Classification of Finite Groups

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Groups of Order pq

Setup

Let G be a group of order pq where p and q are distinct prime numbers.

Without loss of generality, take $p > q$.

Sylow Subgroups

Let n_p and n_q denote the number of Sylow p-subgroups and Sylow q-subgroups of G respectively.

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q$$

So G has a unique Sylow p-subgroup, say $P \trianglelefteq G$, and at least one Sylow q-subgroup, $Q \leqslant G$.

$$P \cong C_p, \qquad Q \cong C_q$$

Pick generators: $\langle\, x \,\rangle = P$ and $\langle\, y \,\rangle = Q$.

Sylow q-subgroup

Because $n_q \equiv 1 \pmod{q}$, we have:

$$n_q = 1,\ q + 1,\ 2q + 1,\ \dots$$

And:

$$n_q \mid p$$

So we have two cases:

$$q \nmid p - 1 \qquad \text{or} \qquad q \mid p - 1$$

Case 1: $q \nmid p - 1$

$q \nmid p - 1$

Here, $n_q = 1$ and so $Q \trianglelefteq G$.

So:

$$G = P \times Q \cong C_{pq}$$

Case 2: $q \mid p - 1$

$q \mid p - 1$

That was the easy one! Still have $n_q = 1$, but now also the other possibilities!

By Lagrange's Theorem: $P \cap Q = 1$.

By Lemma in project: $|PQ| = pq$.

So $G = PQ$. Hence:

$$G = P \rtimes Q$$

This is unique!

Presentation

Describe $P \rtimes Q$ by a presentation.

$$G = \langle\, x,\ y \mid x^p = y^q = 1,\ y^{-1}xy = x^a \,\rangle$$

where a is a generator for the subgroup of order q in $(\mathbb{Z}/p\mathbb{Z})^\times$.

Classification

For distinct primes p and q, any group of order pq is isomorphic to one of:

$$C_{pq}$$

$$\langle\, x, y \mid x^p = y^q = 1,\ y^{-1}xy = x^a \,\rangle \quad \text{additionally, if } q \mid p-1$$

Examples