Interim Report

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Contents

1 Theorems and Lemmas

1.1 Sylow Theorems

Let G be a group of order $p^n m$ where p is a prime and $p \nmid m$.

Theorem 1.1 (1st Sylow Theorem)**.** *G has a Sylow p-subgroup, i.e. a subgroup of order p^n.*

Theorem 1.2 (2nd Sylow Theorem)**.** *All Sylow p-subgroups of G are conjugate to each other.*

Corollary 1.2.1. *If $n_p = 1$ then the Sylow p-subgroup is normal in G.*

Theorem 1.3 (3rd Sylow Theorem)**.** *Let n_p denote the number of Sylow p-subgroups of G. Then:*

i) $n_p \mid m$

ii) $n_p \equiv 1 \pmod p$

Lemma 1.4. *For a group G with $N \leqslant G$ and $H \leqslant G$, then*

$$|NH| = |\{nh \mid n \in N, h \in H\}| = \frac{|N| \cdot |H|}{|N \cap H|}$$

2 Groups of order 6

The prime factorisation of $6 = 2 \cdot 3$, so we can construct groups with products of C_2 and C_3. The automorphism groups of C_2, $\operatorname{Aut} C_2 = \{id\}$, containing just the identity map. So any meaningful products will look like $C_3 \rtimes C_2$.

$\operatorname{Aut} C_3 = \{\mathrm{id}, \psi\}$ where $x\psi = x^{-1}$. So we have two possible products: $C_3 \rtimes_{\mathrm{id}} C_2$ and $C_3 \rtimes_{\varphi} C_2$ where φ is the homomorphism $\varphi : C_2 \to \operatorname{Aut} C_3$ mapping $1 \mapsto \mathrm{id}$ and $x \mapsto \psi$.

2.1 $C_3 \rtimes_{\mathbf{id}} C_2$

By the Fundamental Theorem of Finite Abelian Groups, we know $C_3 \rtimes_{\mathrm{id}} C_2 \cong C_3 \times C_2 \cong C_6$.

2.2 $C_3 \rtimes_{\varphi} C_2$

So the group operation is

$$(a_1, b_1)(a_2, b_2) = (a_1 \cdot a_2\varphi_{b_1}, b_1 \cdot b_2)$$

Investigating elements:

$$(1, x)(1, x) = (1 \cdot 1\varphi_x, x \cdot x) = (1 \cdot 1^{-1}, x^2) = (1, 1)$$

So $(1, x)$ is of order 2.

$$(x, 1)(x, 1) = (x \cdot x\varphi_1, 1 \cdot 1) = (x \cdot x, 1) = (x^2, 1)$$
$$(x^2, 1)(x, 1) = (x^2 \cdot x\varphi_1, 1 \cdot 1) = (x^3, 1) = (1, 1)$$

So $(x, 1)$ is of order 3.

$$(x, 1)(x, x) = (x \cdot x\varphi_1, 1 \cdot x) = (x, x)$$
$$(x, x)(x, 1) = (x \cdot x\varphi_x, x \cdot 1) = (xx^{-1}, x) = (1, x)$$

Hence, $C_3 \rtimes_{\varphi} C_2$ is non-abelian.

3 Groups of Order 6 (Attempt 2)

Let G be a group of order 6, and n_3 denote the number of Sylow 3-subgroups of G. Then by Theorem 1.3:

$$n_3 \equiv 1 \pmod 3 \text{ and } n_3 \mid 2 \implies n_3 = 1$$

So G has one Sylow 3-subgroup, and because 3 is prime, it is isomorphic to C_3, i.e.

$$C_3 \trianglelefteq G$$

Any Sylow 2-subgroup of G will have order 2, and so $C_2 \leqslant G$.

Lagrange's Theorem tells us that C_3 has elements of orders 1 and 3, and C_2 has elements of order 1 and 2 hence:

$$C_3 \cap C_2 = \mathbf{1}$$

By Lemma 1.4:

$$|C_3C_2| = \frac{|C_3| \cdot |C_2|}{|C_3 \cap C_2|} = \frac{3 \cdot 2}{1} = 6$$

So $G = C_3C_2$, $C_3 \trianglelefteq G$ and $C_3 \cap C_2 = \mathbf{1} \implies G = C_3 \rtimes C_2$

Now we need to determine $\operatorname{Aut} C_3$. $C_3 = \{1, x, x^2 = x^{-1}\}$ and so $\operatorname{Aut} C_3 = \{\mathrm{id}, \psi : x \mapsto x^{-1}\} \cong C_2$. So if $C_3 = \langle x \rangle$ and $C_2 = \langle y \rangle$, then we have two possibilities for G:

Case 1:

$$\begin{aligned} G &= \langle\, x, y \mid x^3 = y^2 = 1,\ y^{-1}xy = x \,\rangle \\ &= \langle\, x, y \mid x^3 = y^2 = 1,\ xy = yx \,\rangle \\ &= C_3 \times C_2 \cong C_6 \end{aligned}$$

Case 2:

$$\begin{aligned} G &= \langle\, x, y \mid x^3 = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle \\ &\cong D_6 \end{aligned}$$

Hence G is isomorphic to either C_6 or D_6.

4 Generalisation to Groups of Order $2p$

Let G be a group of order $2p$ where p is a prime number, and n_p denote the number of Sylow p-subgroups of G. Then by Theorem 1.3:

$$n_p \equiv 1 \text{ (mod p) and } n_p \mid 2 \implies n_p = 1$$

So G has one Sylow p-subgroup, it is isomorphic to $C_p = \langle x \rangle$ hence:

$$C_p \trianglelefteq G$$

A Sylow 2-subgroup of G will have order 2 so $C_2 = \langle y \rangle \leqslant G$.

Lagrange's Theorem tells us that C_p has elements of orders 1 and p, and C_2 has elements of order 1 and 2 hence:

$$C_p \cap C_2 = \mathbf{1}$$

By Lemma 1.4:

$$|C_pC_2| = \frac{|C_p| \cdot |C_2|}{|C_p \cap C_2|} = \frac{p \cdot 2}{1} = 2p$$

So $G = C_pC_2$, $C_p \trianglelefteq G$ and $C_p \cap C_2 = \mathbf{1} \implies G = C_p \rtimes C_2$

For an automorphism φ of C_p, $x^i\varphi = (x\varphi)^i$ so φ is determined by it's effect on x. φ is surjective, so it must send x to another generator of C_p. Lagrange's Theorem tells us every element of C_p has order either 1 or p so there are $p-1$ generators. So we have $p-1$ choices for $x\varphi$, hence:

$$|\operatorname{Aut} C_p| = p - 1$$

The automorphism $\beta : x \mapsto x^{-1}$ has order two, so $C_2\varphi$ could be $\mathbf{1}$ or $\langle \beta \rangle \cong C_2$. This gives us two possibilities:

$$y\varphi = x \mapsto x$$

$$y\varphi = x \mapsto x^2$$

Case 1:

$$\begin{aligned} G &= \langle\, x, y \mid x^p = y^2 = 1,\ y^{-1}xy = x \,\rangle \\ &= \langle\, x, y \mid x^p = y^2 = 1,\ xy = yx \,\rangle \\ &= C_p \times C_2 \cong C_{2p} \end{aligned}$$

Case 2:

$$\begin{aligned} G &= \langle\, x, y \mid x^p = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle \\ &\cong D_{2p} \end{aligned}$$

Hence a group of order $2p$ is isomorphic to C_{2p} or D_{2p}.

I need to show $\operatorname{Aut} C_p \cong C_{p-1}$ and that $\beta : x \mapsto x^{-1}$ is the only element of order 2 to show that these are the only two possible groups of order $2p$.

5 Groups of order 4

From the Fundamental Theorem of Finite Abelian Groups, we know that a group G of order 4 could be isomorphic to one of C_4 and $C_2 \times C_2$. Now to show that these are the only possibilities, i.e. a group of order 4 must be abelian.

The Sylow theorems are not so helpful here, because $4 = 2^2$ so any Sylow 2-subgroup will be of order 4, which is just G.

6 Groups of order 9 (Might skip)

7 Generalisation to Groups of Order p^2

8 Groups of order 12

Let G be a group of order 12, and n_3 and n_2 denote the number of Sylow 3-subgroups and Sylow 2-subgroups of G respectively. By Theorem 1.3:

$$n_3 \equiv 1 \pmod 3 \text{ and } n_3 \mid 4 \implies n_3 = 1$$

$$n_2 \equiv 1 \pmod 2 \text{ and } n_2 \mid 3 \implies n_2 = 1$$

So G has a unique Sylow 3-subgroup, isomorphic to $C_3 = \langle x \rangle$, and so $C_3 \trianglelefteq G$. Likewise, G has a unique Sylow 2-subgroup, say $H \trianglelefteq G$ and we know already a group of order 4 is isomorphic to either C_4 or $C_2 \times C_2 \cong V_4$ which gives us 2 cases. $|H| = 4$ because $12 = 2^2 \cdot 3$.

Lagrange's Theorem tells us $H \cap C_3 = \mathbf{1}$, and Lemma 1.4 tells us:

$$|HC_3| = \frac{|H| \cdot |C_3|}{|H \cap C_3|} = 12$$

Hence $G = HC_3$, $C_3 \trianglelefteq G$, $H \leqslant G$, and $H \cap C_3 = \mathbf{1} \implies G = C_3 \rtimes H$ and $G = H \ltimes C_3$.

We have seen already that $\operatorname{Aut} C_3 \cong C_2$. Similarly, any automorphism, φ of C_4 must map generators to generators, so $\operatorname{Aut} C_4 \cong C_2$. An automorphism of V_4 corresponds to a permutation of the three non-identity elements, hence $\operatorname{Aut} V_4 \cong S_3$.

Case 1: $C_3 \rtimes C_4$

An homomorphism $\psi : C_4 \to \operatorname{Aut} C_3 \cong C_2$ is determined by $y\psi$. Either:

$$y\psi = x \mapsto x \quad \text{or} \quad y\psi = x \mapsto x^{-1}$$