

Interim Report

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1 Theorems and Lemmas

1.1 Sylow Theorems

Let G be a group of order $p^n m$ where p is a prime and $p \nmid m$.

Theorem 1.1 (1st Sylow Theorem). *G has a Sylow p -subgroup, i.e. a subgroup of order p^n .*

Theorem 1.2 (2nd Sylow Theorem). *All Sylow p -subgroups of G are conjugate to each other.*

Corollary 1.2.1. *If $n_p = 1$ then the Sylow p -subgroup is normal in G .*

Theorem 1.3 (3rd Sylow Theorem). *Let n_p denote the number of Sylow p -subgroups of G . Then:*

i) $n_p \mid m$

ii) $n_p \equiv 1 \pmod{p}$

1.2 Isomorphism Theorems

Theorem 1.4.

Theorem 1.5.

Theorem 1.6.

Lemma 1.7. For a group G with $N \leq G$ and $H \leq G$, then

$$|NH| = |\{nh \mid n \in N, h \in H\}| = \frac{|N| \cdot |H|}{|N \cap H|}$$

Lemma 1.8. The automorphism group of C_n is isomorphic to the multiplicative group of integers mod n .

i.e. $\text{Aut } C_n \cong \mathbb{Z}/n\mathbb{Z}^*$

Proof. Any automorphism, φ of C_n has the property:

$$(x^i)\varphi = (x\varphi)^i$$

Hence φ is determined by its effect on a generator, x , and preserves element order. In particular, φ sends generators to generators.

So for a generator, x , $x\varphi = x^k$ is surjective if x^k generates C_n . x^k generates C_n if $o(x^k) = n$ which is when $\gcd(n, k) = 1$.

Denote $\varphi_k : x \mapsto x^k$.

Consider:

$$x\varphi_k\varphi_l = (x^k)\varphi_l = (x^k)^l = x^{kl} = x^{lk} = (x^l)^k = x\varphi_l\varphi_k$$

So we see that $\text{Aut } C_n$ is abelian. Moreover, $x\varphi_k\varphi_l = x\varphi_{kl}$.

Now consider $\theta : \text{Aut } C_n \rightarrow \mathbb{Z}/n\mathbb{Z}^*$ defined by $\varphi_k\theta = k$. We will show θ is an isomorphism.

θ is surjective because every $k \in \mathbb{Z}/n\mathbb{Z}^*$ is coprime to n and so x^k is a generator of C_n , hence $\exists \varphi_k \in \text{Aut } C_n$ such that $\varphi_k\theta = k$.

θ is also injective because if $\varphi_k, \varphi_l \in \text{Aut } C_n$ such that $\varphi_k\theta = \varphi_l\theta$ then $k = l$.

Finally, θ is a homomorphism because $(\varphi_k\varphi_l)\theta = \varphi_{kl}\theta = kl = (\varphi_k\theta)(\varphi_l\theta)$. So $\theta : \text{Aut } C_n \rightarrow \mathbb{Z}/n\mathbb{Z}^*$ is an isomorphism. □

2 Groups of Order 6

Let G be a group of order 6, and n_3 denote the number of Sylow 3-subgroups of G . Then by Theorem 1.3:

$$n_3 \equiv 1 \pmod{3} \text{ and } n_3 \mid 2 \implies n_3 = 1$$

So G has one Sylow 3-subgroup, and because 3 is prime, it is isomorphic to C_3 , i.e.

$$C_3 \trianglelefteq G$$

Any Sylow 2-subgroup of G will have order 2, and so $C_2 \leq G$.

Lagrange's Theorem tells us that C_3 has elements of orders 1 and 3, and C_2 has elements of order 1 and 2 hence:

$$C_3 \cap C_2 = \mathbf{1}$$

By Lemma 1.7:

$$|C_3C_2| = \frac{|C_3| \cdot |C_2|}{|C_3 \cap C_2|} = \frac{3 \cdot 2}{1} = 6$$

So $G = C_3C_2$, $C_3 \trianglelefteq G$ and $C_3 \cap C_2 = \mathbf{1} \implies G = C_3 \rtimes C_2$

Now we need to determine $\text{Aut } C_3$. $C_3 = \{1, x, x^2 = x^{-1}\}$ and so $\text{Aut } C_3 = \{\text{id}, \psi : x \mapsto x^{-1}\} \cong C_2$. So if $C_3 = \langle x \rangle$ and $C_2 = \langle y \rangle$, then we have two possibilities for G :

Case 1:

$$\begin{aligned} G &= \langle x, y \mid x^3 = y^2 = 1, y^{-1}xy = x \rangle \\ &= \langle x, y \mid x^3 = y^2 = 1, xy = yx \rangle \\ &= C_3 \times C_2 \cong C_6 \end{aligned}$$

Case 2:

$$G = \langle x, y \mid x^3 = y^2 = 1, y^{-1}xy = x^{-1} \rangle \\ \cong D_6$$

Hence G is isomorphic to either C_6 or D_6 .

3 Generalisation to Groups of Order $2p$

Let G be a group of order $2p$ where p is a prime number, and n_p denote the number of Sylow p -subgroups of G . Then by Theorem 1.3:

$$n_p \equiv 1 \pmod{p} \text{ and } n_p \mid 2 \implies n_p = 1$$

So G has one Sylow p -subgroup, it is isomorphic to $C_p = \langle x \rangle$ hence:

$$C_p \trianglelefteq G$$

A Sylow 2-subgroup of G will have order 2 so $C_2 = \langle y \rangle \leq G$.

Lagrange's Theorem tells us that C_p has elements of orders 1 and p , and C_2 has elements of order 1 and 2 hence:

$$C_p \cap C_2 = \mathbf{1}$$

By Lemma 1.7:

$$|C_p C_2| = \frac{|C_p| \cdot |C_2|}{|C_p \cap C_2|} = \frac{p \cdot 2}{1} = 2p$$

So $G = C_p C_2$, $C_p \trianglelefteq G$ and $C_p \cap C_2 = \mathbf{1} \implies G = C_p \rtimes C_2$

We want a homomorphism $\varphi : \text{Aut } C_p \rightarrow C_2$. By Lemma 1.8, $\text{Aut } C_p \cong \mathbb{Z}/p\mathbb{Z}^*$, so now we need to find elements of order 2 in $\mathbb{Z}/p\mathbb{Z}^*$.

An element $x \in \mathbb{Z}/p\mathbb{Z}^*$ of order 2 satisfies:

$$x^2 = 1 \implies x^2 - 1 = 0 \implies (x - 1)(x + 1) = 0$$

Hence $x = 1$ or -1 . But 1 has order 1 so x can only be -1 .

So $C_2\varphi$ could be $\mathbf{1}$ or $\langle \beta : x \mapsto x^{-1} \rangle$. This gives us two possibilities:

$$y\varphi = x \mapsto x \quad \text{or} \quad y\varphi = x \mapsto x^{-1}$$

Case 1:

$$G = \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x \rangle \\ = C_p \times C_2 \cong C_{2p}$$

Case 2:

$$G = \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x^{-1} \rangle \\ \cong D_{2p}$$

Hence a group of order $2p$ is isomorphic to C_{2p} or D_{2p} .

4 Groups of order 4

From the Fundamental Theorem of Finite Abelian Groups, we know that a group G of order 4 could be isomorphic to one of C_4 and $C_2 \times C_2$. Now to show that these are the only possibilities, i.e. a group of order 4 must be abelian.

The Sylow theorems are not so helpful here, because $4 = 2^2$ so any Sylow 2-subgroup will be of order 4, which is just G .

5 Groups of order 9 (Might skip)

6 Generalisation to Groups of Order p^2

Let G be a group of order p^2 and consider $Z(G) \trianglelefteq G$. By Lagrange's Theorem, $Z(G)$ has order 1, p or p^2 .

7 Groups of order 12

Let G be a group of order $12 = 2^2 \cdot 3$, and n_3 and n_2 denote the number of Sylow 3-subgroups and Sylow 2-subgroups of G respectively. By Theorem 1.3:

$$n_2 \equiv 1 \pmod{2} \text{ and } n_2 \mid 3 \implies n_2 = 1$$

$$n_3 \equiv 1 \pmod{3} \text{ and } n_3 \mid 4 \implies n_3 = 1 \text{ or } 4$$

G has a unique Sylow 2-subgroup of order $2^2 = 4$, say $H \trianglelefteq G$, and we have already classified groups of order 4, so either C_4 or $V_4 \trianglelefteq G$. A Sylow 3-subgroup of G will have order 3, so $C_3 \leq G$, and for some groups, $C_3 \trianglelefteq G$.

Lagrange's Theorem tells us H has elements of order 1, 2, and 4, and C_3 has elements of order 1 and 3. Hence $H \cap C_3 = \mathbf{1}$.

Lemma 1.7 tells us:

$$|HC_3| = \frac{|H| \cdot |C_3|}{|H \cap C_3|} = 12$$

Hence $G = HC_3$, $C_3 \leq G$, $H \trianglelefteq G$, and $H \cap C_3 = \mathbf{1} \implies G = H \rtimes C_3$.

Since an automorphism, φ , must map generators to generators, $\text{Aut } C_4 \cong C_2$ because the generators of C_4 are x and x^{-1} . An automorphism of V_4 corresponds to a permutation of the three non-identity elements, hence $\text{Aut } V_4 \cong S_3$.

Case 1: $H = C_4$ i.e. $G = C_4 \rtimes C_3$.

A homomorphism $\psi : C_3 \rightarrow \text{Aut } C_4 \cong C_2$, preserves order and together with Lagrange's Theorem means that the only possibility for ψ is trivial, i.e. $C_3\psi = \mathbf{1}$.

Hence $G \cong C_4 \times C_3 \cong C_{12}$.

Case 2: $H = V_4$ i.e. $G = (C_2 \times C_2) \rtimes C_3$.

A trivial homomorphism $C_3\psi = \mathbf{1}$ yields the direct product $G \cong C_2 \times C_2 \times C_3 \cong C_2 \times C_6$.

S_3 has one subgroup of order 3, hence there is essentially only one homomorphism $\psi : C_3 \rightarrow \text{Aut } V_4$.

Still need to show this is A_4 .

If we instead consider G where $C_3 \trianglelefteq G$, i.e. $G = C_3 \rtimes H$, then we again have two cases:

Case 1: $H = C_4$ i.e. $G = C_3 \rtimes C_4$.

Say $C_3 = \langle x \rangle$ and $C_4 = \langle y \rangle$. We know $\text{Aut } C_3 \cong C_2$ so a homomorphism ψ maps C_4 to the trivial group, $\mathbf{1}$ or to $\langle \beta : x \mapsto x^{-1} \rangle$.

If $C_4\psi = \mathbf{1}$ then $G = C_3 \times C_4 \cong C_4 \times C_3$, which we have already seen.

If $C_4\psi = \langle \beta \rangle$ then we have:

$$G = \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$

Case 2: $H = V_4$ i.e. $G = C_3 \rtimes (C_2 \times C_2)$.

If $\psi : (C_2 \times C_2) \rightarrow \text{Aut } C_3$ is trivial then we obtain $G = C_3 \times C_2 \times C_2 \cong C_2 \times C_6$ which we have seen before.

The image of a non-trivial homomorphism $\psi : (C_2 \times C_2) \rightarrow \text{Aut } C_3$ is C_2 , so by Theorem 1.4: $\ker \theta = C_2$.

Choose $a, b \in C_2 \times C_2$ with $a, b \neq 1$ such that $a\theta = \beta : x \mapsto x^{-1}$ and $b\theta = \text{id} : x \mapsto x$. Then:

$$G = \langle x, a, b \mid x^3 = a^2 = b^2 = 1, ab = ba, a^{-1}xa = x^{-1}, b^{-1}xb = x \rangle$$

Let $y = xb$. The order of $y = \text{lcm}(\text{o}(x), \text{o}(b)) = \text{lcm}(2, 3) = 6$ because x and b commute. $y^3 = x^3b^3 = b$ so:

$$a^{-1}ya = a^{-1}xba = a^{-1}xab = x^{-1}b = x^2b = y^2y^3 = y^{-1}$$

Hence:

$$G = \langle a, y \mid y^6 = a^2 = 1, a^{-1}ya = y^{-1} \rangle \cong D_{12}$$

So a group G of order 12 is isomorphic to one of:

$$C_{12}, \quad C_2 \times C_6 \quad A_4 \quad D_{12} \quad \text{or} \quad \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$