Classification of Finite Groups

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MT4599

Groups of Order pq

Setup

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Without loss of generality, take $p > q$.

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$$P \cong C_p, \qquad Q \cong C_q$$

Pick generators: $\langle x \rangle = P$ and $\langle y \rangle = Q$.

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So we have two cases:

$$q \nmid p-1 \qquad \text{or} \qquad q \mid p-1$$

Case 1: $q \nmid p - 1$

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So:

$$G = P \times Q \cong C_{pq}$$

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This is unique!

Presentation

Describe $P \rtimes Q$ by a *presentation*.

$$G = \langle\, x,\ y \mid x^p = y^q = 1,\ y^{-1}xy = x^a \,\rangle$$

where a is a generator for the subgroup of order q in $(\mathbb{Z}/p\mathbb{Z})^\times$.

Examples