

# Classification of Finite Groups

Daniel Laing

February 2, 2023

## Contents

<b>I</b>	<b>Done</b>	<b>2</b>
1	Groups of Order 6	2
2	Generalisation to Groups of Order $2p$	2
3	Groups of order 4	3
4	Generalisation to Groups of Order $p^2$	3
5	Groups of Order $pq$	4
<b>II</b>	<b>In Progress</b>	<b>4</b>
6	Theorems and Lemmas	5
6.1	Sylow Theorems . . . . .	5
6.2	Isomorphism Theorems . . . . .	5
7	Groups of order 12	6
8	Generalisation to Groups of order $4p$	7
9	Groups of Order 30	7
<b>III</b>	<b>To Do</b>	<b>8</b>
10	Semi-Direct Product	8
11	Groups of order 9 (Might skip)	8
12	Groups of Order 18	8
12.1	Groups of Order $p^2q$ . . . . .	8
13	Groups of Order $p^3$	8
13.1	Groups of Order 8 . . . . .	8
13.2	Groups of Order 27 . . . . .	8
13.3	General Case? . . . . .	8

# Part I

# Done

## 1 Groups of Order 6

Let  $G$  be a group of order 6, and  $n_3$  denote the number of Sylow 3-subgroups of  $G$ . Then by Theorem 6.3:

$$n_3 \equiv 1 \pmod{3} \quad \text{and} \quad n_3 \mid 2 \implies n_3 = 1$$

So  $G$  has one Sylow 3-subgroup,  $N$ , and because 3 is prime, it is isomorphic to  $C_3$ . Let  $N = \langle x \rangle$ . Any Sylow 2-subgroup,  $H \leq G$ , will have order 2, and so  $H \cong C_2$ . Let  $H = \langle y \rangle$ . Lagrange's Theorem tells us that  $N$  has elements of orders 1 and 3, and  $H$  has elements of order 1 and 2 hence:

$$N \cap H = \mathbf{1}$$

By Lemma 6.7:

$$|NH| = \frac{|N| \cdot |H|}{|N \cap H|} = \frac{3 \cdot 2}{1} = 6$$

So  $G = NH$ ,  $N \trianglelefteq G$  and  $N \cap H = \mathbf{1}$ , which means  $G = N \rtimes H$ , the semidirect product of  $N$  by  $H$ .

Now we need to determine  $\text{Aut } N$ . An automorphism,  $\varphi$  of  $N$  preserves element order. In particular,  $\varphi$  maps generators to generators. Hence,  $x\varphi = x$  or  $x^2$  because they are the generators of  $N$ . So  $\text{Aut } N \cong C_2$ .

Now we want a homomorphism  $\psi : H \rightarrow \text{Aut } N$ . If  $\psi$  is trivial, then it maps  $H$  to the trivial group, so every element of  $H$  gets sent to the trivial automorphism. If  $\psi$  is not trivial, then at least one element of  $H$  is not sent to the trivial automorphism. It cannot be 1 because then element order is not preserved, so it must be the generator,  $y$ . Hence we obtain 2 possibilities for  $G$ :

**Case 1:**

$$\begin{aligned} G &= \langle x, y \mid x^3 = y^2 = 1, y^{-1}xy = x \rangle \\ &= \langle x, y \mid x^3 = y^2 = 1, xy = yx \rangle \\ &= C_3 \times C_2 \cong C_6 \end{aligned}$$

**Case 2:**

$$\begin{aligned} G &= \langle x, y \mid x^3 = y^2 = 1, y^{-1}xy = x^{-1} \rangle \\ &\cong D_6 \end{aligned}$$

These are clearly not isomorphic, because  $C_6$  is abelian, and  $D_6$  is not.

Hence  $G$  is isomorphic one of:

$$C_6 \quad \text{or} \quad D_6$$

## 2 Generalisation to Groups of Order $2p$

Now that we have seen groups of order 6, let's try and work towards a more general case: groups of order 2 times a prime number. So let  $G$  be a group of order  $2p$  where  $p$  is a prime number, and  $n_p$  denote the number of Sylow  $p$ -subgroups of  $G$ . Then by Theorem 6.3:

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid 2 \implies n_p = 1$$

So  $G$  has one Sylow  $p$ -subgroup, say  $N$ , and it is isomorphic to  $C_p$ . Let  $N = \langle x \rangle$ . A Sylow 2-subgroup,  $H \leq G$  will have order 2 so  $H \cong C_2$ . Let  $H = \langle y \rangle$ . Lagrange's Theorem tells us that  $N$  has elements of orders 1 and  $p$ , and  $H$  has elements of order 1 and 2 hence:

$$N \cap H = 1$$

By Lemma 6.7:

$$|NH| = \frac{|N| \cdot |H|}{|N \cap H|} = \frac{p \cdot 2}{1} = 2p$$

So  $G = N \rtimes H$  as before.

We know by Lemma 6.8 that  $\text{Aut } N \cong \mathbb{Z}/p\mathbb{Z}^*$ , so let's look for the elements of order 2. An element  $x \in \mathbb{Z}/p\mathbb{Z}^*$  of order 2 satisfies  $x^2 = 1$ , hence  $x = 1$  or  $-1$ . But 1 has order 1, so  $x$  can only be  $-1$ . From the proof of Lemma 6.8, this corresponds to the inverse map  $\beta : x \mapsto x^{-1}$ .

Now we want a homomorphism  $\psi : H \rightarrow \text{Aut } N$ . By the same argument as for groups of order 6, we have two possibilities for  $G$ :

**Case 1:**

$$\begin{aligned} G &= \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x \rangle \\ &= C_p \times C_2 \cong C_{2p} \end{aligned}$$

**Case 2:**

$$\begin{aligned} G &= \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x^{-1} \rangle \\ &\cong D_{2p} \end{aligned}$$

Again, these are clearly not isomorphic, because  $C_{2p}$  is abelian, and  $D_{2p}$  is not. Hence a group of order  $2p$  is isomorphic to one of:

$$C_{2p} \quad \text{or} \quad D_{2p}$$

### 3 Groups of order 4

From the Fundamental Theorem of Finite Abelian Groups, we know that a group  $G$  of order 4 could be isomorphic to one of  $C_4$  and  $C_2 \times C_2$ . Now to show that these are the only possibilities. The Sylow theorems are not so helpful here, because any Sylow 2-subgroup will be of order 4, which is just  $G$ . Lagrange's Theorem tells us every element of  $G$  has order 1, 2 or 4.

If  $x \in G$  has order 4, then  $x$  generates  $G$  so  $G \cong C_4$ .

If instead there is no element of order 4 in  $G$ , then every  $x \in G$  except the identity is of order 2. Consider  $a, b \in G$  with  $a \neq b$ , and their product,  $ab$ . It must be that  $ab$  is the third element of order 2, otherwise we reach a contradiction. So it is easy to see that  $G \cong C_2 \times C_2$ .

So any group of order 4 is isomorphic to one of:

$$C_4 \quad \text{or} \quad C_2 \times C_2$$

### 4 Generalisation to Groups of Order $p^2$

Let  $G$  be a group of order  $p^2$ . By Lagrange's Theorem, the elements of  $G$  have order 1,  $p$  or  $p^2$ .

If  $x \in G$  has order  $p^2$ , then  $x$  generates  $G$  so  $G \cong C_{p^2}$ .

If  $G$  does not have an element of order  $p^2$  then all elements, except the identity, have order  $p$ . We know that  $G$  must have a subgroup of order  $p$ ,  $P$ , and because  $p$  is prime,  $P \cong C_p$ . Pick a generator for  $P$ , say  $x$  and an element  $y \in G$  such that  $y \notin P$ . Then  $y \neq x^i$  for any  $i$ .

If  $y^j = x^i$  for some  $i$  and  $j$ , then:

$$(y^j)^{1-j} = (x^i)^{1-j} = y^{j-j+1} = y = x^{i(1-j)} = x^k \quad \text{for some } k, \text{ a contradiction.}$$

So no power of  $y$  is equal to any power of  $x$ . Because  $y$  has order  $p$ , it generates a subgroup of order  $p$ ,  $\bar{P}$  with  $P \cap \bar{P} = \mathbf{1}$ . By Lemma 6.7,  $|P\bar{P}| = p^2 = |G|$  so:

$$G = P \times \bar{P} \cong C_p \times C_p$$

If  $G$  has no elements of order  $p$  or  $p^2$ , then it only has elements of order 1, which is the trivial group.

Hence any group of order  $p^2$  is isomorphic to one of:

$$C_{p^2} \quad \text{or} \quad C_p \times C_p$$

## 5 Groups of Order $pq$

Let  $G$  be a group of order  $pq$  where  $p, q$  are prime numbers with  $p > q$ , and let  $n_p$  and  $n_q$  denote the number of Sylow  $p$ -subgroups and Sylow  $q$ -subgroups of  $G$  respectively. Then by Theorem 6.3:

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q \implies n_p = 1$$

$$n_q \equiv 1 \pmod{q} \implies n_q = 1, q+1, 2q+1, \dots \quad \text{and} \quad n_q \mid p$$

So  $G$  has a unique Sylow  $p$ -subgroup, say  $P \trianglelefteq G$ , and a Sylow  $q$ -subgroup,  $Q \leq G$ . Because  $p$  and  $q$  are prime numbers,  $P \cong C_p$  and  $Q \cong C_q$ . Pick generators for each, say  $\langle x \rangle = P$  and  $\langle y \rangle = Q$ . We have 2 possibilities for  $n_q$ :  $p-1$  is a multiple of  $q$  or 1.

**Case 1:**  $q \nmid p-1$ .

If  $p-1$  is not a multiple of  $q$  then  $n_q = 1$  and  $Q \trianglelefteq G$ , hence:

$$G = P \times Q \cong C_{pq}$$

**Case 2:**  $q \mid p-1$ .

If  $p-1$  is a multiple of  $q$  then  $n_q = p$  and so  $Q$  is not normal in  $G$ . By Lagrange's Theorem,  $P \cap Q = \mathbf{1}$  and by Lemma 6.7,  $|PQ| = pq$ . Hence, as well as the direct product, we have  $G = P \rtimes Q$ , some non-trivial semidirect product.

By Lemma 6.8,  $\text{Aut } C_p \cong \mathbb{Z}/p\mathbb{Z}^* \cong C_{p-1}$ . So if  $\nu \in \mathbb{Z}/p\mathbb{Z}^*$ , then  $x \mapsto x^\nu$  is an automorphism. We know also that  $C_{p-1}$  has a unique subgroup of order  $q$ , hence  $G$  has the presentation:

$$G = \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^\alpha \rangle$$

where  $\alpha$  is a generator for the subgroup of order  $q$  in  $\mathbb{Z}/p\mathbb{Z}^*$ .

Notice that picking different generators are equivalent up to isomorphism because the composition of two isomorphisms is an isomorphism.

So any group of order  $pq$  is isomorphic to either:

$$\begin{array}{ll} C_{pq} & \text{or} \quad \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^\alpha \rangle \quad \text{if } q \mid p-1 \\ & C_{pq} \quad \text{if } q \nmid p-1 \end{array}$$

# Part II

## In Progress

### 6 Theorems and Lemmas

#### 6.1 Sylow Theorems

Let  $G$  be a group of order  $p^n m$  where  $p$  is a prime and  $p \nmid m$ .

**Theorem 6.1** (1<sup>st</sup> Sylow Theorem).  $G$  has a Sylow  $p$ -subgroup, i.e. a subgroup of order  $p^n$ .

**Theorem 6.2** (2<sup>nd</sup> Sylow Theorem). All Sylow  $p$ -subgroups of  $G$  are conjugate to each other. In particular, if  $G$  has a unique Sylow  $p$ -subgroup, then it is a normal subgroup.

**Theorem 6.3** (3<sup>rd</sup> Sylow Theorem). Let  $n_p$  denote the number of Sylow  $p$ -subgroups of  $G$ . Then:

$$(i) \ n_p \mid m$$

$$(ii) \ n_p \equiv 1 \pmod{p}$$

#### 6.2 Isomorphism Theorems

**Theorem 6.4.**

**Theorem 6.5.**

**Theorem 6.6.**

**Lemma 6.7.** For a group  $G$  with  $N \leq G$  and  $H \leq G$ , then

$$|NH| = |\{nh \mid n \in N, h \in H\}| = \frac{|N| \cdot |H|}{|N \cap H|}$$

**Lemma 6.8.** The automorphism group of  $C_n$  is isomorphic to the multiplicative group of integers mod  $n$ .

$$\text{i.e. } \text{Aut } C_n \cong \mathbb{Z}/n\mathbb{Z}^*$$

*Proof.* Let  $C_n = \langle x \rangle$ . Any automorphism,  $\varphi$  of  $C_n$  has the property:

$$(x^i)\varphi = (x\varphi)^i$$

Hence  $\varphi$  is determined by its effect on a generator,  $x$ , and preserves element order. In particular,  $\varphi$  sends generators to generators. So for  $\varphi$  to be an automorphism, it must send  $x$  to another generator, say  $x^k$ . An element  $x^k$  generates  $C_n$  if  $x^k$  has order  $n$ , i.e. when  $k$  and  $n$  are co-prime. Denote the automorphism sending  $x$  to  $x^k$  by  $\varphi_k$ .

Let's now investigate how these automorphisms behave. Consider:

$$x\varphi_k\varphi_l = (x^k)\varphi_l = (x^k)^l = x^{(kl)} = x\varphi_{kl} \pmod{n}$$

Because multiplication modulo  $n$  is commutative,  $x^{kl} = x^{lk}$ , so  $\text{Aut } C_n$  is abelian.

Now consider  $\theta : \text{Aut } C_n \rightarrow \mathbb{Z}/n\mathbb{Z}^*$  defined by  $\varphi_k\theta = k$ . We will show  $\theta$  is an isomorphism. Every  $k \in \mathbb{Z}/n\mathbb{Z}^*$  is co-prime to  $n$  and so  $x^k$  is a generator of  $C_n$ , hence there is some  $\varphi_k \in \text{Aut } C_n$  such that  $\varphi_k\theta = k$ . So  $\theta$  is surjective. If  $\varphi_k, \varphi_l \in \text{Aut } C_n$  such that  $\varphi_k\theta = \varphi_l\theta$  then  $k = l$ , so  $\theta$  is also injective. Finally,  $\theta$  is a homomorphism because

$$(\varphi_k\varphi_l)\theta = \varphi_{kl}\theta = kl = (\varphi_k\theta)(\varphi_l\theta)$$

So  $\theta : \text{Aut } C_n \rightarrow \mathbb{Z}/n\mathbb{Z}^*$  is an isomorphism. □

## 7 Groups of order 12

Let  $G$  be a group of order  $12 = 2^2 \cdot 3$ , and  $n_3$  and  $n_2$  denote the number of Sylow 3-subgroups and Sylow 2-subgroups of  $G$  respectively. By Theorem 6.3:

$$n_2 \equiv 1 \pmod{2} \text{ and } n_2 \mid 3 \implies n_2 = 1$$

$$n_3 \equiv 1 \pmod{3} \text{ and } n_3 \mid 4 \implies n_3 = 1 \text{ or } 4$$

So  $G$  has a unique Sylow 2-subgroup of order 4, say  $H \trianglelefteq G$ , and we have already classified groups of order 4, so  $H$  is isomorphic to either  $V_4$  (the Klein 4 group) or  $C_4$ . A Sylow 3-subgroup,  $K \leq G$  will have order 3, so  $K \cong C_3$ .

Lagrange's Theorem tells us  $H$  has elements of order 1, 2, and 4, and  $K$  has elements of order 1 and 3. Hence  $H \cap K = \mathbf{1}$ . Lemma 6.7 tells us:

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|} = 12$$

Hence  $G = HK$ ,  $H \trianglelefteq G$ , and  $H \cap K = \mathbf{1}$ . If we consider groups with 4 Sylow 3-subgroups then we can conclude that they are some semidirect product,  $G = H \rtimes K$ .

Since an automorphism,  $\varphi$ , must map generators to generators,  $\text{Aut } C_4 \cong C_2$  because  $C_4$  has two generators. An automorphism of  $V_4$  corresponds to a permutation of the three non-identity elements, hence  $\text{Aut } V_4 \cong S_3$ .

**Case 1:**  $H \cong C_4$  i.e.  $G \cong C_4 \rtimes C_3$ .

A homomorphism  $\psi : K \rightarrow \text{Aut } H \cong C_2$ , preserves order and together with Lagrange's Theorem means that the only possibility for  $\psi$  is trivial, i.e.  $K\psi = \mathbf{1}$ .

Hence  $G \cong C_4 \times C_3 \cong C_{12}$ .

**Case 2:**  $H \cong V_4$  i.e.  $G \cong (C_2 \times C_2) \rtimes C_3$ .

A trivial homomorphism  $K\psi = \mathbf{1}$  yields the direct product.

Still need to show this is  $A_4$ .

If we instead consider  $G$  where  $K \trianglelefteq G$ , i.e.  $G = K \rtimes H$ , then we again have two cases:

**Case 1:**  $H \cong C_4$  i.e.  $G \cong C_3 \rtimes C_4$ .

Say  $K = \langle x \rangle$  and  $H = \langle y \rangle$ . We know  $\text{Aut } C_3 \cong C_2$  so a homomorphism  $\psi$  maps  $H$  to the trivial group or to  $\langle \beta : x \mapsto x^{-1} \rangle$ .

If  $H\psi = \mathbf{1}$  then  $G = K \times H \cong C_4 \times C_3$ , which we have already seen.

If  $H\psi = \langle \beta \rangle$  then we have:

$$G = \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$

**Case 2:**  $H \cong V_4$  i.e.  $G \cong C_3 \rtimes (C_2 \times C_2)$ .

If  $\psi : H \rightarrow \text{Aut } K$  is trivial then we obtain the direct product again.

The image of a non-trivial homomorphism  $\psi : H \rightarrow \text{Aut } K$  is isomorphic to  $C_2$ , so by Theorem 6.4:  $\ker \theta \cong C_2$ .

Choose  $a, b \in H$  with  $a, b \neq 1$  such that  $a\theta = \beta : x \mapsto x^{-1}$  and  $b\theta = \text{id} : x \mapsto x$ . Then:

$$G = \langle x, a, b \mid x^3 = a^2 = b^2 = 1, ab = ba, a^{-1}xa = x^{-1}, b^{-1}xb = x \rangle$$

Let  $y = xb$ . The order of  $y = \text{lcm}(\text{o}(x), \text{o}(b)) = \text{lcm}(2, 3) = 6$  because  $x$  and  $b$  commute. So:

$$y^3 = x^3b^3 = b$$

and

$$a^{-1}ya = a^{-1}xba = a^{-1}xab = x^{-1}b = x^2b = y^2y^3 = y^{-1}$$

Hence:

$$G = \langle a, y \mid y^6 = a^2 = 1, a^{-1}ya = y^{-1} \rangle \cong D_{12}$$

So a group  $G$  of order 12 is isomorphic to one of:

$$C_{12}, \quad C_2 \times C_6, \quad A_4, \quad D_{12}, \quad \text{or} \quad \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$

## 8 Generalisation to Groups of order $4p$

Suppose  $G$  is a group of order  $4p$  where  $p$  is a prime number. Let  $n_2$  denote the number of Sylow 2-subgroups.

## 9 Groups of Order 30

Let  $G$  be a group of order  $30 = 2 \cdot 3 \cdot 5$ , and let  $n_3$  and  $n_5$  denote the number of Sylow 3-subgroups and Sylow 5-subgroups of  $G$  respectively. Then by Theorem 6.3:

$$n_3 = 1 \text{ or } 10 \quad \text{and} \quad n_5 = 1 \text{ or } 6$$

If  $n_3 = 10$ , then there are 20 elements of order 3, and if  $n_5 = 6$  then there are 24 elements of order 5 in  $G$ .  $G$  only has 30 elements, so then either:

$$n_3 = 1 \text{ and } n_5 = 6, \quad n_3 = 10 \text{ and } n_5 = 1 \quad \text{or} \quad n_3 = n_5 = 1$$

Hence either  $C_3 \trianglelefteq G$  or  $C_5 \trianglelefteq G$ .

Let  $H = C_3C_5$  and by Lagrange's Theorem,  $C_3 \cap C_5 = \mathbf{1}$ , hence  $|H| = 15$  by Lemma 6.7. We know from our classification of groups of order  $pq$  that  $H \cong C_{15}$ . Notice that  $C_2$  is a Sylow 2-subgroup of  $G$ , and by the same argument,  $C_2 \cap C_{15} = \mathbf{1}$  and  $|C_2C_{15}| = 30$ . Hence  $G = C_2C_{15}$ .

Because  $|C_{15}| = 15 = \frac{30}{2}$ , the index of  $C_{15}$  in  $G$  is 2, and we know a subgroup of index 2 is normal, so  $C_{15} \trianglelefteq G$ . Moreover,  $G = C_{15} \rtimes C_2$ .

By Lemma 6.8:

$$\text{Aut } C_{15} = \mathbb{Z}/15\mathbb{Z}^* \cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})^* \cong \mathbb{Z}/3\mathbb{Z}^* \times \mathbb{Z}/5\mathbb{Z}^* \cong C_2 \times C_4$$

A homomorphism,  $\psi : C_2 \rightarrow C_2 \times C_4$  preserves element order, and there are 3 elements of order 2 in  $C_2 \times C_4$ :  $(x, 1)$ ,  $(1, y^2)$  and  $(x, y^2)$  where  $\langle x, y \rangle = C_2 \times C_4$ . We know  $\psi$  is determined by its effect on a generator, so if  $\langle z \rangle = C_2$  then  $z\psi$  has four possibilities:

**Case 1:**  $z\psi = (1, 1)$ .

When  $z\psi = (1, 1)$ , then  $\psi$  is the trivial homomorphism, and so we obtain:

$$G \cong C_2 \times C_{15} \cong C_{30}$$

**Case 2:**  $z\psi = (x, 1)$ .

**Case 3:**  $z\psi = (1, y^2)$ .

Case 4:  $z\psi = (x, y^2)$ .

## Part III

# To Do

10 Semi-Direct Product

11 Groups of order 9 (Might skip)

12 Groups of Order 18

12.1 Groups of Order  $p^2q$

13 Groups of Order  $p^3$

13.1 Groups of Order 8

13.2 Groups of Order 27

13.3 General Case?

14 Groups of Order 24

15 Groups of Order 16