

Classification of Finite Groups

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Part I

Doing

1 Introduction

2 Preliminaries

To start, let's solidify the notation used in this report.

We shall denote groups and sets with capital letters, like G , H , and elements of those groups with lower case letters, like g , h . Greek letters shall denote mappings, generally ϕ , ψ , etc. with ι reserved for the identity map, and we will write mappings on the right.

We will use \mathbb{N} to denote the natural numbers (not including 0), \mathbb{Z} to denote the integers, and \mathbb{R} to denote the real numbers.

To denote the cyclic group of order n we will use C_n , D_{2n} to denote the cyclic group of order $2n$, A_n to denote the alternating group over n elements, S_n to denote the symmetric group over n elements, and Q_8 to denote the quaternion group. The trivial group, $\{1\}$ is denoted by $\mathbf{1}$.

Let's move on to review some facts and theorems which will be valuable later on, as well as introduce some new concepts and prove some new results!

Definition 2.1. If G and H are groups with elements $g_1, g_2 \in G$, then a map:

$$\phi : G \rightarrow H$$

is a homomorphism if:

$$(g_1 g_2) \phi = (g_1 \phi)(g_2 \phi)$$

If ϕ is bijective, then we call it an isomorphism, with $G \cong H$ denoting that G is isomorphic to H . And if ϕ is an isomorphism from G to itself, then we call it an automorphism of G .

Lemma 2.2. *The set of all automorphisms of a group G form a group under composition. Indeed, this is called the automorphism group of G , denoted $\text{Aut } G$.*

Proof. Let $A = \text{Aut } G = \{ \phi : G \rightarrow G \mid \phi \text{ is an isomorphism} \}$, and let $\phi \in A$. Denote an element of G by g .

We know already that the composition of two isomorphisms is an isomorphism, so A is closed under composition.

The identity map, $\iota : g \mapsto g$, is certainly an automorphism of G and so $A \neq \emptyset$.

Indeed, $\iota : g \mapsto g$ is the identity of A , since:

$$g\phi\iota = (g\phi)\iota = g\phi \quad \text{and} \quad g\iota\phi = (g\iota)\phi = g\phi$$

And inverses clearly exist, because automorphisms are bijections, and bijections are invertible. Hence $A = \text{Aut } G$ is a group. \square

Lemma 2.3. *The automorphism group of C_n is isomorphic to the multiplicative group of integers mod n .*

$$\text{i.e. } \text{Aut } C_n \cong (\mathbb{Z}/n\mathbb{Z})^\times$$

Proof. Let $C_n = \langle x \rangle$. Any automorphism, φ of C_n has the property:

$$(x^i)\varphi = (x\varphi)^i$$

Hence φ is determined by its effect on a generator, x , and preserves element order. In particular, φ sends generators to generators. So for φ to be an automorphism, it must send x to another generator,

say x^k . An element x^k generates C_n if x^k has order n , i.e. when k and n are co-prime. Denote the automorphism sending x to x^k by φ_k .

Let's now investigate how these automorphisms behave. Let $\varphi_k, \varphi_l \in \text{Aut } C_n$, and consider:

$$x\varphi_k\varphi_l = (x^k)\varphi_l = (x^k)^l = x^{(kl)} = x\varphi_{kl} \pmod{n}$$

Because multiplication modulo n is commutative, $x^{kl} = x^{lk}$, so $\text{Aut } C_n$ is abelian.

Now consider $\theta : \text{Aut } C_n \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$ defined by $\varphi_k\theta = k$. We will show θ is an isomorphism. Every $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ is co-prime to n and so x^k is a generator of C_n , hence there is some $\varphi_k \in \text{Aut } C_n$ such that $\varphi_k\theta = k$. So θ is surjective. If $\varphi_k\theta = \varphi_l\theta$ then $k = l$, so θ is also injective. Finally, θ is a homomorphism because:

$$(\varphi_k\varphi_l)\theta = \varphi_{kl}\theta = kl = (\varphi_k\theta)(\varphi_l\theta)$$

So $\theta : \text{Aut } C_n \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$ is an isomorphism. □

Lemma 2.4. *Let G be a group with normal subgroup H . And let K be the unique subgroup of its order in H (we call K characteristic in H). Then K is a normal subgroup of G .*

Proof. Consider the map $\varphi_g : G \rightarrow G$ defined by $\varphi_g : x \mapsto g^{-1}xg$ for elements $x, g \in G$. We will show that this is an automorphism of G . For $x, y \in G$:

$$x\varphi_g y\varphi_g = (g^{-1}xg)(g^{-1}yg) = g^{-1}(xy)g = (xy)\varphi_g$$

Hence φ_g is a homomorphism. Moreover, φ_g is invertible with inverse $\varphi_{g^{-1}}$. So φ_g is indeed an automorphism of G .

Because H is normal, $H\varphi_g = H$. So φ_g is an automorphism of H too. And so φ_g maps K to itself, because it is the unique subgroup of its order. Hence the set

$$\{g^{-1}kg \mid k \in K\} = K$$

So K is normal in G . □

2.1 Semidirect Product

We already know about the direct product:

Definition 2.5 (Direct Product). For groups N and H , the direct product, $G = N \times H$ is a group of ordered pairs of elements (n, h) where $n \in N$ and $h \in H$ with the operation:

$$(n_1, h_1)(n_2, h_2) = (n_1n_2, h_1h_2)$$

Moreover, if $\bar{N} = N \times \mathbf{1}$ and $\bar{H} = \mathbf{1} \times H$, then:

- (i) $\bar{N} \trianglelefteq G$ and $\bar{H} \trianglelefteq G$
- (ii) $\bar{N} \cap \bar{H} = \mathbf{1}$
- (iii) $\bar{N}\bar{H} = \{nh \mid n \in N, h \in H\} = G$

Now let's seek a slightly more general way to combine groups, by relaxing that H must be normal. So we have:

$$N \trianglelefteq G, H \leq G, NH = G, \quad \text{and} \quad N \cap H = \mathbf{1}$$

Consider the set, (not the direct product):

$$N \times H = \{(n, h) \mid n \in N, h \in H\}$$

and a map

$$\phi : N \times H \rightarrow G \quad \text{defined by} \quad (n, h) \mapsto nh$$

We want ϕ to be an isomorphism.

To show ϕ is injective, take $n_1, n_2 \in N$ and $h_1, h_2 \in H$, and assume $n_1 h_1 = n_2 h_2$. Then multiplying on the left by n_2^{-1} and on the right by h_1^{-1} gives:

$$n_2^{-1} n_1 = h_2 h_1^{-1}$$

On the left we have an element of N and on the right, an element of H , so $n_2^{-1} n_1 = h_2 h_1^{-1} \in N \cap H$. But $N \cap H = \mathbf{1}$ so then $n_2^{-1} n_1 = h_2 h_1^{-1} = 1$. Hence:

$$n_1 = n_2 \quad \text{and} \quad h_1 = h_2$$

To show ϕ is surjective, consider the image, $\text{im } \phi = \{nh \mid n \in N, h \in H\}$. This is by definition $NH = G$, so ϕ is surjective, and hence a bijection.

For ϕ to be a homomorphism, we need:

$$\begin{aligned} [(n_1, h_1)(n_2, h_2)]\phi &= (n_1, h_1)\phi(n_2, h_2)\phi \\ &= n_1 h_1 n_2 h_2 \\ &= n_1 h_1 n_2 h_1^{-1} h_1 h_2 \\ &= (n_1 h_1 n_2 h_1^{-1})(h_1 h_2) \end{aligned}$$

But N is normal in G so $h_1 n_2 h_1^{-1}$ is just another element in N , say n_3 . So:

$$[(n_1, h_1)(n_2, h_2)]\phi = (n_1 n_3)(h_1 h_2) = (n_1 n_3, h_1 h_2)\phi$$

We know that ϕ is injective, so then:

$$(n_1, h_1)(n_2, h_2) = (n_1 n_3, h_1 h_2)$$

This tells us the multiplication that will make NH a group. Because $N \trianglelefteq G$, the map

$$\varphi_{h_1} : n_2 \mapsto h_1 n_2 h_1^{-1} = n_3$$

is an automorphism of N . This gives rise to the definition:

Definition 2.6 (Semidirect Product).

- (i) For a group G with normal subgroup N and subgroup H with $NH = G$ and $N \cap H = \mathbf{1}$, G is the internal semidirect product of N by H , written $G = N \rtimes H$.
- (ii) For groups N and H , and a homomorphism $\psi : H \rightarrow \text{Aut } N$, the external semidirect product of N by H via ψ is the set:

$$N \rtimes H = \{ (n, h) \mid n \in N, h \in H \}$$

with multiplication:

$$(n_1, h_1)(n_2, h_2) = (n_1(n_2^{h_1\psi}), h_1 h_2)$$

denoted:

$$N \rtimes_{\psi} H$$

We use the notation $n_2^{h_1\psi}$ both because it indicates conjugation, and is clearer.

Lemma 2.7. For a group G with $N \trianglelefteq G$ and $H \leq G$, with $N \cap H = \mathbf{1}$ then:

$$|NH| = |\{nh \mid n \in N, h \in H\}| = |N| \cdot |H|$$

Proof. We just saw above that for elements $n \in N$ and $h \in H$, the map:

$$\phi : N \times H \rightarrow NH \quad \text{defined by} \quad (n, h) \mapsto nh$$

is a bijection. The result follows immediately from this. \square

Lemma 2.8. *Let N and H be groups, and $\alpha \in \text{Aut } H$. Then the semidirect products via the homomorphism ψ , $N \rtimes_{\psi} H$, and via the homomorphism $\alpha\psi$, $N \rtimes_{\alpha\psi} H$, are isomorphic.*

That is, we can apply any automorphism to H and the resulting semidirect product remains in the same isomorphism class.

Proof. Let $G = N \rtimes_{\psi} H$ and $\bar{G} = N \rtimes_{\alpha\psi} H$, and define:

$$\vartheta : G \rightarrow \bar{G} \quad \text{by} \quad \vartheta : (n, h) \mapsto (n, h\alpha)$$

We will show that ϑ is an isomorphism.

First, ϑ^{-1} exists because α^{-1} exists, and is given by:

$$\vartheta^{-1} : (n, h) \mapsto (n, h\alpha^{-1})$$

Hence ϑ is a bijection.

Now for two elements, $(n_1, h_1), (n_2, h_2) \in G$, consider:

$$\begin{aligned} (n_1, h_1)\vartheta (n_2, h_2)\vartheta &= (n_1, h_1\alpha)(n_2, h_2\alpha) \\ &= (n_1 n_2^{(h_1\alpha)\psi}, h_1\alpha h_2\alpha) \\ &= (n_1 n_2^{h_1(\alpha\psi)}, (h_1 h_2)\alpha) \\ &= ((n_1, h_1)(n_2, h_2))\vartheta \end{aligned}$$

So ϑ is an isomorphism. \square

2.2 Group Actions

Some snazzy introduction.

Definition 2.9. Let G be a group, and Ω be a set, with elements $g \in G$ and $\omega \in \Omega$. Consider a map $\mu : \Omega \times G \rightarrow \Omega$, and write ω^g for the image of (ω, g) under μ . So we have:

$$\mu : \Omega \times G \rightarrow \Omega \quad \text{defined by} \quad (\omega, g) \mapsto \omega^g$$

We say G acts on Ω if for all $g_1, g_2 \in G$ and all $\omega \in \Omega$:

$$(i) \quad (\omega^{g_1})^{g_2} = \omega^{(g_1 g_2)}$$

$$(ii) \quad \omega^1 = \omega$$

We call μ the group action of G on Ω .

This might remind you of a homomorphism. Indeed we have a result:

Lemma 2.10. *A group action induces a homomorphism. Specifically, let G be a group which acts on a set Ω , with $g \in G$ and $\omega \in \Omega$, and define:*

$$\rho_g : \Omega \rightarrow \Omega \quad \text{by} \quad \omega \mapsto \omega^g$$

Then:

$$\rho : G \rightarrow \text{Sym } \Omega \quad \text{defined by} \quad g \mapsto \rho_g$$

is a homomorphism.

Proof. Firstly, ρ_g is indeed a permutation of Ω because it is invertible (and therefore a bijection), with:

$$(\rho_g)^{-1} = \rho_{g^{-1}}$$

Consider $g, h \in G$ and their corresponding maps, $\rho_g, \rho_h \in \text{Sym } \Omega$. Then:

$$\omega(g\rho)(h\rho) = \omega\rho_g\rho_h = (\omega^g)^h = \omega^{(gh)} = \omega\rho_{gh} = \omega(gh)\rho$$

Thus ρ is a homomorphism. □

A group acting on the set its cosets will be very useful:

Definition 2.11. For a group G with $H \leq G$, let $\Omega = \{Hg \mid g \in G\}$, i.e. the set of cosets of H in G . If $x \in G$, define a group action:

$$\Omega \times G \rightarrow \Omega \quad \text{by} \quad (Hg, x) \mapsto Hgx$$

Lemma 2.12. *The action above is well defined, meaning the action is independent of our choice of representative g .*

Proof. □

3 First Classifications

Let's start with the easiest case: groups of order 1. Any group G must have an identity element, and so that's all our possible elements used up! All groups of order 1 are isomorphic to the trivial group, 1 .

What about groups of prime order? Let G be a group of order p , where p is a prime number. Then Lagrange's Theorem tell us all elements must have order 1 or p . Pick some $x \in G$ with x having order p . Then $\langle x \rangle = G$ so G is cyclic and $G \cong C_p$.

4 Groups of Order pq

Let G be a group of order pq where p, q are prime numbers with $p > q$, and let n_p and n_q denote the number of Sylow p -subgroups and Sylow q -subgroups of G respectively. Then by Sylow's Theorems:

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q \implies n_p = 1$$

$$n_q \equiv 1 \pmod{q} \implies n_q = 1, q+1, 2q+1, \dots \quad \text{and} \quad n_q \mid p$$

So G has a unique Sylow p -subgroup, say $P \trianglelefteq G$, and a Sylow q -subgroup, $Q \leq G$. Because p and q are prime numbers, $P \cong C_p$ and $Q \cong C_q$. Pick generators for each, say $\langle x \rangle = P$ and $\langle y \rangle = Q$. We have 2 possibilities for n_q : $p-1$ is a multiple of q or 1.

Case 1: $q \nmid p-1$.

If $p-1$ is not a multiple of q then $n_q = 1$ and $Q \trianglelefteq G$, hence:

$$G = P \times Q \cong C_{pq}$$

Case 2: $q \mid p-1$.

If $p-1$ is a multiple of q then $n_q = p$ and so Q is not normal in G . By Lagrange's Theorem, $P \cap Q = 1$ and by Lemma 2.7, $|PQ| = pq$. Hence, as well as the direct product, we have $G = P \rtimes Q$, some non-trivial semidirect product.

By Lemma 2.3, $\text{Aut } C_p \cong (\mathbb{Z}/p\mathbb{Z})^\times \cong C_{p-1}$. So if $\nu \in (\mathbb{Z}/p\mathbb{Z})^\times$, then $x \mapsto x^\nu$ is an automorphism. We know also that C_{p-1} has a unique subgroup of order q , hence G has the presentation:

$$G = \langle x, y, \mid x^p = y^q = 1, y^{-1}xy = x^a \rangle$$

where a is a generator for the subgroup of order q in $(\mathbb{Z}/p\mathbb{Z})^\times$.

Notice that picking different generators are equivalent up to isomorphism because the composition of two isomorphisms is an isomorphism.

So any group of order pq is isomorphic to either:

$$C_{pq} \quad \text{or} \quad \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^a \rangle \quad \begin{array}{l} \text{if } q \mid p-1 \\ C_{pq} \quad \text{if } q \nmid p-1 \end{array}$$

4.1 Groups of Order $2p$

To illustrate an example of groups of order pq , let's take $q = 2$. Because every prime greater than 2 is odd, $p-1$ is an even number, and so $2 \mid p-1$.

An element $\alpha \in (\mathbb{Z}/p\mathbb{Z})^\times$ of order 2 satisfies $\alpha^2 = 1$, hence $\alpha = 1$ or -1 . But 1 has order 1, so α can only be -1 . Side-note: from the proof of Lemma 2.3, this corresponds to the inverse map.

So, in addition to C_{2p} , we have:

$$G \cong \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x^{-1} \rangle$$

Which is the presentation for the dihedral group of order $2p$, D_{2p} .

Hence a group of order $2p$ is isomorphic to one of:

$$C_{2p} \quad \text{or} \quad D_{2p}$$

5 Groups of Order p^2

Let G be a group of order p^2 . First, we will prove a useful lemma:

Lemma 5.1. *If G is a p -group (i.e. a group of prime power order), then every subgroup of index p is normal.*

Proof. Let H be a subgroup of G , with index p . We know kernels are normal subgroups, so we will show that H is the kernel of some homomorphism. Let Ω be the set of all cosets of H . So by definition, $|\Omega| = p$. By Lemma 2.10, there is a homomorphism:

$$\rho : G \rightarrow S_p$$

Let's investigate the kernel of ρ . If we have $x \in \ker \rho$, then:

$$(H1)x = H1 = H$$

which means $x \in H$. So the kernel of ρ is H . Hence, $H \trianglelefteq G$. □

By Lagrange's Theorem, the elements of G have order 1, p or p^2 .

If $x \in G$ has order p^2 , then x generates G so $G \cong C_{p^2}$.

If G does not have an element of order p^2 then all elements, except the identity, have order p . We know that G must have a subgroup of order p , P , and because p is prime, $P \cong C_p$. Pick a generator for P , say x and an element $y \in G$ such that $y \notin P$. Then $y \neq x^i$ for any i .

If $y^j = x^i$ for some i and j , then:

$$(y^j)^{1-j} = (x^i)^{1-j} = y^{j-j+1} = y = x^{i(1-j)} = x^k \quad \text{for some } k, \text{ a contradiction.}$$

So no power of y is equal to any power of x . Because y has order p , it generates a subgroup of order p , \bar{P} , with $P \cap \bar{P} = \mathbf{1}$. The lemma tells us that both P and \bar{P} are normal, and by Lemma 2.7, $|P\bar{P}| = p^2 = |G|$, so:

$$G = P \times \bar{P} \cong C_p \times C_p$$

If G has no elements of order p or p^2 , then it only has elements of order 1, which is the trivial group.

Hence any group of order p^2 is isomorphic to one of:

$$C_{p^2} \quad \text{or} \quad C_p \times C_p$$

6 Groups of order 12

We will see later, that we need groups of order 12 in order to classify groups of other orders.

Let G be a group of order $12 = 2^2 \cdot 3$, and n_3 and n_2 denote the number of Sylow 3-subgroups and Sylow 2-subgroups of G respectively. By Sylow's Theorems:

$$n_2 \equiv 1 \pmod{2} \text{ and } n_2 \mid 3 \implies n_2 = 1$$

$$n_3 \equiv 1 \pmod{3} \text{ and } n_3 \mid 4 \implies n_3 = 1 \text{ or } 4$$

So G has a unique Sylow 2-subgroup of order 4, say $H \trianglelefteq G$, and we have already classified groups of order 4, so H is isomorphic to either V_4 (the Klein 4 group) or C_4 . A Sylow 3-subgroup, $K \leq G$ will have order 3, so $K \cong C_3$. Say $K = \langle x \rangle$.

Lagrange's Theorem tells us H has elements of order 1, 2, and 4, and K has elements of order 1 and 3. Hence $H \cap K = \mathbf{1}$. Lemma 2.7 tells us:

$$|HK| = |H| \cdot |K| = 12$$

Hence $G = HK$, $H \trianglelefteq G$, and $H \cap K = \mathbf{1}$. If we consider groups with 4 Sylow 3-subgroups then we can conclude that they are some semidirect product, $G = H \rtimes K$.

Since an automorphism, φ , must map generators to generators, $\text{Aut } C_4 \cong C_2$ because C_4 has two generators. An automorphism of V_4 corresponds to a permutation of the three non-identity elements, hence $\text{Aut } V_4 \cong S_3$.

Case 1: $H \cong C_4$ i.e. $G \cong C_4 \rtimes C_3$.

Let $H = \langle y \rangle$.

A homomorphism $\psi : K \rightarrow \text{Aut } H \cong C_2$, preserves order and together with Lagrange's Theorem means that the only possibility for ψ is trivial, i.e. $K\psi = \mathbf{1}$.

Hence $G \cong C_4 \times C_3 \cong C_{12}$.

Case 2: $H \cong V_4$ i.e. $G \cong (C_2 \times C_2) \rtimes C_3$.

Let $H = \langle y, z \rangle$.

A trivial homomorphism $K\psi = \mathbf{1}$ yields the direct product. What non-trivial homomorphisms are there? The automorphism group, $\text{Aut } H \cong S_3$ is of order 6, and so has a unique subgroup of order 3, by Sylow's Theorems. We know that a homomorphism $\psi : K \rightarrow \text{Aut } H$ is determined by where it sends the generator x , so for ψ to be non-trivial, it must send x to an element of order 3 in $\text{Aut } H$.

There are 2 such elements. Because $\text{Aut } H \cong S_3$, we will think of them as the permutations of order 3 of the set $\{1, 2, 3\}$. Denote them $a = (1\ 2\ 3)$ and $b = (1\ 3\ 2)$. Notice that $b = a^{-1}$, so we have homomorphisms:

$$\psi_1 : x \mapsto a \quad \text{and} \quad \psi_2 : x \mapsto a^{-1}$$

It appears we have 2 choices, but this is not the case. The inverse map, $\beta : x \mapsto x^{-1}$, is an automorphism of K , and so by Lemma 2.8, the corresponding semidirect products of ψ_1 and ψ_2 are isomorphic. Hence (up to isomorphism) there is one non-trivial homomorphism $\psi : K \rightarrow \text{Aut } H$. So $x \in K$ acts by permuting the 3 non-identity elements of H .

We will show that in this case, $G \cong A_4$. First, let's check A_4 has the same subgroup structure as G . There is a subgroup isomorphic to C_3 in A_4 , generated by the 3-cycle $(1\ 2\ 3)$:

$$\bar{K} = \langle (1\ 2\ 3) \rangle$$

We can also find a subgroup isomorphic to V_4 :

$$\bar{H} = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

Indeed, we can check that \bar{H} is normal in A_4 . We can see that $\bar{H} \cap \bar{K} = \mathbf{1}$ because \bar{H} contains no 3-cycles, and that $\bar{H}\bar{K} = A_4$. So we can conclude that $A_4 = \bar{H} \rtimes \bar{K}$.

Let's investigate how conjugation behaves. If we let $\alpha = (1\ 2)(3\ 4)$, $\beta = (1\ 4)(2\ 3)$ and $\gamma = (1\ 2\ 3)$, then we can write an element of A_4 as $\alpha^i \beta^j \gamma^k$ for some i, j and k . Define $\phi : A_4 \rightarrow G$ by $\phi : \alpha^i \beta^j \gamma^k \mapsto x^i y^j z^k$. Then:

$$\beta\phi = (\gamma^{-1}\alpha\gamma)\phi = c^{-1}ac = b$$

So conjugation acts in the same way. Hence we can conclude that $G \cong A_4$.

If we instead consider G where $K \trianglelefteq G$, i.e. $G = K \rtimes H$, then we again have two cases:

Case 1: $H \cong C_4$ i.e. $G \cong C_3 \rtimes C_4$.

Let $H = \langle y \rangle$.

We know $\text{Aut } C_3 \cong C_2$ so a homomorphism ψ maps H to the trivial group or to $\langle \beta : x \mapsto x^{-1} \rangle$.

If $H\psi = \mathbf{1}$ then $G = K \times H \cong C_4 \times C_3$, which we have already seen.

If $H\psi = \langle \beta \rangle$ then we have:

$$G = \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$

Now let $a = xy^2$. And remember, $y^{-1}xy = x^{-1}$ means x commutes with y^2 . So now:

$$a^3 = xy^2xy^2xy^2 = x^3y^6 = y^2$$

and

$$y^{-1}ay = y^{-1}xy^2y = (y^{-1}xy)y^2 = x^{-1}y^2 = y^2x^{-1} = a^{-1}$$

So:

$$G = \langle a, y \mid a^6 = 1, a^3 = y^2, y^{-1}ay = a^{-1} \rangle$$

This is known as the binary dihedral or dicyclic group, denoted Dic_{12} . This group is also sometimes denoted by T .

Case 2: $H \cong V_4$ i.e. $G \cong C_3 \rtimes (C_2 \times C_2)$.

Let $H = \langle y, z \rangle$.

If $\psi : H \rightarrow \text{Aut } K$ is trivial then we obtain the direct product again.

The image of a non-trivial homomorphism $\psi : H \rightarrow \text{Aut } K$ is isomorphic to C_2 , so by the first Isomorphism Theorem, $\ker \psi \cong C_2$.

We can choose ψ such that $y\psi = \beta : x \mapsto x^{-1}$ and $z\psi = \iota : x \mapsto x$. Then:

$$G = \langle x, y, z \mid x^3 = y^2 = z^2 = 1, yz = zy, y^{-1}xy = x^{-1}, z^{-1}xz = x \rangle$$

Let $a = xz$. The order of $a = \text{lcm}(\text{o}(x), \text{o}(z)) = \text{lcm}(2, 3) = 6$ because x and z commute. So:

$$a^3 = x^3 z^3 = z$$

and

$$y^{-1}ay = y^{-1}xzy = y^{-1}xyz = x^{-1}z = x^2z = a^2a^3 = a^{-1}$$

Hence:

$$G = \langle y, a \mid a^6 = y^2 = 1, a^{-1}ay = a^{-1} \rangle \cong D_{12}$$

So a group G of order 12 is isomorphic to one of:

$$C_{12}, \quad C_2 \times C_6, \quad A_4, \quad D_{12}, \quad \text{or} \quad \text{Dic}_{12}$$

7 Groups of Order p^2q

Let p and q be distinct prime numbers, and G be a group of order p^2q . To classify G in full generality is beyond this report, so we will focus on the cases $|G| = 4q$ and $|G| = 3p^2$.

7.1 $4q$

7.2 $3p^2$

8 Groups of Order 24

Let G be a group of order 24, and let H be a Sylow 3-subgroup of G , so $H \cong C_3$. Let T be a Sylow 2-subgroup of G , so T has order 8. By Lagrange's Theorem, $H \cap T = \mathbf{1}$ and then applying Lemma 2.7, $|HT| = 24$. Now let n_3 denote the number of Sylow 3-subgroups, and by Sylow's Theorems:

$$n_3 \equiv 1 \pmod{3} \quad \text{and} \quad n_3 \mid 8$$

Hence n_3 is either 1 or 4.

If $n_3 = 1$, then H is normal in G . Thus $G = H \rtimes T$. We'll want a homomorphism $\psi : \text{Aut } T \rightarrow H$. From our classification of groups of order 8, we have 5 possibilities:

Case 1: $T \cong C_8$ i.e. $G \cong C_3 \rtimes C_8$

1 group

Case 2: $T \cong (C_4 \times C_2)$ i.e. $G \cong C_3 \rtimes (C_4 \times C_2)$

2 groups

Case 3: $T \cong (C_2 \times C_2 \times C_2)$ i.e. $G \cong C_3 \rtimes (C_2 \times C_2 \times C_2)$

1 group

Case 4: $T \cong D_8$ i.e. $G \cong C_3 \rtimes D_8$

2 groups

Case 5: $T \cong Q_8$ i.e. $G \cong C_3 \rtimes Q_8$

1 group — binary dihedral

If $n_3 = 4$ then H is not normal. Now let G act by conjugation on the set of its Sylow 3-subgroups, $\Omega = \{ H \mid H \text{ is a Sylow 3-subgroup of } G \}$:

$$H^x = x^{-1}Hx = \{ x^{-1}hx \mid h \in H \} \quad \text{for } x \in G$$

This is indeed a group action because for $x, y \in G$:

$$(H^x)^y = (x^{-1}Hx)^y = (y^{-1}x^{-1})H(xy) = (xy)^{-1}H(xy) = H^{(xy)}$$

and:

$$H^1 = 1^{-1}H1 = H$$

Hence we obtain a homomorphism $\rho : G \rightarrow S_4$. The group G is acting on a set of 4 elements, so the kernel must have order dividing 6. SOMETHING TO DO WITH NORMALISERS.

The kernel cannot be of order 3, because G has no normal subgroup of order 3 (because kernels of homomorphism are normal in the domain group). Likewise the kernel cannot be of order 6 because every group of order 6 has a unique Sylow 3-subgroup, which would be normal in G as well, by Lemma 2.4. Hence the kernel must have order 1 or 2.

If the kernel is of order 1, then ρ is actually an isomorphism, so $G \cong S_4$.

If the kernel is of order 2, then we know that $G/\ker \rho \cong \text{im } \rho$, so then $\text{im } \rho$ must have order 12. It also cannot have a normal Sylow 3-subgroup, so looking at our classification of groups of order 12, this must be isomorphic to A_4 . We know that A_4 has a normal subgroup of order 4, and so by the Correspondence Theorem, G must contain a normal subgroup of order 8, say T . By Lagrange's Theorem and Lemma 2.7, we can conclude that $G = T \rtimes H$. Again, we have 5 cases, but this time we'll exclude the trivial homomorphism, because that will just give us the direct product which we have already seen:

Case 1: $T \cong C_8$ i.e. $G \cong C_8 \rtimes C_3$

An automorphism of T , φ , maps generators to generators, so say $\langle x \rangle = T$. Then $x\varphi$ could be x, x^3, x^5 or x^7 . Notice that each of these, apart from $\varphi : x \mapsto x$, has order 2. Hence, and Lagrange's Theorem tells us that there are no non-trivial homomorphisms $\psi : H \rightarrow \text{Aut } T$. As a bonus: $\text{Aut } C_8 \cong V_4$.

Case 2: $T \cong (C_4 \times C_2)$ i.e. $G \cong (C_4 \times C_2) \rtimes C_3$

An automorphism of T , ψ preserves element order, so if $\langle x, y \mid x^4 = y^2 = 1, xy = yx \rangle = T$, then $x\psi$ must be of order 4, and $y\psi$ must be of order 2. Moreover, $y\psi$ cannot be in $\langle x\psi \rangle$ because ψ is injective.

So we are reduced to 2 possible choices for $y\psi$, and 4 possible choices for $x\psi$. Because an automorphism is determined by its effect on generators, this gives us 8 possible automorphisms. Hence $|\text{Aut } T| = 8$. Lagrange's Theorem tells us that there are no non-trivial homomorphisms $\psi : H \rightarrow \text{Aut } T$.

Case 3: $T \cong (C_2 \times C_2 \times C_2)$ i.e. $G \cong (C_2 \times C_2 \times C_2) \rtimes C_3$

To determine $\text{Aut } T$ it is helpful to think of C_2 as the finite field with two elements. Then T is isomorphic a 3 dimensional vector space over two elements. So an automorphism of that vector space is just any linear map, with non-zero determinant. Thus, $\text{Aut } T \cong \text{GL}_3(2)$.

We can determine that $|\mathrm{GL}_3(2)| = 168 = 2^3 \cdot 3 \cdot 7$, so $\mathrm{Aut} T$ has a Sylow 3-subgroup of order 3, isomorphic to C_3 .

Sylow's Theorems tells us that all subgroups of order 3 are conjugate, so Lemma 2.8 tells us there is only one unique action (up to isomorphism) of H on T . So let's pick a homomorphism, ψ , which will let us easily classify the resulting semidirect product.

Write $T = A \times B$ where $A \cong C_2$ and $B \cong C_2 \times C_2$. Then let ψ map H to the subgroup generated by the automorphism which fixes A and permutes the non-identity elements of B in a 3-cycle. This automorphism has order 3 by construction, so we can write:

$$G \cong C_2 \times (V_4 \rtimes C_3)$$

We know already that $V_4 \rtimes C_3 \cong A_4$ so $G \cong C_2 \times A_4$.

Case 4: $T \cong D_8$ i.e. $G \cong D_8 \rtimes C_3$

Let $\langle s, r \mid s^2 = r^4 = 1, s^{-1}rs = r^{-1} \rangle = T$. An automorphism, ψ , of T preserves element order, so for $r\psi$ we have two choices, r or r^{-1} . We can send $s\psi$ to any element of order 2 which is not in $\langle r\psi \rangle$. This leaves only reflections, of which there are 4: s, rs, r^2s and r^3s . Hence there are 8 possible automorphisms of D_8 , so $|\mathrm{Aut} D_8| = 8$. Lagrange's Theorem tells us that there are no non-trivial homomorphisms $\psi : H \rightarrow \mathrm{Aut} T$.

Case 5: $T \cong Q_8$ i.e. $G \cong Q_8 \rtimes C_3$

Firstly, because of the multiplication structure of the quaternions, the image of k is determined by the images of i and j ; it is forced. This reduces the possibilities for an automorphism. Additionally, ± 1 are fixed by an automorphism, because they are the only elements of their order. So an automorphism could send i to any of the remaining 6 elements of order 4. The image of j cannot be in the subgroup generated by the image of i , otherwise we wouldn't have an automorphism. Thus there are 4 choices for the image of j , giving us 24 possible automorphisms altogether.

So $\mathrm{Aut} T$ will have a Sylow subgroup of order 3.

1 group — binary tetrahedral

9 Groups of Order 30

Let G be a group of order $30 = 2 \cdot 3 \cdot 5$. So then G has a Sylow 3-subgroup, T , and a Sylow 5-subgroup, F . Let $H = TF$ and by Lagrange's Theorem, $T \cap F = \mathbf{1}$, hence $|H| = 15$ by Lemma 2.7. We know from our classification of groups of order pq that $H \cong C_{15}$. Because $|H| = 15 = \frac{30}{2}$, the index of H in G is 2, and we know a subgroup of index 2 is normal, so $H \trianglelefteq G$.

Notice that a Sylow 2-subgroup $K \leq G$ has order 2, so $K \cong C_2$. Let $\langle k \rangle = K$ and $\langle h \rangle = H$. By the same argument as above, $H \cap K = \mathbf{1}$ and $|HK| = 30$. Hence $G = HK$. Moreover, $G = H \rtimes K$.

By Lemma 2.3:

$$\mathrm{Aut} C_{15} = (\mathbb{Z}/15\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times \cong C_2 \times C_4$$

A homomorphism, $\psi : C_2 \rightarrow C_2 \times C_4$ preserves element order and we know ψ is determined by its effect on a generator. So then $k\psi$ has four possibilities: either the identity, or one of the three elements of order 2.

Additionally, ψ preserves the Sylow subgroups of H . So write $H = \langle h^3 \rangle \times \langle h^5 \rangle$, the direct product of its Sylow subgroups.

So the action of K on H is either trivial or by inversion on each of the Sylow subgroups of H , giving us 4 possibilities:

Case 1: Trivial action on both Sylow subgroups.

In this case, because the action is trivial on all of H , we recover the direct product, $G = H \times K \cong C_{30}$.

Case 2: Inversion on both Sylow subgroups.

Here, K acts on all of H , so we obtain:

$$G = \langle h, k \mid h^{15} = k^2 = 1, k^{-1}hk = h^{-1} \rangle$$

which we recognise as D_{30} .

Case 3: Inversion on $\langle h^5 \rangle$.

We know already, from our classification of groups of order $2p$, that $C_3 \rtimes C_2 \cong D_6$. So then because the action on $\langle h^3 \rangle$ is trivial:

$$G = \langle h^3 \rangle \times (\langle h^5 \rangle \rtimes K) \cong C_5 \times D_6$$

Case 4: Inversion on $\langle h^3 \rangle$.

Similar to above, we obtain:

$$G = \langle h^5 \rangle \times (\langle h^3 \rangle \rtimes K) \cong C_3 \times D_{10}$$

Hence any group of order 30 is isomorphic to one of:

$$C_{30}, \quad D_{15}, \quad C_5 \times D_6, \quad \text{or} \quad C_3 \times D_{10}$$

Part II

To Do

10 Groups of Order p^3

10.1 Groups of Order 8

10.2 Groups of Order 27

10.3 General Case?

11 Groups of Order 16