

# Classification of Finite Groups

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# Part I

# Done

## 1 Groups of Order 6

Let  $G$  be a group of order 6, and  $n_3$  denote the number of Sylow 3-subgroups of  $G$ . Then by Theorem 5.3:

$$n_3 \equiv 1 \pmod{3} \text{ and } n_3 \mid 2 \implies n_3 = 1$$

So  $G$  has one Sylow 3-subgroup, and because 3 is prime, it is isomorphic to  $C_3$ , i.e.

$$C_3 \trianglelefteq G$$

Any Sylow 2-subgroup of  $G$  will have order 2, and so  $C_2 \leq G$ .

Lagrange's Theorem tells us that  $C_3$  has elements of orders 1 and 3, and  $C_2$  has elements of order 1 and 2 hence:

$$C_3 \cap C_2 = \mathbf{1}$$

By Lemma 5.7:

$$|C_3 C_2| = \frac{|C_3| \cdot |C_2|}{|C_3 \cap C_2|} = \frac{3 \cdot 2}{1} = 6$$

So  $G = C_3 C_2$ ,  $C_3 \trianglelefteq G$  and  $C_3 \cap C_2 = \mathbf{1} \implies G = C_3 \rtimes C_2$

Now we need to determine  $\text{Aut } C_3$ .  $C_3 = \{1, x, x^2 = x^{-1}\}$  and so  $\text{Aut } C_3 = \{\text{id}, \psi : x \mapsto x^{-1}\} \cong C_2$ . So if  $C_3 = \langle x \rangle$  and  $C_2 = \langle y \rangle$ , then we have two possibilities for  $G$ :

**Case 1:**

$$\begin{aligned} G &= \langle x, y \mid x^3 = y^2 = 1, y^{-1}xy = x \rangle \\ &= \langle x, y \mid x^3 = y^2 = 1, xy = yx \rangle \\ &= C_3 \times C_2 \cong C_6 \end{aligned}$$

**Case 2:**

$$\begin{aligned} G &= \langle x, y \mid x^3 = y^2 = 1, y^{-1}xy = x^{-1} \rangle \\ &\cong D_6 \end{aligned}$$

Hence  $G$  is isomorphic one of:

$$C_6 \quad \text{or} \quad D_6$$

## 2 Generalisation to Groups of Order $2p$

Let  $G$  be a group of order  $2p$  where  $p$  is a prime number, and  $n_p$  denote the number of Sylow  $p$ -subgroups of  $G$ . Then by Theorem 5.3:

$$n_p \equiv 1 \pmod{p} \text{ and } n_p \mid 2 \implies n_p = 1$$

So  $G$  has one Sylow  $p$ -subgroup, it is isomorphic to  $C_p = \langle x \rangle$  hence:

$$C_p \trianglelefteq G$$

A Sylow 2-subgroup of  $G$  will have order 2 so  $C_2 = \langle y \rangle \leq G$ .

Lagrange's Theorem tells us that  $C_p$  has elements of orders 1 and  $p$ , and  $C_2$  has elements of order 1 and 2 hence:

$$C_p \cap C_2 = \mathbf{1}$$

By Lemma 5.7:

$$|C_p C_2| = \frac{|C_p| \cdot |C_2|}{|C_p \cap C_2|} = \frac{p \cdot 2}{1} = 2p$$

So  $G = C_p C_2$ ,  $C_p \trianglelefteq G$  and  $C_p \cap C_2 = \mathbf{1} \implies G = C_p \rtimes C_2$

We want a homomorphism  $\varphi : \text{Aut } C_p \rightarrow C_2$ . By Lemma 5.8,  $\text{Aut } C_p \cong \mathbb{Z}/p\mathbb{Z}^*$ , so now we need to find elements of order 2 in  $\mathbb{Z}/p\mathbb{Z}^*$ .

An element  $x \in \mathbb{Z}/p\mathbb{Z}^*$  of order 2 satisfies:

$$x^2 = 1 \implies x^2 - 1 = 0 \implies (x - 1)(x + 1) = 0$$

Hence  $x = 1$  or  $-1$ . But 1 has order 1 so  $x$  can only be  $-1$ .

So  $C_2\varphi$  could be  $\mathbf{1}$  or  $\langle \beta : x \mapsto x^{-1} \rangle$ . This gives us two possibilities:

$$y\varphi = x \mapsto x \quad \text{or} \quad y\varphi = x \mapsto x^{-1}$$

**Case 1:**

$$\begin{aligned} G &= \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x \rangle \\ &= C_p \times C_2 \cong C_{2p} \end{aligned}$$

**Case 2:**

$$\begin{aligned} G &= \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x^{-1} \rangle \\ &\cong D_{2p} \end{aligned}$$

Hence a group of order  $2p$  is isomorphic to one of:

$$C_{2p} \quad \text{or} \quad D_{2p}$$

### 3 Groups of order 4

From the Fundamental Theorem of Finite Abelian Groups, we know that a group  $G$  of order 4 could be isomorphic to one of  $C_4$  and  $C_2 \times C_2$ . Now to show that these are the only possibilities.

The Sylow theorems are not so helpful here, because  $4 = 2^2$  so any Sylow 2-subgroup will be of order 4, which is just  $G$ .

Lagrange's Theorem tells us every element of  $G$  has order 1, 2 or 4.

If  $x \in G$  has order 4, then  $\langle x \rangle = G$  and so  $G$  is cyclic, i.e.  $G \cong C_4$ .

If  $\nexists x \in G$  with order 4, then every  $x \in G$ ,  $x \neq 1$  is of order 2. Consider  $a, b \in G$  with  $a \neq b$ , and the product  $ab$ .  $ab \neq 1$  because then  $b = a^{-1} = a$ , a contradiction.  $ab \neq a$  or  $b$  because then  $a = 1$  or  $b = 1$  respectively. Hence it must be that  $ab$  is the third element of order 2.

It is easy to see that  $G \cong C_2 \times C_2$ .

So any group of order 4 is isomorphic to one of:

$$C_4 \quad \text{or} \quad C_2 \times C_2$$

## 4 Groups of Order $pq$

Let  $G$  be a group of order  $pq$  where  $p, q$  are prime numbers with  $p > q$ , and let  $n_p$  and  $n_q$  denote the number of Sylow  $p$ -subgroups and Sylow  $q$ -subgroups of  $G$  respectively. Then by Theorem 5.3:

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q \implies n_p = 1$$

$$n_q \equiv 1 \pmod{q} \implies n_q = 1, q + 1, 2q + 1, \dots \quad \text{and} \quad n_q \mid p$$

So  $C_p \trianglelefteq G$  and we have 2 possibilities for  $C_q$ :  $p - 1$  is a multiple of  $q$  or 1.

**Case 1:**  $q \nmid p - 1$ .

If  $p - 1$  is not a multiple of  $q$  then  $n_q = 1$  and  $C_q \trianglelefteq G$ , hence:

$$G = C_p \times C_q \cong C_{pq}$$

**Case 2:**  $q \mid p - 1$ .

If  $p - 1$  is a multiple of  $q$  then  $n_q = p$  and so  $C_q \leq G$ . By Lagrange's Theorem,  $C_p \cap C_q = \mathbf{1}$  and by Lemma 5.7,  $|C_p C_q| = pq$ , hence, as well as the direct product, we have  $G = C_p \rtimes C_q$ .

By Lemma 5.8,  $\text{Aut } C_p \cong C_{p-1}$  and so a non-trivial homomorphism  $\varphi : C_q \mapsto \langle \beta : x \mapsto x^{-1} \rangle$ . So if  $\langle x \rangle = C_p$  and  $\langle y \rangle = C_q$ , we obtain:

$$G = \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^{-1} \rangle$$

So any group of order  $pq$  is isomorphic to either:

$$\begin{array}{ll} C_{pq} & \text{or} \quad \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^{-1} \rangle \quad \text{if } q \mid p - 1 \\ & C_{pq} \quad \text{if } q \nmid p - 1 \end{array}$$

## Part II

# In Progress

## 5 Theorems and Lemmas

### 5.1 Sylow Theorems

Let  $G$  be a group of order  $p^n m$  where  $p$  is a prime and  $p \nmid m$ .

**Theorem 5.1** (1<sup>st</sup> Sylow Theorem).  $G$  has a Sylow  $p$ -subgroup, i.e. a subgroup of order  $p^n$ .

**Theorem 5.2** (2<sup>nd</sup> Sylow Theorem). All Sylow  $p$ -subgroups of  $G$  are conjugate to each other.

**Corollary 5.2.1.** If  $n_p = 1$  then the Sylow  $p$ -subgroup is normal in  $G$ .

**Theorem 5.3** (3<sup>rd</sup> Sylow Theorem). Let  $n_p$  denote the number of Sylow  $p$ -subgroups of  $G$ . Then:

$$i) \quad n_p \mid m$$

$$ii) \quad n_p \equiv 1 \pmod{p}$$

## 5.2 Isomorphism Theorems

**Theorem 5.4.**

**Theorem 5.5.**

**Theorem 5.6.**

**Lemma 5.7.** For a group  $G$  with  $N \leq G$  and  $H \leq G$ , then

$$|NH| = |\{nh \mid n \in N, h \in H\}| = \frac{|N| \cdot |H|}{|N \cap H|}$$

**Lemma 5.8.** The automorphism group of  $C_n$  is isomorphic to the multiplicative group of integers mod  $n$ .

$$\text{i.e. } \text{Aut } C_n \cong \mathbb{Z}/n\mathbb{Z}^*$$

*Proof.* Any automorphism,  $\varphi$  of  $C_n$  has the property:

$$(x^i)\varphi = (x\varphi)^i$$

Hence  $\varphi$  is determined by its effect on a generator,  $x$ , and preserves element order. In particular,  $\varphi$  sends generators to generators.

So for a generator,  $x$ ,  $x\varphi = x^k$  is surjective if  $x^k$  generates  $C_n$ .  $x^k$  generates  $C_n$  if  $o(x^k) = n$  which is when  $\gcd(n, k) = 1$ .

Denote  $\varphi_k : x \mapsto x^k$ .

Consider:

$$x\varphi_k\varphi_l = (x^k)\varphi_l = (x^k)^l = x^{kl} = x^{lk} = (x^l)^k = x\varphi_l\varphi_k$$

So we see that  $\text{Aut } C_n$  is abelian. Moreover,  $x\varphi_k\varphi_l = x\varphi_{kl}$ .

Now consider  $\theta : \text{Aut } C_n \rightarrow \mathbb{Z}/n\mathbb{Z}^*$  defined by  $\varphi_k\theta = k$ . We will show  $\theta$  is an isomorphism.

$\theta$  is surjective because every  $k \in \mathbb{Z}/n\mathbb{Z}^*$  is coprime to  $n$  and so  $x^k$  is a generator of  $C_n$ , hence  $\exists \varphi_k \in \text{Aut } C_n$  such that  $\varphi_k\theta = k$ .

$\theta$  is also injective because if  $\varphi_k, \varphi_l \in \text{Aut } C_n$  such that  $\varphi_k\theta = \varphi_l\theta$  then  $k = l$ .

Finally,  $\theta$  is a homomorphism because  $(\varphi_k\varphi_l)\theta = \varphi_{kl}\theta = kl = (\varphi_k\theta)(\varphi_l\theta)$ . So  $\theta : \text{Aut } C_n \rightarrow \mathbb{Z}/n\mathbb{Z}^*$  is an isomorphism. □

## 6 Generalisation to Groups of Order $p^2$

Let  $G$  be a group of order  $p^2$  and consider  $Z(G) \trianglelefteq G$ . By Lagrange's Theorem,  $Z(G)$  has order 1,  $p$  or  $p^2$ .

If  $|Z(G)| = p^2$  then  $G$  is abelian.

Assume  $|Z(G)| \neq p^2$  and consider an element  $x \in G$  but  $x \notin Z(G)$ , and its centraliser,  $C_G(x)$ . We know  $C_G(x) \leq G$  and that  $x \in C_G(x)$ , so  $|C_G(x)| \neq 1$ , and so by Lagrange's Theorem, it must be that  $|C_G(x)| = p$ . So:

$$|x^G| = |G : C_G(x)| = \frac{p^2}{p} = p$$

The Class Equation,  $|G| = |Z(G)| + \sum_{i=1}^k |x_i^G|$ , tells us  $|Z(G)|$  must be a multiple of  $p$  because both  $|G|$  and  $|x^G|$  are multiples of  $p$ . Hence  $|Z(G)| = p$ .

So then  $|G : Z(G)| = p$ , which means  $G/Z(G) \cong C_p$ .

## Sketch

- Show  $G$  must be abelian. Result follows from FTFAB.
- $Z(G)$  has order 1,  $p$ , or  $p^2$  by Lagrange.
- If  $p^2$  then done.
- Size of congruency classes is multiple of  $p$ .
- Class eqn  $\Rightarrow$  order of centre is multiple of  $p$ . (and so is not 1)
- Quotient with  $G$  is cyclic.
- MT4003 showed then  $G$  must be abelian.

## 7 Groups of order 12

Let  $G$  be a group of order  $12 = 2^2 \cdot 3$ , and  $n_3$  and  $n_2$  denote the number of Sylow 3-subgroups and Sylow 2-subgroups of  $G$  respectively. By Theorem 5.3:

$$n_2 \equiv 1 \pmod{2} \text{ and } n_2 \mid 3 \implies n_2 = 1$$

$$n_3 \equiv 1 \pmod{3} \text{ and } n_3 \mid 4 \implies n_3 = 1 \text{ or } 4$$

$G$  has a unique Sylow 2-subgroup of order  $2^2 = 4$ , say  $H \trianglelefteq G$ , and we have already classified groups of order 4, so either  $C_4$  or  $V_4 \trianglelefteq G$ . A Sylow 3-subgroup of  $G$  will have order 3, so  $C_3 \leq G$ , and for some groups,  $C_3 \trianglelefteq G$ .

Lagrange's Theorem tells us  $H$  has elements of order 1, 2, and 4, and  $C_3$  has elements of order 1 and 3. Hence  $H \cap C_3 = \mathbf{1}$ .

Lemma 5.7 tells us:

$$|HC_3| = \frac{|H| \cdot |C_3|}{|H \cap C_3|} = 12$$

Hence  $G = HC_3$ ,  $C_3 \leq G$ ,  $H \trianglelefteq G$ , and  $H \cap C_3 = \mathbf{1} \implies G = H \rtimes C_3$ .

Since an automorphism,  $\varphi$ , must map generators to generators,  $\text{Aut } C_4 \cong C_2$  because the generators of  $C_4$  are  $x$  and  $x^{-1}$ . An automorphism of  $V_4$  corresponds to a permutation of the three non-identity elements, hence  $\text{Aut } V_4 \cong S_3$ .

**Case 1:**  $H = C_4$  i.e.  $G = C_4 \rtimes C_3$ .

A homomorphism  $\psi : C_3 \rightarrow \text{Aut } C_4 \cong C_2$ , preserves order and together with Lagrange's Theorem means that the only possibility for  $\psi$  is trivial, i.e.  $C_3\psi = \mathbf{1}$ .

Hence  $G \cong C_4 \times C_3 \cong C_{12}$ .

**Case 2:**  $H = V_4$  i.e.  $G = (C_2 \times C_2) \rtimes C_3$ .

A trivial homomorphism  $C_3\psi = \mathbf{1}$  yields the direct product  $G \cong C_2 \times C_2 \times C_3 \cong C_2 \times C_6$ .

$S_3$  has one subgroup of order 3, hence there is essentially only one homomorphism  $\psi : C_3 \rightarrow \text{Aut } V_4$ .

Still need to show this is  $A_4$ .

If we instead consider  $G$  where  $C_3 \trianglelefteq G$ , i.e.  $G = C_3 \rtimes H$ , then we again have two cases:

**Case 1:**  $H = C_4$  i.e.  $G = C_3 \rtimes C_4$ .

Say  $C_3 = \langle x \rangle$  and  $C_4 = \langle y \rangle$ . We know  $\text{Aut } C_3 \cong C_2$  so a homomorphism  $\psi$  maps  $C_4$  to the trivial group,  $\mathbf{1}$  or to  $\langle \beta : x \mapsto x^{-1} \rangle$ .

If  $C_4\psi = \mathbf{1}$  then  $G = C_3 \times C_4 \cong C_4 \times C_3$ , which we have already seen.

If  $C_4\psi = \langle \beta \rangle$  then we have:

$$G = \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$

**Case 2:**  $H = V_4$  i.e.  $G = C_3 \rtimes (C_2 \times C_2)$ .

If  $\psi : (C_2 \times C_2) \rightarrow \text{Aut } C_3$  is trivial then we obtain  $G = C_3 \times C_2 \times C_2 \cong C_2 \times C_6$  which we have seen before.

The image of a non-trivial homomorphism  $\psi : (C_2 \times C_2) \rightarrow \text{Aut } C_3$  is  $C_2$ , so by Theorem 5.4:  $\ker \theta = C_2$ .

Choose  $a, b \in C_2 \times C_2$  with  $a, b \neq 1$  such that  $a\theta = \beta : x \mapsto x^{-1}$  and  $b\theta = \text{id} : x \mapsto x$ . Then:

$$G = \langle x, a, b \mid x^3 = a^2 = b^2 = 1, ab = ba, a^{-1}xa = x^{-1}, b^{-1}xb = x \rangle$$

Let  $y = xb$ . The order of  $y = \text{lcm}(\text{o}(x), \text{o}(b)) = \text{lcm}(2, 3) = 6$  because  $x$  and  $b$  commute.  $y^3 = x^3b^3 = b$  so:

$$a^{-1}ya = a^{-1}xba = a^{-1}xab = x^{-1}b = x^2b = y^2y^3 = y^{-1}$$

Hence:

$$G = \langle a, y \mid y^6 = a^2 = 1, a^{-1}ya = y^{-1} \rangle \cong D_{12}$$

So a group  $G$  of order 12 is isomorphic to one of:

$$C_{12}, \quad C_2 \times C_6, \quad A_4, \quad D_{12}, \quad \text{or} \quad \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$

## 8 Generalisation to Groups of order $4p$

Suppose  $G$  is a group of order  $4p$  where  $p$  is a prime number. Let  $n_2$  denote the number of Sylow 2-subgroups.

## 9 Groups of Order 30

Let  $G$  be a group of order  $30 = 2 \cdot 3 \cdot 5$ , and let  $n_3$  and  $n_5$  denote the number of Sylow 3-subgroups and Sylow 5-subgroups of  $G$  respectively. Then by Theorem 5.3:

$$n_3 = 1 \text{ or } 10 \quad \text{and} \quad n_5 = 1 \text{ or } 6$$

If  $n_3 = 10$ , then there are 20 elements of order 3, and if  $n_5 = 6$  then there are 24 elements of order 5 in  $G$ .  $G$  only has 30 elements, so then either:

$$n_3 = 1 \text{ and } n_5 = 6, \quad n_3 = 10 \text{ and } n_5 = 1 \quad \text{or} \quad n_3 = n_5 = 1$$

Hence either  $C_3 \trianglelefteq G$  or  $C_5 \trianglelefteq G$ .

Let  $H = C_3C_5$  and by Lagrange's Theorem,  $C_3 \cap C_5 = \mathbf{1}$ , hence  $|H| = 15$  by Lemma 5.7. We know from our classification of groups of order  $pq$  that  $H \cong C_{15}$ . Notice that  $C_2$  is a Sylow 2-subgroup of  $G$ , and by the same argument,  $C_2 \cap C_{15} = \mathbf{1}$  and  $|C_2C_{15}| = 30$ . Hence  $G = C_2C_{15}$ .

Because  $|C_{15}| = 15 = \frac{30}{2}$ , the index of  $C_{15}$  in  $G$  is 2, and we know a subgroup of index 2 is normal, so  $C_{15} \trianglelefteq G$ . Moreover,  $G = C_{15} \rtimes C_2$ .

By Lemma 5.8:

$$\text{Aut } C_{15} = \mathbb{Z}/15\mathbb{Z}^* \cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})^* \cong \mathbb{Z}/3\mathbb{Z}^* \times \mathbb{Z}/5\mathbb{Z}^* \cong C_2 \times C_4$$

A homomorphism,  $\psi : C_2 \rightarrow C_2 \times C_4$  preserves element order, and there are 3 elements of order 2 in  $C_2 \times C_4$ :  $(x, 1)$ ,  $(1, y^2)$  and  $(x, y^2)$  where  $\langle x, y \rangle = C_2 \times C_4$ . We know  $\psi$  is determined by its effect on a generator, so if  $\langle z \rangle = C_2$  then  $z\psi$  has four possibilities:

**Case 1:**  $z\psi = (1, 1)$ .

When  $z\psi = (1, 1)$ , then  $\psi$  is the trivial homomorphism, and so we obtain:

$$G \cong C_2 \times C_{15} \cong C_{30}$$

**Case 2:**  $z\psi = (x, 1)$ .

**Case 3:**  $z\psi = (1, y^2)$ .

**Case 4:**  $z\psi = (x, y^2)$ .

## Part III

# To Do

## 10 Semi-Direct Product

## 11 Groups of order 9 (Might skip)

## 12 Groups of Order 18

### 12.1 Groups of Order $p^2q$

## 13 Groups of Order $p^3$

### 13.1 Groups of Order 8

### 13.2 Groups of Order 27

### 13.3 General Case?

## 14 Groups of Order 24

## 15 Groups of Order 16