

Classification of Finite Groups

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Part I

Doing

1 Introduction

2 Preliminaries

To start, let's solidify the notation used in this report.

We shall denote groups and sets with capital letters, like G , H , and elements of those groups with lower case letters, like g , h . Greek letters shall denote mappings, generally ϕ , ψ , etc. with ι reserved for the identity map, and we will write mappings on the right.

We will use \mathbb{N} to denote the natural numbers (not including 0), \mathbb{Z} to denote the integers, and \mathbb{R} to denote the real numbers.

To denote the cyclic group of order n we will use C_n , D_{2n} to denote the cyclic group of order $2n$, A_n to denote the alternating group over n elements, S_n to denote the symmetric group over n elements, and Q_8 to denote the quaternion group. The trivial group, $\{1\}$ is denoted by $\mathbf{1}$.

Let's move on to review some facts and theorems which will be valuable later on, as well as introduce some new concepts and prove some new results!

Definition 2.1. A permutation of a set X is a bijection from X to X . The symmetric group X is the set of all permutations of X under composition. We write $\text{Sym } X$ to denote this. It is easy to show $\text{Sym } X$ is a group.

Definition 2.2. If G is a group, and $H \subseteq G$, then H is a subgroup of G if it is a group in its own right with the multiplication from G . We write $H \leq G$ to mean H is a subgroup of G .

If H is closed under conjugation, i.e. for all $g \in G$ and $h \in H$, $g^{-1}hg \in H$, then we say H is a normal subgroup of G . We write $H \trianglelefteq G$ to mean H is a normal subgroup of G .

Definition 2.3. If G is a group and $X \subseteq G$, then the subgroup generated by X is the intersection of all subgroups of G containing X . This is denoted $\langle X \rangle$. The proof that $\langle X \rangle$ is a subgroup of G is omitted. The elements of X are called generators of G .

Definition 2.4. If G is a group with subgroup H then the right coset of H in G with representative $g \in G$ is:

$$Hg = \{ hg \mid h \in H \}$$

Definition 2.5. The order of a group, G , is the number of elements in G , denoted $|G|$. The order of an element $g \in G$ is the smallest $i \in \mathbb{N}$ such that $g^i = 1$.

Definition 2.6. If G and H are groups with elements $g_1, g_2 \in G$, then a map:

$$\phi : G \rightarrow H$$

is a homomorphism if:

$$(g_1g_2)\phi = (g_1\phi)(g_2\phi)$$

If ϕ is bijective, then we call it an isomorphism, with $G \cong H$ denoting that G is isomorphic to H . And if ϕ is an isomorphism from G to itself, then we call it an automorphism of G .

Lemma 2.7. *The set of all automorphisms of a group G form a group under composition. Indeed, this is called the automorphism group of G , denoted $\text{Aut } G$.*

Proof. Let $A = \text{Aut } G = \{ \phi : G \rightarrow G \mid \phi \text{ is an isomorphism} \}$, and let $\phi \in A$. Denote an element of G by g .

We know already that the composition of two isomorphisms is an isomorphism, so A is closed under composition.

The identity map, $\iota : g \mapsto g$, is certainly an automorphism of G and so $A \neq \emptyset$.

Indeed, $\iota : g \mapsto g$ is the identity of A , since:

$$g\phi\iota = (g\phi)\iota = g\phi \quad \text{and} \quad g\iota\phi = (g\iota)\phi = g\phi$$

And inverses clearly exists, because automorphisms are bijections, and bijections are invertible. Hence $A = \text{Aut } G$ is a group. \square

Lemma 2.8. *The automorphism group of C_n is isomorphic to the multiplicative group of integers mod n .*

i.e. $\text{Aut } C_n \cong (\mathbb{Z}/n\mathbb{Z})^\times$

Proof. Let $C_n = \langle x \rangle$. Any automorphism, φ of C_n has the property:

$$(x^i)\varphi = (x\varphi)^i$$

Hence φ is determined by it's effect on a generator, x , and preserves element order. In particular, φ sends generators to generators. So for φ to be an automorphism, it must send x to another generator, say x^k . An element x^k generates C_n if x^k has order n , i.e. when k and n are co-prime. Denote the automorphism sending x to x^k by φ_k .

Let's now investigate how these automorphisms behave. Let $\varphi_k, \varphi_l \in \text{Aut } C_n$, and consider:

$$x\varphi_k\varphi_l = (x^k)\varphi_l = (x^k)^l = x^{(kl)} = x\varphi_{kl} \pmod{n}$$

Because multiplication modulo n is commutative, $x^{kl} = x^{lk}$, so $\text{Aut } C_n$ is abelian.

Now consider $\theta : \text{Aut } C_n \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$ defined by $\varphi_k\theta = k$. We will show θ is an isomorphism. Every $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ is co-prime to n and so x^k is a generator of C_n , hence there is some $\varphi_k \in \text{Aut } C_n$ such that $\varphi_k\theta = k$. So θ is surjective. If $\varphi_k\theta = \varphi_l\theta$ then $k = l$, so θ is also injective. Finally, θ is a homomorphism because:

$$(\varphi_k\varphi_l)\theta = \varphi_{kl}\theta = kl = (\varphi_k\theta)(\varphi_l\theta)$$

So $\theta : \text{Aut } C_n \rightarrow (\mathbb{Z}/n\mathbb{Z})^\times$ is an isomorphism. \square

Theorem 2.9 (Lagrange's Theorem).

2.1 Sylow Theorems

This collection of theorems is extremely useful for describing subgroup structure. Hopefully these ring some bells. Let G be a group of order $p^n m$ where p is a prime and $p \nmid m$.

Theorem 2.10 (1st Sylow Theorem). *G has a Sylow p -subgroup, i.e. a subgroup of order p^n .*

Theorem 2.11 (2nd Sylow Theorem). *All Sylow p -subgroups of G are conjugate to each other. In particular, if G has a unique Sylow p -subgroup, then it is a normal subgroup.*

Theorem 2.12 (3rd Sylow Theorem). *Let n_p denote the number of Sylow p -subgroups of G . Then:*

$$(i) \quad n_p \mid m$$

$$(ii) \quad n_p \equiv 1 \pmod{p}$$

2.2 Isomorphism Theorems

Theorem 2.13.

Theorem 2.14.

Theorem 2.15.

2.3 Semidirect Product

We already know about the direct product:

Definition 2.16 (Direct Product). For groups N and H , the direct product, $G = N \times H$ is a group of ordered pairs of elements (n, h) where $n \in N$ and $h \in H$ with the operation:

$$(n_1, h_1)(n_2, h_2) = (n_1n_2, h_1h_2)$$

Moreover, if $\bar{N} = N \times \mathbf{1}$ and $\bar{H} = \mathbf{1} \times H$, then:

- (i) $\bar{N} \trianglelefteq G$ and $\bar{H} \trianglelefteq G$
- (ii) $\bar{N} \cap \bar{H} = \mathbf{1}$
- (iii) $\bar{N}\bar{H} = \{nh \mid n \in N, h \in H\} = G$

Now let's seek a slightly more general way to combine groups, by relaxing that H must be normal. So we have:

$$N \trianglelefteq G, H \leq G, NH = G, \quad \text{and} \quad N \cap H = \mathbf{1}$$

Consider the set, (not the direct product):

$$N \times H = \{ (n, h) \mid n \in N, h \in H \}$$

and a map

$$\phi : N \times H \rightarrow G \quad \text{defined by} \quad (n, h) \mapsto nh$$

We want ϕ to be an isomorphism.

To show ϕ is injective, take $n_1, n_2 \in N$ and $h_1, h_2 \in H$, and assume $n_1h_1 = n_2h_2$. Then multiplying on the left by n_2^{-1} and on the right by h_1^{-1} gives:

$$n_2^{-1}n_1 = h_2h_1^{-1}$$

On the left we have an element of N and on the right, an element of H , so $n_2^{-1}n_1 = h_2h_1^{-1} \in N \cap H$. But $N \cap H = \mathbf{1}$ so then $n_2^{-1}n_1 = h_2h_1^{-1} = 1$. Hence:

$$n_1 = n_2 \quad \text{and} \quad h_1 = h_2$$

To show ϕ is surjective, consider the image, $\text{im } \phi = \{nh \mid n \in N, h \in H\}$. This is by definition $NH = G$, so ϕ is surjective, and hence a bijection.

For ϕ to be a homomorphism, we need:

$$\begin{aligned} [(n_1, h_1)(n_2, h_2)]\phi &= (n_1, h_1)\phi(n_2, h_2)\phi \\ &= n_1h_1n_2h_2 \\ &= n_1h_1n_2h_1^{-1}h_1h_2 \\ &= (n_1h_1n_2h_1^{-1})(h_1h_2) \end{aligned}$$

But N is normal in G so $h_1 n_2 h_1^{-1}$ is just another element in N , say n_3 . So:

$$[(n_1, h_1)(n_2, h_2)]\phi = (n_1 n_3)(h_1 h_2) = (n_1 n_3, h_1 h_2)\phi$$

We know that ϕ is injective, so then:

$$(n_1, h_1)(n_2, h_2) = (n_1 n_3, h_1 h_2)$$

This tells us the multiplication that will make NH a group. Because $N \trianglelefteq G$, the map

$$\varphi_{h_1} : n_2 \mapsto h_1 n_2 h_1^{-1} = n_3$$

is an automorphism of N . This gives rise to the definition:

Definition 2.17 (Semidirect Product).

- (i) For a group G with normal subgroup N and subgroup H with $NH = G$ and $N \cap H = \mathbf{1}$, G is the internal semidirect product of N by H , written $G = N \rtimes H$.
- (ii) For groups N and H , and a homomorphism $\psi : H \rightarrow \text{Aut } N$, the external semidirect product of N by H via ψ is the set:

$$N \rtimes H = \{ (n, h) \mid n \in N, h \in H \}$$

with multiplication:

$$(n_1, h_1)(n_2, h_2) = (n_1(n_2\phi_{h_1}), h_1 h_2)$$

denoted:

$$N \rtimes_{\psi} H$$

Lemma 2.18. For a group G with $N \leq G$ and $H \leq G$, with $N \cap H = \mathbf{1}$ then:

$$|NH| = |\{nh \mid n \in N, h \in H\}| = |N| \cdot |H|$$

Proof. We just saw above that for elements $n \in N$ and $h \in H$, the map:

$$\phi : N \times H \rightarrow NH \quad \text{defined by} \quad (n, h) \mapsto nh$$

is a bijection. The result follows immediately from this. □

2.4 Group Actions

Some snazzy introduction.

Definition 2.19. Let G be a group, and Ω be a set, with elements $g \in G$ and $\omega \in \Omega$. Consider a map $\mu : \Omega \times G \rightarrow \Omega$, and write ω^g for the image of (ω, g) under μ . So we have:

$$\mu : \Omega \times G \rightarrow \Omega \quad \text{defined by} \quad (\omega, g) \mapsto \omega^g$$

We say G acts on Ω if for all $g_1, g_2 \in G$ and all $\omega \in \Omega$:

$$(i) \quad (\omega^{g_1})^{g_2} = \omega^{(g_1 g_2)}$$

$$(ii) \quad \omega^1 = \omega$$

We call μ the group action of G on Ω .

This might remind you of a homomorphism. Indeed we have a result:

Lemma 2.20. *A group action induces a homomorphism. Specifically, let G be a group which acts on a set Ω , with $g \in G$ and $\omega \in \Omega$, and define:*

$$\rho_g : \Omega \rightarrow \Omega \quad \text{by} \quad \omega \mapsto \omega^g$$

Then:

$$\rho : G \rightarrow \text{Sym } \Omega \quad \text{defined by} \quad g \mapsto \rho_g$$

is a homomorphism.

Proof. Firstly, ρ_g is indeed a permutation of Ω because it is invertible (and therefore a bijection), with:

$$(\rho_g)^{-1} = \rho_{g^{-1}}$$

Consider $g, h \in G$ and their corresponding maps, $\rho_g, \rho_h \in \text{Sym } \Omega$. Then:

$$\omega(g\rho)(h\rho) = \omega\rho_g\rho_h = (\omega^g)^h = \omega^{(gh)} = \omega\rho_{gh} = \omega(gh)\rho$$

Thus ρ is a homomorphism. □

A group acting on the set its cosets will be very useful:

Definition 2.21. For a group G with $H \leq G$, let $\Omega = \{Hg \mid g \in G\}$, i.e. the set of cosets of H in G . If $x \in G$, define a group action:

$$\Omega \times G \rightarrow \Omega \quad \text{by} \quad (Hg, x) \mapsto Hgx$$

Lemma 2.22. *The action above is well defined, meaning the action is independent of our choice of representative g .*

Proof. □

3 First Classifications

Let's start with the easiest case: groups of order 1. Any group G must have an identity element, and so that's all our possible elements used up! All groups of order 1 are isomorphic to the trivial group, $\mathbf{1}$.

What about groups of prime order? Let G be a group of order p , where p is a prime number. Then Lagrange's Theorem tell us all elements must have order 1 or p . Pick some $x \in G$ with x having order p . Then $\langle x \rangle = G$ so G is cyclic and $G \cong C_p$.

4 Groups of Order 6

Let G be a group of order 6, and n_3 denote the number of Sylow 3-subgroups of G . Then by Theorem 2.12:

$$n_3 \equiv 1 \pmod{3} \quad \text{and} \quad n_3 \mid 2 \implies n_3 = 1$$

So G has one Sylow 3-subgroup, N , and because 3 is prime, it is isomorphic to C_3 . Let $N = \langle x \rangle$. Any Sylow 2-subgroup, $H \leq G$, will have order 2, and so $H \cong C_2$. Let $H = \langle y \rangle$. Lagrange's Theorem tells us that N has elements of orders 1 and 3, and H has elements of order 1 and 2 hence:

$$N \cap H = \mathbf{1}$$

By Lemma 2.18:

$$|NH| = |N| \cdot |H| = 6$$

So $G = NH$, $N \trianglelefteq G$ and $N \cap H = \mathbf{1}$, which means $G = N \rtimes H$, the semidirect product of N by H .

Now we need to determine $\text{Aut } N$. An automorphism, φ of N preserves element order. In particular, φ maps generators to generators. Hence, $x\varphi = x$ or x^2 because they are the generators of N . So $\text{Aut } N \cong C_2$.

Now we want a homomorphism $\psi : H \rightarrow \text{Aut } N$. If ψ is trivial, then it maps H to the trivial group, so every element of H gets sent to the trivial automorphism. If ψ is not trivial, then at least one element of H is not sent to the trivial automorphism. It cannot be 1 because then element order is not preserved, so it must be the generator, y . Hence we obtain 2 possibilities for G :

Case 1:

$$\begin{aligned} G &= \langle x, y \mid x^3 = y^2 = 1, y^{-1}xy = x \rangle \\ &= \langle x, y \mid x^3 = y^2 = 1, xy = yx \rangle \\ &= C_3 \times C_2 \cong C_6 \end{aligned}$$

Case 2:

$$\begin{aligned} G &= \langle x, y \mid x^3 = y^2 = 1, y^{-1}xy = x^{-1} \rangle \\ &\cong D_6 \end{aligned}$$

These are clearly not isomorphic, because C_6 is abelian, and D_6 is not.

Hence G is isomorphic one of:

$$C_6 \quad \text{or} \quad D_6$$

5 Generalisation to Groups of Order $2p$

Now that we have seen groups of order 6, let's try and work towards a more general case: groups of order 2 times a prime number. So let G be a group of order $2p$ where p is a prime number, and n_p denote the number of Sylow p -subgroups of G . Then by Theorem 2.12:

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid 2 \implies n_p = 1$$

So G has one Sylow p -subgroup, say N , and it is isomorphic to C_p . Let $N = \langle x \rangle$. A Sylow 2-subgroup, $H \leq G$ will have order 2 so $H \cong C_2$. Let $H = \langle y \rangle$. Lagrange's Theorem tells us that N has elements of orders 1 and p , and H has elements of order 1 and 2 hence:

$$N \cap H = \mathbf{1}$$

By Lemma 2.18:

$$|NH| = |N| \cdot |H| = 2p$$

So $G = N \rtimes H$ as before.

We know by Lemma 2.8 that $\text{Aut } N \cong (\mathbb{Z}/p\mathbb{Z})^\times$, so let's look for the elements of order 2. An element $x \in (\mathbb{Z}/p\mathbb{Z})^\times$ of order 2 satisfies $x^2 = 1$, hence $x = 1$ or -1 . But 1 has order 1, so x can only be -1 . From the proof of Lemma 2.8, this corresponds to the inverse map $\beta : x \mapsto x^{-1}$.

Now we want a homomorphism $\psi : H \rightarrow \text{Aut } N$. By the same argument as for groups of order 6, we have two possibilities for G :

Case 1:

$$\begin{aligned} G &= \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x \rangle \\ &= C_p \times C_2 \cong C_{2p} \end{aligned}$$

Case 2:

$$\begin{aligned} G &= \langle x, y \mid x^p = y^2 = 1, y^{-1}xy = x^{-1} \rangle \\ &\cong D_{2p} \end{aligned}$$

Again, these are clearly not isomorphic, because C_{2p} is abelian, and D_{2p} is not.

Hence a group of order $2p$ is isomorphic to one of:

$$C_{2p} \quad \text{or} \quad D_{2p}$$

6 Groups of Order pq

Let G be a group of order pq where p, q are prime numbers with $p > q$, and let n_p and n_q denote the number of Sylow p -subgroups and Sylow q -subgroups of G respectively. Then by Theorem 2.12:

$$n_p \equiv 1 \pmod{p} \quad \text{and} \quad n_p \mid q \implies n_p = 1$$

$$n_q \equiv 1 \pmod{q} \implies n_q = 1, q + 1, 2q + 1, \dots \quad \text{and} \quad n_q \mid p$$

So G has a unique Sylow p -subgroup, say $P \trianglelefteq G$, and a Sylow q -subgroup, $Q \leq G$. Because p and q are prime numbers, $P \cong C_p$ and $Q \cong C_q$. Pick generators for each, say $\langle x \rangle = P$ and $\langle y \rangle = Q$. We have 2 possibilities for n_q : $p - 1$ is a multiple of q or 1.

Case 1: $q \nmid p - 1$.

If $p - 1$ is not a multiple of q then $n_q = 1$ and $Q \trianglelefteq G$, hence:

$$G = P \times Q \cong C_{pq}$$

Case 2: $q \mid p - 1$.

If $p - 1$ is a multiple of q then $n_q = p$ and so Q is not normal in G . By Lagrange's Theorem, $P \cap Q = \mathbf{1}$ and by Lemma 2.18, $|PQ| = pq$. Hence, as well as the direct product, we have $G = P \rtimes Q$, some non-trivial semidirect product.

By Lemma 2.8, $\text{Aut } C_p \cong (\mathbb{Z}/p\mathbb{Z})^\times \cong C_{p-1}$. So if $\nu \in (\mathbb{Z}/p\mathbb{Z})^\times$, then $x \mapsto x^\nu$ is an automorphism. We know also that C_{p-1} has a unique subgroup of order q , hence G has the presentation:

$$G = \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^a \rangle$$

where a is a generator for the subgroup of order q in $(\mathbb{Z}/p\mathbb{Z})^\times$.

Notice that picking different generators are equivalent up to isomorphism because the composition of two isomorphisms is an isomorphism.

So any group of order pq is isomorphic to either:

$$C_{pq} \quad \text{or} \quad \langle x, y \mid x^p = y^q = 1, y^{-1}xy = x^a \rangle \quad \begin{array}{l} \text{if } q \mid p - 1 \\ C_{pq} \quad \text{if } q \nmid p - 1 \end{array}$$

7 Groups of order 4

The Sylow theorems are not so helpful here, because any Sylow 2-subgroup will be of order 4, which is just G . Lagrange's Theorem tells us every element of G has order 1, 2 or 4.

If $x \in G$ has order 4, then x generates G so $G \cong C_4$.

If instead there is no element of order 4 in G , then every $x \in G$ except the identity is of order 2. Consider $a, b \in G$ with $a \neq b$, and their product, ab . It must be that ab is the third element of order 2, otherwise we reach a contradiction. So it is easy to see that $G \cong C_2 \times C_2$.

So any group of order 4 is isomorphic to one of:

$$C_4 \quad \text{or} \quad C_2 \times C_2$$

8 Generalisation to Groups of Order p^2

Let G be a group of order p^2 . First, we will prove a useful lemma:

Lemma 8.1. *If G is a p -group (i.e. a group of prime power order), then every subgroup of index p is normal.*

Proof. Let H be a subgroup of G , with index p , and let Ω be the set of all cosets of H . So by definition, $|\Omega| = p$. By Lemma 2.20, there is a homomorphism:

$$\rho : G \rightarrow S_p$$

If we have $x \in \ker \rho$, then:

$$(H1)x = H1 = H$$

which means $x \in H$. So the kernel of ρ is just H , which means the quotient $\frac{G}{\ker \rho}$ is of order p . By Theorem 2.13, this means the image of ρ is a subgroup of S_p with order p . Because kernels are normal subgroups, $H \trianglelefteq G$. \square

By Lagrange's Theorem, the elements of G have order 1, p or p^2 .

If $x \in G$ has order p^2 , then x generates G so $G \cong C_{p^2}$.

If G does not have an element of order p^2 then all elements, except the identity, have order p . We know that G must have a subgroup of order p , P , and because p is prime, $P \cong C_p$. Pick a generator for P , say x and an element $y \in G$ such that $y \notin P$. Then $y \neq x^i$ for any i .

If $y^j = x^i$ for some i and j , then:

$$(y^j)^{1-j} = (x^i)^{1-j} = y^{j-j+1} = y = x^{i(1-j)} = x^k \quad \text{for some } k, \text{ a contradiction.}$$

So no power of y is equal to any power of x . Because y has order p , it generates a subgroup of order p , \bar{P} with $P \cap \bar{P} = \mathbf{1}$. The lemma tells us that both P and \bar{P} are normal, and by Lemma 2.18, $|P\bar{P}| = p^2 = |G|$ so:

$$G = P \times \bar{P} \cong C_p \times C_p$$

If G has no elements of order p or p^2 , then it only has elements of order 1, which is the trivial group.

Hence any group of order p^2 is isomorphic to one of:

$$C_{p^2} \quad \text{or} \quad C_p \times C_p$$

9 Groups of order 12

Let G be a group of order $12 = 2^2 \cdot 3$, and n_3 and n_2 denote the number of Sylow 3-subgroups and Sylow 2-subgroups of G respectively. By Theorem 2.12:

$$n_2 \equiv 1 \pmod{2} \text{ and } n_2 \mid 3 \implies n_2 = 1$$

$$n_3 \equiv 1 \pmod{3} \text{ and } n_3 \mid 4 \implies n_3 = 1 \text{ or } 4$$

So G has a unique Sylow 2-subgroup of order 4, say $H \trianglelefteq G$, and we have already classified groups of order 4, so H is isomorphic to either V_4 (the Klein 4 group) or C_4 . A Sylow 3-subgroup, $K \leq G$ will have order 3, so $K \cong C_3$. Say $K = \langle x \rangle$.

Lagrange's Theorem tells us H has elements of order 1, 2, and 4, and K has elements of order 1 and 3. Hence $H \cap K = \mathbf{1}$. Lemma 2.18 tells us:

$$|HK| = |H| \cdot |K| = 12$$

Hence $G = HK$, $H \trianglelefteq G$, and $H \cap K = \mathbf{1}$. If we consider groups with 4 Sylow 3-subgroups then we can conclude that they are some semidirect product, $G = H \rtimes K$.

Since an automorphism, φ , must map generators to generators, $\text{Aut } C_4 \cong C_2$ because C_4 has two generators. An automorphism of V_4 corresponds to a permutation of the three non-identity elements, hence $\text{Aut } V_4 \cong S_3$.

Case 1: $H \cong C_4$ i.e. $G \cong C_4 \rtimes C_3$.

Let $H = \langle y \rangle$.

A homomorphism $\psi : K \rightarrow \text{Aut } H \cong C_2$, preserves order and together with Lagrange's Theorem means that the only possibility for ψ is trivial, i.e. $K\psi = \mathbf{1}$.

Hence $G \cong C_4 \times C_3 \cong C_{12}$.

Case 2: $H \cong V_4$ i.e. $G \cong (C_2 \times C_2) \rtimes C_3$.

Let $H = \langle y, z \rangle$.

A trivial homomorphism $K\psi = \mathbf{1}$ yields the direct product. What non-trivial homomorphisms are there? The automorphism group, $\text{Aut } H \cong S_3$ is of order 6, and so has a unique subgroup of order 3, by Theorem 2.12. We know already that a homomorphism $\psi : K \rightarrow \text{Aut } H$ is determined by where it sends the generator x , so for ψ to be non-trivial, it must send x to an element of order 3 in $\text{Aut } H$.

There are 2 such elements, and we will think of them as the permutations of order 3 of the set $\{1, 2, 3\}$. Denote them $a = (1\ 2\ 3)$ and $b = (1\ 3\ 2)$. Notice that $b = a^{-1}$, so we have homomorphisms:

$$\psi_1 : x \mapsto a \quad \text{and} \quad \psi_2 : x \mapsto a^{-1}$$

It appears we have 2 choices, but this is not the case. If we define $\theta : K \rightarrow K$ by $x\theta = x^{-1}$ then $\theta\psi_1 = \psi_2$. And notice that θ is an automorphism of K , so the semidirect products with ψ_1 and ψ_2 are isomorphic. Hence (up to isomorphism) there is one non-trivial homomorphism $\psi : K \rightarrow \text{Aut } H$. So the x acts by permuting the 3 non-identity elements of H .

We will show that in this case, $G \cong A_4$. First, let's check A_4 has the same subgroup structure as G . There is a subgroup isomorphic to C_3 in A_4 , generated by the 3-cycle $(1\ 2\ 3)$:

$$\bar{K} = \langle (1\ 2\ 3) \rangle$$

We can also find a subgroup isomorphic to V_4 :

$$\bar{H} = \{ 1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3) \}$$

Indeed, \bar{H} is normal in A_4 . We can see that $\bar{H} \cap \bar{K} = \mathbf{1}$ because \bar{H} contains no 3-cycles, and that $\bar{H}\bar{K} = A_4$. So we can conclude that $A_4 = \bar{H} \rtimes \bar{K}$.

Let's investigate how If we let $\alpha = (1\ 2)(3\ 4)$, $\beta = (1\ 4)(2\ 3)$ and $\gamma = (1\ 2\ 3)$, then we can write an element of A_4 as $\alpha^i \beta^j \gamma^k$ for some i, j and k . Define $\phi : A_4 \rightarrow G$ by $\phi : \alpha^i \beta^j \gamma^k \mapsto x^i y^j z^k$. Then:

$$\beta\phi = (\gamma^{-1}\alpha\gamma)\phi = c^{-1}ac = b$$

So conjugation acts in the same way. Hence we can conclude that $G \cong A_4$.

If we instead consider G where $K \trianglelefteq G$, i.e. $G = K \rtimes H$, then we again have two cases:

Case 1: $H \cong C_4$ i.e. $G \cong C_3 \rtimes C_4$.

Let $H = \langle y \rangle$.

We know $\text{Aut } C_3 \cong C_2$ so a homomorphism ψ maps H to the trivial group or to $\langle \beta : x \mapsto x^{-1} \rangle$.

If $H\psi = \mathbf{1}$ then $G = K \times H \cong C_4 \times C_3$, which we have already seen.

If $H\psi = \langle \beta \rangle$ then we have:

$$G = \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$

Case 2: $H \cong V_4$ i.e. $G \cong C_3 \rtimes (C_2 \times C_2)$.

Let $H = \langle y, z \rangle$.

If $\psi : H \rightarrow \text{Aut } K$ is trivial then we obtain the direct product again.

The image of a non-trivial homomorphism $\psi : H \rightarrow \text{Aut } K$ is isomorphic to C_2 , so by Theorem 2.13: $\ker \psi \cong C_2$.

We can choose ψ such that $y\psi = \beta : x \mapsto x^{-1}$ and $z\psi = \iota : x \mapsto x$. Then:

$$G = \langle x, y, z \mid x^3 = y^2 = z^2 = 1, yz = zy, y^{-1}xy = x^{-1}, z^{-1}xz = x \rangle$$

Let $a = xz$. The order of $a = \text{lcm}(\text{o}(x), \text{o}(z)) = \text{lcm}(2, 3) = 6$ because x and z commute. So:

$$a^3 = x^3z^3 = z$$

and

$$y^{-1}ay = y^{-1}xzy = y^{-1}xyz = x^{-1}z = x^2z = a^2a^3 = a^{-1}$$

Hence:

$$G = \langle y, a \mid a^6 = y^2 = 1, a^{-1}ay = a^{-1} \rangle \cong D_{12}$$

So a group G of order 12 is isomorphic to one of:

$$C_{12}, \quad C_2 \times C_6, \quad A_4, \quad D_{12}, \quad \text{or} \quad \langle x, y \mid x^3 = y^4 = 1, y^{-1}xy = x^{-1} \rangle$$

10 Groups of Order p^2q

Let p and q be distinct prime numbers, and G be a group of order p^2q . We shall consider the cases $p < q$ and $p > q$ separately.

10.1 $p < q$

Let $p < q$, and let n_q denote the number of Sylow q -subgroups. Then by Theorem 2.12:

$$n_q \mid p^2$$

so n_q could be 1, p or p^2 . Also:

$$n_q \equiv 1 \pmod{q}$$

If $n_q = p$ then p must be congruent to 1 mod q , which is a contradiction since $p < q$. If $n_q = p^2$ then we must have:

$$q \mid (p^2 - 1)$$

Factorising gives:

$$q \mid (p+1)(p-1)$$

So either:

$$q \mid (p+1) \quad \text{or} \quad q \mid (p-1) \quad \text{or both}$$

However, q cannot divide $p-1$ because $p < q$ so q must divide $p+1$. This is only the case when $p=2$ and $q=3$, so $|G|=12$ which we have already classified.

Hence if $|G|=p^2q \neq 12$ with $p < q$, then G possesses a unique Sylow q -subgroup, $Q \cong C_q$, which is normal in G . A Sylow p -subgroup, $P \leq G$, will have order p^2 and by Lagrange's Theorem, intersects trivially with Q . And by applying Lemma 2.18:

$$|PQ| = |P| \cdot |Q| = p^2q$$

So we can conclude that $G = Q \rtimes P$.

We know, by Lemma 2.8, that $\text{Aut } Q \cong (\mathbb{Z}/q\mathbb{Z})^\times$, so we want a homomorphism, $\psi : P \rightarrow \text{Aut } Q$. We have two possibilities for a group of order p^2 :

Case 1: $P \cong C_{p^2}$ i.e. $G \cong C_q \rtimes C_{p^2}$.

Case 2: $P \cong C_p \times C_p$ i.e. $G \cong C_q \rtimes (C_p \times C_p)$.

11 Groups of Order 30

Let G be a group of order $30 = 2 \cdot 3 \cdot 5$, and let n_3 and n_5 denote the number of Sylow 3-subgroups and Sylow 5-subgroups of G respectively. Then by Theorem 2.12:

$$n_3 = 1 \text{ or } 10 \quad \text{and} \quad n_5 = 1 \text{ or } 6$$

If $n_3 = 10$, then there are 20 elements of order 3, and if $n_5 = 6$ then there are 24 elements of order 5 in G . G only has 30 elements, so then either:

$$n_3 = 1 \text{ and } n_5 = 6, \quad n_3 = 10 \text{ and } n_5 = 1 \quad \text{or} \quad n_3 = n_5 = 1$$

So if T is a Sylow 3-subgroup of G and F is a Sylow 5-subgroup, then at least one must be normal in G . So $T \trianglelefteq G$ or $F \trianglelefteq G$ or both.

Let $H = TF$ and by Lagrange's Theorem, $T \cap F = \mathbf{1}$, hence $|H| = 15$ by Lemma 2.18. We know from our classification of groups of order pq that $H \cong C_{15}$. Notice that a Sylow 2-subgroup $K \leq G$ has order 2, so $K \cong C_2$. By the same argument as above, $H \cap K = \mathbf{1}$ and $|HK| = 30$. Hence $G = HK$.

Because $|H| = 15 = \frac{30}{2}$, the index of H in G is 2, and we know a subgroup of index 2 is normal, so $H \trianglelefteq G$. Moreover, $G = H \rtimes K$.

By Lemma 2.8:

$$\text{Aut } C_{15} = (\mathbb{Z}/15\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times \cong C_2 \times C_4$$

Let $\langle x, y \rangle = C_2 \times C_4$. A homomorphism, $\psi : C_2 \rightarrow C_2 \times C_4$ preserves element order, and there are 3 elements of order 2 in $C_2 \times C_4$: $(x, 1)$, $(1, y^2)$ and (x, y^2) . We know ψ is determined by its effect on a generator, so if $\langle z \rangle = K$ then $z\psi$ has four possibilities:

Case 1: $z\psi = (1, 1)$.

When $z\psi = (1, 1)$, then ψ is the trivial homomorphism, and so we obtain:

$$G \cong C_2 \times C_{15} \cong C_{30}$$

Case 2: $z\psi = (x, 1)$.

Case 3: $z\psi = (1, y^2)$.

Case 4: $z\psi = (x, y^2)$.

Part II

To Do

12 Groups of Order p^3

12.1 Groups of Order 8

12.2 Groups of Order 27

12.3 General Case?

13 Groups of Order 24

14 Groups of Order 16