$ is called the **inverse Fourier transform** of the function $\bar{f}(s)$ and we write $F^{-1}(\bar{f}(s)) = F^{-1}(F(f(x)) = f(x)$.

The constants 1 and $\frac{1}{2\pi}$ preceding the integrals $\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$ and $\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds$ could be replaced by any constants whose product is $\frac{1}{2\pi}$. In our discussion, we shall stick to the above choice.

Remark 1. The Fourier transform of a function will in general be a complex-valued function.

Remark 2. In finding the Fourier transform of a function, we may require the following 'general rule of integration by parts' :

$$\int \mathbf{uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''\, v_4 + \ldots\ldots},$$

where dashes denote the successive differentiation and suffixes denote the successive integration.

5.4. SHIFTING PROPERTY OF FOURIER TRANSFORMS

Theorem I. *If the Fourier transform of the function f(x) is* $\bar{f}(s)$ *then prove that the Fourier transform of the function f(x – a) is* $e^{-ias}\, \bar{f}(s)$.

Proof. We have

$$F(f(x)) = \bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx.$$

$$\therefore \qquad F(f(x-a)) = \int_{-\infty}^{\infty} f(x-a)\, e^{-isx}\, dx$$

Let $t = x - a$. $\therefore$ $dt = dx$

$$\therefore \qquad F(f(x-a)) = \int_{-\infty}^{\infty} f(t)\, e^{-is(t+a)}\, dt = \int_{-\infty}^{\infty} f(t)\,(e^{-ist}\,.\,e^{-ias})\, dt$$

$$= e^{-ias}\int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt = e^{-ias}\, \bar{f}(s).$$

Theorem II. *If the Fourier transform of the function f(x) is* $\bar{f}(s)$ *then prove that the Fourier transform of the function* e^{iax} *f(x) is* $\bar{f}(s-a)$.

Proof. We have $F(f(x)) = \bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx.$

$$\therefore \quad F(e^{iax} f(x)) = \int_{-\infty}^{\infty} (e^{iax} f(x))\, e^{-isx}\, dx = \int_{-\infty}^{\infty} f(x)\, e^{-i(s-a)x}\, dx$$

$$= \bar{f}(s-a).$$

5.5. MODULATION PROPERTY OF FOURIER TRANSFORMS

Theorem. *If the Fourier transform of the function f(x) is* $\bar{f}(s)$ *then prove that the Fourier transform of the function f(x) cos ax is* $\frac{1}{2}\bar{f}(s-a) + \frac{1}{2}\bar{f}(s+a)$.

Proof. We have $F(f(x)) = \bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx.$

$$\therefore \quad F(f(x)\cos ax) = \int_{-\infty}^{\infty} (f(x)\cos ax)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2}\right) e^{-isx}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{iax}\, e^{-isx}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{-iax}\, e^{-isx}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{-i(s-a)x}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\, e^{-i(s+a)x}\, dx$$

$$= \frac{1}{2}\bar{f}(s-a) + \frac{1}{2}\bar{f}(s+a).$$

5.6. CONVOLUTION THEOREM

The convolution theorem is used to find the inverse Fourier transform of the product of two functions with known inverse Fourier transforms of the factors of the product. This theorem is also used in solving partial differential equations.

The **convolution** of the functions $f(x)$ and $g(x)$ is defined by the integral $\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$ and denoted by $f * g$.

$$\therefore \quad (f * g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \qquad \text{...(1)}$$

Let $v = x - u$. $\therefore$ $dv = -du$

$$\therefore \ (1) \Rightarrow \quad (f * g)(x) = \int_{\infty}^{-\infty} f(x-v)\, g(v)\, (-dv)$$

$$= -\int_{\infty}^{-\infty} g(v)\, f(x-v)\, dv$$

$$= \int_{-\infty}^{\infty} g(v)\, f(x-v)\, dv = \int_{-\infty}^{\infty} g(u)\, f(x-u)\, du$$

(Replacing v by u)

$$= (g * f)(x)$$

$\therefore \quad f * g = g * f.$

Convolution Theorem. Let f(x) and g(x) be functions such that their Fourier transforms exist. Prove that

$$\mathbf{F(f * g) = F(f)\ F(g).}$$

Proof. It can be proved mathematically that the Fourier transform of the convolution of f and g (*i.e.*, $f * g$) exists.

By definition, $\quad F(f * g) = \int_{-\infty}^{\infty} (f * g)(x)\, e^{-isx}\, dx$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \right) e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} g(x-u)\, e^{-isx}\, dx \right) du$$

(By changing the order of integration)

Let $\quad v = x - u. \quad \therefore \quad dv = dx$

$$\therefore \quad \int_{-\infty}^{\infty} g(x-u)\, e^{-isx}\, dx = \int_{-\infty}^{\infty} g(v)\, e^{-is(u+v)}\, dv$$

$$\therefore \quad F(f * g) = \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} g(v)\, e^{-is(u+v)}\, dv \right) du$$

$$= \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} g(v)\, e^{-isu}\, e^{-isv}\, dv \right) du$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-isu} \left(\int_{-\infty}^{\infty} g(v)\, e^{-isv}\, dv \right) du$$

$$= \left(\int_{-\infty}^{\infty} f(u)\, e^{-isu}\, du \right) \left(\int_{-\infty}^{\infty} g(v)\, e^{-isv}\, dv \right)$$

$$= F(f)\, F(g)$$

$\therefore \quad \mathbf{F(f * g) = F(f)\ F(g).}$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *If* $A(s) = \dfrac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos sv\, dv$ *and* $B(s) = \dfrac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin sv\, dv$ *then*

$$f(x) = \int_{0}^{\infty} (A(s) \cos sx + B(s) \sin sx)\, ds.$$

Rule II. $f(x) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(v)\, e^{is(x-v)}\, dv \right) ds.$

Rule III. *The Fourier transform of the function f(x) is given by :*

$$\bar{f}(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx.$$

Rule IV. *The inverse Fourier transform of the function g(s) is given by*

$$F^{-1}(g(s)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)\, e^{isx}\, ds.$$

Rule V. $F^{-1}(\bar{f}(s)) = f(x)$.

Rule VI. (*i*) $F(f(x)) = \bar{f}(s) \Rightarrow F(f(x-a)) = e^{-ias}\bar{f}(s)$

(*ii*) $F(f(x)) = \bar{f}(s) \Rightarrow F(e^{iax} f(x)) = \bar{f}(s-a)$.

Rule VII. $F(f(x)) = \bar{f}(s) \Rightarrow F(f(x) \cos ax) = \dfrac{1}{2}\bar{f}(s-a) + \dfrac{1}{2}\bar{f}(s+a)$.

Rule VIII. (*i*) $(f * g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$

(*ii*) $F(f * g) = F(f)\, F(g)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} \lambda, & \text{if } 0 < x < a \\ 0, & \text{otherwise} \end{cases}.$$

Sol.

$$\begin{aligned} F(f(x)) &= \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx \\ &= \int_{-\infty}^{0} f(x)\, e^{-isx}\, dx + \int_{0}^{a} f(x)\, e^{-isx}\, dx + \int_{a}^{\infty} f(x)\, e^{-isx}\, dx \\ &= \int_{-\infty}^{0} 0\,.\, e^{-isx}\, dx + \int_{0}^{a} \lambda e^{-isx}\, dx + \int_{a}^{\lambda} 0\,.\, e^{-isx}\, dx \\ &= 0 + \lambda \int_{0}^{a} e^{-isx}\, dx + 0 = \lambda \left(\frac{e^{-isx}}{-is} \Bigg|_{0}^{a} \right) = \mathbf{\frac{i\lambda}{s}(e^{-isa} - 1).} \end{aligned}$$

Example 2. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} e^{x}, & \text{if } -a < x < a \\ 0, & \text{otherwise} \end{cases}.$$

Sol.

$$\begin{aligned} F(f(x)) &= \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx \\ &= \int_{-\infty}^{-a} f(x)\, e^{-isx}\, dx + \int_{-a}^{a} f(x)\, e^{-isx}\, dx + \int_{a}^{\infty} f(x)\, e^{-isx}\, dx \\ &= \int_{-\infty}^{-a} 0\,.\, e^{-isx}\, dx + \int_{-a}^{a} e^{x}\, e^{-isx}\, dx + \int_{a}^{\infty} 0\,.\, e^{-isx}\, dx \\ &= 0 + \int_{-a}^{a} e^{(1-is)x}\, dx + 0 \\ &= \frac{e^{(1-is)x}}{1-is} \Bigg|_{-a}^{a} = \mathbf{\frac{1}{1-is}\,[e^{(1-is)a} - e^{-(1-is)a}].} \end{aligned}$$

Example 3. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} 5x, & \text{if } 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

Sol.

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} f(x)\, e^{-isx}\, dx + \int_{0}^{3} f(x)\, e^{-isx}\, dx + \int_{3}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} 0\,.\, e^{-isx}\, dx + \int_{0}^{3} 5xe^{-isx}\, dx + \int_{3}^{\infty} 0\,.\, e^{-isx}\, dx$$

$$= 0 + 5 \int_{0}^{3} \underset{\text{I}}{x}\,.\,\underset{\text{II}}{e^{-isx}}\, dx + 0$$

$$= 5\left[x\, \frac{e^{-isx}}{-is} \Bigg|_{0}^{3} - \int_{0}^{3} 1\,.\,\frac{e^{-isx}}{-is}\, dx \right] = \frac{5i}{s}\ [3e^{-3is} - 0] + \frac{5}{is} \cdot \frac{e^{-isx}}{-is} \Bigg|_{0}^{3}$$

$$= \frac{15ie^{-3is}}{s} + \frac{5}{s^2}\ [e^{-3is} - 1]$$

$$= \frac{5}{s^2}\ [3is\, e^{-3is} + e^{-3is} - 1] = \boldsymbol{\frac{5}{s^2}\,[e^{-3is}\,(3is + 1) - 1].}$$

Example 4. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} xe^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}.$$

Sol. $\quad F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$

$$= \int_{-\infty}^{0} f(x)\, e^{-isx} dx + \int_{0}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} 0\,.\, e^{-isx} dx + \int_{0}^{\infty} xe^{-x}\,.\, e^{-isx}\, dx = 0 + \int_{0}^{\infty} \underset{\text{I}}{x}\,.\,\underset{\text{II}}{e^{-(1+is)x}}\, dx$$

$$= \left[x\,.\,\frac{e^{-(1+is)x}}{-(1+is)} - 1\,.\,\frac{e^{-(1+is)x}}{(-(1+is))^2} \right] \Bigg|_{0}^{\infty}$$

$$= -\frac{1}{1+is}\,.\,\frac{x}{e^{x}(\cos sx + i \sin sx)} \Bigg|_{0}^{\infty} - \frac{1}{(1+is)^2}\,.\,\frac{1}{e^{x}(\cos sx + i \sin sx)} \Bigg|_{0}^{\infty}$$

$$= -\frac{1}{1+is}\left[\lim_{x \to \infty} \frac{x}{e^x}\,.\, \lim_{x \to \infty} \frac{1}{\cos sx + i \sin sx} - \frac{0}{e^0(\cos 0 + i \sin 0)} \right]$$

$$- \frac{1}{(1+is)^2}\left[\lim_{x \to \infty} \frac{1}{e^x(\cos sx + i \sin sx)} - \frac{1}{e^0(\cos 0 + i \sin 0)} \right]$$

$$= -\frac{1}{1+is}\ [0 - 0] - \frac{1}{(1+is)^2}\ [0 - 1]$$

$$\left(\because\ \lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{(x)'}{(e^x)'} = \lim_{x \to \infty} \frac{1}{e^x} = 0 \right)$$

$$= \boldsymbol{\frac{1}{(1+is)^2}}.$$

Example 5. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} -1, & \text{if } -a < x < 0 \\ 1, & \text{if } 0 < x < a \\ 0, & \text{otherwise} \end{cases}.$$

Sol.

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-a} f(x)\, e^{-isx}\, dx + \int_{-a}^{0} f(x)\, e^{-isx}\, dx + \int_{0}^{a} f(x)\, e^{-isx}\, dx + \int_{a}^{\infty} f(x) e^{-isx}\, dx$$

$$= \int_{-\infty}^{-a} 0 \,.\, e^{-isx}\, dx + \int_{-a}^{0} -1 \,.\, e^{-isx}\, dx + \int_{0}^{a} 1 \,.\, e^{-isx}\, dx + \int_{a}^{\infty} 0 \,.\, e^{-isx}\, dx$$

$$= 0 + (-1)\,.\, \left.\frac{e^{-isx}}{-is}\right|_{-a}^{0} + \left.\frac{e^{-isx}}{-is}\right|_{0}^{a} + 0$$

$$= -\frac{i}{s}\,(1 - e^{isa}) + \frac{i}{s}(e^{-isa} - 1)$$

$$= \frac{i}{s}\,(-1 + e^{isa} + e^{-isa} - 1) = \frac{i}{s}\,(2\cos sa - 2)$$

$$= \mathbf{\frac{2i}{s}\,(\cos sa - 1).}$$

Example 6. *Find the Fourier transform of the function*

$$f(t) = \begin{cases} 1 - \dfrac{t}{a}, & \text{if } 0 < t < a \\ 1 + \dfrac{t}{a}, & \text{if } -a < t < 0 \\ 0, & \text{otherwise} \end{cases}.$$

Sol. $F(f(t)) = \displaystyle\int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$

$$= \int_{-\infty}^{-a} f(t)\, e^{-ist}\, dt + \int_{-a}^{0} f(t)\, e^{-ist}\, dt + \int_{0}^{a} f(t)\, e^{-ist}\, dt + \int_{a}^{\infty} f(t)\, e^{-ist}\, dt$$

$$= \int_{-\infty}^{-a} 0 \,.\, e^{-ist}\, dt + \int_{-a}^{0} \left(1 + \frac{t}{a}\right) e^{-ist}\, dt + \int_{0}^{a} \left(1 - \frac{t}{a}\right) e^{-ist}\, dt + \int_{a}^{\infty} 0.\, e^{-ist}\, dt$$

$$= 0 + \left.\left(\left(1 + \frac{t}{a}\right).\frac{e^{-ist}}{-is} - \frac{1}{a}.\frac{e^{-ist}}{(-is)^2}\right)\right|_{-a}^{0} + \left.\left(\left(1 - \frac{t}{a}\right).\frac{e^{-ist}}{-is} - \left(-\frac{1}{a}\right).\frac{e^{-ist}}{(-is)^2}\right)\right|_{0}^{a} + 0$$

$$= -\frac{1}{is}\,(1 - 0) + \frac{1}{as^2}\,(1 - e^{ias}) - \frac{1}{is}\,(0 - 1) - \frac{1}{as^2}\,(e^{-ias} - 1)$$

$$= \frac{1}{as^2}\,(1 - e^{ias} - e^{-ias} + 1) = \frac{1}{as^2}\,(2 - 2\cos as) = \frac{1}{as^2}\,.\, 4\sin^2 \frac{as}{2}$$

$$= \mathbf{\frac{4}{as^2} \sin^2 \frac{as}{2}.}$$

Example 7. *Find the Fourier transform of the function*

$$f(u) = \begin{cases} u^2, & \text{if } |u| < u_0 \\ 0, & \text{if } |u| > u_0 \end{cases}.$$

Sol. We have
$$f(u) = \begin{cases} u^2, & \text{if } -u_0 < u < u_0 \\ 0, & \text{if } u < -u_0 \text{ or } u > u_0 \end{cases}$$

$$\begin{aligned}
F(f(u)) &= \int_{-\infty}^{\infty} f(u)\, e^{-isu}\, du \\
&= \int_{-\infty}^{-u_0} f(u)\, e^{-isu}\, du + \int_{-u_0}^{u_0} f(u)\, e^{-isu}\, du + \int_{u_0}^{\infty} f(u)\, e^{-isu}\, du \\
&= \int_{-\infty}^{-u_0} 0 \,.\, e^{-isu}\, du + \int_{-u_0}^{u_0} u^2\, e^{-isu}\, du + \int_{u_0}^{\infty} 0 \,.\, e^{-isu}\, du \\
&= 0 + \left(u^2 \,.\, \frac{e^{-isu}}{-is} - 2u \,.\, \frac{e^{-isu}}{(-is)^2} + 2 \,.\, \frac{e^{-isu}}{(-is)^3} \right)\Bigg|_{-u_0}^{u_0} + 0 \\
&= -\frac{1}{is}(u_0^2\, e^{-isu_0} - u_0^2\, e^{isu_0}) + \frac{2}{s^2}(u_0 e^{-isu_0} + u_0 e^{isu_0}) + \frac{2}{is^3}(e^{-isu_0} - e^{isu_0}) \\
&= \frac{2u_0^2}{s}\left(\frac{e^{isu_0} - e^{-isu_0}}{2i}\right) + \frac{4u_0}{s^2}\left(\frac{e^{isu_0} + e^{-isu_0}}{2}\right) - \frac{4}{s^3}\left(\frac{e^{isu_0} - e^{-isu_0}}{2i}\right) \\
&= \frac{2u_0^2}{s} \sin su_0 + \frac{4u_0}{s^2} \cos su_0 - \frac{4}{s^3} \sin su_0 \\
&= \boldsymbol{\frac{2}{s^3}[(u_0^2 s^2 - 2) \sin su_0 + 2u_0 s \cos su_0].}
\end{aligned}$$

Example 8. *Find the Fourier transform of* $f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$ *and hence evaluate :*

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s}\, ds$ *and* *(ii)* $\displaystyle\int_{0}^{\infty} \frac{\sin s}{s}\, ds$.

Sol. We have
$$f(x) = \begin{cases} 1, & \text{if } -a < x < a \\ 0, & \text{if } x < -a \text{ or } x > -a \end{cases}$$

$$\begin{aligned}
F(f(x)) &= \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx \\
&= \int_{-\infty}^{-a} f(x) e^{-isx}\, dx + \int_{-a}^{a} f(x) e^{-isx}\, dx + \int_{a}^{\infty} f(x) e^{-isx}\, dx \\
&= \int_{-\infty}^{-a} 0 \,.\, e^{-isx}\, dx + \int_{-a}^{a} 1 \,.\, e^{-isx}\, dx + \int_{a}^{\infty} 0 . e^{-isx}\, dx \\
&= 0 + \frac{e^{-isx}}{-is}\Bigg|_{-a}^{a} + 0 = -\frac{1}{is}(e^{-isa} - e^{isa}) \\
&= \frac{2}{s}\left(\frac{e^{isa} - e^{-isa}}{2i}\right) = \frac{2}{s} \sin sa
\end{aligned}$$

$\therefore$ Fourier transform of $f(x)$

$$= \bar{f}(s) = \mathrm{F}(f(x)) = \mathbf{\frac{2 \sin as}{s}}.$$

(*i*) By inverse Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s) e^{isx}\, ds.$$

$$\Rightarrow \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2 \sin as}{s} (\cos sx + i \sin sx)\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin as \cos sx}{s} + i\, \frac{\sin as \sin sx}{s}\right) ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \sin sx}{s}\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds + \frac{i}{\pi}\,.\,0$$

$$\left(\because \ \frac{\sin as \sin sx}{s} \text{ is an odd function of } s\right)$$

$$\therefore \qquad f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$$

$$\therefore \quad \int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \pi f(x) = \begin{cases} \pi, & \textbf{if } |\mathbf{x}| < \mathbf{a} \\ \mathbf{0}, & \textbf{if } |\mathbf{x}| > \mathbf{a} \end{cases}.$$

(*ii*) We have $\displaystyle\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds = \begin{cases} \pi, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$

In particular let $x = 0$ and $a = 1$.

$$\therefore \qquad \int_{-\infty}^{\infty} \frac{\sin (1\,.\,s) \cos (s\,.\,0)}{s}\, ds = \pi \qquad (\because \ |x| = |0| = 0 \ \text{ and } \ a = 1)$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds = \pi \ \Rightarrow \ 2\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \pi$$

$$\therefore \qquad \int_{0}^{\infty} \frac{\sin s}{s}\, ds = \mathbf{\frac{\pi}{2}}.$$

Example 9. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} 1 - x^2, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

and hence evaluate $\displaystyle\int_{0}^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2}\, dx$.

Sol. We have $\qquad f(x) = \begin{cases} 1 - x^2, & \text{if } -1 < x < 1 \\ 0, & \text{if } x < -1 \text{ or } x > 1 \end{cases}.$

$$\mathrm{F}(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-1} f(x)\, e^{-isx}\, dx + \int_{-1}^{1} f(x)\, e^{-isx}\, dx + \int_{1}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-1} 0 \,.\, e^{-isx}\, dx + \int_{-1}^{1} (1-x^2)\, e^{-isx}\, dx + \int_{1}^{\infty} 0 \,.\, e^{-isx}\, dx$$

$$= 0 + \left[(1-x^2) \,.\, \frac{e^{-isx}}{-is} - (-2x) \,.\, \frac{e^{-isx}}{(-is)^2} + (-2) \,.\, \frac{e^{-isx}}{(-is)^3} \right]_{-1}^{1} + 0$$

(By using general rule of integration by parts)

$$= 0 - 0 + \left[-\frac{2x}{s^2} e^{-isx} + \frac{2i}{s^3} e^{-isx} \right]_{-1}^{1}$$

$$= -\frac{2}{s^2} (e^{-is} + e^{is}) + \frac{2i}{s^3} (e^{-is} - e^{is})$$

$$= -\frac{4}{s^2} \left(\frac{e^{is} + e^{-is}}{2} \right) - \frac{4i^2}{s^3} \left(\frac{e^{is} - e^{-is}}{2i} \right)$$

$$= -\frac{4}{s^2} \cos s + \frac{4}{s^3} \sin s = \frac{4}{s^3} (\sin s - s \cos s)$$

$\therefore$ Fourier transform of $f(x)$

$$= \bar{f}(s) = \mathrm{F}(f(x)) = \mathbf{\frac{4}{s^3} (\sin s - s \cos s).}$$

By Inverse Fourier transform, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds$$

$$\Rightarrow \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{s^3} (\sin s - s \cos s)\, e^{isx}\, ds = f(x)$$

$$\Rightarrow \quad \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{s^3} (\sin s - s \cos s)(\cos sx + i \sin sx)\, ds = \begin{cases} 1 - x^2, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \left(\frac{(\sin s - s \cos s) \cos sx}{s^3} + i \frac{(\sin s - s \cos s) \sin sx}{s^3} \right) ds = \begin{cases} \frac{\pi}{2} (1 - x^2), & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

Equating the real parts, we have

$$\int_{-\infty}^{\infty} \frac{(\sin s - s \cos s) \cos sx}{s^3}\, ds = \begin{cases} \frac{\pi}{2} (1 - x^2), & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

$$\Rightarrow \quad 2 \int_{0}^{\infty} \frac{(\sin s - s \cos s) \cos sx}{s^3}\, ds = \begin{cases} \frac{\pi}{2} (1 - x^2), & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

($\because$ The integrand is an even function of s)

$$\Rightarrow \quad \int_{0}^{\infty} \frac{(s \cos s - \sin s) \cos sx}{s^3}\, ds = \begin{cases} \frac{\pi}{4} (x^2 - 1), & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

Taking $x = \frac{1}{2}$, we get

$$\int_0^\infty \frac{(s\cos s - \sin s)\cos s/2}{s^3}\,ds = \frac{\pi}{4}\left(\left(\frac{1}{2}\right)^2 - 1\right) = -\frac{3\pi}{16} \qquad \left(\because \left|\frac{1}{2}\right| < 1\right)$$

$\therefore$ By changing the variable of integration, we get

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx = -\frac{\mathbf{3\pi}}{\mathbf{16}}.$$

Remark. In the above question, instead of comparing real and imaginary parts, we can also use the fact that $\dfrac{(\sin s - s\cos s)\sin sx}{s^3}$ is an odd function of s.

TEST YOUR KNOWLEDGE

Find the Fourier transform of the following functions (Q. 1—10) :

1. $f(x) = \begin{cases} 4, & \text{if } 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

2. $f(x) = \begin{cases} 1, & \text{if } 3 < x < 9 \\ 0, & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} 2x, & 0 < x < 7 \\ 0, & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} x, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$

5. $f(x) = \begin{cases} -1, & \text{if } -4 < x < 0 \\ 1, & \text{if } 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$

6. $f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$

7. $f(x) = \begin{cases} \dfrac{\sqrt{2\pi}}{2l}, & \text{if } |x| < l \\ 0, & \text{if } |x| > l \end{cases}$

8. $f(x) = \begin{cases} \dfrac{1}{2\varepsilon}, & \text{if } |x| \le \varepsilon \\ 0, & \text{if } |x| > \varepsilon \end{cases}$

9. $f(x) = e^{-|x|}$

10. $f(x) = e^{-a|x|}$, $a > 0$.

11. Find the Fourier transform of $f(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$. Hence evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

Answers

1. $\dfrac{4i}{s}(e^{-2is} - 1)$

2. $\dfrac{i}{s}(e^{-9is} - e^{-3is})$

3. $\dfrac{2(e^{-7is}(1 + 7is) - 1)}{s^2}$

4. $\dfrac{e^{-ias}(1 + ias) - 1}{s^2}$

5. $\dfrac{2i}{s}(\cos s - 1)$

6. $\dfrac{2\sin as}{s}$

7. $\dfrac{\sqrt{2\pi}}{ls}\sin ls$

8. $\dfrac{\sin \varepsilon s}{\varepsilon s}$

9. $\dfrac{2}{1 + s^2}$

10. $\dfrac{2a}{a^2 + s^2}$

11. $\dfrac{2\sin s}{s}, \dfrac{\pi}{2}$.

5.7. FOURIER SINE AND COSINE TRANSFORMS

Let f be a real valued function of the real variable x such that $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval and the integral of $|f(x)|$ exists from $-\infty$ to ∞. The integral

$\int_0^\infty f(x)\sin sx\,dx$ is called the **Fourier sine transform** of the function $f(x)$ and we write this as $F_S(f(x))$ or as $\bar{f}_S(s)$. Thus, $\mathbf{\bar{f}_S(s) = F_S(f(x)) = \int_0^\infty f(x)\sin sx\,dx}$.

Similarly, the integral $\int_0^\infty f(x)\cos sx\,dx$ is called the **Fourier cosine transform** of the function $f(x)$ and we write this as $F_C(f(x))$ or as $\bar{f}_C(s)$.

Thus, $\mathbf{\bar{f}_C(s) = F_C(f(x)) = \int_0^\infty f(x)\cos sx\,dx}$.

By Fourier integral expression of $f(x)$, we have

$$f(x) = \int_0^\infty (A(s)\cos sx + B(s)\sin sx)\,ds, \qquad \text{...(1)}$$

where $$A(s) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos sv\,dv \quad \text{and} \quad B(s) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin sv\,dv.$$

Case I. f(x) is an odd function.

$\therefore$ $f(v)\cos sv$ is an odd function of v.

$$\therefore \quad \int_{-\infty}^\infty f(v)\cos sv\,dv = 0$$

$$\therefore \quad A(s) = \frac{1}{\pi}\,.\,0 = 0$$

Also, $f(v)$ sin sv is an even function of v.

$$\therefore \quad \int_{-\infty}^\infty f(v)\sin sv\,dv = 2\int_0^\infty f(v)\sin sv\,dv$$

$$\therefore \quad B(s) = \frac{2}{\pi}\int_0^\infty f(v)\sin sv\,dv$$

$$\therefore \ (1) \Rightarrow \quad f(x) = \int_0^\infty \left(0\,.\cos sx + \left(\frac{2}{\pi}\int_0^\infty f(v)\sin sv\,dv\right)\sin sx\right)ds$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(v)\sin sv\,dv\right)\sin sx\,ds$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}_S(s)\sin sx\,ds.$$

The function $\frac{2}{\pi}\int_0^\infty \bar{f}_S(s)\sin sx\,ds$ *i.e.*, $f(x)$ is called the **inverse Fourier sine transform** of the function $\bar{f}_S(s)$ and we write $F_S^{-1}(\bar{f}_S(s)) = F_S^{-1}(F_S(f(x)) = f(x)$.

Case II. f(x) is an even function.

$\therefore$ $f(v)\cos sv$ is an even function of v.

$$\therefore \quad \int_{-\infty}^\infty f(v)\cos sv\,dv = 2\int_0^\infty f(v)\cos sv\,dv$$

$$\therefore \quad A(s) = \frac{2}{\pi}\int_0^\infty f(v)\cos sv\,dv$$

Also, $f(v)$ sin sv is an odd function of v.

$$\therefore \quad \int_{-\infty}^{\infty} f(v) \sin sv \, dv = 0$$

$$\therefore \quad B(s) = \frac{1}{\pi} \, . \, 0 = 0$$

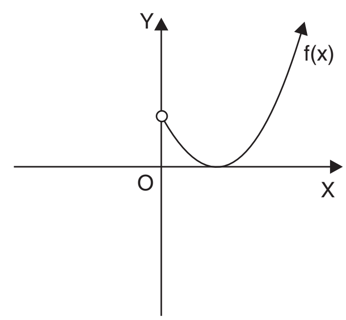
$$\therefore \quad (1) \Rightarrow \quad f(x) = \int_0^{\infty} \left(\left(\frac{2}{\pi} \int_0^{\infty} f(v) \cos sv \, dv \right) \cos sx + 0 \, . \sin sx \right) ds$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(v) \cos sv \, dv \right) \cos sx \, ds$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_C(s) \cos sx \, ds.$$

The function $\frac{2}{\pi} \int_0^{\infty} \bar{f}_C(s) \cos sx \, ds$ *i.e.*, $f(x)$ is called the **inverse Fourier cosine transform** of the function $\bar{f}_C(s)$ and we write

$$F_C^{-1}(\bar{f}_C(s)) = F_C^{-1}(F_C(f(x)) = f(x).$$



Important Note. If a function is defined on the interval $(0, \infty)$ then it can be defined on the interval $(-\infty, 0)$ so that the extension of the function under consideration is an even (resp. odd) function. Let $f(x)$ be a function defined on $(0, \infty)$ with graph as shown in the figure. The function shown in the figure I and figure II are respectively the even extension of $f(x)$ and the odd extension of $f(x)$.

$\therefore$ Any function defined on $(0, \infty)$ can be considered to be an even function as well as an odd function.

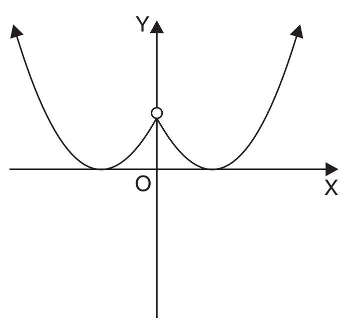


Figure I. Even extension of f(x)

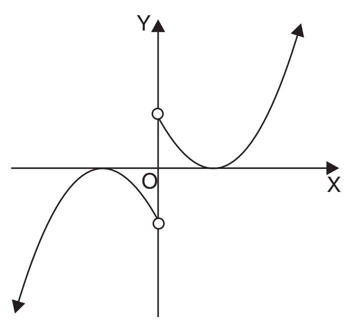


Figure II. Odd extension of f(x)

$\therefore$ If a function is defined on $(0, \infty)$ then the formulae regarding inverse Fourier sine transform and inverse Fourier cosine transform are applicable.

Remark. We know that

$$\int e^{ax} \sin (bx + c) \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin (bx + c)) - (b \cos (bx + c)) + C$$

and

$$\int e^{ax} \cos (bx + c) \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos (bx + c) + b \sin (bx + c)) + C.$$

$\therefore \quad \displaystyle\int_0^{\infty} e^{-ax} \sin bx\, dx = \frac{e^{-ax}}{a^2+b^2}(-a\sin bx - b\cos bx)\Big|_0^{\infty}$

$$= \lim_{x\to\infty} \frac{1}{e^{ax}} \cdot \frac{-(a\sin bx + b\cos bx)}{a^2+b^2} - \frac{1}{a^2+b^2}(-a \,.\, 0 - b \,.\, 1)$$

$$= 0 + \frac{b}{a^2+b^2} = \mathbf{\frac{b}{a^2+b^2}}.$$

Similarly, $\displaystyle\int_0^{\infty} e^{-ax} \cos bx\, dx = \mathbf{\frac{a}{a^2+b^2}}.$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. (*i*) *The Fourier sine transform of the function f(x) is given by*

$$\bar{f}_S(s) = F_S(f(x)) = \int_0^{\infty} f(x) \sin sx\, dx.$$

(*ii*) *The Fourier cosine transform of the function f(x) is given by*

$$\bar{f}_C(s) = F_C(f(x)) = \int_0^{\infty} f(x) \cos sx\, dx.$$

Rule II. (*i*) *The inverse Fourier sine transform of the function g(s) is given by*

$$F_S^{-1}(g(s)) = \frac{2}{\pi}\int_0^{\infty} g(s) \sin sx\, ds.$$

(*ii*) *The inverse Fourier cosine transform of the function g(s) is given by*

$$F_C^{-1}(g(s)) = \frac{2}{\pi}\int_0^{\infty} g(s) \cos sx\, ds.$$

Rule III. (*i*) *If f(x) is either an odd function of x or defined on (0, ∞), then*

$$F_S^{-1}(\bar{f}_S(s)) = f(x).$$

(*ii*) *If f(x) is either an even function of x or defined on (0, ∞), then*

$$F_C^{-1}(\bar{f}_C(s)) = f(x).$$

TYPE I. Problems Based on Fourier Sine and Cosine Transforms

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier sine and cosine transforms of the function e^{-ax}, $a > 0$.*

Sol. Let $f(x) = e^{-ax}$, $a > 0$.

$\therefore \quad \displaystyle F_S(f(x)) = \int_0^{\infty} f(x) \sin sx\, dx$

$$= \int_0^{\infty} e^{-ax} \sin sx\, dx = \mathbf{\frac{s}{a^2+s^2}}. \qquad \left(\text{Using } \int_0^{\infty} e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}\right)$$

Also, $\displaystyle F_C(f(x)) = \int_0^{\infty} f(x) \cos sx\, dx$

$$= \int_0^{\infty} e^{-ax} \cos sx\, dx = \mathbf{\frac{a}{a^2+s^2}}. \qquad \left(\text{Using } \int_0^{\infty} e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}\right)$$

Example 2. *Find the Fourier sine transform and Fourier cosine transform of the function $2e^{-5x} + 5e^{-2x}$.*

Sol. Let $f(x) = 2e^{-5x} + 5e^{-2x}$.

$$\therefore \quad \bar{f}_S(s) = \int_0^\infty f(x)\sin sx\,dx = \int_0^\infty (2e^{-5x} + 5e^{-2x})\sin sx\,dx$$

$$= 2\int_0^\infty e^{-5x}\sin sx\,dx + 5\int_0^\infty e^{-2x}\sin sx\,dx$$

$$= 2\,.\,\frac{s}{(5)^2 + s^2} + 5\,.\,\frac{s}{(2)^2 + s^2} \qquad \left(\text{Using } \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2 + b^2}\right)$$

$$= \mathbf{\frac{2s}{25 + s^2} + \frac{5s}{4 + s^2}}.$$

$$\bar{f}_C(s) = \int_0^\infty f(x)\cos sx\,dx = \int_0^\infty (2e^{-5x} + 5e^{-2x})\cos sx\,dx$$

$$= 2\int_0^\infty e^{-5x}\cos sx\,dx + 5\int_0^\infty e^{-2x}\cos sx\,dx$$

$$= 2\,.\,\frac{5}{(5)^2 + s^2} + 5\,.\,\frac{2}{(2)^2 + s^2} \qquad \left(\text{Using } \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2 + b^2}\right)$$

$$= \mathbf{10\left(\frac{1}{25 + s^2} + \frac{1}{4 + s^2}\right)}.$$

Example 3. *Find the Fourier sine transform and Fourier cosine transform of the function*

$$f(x) = \begin{cases} \sin x\,, & \text{if } 0 < x < a \\ 0\,, & \text{if } \quad x > a. \end{cases}$$

Sol. Fourier sine transform of $f(x)$

$$= \int_0^\infty f(x)\sin sx\,dx = \int_0^a f(x)\sin sx\,dx + \int_a^\infty f(x)\sin sx\,dx$$

$$= \int_0^a \sin x \sin sx\,dx + \int_a^\infty 0\,.\sin sx\,dx$$

$$= \frac{1}{2}\int_0^a 2\sin sx \sin x\,dx + 0 = \frac{1}{2}\int_0^a (\cos (s-1)x - \cos (s+1)x\,dx$$

$$= \frac{1}{2}\left[\frac{\sin (s-1)x}{s-1} - \frac{\sin (s+1)x}{s+1}\right]\Bigg|_0^a$$

$$= \mathbf{\frac{1}{2}\left[\frac{\sin (s-1)a}{s-1} - \frac{\sin (s+1)a}{s+1}\right]}.$$

Fourier cosine transform of $f(x)$

$$= \int_0^\infty f(x)\cos sx\,dx$$

$$= \int_0^a f(x)\cos sx\,dx + \int_a^\infty f(x)\cos sx\,dx$$

$$= \int_0^a \sin x \cos sx\, dx + \int_a^\infty 0 \,.\cos sx\, dx$$

$$= \frac{1}{2}\int_0^a 2\cos sx \sin x\, dx + 0 = \frac{1}{2}\int_0^a (\sin (s+1)x - \sin (s-1)x)\, dx$$

$$= \frac{1}{2}\left[-\frac{\cos (s+1)x}{s+1} + \frac{\cos (s-1)x}{s-1}\right]\Bigg|_0^a$$

$$= \frac{1}{2}\left[-\frac{\cos (s+1)a}{s+1} + \frac{\cos (s-1)a}{s-1} + \frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \mathbf{\frac{1}{2}\left[\frac{\cos (s-1)a}{s-1} - \frac{\cos (s+1)a}{s+1}\right] - \frac{1}{s^2-1}}.$$

Example 4. *Find the Fourier sine and cosine transforms of the function x^{m-1}, $m > 0$.*

Sol. Let $f(x) = x^{m-1}$, $m > 0$.

$$\therefore \quad \bar{f}_S(s) = \int_0^\infty f(x) \sin sx\, dx = \int_0^\infty x^{m-1} \sin sx\, dx \qquad ...(1)$$

and

$$\bar{f}_C(s) = \int_0^\infty f(x) \cos sx\, dx = \int_0^\infty x^{m-1} \cos sx\, dx \qquad ....(2)$$

Let $g(s) = \int_0^\infty x^{m-1} e^{-isx}\, dx$.

$y = isx \Rightarrow dy = is\, dx$

$$\therefore \quad g(s) = \int_0^\infty \left(\frac{y}{is}\right)^{m-1} e^{-y} \frac{dy}{is} = \frac{1}{(is)^m}\int_0^\infty e^{-y} y^{m-1}\, dy = \frac{1}{(is)^m}\,\Gamma(m)$$

$$= \frac{1}{s^m (e^{i\pi/2})^m}\,\Gamma(m) = \frac{\Gamma(m)}{s^m} e^{-im\pi/2} \qquad \left(\because\ e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i\right)$$

$$\therefore \quad \int_0^\infty x^{m-1} e^{-isx}\, dx = \frac{\Gamma(m)}{s^m} e^{-im\pi/2}$$

$$\Rightarrow \quad \int_0^\infty x^{m-1}(\cos(-sx) + i\sin(-sx))\, dx = \frac{\Gamma(m)}{s^m}\left(\cos\left(\frac{-m\pi}{2}\right) + i\sin\left(\frac{-m\pi}{2}\right)\right)$$

$$\Rightarrow \quad \int_0^\infty x^{m-1}\cos sx\, dx - i\int_0^\infty x^{m-1}\sin sx\, dx = \frac{\Gamma(m)}{s^m}\cos\frac{m\pi}{2} - i\frac{\Gamma(m)}{s^m}\sin\frac{m\pi}{2}$$

Comparing real and imaginary parts, we get

$$\int_0^\infty x^{m-1}\cos sx\, dx = \frac{\Gamma(m)}{s^m}\cos\frac{m\pi}{2} \quad \text{and} \quad \int_0^\infty x^{m-1}\sin sx\, dx = \frac{\Gamma(m)}{s^m}\sin\frac{m\pi}{2}.$$

$$\therefore \quad (1) \Rightarrow \quad \bar{f}_S(s) = \mathbf{\frac{\Gamma(m)}{s^m}\sin\frac{m\pi}{2}} \quad \text{and} \quad (2) \Rightarrow \quad \bar{f}_C(s) = \mathbf{\frac{\Gamma(m)}{s^m}\cos\frac{m\pi}{2}}.$$

Example 5. *Find the Fourier cosine transform of the function* e^{-x^2}.

Sol. $$F_C\left(e^{-x^2}\right) = \int_0^\infty e^{-x^2} \cos sx \, dx = \text{I, say}$$

$$\therefore \quad I = \int_0^\infty e^{-x^2} \cos sx \, dx \qquad ...(1)$$

Differentiating w.r.t. s, using Leibnitz's rule of differentiation under integral sign, we have

$$\frac{dI}{ds} = \int_0^\infty e^{-x^2} (-x \sin sx) \, dx$$

$$\therefore \quad \frac{dI}{ds} = \frac{1}{2}\int_0^\infty \underset{\text{I}}{(\sin sx)} \underset{\text{II}}{(-2x e^{-x^2})} \, dx$$

$$= \frac{1}{2}\left[\sin sx \,.\, e^{-x^2} - \int s \cos sx \,.\, e^{-x^2} \, dx\right]_0^\infty$$

$$= \frac{1}{2}(0 - 0) - \frac{s}{2}\int_0^\infty e^{-x^2} \cos sx \, dx = -\frac{s}{2} I$$

$$\Rightarrow \quad \frac{dI}{ds} + \frac{s}{2} I = 0 \quad \Rightarrow \quad \frac{dI}{I} = -\frac{s}{2} ds \quad \Rightarrow \quad \log I = -\frac{s^2}{4} + \log c.$$

$$\Rightarrow \quad \log \frac{I}{c} = -\frac{s^2}{4} \quad \Rightarrow \quad \frac{I}{c} = e^{-s^2/4}$$

$$\therefore \quad I = ce^{-s^2/4} \qquad ...(2)$$

Putting $s = 0$ in (1) and (2), we get $\int_0^\infty e^{-x^2} \cos 0 \, dx = ce^0$

$$\Rightarrow \quad \int_0^\infty e^{-x^2} dx = c \quad \Rightarrow \quad \frac{\sqrt{\pi}}{2} = c \qquad \left(\because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right)$$

$$\therefore \ (2) \Rightarrow \quad I = \frac{\sqrt{\pi}}{2} e^{-s^2/4} \quad \textit{i.e.,} \quad F_C(e^{-x^2}) = \boldsymbol{\frac{\sqrt{\pi}}{2} e^{-s^2/4}}.$$

Example 6. *Find the Fourier cosine transform of the function* $\dfrac{1}{1+x^2}$. *Hence deduce the Fourier sine transform of the function* $\dfrac{x}{1+x^2}$.

Sol. $$F_C\left(\frac{1}{1+x^2}\right) = \int_0^\infty \frac{1}{1+x^2} \cos sx \, dx = \text{I, say}$$

$$\therefore \quad I = \int_0^\infty \frac{\cos sx}{1+x^2} dx \qquad ...(1)$$

Differentiating w.r.t. s, using the Leibnitz's rule of differentiation under integral sign, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{-x \sin sx}{1+x^2}\, dx \qquad ...(2)$$

We have
$$-\frac{x}{1+x^2} = -\frac{x^2}{x(1+x^2)} = -\frac{(1+x^2)-1}{x(1+x^2)} = -\frac{1}{x} + \frac{1}{x\,(1+x^2)}$$

$\therefore \quad (2) \quad \Rightarrow$
$$\frac{dI}{ds} = \int_0^\infty \left(-\frac{1}{x} + \frac{1}{x(1+x^2)}\right) \sin sx\, dx$$

$$= -\int_0^\infty \frac{\sin sx}{x}\, dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\, dx$$

$\therefore$
$$\frac{dI}{ds} = -\frac{\pi^*}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\, dx \qquad ...(3)$$

Differentiating again w.r.t. s, we get

$$\frac{d^2I}{ds^2} = 0 + \int_0^\infty \frac{x \cos sx}{x(1+x^2)}\, dx = \int_0^\infty \frac{\cos sx}{1+x^2}\, dx = I$$

$\Rightarrow \qquad (D^2 - 1)I = 0, \quad \text{where} \quad D \equiv \frac{d}{ds}$

$$D^2 - 1 = 0 \quad \Rightarrow \quad D = \pm 1$$

$\therefore \qquad I = c_1\, e^{1.s} + c_2\, e^{-1.s} = c_1 e^s + c_2\, e^{-s}$

$\Rightarrow \qquad \dfrac{dI}{ds} = c_1 e^s - c_2 e^{-s}$

Putting $s = 0$ in (1), we get

$$c_1 e^0 + c_2 e^0 = \int_0^\infty \frac{\cos 0}{1+x^2}\, dx = \int_0^\infty \frac{1}{1+x^2}\, dx = \tan^{-1} x \Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\therefore \qquad c_1 + c_2 = \dfrac{\pi}{2} \qquad ...(4)$

Putting $s = 0$ in (3), we get

$$c_1 e^0 - c_2 e^0 = -\frac{\pi}{2} + \int_0^\infty \frac{\sin 0}{x(1+x^2)}\, dx = -\frac{\pi}{2} + \int_0^\infty 0\, dx = -\frac{\pi}{2}$$

$\therefore \qquad c_1 - c_2 = -\dfrac{\pi}{2} \qquad ...(5)$

Solving (4) and (5), we get $\quad c_1 = 0 \quad$ and $\quad c_2 = \dfrac{\pi}{2}$.

$\therefore \qquad I = 0.\, e^s + \dfrac{\pi}{2}\, e^{-s} = \dfrac{\pi}{2} e^{-s}$

---

*__Why this step.__ We know that $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

Let $\qquad y = sx,\ s > 0.$

$\therefore \qquad \displaystyle\int_0^\infty \frac{\sin sx}{x}\, dx = \int_0^\infty \frac{\sin y}{y/s} \frac{dy}{s} = \int_0^\infty \frac{\sin y}{y}\, dy = \frac{\pi}{2}.$

$\therefore$ $$F_C\left(\frac{1}{1+x^2}\right) = \frac{\pi}{\mathbf{2}}\mathbf{e^{-s}}.$$

Also $$F_S\left(\frac{x}{1+x^2}\right) = \int_0^\infty \frac{x}{1+x^2}\sin sx\,dx = -\frac{dI}{ds} \qquad \text{(Using (2))}$$

$$= -(c_1e^s - c_2e^{-s}) = -\left(0.e^s - \frac{\pi}{2}e^{-s}\right) = \frac{\pi}{\mathbf{2}}\mathbf{e^{-s}}.$$

Example 7. *Find the Fourier sine transform of the function* $\dfrac{1}{x(x^2+a^2)}$.

Sol. $$F_S\left(\frac{1}{x(x^2+a^2)}\right) = \int_0^\infty \frac{1}{x(x^2+a^2)}\sin sx\,dx = I, \text{ say}$$

$\therefore$ $$I = \int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\,dx \qquad \text{...(1)}$$

Differentiating w.r.t. *s*, using Leibnitz's rule of differentiation under integral sign, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{x\cos sx}{x(x^2+a^2)}\,dx = \int_0^\infty \frac{\cos sx}{x^2+a^2}\,dx \qquad \text{...(2)}$$

Differentiating again w.r.t. *s*, we get

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{-x\sin sx}{x^2+a^2}\,dx$$

We have $$-\frac{x}{x^2+a^2} = -\frac{x^2}{x(x^2+a^2)} = -\frac{(x^2+a^2)-a^2}{x(x^2+a^2)} = -\frac{1}{x} + \frac{a^2}{x(x^2+a^2)}$$

$\therefore$ $$\frac{d^2I}{ds^2} = \int_0^\infty \left(-\frac{1}{x} + \frac{a^2}{x(x^2+a^2)}\right)\sin sx\,dx$$

$$= -\int_0^\infty \frac{\sin sx}{x}\,dx + a^2\int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\,dx = -\frac{\pi}{2} + a^2 I$$

$\therefore$ $$\frac{d^2I}{ds^2} - a^2 I = -\frac{\pi}{2} \qquad \left(\text{Using } \int_0^\infty \frac{\sin \lambda x}{x}\,dx = \frac{\pi}{2}, \text{ if } \lambda > 0\right)$$

$\Rightarrow$ $$(D^2 - a^2)I = -\frac{\pi}{2}$$

$$D^2 - a^2 = 0 \quad \Rightarrow \quad D = \pm a \quad \therefore \quad \text{C.F.} = c_1e^{as} + c_2\,e^{-as}$$

$$\text{P.I.} = \frac{1}{D^2-a^2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}\cdot\frac{1}{D^2-a^2}e^{0s} = -\frac{\pi}{2}\cdot\frac{1}{0^2-a^2}e^{0s} = \frac{\pi}{2a^2}$$

$\therefore$ $$I = \text{C.F.} + \text{P.I.} = c_1e^{as} + c_2e^{-as} + \frac{\pi}{2a^2}$$

$\Rightarrow$ $$\frac{dI}{ds} = ac_1e^{as} - ac_2e^{-as}$$

Putting $s = 0$ in (1), we get

$$\int_0^\infty \frac{\sin 0}{x(x^2+a^2)}\,dx = c_1e^0 + c_2e^0 + \frac{\pi}{2a^2}.$$

$$\Rightarrow \qquad 0 = c_1 + c_2 + \frac{\pi}{2a^2} \quad \Rightarrow \quad c_1 + c_2 = -\frac{\pi}{2a^2} \qquad ...(3)$$

Putting $s = 0$ in (2), we get

$$\int_0^{\infty} \frac{\cos 0}{x^2 + a^2}\, dx = ac_1 e^0 - ac_2 e^0.$$

$$\Rightarrow \qquad \frac{1}{a} \tan^{-1} \frac{x}{a} \Big|_0^{\infty} = ac_1 - ac_2 \quad \Rightarrow \quad \frac{1}{a}\left(\frac{\pi}{2} - 0\right) = a(c_1 - c_2)$$

$$\Rightarrow \qquad c_1 - c_2 = \frac{\pi}{2a^2} \qquad ...(4)$$

Solving (3) and (4), we get $c_1 = 0$ and $c_2 = -\dfrac{\pi}{2a^2}$.

$$\therefore \qquad I = 0.e^{as} + \left(-\frac{\pi}{2a^2}\right) e^{-as} + \frac{\pi}{2a^2} = \frac{\pi}{2a^2}(1 - e^{-as})$$

$$\therefore \qquad F_S\left(\frac{1}{x(x^2 + a^2)}\right) = \boldsymbol{\frac{\pi}{2a^2}(1 - e^{-as}).}$$

TEST YOUR KNOWLEDGE

Find the Fourier sine and cosine transforms of the following functions (Q. 1– 6) :

1. $f(x) = e^{-7x}$

2. $f(x) = 3e^{-2x} + 8e^{-11x}$

3. $f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & \text{if } \quad x > 1 \end{cases}$

4. $f(x) = \begin{cases} 0, & \text{if } 0 < x < a \\ x, & \text{if } a \le x \le b \\ 0, & \text{if } \quad x > b \end{cases}$

5. $f(x) = \begin{cases} x, & \text{if } 0 < x < 1 \\ 2 - x, & \text{if } 1 < x < 2 \\ 0, & \text{if } \quad x > 2 \end{cases}$

6. $f(x) = \begin{cases} \cos x, & \text{if } 0 < x < a \\ 0, & \text{if } \quad x > a \end{cases}$.

7. Find the Fourier sine transform of the function $\dfrac{1}{x}$.

8. Find the Fourier sine transform of the function $\dfrac{e^{-ax}}{x}$, $a > 0$.

9. If $\bar{f}_S(s)$ and $\bar{f}_C(s)$ be the Fourier sine and cosine transforms of the function $f(x)$ respectively, then show that :

(*i*) $F_S(f(x) \sin ax) = \dfrac{1}{2}[\bar{f}_C(s - a) - \bar{f}_C(s + a)]$ (*ii*) $F_S(f(x) \cos ax) = \dfrac{1}{2}[\bar{f}_S(s + a) + \bar{f}_S(s - a)]$

(*iii*) $F_C(f(x) \sin ax) = \dfrac{1}{2}[\bar{f}_S(s + a) - \bar{f}_S(s - a)]$ (*iv*) $F_C(f(x) \cos ax) = \dfrac{1}{2}[\bar{f}_C(s + a) + \bar{f}_C(s - a)]$.

Answers

1. $\bar{f}_S(s) = \dfrac{s}{49 + s^2}, \ \bar{f}_C(s) = \dfrac{7}{49 + s^2}$

2. $\bar{f}_S(s) = s\left(\dfrac{3}{4+s^2} + \dfrac{8}{121+s^2}\right)$, $\bar{f}_C(s) = 2\left(\dfrac{3}{4+s^2} + \dfrac{44}{121+s^2}\right)$

3. $\bar{f}_S(s) = \dfrac{1-\cos s}{s}$, $\bar{f}_C(s) = \dfrac{\sin s}{s}$

4. $\bar{f}_S(s) = \dfrac{1}{s}\,(a\cos as - b\cos bs) + \dfrac{1}{s^2}\,(\sin bs - \sin as)$,

$\bar{f}_C(s) = \dfrac{1}{s}\,(b\sin bs - a\sin as) + \dfrac{1}{s^2}(\cos bs - \cos as)$

5. $\bar{f}_S(s) = \dfrac{2\sin s(1-\cos s)}{s^2}$, $\bar{f}_C(s) = \dfrac{1}{s^2}\,(2\cos s - \cos 2s - 1)$

6. $\bar{f}_S(s) = \dfrac{s}{s^2-1} - \dfrac{1}{2}\left[\dfrac{\cos(s+1)a}{s+1} + \dfrac{\cos(s-1)a}{s-1}\right]$, $\bar{f}_C(s) = \dfrac{1}{2}\left[\dfrac{\sin(s+1)a}{s+1} + \dfrac{\sin(s-1)a}{s-1}\right]$

7. $\dfrac{\pi}{2}$ **8.** $\tan^{-1}\dfrac{s}{a}$.

Hint

9. (*i*) $F_S(f(x)\sin ax) = \displaystyle\int_0^\infty (f(x)\sin ax)\sin sx\,dx$

$$= \frac{1}{2}\int_0^\infty f(x)\,(2\sin sx\sin ax)\,dx$$

$$= \frac{1}{2}\int_0^\infty f(x)\,(\cos(s-a)x - \cos(s+a)x)\,dx$$

$$= \frac{1}{2}\left[\int_0^\infty f(x)\cos(s-a)x\,dx - \int_0^\infty f(x)\cos(s+a)x\,dx\right]$$

$$= \frac{1}{2}\left[\bar{f}_C(s-a) - \bar{f}_C(s+a)\right].$$

TYPE II. Problems Based on Inverse Fourier Sine and Cosine Transforms

ILLUSTRATIVE EXAMPLES

Example 1. *Find the function whose Fourier cosine transform is* $\dfrac{\sin as}{s}$, $a > 0$.

Sol. Let $\dfrac{\sin as}{s}$ be the Fourier cosine transform of $f(x)$, $x > 0$.

$$\therefore \quad f(x) = F_C^{-1}\left(\frac{\sin as}{s}\right) = \frac{2}{\pi}\int_0^\infty \frac{\sin as}{s}\cos sx\,ds$$

$$= \frac{1}{\pi}\int_0^\infty \frac{2\sin as\cos sx}{s}\,ds = \frac{1}{\pi}\int_0^\infty \frac{\sin s(a+x) + \sin s(a-x)}{s}\,ds$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin s(a+x)}{s}\,ds + \frac{1}{\pi}\int_0^\infty \frac{\sin s(a-x)}{s}\,ds$$

$$= \begin{cases} \dfrac{1}{\pi}\cdot\dfrac{\pi}{2}+\dfrac{1}{\pi}\cdot\dfrac{\pi}{2} & \text{if } a-x>0 \\ \dfrac{1}{\pi}\cdot\dfrac{\pi}{2}+\dfrac{1}{\pi}\cdot\left(-\dfrac{\pi}{2}\right) & \text{if } a-x<0 \end{cases} \qquad (\because \quad x>0, a>0 \quad \Rightarrow \quad x+a>0)$$

$$= \begin{cases} 1, & \text{if } x<a \\ 0, & \text{if } x>a \end{cases}$$

$$\therefore \qquad f(x) = \begin{cases} \mathbf{1}, & \mathbf{if\ 0<x<a} \\ \mathbf{0}, & \mathbf{if\ \quad x>a} \end{cases}. \qquad (\because \quad x>0)$$

Remark. If $\lambda < 0$, then

$$\int_0^\infty \frac{\sin \lambda x}{x}\, dx = \int_0^\infty \frac{-\sin(-\lambda)x}{x}\, dx = -\int_0^\infty \frac{\sin(-\lambda)x}{x}\, dx = -\frac{\pi}{2}.$$

Example 2. *Find the function f(x) if* $F_C(f(x)) = \begin{cases} \dfrac{1}{2\pi}\left(a-\dfrac{s}{2}\right), & \text{if } 0<s<2a \\ 0, & \text{if } \quad s \ge 2a \end{cases}.$

Sol. We have $\quad F_C(f(x)) = \begin{cases} \dfrac{1}{2\pi}\left(a-\dfrac{s}{2}\right), & \text{if } 0<s<2a \\ 0, & \text{if } \quad s \ge 2a \end{cases}.$

$$\therefore \qquad f(x) = F_C^{-1}(F_C(f(x))) = \frac{2}{\pi}\int_0^\infty F_C(f(x)) \cos sx\, ds$$

$$= \frac{2}{\pi}\left[\int_0^{2a} F_C(f(x))\cos sx\, ds + \int_{2a}^\infty F_C(f(x))\cos sx\, ds\right]$$

$$= \frac{2}{\pi}\left[\int_0^{2a} \frac{1}{2\pi}\left(a-\frac{s}{2}\right)\cos sx\, ds + \int_{2a}^\infty 0\,.\cos sx\, ds\right]$$

$$= \frac{1}{\pi^2}\int_0^{2a}\left(a-\frac{s}{2}\right)\cos sx\, ds + 0$$

$$= \frac{1}{\pi^2}\left[\left(a-\frac{s}{2}\right)\frac{\sin sx}{x} - \int\left(-\frac{1}{2}\right)\frac{\sin sx}{x}\, ds\right]\Bigg|_0^{2a}$$

$$= \frac{1}{\pi^2}\left[\frac{2a-s}{2x}\sin sx - \frac{\cos sx}{2x^2}\right]_0^{2a} = \frac{1}{\pi^2}\left[0 - \frac{\cos 2ax}{2x^2} - 0 + \frac{\cos 0}{2x^2}\right]$$

$$= \frac{1}{2\pi^2 x^2}(1-\cos 2ax) = \frac{\mathbf{\sin^2 ax}}{\mathbf{\pi^2 x^2}}.$$

Example 3. *Find the inverse Fourier sine transform of the function e^{-as}/s, $a > 0$, $s > 0$. Hence deduce the value of $F_S^{-1}(1/s)$.*

Sol. $$F_S^{-1}\left(\frac{e^{-as}}{s}\right) = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s} \sin sx\, ds = f(x), \text{ say}$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s} \sin sx\, ds \qquad \text{...(1)}$$

Differentiating w.r.t. x, we get

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s} . s \cos sx\, ds = \frac{2}{\pi}\int_0^\infty e^{-as} \cos sx\, ds$$

$$= \frac{2}{\pi} \cdot \frac{a}{a^2 + x^2} \qquad \left(\because \int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}\right)$$

Integrating w.r.t. x, we get

$$f(x) = \frac{2a}{\pi} \cdot \frac{1}{a} \tan^{-1} \frac{x}{a} + c = \frac{2}{\pi} \tan^{-1} \frac{x}{a} + c \qquad \text{...(2)}$$

Putting $x = 0$ in (1) and (2) and equating, we get

$$\frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s} \sin 0\, ds = \frac{2}{\pi} \tan^{-1} \frac{0}{a} + c$$

$$\Rightarrow \quad \frac{2}{\pi} . 0 = 0 + c \quad \Rightarrow \quad c = 0$$

$$\therefore \quad (2) \quad \Rightarrow \quad f(x) = \frac{2}{\pi} \tan^{-1} \frac{x}{a}$$

$$\therefore \quad F_S^{-1}\left(\frac{e^{-as}}{s}\right) = \mathbf{\frac{2}{\pi} \tan^{-1} \frac{x}{a}}.$$

Let $a = 0$.

$$\therefore \quad F_S^{-1}\left(\frac{e^{-0}}{s}\right) = \frac{2}{\pi} \tan^{-1} \frac{x}{0} \quad \text{or} \quad F_S^{-1}\left(\frac{1}{s}\right) = \frac{2}{\pi} \cdot \frac{\pi}{2} = \mathbf{1}.$$

Example 4. *Find the inverse Fourier cosine transform of the function $\frac{1}{1+s^2}$.*

Sol. $$F_C^{-1}\left(\frac{1}{1+s^2}\right) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+s^2} \cos sx\, ds = f(x), \text{ say}$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos sx}{1+s^2}\, ds \qquad \text{...(1)}$$

Differentiating w.r.t. x, we get

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \frac{-s\sin sx}{1+s^2}\,ds$$

We have $$-\frac{s}{1+s^2} = -\frac{s^2}{s(1+s^2)} = -\frac{(1+s^2)-1}{s(1+s^2)} = -\frac{1}{s} + \frac{1}{s(1+s^2)}$$

$$\therefore \quad \frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \left(-\frac{1}{s} + \frac{1}{s(1+s^2)}\right)\sin sx\,ds$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s}\,ds + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\,ds$$

$$= -\frac{2}{\pi}\cdot\frac{\pi}{2} + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\,ds \qquad \text{(Assuming } x > 0\text{)}$$

$$\therefore \quad \frac{df}{dx} = -1 + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\,ds \qquad ...(2)$$

Differentiating again w.r.t. x, we get

$$\frac{d^2f}{dx^2} = 0 + \frac{2}{\pi}\int_0^\infty \frac{s\cos sx}{s(1+s^2)}\,ds = \frac{2}{\pi}\int_0^\infty \frac{\cos sx}{1+s^2}\,ds = f(x) \qquad \text{(By using (1))}$$

$$\Rightarrow \quad \frac{d^2f}{dx^2} - f(x) = 0 \quad \Rightarrow \quad (D^2 - 1)f = 0$$

$$\therefore \quad f(x) = c_1e^{1.x} + c_2e^{-1.x} = c_1e^x + c_2e^{-x} \qquad ...(3)$$

$$\Rightarrow \quad \frac{df}{dx} = c_1e^x - c_2e^{-x} \qquad ...(4)$$

Putting $x = 0$ in (1) and (3) and equating, we get

$$\frac{2}{\pi}\int_0^\infty \frac{\cos 0}{1+s^2}\,ds = c_1e^0 + c_2e^0$$

$$\Rightarrow \quad \frac{2}{\pi}\tan^{-1}s\Big|_0^\infty = c_1 + c_2 \quad \Rightarrow \quad \frac{2}{\pi}\left(\frac{\pi}{2} - 0\right) = c_1 + c_2$$

$$\Rightarrow \quad c_1 + c_2 = 1 \qquad ...(5)$$

Putting $x = 0$ in (2) and (4) and equating, we get

$$-1 + \frac{2}{\pi}\int_0^\infty \frac{\sin 0}{s(1+s^2)}\,ds = c_1e^0 - c_2e^0$$

$$\Rightarrow \quad -1 + \frac{2}{\pi}\,.\,0 = c_1 - c_2 \quad \Rightarrow \quad c_1 - c_2 = -1 \qquad ...(6)$$

Solving (5) and (6), we get $c_1 = 0$, $c_2 = 1$

$\therefore \qquad f(x) = 0.e^x + 1 \,.\, e^{-x} = e^{-x}$

$$\therefore \qquad F_C^{-1}\left(\frac{1}{1+s^2}\right) = \mathbf{e^{-x}, \; x > 0.}$$

TEST YOUR KNOWLEDGE

1. Find the inverse Fourier sine and cosine transforms of the function $e^{-\pi s}$.
2. Find the function $F_C^{-1}\left(\dfrac{\sin 7s}{s}\right)$.
3. Find the function $f(x)$ if $F_C(f(x)) = \begin{cases} \dfrac{4-s}{4\pi}, & \text{if } 0 < s < 4 \\ 0, & \text{if } \quad s \geq 4. \end{cases}$
4. Find the inverse Fourier sine transform of the function e^{-7s}/s, $s > 0$.
5. Find the inverse Fourier sine transform of the function $\dfrac{s}{1+s^2}$.

Answers

1. $F_S^{-1}(e^{-\pi s}) = \dfrac{2x}{\pi(\pi^2 + x^2)}$, $F_C^{-1}(e^{-\pi s}) = \dfrac{2}{\pi^2 + x^2}$
2. 1 if $0 < x < 7$ and 0 if $x > 7$
3. $\dfrac{\sin^2 2x}{\pi^2 x^2}$ **4.** $\dfrac{2}{\pi}\tan^{-1}\dfrac{x}{7}$ **5.** e^{-x}.

TYPE III. Problems Based on Integrals and Integral Equations

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the* **Laplace integral** $\displaystyle\int_0^\infty \frac{s \sin sx}{a^2 + s^2}\, ds$ *, x > 0, a > 0 is equal to* $\dfrac{\pi}{2} e^{-ax}$.

Sol. Let $\qquad f(x) = e^{-ax}, \quad x > 0, a > 0.$ (Note this step)

$$\therefore \qquad F_S(f(x)) = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \frac{s}{a^2 + s^2} \qquad \left(\text{Using } \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}\right)$$

$$\therefore \qquad f(x) = F_S^{-1}\left(\frac{s}{a^2 + s^2}\right) \qquad \text{(Note that } f(x) \text{ is defined only for } x > 0)$$

$$\Rightarrow \qquad e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{s}{a^2 + s^2} \sin sx \, ds$$

$$\Rightarrow \qquad \int_0^\infty \frac{s \sin sx}{a^2 + s^2}\, ds = \frac{\pi}{2} \mathbf{e^{-ax}, \; x > 0, \; a > 0.}$$

Example 2. *Show that :*

$$\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin x\lambda \, d\lambda = \begin{cases} \pi/2, & \text{if } 0 < x < \pi \\ 0, & \text{if } \quad x > \pi \end{cases}.$$

Sol. Let
$$f(x) = \begin{cases} \pi/2, & \text{if } 0 < x < \pi \\ 0, & \text{if } \quad x > \pi \end{cases}.$$

$\therefore$
$$F_S(f(x)) = \int_0^\infty f(x) \sin sx \, dx$$

$$= \int_0^\pi f(x) \sin sx \, dx + \int_\pi^\infty f(x) \sin sx \, dx$$

$$= \int_0^\pi \frac{\pi}{2} \sin sx \, dx + \int_\pi^\infty 0 \,.\, \sin sx \, dx$$

$$= \frac{\pi}{2}\left(-\frac{\cos sx}{s}\right)\Bigg|_0^\pi + 0$$

$$= \frac{\pi}{2s} \, (-\cos \pi s + \cos 0) = \frac{\pi(1-\cos \pi s)}{2s}$$

$\therefore$
$$f(x) = F_S^{-1}(F_S(f(x))) = F_S^{-1}\left(\frac{\pi(1-\cos \pi s)}{2s}\right) = \frac{2}{\pi}\int_0^\infty \frac{\pi(1-\cos \pi s)}{2s} \sin sx \, ds$$

$$= \int_0^\infty \frac{1-\cos \pi s}{s} \sin sx \, ds = \int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin \lambda x \, d\lambda$$

(Changing s by λ)

$\therefore$
$$\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin \lambda x \, d\lambda = f(x) = \begin{cases} \boldsymbol{\pi/2}, & \textbf{if } \quad \mathbf{0 < x < \pi} \\ \mathbf{0}, & \textbf{if } \quad \mathbf{x > \pi} \end{cases}.$$

Example 3. *Using the Fourier sine transform of the function $e^{-ax} - e^{-bx}$, $a, b > 0, x > 0$, show that :*

$$\int_0^\infty \frac{u \sin ux}{(u^2 + a^2)(u^2 + b^2)} \, du = \frac{\pi(e^{-ax} - e^{-bx})}{2(b^2 - a^2)}, \; x > 0.$$

Sol. Let
$$f(x) = e^{-ax} - e^{-bx}, \; x > 0.$$

$\therefore$
$$\bar{f}_S(s) = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty (e^{-ax} - e^{-bx}) \sin sx \, dx$$

$$= \int_0^\infty e^{-ax} \sin sx \, dx - \int_0^\infty e^{-bx} \sin sx \, dx$$

$$= \frac{s}{a^2 + s^2} - \frac{s}{b^2 + s^2} \qquad \left(\text{Using } \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}\right)$$

$$= \frac{s(b^2 + s^2 - a^2 - s^2)}{(a^2 + s^2)(b^2 + s^2)} = \frac{s(b^2 - a^2)}{(a^2 + s^2)(b^2 + s^2)}$$

$$\therefore \qquad \bar{f}_S(s) = \frac{s(b^2 - a^2)}{(a^2 + s^2)(b^2 + s^2)}$$

Since the function $f(x)$ is defined only for $(0, \infty)$, we have $F_S^{-1}(\bar{f}_S(s)) = f(x)$.

$$\therefore \qquad \frac{2}{\pi}\int_0^\infty \bar{f}_S(s) \sin sx \, ds = f(x)$$

$$\Rightarrow \qquad \frac{2}{\pi}\int_0^\infty \frac{s(b^2 - a^2)}{(a^2 + s^2)(b^2 + s^2)} \sin sx \, ds = f(x)$$

$$\Rightarrow \qquad \frac{2(b^2 - a^2)}{\pi}\int_0^\infty \frac{s \sin sx}{(a^2 + s^2)(b^2 + s^2)} \, ds = e^{-ax} - e^{-bx}$$

$$\Rightarrow \qquad \int_0^\infty \frac{s \sin sx}{(a^2 + s^2)(b^2 + s^2)} \, ds = \frac{\pi(e^{-ax} - e^{-bx})}{2(b^2 - a^2)}$$

$$\therefore \qquad \int_0^\infty \frac{u \sin ux}{(a^2 + u^2)(b^2 + u^2)} \, du = \boldsymbol{\frac{\pi(e^{-ax} - e^{-bx})}{2(b^2 - a^2)}, \; x > 0.} \qquad \text{(Replacing } s \text{ by } u\text{)}$$

Example 4. *Using the Fourier cosine transform of* $e^{-x} \cos x$, $x > 0$, *show that* $\int_0^\infty \frac{(u^2 + 2) \cos ux}{u^4 + 4} \, du = \frac{\pi}{2} e^{-x} \cos x$, $x > 0$.

Sol. Let $\qquad f(x) = e^{-x} \cos x, \quad x > 0.$

$$\therefore \qquad \bar{f}_C(s) = \int_0^\infty f(x) \cos sx \, dx = \int_0^\infty e^{-x} \cos x \cos sx \, dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x} (2 \cos sx \cos x) \, dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x} (\cos (s + 1)x + \cos (s - 1)x) \, dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x} \cos (s + 1)x \, dx + \frac{1}{2}\int_0^\infty e^{-x} \cos (s - 1)x \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{1 + (s + 1)^2} + \frac{1}{2} \cdot \frac{1}{1 + (s - 1)^2}$$

$$\left(\text{Using } \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{s^2+2s+2} + \frac{1}{s^2-2s+2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-2s+2+s^2+2s+2}{(s^2+2)^2-4s^2}\right] = \frac{1}{2}\left[\frac{2s^2+4}{s^4+4}\right] = \frac{s^2+2}{s^4+4}$$

$\therefore$ $$\bar{f}_C(s) = \frac{s^2+2}{s^4+4}$$

Since the function $f(x)$ is defined only for $(0, \infty)$, we have $F_C^{-1}(\bar{f}_C(s)) = f(x)$.

$\therefore$ $$\frac{2}{\pi}\int_0^\infty \bar{f}_C(s) \cos sx \, ds = f(x)$$

$\Rightarrow$ $$\frac{2}{\pi}\int_0^\infty \frac{s^2+2}{s^4+4} \cos sx \, ds = f(x)$$

$\Rightarrow$ $$\int_0^\infty \frac{(s^2+2)\cos sx}{s^4+4} ds = \frac{\pi}{2} \,.\, e^{-x} \cos x$$

$\therefore$ $$\int_0^\infty \frac{(u^2+2)\cos ux}{u^4+4} du = \frac{\pi}{\mathbf{2}} \; \boldsymbol{e^{-x} \cos x, \; x > 0.}$$ (Replacing s by u)

Example 5. *Solve the integral equation :*

$$\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}.$$

Sol. We have $$\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}.$$

$\Rightarrow$ $$\int_0^\infty f(x) \cos sx \, dx = e^{-s}$$

$\Rightarrow$ $$F_C(f(x)) = e^{-s}$$

$\therefore$ $$f(x) = F_C^{-1}(e^{-s}) = \frac{2}{\pi}\int_0^\pi e^{-s} \cos sx \, ds$$

$$= \frac{2}{\pi}\frac{1}{1+x^2} \qquad \left(\because \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}\right)$$

$\therefore$ $$f(x) = \frac{\mathbf{2}}{\boldsymbol{\pi(1+x^2)}}.$$

Example 6. *Solve the integral equation :*

$$\int_0^\infty f(x) \sin sx \, dx = \begin{cases} 1, & \text{if } 0 \le s < 1 \\ 2, & \text{if } 1 \le s < 2 \\ 0, & \text{if } s > 2 \end{cases}.$$

Sol. We have $\displaystyle\int_0^\infty f(x)\sin sx\,dx = \begin{cases} 1, & \text{if } 0 \le s < 1 \\ 2, & \text{if } 1 \le s < 2 \\ 0, & \text{if } s > 2 \end{cases}.$

$$\Rightarrow \qquad F_S(f(x)) = \begin{cases} 1, & \text{if } 0 \le s < 1 \\ 2, & \text{if } 1 \le s < 2 \\ 0, & \text{if } s > 2 \end{cases}$$

$$\therefore \qquad f(x) = F_S^{-1}(F_S(f(x))) = \frac{2}{\pi}\int_0^\infty F_S(f(x))\sin sx\,ds$$

$$= \frac{2}{\pi}\left[\int_0^1 1.\sin sx\,ds + \int_1^2 2.\sin sx\,ds + \int_2^\infty 0.\sin sx\,ds\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos sx}{x}\Bigg|_0^1 + \frac{2(-\cos sx)}{x}\Bigg|_1^2 + 0\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{x} + \frac{\cos x}{x} - \frac{2\cos 2x}{x}\right] = \boldsymbol{\frac{2}{\pi x}(1 + \cos x - 2\cos 2x).}$$

TEST YOUR KNOWLEDGE

1. Show that the *Laplace integral* $\displaystyle\int_0^\infty \frac{\cos sx}{a^2+s^2}\,ds$, $x > 0$, $a > 0$ is equal to $\dfrac{\pi}{2a}\,e^{-ax}$.

2. Show that $\displaystyle\int_0^\infty \frac{\cos\lambda x}{\lambda^2+1}\,d\lambda = \frac{\pi}{2}\,e^{-x}, \quad x \ge 0.$

3. Find the Fourier sine transform of the function $e^{-|x|}$, $x > 0$. Hence show that :

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m},\ m > 0.$$

4. Using the Fourier cosine transform of the function $f(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{if } x > 1 \end{cases}$, evaluate the integral

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda.$$

5. Solve the integral equation : $\displaystyle\int_0^\infty f(x)\sin\alpha x\,dx = \begin{cases} 1-\alpha, & \text{if } 0 \le \alpha \le 1 \\ 0, & \text{if } \alpha > 1 \end{cases}.$

6. Solve the integral equation : $\displaystyle\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda, & \text{if } 0 \le \lambda \le 1 \\ 0, & \text{if } \lambda > 1 \end{cases}.$

Hence deduce that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}.$

Answers

3. $\dfrac{s}{1+s^2}$ **4.** $\dfrac{\pi}{2}$ if $0 < x < 1$ and 0 if $x > 1$ **5.** $f(x) = \dfrac{2(x - \sin x)}{\pi x^2}$

6. $f(x) = \dfrac{2(1-\cos x)}{\pi x^2}$.

Hint

6. We have $\displaystyle\int_0^\infty \frac{2(1-\cos x)}{\pi x^2} \cos \lambda x \, dx = \begin{cases} 1-\lambda, & if \quad 0 \le \lambda \le 1 \\ 0, & if \quad \lambda > 1 \end{cases}$.

Taking limits as $\lambda \to 0$, we get $\displaystyle\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2} . 1\, dx = 1$

$$\Rightarrow \qquad \frac{4}{\pi}\int_0^\infty \frac{\sin^2 \frac{x}{2}}{x^2}\, dx = 1$$

$$x/2 = t \quad \Rightarrow \quad dx = 2\, dt$$

$$\therefore \qquad \frac{4}{\pi}\int_0^\infty \frac{\sin^2 t}{4t^2}\, 2\, dt = 1.$$

5.8. LINEARITY OF TRANSFORMS

In this section, we shall prove that the Fourier transform, the Fourier sine and cosine transforms are linear operations.

Theorem I. *If f(x) and g(x) be any functions whose Fourier transforms exist then for any constants a and b then prove that*

$$F(af(x) + bg(x)) = aF(f(x)) + bF(g(x)).$$

Proof. By definition,

$$\begin{aligned} F(af(x) + bg(x)) &= \int_{-\infty}^{\infty} (af(x) + bg(x))e^{-isx}\, dx \\ &= a\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx + b\int_{-\infty}^{\infty} g(x)\, e^{-isx}\, dx \\ &= aF(f(x)) + bF(g(x)). \end{aligned}$$

Theorem II. *If f(x) and g(x) be any functions whose Fourier sine transforms exist then for any constants a and b prove that*

$$F_S(af(x) + bg(x)) = aF_S(f(x)) + bF_S(g(x)).$$

Proof. By definition,

$$\begin{aligned} F_S\, (af(x) + bg(x)) &= \int_0^\infty (af(x) + bg(x)) \sin sx\, dx \\ &= a\int_0^\infty f(x) \sin sx\, dx + b\int_0^\infty g(x) \sin sx\, dx \\ &= aF_S(f(x)) + bF_S(g(x)). \end{aligned}$$

Theorem III. *If f(x) and g(x) be any functions whose Fourier cosine transforms exist then for any constants a and b then prove that*

$$F_C\, (af(x) + bg(x)) = aF_C(f(x)) + bF_C(g(x)).$$

Proof. By definition,

$$\begin{aligned} F_C(af(x) + bg(x)) &= \int_0^{\infty} (af(x) + bg(x)) \cos sx\, dx \\ &= a\int_0^{\infty} f(x) \cos sx\, dx + b\int_0^{\infty} g(x) \cos sx\, dx \\ &= aF_C(f(x)) + bF_C(g(x)). \end{aligned}$$

Remark. On the same lines, we can also prove that the inverse Fourier transform, the inverse Fourier sine and cosine transforms are also linear operations.

5.9. CHANGE OF SCALE PROPERTY OF TRANSFORMS

Theorem I. *If the Fourier transform of the function f(x) is $\bar{f}(s)$ then prove that the Fourier transform of f(ax) is $\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$, where a > 0.*

Proof. We have $\bar{f}(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$.

$\therefore$ $$F(f(ax)) = \int_{-\infty}^{\infty} f(ax)\, e^{-isx}\, dx$$

Let $t = ax$. $\therefore$ $dt = adx$

$\therefore$ $$\begin{aligned} F(f(ax)) &= \int_{-\infty}^{\infty} f(t)\, e^{-is(t/a)} \frac{dt}{a} \qquad (\because \ a > 0) \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(t)\, e^{-i(s/a)t}\, dt = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right). \end{aligned}$$

Theorem II. *If the Fourier sine transform of the function f(x) is $\bar{f}_S(s)$ then prove that the Fourier sine transform of f(ax) is $\frac{1}{a}\bar{f}_S\left(\frac{s}{a}\right)$, where a > 0.*

Proof. We have $\bar{f}_S(s) = F_S(f(x)) = \int_0^{\infty} f(x) \sin sx\, dx$.

$\therefore$ $$F_S(f(ax)) = \int_0^{\infty} f(ax) \sin sx\, dx$$

Let $t = ax$. $\therefore$ $dt = adx$

$\therefore$ $$\begin{aligned} F_S(f(ax)) &= \int_0^{\infty} f(t) \sin (s(t/a)) \frac{dt}{a} \qquad (\because \ a > 0) \\ &= \frac{1}{a}\int_0^{\infty} f(t) \sin ((s/a)t)\, dt = \frac{1}{a}\bar{f}_S\left(\frac{s}{a}\right). \end{aligned}$$

Theorem III. *If the Fourier cosine transform of the function f(x) is $\bar{f}_C(s)$ then prove that the Fourier cosine transform of f(ax) is $\frac{1}{a}\bar{f}_C\left(\frac{s}{a}\right)$, where a > 0.*

Proof. We have $\bar{f}_C(s) = F_C(f(x)) = \int_0^{\infty} f(x) \cos sx\, dx$.

$$\therefore \qquad F_C(f(ax)) = \int_0^\infty f(ax)\cos sx\,dx$$

Let $t = ax$. $\therefore$ $dt = a\,dx$

$$\therefore \qquad F_C(f(ax)) = \int_0^\infty f(t)\cos(s(t/a))\,\frac{dt}{a} \qquad (\because \ a > 0)$$

$$= \frac{1}{a}\int_0^\infty f(t)\cos((s/a)t)\,dt = \frac{1}{a}\bar{f}_C\left(\frac{s}{a}\right).$$

5.10. TRANSFORMS OF DERIVATIVES

In this section, we shall derive the formulae to find the Fourier transform and the Fourier sine and cosine transforms of the derivatives of a function.

Theorem I. *If f(x) is any function for which the Fourier transforms of f(x) and f ′(x) exist and f(x) → 0 as* $|x| \to \infty$ *then prove that*

$$F(f'(x)) = is\,F(f(x)).$$

Proof. By definition,

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-isx}\,dx \qquad \text{...(1)}$$

and

$$F(f'(x)) = \int_{-\infty}^{\infty} f'(x)e^{-isx}\,dx \qquad \text{...(2)}$$

$$(2) \Rightarrow \qquad F(f'(x)) = \int_{-\infty}^{\infty} \underset{\text{I}}{e^{-isx}}\ \underset{\text{II}}{f'(x)}\,dx$$

Integrating by parts, we get

$$F(f'(x)) = e^{-isx} f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(-is)\,e^{-isx} f(x)\,dx$$

$$= 0 - 0 + is\int_{-\infty}^{\infty} f(x)\,e^{-isx}\,dx = is\,F(f(x)).$$

$$(\because \ f(x) \to 0 \text{ as } |x| \to \infty)$$

Remark. By repeated application of the above theorem, we have

$$F(f''(x)) = is\,F(f'(x)) = (is).(is)F(f(x)) = -s^2F(f(x))$$

$\therefore$ $\mathbf{F(f''(x)) = -s^2F(f(x))}$.

By using **P.M.I.,** we can prove that

$$\mathbf{F(f^{(n)}(x)) = (is)^nF(f(x)),\ n \in N.}$$

Theorem II. *If f(x) is any function for which the Fourier sine and cosine transforms of f(x) and f ′(x) exist and f(x) → 0 as x → ∞ then prove that*

(*i*) $F_S(f'(x)) = -sF_C(f(x))$ (*ii*) $F_C(f'(x)) = sF_S(f(x)) - f(0)$.

Proof. By definition $F_S(f(x)) = \int_0^\infty f(x)\sin sx\,dx$

and

$$F_C(f(x)) = \int_0^\infty f(x)\cos sx\,dx.$$

(*i*) $$F_S(f'(x)) = \int_0^\infty f'(x) \sin sx \, dx = \int_0^\infty \underset{\text{I}}{\sin sx} \, . \underset{\text{II}}{f'(x)} \, dx$$

$$= \sin sx \, f(x) \Big|_0^\infty - \int_0^\infty s \cos sx \, f(x) \, dx$$

$$= 0 - 0 - s\int_0^\infty f(x) \cos sx \, dx = - sF_C(f(x)).$$

$(\because \quad f(x) \to \quad 0 \text{ as } x \to \infty)$

(*ii*) $$F_C(f'(x)) = \int_0^\infty f'(x) \cos sx \, dx = \int_0^\infty \underset{\text{I}}{\cos sx} \, . \underset{\text{II}}{f'(x)} \, dx$$

$$= \cos sx \, f(x) \Big|_0^\infty - \int_0^\infty - s \sin sx \, f(x) \, dx$$

$$= 0 - (\cos 0) f(0) + s\int_0^\infty f(x) \sin sx \, dx \quad (\because \quad f(x) \to \quad 0 \text{ as } x \to \infty)$$

$$= - f(0) + sF_S \, (f(x)) = sF_S \, (f(x)) - f(0).$$

Remarks. (*i*) $F_S(f''(x)) = - sF_C(f'(x)) = - s[sF_S(f(x)) - f(0)] = - s^2F_S(f(x)) + sf(0)$.

(*ii*) $F_C(f''(x)) = sF_S(f'(x)) - f'(0) = s[- sF_C \, (f(x))] - f'(0) = - s^2 \, F_C \, (f(x)) - f'(0)$.

$\therefore$ $\mathbf{F_S(f''(x)) = - s^2 \, F_S(f(x)) + sf(0)}$ and $\mathbf{F_C(f''(x)) = - s^2 \, F_C(f(x)) - f'(0)}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier sine and cosine transforms of the function e^{-mx}, $m > 0$ by using its second derivative.*

Sol. Let $$f(x) = e^{-mx} , \; m > 0.$$

$\therefore$ $$f'(x) = - me^{-mx} \quad \text{and} \quad f''(x) = (- m) \, (- m) \, e^{-mx} = m^2 \, e^{-mx}$$

We have $$F_S(f''(x)) = - s^2 \, F_S \, (f(x)) + sf(0).$$

$\therefore$ $$F_S(m^2 \, e^{-mx}) = - s^2 \, F_S(e^{-mx}) + se^0$$

$\Rightarrow$ $$m^2F_S(e^{-mx}) + s^2F_S(e^{-mx}) = s$$

$\Rightarrow$ $$(m^2 + s^2) \, F_S \, (e^{-mx}) = s \quad \Rightarrow \quad F_S(e^{-mx}) = \frac{\mathbf{s}}{\mathbf{m^2 + s^2}}.$$

Also, $$F_C(f''(x)) = - s^2 \, F_C(f(x)) - f'(0).$$

$\Rightarrow$ $$F_C \, (m^2e^{-mx}) = - s^2 \, F_C(e^{-mx}) - (- me^0)$$

$\Rightarrow$ $$m^2F_C(e^{-mx}) + s^2F_C(e^{-mx}) = m$$

$\Rightarrow$ $$(m^2 + s^2) \, F_C(e^{-mx}) = m \quad \Rightarrow \quad F_C(e^{-mx}) = \frac{\mathbf{m}}{\mathbf{m^2 + s^2}}.$$

5.11. PARSEVAL'S IDENTITIES

Parseval's identities are for Fourier transforms and also for Fourier sine and cosine transforms. These identities are proved in the following theorems.

Theorem I. *If $\bar{f}(s)$ and $\bar{g}(s)$ are the Fourier transforms of the functions f(x) and g(x) respectively then prove that*

$$\int_{-\infty}^{\infty} f(x)\, g(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, \overline{\bar{g}(s)}\, ds \quad \textit{and} \quad \int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2\, ds,$$

where $\overline{\bar{g}(s)}$ *represents the complex conjugate of the function* $\bar{g}(s)$.

Proof. $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \bar{f}(s)\,\overline{\bar{g}(s)}\, ds = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\left(\overline{\int_{-\infty}^{\infty} g(x)\, e^{-isx}\, dx}\right) ds$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\left(\int_{-\infty}^{\infty} \overline{g(x)\, e^{-isx}}\, dx\right) ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\left(\int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx\right) ds \quad (\because \; g(x) \text{ is a real valued function})$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\left(\int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds\right) dx$$

(By changing the order of integration)

$$= \int_{-\infty}^{\infty} g(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds\right) dx$$

$$= \int_{-\infty}^{\infty} g(x)\, f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

$$\therefore \quad \int_{-\infty}^{\infty} \mathbf{f(x)\, g(x)\, dx} = \frac{\mathbf{1}}{\mathbf{2\pi}}\int_{-\infty}^{\infty} \mathbf{\bar{f}(s)\, \overline{\bar{g}(s)}\, ds}. \qquad \ldots(1)$$

In particular, let $g(x) = f(x)$.

$$\therefore \quad (1) \Rightarrow \quad \int_{-\infty}^{\infty} f(x)\, f(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, \overline{\bar{f}(s)}\, ds.$$

$$\therefore \quad \int_{-\infty}^{\infty} \mathbf{|f(x)|^2\, dx} = \frac{\mathbf{1}}{\mathbf{2\pi}}\int_{-\infty}^{\infty} \mathbf{|\bar{f}(s)|^2\, ds}. \qquad (\because \; (f(x))^2 = |f(x)|^2)$$

Theorem II. *If* $\bar{f}_S(s)$ *and* $\bar{g}_S(s)$ *are the Fourier sine transforms of the functions f(x) and g(x) respectively then prove that*

$$\int_0^{\infty} f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^{\infty} \bar{f}_S(s)\, \bar{g}_S(s)\, ds \quad \textit{and} \quad \int_0^{\infty} (f(x))^2\, dx = \frac{2}{\pi}\int_0^{\infty} (\bar{f}_S(s))^2\, ds.$$

Proof. $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} \bar{f}_S(s)\, \bar{g}_S(s)\, ds = \dfrac{2}{\pi}\int_0^{\infty} \bar{f}_S(s)\left(\int_0^{\infty} g(x) \sin sx\, dx\right) ds$

$$= \frac{2}{\pi}\int_0^{\infty} g(x)\left(\int_0^{\infty} \bar{f}_S(s) \sin sx\, ds\right) dx$$

(By changing the order of integration)

$$= \int_0^{\infty} g(x)\left(\frac{2}{\pi}\int_0^{\infty} \bar{f}_S(s) \sin sx\, ds\right) dx$$

$$= \int_0^{\infty} g(x)\, f(x)\, dx = \int_0^{\infty} f(x)\, g(x)\, dx$$

$$\therefore \quad \int_0^{\infty} \mathbf{f(x)\, g(x)\, dx} = \frac{\mathbf{2}}{\pi}\int_0^{\infty} \mathbf{\bar{f}_S(s)\, \bar{g}_S(s)\, ds}. \qquad \ldots(1)$$

In particular, let $g(x) = f(x)$.

$$\therefore \quad \int_0^{\infty} \mathbf{(f(x))^2\, dx} = \frac{\mathbf{2}}{\pi}\int_0^{\infty} \mathbf{(\bar{f}_S(s))^2\, ds}.$$

Theorem III. *If $\bar{f}_C(s)$ and $\bar{g}_C(s)$ are the Fourier cosine transforms of the functions f(x) and g(x) respectively then prove that*

$$\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\bar{g}_C(s)\, ds \quad and \quad \int_0^\infty (f(x))^2\, dx = \frac{2}{\pi}\int_0^\infty (\bar{f}_C(s))^2\, ds.$$

Proof. $$\frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\,\bar{g}_C(s)\, ds = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\left(\int_0^\infty g(x)\cos sx\, dx\right) ds$$

$$= \frac{2}{\pi}\int_0^\infty g(x)\left(\int_0^\infty \bar{f}_C(s)\cos sx\, ds\right) dx$$

(By changing the order of integration)

$$= \int_0^\infty g(x)\left(\frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\cos sx\, ds\right) dx$$

$$= \int_0^\infty g(x)\, f(x)\, dx = \int_0^\infty f(x)\, g(x)\, dx$$

$\therefore$ $$\int_0^\infty \mathbf{f(x)\, g(x)\, dx} = \frac{\mathbf{2}}{\pi}\int_0^\infty \mathbf{\bar{f}_C(s)\,\bar{g}_C(s)\, ds}. \quad ...(1)$$

In particular, let $g(x) = f(x)$.

$\therefore$ (1) $\Rightarrow$ $$\int_0^\infty f(x)\, f(x)\, dx = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\,\bar{f}_C(s)\, ds$$

$\therefore$ $$\int_0^\infty \mathbf{(f(x))^2\, dx} = \frac{\mathbf{2}}{\pi}\int_0^\infty \mathbf{(\bar{f}_C(s))^2\, ds}.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. (*i*) $F(f'(x)) = isF(f(x))$

(*ii*) $F(f^{(n)}(x)) = (is)^n F(f(x)),\ n \in \mathbf{N}$

Rule II. (*i*) $F_S(f'(x)) = -\, sF_C(f(x))$

(*ii*) $F_S(f''(x)) = -\, s^2F_S(f(x)) + sF(0)$

Rule III. (*i*) $F_C(f'(x)) = sF_S(f(x)) - f(0)$

(*ii*) $F_C(f''(x)) = -\, s^2F_C(f(x)) - f'(0)$

Rule IV. (*i*) $$\int_{-\infty}^\infty f(x)\, g(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^\infty F(f(x))\,\overline{F(g(x))}\, ds$$

(*ii*) $$\int_{-\infty}^\infty |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^\infty |F(f(x))|^2\, ds$$

Rule V. (*i*) $$\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty F_S(f(x))\, F_S(g(x))\, ds$$

(*ii*) $$\int_0^\infty (f(x))^2\, dx = \frac{2}{\pi}\int_0^\infty (F_S(f(x)))^2\, ds$$

Rule VI. (*i*) $$\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty F_C(f(x))F_C(g(x))\, ds$$

(*ii*) $$\int_0^\infty (f(x))^2\, dx = \frac{2}{\pi}\int_0^\infty (F_C(f(x)))^2\, ds.$$

ILLUSTRATIVE EXAMPLES

Example 1. *If* $f(x) = \begin{cases} 1, & if \quad |x| < a \\ 0, & if \quad |x| > a \end{cases}$, *use the Parseval's identity for Fourier transform of f(x) to show that* $\int_0^\infty \frac{\sin^2 ax}{x^2} dx = \frac{\pi a}{2}$.

Sol. We have $\qquad f(x) = \begin{cases} 1, & \text{if} \quad |x| < a \\ 0, & \text{if} \quad |x| > a \end{cases}$.

$$\therefore \qquad F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-a} 0 \,.\, e^{-isx}\, dx + \int_{-a}^{a} 1 \,.\, e^{-isx}\, dx + \int_{a}^{\infty} 0 \,.\, e^{-isx}\, dx$$

$$= 0 + \left. \frac{e^{-isx}}{-is} \right|_{-a}^{a} + 0$$

$$= -\frac{1}{is}(e^{-isa} - e^{isa}) = \frac{2}{s}\left(\frac{e^{isa} - e^{-isa}}{2i}\right) = \frac{2}{s} \sin sa$$

By Parseval's identity for Fourier transforms, we have

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(f(x))|^2\, ds.$$

$$\Rightarrow \quad \int_{-\infty}^{-a} (0)^2\, dx + \int_{-a}^{a} (1)^2\, dx + \int_{a}^{\infty} (0)^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| \frac{2}{s} \sin sa \right|^2 ds$$

$$\Rightarrow \quad 0 + (a - (-a)) + 0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4 \sin^2 as}{s^2}\, ds$$

$$\Rightarrow \quad \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2}\, dx = 2a \qquad \text{(Replacing } s \text{ by } x\text{)}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2}\, dx = \pi a \quad \Rightarrow \quad 2\int_{0}^{\infty} \frac{\sin^2 ax}{x^2}\, dx = \pi a$$

$$\left(\because \frac{\sin^2 ax}{x^2} \text{ is an even function} \right)$$

$$\therefore \quad \int_{0}^{\infty} \frac{\sin^2 ax}{x^2}\, dx = \frac{\pi \mathbf{a}}{\mathbf{2}}.$$

Example 2. *Using a Parseval's identity, show that :*

$$\int_0^\infty \frac{dx}{(a^2 + x^2)(b^2 + x^2)} = \frac{\pi}{2ab(a+b)}, \quad a, b > 0.$$

Sol. Let $\qquad f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. $\qquad$ (Note this step)

$$\therefore \qquad \bar{f}_C(s) = \int_0^\infty e^{-ax} \cos sx\, dx = \frac{a}{a^2 + s^2}$$

and

$$\bar{g}_C(s) = \int_0^\infty e^{-bx} \cos sx\, dx = \frac{b}{b^2 + s^2}$$

$\therefore$ By Parseval's identity for Fourier cosine transforms, we have

$$\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\bar{g}_C(s)\, ds.$$

$$\Rightarrow \qquad \int_0^\infty e^{-ax}\, e^{-bx}\, dx = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2}\cdot\frac{b}{b^2+s^2}\, ds$$

$$\Rightarrow \qquad \int_0^\infty e^{-(a+b)x}\, dx = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)}$$

$$\Rightarrow \qquad \frac{\pi}{2ab}\cdot\frac{e^{-(a+b)x}}{-(a+b)}\Bigg|_0^\infty = \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} \qquad \text{(Changing } s \text{ by } x\text{)}$$

$$\therefore \qquad \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = -\frac{\pi}{2ab(a+b)}\,(0-1) = \frac{\pi}{\mathbf{2ab(a+b)}}.$$

Example 3. *Using a Parseval's identity, show that :*

$$\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}\, dx = \frac{\pi}{2}\left(\frac{1-e^{-a^2}}{a^2}\right),\ a > 0.$$

Sol. Let $f(x) = e^{-ax},\ x > 0,\ a > 0$

and
$$g(x) = \begin{cases} 1, & \text{if } \quad 0 < x \le a \\ 0, & \text{if } \quad x > a \end{cases}.$$

$$\therefore \qquad \bar{f}_C(s) = \int_0^\infty e^{-ax}\cos sx\, dx = \frac{a}{a^2+s^2}$$

and
$$\bar{g}_C(s) = \int_0^\infty g(x)\cos sx\, dx$$
$$= \int_0^a 1\,.\cos sx\, dx + \int_a^\infty 0\,.\cos sx\, dx$$
$$= \frac{\sin sx}{s}\Bigg|_0^a + 0 = \frac{1}{s}\,(\sin sa - \sin 0) = \frac{\sin as}{s}$$

$\therefore$ By Parseval's identity for Fourier cosine transforms, we have

$$\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\, \bar{g}_C(s)\, ds\,.$$

$$\Rightarrow \qquad \int_0^a f(x)\, g(x)\, dx + \int_a^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2}\,.\,\frac{\sin as}{s}\, ds$$

$$\Rightarrow \qquad \int_0^a e^{-ax}\,.\,1\, dx + \int_a^\infty e^{-ax}\,.\,0\, dx = \frac{2a}{\pi}\int_0^\infty \frac{\sin as}{s(a^2+s^2)}\, ds$$

$$\Rightarrow \qquad \frac{e^{-ax}}{-a}\Bigg|_0^a + 0 = \frac{2a}{\pi}\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}\, dx \qquad \text{(Replacing } s \text{ by } x\text{)}$$

$$\Rightarrow \qquad \frac{1}{-a}\,[e^{-a^2} - 1] = \frac{2a}{\pi}\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}\, dx$$

$$\therefore \qquad \int_0^\infty \frac{\sin ax}{x(a^2+x^2)}\, dx = \frac{\pi}{\mathbf{2}}\left(\frac{\mathbf{1-e^{-a^2}}}{\mathbf{a^2}}\right).$$

TEST YOUR KNOWLEDGE

1. Find the Fourier sine and cosine transforms of the function $e^{-3x/2}$ by using its second derivative.
2. Using a Parseval's identity, show that :

   (*i*) $\displaystyle\int_0^\infty \frac{dx}{(x^2+4)(x^2+9)} = \frac{\pi}{60}$ (*ii*) $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)(x^2+16)} = \frac{\pi}{80}$.
3. Using a Parseval's identity, show that :

   (*i*) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ (*ii*) $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{4}$.
4. Use Parseval's identity for Fourier sine transform of the function $f(x) = \begin{cases} 1, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x > 1 \end{cases}$ to

   show that $\displaystyle\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx = \frac{\pi}{2}$.
5. Use Parseval's identity for Fourier cosine transform of the function $f(x) = \begin{cases} 1, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x > 1 \end{cases}$

   to show that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$.

Answer

1. $\dfrac{4s}{9+4s^2}, \dfrac{6}{9+4s^2}$.

Hint

3. (*ii*) Use Parseval's identity for the Fourier sine transform of the function e^{-x}, $x > 0$.

Solution of Differential Equations Using Fourier Transforms

6

6.1. INTRODUCTION

When the function involved in a physical (or geometrical) problem involves more than one independent variable, a partial differential equation is obtained. The solution of this partial differential equation gives the required function. Fourier transforms are a very important tool in solving such partial differential equations.

6.2. PARTIAL DIFFERENTIAL EQUATION

An equation containing one or more partial derivatives of an unknown function of more than one independent variable is called a **partial differential equation.**

For example, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is a partial differential equation. The **order** of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation. The order of the above partial differential equation is two.

A **solution** of a partial differential equation is a relation between the variables by means of which and the partial derivatives derived there from the given partial differential equation is satisfied.

In the present chapter, we shall confine only to the application of Fourier transforms in solving partial differential equations of the following types :

(*i*) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (**One dimensional heat equation**)

(*ii*) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (**One dimensional wave equation**)

(*iii*) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (**Two dimensional Laplace equation**)

Here c is a constant, t is time, x, y are cartesian coordinates and *dimension* is the number of the coordinates in the equation.

The general methods of solving partial differential equations would be taken up later on.

6.3. METHOD OF SOLVING PARTIAL DIFFERENTIAL EQUATION BY USING FOURIER TRANSFORMS

Let the unknown function in the given partial differential equation be $u(x, t)$. Fourier transform or Fourier sine (or cosine) transform is taken w.r.t. x on both side of the given equation. This equation is simplied by using the following formulae :

1. (*i*) $F(f'(x)) = is F(f(x))$ (Assuming $f(x) \to 0$ as $|x| \to \infty$)

(*ii*) $F(f''(x)) = -s^2 F(f(x))$ (Assuming $f(x), f'(x) \to 0$ as $|x| \to \infty$)

2. (*i*) $F_S(f'(x)) = -sF_C(f(x))$ (Assuming $f(x) \to 0$ as $x \to \infty$)

(*ii*) $F_S(f''(x)) = -s^2 F_S(f(x)) + sf(0)$ (Assuming $f(x), f'(x) \to 0$ as $x \to \infty$)

3. (*i*) $F_C(f'(x)) = sF_S(f(x)) - f(0)$ (Assuming $f(x) \to 0$ as $x \to \infty$)

(*ii*) $F_C(f''(x)) = -s^2 F_C(f(x)) - f'(0)$ (Assuming $f(x), f'(x) \to 0$ as $x \to \infty$)

Formula **2**(*ii*) is used if $f(0)$ is given and formula **3**(*ii*) is used if $f'(0)$ is given.

In applying the above formulae to the function $u(x, t)$, the requirement of $u(x, t) \to 0$ or $u(x, t) \to 0$, $u_x(x, t) \to 0$ both as $x \to \infty$ or $|x| \to \infty$ are generally given with the problem. If such requirements are not given then these are assumed to have been given with the problem.

Remarks. 1. If $u(0, t)$ is given then generally Fourier sine transform w.r.t. x of the given equation is taken.

2. If $u_x(0, t)$ is given then generally Fourier cosine transform w.r.t. x of the given equation is taken.

3. The solution of the equation $\frac{dy}{dx} = ky$ is $y = ce^{kx}$. ($\because$ $D - k = 0 \Rightarrow D = k$)

4. The solution of the equation $\frac{dy}{dx} + Py = Q$ is $y(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$, where I.F. $= e^{\int P\,dx}$.

ILLUSTRATIVE EXAMPLES

Example 1. *If the flow of heat is linear so that the variation of* θ*(temperature) w.r.t. y and z may be neglected and if it is assumed that no heat is generated in the medium, then solve the one dimensional heat equation* $\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2}$, *where* $-\infty < x < \infty$ *and* $\theta = f(x)$ *when* $t = 0$, $f(x)$ *being a given function of x.*

Sol. The given equation is

$$\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2} \qquad ...(1)$$

The given condition is :

$$\theta(x, 0) = f(x), \quad -\infty < x < \infty.$$

Taking Fourier transform of (1) w.r.t. x, we get

$$F\left(\frac{\partial \theta}{\partial t}\right) = F\left(k\frac{\partial^2 \theta}{\partial x^2}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial \theta}{\partial t} e^{-isx}\,dx = kF\left(\frac{\partial^2 \theta}{\partial x^2}\right)$$

$$\Rightarrow \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \theta\, e^{-isx}\,dx = k\,[-s^2 F(\theta)] \qquad (\text{Using } F(f''(x)) = -s^2 F(f(x)))$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}(\bar{\theta}) + ks^2\bar{\theta} = 0 \qquad ...(2)$$

The solution of (2) is $\bar{\theta} = ce^{-ks^2 t}$ ...(3)

Putting $t = 0$ in (3), we get $\bar{\theta}(s, 0) = ce^0 = c$.

$$\Rightarrow \quad \int_{-\infty}^{\infty} \theta(x, 0)\, e^{-isx}\, dx = c \quad \Rightarrow \quad \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = c \quad \Rightarrow \quad \bar{f}(s) = c$$

$$\therefore \qquad c = \bar{f}(s)$$

$$\therefore \quad (3) \quad \Rightarrow \qquad \bar{\theta} = \bar{f}(s)\, e^{-ks^2 t}$$

$$\therefore \qquad \theta(x, t) = \mathrm{F}^{-1}(\bar{\theta}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\theta}\, e^{isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{-ks^2 t}\, e^{isx}\, ds = \frac{\mathbf{1}}{\mathbf{2\pi}}\int_{-\infty}^{\infty} \bar{\mathbf{f}}(\mathbf{s})\, \mathbf{e}^{-(\mathbf{ks^2 t} - \mathbf{isx})}\, \mathbf{ds}.$$

Example 2. *If the initial temperature of an infinite bar is given by*

$$\theta(x) = \begin{cases} \theta_0, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases},$$

determine the temperature at any point x and at any instant t.

Sol. To find the temperature $\theta(x, t)$, the required heat-flow equation is

$$\frac{\partial \theta}{\partial t} = c^2 \frac{\partial^2 \theta}{\partial x^2}, \ t > 0 \qquad ...(1)$$

The given condition is

$$\theta(x, 0) = \begin{cases} \theta_0, & \text{for } |x| < a \\ 0, & \text{for } |x| > a. \end{cases}$$

Taking Fourier transform of (1) w.r.t. x, we get

$$\int_{-\infty}^{\infty} \frac{\partial \theta}{\partial t}\, e^{-isx}\, dx = c^2 \mathrm{F}\left(\frac{\partial^2 \theta}{\partial x^2}\right)$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \theta\, e^{-isx}\, dx = c^2(-s^2\bar{\theta}) \qquad (\text{Using } \mathrm{F}(f''(x)) = -s^2\mathrm{F}(f(x)))$$

$$\Rightarrow \qquad \frac{\partial \bar{\theta}}{\partial t} + c^2 s^2 \bar{\theta} = 0 \qquad ...(2)$$

The solution of (2) is $\bar{\theta} = \mathrm{K}e^{-c^2 s^2 t}$ ...(3)

Putting $t = 0$ in (3), we get $\bar{\theta}(s, 0) = \mathrm{K}e^0$ or $\bar{\theta}(s, 0) = \mathrm{K}$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \theta(x, 0)\, e^{-isx}\, dx = \mathrm{K}$$

$$\Rightarrow \qquad \int_{-\infty}^{-a} 0 \,.\, e^{-isx}\, dx + \int_{-a}^{a} \theta_0 e^{-isx}\, dx + \int_{a}^{\infty} 0 .\, e^{-isx}\, dx = \mathrm{K}$$

$$\Rightarrow \qquad 0 + \theta_0 \left. \frac{e^{-isx}}{-is} \right|_{-a}^{a} + 0 = \mathrm{K}$$

$$\Rightarrow \qquad -\frac{\theta_0}{is}(e^{-isa} - e^{isa}) = \mathrm{K} \quad \Rightarrow \quad \frac{2\theta_0}{s}\,\frac{e^{ias} - e^{-ias}}{2i} = \mathrm{K} \quad \Rightarrow \quad \mathrm{K} = \frac{2\theta_0}{s}\sin as$$

$\therefore$ (3) $\Rightarrow$ $\bar{\theta} = \dfrac{2\theta_0 \sin as}{s} e^{-c^2s^2t}$

$$\therefore \quad \theta(x, t) = F^{-1}(\bar{\theta}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\theta}\, e^{isx} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\theta_0 \sin as}{s} e^{-c^2s^2t} e^{isx}\, ds$$

$$= \frac{\theta_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-c^2s^2t} (\cos sx + i \sin sx)\, ds$$

$$= \frac{\theta_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-c^2s^2t} \cos sx\, ds + \frac{i\theta_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-c^2s^2t} \sin sx\, ds$$

$$= \frac{2\theta_0}{\pi} \int_0^{\infty} \frac{\sin as}{s} e^{-c^2s^2t} \cos sx\, ds + 0$$

$$= \frac{\theta_0}{\pi} \int_0^{\infty} \frac{e^{-c^2s^2t}}{s} (2 \sin as \cos sx)\, ds$$

$$\therefore \quad \theta(x, t) = \boldsymbol{\frac{\theta_0}{\pi} \int_0^{\infty} \frac{e^{-c^2s^2t}}{s} (\sin (a + x)s + \sin (a - x)s)\, ds.}$$

Example 3. *An infinite string is initially at rest and that the initial displacement is f(x), (– ∞ < x < ∞). Determine the displacement y(x, t) of the string.*

Sol. The equation for the vibration of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The given conditions are :

(*i*) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ ($\because$ String is initially at rest)

(*ii*) $y(x, 0) = f(x), -\infty < x < \infty$ ($\because$ Initial displacement is $f(x)$)

Taking Fourier transform of (1) w.r.t. x, we get

$$F\left(\frac{\partial^2 y}{\partial t^2}\right) = F\left(c^2 \frac{\partial^2 y}{\partial x^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2} (F(y)) = c^2 F\left(\frac{\partial^2 y}{\partial x^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2} (F(y)) = c^2[- s^2 F(y)] \qquad \text{(Using } F(f''(x)) = - s^2 F(f(x))\text{)}$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2} \bar{y} + c^2s^2 \bar{y} = 0 \qquad ...(2)$$

The roots of A.E. of (2) are $\pm csi$.

$\therefore$ The solution of (1) is

$$\bar{y} = c_1 \cos cst + c_2 \sin cst \qquad ...(3)$$

We have $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0.$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \left(\frac{\partial y}{\partial t}\right)_{t=0} e^{-isx}\, dx = 0 \quad \Rightarrow \quad \left(\int_{-\infty}^{\infty} \frac{\partial y}{\partial t} e^{-isx}\, dx\right)\Bigg|_{t=0} = 0$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial t}\int_{-\infty}^{\infty} y\, e^{-isx}\, dx\right)\Bigg|_{t=0} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}(\bar{y})\Bigg|_{t=0} = 0$$

$$\therefore \quad (3) \quad \Rightarrow \quad (-c_1\, cs \sin cst + c_2\, cs \cos cst)\Bigg|_{t=0} = 0$$

$$\Rightarrow \quad -c_1\, cs \sin 0 + c_2\, cs \cos 0 = 0 \quad \Rightarrow \quad 0 + c_2\, cs = 0 \quad \Rightarrow \quad c_2 = 0$$

$$\therefore \quad (3) \quad \Rightarrow \quad \bar{y} = c_1 \cos cst \qquad ...(4)$$

Putting $t = 0$ in (4), we get

$$\bar{y}(s, 0) = c_1 \cos(cs \,.\, 0) = c_1$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} y(x, 0)\, e^{-isx}\, dx = c_1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = c_1 \quad \Rightarrow \quad \bar{f}(s) = c_1$$

$$\therefore \quad c_1 = \bar{f}(s)$$

$$\therefore \quad (4) \quad \Rightarrow \quad \bar{y} = \bar{f}(s) \cos cst$$

$$\therefore \quad y = \mathrm{F}^{-1}(\bar{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{y}\, e^{isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (\bar{f}(s) \cos cst)\, e^{isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\left(\frac{e^{icst} + e^{-icst}}{2}\right) e^{isx}\, ds$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} (\bar{f}(s)\, e^{is(x+ct)} + \bar{f}(s)\, e^{is(x-ct)})\, ds$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{is(x+ct)}\, ds + \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{is(x-ct)}\, ds\right]$$

$$\therefore \quad y(x, t) = \mathbf{\frac{1}{2}\, [f(x + ct) + f(x - ct)].}$$

Example 4. *Let u(x, t) denote the temperature at distance x and time t. Solve the heat-flow equation* $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$*,* $x \geq 0$*,* $t \geq 0$ *under the boundary condition* $u = u_0$ *when* $x = 0$*,* $t > 0$ *and the initial condition* $u = 0$ *when* $x > 0$*,* $t = 0$*.*

Sol. We have $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Taking Fourier sine transform of (1) w.r.t. x, we get

$$F_S\left(\frac{\partial u}{\partial t}\right) = F_S\left(k\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \quad \int_0^\infty \frac{\partial u}{\partial t} \sin sx\, dx = k\, F_S\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\int_0^\infty u \sin sx\, dx = k\,(-s^2\, F_S(u) + s.u(0, t))$$

(Using $F_S(f''(x)) = -s^2F_S(f(x)) + sf(0)$)

$$\Rightarrow \quad \frac{\partial}{\partial t}(\overline{u}_S) = -ks^2\overline{u}_S + ksu_0 \qquad (\because \quad u(0, t) = u_0)$$

$$\therefore \quad \frac{\partial}{\partial t}\overline{u}_S + ks^2\overline{u}_S = ksu_0 \qquad ...(2)$$

This is a linear differential equation.

Here $\qquad$ I.F. $= e^{\int ks^2 dt} = e^{ks^2 t}$

$\therefore$ The solution of (2) is

$$\overline{u}_S\, e^{ks^2t} = \int ksu_0 e^{ks^2t}\, dt + c$$

$$\Rightarrow \quad \overline{u}_S\, e^{ks^2t} = ksu_0\,\frac{e^{ks^2t}}{ks^2} + c$$

$$\Rightarrow \quad \overline{u}_S\, e^{ks^2t} = \frac{u_0}{s}\, e^{ks^2t} + c$$

$$\Rightarrow \quad \overline{u}_S = \frac{u_0}{s} + ce^{-ks^2t} \qquad ...(3)$$

Putting $t = 0$ in (3), we get

$$\overline{u}_S(s, 0) = \frac{u_0}{s} + ce^0$$

$$\Rightarrow \quad \overline{u}_S(s, 0) = \frac{u_0}{s} + c$$

$$\Rightarrow \quad \int_0^\infty u(x, 0) \sin sx\, dx = \frac{u_0}{s} + c$$

$$\Rightarrow \quad \int_0^\pi 0\,.\sin sx\, dx = \frac{u_0}{s} + c \qquad (\because \quad u(x, 0) = 0)$$

$$\Rightarrow \quad 0 = \frac{u_0}{s} + c \quad \Rightarrow \quad c = -\frac{u_0}{s}$$

$$\therefore \quad (3) \quad \Rightarrow \quad \overline{u}_S = \frac{u_0}{s} - \frac{u_0}{s}\, e^{-ks^2t} = \frac{u_0}{s}(1 - e^{-ks^2t})$$

$$\therefore \quad u = F_S^{-1}(\overline{u}_S) = \frac{2}{\pi}\int_0^\infty \overline{u}_S \sin sx\, ds$$

$$\therefore \quad u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{u_0}{s}(1 - e^{-ks^2t}) \sin sx\, ds$$

$$= \frac{2u_0}{\pi}\left[\int_0^\infty \frac{\sin sx}{s}\,ds - \int_0^\infty \frac{1-e^{-ks^2t}}{s}\sin sx\,ds\right]$$

$$= \frac{2u_0}{\pi}\left[\frac{\pi}{2} - \int_0^\infty \frac{1-e^{-ks^2t}}{s}\sin sx\,ds\right] \qquad \left(\because \int_0^\infty \frac{\sin sx}{s}\,ds = \frac{\pi}{2} \text{ if } x > 0\right)$$

$\therefore$
$$u(x, t) = \mathbf{u_0}\left[\mathbf{1} - \frac{\mathbf{2}}{\pi}\int_{\mathbf{0}}^\infty \frac{\mathbf{1-e^{-ks^2t}}}{\mathbf{s}}\,\mathbf{\sin sx\,ds}\right].$$

Example 5. *The temperature u in a semi-infinite rod is given by the heat-flow equation*

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2},\ 0 \le x < \infty$$

subject to the conditions :

(*i*) $u = 0$ *when* $t = 0,\ x \ge 0$

(*ii*) $\dfrac{\partial u}{\partial x} = -\mu$ *(a constant) when* $x = 0$

(*iii*) $\dfrac{\partial u}{\partial x} \to 0$ *as* $x \to \infty$.

Making use of the Fourier cosine transform, show that the temperature u(x, t) is given by

$$\frac{2\mu}{\pi}\int_0^\infty \frac{1-e^{-s^2c^2t}}{s^2}\cos sx\,ds.$$

Sol. We have
$$\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Taking Fourier cosine transform of (1) w.r.t. x, we get

$$F_C\left(\frac{\partial u}{\partial t}\right) = F_C\left(c^2\,\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \quad \int_0^\infty \frac{\partial u}{\partial t}\cos sx\,dx = c^2 F_C\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\int_0^\infty u\cos sx\,dx = c^2\left(-s^2F_C(u) - \left(\frac{\partial u}{\partial x}\text{ at } x = 0\right)\right)$$

(Using $F_C(f''(x)) = -s^2 F_C(f(x)) - (f'(0))$)

$$\Rightarrow \quad \frac{\partial}{\partial t}(\overline{u}_C) = -c^2s^2\,\overline{u}_C - c^2(-\mu)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\overline{u}_C) + s^2c^2\,\overline{u}_C = \mu c^2 \qquad ...(2)$$

This is a linear differential equation.

Here
$$\text{I.F.} = e^{\int s^2c^2\,dt} = e^{s^2c^2t}$$

$\therefore$ The solution of (2) is

$$\bar{u}_C\, e^{s^2c^2t} = \int \mu c^2\, e^{s^2c^2t}\, dt + \text{K}$$

$$\Rightarrow \qquad \bar{u}_C\, e^{s^2c^2t} = \mu c^2 \frac{e^{s^2c^2t}}{s^2c^2} + \text{K}$$

$$\Rightarrow \qquad \bar{u}_C = \frac{\mu}{s^2} + \text{K}e^{-s^2c^2t} \qquad \text{...(3)}$$

Putting $t = 0$ in (3), we get $\bar{u}_C(s, 0) = \dfrac{\mu}{s^2} + \text{K}e^0.$

$$\Rightarrow \qquad \bar{u}_C(s, 0) = \frac{\mu}{s^2} + \text{K}$$

$$\Rightarrow \qquad \int_0^\infty u(x, 0)\cos sx\, dx = \frac{\mu}{s^2} + \text{K} \quad \Rightarrow \quad \int_0^\infty 0 \,.\cos sx\, dx = \frac{\mu}{s^2} + \text{K}$$

$$\Rightarrow \qquad 0 = \frac{\mu}{s^2} + \text{K} \quad \Rightarrow \quad \text{K} = -\frac{\mu}{s^2}$$

$$\therefore \quad (3) \quad \Rightarrow \qquad \bar{u}_C = \frac{\mu}{s^2} - \frac{\mu}{s^2} e^{-s^2c^2t} = \frac{\mu}{s^2}(1 - e^{-s^2c^2t})$$

$$\therefore \qquad u = \text{F}_C^{-1}(\bar{u}_C) = \frac{2}{\pi}\int_0^\infty \bar{u}_C \cos sx\, ds$$

$$\therefore \qquad u(s, t) = \frac{2}{\pi}\int_0^\infty \frac{\mu}{s^2}(1 - e^{-s^2c^2t})\cos sx\, ds$$

$$= \boldsymbol{\frac{2\mu}{\pi}\int_0^\infty \frac{1 - e^{-s^2c^2t}}{s^2}\cos sx\, ds.}$$

Example 6. *Use Fourier cosine transform to show that the steady temperature in the semi-infinite solid y > 0 when the temperature on the surface y = 0 is kept at unity over the strip | x | < a and at zero outside the strip is*

$$\frac{1}{\pi}\left[\tan^{-1}\frac{a + x}{y} + \tan^{-1}\frac{a - x}{y}\right].$$

The result $\displaystyle\int_0^\infty \frac{e^{-bx}\sin cx}{x}\, dx = \tan^{-1}\frac{c}{b}$, *b > 0, c > 0 may be used.*

Sol. Let $u(x, y)$ denote the steady temperature at the point (x, y), on the solid under consideration.

The steady temperature $u(x, y)$ satisfies the two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty,\ 0 \le y < \infty \qquad \text{...(1)}$$

subject to the boundary conditions :

(*i*) $u = 1$ when $y = 0, -a < x < a$

(*ii*) $u = 0$ when $y = 0,\ x \le -a$ or $x \ge a$.

Taking Fourier cosine transform of (1) w.r.t. x, we get

$$F_C\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

$$\Rightarrow \qquad F_C\left(\frac{\partial^2 u}{\partial x^2}\right) + F_C\left(\frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\Rightarrow \quad \left[-s^2 F_C(u) - \left(\frac{\partial u}{\partial x}\, at\ x = 0\right)\right] + \frac{\partial^2}{\partial y^2} F_C(u) = 0 \quad (\text{Using } F_C(f''(x)) = -s^2 F_C(f(x)) - f'(0))$$

$$\Rightarrow \qquad -s^2 \bar{u}_C - 0 + \frac{\partial^2}{\partial y^2}(\bar{u}_C) = 0 \qquad \left(\text{Assuming } \frac{\partial u}{\partial x} = 0\ at\ x = 0\right)$$

$$\Rightarrow \qquad \frac{\partial^2}{\partial y^2}(\bar{u}_C) - s^2 \bar{u}_C = 0$$

The solution of this equation is $\bar{u}_C = c_1 e^{sy} + c_2 e^{-sy}$.

$\bar{u}_C$ being the value of a definite integral is a finite quantity.

$\therefore$ $c_1 = 0$ for otherwise $\bar{u}_C$ becomes infinite as $y \to \infty$.

$\therefore$ $\bar{u}_C = c_2\, e^{-sy}$ ...(3)

Putting $y = 0$ in (3), we get

$$\bar{u}_C(x, 0) = c_2 e^0 = c_2$$

$$\Rightarrow \quad \int_0^\infty u(x, 0) \cos sx\, dx = c_2$$

$$\Rightarrow \quad \int_0^a 1 . \cos sx\, dx + \int_a^\infty 0 . \cos sx\, dx = c_2$$

$$\Rightarrow \qquad \left.\frac{\sin sx}{s}\right|_0^a + 0 = c_2 \quad \Rightarrow \quad c_2 = \frac{\sin sa}{s}$$

$$\therefore \quad (3) \quad \Rightarrow \qquad \bar{u}_C = \frac{\sin sa}{s} e^{-sy} \qquad ...(4)$$

$$\therefore \qquad u = F_C^{-1}(\bar{u}_C) = \frac{2}{\pi}\int_0^\infty \bar{u}_C \cos sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin sa}{s} e^{-sy} \cos sx\, ds$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{-sy}}{s}(2 \sin sa \cos sx)\, ds$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{-sy}}{s}(\sin (a + x)s + \sin (a - x)s)\, ds$$

$$= \frac{1}{\pi}\left[\int_0^\infty \frac{e^{-sy} \sin (a + x)s}{s}\, ds + \int_0^\infty \frac{e^{-sy} \sin (a - x)s}{s}\, ds\right]$$

$$\therefore \qquad u(x, y) = \frac{\mathbf{1}}{\pi}\left[\mathbf{\tan^{-1}\frac{a+x}{y} + \tan^{-1}\frac{a-x}{y}}\right].$$

$$\left(\text{Using } \int_0^\infty \frac{e^{-bx}\sin cx}{x}\,dx = \tan^{-1}\frac{c}{b}\right)$$

Example 7. *Solve the equation* $\frac{\partial^4 V}{\partial x^4} + \frac{\partial^2 V}{\partial y^2} = 0$, $-\infty < x < \infty$, $y \geq 0$ *subject to the following conditions :*

(i) *V and its partial derivatives tend to zero as* $|x| \to \infty$

(ii) $V = f(x)$, $\frac{\partial V}{\partial y} = 0$ *when* $y = 0$.

Sol. We have $\frac{\partial^4 V}{\partial x^4} + \frac{\partial^2 V}{\partial y^2} = 0.$ ...(1)

The given conditions are :

(i) V and its partial derivatives tend to zero as $|x| \to \infty$

(ii) $V(x, 0) = f(x)$, $V_y(x, 0) = 0$.

Taking Fourier transform of (1) w.r.t. x, we get

$$F\left(\frac{\partial^4 V}{\partial x^4}\right) + F\left(\frac{\partial^2 V}{\partial y^2}\right) = 0.$$

$$\Rightarrow \qquad (is)^4\, F(V) + \int_{-\infty}^{\infty} \frac{\partial^2 V}{\partial y^2}\, e^{-isx}\, dx = 0$$

$$\Rightarrow \qquad s^4 F(V) + \frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} V\, e^{-isx}\, dx = 0$$

$$\Rightarrow \qquad s^4 F(V) + \frac{\partial^2}{\partial y^2}\, F(V) = 0$$

$$\Rightarrow \qquad \frac{\partial^2}{\partial y^2}(\overline{V}) + s^4\, \overline{V} = 0 \qquad (\text{Putting } F(V) = \overline{V})$$

This is a linear differential equation.

The A.E. is $D^2 + s^4 = 0$. $\therefore$ $D = \pm\, is^2$

$\therefore$ The solution is $\overline{V} = c_1 \cos s^2 y + c_2 \sin s^2 y$ ...(2)

Putting $y = 0$ in (2), we get

$$\overline{V}(s, 0) = c_1 \cos 0 + c_2 \sin 0 = c_1$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} V(x, 0)\, e^{-isx}\, dx = c_1$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = c_1 \quad \Rightarrow \quad \bar{f}(s) = c_1$$

$$\therefore \quad (2) \quad \Rightarrow \quad \overline{V} = \bar{f}(s) \cos s^2 y + c_2 \sin s^2 y$$

$$\Rightarrow \quad \frac{\partial \overline{V}}{\partial y} = -s^2 \bar{f}(s) \sin s^2 y + c_2 s^2 \cos s^2 y \qquad ...(3)$$

Now,
$$\frac{\partial \overline{V}}{\partial y} = \frac{\partial}{\partial y} \int_{-\infty}^{\infty} V\, e^{-isx}\, dx = \int_{-\infty}^{\infty} \frac{\partial V}{\partial y}\, e^{-isx}\, dx$$

$$\therefore \quad \left.\frac{\partial \overline{V}}{\partial y}\right|_{y=0} = \left.\left(\int_{-\infty}^{\infty} \frac{\partial V}{\partial y}\, e^{-isx}\, dx\right)\right|_{y=0} = \int_{-\infty}^{\infty} \left(\left.\frac{\partial V}{\partial y}\right|_{y=0}\right) e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} 0 \,.\, e^{-isx}\, dx = 0$$

$$\therefore \quad \left.\frac{\partial \overline{V}}{\partial y}\right|_{y=0} = 0$$

$$\therefore \quad (3) \quad \Rightarrow \quad -s^2 \bar{f}(s) \sin 0 + c_2 s^2 \cos 0 = 0 \quad \Rightarrow \quad c_2 s^2 = 0 \quad \Rightarrow \quad c_2 = 0$$

$$\therefore \quad \overline{V} = \bar{f}(s) \cos s^2 y \qquad (\because \quad c_2 = 0)$$

$$\therefore \quad V(x, y) = F^{-1}(\overline{V}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{V}\, e^{isx}\, ds$$

$$= \boldsymbol{\frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s) \cos s^2 y\, e^{isx}\, ds.}$$

TEST YOUR KNOWLEDGE

1. Determine the distribution of temperature in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.

2. Solve the equation $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}$ if :

(*i*) $u(0, t) = 0$ (*ii*) $u(x, 0) = e^{-x}$, $x > 0$

(*iii*) $u(x, t)$ is bounded when $x > 0$, $t > 0$.

3. Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$ subject to the conditions :

(*i*) $u(0, t) = 0$ when $t > 0$ (*ii*) $u(x, 0) = \begin{cases} 1, & \text{if } \; 0 < x < 1 \\ 0, & \text{if } \; x \geq 1 \end{cases}$

(*iii*) $u(x, t)$ is bounded.

4. Solve the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ subject to the conditions :

(*i*) $u_x(0, t) = 0$ (*ii*) $u(x, 0) = \begin{cases} x, & \text{if } \; 0 \leq x \leq 1 \\ 0, & \text{if } \; x > 1 \end{cases}$

(*iii*) $u(x, t)$ is bounded when $x > 0$, $t > 0$.

Answers

1. $u(x, t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \bar{f}_S(s) e^{-c^2 s^2 t} \sin sx \, ds$, where $u(x, t)$ represents the temperature

2. $u(x, t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{s}{1+s^2}\, e^{-2s^2 t} \sin sx \, ds$

3. $u(x, t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \dfrac{1-\cos s}{s}\, e^{-s^2 t} \sin sx \, ds$

4. $u(x, t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \left(\dfrac{\sin s}{s} + \dfrac{\cos s - 1}{s^2}\right) e^{-s^2 t} \cos sx \, ds$.

Hint

1. If $u(x, t)$ represents the temperature then we are to solve the one dimensional heat-flow equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \; x > 0, \; t > 0$$

subject to the conditions :

(*i*) $u(0, t) = 0$ (*ii*) $u(x, 0) = f(x)$.

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