# CLASSIFICATION
## OF
# FINITE GROUPS

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2023

*I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment.*

# Contents

**Abstract**

This report covers the full classification, including proofs, of all groups of order less than or equal to 31. In some cases, a classification of the general case is given; namely $p$-groups up to $p^3$, and groups of order $pq$, $4p$ and $2p^2$, where $p$ and $q$ are distinct primes. Where possible, the names of specific, and families of groups are given.

A table of the full classification is found in the Appendix. Note: not all presentations are included, specifically long ones, and those not especially relevant.

# Chapter I

# Introduction

The study of groups is an important area of mathematics, and as such, it's quite useful to have quick examples of groups 'in your back pocket' so to speak; even more so, to have an exhaustive list of the possible behaviours of groups. We do this by the notion of an isomorphism class, a kind of equivalence between groups which are essentially the same. However as groups increase in size, this becomes increasingly hard to do, especially for so called '$p$-groups': groups which have order the power of a prime. For example, up to isomorphism, there are 51 possible groups of order 32 — wow!

This report will cover the classification, and proof thereof, of groups of order up to, and including, 31. In doing so, we will further our understanding of constructing groups, and where possible, find out the names given to the groups we come across by the wider world of group theory.

To start, let's solidify the notation used in this report. We shall denote groups and sets with capital letters, like $G$, $H$, and elements of those groups with lower case letters, like $g$, $h$. Greek letters shall denote mappings, generally $\phi$, $\psi$, etc. with $\iota$ reserved for the identity map, and we will write mappings on the right.

To denote the cyclic group of order $n$ we will use $C_n$, $D_{2n}$ to denote the dihedral group of order $2n$, $A_n$ to denote the alternating group over $n$ elements, $S_n$ to denote the symmetric group over $n$ elements, and $Q_8$ to denote the quaternion group. The trivial group, $\{1\}$ is denoted by $\mathbf{1}$. We will meet other groups as we go on our journey of classification!

This report consists of 4 main movements: first we will review and expand our knowledge of group theory, and then in the subsequent 3 chapters, we will classify our groups. We will first tackle $p$-groups up to $p^3$, then groups which have composite order, specifically $pq$, $4q$ and $2p^2$, before filling in the gaps we missed, classifying groups of orders 12, 24, 30 and 16 in the final chapter.

# Chapter II

# Preliminaries

The content of *MT4003* will be assumed knowledge, however this material can be found in group theory textbooks, particularly Robinson's *A Course in the Theory of Groups*.[1] Let's move on to review some facts and theorems which will be valuable later on, as well as introduce some new concepts and prove some new results!

## 2.1 Automorphisms

In addition to in Robinson, this section's material is commonly found in group theory textbooks, including Chapter 6 of Burnside's *Theory of Groups*[2] and Chapter 4 of Ledermann's *Introduction to the Theory of Finite Groups*[3].

**Definition 2.1.** If $G$ and $H$ are groups with elements $g_1$, $g_2 \in G$, then a map:

$$\phi : G \to H$$

is a homomorphism if:

$$(g_1 g_2)\phi = (g_1\phi)(g_2\phi)$$

If $\phi$ is bijective, then we call it an isomorphism, with $G \cong H$ denoting that $G$ is isomorphic to $H$. And if $\phi$ is an isomorphism from $G$ to itself, then we call it an automorphism of $G$.

**Lemma 2.2.** *The set of all automorphisms of a group $G$ form a group under composition. Indeed, this is called the automorphism group of $G$, denoted* $\operatorname{Aut} G$.

*Proof.* Let $A = \operatorname{Aut} G = \{\, \phi : G \to G \mid \phi \text{ is an isomorphism} \,\}$, and let $\phi \in A$. Denote an element of $G$ by $g$.

1. Derek J. S. Robinson, A Course in the Theory of Groups (Springer-Verlag New York Heidelberg Berlin, 1982), ISBN: 0-387-90600-2.

2. W. Burnside, Theory of Groups of Finite Order, Second Editon (Cambridge University Press, 1911), Ch. 6.

3. Walter Ledermann, Introduction to the Theory of Finite Groups, Fourth Edition (Oliver / Boyd Edinburgh / London, 1961), Ch. 4.

We know already that the composition of two isomorphisms is an isomorphism, so $A$ is closed under composition.

The identity map, $\iota : g \mapsto g$, is certainly an automorphism of $G$ and so $A$ is non-empty.

Indeed, $\iota : g \mapsto g$ is the identity of $A$, since:

$$g\phi\iota = (g\phi)\iota = g\phi \quad \text{and} \quad g\iota\phi = (g\iota)\phi = g\phi$$

And inverses clearly exist, because automorphisms are bijections, and bijections are invertible. Hence $A = \operatorname{Aut} G$ is a group. □

**Lemma 2.3.** *The automorphism group of $C_n$ is isomorphic to the multiplicative group of integers mod $n$.*

*i.e.* $\operatorname{Aut} C_n \cong (\mathbb{Z}/n\mathbb{Z})^\times$

*Proof.* Let $C_n = \langle x \rangle$. Any automorphism, $\phi$ of $C_n$ has the property:

$$(x^i)\phi = (x\phi)^i$$

Hence $\phi$ is determined by its effect on a generator, $x$, and preserves element order. In particular, $\phi$ sends generators to generators. So for $\phi$ to be an automorphism, it must send $x$ to another generator, say $x^k$. An element $x^k$ generates $C_n$ if $x^k$ has order $n$, i.e. when $k$ and $n$ are co-prime. Denote the automorphism sending $x$ to $x^k$ by $\phi_k$.

Let's now investigate how these automorphisms behave. Let $\phi_k, \phi_l \in \operatorname{Aut} C_n$, and consider:

$$x\phi_k\phi_l = (x^k)\phi_l = (x^k)^l = x^{(kl)} = x\phi_{kl} \pmod{n}$$

Because multiplication modulo $n$ is commutative, $x^{kl} = x^{lk}$, so $\operatorname{Aut} C_n$ is abelian.

Now consider $\theta : \operatorname{Aut} C_n \to (\mathbb{Z}/n\mathbb{Z})^\times$ defined by $\phi_k\theta = k$. We will show $\theta$ is an isomorphism. Every $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ is co-prime to $n$ and so $x^k$ is a generator of $C_n$, hence there is some $\phi_k \in \operatorname{Aut} C_n$ such that $\phi_k\theta = k$. So $\theta$ is surjective. If $\phi_k\theta = \phi_l\theta$ then $k = l$, so $\theta$ is also injective. Finally, $\theta$ is a homomorphism because:

$$(\phi_k\phi_l)\theta = \phi_{kl}\theta = kl = (\phi_k\theta)(\phi_l\theta)$$

So $\theta : \operatorname{Aut} C_n \to (\mathbb{Z}/n\mathbb{Z})^\times$ is an isomorphism. □

**Definition 2.4.** A subgroup $H$ of a group $G$ is called characteristic if it is fixed by all automorphisms of $G$.

i.e. for an automorphism $\phi$ of $G$, $H\phi = H$.

**Lemma 2.5.** *Let $G$ be a group with normal subgroup $H$, and let $K$ be characteristic in $H$. Then $K$ is a normal subgroup of $G$.*

*Proof.* Consider the map $\phi_g : G \to G$ defined by $\phi_g : x \mapsto g^{-1}xg$ for elements $x$, $g \in G$. We will show that this is an automorphism of $G$. For $x$, $y \in G$:

$$x\phi_g y\phi_g = (g^{-1}xg)(g^{-1}yg) = g^{-1}(xy)g = (xy)\phi_g$$

Hence $\phi_g$ is a homomorphism. Moreover, we can check that $\phi_g$ is invertible with inverse $\phi_{g^{-1}}$. So $\phi_g$ is indeed an automorphism of G.

Because $H$ is normal, $H\phi_g = H$. So $\phi_g$ is an automorphism of $H$ too. And so $\phi_g$ maps $K$ to itself, because it is characteristic. Hence:

$$\{\, g^{-1}kg \mid k \in K \,\} = K$$

So $K$ is normal in $G$. □

## 2.2 Semidirect Product

A way to combine, or describe combinations of groups will be among the most used tools in this report! The learning from this section is mostly from *MT5864, Advanced Group Theory*, and can be found in Robinson.[4] The direct product is used commonly throughout group theory:

**Definition 2.6.** For groups $N$ and $H$, the direct product, $G = N \times H$ is a group of ordered pairs of elements $(n, h)$ where $n \in N$ and $h \in H$ with the operation:

$$(n_1, h_1)(n_2, h_2) = (n_1n_2, h_1h_2)$$

Furthermore, we recall its properties:

**Lemma 2.7.** *If* $\bar{N} = N \times \mathbf{1}$ *and* $\bar{H} = \mathbf{1} \times H$*, then:*

*(i)* $\bar{N} \trianglelefteq G$ *and* $\bar{H} \trianglelefteq G$

*(ii)* $\bar{N} \cap \bar{H} = \mathbf{1}$

*(iii)* $\bar{N}\bar{H} = \{\, nh \mid n \in N,\ h \in H \,\} = G$

Now let's seek a slightly more general way to combine groups, by relaxing that $H$ must be normal. So if we have a group $G$ with the following:

$$N \trianglelefteq G,\ H \leqslant G,\ NH = G, \quad \text{and} \quad N \cap H = \mathbf{1}$$

consider the set, (not the direct product):

$$N \times H = \{\, (n,\, h) \mid n \in N,\ h \in H \,\}$$

We want to give this set a group structure, such that it is isomorphic to $G$. In other words, we want to construct an isomorphism:

$$\phi : N \times H \to G \quad \text{defined by} \quad (n,\, h) \mapsto nh$$

4. Robinson, A Course in the Theory of Groups.

and in doing so, find a suitable group structure for $N \times H$.

To show $\phi$ is injective, take $n_1$, $n_2 \in N$ and $h_1$, $h_2 \in H$, and assume $n_1 h_1 = n_2 h_2$. Then multiplying on the left by $n_2^{-1}$ and on the right by $h_1^{-1}$ gives:

$$n_2^{-1} n_1 = h_2 h_1^{-1}$$

On the left we have an element of $N$ and on the right, an element of $H$, so:

$$n_2^{-1} n_1 = h_2 h_1^{-1} \in N \cap H$$

But $N \cap H = \mathbf{1}$ so then $n_2^{-1} n_1 = h_2 h_1^{-1} = 1$. Hence:

$$n_1 = n_2 \quad \text{and} \quad h_1 = h_2$$

To show $\phi$ is surjective, consider the image, $\operatorname{im} \phi = \{\, nh \mid n \in N,\ h \in H \,\}$. This is by definition $NH = G$, so $\phi$ is surjective, and hence a bijection.

For $\phi$ to be a homomorphism, we need:

$$\begin{aligned} [(n_1,\, h_1)(n_2,\, h_2)]\phi &= (n_1,\, h_1)\phi\,(n_2,\, h_2)\phi \\ &= n_1 h_1 n_2 h_2 \\ &= n_1 h_1 n_2 (h_1^{-1} h_1) h_2 \\ &= (n_1 h_1 n_2 h_1^{-1})(h_1 h_2) \end{aligned}$$

But $N$ is normal in $G$ so $h_1 n_2 h_1^{-1}$ is just another element in $N$, say $n_3$. So:

$$[(n_1,\, h_1)(n_2,\, h_2)]\phi = (n_1 n_3)(h_1 h_2) = (n_1 n_3,\, h_1 h_2)\phi$$

We know that $\phi$ is injective, so then:

$$(n_1,\, h_1)(n_2,\, h_2) = (n_1 n_3,\, h_1 h_2) \qquad (\dagger)$$

This tells us the multiplication that will make $NH$ a group. Because $N \trianglelefteq G$, the map

$$n_2 \mapsto h_1 n_2 h_1^{-1} = n_3$$

is an automorphism of $N$.

This gives rise to the formal definition:

**Definition 2.8** (Semidirect Product)**.**

(i) For a group $G$ with normal subgroup $N$ and subgroup $H$ with $NH = G$ and $N \cap H = \mathbf{1}$, $G$ is the internal semidirect product of $N$ by $H$, written $G = N \rtimes H$.

(ii) For groups $N$ and $H$, and a homomorphism $\psi : H \to \operatorname{Aut} N$, the external semidirect product of $N$ by $H$ via $\psi$ is the set:

$$N \times H = \{\, (n,\, h) \mid n \in N,\ h \in H \,\}$$

with multiplication:

$$(n_1,\, h_1)(n_2,\, h_2) = (n_1(n_2^{h_1\psi}),\, h_1h_2)$$

denoted:

$$N \rtimes_\psi H$$

We use the notation $n_2^{h_1\psi}$ to mean the image of $n_2$ under the automorphism $h_1\psi$, both because it indicates conjugation, and is clearer.

We can verify that $N \times H$ with this multiplication is indeed a group. We have already seen that $N \times H$ is closed under (†). The identity element is $(1,\ 1)$:

$$(n,\, h)(1,\, 1) = (n1^{h\psi},\, h1) = (n,\, h)$$

$$(1,\, 1)(n,\, h) = (1n^{1\psi},\, 1h) = (n,\, h)$$

And finally, inverses exist, with $(n,\, h)^{-1} = \left((n^{-1})^{(h\psi)^{-1}},\, h^{-1}\right)$:

$$\begin{aligned}(n,\, h)\left((n^{-1})^{(h\psi)^{-1}},\, h^{-1}\right) &= \left(n\left((n^{-1})^{(h\psi)^{-1}}\right)^{h\psi},\, hh^{-1}\right)\\ &= (nn^{-1},\, hh^{-1})\\ &= (1,\, 1)\end{aligned}$$

$$\begin{aligned}\left((n^{-1})^{(h\psi)^{-1}},\, h^{-1}\right)(n,\, h) &= \left((n^{-1})^{(h\psi)^{-1}}n^{(h^{-1})\psi},\, h^{-1}h\right)\\ &= \left((n^{-1})^{(h\psi)^{-1}}n^{(h\psi)^{-1}},\, h^{-1}h\right)\\ &= \left((n^{-1}n)^{(h\psi)^{-1}},\, h^{-1}h\right)\\ &= \left(1^{(h\psi)^{-1}},\, 1\right)\\ &= (1,\, 1)\end{aligned}$$

So we have a group.

**Lemma 2.9.** *For a group $G$ with $N \leqslant G$ and $H \leqslant G$, with $N \cap H = \mathbf{1}$ then:*

$$|NH| = |\{nh \mid n \in N, h \in H\}| = |N| \cdot |H|$$

*Proof.* We just saw above that if $N \trianglelefteq G$, for elements $n \in N$ and $h \in H$, the map:

$$\phi : N \times H \to NH \quad \text{defined by} \quad (n,\, h) \mapsto nh$$

is a bijection. We only used the normality of $N$ for the homomorphism property of $\phi$, which we don't need here, hence $\phi$ is still a bijection if $N \leqslant G$. The result follows immediately from this. □

**Lemma 2.10.** *Let $N$ and $H$ be groups, and $\alpha \in \operatorname{Aut} H$. Then the semidirect products via the homomorphism $\phi$, $N \rtimes_\phi H$, and via the homomorphism $\psi$, $N \rtimes_\psi H$, are isomorphic if for $\alpha \in \operatorname{Aut} N$ and $\beta \in \operatorname{Aut} H$, we have:*

$$h^\beta \psi = \alpha^{-1} h \phi \alpha \qquad \text{(for all } h \in H\text{)}$$

*That is, we can apply any automorphism to $H$ and conjugate $\operatorname{Aut} N$, and the resulting semidirect product remains in the same isomorphism class.*

*Proof.* Let $G = N \rtimes_\phi H$ and $\bar{G} = N \rtimes_\psi H$, and define:

$$\theta : G \to \bar{G} \quad \text{by} \quad \theta : (n,\, h) \mapsto (n^\alpha,\, h^\beta)$$

We will show that $\theta$ is an isomorphism.

First, $\theta^{-1}$ exists because both $\alpha^{-1}$ and $\beta^{-1}$ exist. We can check that the inverse is given by:

$$\theta^{-1} : (n,\, h) \mapsto (n^{\alpha^{-1}},\, h^{\beta^{-1}})$$

Hence $\theta$ is a bijection. We also have that:

$$h^\beta \psi = \alpha^{-1} h \phi \alpha$$

which implies:

$$\alpha h^\beta \psi = h \phi \alpha \qquad (*)$$

Now for two elements, $(n_1,\, h_1),\ (n_2,\, h_2) \in G$, consider:

$$\begin{aligned}
(n_1,\, h_1)\theta\ (n_2,\, h_2)\theta &= (n_1^\alpha,\, h_1^\beta)(n_2^\alpha,\, h_2^\beta) \\
&= (n_1^\alpha (n_2^\alpha)^{h_1^\beta \psi},\, h_1^\beta h_2^\beta) \\
&= (n_1^\alpha (n_2^{h_1 \phi})^\alpha,\, h_1^\beta h_2^\beta) \qquad \text{(by } *\text{)} \\
&= ((n_1 (n_2)^{h_1 \phi})^\alpha,\, (h_1 h_2)^\beta) \\
&= (n_1 (n_2)^{h_1 \phi},\, h_1 h_2)\theta \\
&= (n_1,\, h_1)(n_2,\, h_2)\theta
\end{aligned}$$

So $\theta$ is an isomorphism. □

## 2.3 Presentations

To encode groups succinctly, we will use group presentations. Their notation is like a combination of subgroup generators and set builder notation. Presentations are formally defined in terms of subgroups of free groups, which is tangential to this report. Moreover, presentations in general are a bit difficult to work with.

We can read further on computation with group presentations in Sims' *Computation with finitely presented groups*[5]. Indeed, in section 1.5 we read that it's algorithmically impossible to determine whether two group presentations are the same! For our purposes, a loose definition will be enough.

**Definition 2.11** (Informal)**.** In this report, a group presentation has 3 main parts:

(i) A list of generators.

(ii) How the generators themselves behave.

(iii) How the generators interact.

The presentation defines the largest group fulfilling the criteria.

Let's illustrate this with a couple of examples. First, let's look at a simple presentation, and understand how it works.

$$\langle\, x \mid x^4 = 1 \,\rangle$$

We see that to the left of the mid-line, we have a generator $x$, and to the right, $x^4 = 1$ tells us that $x$ has order 4. There is only a single generator, so there are no other generators to specify how they interact. So this group is cyclic, and has order 4, which we know to be $C_4$. Both $C_2$ and $\mathbf{1}$ also satisfy this presentation, but $C_4$ is the largest.

For the second example, let's build a presentation for $D_8$. We know that the dihedral group has 2 generators, a rotation $r$, and a flip $s$. So we start writing:

$$\langle\, r,\, s$$

The first generator, $r$, has order 4, and $s$ has order 2. Let's record that:

$$\langle\, r,\, s \mid r^4 = s^2 = 1$$

Finally, we know how the relate, namely $s^{-1}rs = r^{-1}$. So we finish the presentation:

$$D_8 = \langle\, r,\, s \mid r^4 = s^2 = 1,\ s^{-1}rs = r^{-1} \,\rangle$$

Presentations will be especially useful for encoding semidirect products, because we can specify the structure of the groups in the product and then the effect of conjugation. That is, if we have $N \rtimes_\psi H$, we can record this with a presentation. We write the generators, then after the mid-line, the structure of $N$, the structure of $H$ and then the effect of the homomorphism, $\psi$.

5. Charles C. Sims, Computation with finitely presented groups (Cambridge University Press, 1994), ISBN: 0-521-4323-8.

## 2.4 Group Actions

Our last piece of new group theory technology will be group actions. As above, much of this section is based on *MT5864*, and can be found in Chapter 1 of Robinson.[6]

**Definition 2.12.** Let $G$ be a group, and $\Omega$ be a set, with elements $g \in G$ and $\omega \in \Omega$. Consider a map $\mu : \Omega \times G \to \Omega$, and write $\omega^g$ for the image of $(\omega, g)$ under $\mu$. So we have:

$$\mu : \Omega \times G \to \Omega \quad \text{defined by} \quad (\omega, g) \mapsto \omega^g$$

We say $G$ acts on $\Omega$ if for all $g_1, g_2 \in G$ and all $\omega \in \Omega$:

(i) $(\omega^{g_1})^{g_2} = \omega^{(g_1 g_2)}$

(ii) $\omega^1 = \omega$

We call $\mu$ the group action of $G$ on $\Omega$.

This might remind you of a homomorphism. Indeed we have a result:

**Lemma 2.13.** *A group action induces a homomorphism. Specifically, let $G$ be a group which acts on a set $\Omega$, with $g \in G$ and $\omega \in \Omega$, and define:*

$$\rho_g : \Omega \to \Omega \quad \textit{by} \quad \omega \mapsto \omega^g$$

*Then:*

$$\rho : G \to \operatorname{Sym} \Omega \quad \textit{defined by} \quad g \mapsto \rho_g$$

*is a homomorphism.*

*Proof.* Firstly, $\rho_g$ is indeed a permutation of $\Omega$ because it is invertible (and therefore a bijection), with $(\rho_g)^{-1} = \rho_{g^{-1}}$. For $\omega \in \Omega$:

$$\omega \rho_g \rho_{g^{-1}} = (\omega^g)^{g^{-1}} = \omega^{g g^{-1}} = \omega$$

$$\omega \rho_{g^{-1}} \rho_g = (\omega^{g^{-1}})^g = \omega^{g^{-1} g} = \omega$$

Consider $g, h \in G$ and their corresponding maps, $\rho_g, \rho_h \in \operatorname{Sym} \Omega$. Then:

$$\omega(g\rho)(h\rho) = \omega \rho_g \rho_h = (\omega^g)^h = \omega^{(gh)} = \omega \rho_{gh} = \omega(gh)\rho$$

Thus $\rho$ is a homomorphism. □

By this we see that in a semidirect product, $N \rtimes H$, the group $H$ acts on $N$!

A group acting on the set its cosets will be useful:

**Definition 2.14.** For a group $G$ with $H \leqslant G$, let $\Omega = \{ Hg \mid g \in G \}$, i.e. the set of cosets of $H$ in $G$. If $x \in G$, define a group action:

$$\mu : \Omega \times G \to \Omega \quad \text{by} \quad (Hg, x) \mapsto Hgx$$

6. Robinson, A Course in the Theory of Groups.

**Lemma 2.15.** *A group's action on it's cosets is well defined, meaning the action is independent of our choice of representative.*

*Proof.* We will show the lemma is true for right cosets. A similar argument shows the same for left cosets.

Suppose $G$ is a group with non-trivial subgroup $H$, and let $\Omega$ be the set of right cosets of $H$. Let $k, l \in G$ be representatives from the same coset, and we will show that $(Hk)\mu = (Hl)\mu$. Because $k$ and $l$ are in the same coset:

$$Hk = Hl$$

And so $H = H(lk^{-1})$ implies that $lk^{-1} \in H$. Then:

$$(Hl)\mu = H(lx) = H(lk^{-1}kx) = H(kx) = (Hk)\mu$$

Hence $\mu$ is well defined. □

**Definition 2.16.** The orbit of an element $\omega \in \Omega$, is the set:

$$\omega^G = \{\, \omega^x \mid x \in G \,\}$$

The stabiliser of $\omega$ is the set:

$$G_\omega = \{\, x \in G \mid \omega^x = \omega \,\}$$

Finally, there is the orbit-stabiliser theorem:

**Theorem 2.17.** *Let $G$ be a group which acts on a set $\Omega$. Then for $\omega \in \Omega$:*

$$|\omega^G| = |G : G_\omega|$$

*Proof.* Define a mapping:

$$\phi : \{\, G_\omega x \mid x \in G \,\} \to \omega^G \quad \text{by} \quad \phi : G_\omega x \mapsto \omega^x$$

We will show that $\phi$ is a well-defined bijection. First, well-defined:

If $x$ and $y$ are representatives of the same coset, we will show that $(G_\omega x)\phi = (G_\omega y)\phi$. So if we have $G_\omega x = G_\omega y$. Then as before:

$$xy^{-1} \in G_\omega \quad \text{so} \quad \omega^{xy^{-1}} = \omega$$

Hence:

$$(G_\omega x)\phi = \omega^x = \omega^{xy^{-1}y} = \omega^y = (G_\omega y)\phi$$

So $\phi$ is well-defined.

If $\omega^g = \omega^h$ for $g$, $h \in G$ then:

$$\omega^{gh^{-1}} = \omega h h^{-1} = \omega$$

So $gh^{-1} \in G_\omega$ and $G_\omega g = G_\omega h$ and $\phi$ is injective. Finally, $\phi$ is surjective by definition, and so is a bijection. Hence we have proven the result. □

# Chapter III

# Groups of Prime-Power Orders

In this chapter, we will focus on classifying groups which have order $p$, $p^2$ and $p^3$, for a prime number $p$. These three will be proven generally, applying to any prime. However, we will leave the particular case of $16 = 2^4$ for Chapter V because it is significantly harder. First, we will prove a few useful lemmas:

**Lemma 3.1.** *If $G$ is a $p$-group (i.e. a group of prime power order), then every subgroup of index $p$ is normal.*

*Proof.* Let $H$ be a subgroup of $G$, with index $p$. We know kernels are normal subgroups, so we will show that $H$ is the kernel of some homomorphism. Let $\Omega$ be the set of all cosets of $H$. So by definition, $|\Omega| = p$. By Lemma 2.13, there is a homomorphism:

$$\rho : G \to S_p$$

Let's investigate the kernel of $\rho$. If we have $x \in \ker \rho$, then:

$$(H1)x = H1 = H$$

So $x \in H$ and $\ker \rho \leqslant H$.

Because $G$ is a $p$-group, Lagrange's Theorem tells us every non-trivial subgroup has order a multiple of $p$. Similarly, order of $S_p$ is $p!$, and so could have elements of order $p$. The kernel is a non trivial subgroup, and therefore must have order $p$. So $\ker \rho = H$. Hence, $H \trianglelefteq G$. □

**Lemma 3.2.** *If $G$ is a group of prime power order, the centre of $G$ is non-trivial. In particular, $p$ divides the order of the centre.*

*Proof.* Let $Z$ denote the centre of $G$, and consider the action of $G$ on itself by conjugation. The orbit of an element, $g \in G$ is:

$$g^G = \{\, x^{-1}gx \mid x \in G \,\}$$

which is the conjugacy class of $g$. We know that the size of each conjugacy class divides some power of $p$, so the size of each orbit does too, because the are equal. In particular,

the size of each orbit is divisible by $p$. So then the sum of the sizes of all of the conjugacy classes is also divisible by $p$. Looking at the class equation:

$$|G| = |Z| + \sum_{i=1}^{k} |g_i^G|$$

then reducing mod $p$ gives:

$$|G| \equiv |Z| \mod p$$

Because $G$ is non-trivial, it follows that $|Z| \neq 1$.

□

**Lemma 3.3.** *For a group $G$ with centre $\mathrm{Z}(G)$. Then if $G/\mathrm{Z}(G)$ is cyclic, $G$ is abelian.*

*Proof.* Let $x \in G$ be the element such that $x\,\mathrm{Z}(G)$ generates $G/\mathrm{Z}(G)$. Because $G$ is the union of cosets of $\mathrm{Z}(G)$, then indeed:

$$\langle\, x,\, \mathrm{Z}(G) \,\rangle = G$$

The centraliser of $x$ certainly contains $x$, and every element of $\mathrm{Z}(G)$ also commutes with $x$. Hence the centre of $G$ is a subgroup of the centraliser of $x$. The result follows by concluding:

$$G = \langle\, x,\, \mathrm{Z}(G) \,\rangle = \langle\, \mathrm{Z}(G) \,\rangle = \mathrm{Z}(G)$$

□

## 3.1 Prime Order

Let's start with the easiest case: groups of order 1.

**Theorem 3.4.** *Any group of order 1 is isomorphic to the trivial group, $\mathbf{1}$.*

*Proof.* Any group $G$ must have an identity element, and so that's all our possible elements used up! All groups of order 1 are isomorphic to the trivial group, $\mathbf{1}$. □

What about groups of prime order?

**Theorem 3.5.** *For a prime $p$, any group of order $p$ is isomorphic to $C_p$.*

*Proof.* Let $G$ be a group of order $p$, where $p$ is a prime number. Then Lagrange's Theorem tell us all elements must have order 1 or $p$. Pick some $x \in G$ with $x$ having order $p$. Then $\langle x \rangle = G$ so G is cyclic of order $p$. □

## 3.2 Groups of Order $p^2$

Slightly harder, are groups of order $p^2$. Sylow's Theorems, which we will make extensive use of later, are not so useful here. Nonetheless, we have:

**Theorem 3.6.** *For a prime $p$, any group of order $p^2$ is isomorphic to one of:*

$$C_{p^2} \quad or \quad C_p \times C_p$$

*Proof.* Let $G$ be a group of order $p^2$. By Lagrange's Theorem, the elements of $G$ have order 1, $p$ or $p^2$. We will consider these possible cases in turn:

**Case 1:** $G$ has an element of order $p^2$.

If $x \in G$ has order $p^2$, then $x$ generates $G$ so $G \cong C_{p^2}$.

**Case 2:** $G$ has no element of order $p^2$.

If $G$ does not have an element of order $p^2$ then all elements, except the identity, have order $p$. We know that $G$ must have a subgroup of order $p$, $P$, and because $p$ is prime, $P \cong C_p$. Pick a generator for $P$, say $x$ and an element $y \in G$ such that $y \notin P$.

The intersection is a subgroup of both $P$ and $\bar{P}$. It must be a proper subgroup because we chose $y \notin P$. Lagrange's Theorem tells us it must be trivial and Lemma 3.1 tells us that both $P$ and $\bar{P}$ are normal, and by Lemma 2.9, $|P\bar{P}| = p^2 = |G|$, so:

$$G = P \times \bar{P} \cong C_p \times C_p$$

□

## 3.3 Groups of Order $p^3$

Finally, we come to our last general argument for $p$-groups: order $p^3$.

**Theorem 3.7.** *For a prime $p$, any group of order $p^3$ is isomorphic to one of:*

$$C_{p^3}, \quad C_{p^2} \times C_p, \quad C_p \times C_p \times C_p$$

$$D_8 \quad or \quad Q_8 \qquad (\text{when } p = 2)$$

$$\mathrm{UT}_3(p) \quad or \quad C_{p^2} \rtimes C_p \qquad (\text{when } p \geqslant 3)$$

Before we begin the proof, it will be beneficial to briefly recall a couple of definitions.

**Definition 3.8.** The commutator of $a$ and $b$ is $a^{-1}b^{-1}ab$, denoted by $[a, b]$.

The derived subgroup of $G$, $G' = \langle\, [x, y] \mid x, y \in G \,\rangle$, is the smallest normal subgroup such that $G/G'$ is abelian.

The remainder of the section will be the proof of Theorem 3.7. The proof is based on the one found on the Groupprops subwiki[1]. Let $G$ be a group of order $p^3$, where $p$ is a prime number. We will first gain a handle on $G$ by describing its centre, and quotient by it. If $G$ is abelian, we know by the Fundamental Theorem of Finite Abelian Groups that it is isomorphic to one of:

$$C_{p^3}, \quad C_{p^2} \times C_p \quad \text{or} \quad C_p \times C_p \times C_p \tag{$\star$}$$

So from now on, we will focus on the non-abelian groups.

Denote the centre of $G$ by $Z$ and consider its order. Lagrange's Theorem tells us $Z$ must have order dividing $p^3$. It cannot be $p^3$ because $G$ is non-abelian, and Lemma 3.2 tells us that it cannot be 1. If $|Z| = p^2$, then $|G/Z| = p$, so $G/Z \cong C_p$. However Lemma 3.3 says that then $G$ must be abelian, so then $|Z|$ must be $p$. By our previous classification, $G/Z$ is isomorphic to either $C_{p^2}$ or $C_p \times C_p$. Lemma 3.3 tells us that it must by the latter.

This gives us a handle to start investigating the structure of $G$. We saw that $G/Z$ is abelian, so $G' \leqslant Z$, but because $G'$ is non-trivial, we must have equality.

So far, we know $G/Z \cong C_p \times C_p$, and that $G' = Z$. Now pick two elements, $a$ and $b$ so that $aZ$ and $bZ$ generate $G/Z$. So then $G = \langle\, Z,\, a,\, b \,\rangle$.

Let $z = [a, b]$. If $z = 1$ then that means $a$ and $b$ commute. And by definition, $a$ commutes with $Z$, so $a \in Z$, which contradicts our choice of $a$ as a generator of $G/Z$. Hence $z \neq 1$, and in particular, $a$ and $b$ do not commute. Now we know $G' = Z$ which has order $p$, so $Z \cong C_p$. Moreover, $z \in Z$, and $z \neq 1$ so we can conclude that $\langle\, z \,\rangle = Z$. We can see that although $a$ and $b$ are not in $Z$, $a^p$ and $b^p$ are, because $aZ$ and $bZ$ have order $p$ in $G/Z$. Considering the orders of $a$ and $b$ we have 3 cases: both have order $p$, one of order $p$ and the other $p^2$, and both of order $p^2$.

Because when $p$ is odd or even have different behaviours, we will break the remainder of the proof into two sub-lemmas.

**Lemma 3.9.** *For an odd prime $p$, the non-abelian groups of order $p^3$ are:*

$$\mathrm{UT}_3(p) \quad \textit{and} \quad C_{p^2} \rtimes C_p$$

*Proof.* Continuing from above, we have the following relations:

$$[a, b] = z \neq 1, \quad \langle\, z \,\rangle = \mathrm{Z}, \quad a,\, b \notin \mathrm{Z}, \quad \text{and} \quad a^p,\, b^p \in \mathrm{Z}$$

**Case 1:** Both $a$ and $b$ have order $p$.

Writing a presentation from the above relations:

$$G = \langle\, z,\, a,\, b \mid z^p = a^p = b^p = 1,\ az = za,\ bz = zb,\ [a, b] = z \,\rangle \tag{$\dagger$}$$

We can write an arbitrary $g \in G$ as $a^i b^j z^k$ for integers $i$, $j$ and $k$ taken mod $p$. Hence the group definied by ($\dagger$) has order at most $p^3$, but not necessarily realised by a group of order $p^3$.

1. Groupprops, "Classification of groups of prime-cube order," February 24, 2016, accessed February 23, 2023, https://groupprops.subwiki.org/wiki/Classification_of_groups_of_prime-cube_order.

Now consider the set:

$$\left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{F}_p \right\}$$

It can be shown that this is a group under the usual matrix multiplication, and is known as the unitriangular group[2], denoted $\mathrm{UT}_3(p)$. Taking:

$$z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

we can check that $\mathrm{UT}_3(p)$ satisfies (†). (Indeed, it is the standard presentation defining $\mathrm{UT}_3(p)$). Thus (†) is realised by a group of order $p^3$, so there is a single isomorphism class for this case.

We will quickly show that $\mathrm{UT}_3(p)$ has elements of order at most $p$. Any element can be written as $a^i b^j z^k$. Moreover, $a$ and $b$ commute with $z$, and we can swap $a$ and $b$ at the price of introducing a $z$. So an element has order dividing $\mathrm{lcm}(\mathrm{o}(a), \mathrm{o}(b), \mathrm{o}(z)) = p$. Hence an element of $\mathrm{UT}_3(p)$ has order 1 or $p$.

**Case 2:** One element of each order $p$ and $p^2$.

The roles of $a$ and $b$ are interchangeable, so we can take $a$ to have order $p^2$ and $b$ to have order $p$ without loss of generality. So $\langle\, a \,\rangle \cong C_{p^2}$, and has index $p$, so by Lemma 3.1 is normal. We noted that $a^p \neq 1$ is in $Z$, and so $a^p$ is some power of $z$, say $a^p = z^r$, with $r$ taken mod $p$.

We know that $0 < r < p$ and I claim that we can take $r = 1$ without loss of generality. Because $z^r \neq 1$, it has order $p$. Hence $z^r$ generates $Z$. Because $p$ is prime, $Z \cong C_p$ is a finite field. Therefore, there exists some integer $s$, with $0 < s < p$ such that $sr \equiv 1 \pmod{p}$. Then say $\bar{a} = a^s$ so:

$$\bar{a}^p = (a^s)^p = (a^p)^s = (z^r)^s = z \pmod{p}$$

Indeed we can take $\bar{a}^p$ to be any power of $z$ we want using a similar substitution. This is actually a sneaky application of Lemma 2.10, because it will turn out that $G$ can be written as a semidirect product.

Take $a^p = z$. Now because $G = \langle\, z, a, b \,\rangle$, $\langle\, a \,\rangle \cap \langle\, b \,\rangle = \mathbf{1}$. So we can conclude that $G = \langle\, a \,\rangle \rtimes \langle\, b \,\rangle$. How does $b$ conjugate $a$? Consider:

$$a^{-1}b^{-1}ab = [a, b] = z = a^p$$

So:

$$b^{-1}ab = a^{p+1}$$

Thus:

$$G \cong C_{p^2} \rtimes C_p$$

2. Groupprops, "Unitriangular matrix group:UT(3,p)," August 22, 2014, accessed February 23, 2023, https://groupprops.subwiki.org/wiki/Unitriangular_matrix_group:UT(3,%20p).

With presentation:

$$\langle\, a,\, b \mid a^{p^2} = b^p = 1,\ b^{-1}ab = a^{p+1} \,\rangle$$

We can see that this is not isomorphic to the previous case, because $\mathrm{UT}_3(p)$ has no element of order $p^2$.

**Case 3:** Both $a$ and $b$ have order $p^2$.

We will show that by substitutions, this is equivalent to the above. In the previous case, we saw that we can take $a^p$ and $b^p$ to be arbitrary powers of $z$ using substitutions (which is Lemma 2.10 behind the scenes). So take $a^p = z$ and $b^p = z^{-1}$. Let $d = ab$, and consider:

$$d^p = (ab)^p = abab\ldots ab$$

We will collect together the $a$'s and $b$'s, maintaining equality with the commutator, $[a,\, b] = z$:

$$\begin{aligned} d^p &= ababab\ldots ab \\ &= zaabbab\ldots ab \\ &= z^2aababb\ldots ab \\ &= z^3aaabbb\ldots ab \\ &\vdots \\ &= z^{\frac{p(p-1)}{2}}a^pb^p \end{aligned} \tag{‡}$$

However, $z^p = 1$ so:

$$d^p = a^pb^p = zz^{-1} = 1$$

So $d$ has order $p$, and we are back in the previous case.

□

Now we will consider the case when $p = 2$.

**Lemma 3.10.** *The non-abelian groups of order 8 are:*

$$D_8 \quad \textit{and} \quad Q_8$$

*Proof.* Leveraging the work we've already done, let's consider how the 3 cases behave when $p = 2$:

**Case 1:** $G \cong \mathrm{UT}_3(2)$.

The group behaves differently when $p = 2$ because we know that a group whose elements all have order either 1 or 2 is abelian. So the elements cannot have order only 1 or 2. In particular:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

have order 4. We can check that all other non-identity elements have order 2. Thus $\mathrm{UT}_2(2) \cong D_8$.

**Case 2:** $G = \langle\, a,\, b \mid a^{p^2} = b^p = 1,\ b^{-1}ab = a^{p+1} \,\rangle$

When $p = 2$, this reduces to the dihedral group, $D_8$.

**Case 3:** Both $a$ and $b$ have order 4.

If $p = 2$, (‡) does not hold, and the exponent of $z$ is 1 in that case. So we have the relations:

$$a^4 = b^4 = z^2 = 1, \quad a^2 = z, \quad b^2 = z^{-1} \quad \text{and} \quad [a,\, b] = z$$

Therefore:

$$b^{-1}ab = az = a^3 = a^{-1}$$

So we obtain the presentation:

$$G = \langle\, a,\, b \mid a^4 = b^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1} \,\rangle$$

Which we recognise as the quaternion group, $Q_8$.

□

So (⋆), together with the two lemmas, prove Theorem 3.7.

# Chapter IV

# Groups of Composite Orders

Now we will classify groups which have order a composite number, focusing only on numbers which have two distinct prime factors, $p$ and $q$. Groups of order $pq$ will be classified fully, but to attempt the same for $p^2q$ is beyond the scope of this report. Hence we will deal with groups which have order $4q$ and $2p^2$.

## 4.1 Groups of Order $pq$

**Theorem 4.1.** *For distinct primes $p$ and $q$, any group of order $pq$ is isomorphic to one of:*

$$C_{pq}$$

$$C_p \rtimes C_q = \langle\, x, y \mid x^p = y^q = 1,\ y^{-1}xy = x^a \,\rangle \qquad \text{(additionally, if } q \mid p-1\text{)}$$

*where $a$ is a generator for the subgroup of order $q$ in $(\mathbb{Z}/p\mathbb{Z})^\times$.*

We will devote the rest of the section to the proof of this Theorem. Let $G$ be a group of order $pq$ where $p$, $q$ are prime numbers with $p > q$, and let $n_p$ and $n_q$ denote the number of Sylow p-subgroups and Sylow q-subgroups of $G$ respectively. Then by Sylow's Theorems:

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q$$

So $G$ has a unique Sylow $p$-subgroup, say $P \trianglelefteq G$, and a Sylow $q$-subgroup, $Q \leqslant G$. Because $p$ and $q$ are prime numbers, $P \cong C_p$ and $Q \cong C_q$. Pick generators for each, say $x$ and $y$. So:

$$P = \langle\, x \,\rangle \cong C_p \quad \text{and} \quad Q = \langle\, y \,\rangle \cong C_q$$

We have 2 possibilities for $n_q$: $p-1$ is a multiple of $q$ or 1.

**Lemma 4.2.** *For distinct primes $p$ and $q$, with $q \nmid p-1$, any group of order $pq$ is isomorphic to $C_{pq}$.*

*Proof.* If $p-1$ is not a multiple of $q$, then $n_q = 1$ and $Q \trianglelefteq G$, hence:

$$G = P \times Q \cong C_{pq}$$

□

**Lemma 4.3.** *For distinct primes $p$ and $q$, with $q \mid p-1$, any group of order $pq$ is isomorphic to one of:*

$$C_{pq} \quad \text{or}$$

$$C_p \rtimes C_q = \langle\, x, y \mid x^p = y^q = 1,\ y^{-1}xy = x^a \,\rangle$$

*where $a$ is a generator for the subgroup of order $q$ in $(\mathbb{Z}/p\mathbb{Z})^\times$.*

*Proof.* If $p-1$ is a multiple of $q$, then $n_q$ could be $p$ as well as 1. So $Q$ is not necessarily normal in $G$. Therefore we have another group as well as the direct product. Concentrating on this group, Lagrange's Theorem tell us $P \cap Q = \mathbf{1}$ and by Lemma 2.9, $|PQ| = pq$. Hence we have $G = P \rtimes Q$, some non-trivial semidirect product.

By Lemma 2.3, $\operatorname{Aut} C_p \cong (\mathbb{Z}/p\mathbb{Z})^\times \cong C_{p-1}$. So if $\nu \in (\mathbb{Z}/p\mathbb{Z})^\times$, then $x \mapsto x^\nu$ is an automorphism. We know also that $C_{p-1}$ has a unique subgroup of order $q$, hence $G$ has the presentation:

$$G = \langle\, x, y, \mid x^p = y^q = 1,\ y^{-1}xy = x^a \,\rangle$$

where $a$ is a generator for the subgroup of order $q$ in $(\mathbb{Z}/p\mathbb{Z})^\times$.

Notice that picking different generators are equivalent up to isomorphism because the composition of two isomorphisms is an isomorphism. □

Thus Theorem 4.1 is proved.

## 4.2 Groups of Order 2$p$

To illustrate an example of groups of order $pq$, let's take $q = 2$.

**Theorem 4.4.** *Any group of order $2p$ is isomorphic to one of:*

$$C_{2p} \quad \text{or} \quad D_{2p}$$

*Proof.* Because every prime greater than 2 is odd, $p-1$ is an even number, and so $2 \mid p-1$.

An element $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ of order 2 satisfies $a^2 = 1$, hence $a = 1$ or $-1$. But 1 has order 1, so $a$ can only be $-1$. Side-note: from the proof of Lemma 2.3, this corresponds to the inverse map.

So, in addition to $C_{2p}$, we have:

$$G \cong \langle\, x, y \mid x^p = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$$

Which is the presentation for the dihedral group of order $2p$, $D_{2p}$. □

## 4.3 Groups of Order $4q$

As discussed at the beginning of this chapter, we will only classify certain groups of order $p^2q$. First, we will classify groups of order $4q$ for a prime $q > 3$, in the next section, groups of order $2p^2$, then in the next chapter, return to the special case of groups of order 12.

**Theorem 4.5.** *For a prime $q > 3$, any group of order $4p$ is isomorphic to one of:*

$$C_{4q}, \quad C_{2q} \times C_2, \quad D_{4q} \quad \text{or} \quad \mathrm{Dic}_{4q}$$

$$\langle\, x,\, t \mid x^q = t^4 = 1,\, t^{-1}xt = x^a \,\rangle \qquad \text{(additionally, if } 4 \mid q-1\text{)}$$

*where $a$ is the generator of the subgroup of order 4 in $(\mathbb{Z}/q\mathbb{Z})^\times$.*

We will prove Theorem 4.5 in the rest of this section. Let $q > 3$ be a prime number, and $G$ be a group of order $4q$. Denote the number of Sylow $q$-subgroups by $n_q$. Then $n_q$ must divide 4, so could be 1, 2 or 4, and must be congruent to 1 mod $q$.

If $q = 3$, then $G$ could have 4 Sylow $q$-subgroups, so we will classify groups of order 12 later. If $q = 2$, then we have a group of order $p^3$, which we have already classified. This is why we take $q > 3$. So $G$ has a normal Sylow $q$-subgroup, $Q \cong C_q$. Let $x$ generate $Q$.

Lagrange's Theorem, together with Lemma 2.9, tell us that a Sylow 2-subgroup, $T$, intersects trivially with $Q$, and $|QT| = |G|$. Hence, $G = Q \rtimes T$.

We know by Lemma 2.3, that $\operatorname{Aut} Q \cong C_{q-1}$. So we have two cases: $T \cong V_4$ or $T \cong C_4$.

**Lemma 4.6.** *For a prime $q > 3$, let $G$ be a group of order $4q$. Then if $G$ has a Sylow 2-subgroup isomorphic to $V_4$, then $G$ is isomorphic to one of:*

$$C_{2q} \times C_2 \quad \text{or} \quad D_{4q}$$

*Proof.* We saw in our classification of groups of order $2p$, that $(\mathbb{Z}/q\mathbb{Z})^\times$ has a unique element of order 2, corresponding to the inversion map. So Lemma 2.10 tells us that there is a single non-trivial homomorphism $\psi : T \to \operatorname{Aut} Q$.

If $\psi$ is trivial, then we obtain the product:

$$G \cong C_q \times V_4 \cong C_{2q} \times C_2$$

If $\psi$ is non-trivial, it maps $T$ to the subgroup generated by the inversion map, isomorphic to $C_2$. Therefore the kernel is isomorphic to $C_2$, so pick $z$ such that it generates the kernel. Denote the other generator of $T$ by $y$, then we obtain the following presentation:

$$G = \langle\, x,\, y,\, z \mid x^q = y^2 = z^2 = 1,\; yz = zy,\; xz = zx,\; y^{-1}xy = x^{-1} \,\rangle$$

Now let $a = xz$, and we will show that $G \cong D_{4p}$.

Firstly, notice that the order of $a$ is $4q$, and:

$$a^q = x^q z^q = z \quad \text{and} \quad a^{q-1} = x^{q-1} z^{q-1} = x^{q-1}$$

Now consider:

$$y^{-1}ay = y^{-1}xzy = y^{-1}xyz = x^{-1}z = a^{q-1}a^q = a^{2q-1} = a^{-1}$$

Hence:

$$G = \langle\, a,\, y \mid a^{2q} = y^2 = 1,\ y^{-1}ay = a^{-1} \,\rangle$$

which we recognise as $D_{4q}$. □

**Lemma 4.7.** *For a prime $q > 3$ and $4 \nmid q - 1$, let $G$ be a group of order $4q$. Then if $G$ has a Sylow 2-subgroup isomorphic to $C_4$, then $G$ is isomorphic to one of:*

$$C_{4q} \quad \textit{or} \quad \mathrm{Dic}_{4q}$$

*Proof.* Let $t$ generate $T$. If $4 \nmid q - 1$, then $q \equiv 3 \mod 4$. So then $\operatorname{Aut} Q$ has no subgroup of order 4, and a homomorphism, $\psi$ must map $T$ to either the trivial group, or the group generated by the inverse automorphism.

If $T\psi$ is trivial, then we recover the direct product, $C_q \times C_4 \cong C_{4q}$.

If $T\psi$ is non-trivial, then $G$ has the presentation:

$$G = \langle\, x,\, t \mid x^q = t^4 = 1,\ t^{-1}xt = x^{-1} \,\rangle$$

Let $a = xt^2$. Then:

$$a^q = xt^2 \dots xt^2 = x^q t^{2q} = t^{2q}$$

We know $q \equiv 3 \mod 4$, so for some $n$, $q = 4n + 3$. Thus $2q = 8n + 6 = 4(2n + 1) + 2$. So then:

$$a^q = t^{4(2n+1)+2} = t^2$$

Additionally:

$$t^{-1}at = t^{-1}xt^2t = (t^{-1}xt)t^2 = x^{-1}t^2 = t^2x^{-1} = a^{-1}$$

Hence:

$$G = \langle\, a,\, t \mid a^{2q} = 1,\ a^q = t^2,\ t^{-1}at = a^{-1} \,\rangle$$

This is known as the binary dihedral or dicyclic group[1], denoted $\mathrm{Dic}_{4q}$. □

**Lemma 4.8.** *For a prime $q > 3$ and $4 \mid q - 1$, let $G$ be a group of order $4q$. Then if $G$ has a Sylow 2-subgroup isomorphic to $C_4$, then $G$ is isomorphic to one of:*

$$C_{4q}, \quad \mathrm{Dic}_{4q} \quad \textit{or} \quad \langle\, x,\, t \mid x^q = t^4 = 1,\, t^{-1}xt = x^a \,\rangle$$

*Proof.* Our classification for when $4 \nmid q - 1$ still holds, so we could be isomorphic to:

$$C_{4q} \quad \text{or} \quad \mathrm{Dic}_{4q}$$

If $4 \mid q - 1$, i.e. $q \equiv 1 \mod 4$, then $\operatorname{Aut} Q$ contains a unique element of order 4, and so has a unique subgroup generated by it. We know by Lemma 2.3, that $\operatorname{Aut} Q \cong (\mathbb{Z}/q\mathbb{Z})^\times$,

1. Groupprops, "Dicyclic Groups," October 21, 2017, accessed January 19, 2023, https://groupprops.subwiki.org/wiki/Dicyclic_group.

so say $a$ is the generator of the subgroup of order 4 in $(\mathbb{Z}/q\mathbb{Z})^\times$. So we obtain another homomorphism, mapping $T$ to this subgroup, and we get a group with the presentation:

$$G = \langle\, x,\, t \mid x^q = t^4 = 1,\ t^{-1}xt = x^a \,\rangle$$

□

Hence we have proved our result.

## 4.4 Groups of Order $2p^2$

**Theorem 4.9.** *For an odd prime $p$, any group of order $2p^2$ is isomorphic to one of:*

$$C_{2p^2}, \quad D_{2p^2}, \quad C_p \times C_{2p}, \quad C_p \times D_{2p} \quad \textit{or} \quad \mathrm{Dih}(C_p \times C_p)$$

Again, the remainder of this section will prove this. Let $G$ be a group of order $2p^2$, with $p > 2$. Denote the number of Sylow $p$-subgroups by $n_p$. By Sylow's Theorems, $n_p$ divides 2, and is congruent to 1 mod $p$, so must be 1. Hence, $G$ has a normal Sylow $p$-subgroup, $P$ of order $p^2$.

If $T$ is a Sylow 2-subgroup, then by applying Lagrange's Theorem and Lemma 2.9, we can conclude that $G = P \rtimes T$. From our classification of groups of order $p^2$, we have 2 choices for $P$: $C_{p^2}$ or $C_p \times C_p$.

**Lemma 4.10.** *For an odd prime $p$, let $G$ be a group of order $2p^2$. Then if $G$ has a Sylow $p$-subgroup isomorphic to $C_{p^2}$, $G$ is isomorphic to one of:*

$$C_{2p^2} \quad \textit{or} \quad D_{2p^2}$$

*Proof.* From Lemma 2.3, we know $|\mathrm{Aut}\, P| = p^2 - p = p(p-1)$. Because $p$ is prime, $2 \nmid p$, but $2 \mid p-1$, so $\mathrm{Aut}\, P$ has a unique element of order 2. Hence, a homomorphism $\psi : T \to \mathrm{Aut}\, P$ has image isomorphic to either $\mathbf{1}$ or $C_2$.

So in addition to the direct product for a trivial homomorphism, $G \cong C_{2p^2}$, we have $G \cong C_{p^2} \rtimes C_2$, with $C_2$ acting by inversion. If $x$ generates $P$, and $y$ generates $T$, we have the presentation:

$$G = \langle\, x,\, y \mid x^{p^2} = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$$

which we recognise as $D_{2p^2}$, the dihedral group of order $2p^2$. □

**Lemma 4.11.** *For an odd prime $p$, let $G$ be a group of order $2p^2$. Then if $G$ has a Sylow $p$-subgroup isomorphic to $C_p \times C_p$, $G$ is isomorphic to one of:*

$$C_p \times C_{2p}, \quad C_p \times D_{2p} \quad \textit{or} \quad \mathrm{Dih}(C_p \times C_p)$$

*Proof.* Consider $P$ as the product of the subgroups generated by $a$ and $b$, i.e. $P = \langle a \rangle \times \langle b \rangle$. Then the action of $T$ on $P$ can either be trivial on both subgroups, invert one, or invert both.

If the action is trivial on both subgroups, then we obtain the direct product $G \cong C_p \times C_{2p}$.

If the action is non-trivial on just one of the subgroups, then we can consider only one case. This is because they are equivalent up to an isomorphism of $T$, and Lemma 2.10 tells us the resulting semidirect products are isomorphic. So we have:

$$G = \langle a \rangle \times (\langle b \rangle \rtimes T) \cong C_p \times D_{2p}$$

Finally, if we choose to invert both subgroups, then we act on all of $P$ by inversion. So if $a$ and $b$ generate $P$, then:

$$G = \langle\, a,\, b,\, x \mid a^p = b^p = x^2 = 1,\ ab = ba,\ x^{-1}ax = a^{-1},\ x^{-1}bx = b^{-1} \,\rangle$$

Because $C_p$ has all elements of order p, excluding 1, and they are all automorphic to each other (meaning that some automorphism maps one to the other), $x^{-1}gx = g^{-1}$ for all $g \in P$. Hence:

$$G = \langle\, P,\, x \mid x^2 = 1,\ x^{-1}gx = g^{-1} \text{ for all } g \in P \,\rangle$$

which is known as the generalised dihedral group[2] for $P$, denoted $\mathrm{Dih}(P)$. □

And so Theorem 4.9 is proven.

2. Groupprops, "Generalized dihedral group," January 17, 2011, accessed February 15, 2023, https://groupprops.subwiki.org/wiki/Generalized_dihedral_group.

# Chapter V

# Groups of Particular Orders

Finally, let's deal with the groups of particular order which have been missed by our general classifications. Namely these orders are 12, 24, 30, and 16.

## 5.1 Groups of Order 12

We have seen that groups of order 12 have slightly different behaviour to groups of order $4q$ in general, and we will need this classification in order to classify groups of order 24.

**Theorem 5.1.** *Any group of order 12 is isomorphic to one of:*

$$C_{12}, \quad C_2 \times C_6, \quad A_4, \quad D_{12} \quad \textit{or} \quad \mathrm{Dic}_{12}$$

This section will be devoted to the proof of Theorem 5.1. Let $G$ be a group of order $12 = 2^2 \cdot 3$, and $n_3$ denote the number of Sylow 3-subgroups. By Sylow's Theorems:

$$n_3 \equiv 1 \pmod 3 \quad \text{and} \quad n_3 \mid 4$$

Hence:

$$n_3 = 1 \text{ or } 4$$

Let $H$ be a Sylow 2-subgroup and $K$ be a Sylow 3-subgroup of $G$, generated by $x$.

Lagrange's Theorem tells us $H$ has elements of order 1, 2, and 4, and $K$ has elements of order 1 and 3. Hence $H \cap K = \mathbf{1}$. Lemma 2.9 tells us:

$$|HK| = |H| \cdot |K| = 12$$

Hence $G = HK$ and $H \cap K = \mathbf{1}$.

We have 4 cases, two choices for what the Sylow 2-subgroup is like, and 2 for if the Sylow 3-subgroup is normal or not. Let's consider each case in turn, starting with if there is a normal Sylow 3-subgroup.

If $K$ is normal, then we conclude that $G = K \rtimes H$. Let's consider the automorphism groups of the possibilities for $H$. Since an automorphism, $\varphi$, must map generators to generators, $\operatorname{Aut} C_4 \cong C_2$ because $C_4$ has two generators. An automorphism of $V_4$ corresponds to a permutation of the three non-identity elements, hence $\operatorname{Aut} V_4 \cong S_3$.

**Lemma 5.2.** *Let $G$ be a group of order 12, with Sylow 2-subgroup isomorphic to $C_4$ and normal Sylow 3-subgroup. Then $G$ is isomorphic to one of:*

$$C_{12} \quad \textit{or} \quad \mathrm{Dic}_{12}$$

*Proof.* We have $G = K \rtimes H$ for Sylow 3-subgroup $K = \langle x \rangle$ and Sylow 2-subgroup $H = \langle y \rangle$. We know $\operatorname{Aut} C_3 \cong C_2$ so a homomorphism $\psi : H \to \operatorname{Aut} K$ maps $H$ to the trivial group or to $\langle \beta : x \mapsto x^{-1} \rangle$.

If $H\psi = \mathbf{1}$ then $G = K \times H \cong C_3 \times C_4 \cong C_{12}$.

If $H\psi = \langle \beta \rangle$ then we have:

$$G = \langle x, y \mid x^3 = y^4 = 1,\ y^{-1}xy = x^{-1} \rangle$$

Now let $a = xy^2$. And remember, $y^{-1}xy = x^{-1}$ means $x$ commutes with $y^2$. So now:

$$a^3 = xy^2xy^2xy^2 = x^3y^6 = y^2$$

and:

$$y^{-1}ay = y^{-1}xy^2y = (y^{-1}xy)y^2 = x^{-1}y^2 = y^2x^{-1} = a^{-1}$$

So:

$$G = \langle a,\ y \mid a^6 = 1,\ a^3 = y^2,\ y^{-1}ay = a{-}1 \rangle$$

which we recognise as $\mathrm{Dic}_{12}$. This group is also sometimes denoted by $T$. □

**Lemma 5.3.** *Let $G$ be a group of order 12, with Sylow 2-subgroup isomorphic to $V_4$ and normal Sylow 3-subgroup. Then $G$ is isomorphic to one of:*

$$C_2 \times C_6 \quad \textit{or} \quad D_{12}$$

*Proof.* We have $G = K \rtimes H$ for Sylow 3-subgroup $K = \langle x \rangle$ and Sylow 2-subgroup $H = \langle y, z \rangle$. If $\psi : H \to \operatorname{Aut} K$ is trivial then we obtain the direct product $C_2 \times C_6$.

We know $\operatorname{Aut} K \cong C_2$, so there are 3 choices of elements in $H$ to send to it, but they are all equivalent up to isomorphism, by Lemma 2.10, taking $\alpha$ to be the identity map.

Furthermore, we know that $H/\operatorname{im}\psi \cong \ker\psi$, so $\ker\psi$ must be isomorphic to $C_2$. Pick $z$ so that it generates the kernel, and so the remaining generator, $y$ is not in the kernel. Then:

$$G = \langle x,\ y,\ z \mid x^3 = y^2 = z^2 = 1,\ yz = zy,\ xz = zx,\ y^{-1}xy = x \rangle$$

Let $a = xz$. So:

$$a^3 = x^3z^3 = z$$

and:

$$y^{-1}ay = y^{-1}xzy = y^{-1}xyz = x^{-1}z = x^2z = a^{-1}$$

Hence:

$$G = \langle y,\ a \mid a^6 = y^2 = 1,\ y^{-1}ay = a^{-1} \rangle \cong D_{12}$$

□

**Lemma 5.4.** *Let $G$ be a group of order 12, with Sylow 2-subgroup isomorphic to $C_4$ and 4 Sylow 3-subgroups. Then $G$ is isomorphic to $C_{12}$.*

*Proof.* If $G$ has 4 Sylow 3-subgroups, then there are 8 elements of order 3 in $G$. So the remaining 4 must form the Sylow 2-subgroup, hence it is normal. Thus $G = H \rtimes K$.

Let $H = \langle\, y \,\rangle$. A homomorphism $\psi : K \to \operatorname{Aut} H \cong C_2$, preserves order and together with Lagrange's Theorem means that the only possibility for $\psi$ is trivial, i.e. $K\psi = \mathbf{1}$. □

**Lemma 5.5.** *Let $G$ be a group of order 12, with Sylow 2-subgroup isomorphic to $v_4$ and 4 Sylow 3-subgroups. Then $G$ is isomorphic to $A_4$.*

*Proof.* As before, we know the Sylow 2-subgroup is normal, and $G = H \rtimes K$. Let $H = \langle\, y,\, z \,\rangle$. A trivial homomorphism $K\psi = \mathbf{1}$ yields the direct product, $C_2 \times C_6$, but this has a normal (and hence unique) Sylow 3-subgroup. What non-trivial homomorphisms are there?

The automorphism group, $\operatorname{Aut} H \cong S_3$ is of order 6, and so has a unique subgroup of order 3, by Sylow's Theorems. We know that a homomorphism $\psi : K \to \operatorname{Aut} H$ is determined by where it sends the generator $x$, so for $\psi$ to be non-trivial, it must send $x$ to an element of order 3 in $\operatorname{Aut} H$.

There are 2 such elements. Because $\operatorname{Aut} H \cong S_3$, we will think of them as the permutations of order 3 of the set $\{1, 2, 3\}$. Denote them $a = (1\ 2\ 3)$ and $b = (1\ 3\ 2)$. Notice that $b = a^{-1}$, so we have homomorphisms:

$$\psi_1 : x \mapsto a \quad \text{and} \quad \psi_2 : x \mapsto a^{-1}$$

It appears we have 2 choices, but this is not the case. The inverse map, $\beta : x \mapsto x^{-1}$, is an automorphism of K, and so by Lemma 2.10, the corresponding semidirect products of $\psi_1$ and $\psi_2$ are isomorphic. Hence (up to isomorphism) there is one non-trivial homomorphism $\psi : K \to \operatorname{Aut} H$. So $x \in K$ acts by permuting the 3 non-identity elements of $H$.

We will show that in this case, $G \cong A_4$. First, let's check $A_4$ has the same subgroup structure as $G$. There is a subgroup isomorphic to $C_3$ in $A_4$, generated by the 3-cycle $(1\ 2\ 3)$:

$$\bar{K} = \langle\, (1\ 2\ 3) \,\rangle$$

We can also find a subgroup isomorphic to $V_4$:

$$\bar{H} = \{\, 1,\, (1\ 2)(3\ 4),\, (1\ 3)(2\ 4),\, (1\ 4)(2\ 3) \,\}$$

Indeed, we can check that $\bar{H}$ is normal in $A_4$. We can see that $\bar{H} \cap \bar{K} = \mathbf{1}$ because $\bar{H}$ contains no 3-cycles, and that $\bar{H}\bar{K} = A_4$. So we can conclude that $A_4 = \bar{H} \rtimes \bar{K}$.

Let's investigate how conjugation behaves. If we let $\alpha = (1\ 2)(3\ 4)$, $\beta = (1\ 4)(2\ 3)$ and $\gamma = (1\ 2\ 3)$, then we can write an element of $A_4$ as $\alpha^i\beta^j\gamma^k$ for some $i$, $j$ and $k$. Define $\phi : A_4 \to G$ by $\phi : \alpha^i\beta^j\gamma^k \mapsto x^i y^j z^k$. Then:

$$\beta\phi = (\gamma^{-1}\alpha\gamma)\phi = c^{-1}ac = b$$

So conjugation acts in the same way. Hence we can conclude that $G \cong A_4$. □

The collection of these lemmas prove Theorem 5.1.

## 5.2 Groups of Order 24

**Theorem 5.6.** *Any group of order 24 is isomorphic to one of:*

| | | |
|---|---|---|
| $C_{24}$ | $C_3 \rtimes C_8$ | $C_{12} \times C_2$ |
| $\mathrm{Dic}_{12} \times C_2$ | $S_3 \times C_4$ | $C_3 \times C_2 \times C_2 \times C_2$ |
| $S_3 \times C_2 \times C_2$ | $C_3 \times D_8$ | $D_{24}$ |
| $C_3 \rtimes_{V_4} D_8$ | $C_3 \times Q_8$ | $\mathrm{Dic}_{24}$ |
| $S_4$ | $C_2 \times A_4$ | $\mathrm{SL}_2(3)$ |

In the rest of this section, we will prove this result. Let $G$ be a group of order 24, and let $H$ be a Sylow 3-subgroup of $G$, so $H \cong C_3$, and let $h$ generate $H$. Let $T$ by a Sylow 2-subgroup of $G$, so $T$ has order 8. By Lagrange's Theorem, $H \cap T = \mathbf{1}$ and then applying Lemma 2.9, $|HT| = 24$. Now let $n_3$ denote the number of Sylow 3-subgroups, and by Sylow's Theorems:

$$n_3 \equiv 1 \mod 3 \quad \text{and} \quad n_3 \mid 8$$

Hence $n_3$ is either 1 or 4.

If $n_3 = 1$, then $H$ is normal in $G$. Thus $G = H \rtimes T$. We'll want a homomorphism $\psi : T \to \operatorname{Aut} H$. We know $\operatorname{Aut} H \cong C_2$, and from our classification of groups of order 8, we have 5 possibilities. An action of $T$ on $H$ will have image isomorphic to $C_2$, and a kernel isomorphic to a group of order 4. We can classify the possible actions by considering the kernel.

**Lemma 5.7.** *Let $G$ be a group of order 24, with a normal Sylow 3-subgroup, and a Sylow 2-subgroup isomorphic to $C_8$. Then $G$ is isomorphic to one of:*

$$C_{24} \quad \textit{or} \quad C_3 \rtimes C_8$$

*Proof.* We have $G = H \rtimes T$, for the Sylow 3-subgroup $H$, and a Sylow 2-subgroup $T = \langle\, t \,\rangle$, with map $\psi : T \to \operatorname{Aut} H$. If $\psi$ is trivial, then:

$$G = T \times H \cong C_{24}$$

For a non-trivial $\psi$, the kernel is a subgroup of order 4. There is a unique subgroup of order 4 in $T$, generated by $t^2$. Hence $\langle\, t^2 \,\rangle$ is the kernel of a non-trivial $\psi$, so $\psi$ must send $t$ to the inversion map. Hence a non-trivial action of $T$ on $H$ is unique. We obtain:

$$G = \langle\, h, t \mid h^3 = t^8 = 1,\ h^{-1}th = t^{-1} \,\rangle \cong C_3 \rtimes C_8$$

□

**Lemma 5.8.** *Let $G$ be a group of order 24, with a normal Sylow 3-subgroup, and a Sylow 2-subgroup isomorphic to $C_4 \times C_2$. Then $G$ is isomorphic to one of:*

$$C_3 \times C_4 \times C_2, \quad \mathrm{Dic}_{12} \times C_2 \quad \textit{or} \quad S_3 \times C_4$$

*Proof.* We have $G = H \rtimes T$, for the Sylow 3-subgroup $H$, and a Sylow 2-subgroup $T \cong C_4 \times C_2$, with map $\psi : T \to \operatorname{Aut} H$. In this case, $T$ has subgroups isomorphic to both $C_4$ and $C_2 \times C_2$, so we have more possibilities for $\psi$. Firstly, if $\psi$ is trivial, then we obtain the direct product:

$$G \cong C_3 \times C_4 \times C_2$$

Let $T$ be generated by $x$ and $y$, where $x^4 = y^2 = 1$, and consider non-trivial $\psi$. Say the kernel of $\psi$ is isomorphic to $C_2 \times C_2$. So it must be generated by the elements of order 2 in $T$, $x^2$ and $y$. Then $\psi$ must map $x$ to the non-identity element in $\operatorname{Aut} H$: inversion. Hence $\langle\, x \,\rangle$ acts by inversion on $H$, giving:

$$\begin{aligned} G &= (H \rtimes \langle\, x \,\rangle) \times \langle\, y \,\rangle \\ &\cong (C_3 \rtimes C_4) \times C_2 \\ &\cong \mathrm{Dic}_{12} \times C_2 \end{aligned}$$

If instead the kernel is isomorphic to $C_4$, then it must be generated by an element of order 4 from $T$. However, all elements of order 4 are automorphic, and so by Lemma 2.10, we can pick $x$ to generate the kernel, without loss of generality. So then $\psi$ must map $y$ to inversion. Hence $\langle\, x \,\rangle$ acts trivially on $H$, and $\langle\, y \,\rangle$ acts by inversion. Thus:

$$\begin{aligned} G &= (H \rtimes \langle\, y \,\rangle) \times \langle\, x \,\rangle \\ &\cong (C_3 \rtimes C_2) \times C_4 \\ &\cong S_3 \times C_4 \end{aligned}$$

□

**Lemma 5.9.** *Let $G$ be a group of order 24, with a normal Sylow 3-subgroup, and a Sylow 2-subgroup isomorphic to $C_2 \times C_2 \times C_2$. Then $G$ is isomorphic to one of:*

$$C_3 \times C_2 \times C_2 \times C_2 \quad \text{or} \quad S_3 \times C_2 \times C_2$$

*Proof.* We have $G = H \rtimes T$, for the Sylow 3-subgroup $H$, and the specified Sylow 2-subgroup $T$, with map $\psi : T \to \operatorname{Aut} H$.

Let $\langle\, a, b, c \,\rangle = T$. All elements in $T$ have order 1 or 2, so cannot have subgroups isomorphic to $C_4$. However, $T$ does have subgroups isomorphic to $C_2 \times C_2$, which can be generated by 2 of the 3 generators of $T$. This gives us 3 subgroups, but permuting the generators $a$, $b$ and $c$ is an automorphism of $T$, so Lemma 2.10 tells us the resulting semidirect products are isomorphic. So choose $\psi$ such that $b$ and $c$ are in the kernel. Then $a\psi$ is either the identity map or the inversion map. If $\psi$ is trivial, then we obtain the direct product:

$$G \cong C_3 \times C_2 \times C_2 \times C_2$$

If $a\psi$ is inversion, then:

$$G = (H \rtimes \langle\, a \,\rangle) \times \langle\, b \,\rangle \times \langle\, c \,\rangle \cong S_3 \times C_2 \times C_2$$

□

**Lemma 5.10.** *Let $G$ be a group of order 24, with a normal Sylow 3-subgroup, and a Sylow 2-subgroup isomorphic to $D_8$. Then $G$ is isomorphic to one of:*

$$C_3 \times D_8, \quad D_{24} \quad \text{or} \quad C_3 \rtimes_{V_4} D_8$$

*Proof.* We have $G = H \rtimes T$, for the Sylow 3-subgroup $H$, and the specified Sylow 2-subgroup $T$, with map $\psi : T \to \operatorname{Aut} H$.

Let $r$ and $s$ generate $T$ with $r^4 = s^2 = 1$. A trivial homomorphism will yield the direct product:

$$G \cong C_3 \times D_8$$

So for a non trivial homomorphism, firstly assume $\ker \psi \cong C_4$. There is a unique subgroup in $T$ isomorphic to $C_4$, so its generated by an element of order 4. However the choice of generator is the same up to an isomorphism of $T$, so Lemma 2.10 lets us pick $r$ to be the generator, without loss of generality. Hence $s$ cannot be in the kernel, and so $s\psi$ is the inversion map. We obtain the presentation:

$$G = \langle\, h,\, r,\, s \mid h^3 = r^4 = s^2 = 1,\ hr = rh,\ s^{-1}rs = r^{-1},\ s^{-1}hs = h^{-1} \,\rangle$$

Let $a = hr$, and consider:

$$s^{-1}as = s^{-1}hrs = s^{-1}hrs^2s^{-1} = (s^{-1}hs)(srs^{-1}) = h^{-1}r^{-1} = r^{-1}h^{-1} = a^{-1}$$

So we have:

$$G = \langle\, a,\, s \mid a^{12} = s^2 = 1,\ s^{-1}as = a^{-1} \,\rangle$$

Which we recognise as $D_{24}$, the dihedral group of order 24.

If instead we consider $\psi$ with kernel isomorphic to $C_2 \times C_2$, then the kernel is generated by two elements of order 2. However, $T$ only has two elements of order 2, $r^2$ and $s$, so they must generate the kernel. So then $\psi$ must map $r$ to inversion. Hence this action is fully specified. So:

$$G \cong C_3 \rtimes_{V_4} D_8$$

We will use the above notation to mean the unique action with kernel isomorphic to $V_4$. □

**Lemma 5.11.** *Let $G$ be a group of order 24, with a normal Sylow 3-subgroup, and a Sylow 2-subgroup isomorphic to $Q_8$. Then $G$ is isomorphic to one of:*

$$C_3 \times Q_8, \quad \text{or} \quad \mathrm{Dic}_{24}$$

*Proof.* We have $G = H \rtimes T$, for the Sylow 3-subgroup $H$, and a Sylow 2-subgroup $T \cong Q_8$, with map $\psi : T \to \operatorname{Aut} H$.

Let $T$ be generated by $i$ and $j$, with the product denoted by $k$. That is:

$$T = \langle\, i,\, j \mid i^4 = j^4 = 1,\ i^2 = j^2,\ j^{-1}ij = i^{-1} \,\rangle$$

There is a single element of order 2 in $T$, hence $T$ has no subgroup isomorphic to $C_2 \times C_2$. The elements $i$, $j$ and $k$ each generate a cyclic subgroup in $T$. So $\psi$ will send one of them

to the kernel. We know that permuting these is an automorphism of $T$, so Lemma 2.10 tells us the choice results in isomorphic semidirect products.

So take $i \in \ker\psi$. Indeed $\langle\, i \,\rangle = \ker\psi$. Then for a non-trivial homomorphism, we must have $j \notin \ker\psi$. Otherwise:

$$i\psi \; j\psi = (ij)\psi = k\psi \in \ker\psi$$

making $\psi$ trivial.

Thus either $\psi$ is trivial and we obtain:

$$G \cong C_3 \times Q_8$$

or $\psi$ maps $j$ to the inversion map and we obtain the presentation:

$$G = \langle\, h,\, i,\, j \mid h^3 = i^4 = j^4 = 1,\; hi = ih,\; i^2 = j^2,\; j^{-1}hj = h^{-1},\; j^{-1}ij = i^{-1} \,\rangle$$

Now let $a = hi$. So:

$$a^6 = h^6 i^6 = i^2 = j^2$$

And:

$$j^{-1}aj = j^{-1}hij = j^{-1}hji^{-1} = h^{-1}i^{-1} = i^{-1}h^{-1} = a^{-1}$$

Hence:

$$G = \langle\, a,\, j \mid a^{12} = 1,\; a^6 = j^2,\; j^{-1}aj = a^{-1} \,\rangle$$

We recognise this as the dicyclic group of order 24, $\mathrm{Dic}_{24}$. □

Now we will consider groups of order 24 which don't have a normal Sylow 3-subgroup.

**Lemma 5.12.** *Let $G$ be a group of order 24, with a non-normal Sylow 3-subgroup. Then $G$ either has a normal Sylow 2-subgroup, or is isomorphic to $S_4$.*

*If $G$ does have a normal Sylow 2-subgroup, $T$, then it can be expressed as a semidirect product with a Sylow 3-subgroup, $H$, $G = T \rtimes H$.*

*Proof.* If $n_3 = 4$ then $H$ is not normal. We will proceed to show the result in a similar way to Borcherds[1].

The normaliser of $H$, $\mathrm{N}_G(H)$ has index 4. Now let $G$ act on the set of the cosets of $\mathrm{N}_G(H)$ by conjugation. Hence we obtain a homomorphism $\rho : G \to S_4$. The kernel is a subgroup of $\mathrm{N}_G(H)$ so must have order dividing 6 by Lagrange's Theorem.

The kernel cannot be of order 3, because $G$ has no normal subgroup of order 3 (because kernels of homomorphism are normal in the domain group). Likewise the kernel cannot be of order 6 because every group of order 6 has a unique Sylow 3-subgroup, which is characteristic. So by Lemma 2.5, it would be normal in $G$. Hence the kernel must have order 1 or 2.

If the kernel is of order 1, then $\rho$ is an isomorphism, so $G \cong S_4$.

1. Richard E. Borcherds, "Group theory 21: Groups of order 24," June 30, 2020, accessed February 9, 2023, https://www.youtube.com/watch?v=6TWuo2NO8vg.

If the kernel is of order 2, then we know that $G/\ker\rho \cong \operatorname{im}\rho$, so then $\operatorname{im}\rho$ must have order 12. It also cannot have a normal Sylow 3-subgroup, so looking at our classification of groups of order 12, this must be isomorphic to $A_4$. We know that $A_4$ has a normal subgroup of order 4, and so by the Correspondence Theorem, $G$ must contain a normal subgroup of order 8, say $T$.

Considering now the remaining groups of order 24 which will have a normal Sylow 2-subgroup, by Lagrange's Theorem and Lemma 2.9, we can conclude that $G = T \rtimes H$. □

Again, we have 5 cases which we will prove in 5 sub-lemmas, but this time we'll exclude the trivial homomorphism in the proofs, because that will just give us the direct product which we have already seen.

**Lemma 5.13.** *Let $G$ be a group of order 24, with a normal Sylow 2-subgroup, isomorphic to $C_8$. Then $G$ is isomorphic to $C_8 \times C_3$.*

*Proof.* By Lemma 5.12, $G = T \rtimes H$. An automorphism of $T$, $\varphi$, maps generators to generators, so say $\langle x \rangle = T$. Then $x\varphi$ could be $x$, $x^3$, $x^5$ or $x^7$. Hence, and Lagrange's Theorem tells us that there are no non-trivial homomorphisms $\psi : H \to \operatorname{Aut} T$. As a bonus: notice that each of these, apart from the identity, has order 2, so $\operatorname{Aut} C_8 \cong V_4$. □

**Lemma 5.14.** *Let $G$ be a group of order 24, with a normal Sylow 2-subgroup, isomorphic to $C_4 \times C_2$. Then $G$ is isomorphic to $C_4 \times C_2 \times C_3$.*

*Proof.* By Lemma 5.12, $G = T \rtimes H$. An automorphism of $T$, say $\phi$, preserves element order. So if:

$$T = \langle\, x,\, y \mid x^4 = y^2 = 1,\ xy = yx \,\rangle$$

then $x\phi$ must be of order 4, and $y\phi$ must be of order 2. Moreover, $y\phi$ cannot be in $\langle\, x\phi \,\rangle$ because $\phi$ is injective.

So we are reduced to 2 possible choices for $y\phi$, and 4 possible choices for $x\phi$. Because an automorphism is determined by its effect on generators, this gives us 8 possible automorphisms. Hence $|\operatorname{Aut} T| = 8$, and Lagrange's Theorem tells us that there are no non-trivial homomorphisms $\psi : H \to \operatorname{Aut} T$. □

**Lemma 5.15.** *Let $G$ be a group of order 24, with a normal Sylow 2-subgroup, isomorphic to $C_2 \times C_2 \times C_2$. Then $G$ is isomorphic to one of:*

$$C_2 \times C_2 \times C_2 \times C_3 \quad \textit{or} \quad C_2 \times A_4$$

*Proof.* By Lemma 5.12, $G = T \rtimes H$. To determine $\operatorname{Aut} T$ it is helpful to think of $C_2$ as the finite field with two elements. Then $T$ is isomorphic a 3 dimensional vector space over two elements. So an automorphism of that vector space is just any linear map, with non-zero determinant. Thus, $\operatorname{Aut} T \cong \mathrm{GL}_3(2)$.

We can determine that $|\mathrm{GL}_3(2)| = 168 = 2^3 \cdot 3 \cdot 7$, so $\operatorname{Aut} T$ has a Sylow 3-subgroup of order 3, isomorphic to $C_3$. Sylow's Theorems tells us that all subgroups of order 3 are

conjugate, so Lemma 2.10 tells us there is only one unique action (up to isomorphism) of $H$ on $T$. As before, pick a homomorphism, $\psi$, which will let us easily classify the resulting semidirect product.

Write $T = A \times B$ where $A \cong C_2$ and $B \cong C_2 \times C_2$. Then let $\psi$ map $H$ to the subgroup generated by the automorphism which fixes $A$ and permutes the non-identity elements of $B$ in a 3-cycle. This automorphism has order 3 by construction, so we can write:

$$G \cong C_2 \times (V_4 \rtimes C_3)$$

We know already that $V_4 \rtimes C_3 \cong A_4$, so $G \cong C_2 \times A_4$. □

**Lemma 5.16.** *Let $G$ be a group of order 24, with a normal Sylow 2-subgroup, isomorphic to $D_8$. Then $G$ is isomorphic to $D_8 \times C_3$.*

*Proof.* By Lemma 5.12, $G = T \rtimes H$. Let $\langle\, s, r \mid s^2 = r^4 = 1,\ s^{-1}rs = r^{-1} \,\rangle = T$. An automorphism, $\psi$, of $T$ preserves element order, so for $r\psi$ we have two choices, $r$ or $r^{-1}$. We can send $s\psi$ to any element of order 2 which is not in $\langle r\psi \rangle$. This leaves only reflections, of which there are 4: $s$, $rs$, $r^2s$ and $r^3s$. Hence there are 8 possible automorphisms of $D_8$, so $|\mathrm{Aut}\, D_8| = 8$. Lagrange's Theorem tells us that there are no non-trivial homomorphisms $\psi : H \to \mathrm{Aut}\, T$. □

**Lemma 5.17.** *Let $G$ be a group of order 24, with a normal Sylow 2-subgroup, isomorphic to $Q_8$. Then $G$ is isomorphic to one of:*

$$Q_8 \times C_3 \quad \textit{or} \quad \mathrm{SL}_2(3)$$

*Proof.* By Lemma 5.12, $G = T \rtimes H$. Firstly, for an automorphism, $\phi$, because:

$$i\phi j\phi = (ij)\phi$$

the image of $k$ under $\phi$ is determined by the images of $i$ and $j$. This reduces the possibilities for an automorphism. Additionally, $\pm 1$ are fixed by an automorphism, because they are the only elements of their order. So an automorphism could send $i$ to any of the 6 elements of order 4. The image of $j$ cannot be in the subgroup generated by the image of $i$, otherwise we wouldn't have an automorphism. Thus there are 4 choices for the image of $j$, giving us at most 24 possible automorphisms.

An element $g \in T$ can be written as $g = i^l j^u$ for some $0 \leqslant l,\ u \leqslant 3$. So then:

$$\begin{aligned} g\phi &= (i^l j^u)\phi \\ &= (i^l\phi)(j^u\phi) \\ &= (i\phi)^l (j\phi)^u \end{aligned}$$

Hence $g\phi$ is determined by $i\phi$ and $j\phi$, and each of the 24 possible combinations of images extends to a valid automorphism of $T$. Therefore, $|\mathrm{Aut}\, T| = 24$.

So $\mathrm{Aut}\, T$ will have a Sylow subgroup of order 3, and Sylow tells us all subgroups of order 3 are conjugate. If our homomorphism for the semidirect product is $\psi : H \to \mathrm{Aut}\, T$, then

Lemma 2.10 tells us that if $\psi$ maps $H$ to a given subgroup in $\operatorname{Aut} T$, then mapping $H$ to a conjugate subgroup produces an isomorphic semidirect product. Thus the choice of which subgroup of $\operatorname{Aut} T$ (if there is any choice) to map into is equivalent up to isomorphism.

If $t$ generates a subgroup of order 3 in $\operatorname{Aut} T$, then $\psi$ can map $t$ to either $h$ or $h^{-1}$. But again, Lemma 2.10 tells us that by applying $\beta : x \mapsto x^{-1}$ if necessary, the two choices result in isomorphic semidirect products. Hence $T \rtimes H$ is uniquely specified.

We will show that $\mathrm{SL}_2(3)$ is in that isomorphism class. For the sake of brevity, not every fact will be explicitly shown, and we will indicate where the reader may verify assertions made. Firstly, let $e_1$ and $e_2$ be the basis vectors for the vector space. We can find an element of order 3 in $\mathrm{SL}_2(3)$:

$$\bar{h} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

And so $\bar{h}$ generates a cyclic subgroup of order 3 inside $\mathrm{SL}_2(3)$, say $\bar{H}$. It can be shown that the centre of $\mathrm{SL}_2(3)$ (denote by $\bar{Z}$) is:

$$\bar{Z} = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda^2 = 1 \right\} = \{ \pm I \}$$

Now let's find 3 matrices which correspond to $i$, $j$ and $k$ from $Q_8$. We need matrices, $A$, $B$ and $C$ such that $M^2 = -1$. So the minimal polynomial of $M$ is $x^2 + 1$. Without too much trouble, we can find:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and check its minimal polynomial. Conjugating by $\bar{h}$ will give us another:

$$B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

So then their product will give us $C$:

$$C = AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

These calculations can be verified by the reader. With $A$, $B$ and $C$ in hand, let's quotient $\mathrm{SL}_2(3)$ by its centre and find what group we get. It will turn out that $\mathrm{SL}_2(3)/\bar{Z}$ is isomorphic to $A_4$, and we will use this to find a normal copy of $Q_8$ inside $\mathrm{SL}_2(3)$.

In $\mathrm{SL}_2(3)/\bar{Z}$, we have 4 1-dimensional subspaces given by:

$$\begin{aligned} U_1 &= \operatorname{Span}(e_1) \\ U_2 &= \operatorname{Span}(e_2) \\ U_3 &= \operatorname{Span}(e_1 + e_2) \\ U_4 &= \operatorname{Span}(e_1 - e_2) \end{aligned}$$

Let's investigate how the quotient acts on the set of these subspaces. The element $A$ sets $e_1$ to $e_2$ and $e_2$ to $-e_1$, so swaps $U_1$ and $U_2$. And so $e_1 + e_2$ becomes $e_1 - e_2$, and $e_1 - e_2$ becomes $-e_1 - e_2 = -(e_1 + e_2)$. Thus $A$ swaps $U_3$ and $U_4$ and so it corresponds to the permutation (1 2)(3 4).

By a similar argument, we can determine that $B$ corresponds to $(1\ 4)(2\ 3)$, and $C$ corresponds to $(1\ 3)(2\ 4)$. Again, these calculations can be checked. So we can conclude that:

$$\mathrm{SL}_2(3)/\bar{Z} \cong A_4$$

We know that $A_4$ has a normal subgroup isomorphic to $C_2 \times C_2$, and so by applying the Correspondence Theorem, $\mathrm{SL}_2(3)$ has a normal subgroup isomorphic to $Q_8$, which contains the centre, and elements $A$, $B$ and $C$.

We can check that $\bar{H} \cap \bar{T} = \mathbf{1}$ and so $\mathrm{SL}_2(3) = T \rtimes H$. Hence, $Q_8 \rtimes C_3 \cong \mathrm{SL}_2(3)$. □

Together, Lemmas 5.7 to 5.17 prove Theorem 5.6.

## 5.3 Groups of Order 30

**Theorem 5.18.** *Any group of order 30 is isomorphic to one of:*

$$C_{30}, \quad D_{15}, \quad C_5 \times D_6 \quad \textit{or} \quad C_3 \times D_{10}$$

We will devote the rest of this section to the proof. This classification is based on the one given in the cited Stack Exchange post[2]. Let $G$ be a group of order $30 = 2 \cdot 3 \cdot 5$. We will first show that $G$ can be written as one of 4 semidirect products, then we will identify what those products are isomorphic to.

**Lemma 5.19.** *Any group of order 30 can be expressed as a semidirect product of a subgroup of order 15 with a Sylow 2-subgroup. Moreover, there are precisely 4 possible actions for such semidirect products.*

*Proof.* Sylow's Theorems tell us $G$ has a Sylow 3-subgroup, $T$, and a Sylow 5-subgroup, $F$. Let $H = TF$ and by Lagrange's Theorem, $T \cap F = \mathbf{1}$, hence $|H| = 15$ by Lemma 2.9. We know from our classification of groups of order $pq$ that $H \cong C_{15}$. Because $|H| = 15 = \frac{30}{2}$, the index of $H$ in $G$ is 2, and we know a subgroup of index 2 is normal, so $H \trianglelefteq G$.

A Sylow 2-subgroup, $K \leqslant G$, has order 2, so $K \cong C_2$. Let $\langle k \rangle = K$ and $\langle h \rangle = H$. By the same argument as above, $H \cap K = \mathbf{1}$ and $|HK| = 30$. Hence $G = HK$. Moreover, $G = H \rtimes K$.

By Lemma 2.3:

$$\operatorname{Aut} C_{15} = (\mathbb{Z}/15\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times \cong C_2 \times C_4$$

A homomorphism, $\psi : K \to \operatorname{Aut} H$ preserves element order and we know $\psi$ is determined by its effect on a generator. So then $k\psi$ has four possibilities: either the identity, or one of the three elements of order 2.

2. Stack Exchange (user azimut), "Classification of groups of order 30 (duplicate)," December 10, 2020, accessed January 24, 2023, https://math.stackexchange.com/questions/569226/classification-of-groups-of-order-30.

Additionally, an automorphism of $H$ preserves its Sylow subgroups. Write $H$ as the direct product of its Sylow subgroups:

$$H = \langle\, h^3 \,\rangle \times \langle\, h^5 \,\rangle$$

So the action of $K$ on $H$ is either trivial or by inversion on each of the Sylow subgroups of $H$, giving us 4 possibilities. □

**Lemma 5.20.** *The semidirect products specified by Lemma 5.19 are isomorphic to:*

$$C_{30}, \quad D_{15}, \quad C_5 \times D_6 \quad \textit{or} \quad C_3 \times D_{10}$$

*Proof.*

**Case 1:** Trivial action on both Sylow subgroups.

In this case, because the action is trivial on all of $H$, we recover the direct product, $G = H \times K \cong C_{30}$.

**Case 2:** Inversion on both Sylow subgroups.

Here, $K$ acts on all of $H$, so we obtain:

$$G = \langle\, h,\, k \mid h^{15} = k^2 = 1,\ k^{-1}hk = h^{-1} \,\rangle$$

which we recognise as $D_{30}$.

**Case 3:** Inversion on $\langle h^5 \rangle$.

We know already, from our classification of groups of order $2p$, that $C_3 \rtimes C_2 \cong D_6$. So then because the action on $\langle h^3 \rangle$ is trivial:

$$G = \langle h^3 \rangle \times (\langle h^5 \rangle \rtimes K) \cong C_5 \times D_6$$

**Case 4:** Inversion on $\langle h^3 \rangle$.

Similar to above, we obtain:

$$G = \langle h^5 \rangle \times (\langle h^3 \rangle \rtimes K) \cong C_3 \times D_{10}$$

□

These two lemmas prove Theorem 5.18.

## 5.4 Groups of Order 16

And finally, all that remains is to classify groups of order 16. This classification is often not included; *MT4003* classifies groups up to order 15 for example, however it appears in Burnside's *Theory of Groups*[3] on page 145. So to cap off this report, let's tackle one of the hardest cases! For brevity, denote $C_2 \times C_2 \times C_2 \times C_2$ by $(C_2)^4$.

3. Burnside, Theory of Groups of Finite Order, p145.

**Theorem 5.21.** *Any group of order 16 is isomorphic to one of:*

| | | |
|---|---|---|
| $(C_2)^4$ | $C_8 \times C_2$ | $\mathrm{SD}_{16}$ |
| $\mathrm{M}_{16}$ | $D_{16}$ | $\mathrm{Dic}_{16}$ |
| $C_{16}$ | $C_4 \times C_2 \times C_2$ | $D_8 \times C_2$ |
| $(C_4 \times C_2) \rtimes C_2$ | *Pauli Group* | $Q_8 \times C_2$ |
| $C_4 \rtimes C_4$ | $C_4 \times C_4$ | |

The rest of this section is the proof of this classification, and is based of the one by Wild[4]. We begin by proving a lemma:

**Lemma 5.22.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$, then $G$ has a normal subgroup isomorphic to either $C_8$ or $C_4 \times C_2$.*

*Proof.* Firstly, we know that a subgroup with index 2 is normal, so any subgroup of order 8 in $G$ is normal. So it remains to show $G$ possesses such subgroups. If some $x \in G$ has order 8, then $\langle x \rangle \cong C_8$ and we are done. So assume $G$ has no element of order 8.

Because $G \ncong (C_2)^4$, there is at least one element of order 4 in $G$, say $y$. By Lemma 3.2, there is some element $z \in \mathrm{Z}(G)$ which has order 2. Let $H = \langle z \rangle$. Moreover, $H \trianglelefteq G$ because $z$ is in the centre.

If $y^2 \neq z$, then $\langle y \rangle \cap H = \mathbf{1}$, so $\langle y, z \rangle \cong C_4 \times C_2$.

If then all elements of order 4 in $G$ have $y^2 = z$, all elements in $G/H$ have order 2. Thus $G/H \cong (C_2)^3$, and in particular, is abelian. So the conjugacy class of $y$ is a subgroup of $yH$. Hence the centraliser of $y$, $\mathrm{C}_G(y)$, has order 8. Let $g \in \mathrm{C}_G(y) \setminus \langle y \rangle$. If $g$ has order 2, then $\langle y, g \rangle \cong C_4 \times C_2$. If $g$ has order 4, then:

$$g^2 = z \quad \text{and} \quad (yg)^2 = y^2 g^2 = z^2 = 1$$

We can see that $yg \notin \langle y \rangle$ because then, $yg = y^2$ which gives the contradiction $y = g$. So $\langle y, yg \rangle \cong C_4 \times C_2$. □

Since we know already that $(C_2)^4$ is a group of order 16 by the Fundamental Theorem of Finite Abelian Groups, we will concentrate on when $G$ is not isomorphic to $(C_2)^4$. If $N$ is the subgroup of $G$ specified by Lemma 5.22, we will classify the groups by extending $N$ in different ways. Pick some element $a \in G \setminus N$. Then the order of $aN$ is two, (it must be either 1 or 2, and cannot be 1 because that contradicts our choice of $a$) and $a^2 \in N$.

We know already what the possible automorphisms of $C_8$ and $C_4 \times C_2$ are from our classification of groups of order 24. They are summarised in Tables 1 and 2 in the appendix. So now by considering choices for the order of $a$ and automorphisms of $N$, we will classify $G$.

4. Marcel Wild, "The Groups of Order Sixteen Made Easy," The American Mathematical Monthly 112, no. 1 (2005): 20–31, ISSN: 00029890, 19300972, accessed February 27, 2023, http://www.jstor.org/stable/30037381.

**Lemma 5.23.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$. Suppose $G$ has a normal subgroup $N \cong C_8$, and element $a \in G \setminus N$ of order 2. Then $G$ is isomorphic to one of:*

$$C_8 \times C_2, \quad \mathrm{SD}_{16}, \quad \mathrm{M}_{16} \quad \textit{or} \quad D_{16}$$

*Proof.* If $a$ has order 2, then $a^2$ must be 1. So then $N \cap \langle a \rangle = \mathbf{1}$, and $G = N \rtimes \langle a \rangle$. We have 4 possible semidirect products, because each automorphism of $N$ has order 2 (excluding the identity map). Hence we obtain the following 4 groups:

$$\begin{aligned} G_1 &= \langle x, a \mid x^8 = a^2 = 1,\ a^{-1}xa \overset{\phi_1}{=} x \rangle \\ &= \langle x \rangle \times \langle a \rangle \\ &\cong C_8 \times C_2 \end{aligned}$$

$$\begin{aligned} G_2 &= \langle x, a \mid x^8 = a^2 = 1,\ a^{-1}xa \overset{\phi_2}{=} x^3 \rangle \\ &= \langle x \rangle \rtimes_3 \langle a \rangle \\ &\cong \mathrm{SD}_{16} \end{aligned}$$

$$\begin{aligned} G_3 &= \langle x, a \mid x^8 = a^2 = 1,\ a^{-1}xa \overset{\phi_3}{=} x^5 \rangle \\ &= \langle x \rangle \rtimes_5 \langle a \rangle \\ &\cong \mathrm{M}_{16} \end{aligned}$$

$$\begin{aligned} G_4 &= \langle x, a \mid x^8 = a^2 = 1,\ a^{-1}xa \overset{\phi_4}{=} x^7 = x^{-1} \rangle \\ &= \langle x \rangle \rtimes_7 \langle a \rangle \\ &\cong D_{16} \end{aligned}$$

We are subscripting $\rtimes$ with the power of $x$ it gets sent to upon conjugation with $a$. The group $\mathrm{SD}_{16}$ is known as the semidihedral group[5], of order 16, and $\mathrm{M}_{16}$ is called the modular group[6] of order 16. □

**Lemma 5.24.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$. Suppose $G$ has a normal subgroup $N \cong C_8$, and element $a \in G \setminus N$ of order 4. Then $G$ is isomorphic to* $\mathrm{Dic}_{16}$.

*Proof.* Assume all elements in $G \setminus N$ have order at least 4, otherwise we are back in the previous lemma. How does conjugation behave? In particular, what is $a^{-1}ga$ for an element $g \in G$? I claim the only possibility is the automorphism $\phi_4 : x \mapsto x^{-1}$. It cannot be $\phi_2 : x \mapsto x^3$ because then:

$$(xa)(xa) = x(axa^{-1})a^2 \overset{\phi_2}{=} x(x^3)a^2 = x^4a^2 = 1$$

5. Groupprops, "Semidihedral group:SD16," January 20, 2013, accessed February 27, 2023, https://groupprops.subwiki.org/wiki/Semidihedral_group:SD16.

6. David Clausen, "Classifying All Groups of Order 16" (2012), accessed February 27, 2023, http://buzzard.ups.edu/courses/2012spring/projects/clausen-groups-16-ups-434-2012.pdf.

Likewise, it cannot be $\phi_1 : x \mapsto x$ or $\phi_3 : x \mapsto x^5$ because then $x^2a$ will have order 2:

$$(x^2a)^2 = x^2(ax^2a^{-1})a^2 \overset{\phi_1}{=} x^4a^2 = 1$$

$$(x^2a)^2 = x^2(ax^2a^{-1})a^2 \overset{\phi_3}{=} x^2(x^2)^5a^2 = x^4a^2 = 1$$

All of which contradict our assumption. Hence the only posibility for the effect of conjugation by $a$ is the map $\phi_4 : x \mapsto x^7 = x^{-1}$. So we obtain the presentation:

$$G_5 = \langle\, x,\, a \mid x^8 = a^4 = 1,\ a^2 = x^4,\ a^{-1}xa \overset{\phi_4}{=} x^{-1} \,\rangle$$

Which we recognise as the dicyclic group, $\mathrm{Dic}_{16}$. □

**Lemma 5.25.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$. There are no groups $G$ which have a normal subgroup $N \cong C_8$, and element $a \in G \setminus N$ of order 8.*

*Proof.* We have two choices for $a$: either $a^2 = x^2$ or $a^2 = x^6$. Because $x^2$ and $x^6$ are automorphic, we only need to consider one case, say $a^2 = x^2$, and apply Lemma 2.10. If conjugation by $a$ is either $\phi_2 : x \mapsto x^3$ or $\phi_4 : x \mapsto x^7$ then we obtain a contradiction:

$$a^2 = a^{-1}a^2a = a^{-1}x^2a \overset{\phi_2}{=} x^6$$

$$a^2 = a^{-1}a^2a = a^{-1}x^2a \overset{\phi_4}{=} x^{14} = x^6$$

The remaining possibilities for conjugation by $a$ are the maps $\phi_1 : x \mapsto x$ and $\phi_3 : x \mapsto x^5$. For the first:

$$(x^3a)(x^3a) = x^3(ax^3a^{-1})a^2 \overset{\phi_1}{=} x^3x^3a^2 = x^8 = 1$$

a contradiction. And the second:

$$(xa)(xa) = x(axa^{-1})a^2 \overset{\phi_3}{=} x(x^5)a^2 = x^8 = 1$$

another contradiction. Hence we obtain no new groups. □

**Lemma 5.26.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$. Suppose $G$ has a normal subgroup $N \cong C_8$, and element $a \in G \setminus N$ of order 16. Then $G$ is isomorphic to $C_{16}$.*

*Proof.* The only possibility here is $G_6 \cong C_{16}$, generated by $a$. □

Now let's move on to consider when $\langle\, x,\, y \,\rangle = N \cong C_4 \times C_2$. In particular, all elements of $G$ have order less than 8.

**Lemma 5.27.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$. Suppose $G$ has a normal subgroup $N \cong C_4 \times C_2$, and element $a \in G \setminus N$. There are no groups $G$ such that conjugation by $a$ is of order 4.*

*Proof.* The automorphisms of order 4 are $\psi_2$ and $\psi_4$. Consider an element $g \in N$. We can write $g = x^i y^j$ for some $0 \leqslant i \leqslant 3$ and $0 \leqslant j \leqslant 1$. Then on the one hand:

$$a^{-2} g a^2 = g$$

Because $N$ is abelian. On the other:

$$\begin{aligned} a^{-2} g a^2 &= a^{-1}(a^{-1} x^i \, y^j a) a \\ &= a^{-1}(x^i \psi_2 \; y^j \psi_2) a \\ &= a^{-1}(x^{-i} y^i \; x^{2j} y^j) a \\ &= a^{-1}(x^{2j-i} \; y^{i+j}) a \\ &= x^{i-2j} y^{2j-i} \; x^{2i+2j} y^{i+j} \\ &= x^{3i} y^{3j} \\ &= y^j x^{-i} \end{aligned}$$

Which is $g^{-1}$. Hence we have $g = g^{-1}$, meaning all elements in $N$ have order 2. This is a contradiction, because $x \in N$ has order 4. A similar contradiction can be shown for $\psi_4$. □

**Lemma 5.28.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$. Suppose $G$ has a normal subgroup $N \cong C_4 \times C_2$, and element $a \in G \setminus N$ of order 2. Then $G$ is isomorphic to one of:*

$$C_4 \times C_2 \times C_2, \quad D_8 \times C_2, \quad (C_4 \times C_2) \rtimes C_2 \quad \textit{or} \quad \textit{the Pauli Group}$$

*The Pauli Group has presentition:*

$$\langle\, x,\, y,\, a \mid x^4 = y^2 = a^2 = 1,\; xy = yx,\; xa = ax,\; a^{-1} y a = x^2 y \,\rangle$$

*Proof.* If $a$ has order 2, then $a^2 = 1$, so $\langle\, a \,\rangle \cap N = \mathbf{1}$. Hence $G = N \rtimes \langle\, a \,\rangle$. So we can apply Lemma 2.10, and only consider one representative from each conjugacy class. We

have the following presentations:

$$\begin{aligned} G_7 &= \langle\, x,\, y,\, a \mid x^4 = y^2 = a^2 = 1,\ xy = yx,\ a^{-1}xa \overset{\psi_1}{=} x,\ a^{-1}ya \overset{\psi_1}{=} y \,\rangle \\ &= \langle\, x \,\rangle \times \langle\, y \,\rangle \times \langle\, a \,\rangle \\ &\cong C_4 \times C_2 \times C_2 \end{aligned}$$

$$\begin{aligned} G_8 &= \langle\, x,\, y,\, a \mid x^4 = y^2 = a^2 = 1,\ xy = yx,\ a^{-1}xa \overset{\psi_3}{=} x^{-1},\ a^{-1}ya \overset{\psi_3}{=} y \,\rangle \\ &= (\langle\, x \,\rangle \rtimes \langle\, a \,\rangle) \times \langle\, y \,\rangle \\ &\cong D_8 \times C_2 \end{aligned}$$

$$\begin{aligned} G_9 &= \langle\, x,\, y,\, a \mid x^4 = y^2 = a^2 = 1,\ xy = yx,\ a^{-1}xa \overset{\psi_5}{=} xy,\ a^{-1}ya \overset{\psi_5}{=} y \,\rangle \\ &= (\langle\, x \,\rangle \times \langle\, y \,\rangle) \rtimes \langle\, a \,\rangle \\ &\cong (C_4 \times C_2) \rtimes C_2 \qquad \text{(nameless)} \end{aligned}$$

$$G_{10} = \langle\, x,\, y,\, a \mid x^4 = y^2 = a^2 = 1,\ xy = yx,\ a^{-1}xa \overset{\psi_8}{=} x,\ a^{-1}ya \overset{\psi_8}{=} x^2y \,\rangle$$

The final group, $G_{10}$, is sometimes called the Pauli Group[7], because it consists of the Pauli Matrices from quantum mechanics, and their products with powers of $i$.

The group $G_9$ doesn't appear to have any significant name. □

**Lemma 5.29.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$. Suppose $G$ has a normal subgroup $N \cong C_4 \times C_2$, generated by $x$ of order 4, and $y$ of order 2. Further suppose that $a \in G \setminus N$ has order 4 with $a^2 = x^2$. Then $G$ is isomorphic to one of:*

$$Q_8 \times C_2 \quad \text{or} \quad C_4 \rtimes C_4$$

*Proof.* Firstly, let's show what conjugation by $a$ cannot be. If it's $\psi_1$:

$$(xa)(xa) = xa^2(a^{-1}xa) \overset{\psi_1}{=} xa^2x = 1$$

If it's $\psi_6$:

$$(xya)(xya) = xya^2(a^{-1}xya) \overset{\psi_6}{=} xya^2(x^{-1}x^2y) = x^4y^2 = 1$$

If it's $\psi_8$:

$$(x^2ya)(x^2ya) = x^2ya^2(a^{-1}x^2ya) \overset{\psi_8}{=} x^2ya^2(x^2x^2y) = x^8y^2 = 1$$

7. Clausen, "Classifying All Groups of Order 16."

All of which are contradictions to our assumption. So the remaining possibilities are:

$$\begin{aligned} G_{11} &= \langle\, x,\, y,\, a \mid x^4 = y^2 = a^4 = 1,\ xy = yx,\ a^{-1}xa \overset{\psi_3}{=} x^{-1},\ a^{-1}ya \overset{\psi_3}{=} y \,\rangle \\ &= \langle\, x,\, a \,\rangle \times \langle\, y \,\rangle \\ &\cong Q_8 \times C_2 \end{aligned}$$

$$\begin{aligned} G_{12a} &= \langle\, x,\, y,\, a \mid x^4 = y^2 = a^4 = 1,\ xy = yx,\ a^{-1}xa \overset{\psi_5}{=} xy,\ a^{-1}ya \overset{\psi_5}{=} y \,\rangle \\ G_{12b} &= \langle\, x,\, y,\, a \mid x^4 = y^2 = a^4 = 1,\ xy = yx,\ a^{-1}xa \overset{\psi_7}{=} x^{-1}y,\ a^{-1}ya \overset{\psi_7}{=} y \,\rangle \end{aligned}$$

Focusing on the last two groups, we will show they are both isomorphic to $C_4 \rtimes C_4$, with the inversion action. First $G_{12a}$. Consider the element $ax$, and so $\langle\, ax \,\rangle = \{\, 1,\, ax,\, y,\, axy \,\}$. By inspection, $\langle\, x \,\rangle \cap \langle\, ax \,\rangle = \mathbf{1}$. So then:

$$x^{-1}(ax)x = x^{-1}xaxy = axy = (ax)^{-1}$$

Hence, $G_{12a} = \langle\, x \,\rangle \rtimes \langle\, ax \,\rangle \cong C_4 \rtimes C_4$.

Likewise for $G_{12b}$, $\langle\, ax \,\rangle = \{\, 1,\, ax,\, a^2y,\, x^{-1}a^{-1} \,\}$, and again $\langle\, x \,\rangle \cap \langle\, ax \,\rangle = \mathbf{1}$. Additionally:

$$x^{-1}(ax)x = ax^3y = ax^{-1}y = xa$$

Multiplying by $x^4 = 1$ gives:

$$x^4xa = x^{-1}x^2a = x^{-1}a^2a = x^{-1}a^{-1}$$

Hence $G_{12b}$ is isomorphic to the same semidirect product, $C_4 \rtimes C_4$.

We can check that it is indeed valid to write $C_4 \rtimes C_4$. We know $\operatorname{Aut} C_4 \cong C_2$ and so a homomorphism $\varphi : C_4 \to \operatorname{Aut} C_4$ can map the generator to either the identity map or the inverse map. Hence we have only one non-trivial semidirect product. □

**Lemma 5.30.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$. Suppose $G$ has a normal subgroup $N \cong C_4 \times C_2$, generated by $x$ of order 4, and $y$ of order 2. Further suppose that $a \in G \setminus N$ has order 4 with $a^2 = y$. Then $G$ is isomorphic to $C_4 \times C_4$.*

*Proof.* Conjugation by $a$ cannot be $\psi_6$ or $\psi_8$ because then:

$$a^2 = a^{-1}a^2a = a^{-1}ya \overset{\psi_{6,8}}{=} x^2y$$

a contradiction. If it's $\psi_5$ then let $\bar{a} = xa$ and so:

$$(xa)(xa) = x(axa^{-1})a^2 \overset{\psi_5}{=} x^2y^2 = x^2$$

and we are in the previous case. Likewise if conjugation by $a$ is $\psi_3$ then:

$$(x^2a)(x^2a) = x^2(ax^2a^{-1})a^2 \overset{\psi_3}{=} x^2y^2 = x^2$$

Finally, $\psi_7$ gives:

$$(xa)(xa) = xa^2(a^{-1}xa) \overset{\psi_7}{=} xa^2x^3y = x^4y^2 = 1$$

a contradiction, leaving only $\psi_1$ as the final standing possibility, giving:

$$G_{13} = \langle\, x,\, y,\, a \mid x^4 = y^2 = a^4 = 1,\ xy = yx,\ ax \overset{\psi_1}{=} xa,\ ay \overset{\psi_1}{=} ya \,\rangle$$

Letting $b = xy$:

$$G_{13} = \langle\, b,\, a \mid b^4 = a^4 = 1,\ ab = ba \,\rangle \cong C_4 \times C_4$$

□

**Lemma 5.31.** *Let $G$ be a group of order 16, not isomorphic to $(C_2)^4$. Suppose $G$ has a normal subgroup $N \cong C_4 \times C_2$, generated by $x$ of order 4, and $y$ of order 2. Further suppose that $a \in G \setminus N$ has order 4 with $a^2 = x^2y$. Then $G$ will be isomorphic to $C_4 \times C_4$, as in the previous lemma.*

*Proof.* For any group in this case, if we apply the automorphism $\psi_8$, then we have:

$$a^2\psi_8 = (x^2y)\psi_8 = x^2x^2y = y$$

So it will be isomorphic to a group from the previous subcase. □

So we prove our classification with these lemmas. Thus we finish not only the classification of groups of order 16, but this report as a whole.

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# Appendix

Table 1: Automorphisms of $C_8$

| $\operatorname{Aut} C_8$ | $x \mapsto$ |
|---|---|
| $\phi_1$ | $x$ |
| $\phi_2$ | $x^3$ |
| $\phi_3$ | $x^5$ |
| $\phi_4$ | $x^{-1}$ |

Table 2: Automorphisms of $C_4 \times C_2$

| $\operatorname{Aut} C_8$ | $x \mapsto$ | $y \mapsto$ | Order |
|---|---|---|---|
| $\psi_1$ | $x$ | $y$ | 1 |
| $\psi_2$ | $x^3y$ | $x^2y$ | 4 |
| $\psi_3$ | $x^3$ | $y$ | 2 |
| $\psi_4$ | $xy$ | $x^2y$ | 4 |
| $\psi_5$ | $xy$ | $y$ | 2 |
| $\psi_6$ | $x^3$ | $x^2y$ | 2 |
| $\psi_7$ | $x^3y$ | $y$ | 2 |
| $\psi_8$ | $x$ | $x^2y$ | 2 |

## Full Classification of Groups up to Order 31

| Order | Classification | Presentation |
|---|---|---|
| 1 | $\mathbf{1}$ | $\langle\, 1 \,\rangle$ |
| 2 | $C_2$ | $\langle\, x \mid x^2 = 1 \,\rangle$ |
| 3 | $C_3$ | $\langle\, x \mid x^3 = 1 \,\rangle$ |
| 4 | $C_4$ | $\langle\, x \mid x^4 = 1 \,\rangle$ |
| | $C_2 \times C_2$ | $\langle\, x, y \mid x^2 = y^2 = 1,\ xy = yx \,\rangle$ |
| 5 | $C_5$ | $\langle\, x \mid x^5 = 1 \,\rangle$ |
| 6 | $D_6 \cong S_3$ | $\langle\, x, y \mid x^3 = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_6$ | $\langle\, x \mid x^6 = 1 \,\rangle$ |
| 7 | $C_7$ | $\langle\, x \mid x^7 = 1 \,\rangle$ |
| 8 | $C_8$ | $\langle\, x \mid x^8 = 1 \,\rangle$ |
| | $C_4 \times C_2$ | $\langle\, x, y \mid x^4 = y^2 = 1,\ xy = yx \,\rangle$ |
| | $D_8$ | $\langle\, x, y \mid x^4 = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $Q_8$ | $\langle\, x, y \mid x^4 = y^4 = 1,\ x^2 = y^2,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_2 \times C_2 \times C_2$ | $\langle\, x, y, z \mid x^2 = y^2 = z^2 = 1,\ xy = yx,\ xz = zx,\ yz = zy \,\rangle$ |
| 9 | $C_9$ | $\langle\, x \mid x^9 = 1 \,\rangle$ |
| | $C_3 \times C_3$ | $\langle\, x, y \mid x^3 = y^3 = 1,\ xy = yx \,\rangle$ |
| 10 | $D_{10}$ | $\langle\, x, y \mid x^5 = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_{10}$ | $\langle\, x \mid x^{10} = 1 \,\rangle$ |

| Order | Classification | Presentation |
|---|---|---|
| 11 | $C_{11}$ | $\langle\, x \mid x^{11} = 1 \,\rangle$ |
| 12 | $\mathrm{Dic}_{12}$ | $\langle\, x, y \mid x^6 = 1,\ x^3 = y^2,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_{12}$ | $\langle\, x \mid x^{12} = 1 \,\rangle$ |
| | $A_4$ | —— |
| | $D_{12}$ | $\langle\, x, y \mid x^6 = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_6 \times C_2$ | $\langle\, x, y \mid x^6 = y^2 = 1,\ xy = yx \,\rangle$ |
| 13 | $C_{13}$ | $\langle\, x \mid x^{13} = 1 \,\rangle$ |
| 14 | $D_{14}$ | $\langle\, x, y \mid x^7 = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_{14}$ | $\langle\, x \mid x^{14} = 1 \,\rangle$ |
| 15 | $C_{15}$ | $\langle\, x \mid x^{15} = 1 \,\rangle$ |
| 16 | $C_{16}$ | $\langle\, x \mid x^{16} = 1 \,\rangle$ |
| | $C_4 \times C_4$ | $\langle\, x, y \mid x^4 = y^4,\ xy = yx \,\rangle$ |
| | $(C_4 \times C_2) \rtimes C_2$ | $\langle\, x, y, z \mid x^4 = y^2 = z^2 = 1,\ xy = yx,\ yz = yz,\ z^{-1}xz = xy \,\rangle$ |
| | $C_4 \rtimes C_4$ | $\langle\, x, y \mid x^4 = y^4 = 1,\ x^{-1}yx = y^{-1} \,\rangle$ |
| | $C_8 \times C_2$ | $\langle\, x, y \mid x^8 = y^2 = 1,\ xy = yx \,\rangle$ |
| | $\mathrm{M}_{16}$ | $\langle\, x, y \mid x^8 = y^2 = 1,\ y^{-1}xy = x^5 \,\rangle$ |
| | $D_{16}$ | $\langle\, x, y \mid x^8 = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $\mathrm{SD}_{16}$ | $\langle\, x, y \mid x^8 = y^2 = 1,\ y^{-1}xy = x^3 \,\rangle$ |
| | $\mathrm{Dic}_{16}$ | $\langle\, x, y \mid x^8 = 1,\ x^4 = y^2,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_4 \times C_2 \times C_2$ | $\langle\, x, y, z \mid x^4 = y^2 = z^2 = 1,\ xy = yx,\ xz = zx,\ yz = zy \,\rangle$ |
| | $D_8 \times C_2$ | $\langle\, x, y, z \mid x^4 = y^2 = z^2 = 1,\ xy = yx,\ yz = zy,\ z^{-1}xz = x^{-1} \,\rangle$ |
| | $Q_8 \times C_2$ | $\langle\, x, y, z \mid x^4 = y^4 = z^2 = 1,\ xz = zx,\ yz = zy,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | Pauli Group | $\langle\, x, y, z \mid x^4 = y^2 = z^2 = 1,\ xy = yx,\ xz = zx,\ z^{-1}yz = x^2y \,\rangle$ |
| | $(C_2)^4$ | —— |
| 17 | $C_{17}$ | $\langle\, x \mid x^{17} = 1 \,\rangle$ |
| 18 | $D_{18}$ | $\langle\, x, y \mid x^8 = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_{18}$ | $\langle\, x \mid x^{18} = 1 \,\rangle$ |
| | $D_6 \times C_3$ | $\langle\, x, y, z \mid x^3 = y^3 = z^2 = 1,\ xy = yx,\ yz = zy,\ z^{-1}xz = x^{-1} \,\rangle$ |
| | $\mathrm{Dih}(C_p \times C_p)$ | $\langle\, x, y, z \mid x^p = y^p = z^2 = 1,\ xy = yx,\ z^{-1}xz = x^{-1},\ z^{-1}yz = y^{-1} \,\rangle$ |
| | $C_6 \times C_3$ | $\langle\, x, y \mid x^6 = y^3 = 1,\ xy = yx \,\rangle$ |
| 19 | $C_{19}$ | $\langle\, x \mid x^{19} = 1 \,\rangle$ |
| 20 | $\mathrm{Dic}_{20}$ | $\langle\, x, y \mid x^{10} = 1,\ x^5 = y^2,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_{20}$ | $\langle\, x \mid x^{20} = 1 \,\rangle$ |
| | — | $\langle\, x, y \mid x^5 = y^4 = 1,\ y^{-1}xy = x^2 \,\rangle$ |
| | $D_{20}$ | $\langle\, x, y \mid x^{10} = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_{10} \times C_2$ | $\langle\, x, y \mid x^{10} = y^2 = 1,\ xy = yx \,\rangle$ |

| Order | Classification | Presentation |
|---|---|---|
| 21 | $C_7 \rtimes C_3$ | $\langle\, x,\, y \mid x^7 = y^3 = 1,\, y^{-1}xy = x^4 \,\rangle$ |
| | $C_{21}$ | $\langle\, x \mid x^{21} = 1 \,\rangle$ |
| 22 | $D_{20}$ | $\langle\, x,\, y \mid x^{11} = y^2 = 1,\, y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_{22}$ | $\langle\, x \mid x^{22} = 1 \,\rangle$ |
| 23 | $C_{23}$ | $\langle\, x \mid x^{23} = 1 \,\rangle$ |
| 24 | $C_3 \rtimes C_8$ | $\langle\, x,\, y \mid x^3 = y^8 = 1,\, x^{-1}yx = y^{-1} \,\rangle$ |
| | $C_{24}$ | —— |
| | $\mathrm{SL}_2(3)$ | —— |
| | $\mathrm{Dic}_{24}$ | $\langle\, x,\, y \mid x^{12} = 1,\, x^6 = y^2,\, y^{-1}xy = x^{-1} \,\rangle$ |
| | $S_3 \times C_4$ | —— |
| | $D_{24}$ | $\langle\, x,\, y \mid x^{12} = y^2 = 1,\, y^{-1}xy = x^{-1} \,\rangle$ |
| | $\mathrm{Dic}_{12} \times C_2$ | $\langle\, x,\, y,\, z \mid x^6 = z^2 = 1,\, x^3 = y^2,\, y^{-1}xy = x^{-1},\, xz = zx,\, yz = zy \,\rangle$ |
| | $C_3 \rtimes_{V_4} D_8$ | $\langle\, x,\, y,\, z \mid x^3 = y^4 = z^2 = 1,\, y^{-1}xy = x^{-1},\, z^{-1}xz = x^{-1},\, z^{-1}yz = y^{-1} \,\rangle$ |
| | $C_3 \times C_4 \times C_2$ | —— |
| | $D_8 \times C_3$ | —— |
| | $Q_8 \times C_3$ | —— |
| | $S_4$ | —— |
| | $A_4 \times C_2$ | —— |
| | $S_3 \times C_2 \times C_2$ | —— |
| | $C_3 \times (C_2)^3$ | —— |
| 25 | $C_{25}$ | —— |
| | $C_5 \times C_5$ | —— |
| 26 | $D_{26}$ | —— |
| | $C_{26}$ | —— |
| 27 | $C_{27}$ | —— |
| | $C_9 \times C_3$ | —— |
| | $\mathrm{UT}_3(3)$ | —— |
| | $C_9 \rtimes C_3$ | —— |
| | $(C_3)^3$ | —— |
| 28 | $\mathrm{Dic}_{28}$ | $\langle\, x,\, y \mid x^{14} = 1,\, x^7 = y^2,\, y^{-1}xy = x^{-1} \,\rangle$ |
| | $C_{28}$ | —— |
| | $D_{28}$ | —— |
| | $C_{14} \times C_2$ | —— |
| 29 | $C_{29}$ | —— |
| 30 | $C_5 \times D_6$ | —— |
| | $C_3 \times D_{10}$ | —— |
| | $D_{30}$ | —— |
| | $C_{30}$ | —— |
| 31 | $C_{31}$ | —— |