

# Classification of Finite Groups

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Daniel Laing

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# Groups of Order $pq$

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# Setup

Let  $G$  be a group of order  $pq$  where  $p$  and  $q$  are distinct prime numbers.

Without loss of generality, take  $p > q$ .

# Sylow Subgroups

Let  $n_p$  and  $n_q$  denote the number of Sylow  $p$ -subgroups and Sylow  $q$ -subgroups of  $G$  respectively.

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q$$

So  $G$  has a unique Sylow  $p$ -subgroup, say  $P \trianglelefteq G$ , and at least one Sylow  $q$ -subgroup,  $Q \leq G$ .

$$P \cong C_p, \quad Q \cong C_q$$

Pick generators:  $\langle x \rangle = P$  and  $\langle y \rangle = Q$ .

# Sylow $q$ -subgroup

Because  $n_q \equiv 1 \pmod{q}$ , we have:

$$n_q = 1, q + 1, 2q + 1, \dots$$

And:

$$n_q \mid p$$

So we have two cases:

$$q \nmid p - 1 \quad \text{or} \quad q \mid p - 1$$

Case 1:  $q \nmid p - 1$

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$$q \nmid p - 1$$

Here,  $n_q = 1$  and so  $Q \trianglelefteq G$ .

So:

$$G = P \times Q \cong C_{pq}$$

Case 2:  $q \mid p - 1$

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$$q \mid p - 1$$

That was the easy one! Still have  $n_q = 1$ , but now also the other possibilities!

By Lagrange's Theorem:  $P \cap Q = 1$ .

By Lemma in project:  $|PQ| = pq$ .

So  $G = PQ$ . Hence:

$$G = P \rtimes Q$$

This is unique!

# Presentation

Describe  $P \rtimes Q$  by a presentation.

$$G = \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^a \rangle$$

where  $a$  is a generator for the subgroup of order  $q$  in  $(\mathbb{Z}/p\mathbb{Z})^\times$ .

# Classification

For distinct primes  $p$  and  $q$ , any group of order  $pq$  is isomorphic to one of:

$$C_{pq}$$

$$\langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^a \rangle \quad \text{additionally, if } q \mid p - 1$$

# Examples

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