

# Classification of Finite Groups

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# Part I

## Doing

### 1 Introduction

### 2 Preliminaries

To start, let's solidify the notation used in this report.

We shall denote groups and sets with capital letters, like  $G$ ,  $H$ , and elements of those groups with lower case letters, like  $g$ ,  $h$ . Greek letters shall denote mappings, generally  $\phi$ ,  $\psi$ , etc. with  $\iota$  reserved for the identity map, and we will write mappings on the right.

We will use  $\mathbb{N}$  to denote the natural numbers (not including 0),  $\mathbb{Z}$  to denote the integers, and  $\mathbb{R}$  to denote the real numbers.

To denote the cyclic group of order  $n$  we will use  $C_n$ ,  $D_{2n}$  to denote the cyclic group of order  $2n$ ,  $A_n$  to denote the alternating group over  $n$  elements,  $S_n$  to denote the symmetric group over  $n$  elements, and  $Q_8$  to denote the quaternion group. The trivial group,  $\{1\}$  is denoted by  $\mathbf{1}$ .

Let's move on to review some facts and theorems which will be valuable later on, as well as introduce some new concepts and prove some new results!

**Definition 1.** If  $G$  and  $H$  are groups with elements  $g_1, g_2 \in G$ , then a map:

$$\phi : G \rightarrow H$$

is a homomorphism if:

$$(g_1 g_2) \phi = (g_1 \phi)(g_2 \phi)$$

If  $\phi$  is bijective, then we call it an isomorphism, with  $G \cong H$  denoting that  $G$  is isomorphic to  $H$ . And if  $\phi$  is an isomorphism from  $G$  to itself, then we call it an automorphism of  $G$ .

**Lemma 1.** *The set of all automorphisms of a group  $G$  form a group under composition. Indeed, this is called the automorphism group of  $G$ , denoted  $\text{Aut } G$ .*

*Proof.* Let  $A = \text{Aut } G = \{ \phi : G \rightarrow G \mid \phi \text{ is an isomorphism} \}$ , and let  $\phi \in A$ . Denote an element of  $G$  by  $g$ .

We know already that the composition of two isomorphisms is an isomorphism, so  $A$  is closed under composition.

The identity map,  $\iota : g \mapsto g$ , is certainly an automorphism of  $G$  and so  $A \neq \emptyset$ .

Indeed,  $\iota : g \mapsto g$  is the identity of  $A$ , since:

$$g \phi \iota = (g \phi) \iota = g \phi \quad \text{and} \quad g \iota \phi = (g \iota) \phi = g \phi$$

And inverses clearly exist, because automorphisms are bijections, and bijections are invertible. Hence  $A = \text{Aut } G$  is a group.  $\square$

**Lemma 2.** *The automorphism group of  $C_n$  is isomorphic to the multiplicative group of integers mod  $n$ .*

$$\text{i.e. } \text{Aut } C_n \cong (\mathbb{Z}/n\mathbb{Z})^\times$$

*Proof.* Let  $C_n = \langle x \rangle$ . Any automorphism,  $\varphi$  of  $C_n$  has the property:

$$(x^i) \varphi = (x \varphi)^i$$

Hence  $\varphi$  is determined by its effect on a generator,  $x$ , and preserves element order. In particular,  $\varphi$  sends generators to generators. So for  $\varphi$  to be an automorphism, it must send  $x$  to another generator,

say  $x^k$ . An element  $x^k$  generates  $C_n$  if  $x^k$  has order  $n$ , i.e. when  $k$  and  $n$  are co-prime. Denote the automorphism sending  $x$  to  $x^k$  by  $\varphi_k$ .

Let's now investigate how these automorphisms behave. Let  $\varphi_k, \varphi_l \in \text{Aut } C_n$ , and consider:

$$x\varphi_k\varphi_l = (x^k)\varphi_l = (x^k)^l = x^{(kl)} = x\varphi_{kl} \quad \text{mod } n$$

Because multiplication modulo  $n$  is commutative,  $x^{kl} = x^{lk}$ , so  $\text{Aut } C_n$  is abelian.

Now consider  $\theta : \text{Aut } C_n \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$  defined by  $\varphi_k\theta = k$ . We will show  $\theta$  is an isomorphism. Every  $k \in (\mathbb{Z}/n\mathbb{Z})^\times$  is co-prime to  $n$  and so  $x^k$  is a generator of  $C_n$ , hence there is some  $\varphi_k \in \text{Aut } C_n$  such that  $\varphi_k\theta = k$ . So  $\theta$  is surjective. If  $\varphi_k\theta = \varphi_l\theta$  then  $k = l$ , so  $\theta$  is also injective. Finally,  $\theta$  is a homomorphism because:

$$(\varphi_k\varphi_l)\theta = \varphi_{kl}\theta = kl = (\varphi_k\theta)(\varphi_l\theta)$$

So  $\theta : \text{Aut } C_n \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$  is an isomorphism. □

**Definition 2.** A subgroup  $H$  of a group  $G$  is called characteristic if it is fixed by all automorphisms of  $G$ .

i.e. for an automorphism  $\phi$  of  $G$ ,  $H\phi = H$ .

**Lemma 3.** Let  $G$  be a group with normal subgroup  $H$ , and let  $K$  be characteristic in  $H$ . Then  $K$  is a normal subgroup of  $G$ .

*Proof.* Consider the map  $\varphi_g : G \rightarrow G$  defined by  $\varphi_g : x \mapsto g^{-1}xg$  for elements  $x, g \in G$ . We will show that this is an automorphism of  $G$ . For  $x, y \in G$ :

$$x\varphi_g y\varphi_g = (g^{-1}xg)(g^{-1}yg) = g^{-1}(xy)g = (xy)\varphi_g$$

Hence  $\varphi_g$  is a homomorphism. Moreover,  $\varphi_g$  is invertible with inverse  $\varphi_{g^{-1}}$ . So  $\varphi_g$  is indeed an automorphism of  $G$ .

Because  $H$  is normal,  $H\varphi_g = H$ . So  $\varphi_g$  is an automorphism of  $H$  too. And so  $\varphi_g$  maps  $K$  to itself, because it is characteristic. Hence:

$$\{g^{-1}kg \mid k \in K\} = K$$

So  $K$  is normal in  $G$ . □

## 2.1 Semidirect Product

We already know about the direct product:

**Definition 3** (Direct Product). For groups  $N$  and  $H$ , the direct product,  $G = N \times H$  is a group of ordered pairs of elements  $(n, h)$  where  $n \in N$  and  $h \in H$  with the operation:

$$(n_1, h_1)(n_2, h_2) = (n_1n_2, h_1h_2)$$

Moreover, if  $\bar{N} = N \times \mathbf{1}$  and  $\bar{H} = \mathbf{1} \times H$ , then:

- (i)  $\bar{N} \trianglelefteq G$  and  $\bar{H} \trianglelefteq G$
- (ii)  $\bar{N} \cap \bar{H} = \mathbf{1}$
- (iii)  $\bar{N}\bar{H} = \{nh \mid n \in N, h \in H\} = G$

Now let's seek a slightly more general way to combine groups, by relaxing that  $H$  must be normal. So we have:

$$N \trianglelefteq G, H \leq G, NH = G, \quad \text{and} \quad N \cap H = \mathbf{1}$$

Consider the set, (not the direct product):

$$N \times H = \{ (n, h) \mid n \in N, h \in H \}$$

and a map

$$\phi : N \times H \rightarrow G \quad \text{defined by} \quad (n, h) \mapsto nh$$

We want  $\phi$  to be an isomorphism.

To show  $\phi$  is injective, take  $n_1, n_2 \in N$  and  $h_1, h_2 \in H$ , and assume  $n_1 h_1 = n_2 h_2$ . Then multiplying on the left by  $n_2^{-1}$  and on the right by  $h_1^{-1}$  gives:

$$n_2^{-1} n_1 = h_2 h_1^{-1}$$

On the left we have an element of  $N$  and on the right, an element of  $H$ , so  $n_2^{-1} n_1 = h_2 h_1^{-1} \in N \cap H$ . But  $N \cap H = \mathbf{1}$  so then  $n_2^{-1} n_1 = h_2 h_1^{-1} = 1$ . Hence:

$$n_1 = n_2 \quad \text{and} \quad h_1 = h_2$$

To show  $\phi$  is surjective, consider the image,  $\text{im } \phi = \{ nh \mid n \in N, h \in H \}$ . This is by definition  $NH = G$ , so  $\phi$  is surjective, and hence a bijection.

For  $\phi$  to be a homomorphism, we need:

$$\begin{aligned} [(n_1, h_1)(n_2, h_2)]\phi &= (n_1, h_1)\phi (n_2, h_2)\phi \\ &= n_1 h_1 n_2 h_2 \\ &= n_1 h_1 n_2 h_1^{-1} h_1 h_2 \\ &= (n_1 h_1 n_2 h_1^{-1})(h_1 h_2) \end{aligned}$$

But  $N$  is normal in  $G$  so  $h_1 n_2 h_1^{-1}$  is just another element in  $N$ , say  $n_3$ . So:

$$[(n_1, h_1)(n_2, h_2)]\phi = (n_1 n_3)(h_1 h_2) = (n_1 n_3, h_1 h_2)\phi$$

We know that  $\phi$  is injective, so then:

$$(n_1, h_1)(n_2, h_2) = (n_1 n_3, h_1 h_2)$$

This tells us the multiplication that will make  $NH$  a group. Because  $N \trianglelefteq G$ , the map

$$\varphi_{h_1} : n_2 \mapsto h_1 n_2 h_1^{-1} = n_3$$

is an automorphism of  $N$ . This gives rise to the definition:

**Definition 4** (Semidirect Product).

- (i) For a group  $G$  with normal subgroup  $N$  and subgroup  $H$  with  $NH = G$  and  $N \cap H = \mathbf{1}$ ,  $G$  is the internal semidirect product of  $N$  by  $H$ , written  $G = N \rtimes H$ .
- (ii) For groups  $N$  and  $H$ , and a homomorphism  $\psi : H \rightarrow \text{Aut } N$ , the external semidirect product of  $N$  by  $H$  via  $\psi$  is the set:

$$N \times H = \{ (n, h) \mid n \in N, h \in H \}$$

with multiplication:

$$(n_1, h_1)(n_2, h_2) = (n_1(n_2^{h_1\psi}), h_1 h_2)$$

denoted:

$$N \rtimes_{\psi} H$$

We use the notation  $n_2^{h_1\psi}$  both because it indicates conjugation, and is clearer.

**Lemma 4.** For a group  $G$  with  $N \leq G$  and  $H \leq G$ , with  $N \cap H = \mathbf{1}$  then:

$$|NH| = |\{nh \mid n \in N, h \in H\}| = |N| \cdot |H|$$

*Proof.* We just saw above that for elements  $n \in N$  and  $h \in H$ , the map:

$$\phi : N \times H \rightarrow NH \quad \text{defined by} \quad (n, h) \mapsto nh$$

is a bijection. The result follows immediately from this.  $\square$

**Lemma 5.** Let  $N$  and  $H$  be groups, and  $\alpha \in \text{Aut } H$ . Then the semidirect products via the homomorphism  $\psi$ ,  $N \rtimes_{\psi} H$ , and via the homomorphism  $\alpha\psi$ ,  $N \rtimes_{\alpha\psi} H$ , are isomorphic.

That is, we can apply any automorphism to  $H$  and the resulting semidirect product remains in the same isomorphism class.

*Proof.* Let  $G = N \rtimes_{\psi} H$  and  $\bar{G} = N \rtimes_{\alpha\psi} H$ , and define:

$$\vartheta : G \rightarrow \bar{G} \quad \text{by} \quad \vartheta : (n, h) \mapsto (n, h\alpha)$$

We will show that  $\vartheta$  is an isomorphism.

First,  $\vartheta^{-1}$  exists because  $\alpha^{-1}$  exists, and is given by:

$$\vartheta^{-1} : (n, h) \mapsto (n, h\alpha^{-1})$$

Hence  $\vartheta$  is a bijection.

Now for two elements,  $(n_1, h_1), (n_2, h_2) \in G$ , consider:

$$\begin{aligned} (n_1, h_1)\vartheta (n_2, h_2)\vartheta &= (n_1, h_1\alpha)(n_2, h_2\alpha) \\ &= (n_1 n_2^{(h_1\alpha)\psi}, h_1\alpha h_2\alpha) \\ &= (n_1 n_2^{h_1(\alpha\psi)}, (h_1 h_2)\alpha) \\ &= ((n_1, h_1)(n_2, h_2))\vartheta \end{aligned}$$

So  $\vartheta$  is an isomorphism.  $\square$

## 2.2 Group Actions

Some snazzy introduction.

**Definition 5.** Let  $G$  be a group, and  $\Omega$  be a set, with elements  $g \in G$  and  $\omega \in \Omega$ . Consider a map  $\mu : \Omega \times G \rightarrow \Omega$ , and write  $\omega^g$  for the image of  $(\omega, g)$  under  $\mu$ . So we have:

$$\mu : \Omega \times G \rightarrow \Omega \quad \text{defined by} \quad (\omega, g) \mapsto \omega^g$$

We say  $G$  acts on  $\Omega$  if for all  $g_1, g_2 \in G$  and all  $\omega \in \Omega$ :

- (i)  $(\omega^{g_1})^{g_2} = \omega^{(g_1 g_2)}$
- (ii)  $\omega^1 = \omega$

We call  $\mu$  the group action of  $G$  on  $\Omega$ .

This might remind you of a homomorphism. Indeed we have a result:

**Lemma 6.** A group action induces a homomorphism. Specifically, let  $G$  be a group which acts on a set  $\Omega$ , with  $g \in G$  and  $\omega \in \Omega$ , and define:

$$\rho_g : \Omega \rightarrow \Omega \quad \text{by} \quad \omega \mapsto \omega^g$$

Then:

$$\rho : G \rightarrow \text{Sym } \Omega \quad \text{defined by} \quad g \mapsto \rho_g$$

is a homomorphism.

*Proof.* Firstly,  $\rho_g$  is indeed a permutation of  $\Omega$  because it is invertible (and therefore a bijection), with:

$$(\rho_g)^{-1} = \rho_{g^{-1}}$$

Consider  $g, h \in G$  and their corresponding maps,  $\rho_g, \rho_h \in \text{Sym } \Omega$ . Then:

$$\omega(g\rho)(h\rho) = \omega\rho_g\rho_h = (\omega^g)^h = \omega^{(gh)} = \omega\rho_{gh} = \omega(gh)\rho$$

Thus  $\rho$  is a homomorphism. □

A group acting on the set its cosets will be very useful:

**Definition 6.** For a group  $G$  with  $H \leq G$ , let  $\Omega = \{Hg \mid g \in G\}$ , i.e. the set of cosets of  $H$  in  $G$ . If  $x \in G$ , define a group action:

$$\Omega \times G \rightarrow \Omega \quad \text{by} \quad (Hg, x) \mapsto Hgx$$

**Lemma 7.** The action above is well defined, meaning the action is independent of our choice of representative  $g$ .

*Proof.* □

### 3 First Classifications

Let's start with the easiest case: groups of order 1. Any group  $G$  must have an identity element, and so that's all our possible elements used up! All groups of order 1 are isomorphic to the trivial group,  $\mathbf{1}$ .

What about groups of prime order? Let  $G$  be a group of order  $p$ , where  $p$  is a prime number. Then Lagrange's Theorem tell us all elements must have order 1 or  $p$ . Pick some  $x \in G$  with  $x$  having order  $p$ . Then  $\langle x \rangle = G$  so  $G$  is cyclic and  $G \cong C_p$ .

### 4 Groups of Order $pq$

Let  $G$  be a group of order  $pq$  where  $p, q$  are prime numbers with  $p > q$ , and let  $n_p$  and  $n_q$  denote the number of Sylow  $p$ -subgroups and Sylow  $q$ -subgroups of  $G$  respectively. Then by Sylow's Theorems:

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q \implies n_p = 1$$

$$n_q \equiv 1 \pmod{q} \implies n_q = 1, q+1, 2q+1, \dots \quad \text{and} \quad n_q \mid p$$

So  $G$  has a unique Sylow  $p$ -subgroup, say  $P \trianglelefteq G$ , and a Sylow  $q$ -subgroup,  $Q \leq G$ . Because  $p$  and  $q$  are prime numbers,  $P \cong C_p$  and  $Q \cong C_q$ . Pick generators for each, say  $\langle x \rangle = P$  and  $\langle y \rangle = Q$ . We have 2 possibilities for  $n_q$ :  $p-1$  is a multiple of  $q$  or 1.

**Case 1:**  $q \nmid p-1$ .

If  $p-1$  is not a multiple of  $q$  then  $n_q = 1$  and  $Q \trianglelefteq G$ , hence:

$$G = P \times Q \cong C_{pq}$$

**Case 2:**  $q \mid p-1$ .

If  $p-1$  is a multiple of  $q$  then  $n_q = p$  and so  $Q$  is not normal in  $G$ . By Lagrange's Theorem,  $P \cap Q = \mathbf{1}$  and by Lemma 4,  $|PQ| = pq$ . Hence, as well as the direct product, we have  $G = P \rtimes Q$ , some non-trivial semidirect product.

By Lemma 2,  $\text{Aut } C_p \cong (\mathbb{Z}/p\mathbb{Z})^\times \cong C_{p-1}$ . So if  $\nu \in (\mathbb{Z}/p\mathbb{Z})^\times$ , then  $x \mapsto x^\nu$  is an automorphism. We know also that  $C_{p-1}$  has a unique subgroup of order  $q$ , hence  $G$  has the presentation:

$$G = \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^a \rangle$$

where  $a$  is a generator for the subgroup of order  $q$  in  $(\mathbb{Z}/p\mathbb{Z})^\times$ .

Notice that picking different generators are equivalent up to isomorphism because the composition of two isomorphisms is an isomorphism.

So any group of order  $pq$  is isomorphic to either:

$$\begin{array}{ll} C_{pq} & \text{or } \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^a \rangle \quad \text{if } q \mid p-1 \\ & C_{pq} \quad \text{if } q \nmid p-1 \end{array}$$

## 4.1 Groups of Order $2p$

To illustrate an example of groups of order  $pq$ , let's take  $q = 2$ . Because every prime greater than 2 is odd,  $p-1$  is an even number, and so  $2 \mid p-1$ .

An element  $\alpha \in (\mathbb{Z}/p\mathbb{Z})^\times$  of order 2 satisfies  $\alpha^2 = 1$ , hence  $\alpha = 1$  or  $-1$ . But 1 has order 1, so  $\alpha$  can only be  $-1$ . Side-note: from the proof of Lemma 2, this corresponds to the inverse map.

So, in addition to  $C_{2p}$ , we have:

$$G \cong \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x^{-1} \rangle$$

Which is the presentation for the dihedral group of order  $2p$ ,  $D_{2p}$ .

Hence a group of order  $2p$  is isomorphic to one of:

$$C_{2p} \quad \text{or} \quad D_{2p}$$

## 5 Groups of Order $p^2$

Let  $G$  be a group of order  $p^2$ . First, we will prove a useful lemma:

**Lemma 8.** *If  $G$  is a  $p$ -group (i.e. a group of prime power order), then every subgroup of index  $p$  is normal.*

*Proof.* Let  $H$  be a subgroup of  $G$ , with index  $p$ . We know kernels are normal subgroups, so we will show that  $H$  is the kernel of some homomorphism. Let  $\Omega$  be the set of all cosets of  $H$ . So by definition,  $|\Omega| = p$ . By Lemma 6, there is a homomorphism:

$$\rho : G \rightarrow S_p$$

Let's investigate the kernel of  $\rho$ . If we have  $x \in \ker \rho$ , then:

$$(H1)x = H1 = H$$

which means  $x \in H$ . So the kernel of  $\rho$  is  $H$ . Hence,  $H \trianglelefteq G$ . □

By Lagrange's Theorem, the elements of  $G$  have order 1,  $p$  or  $p^2$ .

If  $x \in G$  has order  $p^2$ , then  $x$  generates  $G$  so  $G \cong C_{p^2}$ .

If  $G$  does not have an element of order  $p^2$  then all elements, except the identity, have order  $p$ . We know that  $G$  must have a subgroup of order  $p$ ,  $P$ , and because  $p$  is prime,  $P \cong C_p$ . Pick a generator for  $P$ , say  $x$  and an element  $y \in G$  such that  $y \notin P$ . Then  $y \neq x^i$  for any  $i$ .

If  $y^j = x^i$  for some  $i$  and  $j$ , then:

$$(y^j)^{1-j} = (x^i)^{1-j} = y^{j-j+1} = y = x^{i(1-j)} = x^k \quad \text{for some } k, \text{ a contradiction.}$$

So no power of  $y$  is equal to any power of  $x$ . Because  $y$  has order  $p$ , it generates a subgroup of order  $p$ ,  $\bar{P}$ , with  $P \cap \bar{P} = \mathbf{1}$ . The lemma tells us that both  $P$  and  $\bar{P}$  are normal, and by Lemma 4,  $|P\bar{P}| = p^2 = |G|$ , so:

$$G = P \times \bar{P} \cong C_p \times C_p$$

If  $G$  has no elements of order  $p$  or  $p^2$ , then it only has elements of order 1, which is the trivial group.

Hence any group of order  $p^2$  is isomorphic to one of:

$$C_{p^2} \quad \text{or} \quad C_p \times C_p$$

## 6 Groups of order 12

We will see later, that we need groups of order 12 in order to classify groups of other orders.

Let  $G$  be a group of order  $12 = 2^2 \cdot 3$ , and  $n_3$  and  $n_2$  denote the number of Sylow 3-subgroups and Sylow 2-subgroups of  $G$  respectively. By Sylow's Theorems:

$$n_2 \equiv 1 \pmod{2} \quad \text{and} \quad n_2 \mid 3 \implies n_2 = 1 \text{ or } 3$$

$$n_3 \equiv 1 \pmod{3} \quad \text{and} \quad n_3 \mid 4 \implies n_3 = 1 \text{ or } 4$$

Let  $H$  be a Sylow 2-subgroup and  $K$  be a Sylow 3-subgroup of  $G$ .

Lagrange's Theorem tells us  $H$  has elements of order 1, 2, and 4, and  $K$  has elements of order 1 and 3. Hence  $H \cap K = \mathbf{1}$ . Lemma 4 tells us:

$$|HK| = |H| \cdot |K| = 12$$

Hence  $G = HK$ ,  $H \trianglelefteq G$ , and  $H \cap K = \mathbf{1}$ .

Since an automorphism,  $\varphi$ , must map generators to generators,  $\text{Aut } C_4 \cong C_2$  because  $C_4$  has two generators. An automorphism of  $V_4$  corresponds to a permutation of the three non-identity elements, hence  $\text{Aut } V_4 \cong S_3$ .

If we consider  $G$  where  $K \trianglelefteq G$ , i.e.  $G = K \rtimes H$ , then we have two cases:

**Case 1:**  $H \cong C_4$  i.e.  $G \cong C_3 \rtimes C_4$ .

Let  $H = \langle y \rangle$ .

We know  $\text{Aut } C_3 \cong C_2$  so a homomorphism  $\psi$  maps  $H$  to the trivial group or to  $\langle \beta : x \mapsto x^{-1} \rangle$ .

If  $H\psi = \mathbf{1}$  then  $G = K \times H \cong C_4 \times C_3$ , which we have already seen.

If  $H\psi = \langle \beta \rangle$  then we have:

$$G = \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$

Now let  $a = xy^2$ . And remember,  $y^{-1}xy = x^{-1}$  means  $x$  commutes with  $y^2$ . So now:

$$a^3 = xy^2xy^2xy^2 = x^3y^6 = y^2$$



and

$$y^{-1}ay = y^{-1}xy^2y = (y^{-1}xy)y^2 = x^{-1}y^2 = y^2x^{-1} = a^{-1}$$

So:

$$G = \langle a, y \mid a^6 = 1, a^3 = y^2, y^{-1}ay = a^{-1} \rangle$$

This is known as the binary dihedral or dicyclic group, denoted  $\text{Dic}_{12}$ . This group is also sometimes denoted by  $T$ .

**Case 2:**  $H \cong V_4$  i.e.  $G \cong C_3 \rtimes (C_2 \times C_2)$ .

If  $\psi : H \rightarrow \text{Aut } K$  is trivial then we obtain the direct product again. We saw in our classification of groups of order  $2p$ , that  $\text{Aut } K$  only has a single element of order 2, corresponding to the inverse map. So we have 3 choices of elements in  $H$  to send to it, but they are all equivalent up to isomorphism, by Lemma 5.

We know that  $H/\text{im } \psi \cong \ker \psi$ , so  $\ker \psi$  must be isomorphic to  $C_2$ . Pick  $z$  so that it generates the kernel, and so the remaining generator,  $y$  is not in the kernel. Then:

$$G = \langle x, y, z \mid x^3 = y^2 = z^2 = 1, yz = zy, xz = zx, y^{-1}xy = x \rangle$$

Let  $a = xz$ . The order of  $a = \text{lcm}(\text{o}(x), \text{o}(z)) = \text{lcm}(2, 3) = 6$  because  $x$  and  $z$  commute. So:

$$a^3 = x^3z^3 = z$$

and

$$y^{-1}ay = y^{-1}xzy = y^{-1}xyz = x^{-1}z = x^2z = a^2a^3 = a^{-1}$$

Hence:

$$G = \langle y, a \mid a^6 = y^2 = 1, a^{-1}ay = a^{-1} \rangle \cong D_{12}$$

Instead, if  $G$  has 4 Sylow 3-subgroups, then there are 8 elements of order 3 in  $G$ . So the remaining 4 must form the Sylow 2-subgroup, hence it is normal.

**Case 1:**  $H \cong C_4$  i.e.  $G \cong C_4 \rtimes C_3$ .

Let  $H = \langle y \rangle$ .

A homomorphism  $\psi : K \rightarrow \text{Aut } H \cong C_2$ , preserves order and together with Lagrange's Theorem means that the only possibility for  $\psi$  is trivial, i.e.  $K\psi = \mathbf{1}$ .

Hence  $G \cong C_4 \times C_3 \cong C_{12}$ .

**Case 2:**  $H \cong V_4$  i.e.  $G \cong (C_2 \times C_2) \rtimes C_3$ .

Let  $H = \langle y, z \rangle$ .

A trivial homomorphism  $K\psi = \mathbf{1}$  yields the direct product. What non-trivial homomorphisms are there? The automorphism group,  $\text{Aut } H \cong S_3$  is of order 6, and so has a unique subgroup of order 3, by Sylow's Theorems. We know that a homomorphism  $\psi : K \rightarrow \text{Aut } H$  is determined by where it sends the generator  $x$ , so for  $\psi$  to be non-trivial, it must send  $x$  to an element of order 3 in  $\text{Aut } H$ .

There are 2 such elements. Because  $\text{Aut } H \cong S_3$ , we will think of them as the permutations of order 3 of the set  $\{1, 2, 3\}$ . Denote them  $a = (1\ 2\ 3)$  and  $b = (1\ 3\ 2)$ . Notice that  $b = a^{-1}$ , so we have homomorphisms:

$$\psi_1 : x \mapsto a \quad \text{and} \quad \psi_2 : x \mapsto a^{-1}$$

It appears we have 2 choices, but this is not the case. The inverse map,  $\beta : x \mapsto x^{-1}$ , is an automorphism of  $K$ , and so by Lemma 5, the corresponding semidirect products of  $\psi_1$

and  $\psi_2$  are isomorphic. Hence (up to isomorphism) there is one non-trivial homomorphism  $\psi : K \rightarrow \text{Aut } H$ . So  $x \in K$  acts by permuting the 3 non-identity elements of  $H$ .

We will show that in this case,  $G \cong A_4$ . First, let's check  $A_4$  has the same subgroup structure as  $G$ . There is a subgroup isomorphic to  $C_3$  in  $A_4$ , generated by the 3-cycle  $(1\ 2\ 3)$ :

$$\bar{K} = \langle (1\ 2\ 3) \rangle$$

We can also find a subgroup isomorphic to  $V_4$ :

$$\bar{H} = \{ 1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3) \}$$

Indeed, we can check that  $\bar{H}$  is normal in  $A_4$ . We can see that  $\bar{H} \cap \bar{K} = \mathbf{1}$  because  $\bar{H}$  contains no 3-cycles, and that  $\bar{H}\bar{K} = A_4$ . So we can conclude that  $A_4 = \bar{H} \rtimes \bar{K}$ .

Let's investigate how conjugation behaves. If we let  $\alpha = (1\ 2)(3\ 4)$ ,  $\beta = (1\ 4)(2\ 3)$  and  $\gamma = (1\ 2\ 3)$ , then we can write an element of  $A_4$  as  $\alpha^i \beta^j \gamma^k$  for some  $i, j$  and  $k$ . Define  $\phi : A_4 \rightarrow G$  by  $\phi : \alpha^i \beta^j \gamma^k \mapsto x^i y^j z^k$ . Then:

$$\beta\phi = (\gamma^{-1}\alpha\gamma)\phi = c^{-1}ac = b$$

So conjugation acts in the same way. Hence we can conclude that  $G \cong A_4$ .

So a group  $G$  of order 12 is isomorphic to one of:

$$C_{12}, \quad C_2 \times C_6, \quad A_4, \quad D_{12}, \quad \text{or} \quad \text{Dic}_{12}$$

## 7 Some Groups of Order $p^2q$

Let  $p$  and  $q$  be distinct prime numbers, and  $G$  be a group of order  $p^2q$ . To classify  $G$  in full generality is beyond this report, so we will focus on the cases when  $p = 2$  and when  $q = 2$ .

### 7.1 $4q$

Let  $G$  be a group of order  $4p$ , and require  $p > 3$ . And let  $n_q$  denote the number of Sylow  $q$ -subgroups. The  $n_q$  must divide 4, so could be 1, 2 or 4, and must be congruent to 1 mod  $q$ . If  $q \leq 3$  then  $G$  could have 4 or 2 Sylow  $q$ -subgroups, but we have already classified those orders, which is why we took  $q > 3$ . Hence  $G$  has a normal Sylow  $q$ -subgroup,  $Q \cong C_q$ . Let  $x$  generate  $Q$ .

Lagrange's Theorem, together with Lemma 4, tell us that a Sylow 2-subgroup,  $T$ , intersects trivially with  $Q$ , and  $|QT| = |G|$ . Hence,  $G = Q \rtimes T$ .

We know by Lemma 2, that  $\text{Aut } Q \cong C_{q-1}$ . So we have two cases:

**Case 1:**  $T \cong V_4$  i.e.  $G \cong C_q \rtimes V_4$ .

We saw in our classification of groups of order  $2p$ , that  $(\mathbb{Z}/q\mathbb{Z})^\times$  has a unique element of order 2, corresponding to the inversion map. So Lemma 5 tells us that there is only a single non-trivial homomorphism  $\psi : T \rightarrow \text{Aut } Q$ .

If  $\psi$  is trivial, then we obtain the product:

$$G \cong C_q \times V_4 \cong C_{2q} \times C_2$$

If  $\psi$  is non-trivial, it maps  $T$  to the subgroup generated by the inversion map, isomorphic to  $C_2$ . Therefore the kernel is isomorphic to  $C_2$ , so pick  $z$  such that it generates the kernel. Denote the other generator of  $T$  by  $y$ , then we obtain the following presentation:

$$G = \langle x, y, z \mid x^q = y^2 = z^2 = 1, yz = zy, xz = zx, y^{-1}xy = x^{-1} \rangle$$

Now let  $a = xz$ , and in a similar calculation to when we classified groups of order 12, we will show that  $G \cong D_{4p}$ .

Firstly, notice that the order of  $a$  is  $4q$ , and:

$$a^q = x^q z^q = z \quad \text{and} \quad a^{q-1} = x^{q-1} z^{q-1} = x^{q-1}$$

Now consider:

$$y^{-1}ay = y^{-1}xzy = y^{-1}xyz = x^{-1}z = a^{q-1}a^q = a^{2q-1} = a^{-1}$$

Hence:

$$G = \langle a, y \mid a^{2q} = y^2 = 1, y^{-1}ay = a^{-1} \rangle$$

which we recognise as  $D_{4p}$ .

**Case 2:**  $T \cong C_4$  i.e.  $G \cong C_q \rtimes C_4$ .

Let  $t$  generate  $T$ . Assume  $4 \nmid q-1$ , which means  $q \equiv 3 \pmod{4}$ . So then  $\text{Aut } Q$  has no subgroup of order 4, and a homomorphism,  $\psi$  must map  $T$  to either the trivial group, or the group generated by the inverse automorphism.

If  $T\psi$  is trivial, then we recover the direct product,  $C_q \times C_4 \cong C_{4q}$ .

If  $T\psi$  is non-trivial, then  $G$  has the presentation:

$$G = \langle x, t \mid x^q = t^4 = 1, t^{-1}xt = x^{-1} \rangle$$

Let  $a = xt^2$ . Then:

$$a^q = xt^2 \dots xt^2 = x^q t^{2q} = t^{2q}$$

We know  $q \equiv 3 \pmod{4}$ , so for some  $n$ ,  $q = 4n + 3$ . Thus  $2q = 8n + 6 = 4(2n + 1) + 2$ . So then:

$$a^q = t^{4(2n+1)+2} = t^2$$

Additionally:

$$t^{-1}at = t^{-1}xt^2t = (t^{-1}xt)t^2 = x^{-1}t^2 = t^2x^{-1} = a^{-1}$$

Hence:

$$G = \langle a, t \mid a^{2q} = 1, a^q = t^2, t^{-1}at = a^{-1} \rangle$$

which is the dicyclic group of order  $4q$ ,  $\text{Dic}_{4q}$ .

If  $4 \mid q-1$ , i.e.  $q \equiv 1 \pmod{4}$ , then  $\text{Aut } Q$  contains a unique element of order 4, and so has a unique subgroup generated by it. We know by Lemma 2, that  $\text{Aut } Q \cong (\mathbb{Z}/q\mathbb{Z})^\times$ , so say  $\alpha$  is the generator of the subgroup of order 4 in  $(\mathbb{Z}/q\mathbb{Z})^\times$ . Our homomorphism can map  $T$  to this subgroup, and we get a group with the presentation:

$$G = \langle x, t \mid x^q = t^4 = 1, t^{-1}xt = x^\alpha \rangle$$

## 7.2 $2p^2$

Let  $G$  be a group of order  $2p^2$ , with  $p > 2$ . Denote the number of Sylow  $p$ -subgroups by  $n_p$ . By Sylow's Theorems,  $n_p$  divides 2, and is congruent to 1 mod  $p$ , so must be 1. Hence,  $G$  has a normal Sylow  $p$ -subgroup,  $P$  of order  $p^2$ .

If  $T$  is a Sylow 2-subgroup, then by applying Lagrange's Theorem, and Lemma 4, we can conclude that  $G = P \rtimes T$ . From our classification of groups of order  $p^2$ , we have 2 choices for  $P$ :

**Case 1:**  $P \cong C_{p^2}$  i.e.  $G \cong C_{p^2} \rtimes C_2$ .

From Lemma 2, we know  $|\text{Aut } P| = p^2 - p = p(p - 1)$ . Because  $p$  is prime,  $2 \nmid p$ , but  $2 \mid p - 1$ , so  $\text{Aut } P$  has a unique element of order 2. Hence, in addition to the direct product,  $G \cong C_{2p^2}$ , we have  $G \cong C_{p^2} \rtimes C_2$ , with  $C_2$  acting by inversion. If  $x$  generates  $P$ , and  $y$  generates  $T$ , we have the presentation:

$$G = \langle x, y \mid x^{p^2} = y^2 = 1, y^{-1}xy = x^{-1} \rangle$$

which we recognise as  $D_{2p^2}$ , the dihedral group of order  $2p^2$ .

**Case 2:**  $P \cong C_p \times C_p$  i.e.  $G \cong C_p \times C_p \rtimes C_2$ .

Consider  $P$  as the product of the subgroups generated by  $a$  and  $b$ , i.e.  $P = \langle a \rangle \times \langle b \rangle$ . Then the action of  $T$  on  $P$  can either be trivial on both subgroups, invert one, or invert both.

If the action is trivial on both subgroups, then we recover the direct product  $G \cong C_p \times C_{2p}$ .

If the action is non-trivial on just one of the subgroups, then we can consider only one case. This is because they are equivalent up to an isomorphism of  $T$ , and Lemma 5 tells us the resulting semidirect products are isomorphic. So we have:

$$G = \langle a \rangle \times (\langle b \rangle \rtimes T) \cong C_p \times D_{2p}$$

Finally, if we choose to invert both subgroups, then we act on all of  $P$  by inversion. So if  $a$  and  $b$  generate  $P$ , then:

$$G = \langle a, b, x \mid a^p = b^p = x^2 = 1, ab = ba, x^{-1}ax = a^{-1}, x^{-1}bx = b^{-1} \rangle$$

Because  $C_p$  has all elements of order  $p$ , excluding 1, and they are all automorphic to each other (meaning that some automorphism maps one to the other),  $x^{-1}gx = g^{-1}$  for all  $g \in P$ . Hence:

$$G = \langle P, x \mid x^2 = 1, x^{-1}gx = g^{-1} \forall g \in P \rangle$$

which is known as the generalised dihedral group for  $C_p$ , denoted  $\text{Dih}(C_p)$ .

## 8 Groups of Order 24

Let  $G$  be a group of order 24, and let  $H$  be a Sylow 3-subgroup of  $G$ , so  $H \cong C_3$ , and let  $h$  generate  $H$ . Let  $T$  be a Sylow 2-subgroup of  $G$ , so  $T$  has order 8. By Lagrange's Theorem,  $H \cap T = 1$  and then applying Lemma 4,  $|HT| = 24$ . Now let  $n_3$  denote the number of Sylow 3-subgroups, and by Sylow's Theorems:

$$n_3 \equiv 1 \pmod{3} \quad \text{and} \quad n_3 \mid 8$$

Hence  $n_3$  is either 1 or 4.

If  $n_3 = 1$ , then  $H$  is normal in  $G$ . Thus  $G = H \rtimes T$ . We'll want a homomorphism  $\psi : T \rightarrow \text{Aut } H$ . We know  $\text{Aut } H \cong C_2$ , and from our classification of groups of order 8, we have 5 possibilities. An action of  $T$  on  $H$  will have image isomorphic to  $C_2$ , and a kernel isomorphic to a group of order 4. We can classify the possible actions by considering the kernel.

**Case 1:**  $T \cong C_8$  i.e.  $G \cong C_3 \rtimes C_8$ .

Let  $t$  generate  $T$ , and so its unique subgroup of order 4 is generated by  $t^2$ . Hence  $\langle t^2 \rangle$  is the kernel of  $\psi$ , so  $\psi$  must send  $t$  to the identity or inversion map. Hence a non-trivial action of  $T$  on  $H$  is unique. If the action is trivial, then:

$$G = T \times H \cong C_{24}$$

Otherwise we obtain:

$$G = \langle h, t \mid h^3 = t^8 = 1, h^{-1}th = t^{-1} \rangle \cong C_3 \rtimes C_8$$

**Case 2:**  $T \cong (C_4 \times C_2)$  i.e.  $G \cong C_3 \rtimes (C_4 \times C_2)$ .

In this case,  $T$  has subgroups isomorphic to both  $C_4$  and  $C_2 \times C_2$ , so we have more possibilities for  $\psi$ . Firstly, if  $\psi$  is trivial, then we obtain the direct product:

$$G \cong C_3 \times C_4 \times C_2$$

Let  $T$  be generated by  $x$  and  $y$ , where  $x^4 = y^2 = 1$ , and consider non-trivial  $\psi$ . Say the kernel of  $\psi$  is isomorphic to  $C_2 \times C_2$ . So it must be generated by the elements of order 2 in  $T$ :  $x^2$  and  $y$ . Then  $\psi$  must map  $x$  to the non-identity element in  $\text{Aut } H$ , inversion. Hence  $\langle x \rangle$  acts by inversion on  $H$ , giving:

$$\begin{aligned} G &= (H \rtimes \langle x \rangle) \times \langle y \rangle \\ &\cong (C_3 \rtimes C_4) \times C_2 \\ &\cong \text{Dic}_{12} \times C_2 \end{aligned}$$

If instead the kernel is isomorphic to  $C_4$ , then it must be generated by an element of order 4 from  $T$ . However, all elements of order 4 are automorphic, and so by Lemma 5, we can pick  $x$  to generate the kernel, without loss of generality. So then  $\psi$  must map  $y$  to inversion. Hence  $\langle x \rangle$  acts trivially on  $H$ , and  $\langle y \rangle$  acts by inversion. Thus:

$$\begin{aligned} G &= (H \rtimes \langle y \rangle) \times \langle x \rangle \\ &\cong (C_3 \rtimes C_2) \times C_4 \\ &\cong S_3 \times C_4 \end{aligned}$$

**Case 3:**  $T \cong (C_2 \times C_2 \times C_2)$  i.e.  $G \cong C_3 \rtimes (C_2 \times C_2 \times C_2)$ .

Let  $\langle a, b, c \rangle = T$ . All elements in  $T$  have order 1 or 2, so cannot have subgroups isomorphic to  $C_4$ . However,  $T$  does have subgroups isomorphic to  $C_2 \times C_2$ , which can be generated by 2 of the three generators of  $T$ . This gives us 3 subgroups, but permuting the generators  $a, b$  and  $c$  is an automorphism of  $T$ , so Lemma 5 tells us the resulting semidirect products are isomorphic. So choose  $\psi$  such that  $b$  and  $c$  are in the kernel. Then  $a\psi$  is either the identity map or the inversion map. If  $\psi$  is trivial, then we obtain the direct product:

$$G \cong C_3 \times C_2 \times C_2 \times C_2$$

If  $a\psi$  is inversion, then:

$$G = (C_3 \rtimes \langle a \rangle) \times \langle b \rangle \times \langle c \rangle \cong S_3 \times C_2 \times C_2$$

**Case 4:**  $T \cong D_8$  i.e.  $G \cong C_3 \rtimes D_8$ .

Let  $r$  and  $s$  generate  $T$  with  $r^4 = s^2 = 1$ . A trivial homomorphism will yield the direct product:

$$G \cong C_3 \times D_8$$

So for a non trivial homomorphism, firstly assume  $\ker \psi \cong C_4$ . There is a unique subgroup in  $T$  isomorphic to  $C_4$ , so it's generated by an element of order 4. However the choice of generator is the same up to an isomorphism of  $T$ , so Lemma 5 lets us pick  $r$  to be the generator, without loss of generality. Hence  $s$  cannot be in the kernel, and so  $s\psi$  is the inversion map. We obtain the presentation:

$$G = \langle x, r, s \mid x^3 = r^4 = s^2 = 1, xr = rx, s^{-1}rs = r^{-1}, s^{-1}xs = x^{-1} \rangle$$

Let  $a = xr$ , and consider:

$$s^{-1}as = s^{-1}xrs = s^{-1}xrs^2s^{-1} = (s^{-1}xs)(srs^{-1}) = x^{-1}r^{-1} = r^{-1}x^{-1} = a^{-1}$$

So we have:

$$G = \langle a, s \mid a^{12} = s^2 = 1, s^{-1}as = a^{-1} \rangle$$

Which we recognise as  $D_{24}$ , the dihedral group of order 24.

If instead we consider  $\psi$  with kernel isomorphic to  $C_2 \times C_2$ , then the kernel is generated by two elements of order 2. However,  $T$  only has two elements of order 2,  $r^2$  and  $s$ , so they must generate the kernel. So then  $\psi$  must map  $r$  to inversion. Hence this action is fully specified. So:

$$G \cong C_3 \rtimes_{V_4} D_8$$

We will use the above notation to mean the unique action with kernel isomorphic to  $V_4$ .

**Case 5:**  $T \cong Q_8$  i.e.  $G \cong C_3 \rtimes Q_8$ .

Let  $T$  be generated by  $i$  and  $j$ , with the product denoted by  $k$ . That is:

$$T = \langle i, j \mid i^4 = j^4 = 1, i^2 = j^2, j^{-1}ij = i^{-1} \rangle$$

There is a single element of order 2 in  $T$ , hence  $T$  has no subgroup isomorphic to  $C_2 \times C_2$ . The elements  $i, j$  and  $k$  each generate a cyclic subgroup in  $T$ . So  $\psi$  will send one of them to the kernel. We know that permuting these is an automorphism of  $T$ , so Lemma 5 tells us the choice results in isomorphic semidirect products.

So take  $i \in \ker \psi$ . Indeed  $\langle i \rangle = \ker \psi$ . Then for a non-trivial homomorphism, we must have  $j \notin \ker \psi$ . Otherwise:

$$i\psi \ j\psi = (ij)\psi = k\psi \in \ker \psi$$

making  $\psi$  trivial.

Thus either  $\psi$  is trivial and we obtain:

$$G \cong C_3 \times Q_8$$

or  $\psi$  maps  $j$  to the inversion map and we obtain the presentation:

$$G = \langle x, i, j \mid x^3 = i^4 = j^4 = 1, xi = ix, i^2 = j^2, j^{-1}xj = x^{-1}, j^{-1}ij = i^{-1} \rangle$$

Now let  $a = xi$ . So:

$$a^6 = x^6 i^6 = i^2 = j^2$$

And:

$$j^{-1}aj = j^{-1}xij = j^{-1}xji^{-1} = x^{-1}i^{-1} = i^{-1}x^{-1} = a^{-1}$$

Hence:

$$G = \langle a, j \mid a^{12} = 1, a^6 = j^2, j^{-1}aj = a^{-1} \rangle$$

We recognise this as the dicyclic group of order 24,  $\text{Dic}_{24}$ .

If  $n_3 = 4$  then  $H$  is not normal. So then the normaliser of  $H$ ,  $N_G(H)$  has index 4. Now let  $G$  act on the set of the cosets of  $N_G(H)$  by conjugation. Hence we obtain a homomorphism  $\rho : G \rightarrow S_4$ . The kernel is a subgroup of  $N_G(H)$  so must have order dividing 6 by Lagrange's Theorem.

The kernel cannot be of order 3, because  $G$  has no normal subgroup of order 3 (because kernels of homomorphism are normal in the domain group). Likewise the kernel cannot be of order 6 because every group of order 6 has a unique Sylow 3-subgroup, which is characteristic. So by Lemma 3, it would be normal in  $G$ . Hence the kernel must have order 1 or 2.

If the kernel is of order 1, then  $\rho$  is an isomorphism, so  $G \cong S_4$ .

If the kernel is of order 2, then we know that  $G/\ker \rho \cong \text{im } \rho$ , so then  $\text{im } \rho$  must have order 12. It also cannot have a normal Sylow 3-subgroup, so looking at our classification of groups of

order 12, this must be isomorphic to  $A_4$ . We know that  $A_4$  has a normal subgroup of order 4, and so by the Correspondence Theorem,  $G$  must contain a normal subgroup of order 8, say  $T$ . By Lagrange's Theorem and Lemma 4, we can conclude that  $G = T \rtimes H$ . Again, we have 5 cases, but this time we'll exclude the trivial homomorphism, because that will just give us the direct product which we have already seen:

**Case 1:**  $T \cong C_8$  i.e.  $G \cong C_8 \rtimes C_3$ .

An automorphism of  $T$ ,  $\varphi$ , maps generators to generators, so say  $\langle x \rangle = T$ . Then  $x\varphi$  could be  $x, x^3, x^5$  or  $x^7$ . Notice that each of these, apart from  $\varphi : x \mapsto x$ , has order 2. Hence, and Lagrange's Theorem tells us that there are no non-trivial homomorphisms  $\psi : H \rightarrow \text{Aut } T$ . As a bonus:  $\text{Aut } C_8 \cong V_4$ .

**Case 2:**  $T \cong (C_4 \times C_2)$  i.e.  $G \cong (C_4 \times C_2) \rtimes C_3$ .

An automorphism of  $T$ ,  $\psi$  preserves element order, so if  $\langle x, y \mid x^4 = y^2 = 1, xy = yx \rangle = T$ , then  $x\psi$  must be of order 4, and  $y\psi$  must be of order 2. Moreover,  $y\psi$  cannot be in  $\langle x\psi \rangle$  because  $\psi$  is injective.

So we are reduced to 2 possible choices for  $y\psi$ , and 4 possible choices for  $x\psi$ . Because an automorphism is determined by its effect on generators, this gives us 8 possible automorphisms. Hence  $|\text{Aut } T| = 8$ , and Lagrange's Theorem tells us that there are no non-trivial homomorphisms  $\psi : H \rightarrow \text{Aut } T$ .

**Case 3:**  $T \cong (C_2 \times C_2 \times C_2)$  i.e.  $G \cong (C_2 \times C_2 \times C_2) \rtimes C_3$ .

To determine  $\text{Aut } T$  it is helpful to think of  $C_2$  as the finite field with two elements. Then  $T$  is isomorphic to a 3 dimensional vector space over two elements. So an automorphism of that vector space is just any linear map, with non-zero determinant. Thus,  $\text{Aut } T \cong \text{GL}_3(2)$ .

We can determine that  $|\text{GL}_3(2)| = 168 = 2^3 \cdot 3 \cdot 7$ , so  $\text{Aut } T$  has a Sylow 3-subgroup of order 3, isomorphic to  $C_3$ .

Sylow's Theorems tells us that all subgroups of order 3 are conjugate, so Lemma 5 tells us there is only one unique action (up to isomorphism) of  $H$  on  $T$ . As before, pick a homomorphism,  $\psi$ , which will let us easily classify the resulting semidirect product.

Write  $T = A \times B$  where  $A \cong C_2$  and  $B \cong C_2 \times C_2$ . Then let  $\psi$  map  $H$  to the subgroup generated by the automorphism which fixes  $A$  and permutes the non-identity elements of  $B$  in a 3-cycle. This automorphism has order 3 by construction, so we can write:

$$G \cong C_2 \times (V_4 \rtimes C_3)$$

We know already that  $V_4 \rtimes C_3 \cong A_4$ , so  $G \cong C_2 \times A_4$ .

**Case 4:**  $T \cong D_8$  i.e.  $G \cong D_8 \rtimes C_3$ .

Let  $\langle s, r \mid s^2 = r^4 = 1, s^{-1}rs = r^{-1} \rangle = T$ . An automorphism,  $\psi$ , of  $T$  preserves element order, so for  $r\psi$  we have two choices,  $r$  or  $r^{-1}$ . We can send  $s\psi$  to any element of order 2 which is not in  $\langle r\psi \rangle$ . This leaves only reflections, of which there are 4:  $s, rs, r^2s$  and  $r^3s$ . Hence there are 8 possible automorphisms of  $D_8$ , so  $|\text{Aut } D_8| = 8$ . Lagrange's Theorem tells us that there are no non-trivial homomorphisms  $\psi : H \rightarrow \text{Aut } T$ .

**Case 5:**  $T \cong Q_8$  i.e.  $G \cong Q_8 \rtimes C_3$ .

Firstly, because of the multiplication structure of the quaternions, the image of  $k$  is determined by the images of  $i$  and  $j$ ; it is forced. This reduces the possibilities for an automorphism. Additionally,  $\pm 1$  are fixed by an automorphism, because they are the only elements of their order. So an automorphism could send  $i$  to any of the remaining 6 elements of

order 4. The image of  $j$  cannot be in the subgroup generated by the image of  $i$ , otherwise we wouldn't have an automorphism. Thus there are 4 choices for the image of  $j$ , giving us 24 possible automorphisms altogether.

So  $\text{Aut } T$  will have a Sylow subgroup of order 3.

Somehow show this is  $\text{SL}_2(\mathbb{F}_3)$ .

## 9 Groups of Order 30

Let  $G$  be a group of order  $30 = 2 \cdot 3 \cdot 5$ . So then  $G$  has a Sylow 3-subgroup,  $T$ , and a Sylow 5-subgroup,  $F$ . Let  $H = TF$  and by Lagrange's Theorem,  $T \cap F = \mathbf{1}$ , hence  $|H| = 15$  by Lemma 4. We know from our classification of groups of order  $pq$  that  $H \cong C_{15}$ . Because  $|H| = 15 = \frac{30}{2}$ , the index of  $H$  in  $G$  is 2, and we know a subgroup of index 2 is normal, so  $H \trianglelefteq G$ .

Notice that a Sylow 2-subgroup  $K \leq G$  has order 2, so  $K \cong C_2$ . Let  $\langle k \rangle = K$  and  $\langle h \rangle = H$ . By the same argument as above,  $H \cap K = \mathbf{1}$  and  $|HK| = 30$ . Hence  $G = HK$ . Moreover,  $G = H \rtimes K$ .

By Lemma 2:

$$\text{Aut } C_{15} = (\mathbb{Z}/15\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times \cong C_2 \times C_4$$

A homomorphism,  $\psi : C_2 \rightarrow C_2 \times C_4$  preserves element order and we know  $\psi$  is determined by its effect on a generator. So then  $k\psi$  has four possibilities: either the identity, or one of the three elements of order 2.

Additionally,  $\psi$  preserves the Sylow subgroups of  $H$ . So write  $H = \langle h^3 \rangle \times \langle h^5 \rangle$ , the direct product of its Sylow subgroups.

So the action of  $K$  on  $H$  is either trivial or by inversion on each of the Sylow subgroups of  $H$ , giving us 4 possibilities:

**Case 1:** Trivial action on both Sylow subgroups.

In this case, because the action is trivial on all of  $H$ , we recover the direct product,  $G = H \times K \cong C_{30}$ .

**Case 2:** Inversion on both Sylow subgroups.

Here,  $K$  acts on all of  $H$ , so we obtain:

$$G = \langle h, k \mid h^{15} = k^2 = 1, k^{-1}hk = h^{-1} \rangle$$

which we recognise as  $D_{30}$ .

**Case 3:** Inversion on  $\langle h^5 \rangle$ .

We know already, from our classification of groups of order  $2p$ , that  $C_3 \rtimes C_2 \cong D_6$ . So then because the action on  $\langle h^3 \rangle$  is trivial:

$$G = \langle h^3 \rangle \times (\langle h^5 \rangle \rtimes K) \cong C_5 \times D_6$$

**Case 4:** Inversion on  $\langle h^3 \rangle$ .

Similar to above, we obtain:

$$G = \langle h^5 \rangle \times (\langle h^3 \rangle \rtimes K) \cong C_3 \times D_{10}$$



Hence any group of order 30 is isomorphic to one of:

$$C_{30}, \quad D_{15}, \quad C_5 \times D_6, \quad \text{or} \quad C_3 \times D_{10}$$

## Part II

# To Do

## 10 Groups of Order $p^3$

### 10.1 Groups of Order 8

### 10.2 Groups of Order 27

### 10.3 General Case?

## 11 Groups of Order 16