Laplace Transforms

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)\, e^{-st}\, dt, \quad s \in \mathbb{R}, \quad t > 0$$

$$\mathcal{L}(1) = \frac{1}{s}, \quad s > 0 \qquad \mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}, \quad s > 0$$

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0 \qquad \mathcal{L}(\cos at) = \frac{s}{s^2 + a^2}, \quad s > 0$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad s > 0 \qquad \mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}, \quad s > |a|$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > a \qquad \mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2}, \quad s > |a|$$

Laplace Transforms

① **Linearity**

$$\mathcal{L}(a f(t) + b g(t)) = a \mathcal{L}(f(t)) + b \mathcal{L}(g(t)).$$

② **First Shifting property**

If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at} f(t)) = F(s-a)$

Ex $\mathcal{L}(e^{at} t^n)$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad s > 0$$

$$\mathcal{L}(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}, \quad s > a$$

Ex $\mathcal{L}(e^{at} \sinh bt)$

$$\mathcal{L}(\sinh bt) = \frac{b}{s^2 - b^2}, \quad s > |b|$$

$$\mathcal{L}(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$$

$s - a > |b|$

Laplace Transforms

Ex $\mathcal{L}(e^{at}\cos bt)$

$$\mathcal{L}(\cos bt) = \frac{s}{s^2+b^2}, \quad s>0$$

$$\mathcal{L}(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}, \quad s>a$$

$$\mathcal{L}(\cos 5t) = \frac{s}{s^2+25}, \quad s>0$$

$$\mathcal{L}(\sin 5t) = \frac{5}{s^2+25}, \quad s>0$$

Ex $\mathcal{L}(e^{-3t}(2\cos 5t - 3\sin 5t))$

$$= 2\,\mathcal{L}(e^{-3t}\cos 5t) - 3\,\mathcal{L}(\sin 5t)$$

$$= 2\cdot\frac{s+3}{(s+3)^2+25} - 3\cdot\frac{5}{(s+3)^2+25}$$

$$s>-3$$

Laplace Transforms

Ex) $\mathcal{L}\left(e^{-t}(3\sinh 2t - 5\cosh 2t)\right)$

Sol

$$= 3\mathcal{L}(e^{-t}\sinh 2t) - 5\mathcal{L}(e^{-t}\cosh 2t)$$

$$\mathcal{L}(\sinh 2t) = \frac{2}{s^2-4}, \quad s>2$$

$$\mathcal{L}(\cosh 2t) = \frac{s}{s^2-4}, \quad s>2$$

$$= 3 \cdot \frac{2}{(s+1)^2-4} - 5 \cdot \frac{s+1}{(s+1)^2-4}, \quad s+1>2$$

$$\boxed{s>1}$$

Laplace Transforms

Ex: If $\mathcal{L}(f(t)) = F(s)$, find $\mathcal{L}(\sinh at\, f(t))$

$$= \mathcal{L}\left(\frac{e^{at} - e^{-at}}{2} f(t)\right)$$

$$= \frac{1}{2}\left[\mathcal{L}(e^{at} f(t)) - \mathcal{L}(e^{-at} f(t))\right]$$

$$= \frac{1}{2}\left[F(s-a) - F(s+a)\right]$$

$$= \frac{1}{2} \left[\mathcal{L} \left(e^{at} f(t) \right) - \right.$$

$$\sinh at = \frac{1}{2} \left[F(s-a) - F(\right.$$

$$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$$

$$\cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)$$

Laplace Transforms

Second shifting property

If $\mathcal{L}(f(t)) = F(s)$, $\mathcal{L}(g(t))$: $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

$$\boxed{\mathcal{L}(g(t)) = e^{-as} F(s)}$$

Ex Find $\mathcal{L}(g(t))$: $g(t) = \begin{cases} \cos(t - \frac{\pi}{2}), & t > \frac{\pi}{2} \\ 0, & t < \frac{\pi}{2} \end{cases}$

$\mathcal{L}(\cos t) = \dfrac{s}{s^2+1}$, $s > 0$

$\mathcal{L}(g(t)) = e^{-\frac{\pi}{2}s} \dfrac{s}{s^2+1}$

Laplace Transforms

Second shifting property

Ex. Find $\mathcal{L}(g(t))$: $g(t) = \begin{cases} (t-2)^3 & , t > 2 \\ 0 & , t < 2 \end{cases}$

$$\mathcal{L}(t^3) = \frac{3!}{s^4}, s > 0$$

$$= \frac{6}{s^4}$$

$$\mathcal{L}(g(t)) = e^{-2s} \cdot \frac{6}{s^4}, s > 0$$

Laplace Transforms

Change of Scale

If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$

Ex If $\mathcal{L}\left(\frac{\sin t}{t}\right) = \tan^{-1}\frac{1}{s}$, find $\mathcal{L}\left(\frac{\sin at}{at}\right)$

$$= \frac{1}{a} \tan^{-1}\frac{1}{\frac{s}{a}} = \frac{1}{a}\tan^{-1}\frac{a}{s}$$

Laplace Transforms

Change of Scale

If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$

Ex) If $\mathcal{L}\left(\frac{\sin t}{t}\right) = \tan^{-1}\frac{1}{s}$, find $\mathcal{L}\left(\frac{\sin at}{t}\right)$

$$\mathcal{L}\left(\frac{\sin at}{at}\right) = \frac{1}{a}\tan^{-1}\frac{a}{s}$$

a) $\mathcal{L}\left(\frac{\sin at}{at}\right) = \tan^{-1}\frac{a}{s}$

$\mathcal{L}\left(\frac{\sin at}{t}\right) = \tan^{-1}\frac{a}{s}$

Laplace Transforms

Laplace Transform of derivatives

If $\mathcal{L}(f(t)) = F(s)$, then

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\mathcal{L}(f''(t)) = s^2 F(s) - sf(0) - f'(0)$$

Ex Use $\mathcal{L}(f'(t))$ to compute $\mathcal{L}(t^{n+1})$

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$f(t) = t^{n+1}$, $f(0) = 0$, $f'(t) = (n+1)t^n$

$$\mathcal{L}((n+1)t^n) = s\mathcal{L}(f(t))$$

$$(n+1)\mathcal{L}(t^n) = s\mathcal{L}(f(t))$$

$$(n+1)\frac{n!}{s^{n+1}} = s\mathcal{L}(f(t))$$

$$\mathcal{L}(f(t)) = \frac{(n+1)!}{s^{n+2}}$$

Laplace Transforms

Laplace Transform of derivatives

Ex: $\mathcal{L}(t\sin bt)$ Use $\mathcal{L}(f''(t))$.

Sol: $f(t) = t\sin bt$ $\qquad f(0) = 0$

$f'(t) = bt\cos bt + \sin bt \cdot 1$ $\qquad f'(0) = 0$

$f''(t) = -b^2 t\sin bt + b\cos bt + b\cos bt$

$\mathcal{L}(f''(t)) = -b^2\mathcal{L}(t\sin bt) + 2b\,\mathcal{L}(\cos bt)$ $\rightarrow$ $\mathcal{L}(t\sin t) = \dfrac{2bs}{(s^2+b^2)^2}$

$\parallel$

$s^2\mathcal{L}(t\sin bt)$

$(s^2+b^2)\mathcal{L}(t\sin bt) = 2b\cdot\dfrac{s}{s^2+b^2}$

Laplace Transforms

Laplace Transform of Integral

If $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{F(s)}{s}$

Ex) If $\mathcal{L}\left(\frac{\sin t}{t}\right) = \tan^{-1}\frac{1}{s}$

find $\mathcal{L}\left(\int_0^t \frac{\sin u}{u}\,du\right)$.

$= \frac{1}{s}\tan^{-1}\frac{1}{s}$