Classification of Finite Groups

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1 Introduction

2 Preliminaries

To start, let's solidify the notation used in this report.

We shall denote groups and sets with capital letters, like G, H, and elements of those groups with lower case letters, like g, h. Greek letters shall denote mappings, generally ϕ, ψ, etc. with ι reserved for the identity map, and we will write mappings on the right.

We will use $\mathbb{N}$ to denote the natural numbers (not including 0), $\mathbb{Z}$ to denote the integers, and $\mathbb{R}$ to denote the real numbers.

To denote the cyclic group of order n we will use C_n, D_{2n} to denote the cyclic group of order $2n$, A_n to denote the alternating group over n elements, S_n to denote the symmetric group over n elements, and Q_8 to denote the quaternion group. The trivial group, $\{1\}$ is denoted by $\mathbf{1}$.

Let's move on to review some facts and theorems which will be valuable later on, as well as introduce some new concepts and prove some new results!

Definition 1. If G and H are groups with elements $g_1, g_2 \in G$, then a map:

$$\phi : G \to H$$

is a <u>homomorphism</u> if:

$$(g_1 g_2)\phi = (g_1\phi)(g_2\phi)$$

If ϕ is bijective, then we call it an <u>isomorphism</u>, with $G \cong H$ denoting that G is isomorphic to H. And if ϕ is an isomorphism from G to itself, then we call it an <u>automorphism</u> of G.

Lemma 1. *The set of all automorphisms of a group G form a group under composition. Indeed, this is called the <u>automorphism group</u> of G, denoted* $\operatorname{Aut} G$.

Proof. Let $A = \operatorname{Aut} G = \{\phi : G \to G \mid \phi \text{ is an isomorphism}\}$, and let $\phi \in A$. Denote an element of G by g.

We know already that the composition of two isomorphisms is an isomorphism, so A is closed under composition.

The identity map, $\iota : g \mapsto g$, is certainly an automorphism of G and so $A \neq \varnothing$.

Indeed, $\iota : g \mapsto g$ is the identity of A, since:

$$g\phi\iota = (g\phi)\iota = g\phi \quad \text{and} \quad g\iota\phi = (g\iota)\phi = g\phi$$

And inverses clearly exists, because automorphisms are bijections, and bijections are invertible. Hence $A = \operatorname{Aut} G$ is a group. □

Lemma 2. *The automorphism group of C_n is isomorphic to the multiplicative group of integers mod n.*

i.e. $\operatorname{Aut} C_n \cong (\mathbb{Z}/n\mathbb{Z})^\times$

Proof. Let $C_n = \langle x \rangle$. Any automorphism, φ of C_n has the property:

$$(x^i)\varphi = (x\varphi)^i$$

Hence φ is determined by it's effect on a generator, x, and preserves element order. In particular, φ sends generators to generators. So for φ to be an automorphism, it must send x to another generator, say x^k. An element x^k generates C_n if x^k has order n, i.e. when k and n are co-prime. Denote the automorphism sending x to x^k by φ_k.

Let's now investigate how these automorphisms behave. Let $\varphi_k, \varphi_l \in \operatorname{Aut} C_n$, and consider:

$$x\varphi_k\varphi_l = (x^k)\varphi_l = (x^k)^l = x^{(kl)} = x\varphi_{kl} \qquad \bmod n$$

Because multiplication modulo n is commutative, $x^{kl} = x^{lk}$, so $\operatorname{Aut} C_n$ is abelian.

Now consider $\theta : \operatorname{Aut} C_n \to (\mathbb{Z}/n\mathbb{Z})^\times$ defined by $\varphi_k\theta = k$. We will show θ is an isomorphism. Every $k \in (\mathbb{Z}/n\mathbb{Z})^\times$ is co-prime to n and so x^k is a generator of C_n, hence there is some $\varphi_k \in \operatorname{Aut} C_n$ such that $\varphi_k\theta = k$. So θ is surjective. If $\varphi_k\theta = \varphi_l\theta$ then $k = l$, so θ is also injective. Finally, θ is a homomorphism because:

$$(\varphi_k\varphi_l)\theta = \varphi_{kl}\theta = kl = (\varphi_k\theta)(\varphi_l\theta)$$

So $\theta : \operatorname{Aut} C_n \to (\mathbb{Z}/n\mathbb{Z})^\times$ is an isomorphism. □

Definition 2. A subgroup H of a group G is called <u>characteristic</u> if it is fixed by all automorphisms of G.

i.e. for an automorphism ϕ of G, $H\phi = H$.

Lemma 3. *Let G be a group with normal subgroup H, and let K be characteristic in H. Then K is a normal subgroup of G.*

Proof. Consider the map $\varphi_g : G \to G$ defined by $\varphi_g : x \mapsto g^{-1}xg$ for elements x, $g \in G$. We will show that this is an automorphism of G. For x, $y \in G$:

$$x\varphi_g y\varphi_g = (g^{-1}xg)(g^{-1}yg) = g^{-1}(xy)g = (xy)\varphi_g$$

Hence φ_g is a homomorphism. Moreover, φ_g is invertible with inverse $\varphi_{g^{-1}}$. So φ_g is indeed an automorphism of G.

Because H is normal, $H\varphi_g = H$. So φ_g is an automorphism of H too. And so φ_g maps K to itself, because it is characteristic. Hence:

$$\{\, g^{-1}kg \mid k \in K \,\} = K$$

So K is normal in G. □

2.1 Semidirect Product

We already know about the direct product:

Definition 3 (Direct Product)**.** For groups N and H, the <u>direct product</u>, $G = N \times H$ is a group of ordered pairs of elements (n, h) where $n \in N$ and $h \in H$ with the operation:

$$(n_1, h_1)(n_2, h_2) = (n_1n_2, h_1h_2)$$

Moreover, if $\bar{N} = N \times \mathbf{1}$ and $\bar{H} = \mathbf{1} \times H$, then:

(i) $\bar{N} \trianglelefteq G$ and $\bar{H} \trianglelefteq G$

(ii) $\bar{N} \cap \bar{H} = \mathbf{1}$

(iii) $\bar{N}\bar{H} = \{\, nh \mid n \in N,\ h \in H \,\} = G$

Now let's seek a slightly more general way to combine groups, by relaxing that H must be normal. So we have:

$$N \trianglelefteq G, \; H \leqslant G, \; NH = G, \quad \text{and} \quad N \cap H = \mathbf{1}$$

Consider the set, (not the direct product):

$$N \times H = \{ (n, h) \mid n \in N, \; h \in H \}$$

and a map

$$\phi : N \times H \to G \quad \text{defined by} \quad (n, h) \mapsto nh$$

We want ϕ to be an isomorphism.

To show ϕ is injective, take $n_1, n_2 \in N$ and $h_1, h_2 \in H$, and assume $n_1 h_1 = n_2 h_2$. Then multiplying on the left by n_2^{-1} and on the right by h_1^{-1} gives:

$$n_2^{-1} n_1 = h_2 h_1^{-1}$$

On the left we have an element of N and on the right, an element of H, so $n_2^{-1} n_1 = h_2 h_1^{-1} \in N \cap H$. But $N \cap H = \mathbf{1}$ so then $n_2^{-1} n_1 = h_2 h_1^{-1} = 1$. Hence:

$$n_1 = n_2 \quad \text{and} \quad h_1 = h_2$$

To show ϕ is surjective, consider the image, $\operatorname{im} \phi = \{ nh \mid n \in N, \; h \in H \}$. This is by definition $NH = G$, so ϕ is surjective, and hence a bijection.

For ϕ to be a homomorphism, we need:

$$\begin{aligned}
[(n_1, h_1)(n_2, h_2)]\phi &= (n_1, h_1)\phi \, (n_2, h_2)\phi \\
&= n_1 h_1 n_2 h_2 \\
&= n_1 h_1 n_2 h_1^{-1} h_1 h_2 \\
&= (n_1 h_1 n_2 h_1^{-1})(h_1 h_2)
\end{aligned}$$

But N is normal in G so $h_1 n_2 h_1^{-1}$ is just another element in N, say n_3. So:

$$[(n_1, h_1)(n_2, h_2)]\phi = (n_1 n_3)(h_1 h_2) = (n_1 n_3, \, h_1 h_2)\phi$$

We know that ϕ is injective, so then:

$$(n_1, h_1)(n_2, h_2) = (n_1 n_3, \, h_1 h_2)$$

This tells us the multiplication that will make NH a group. Because $N \trianglelefteq G$, the map

$$\varphi_{h_1} : n_2 \mapsto h_1 n_2 h_1^{-1} = n_3$$

is an automorphism of N. This gives rise to the definition:

Definition 4 (Semidirect Product)**.**

(i) For a group G with normal subgroup N and subgroup H with $NH = G$ and $N \cap H = \mathbf{1}$, G is the internal semidirect product of N by H, written $G = N \rtimes H$.

(ii) For groups N and H, and a homomorphism $\psi : H \to \operatorname{Aut} N$, the external semidirect product of N by H via ψ is the set:

$$N \times H = \{ (n,\, h) \mid n \in N,\ h \in H \}$$

with multiplication:

$$(n_1,\, h_1)(n_2,\, h_2) = (n_1(n_2^{h_1\psi}),\, h_1h_2)$$

denoted:

$$N \rtimes_\psi H$$

We use the notation $n_2^{h_1\psi}$ both because it indicates conjugation, and is clearer.

Lemma 4. *For a group G with $N \leqslant G$ and $H \leqslant G$, with $N \cap H = \mathbf{1}$ then:*

$$|NH| = |\{nh \mid n \in N, h \in H\}| = |N| \cdot |H|$$

Proof. We just saw above that for elements $n \in N$ and $h \in H$, the map:

$$\phi : N \times H \to NH \quad \text{defined by} \quad (n,\, h) \mapsto nh$$

is a bijection. The result follows immediately from this. □

Lemma 5. *Let N and H be groups, and $\alpha \in \operatorname{Aut} H$. Then the semidirect products via the homomorphism ϕ, $N \rtimes_\phi H$, and via the homomorphism ψ, $N \rtimes_\psi H$, are isomorphic if $h\beta\psi = \alpha^{-1}h\phi\alpha$ for all $h \in H$, $\alpha \in \operatorname{Aut} N$ and $\beta \in \operatorname{Aut} H$.*

That is, we can apply any automorphism to H and conjugate N, and the resulting semidirect product remains in the same isomorphism class.

Proof. Let $G = N \rtimes_\phi H$ and $\bar{G} = N \rtimes_\psi H$, and define:

$$\vartheta : G \to \bar{G} \quad \text{by} \quad \vartheta : (n,\, h) \mapsto (n\alpha,\, h\beta)$$

We will show that ϑ is an isomorphism.

First, ϑ^{-1} exists because both α^{-1} and β^{-1} exist, and is given by:

$$\vartheta^{-1} : (n,\, h) \mapsto (n\alpha^{-1},\, h\beta^{-1})$$

Hence ϑ is a bijection.

We also have that:

$$h\beta\psi = \alpha^{-1}h\phi\alpha$$

implies:

$$\alpha h\beta\psi = h\phi\alpha$$

Now for two elements, $(n_1,\, h_1),\ (n_2,\, h_2) \in G$, consider:

$$\begin{aligned}
(n_1,\, h_1)\vartheta\ (n_2,\, h_2)\vartheta &= (n_1\alpha,\, h_1\beta)(n_2\alpha,\, h_2\beta) \\
&= (n_1\alpha\, n_2\alpha^{(h_1\beta)\psi},\, h_1\beta\, h_2\beta) \\
&= (n_1\alpha\, n_2^{(\alpha h_1\beta\psi)},\, h_1\beta\, h_2\beta) \\
&= (n_1\alpha\, n_2^{(h_1\beta\phi\alpha)},\, h_1\beta\, h_2\beta) \\
&= (n_1\alpha\, (n_2^{(h_1\beta\phi)})^{\alpha},\, h_1\beta\, h_2\beta) \\
&= ((n_1n_2^{(h_1\beta\phi)})\alpha,\, (h_1h_2)\beta) \\
&= (n_1n_2^{(h_1\beta\phi)},\, h_1h_2)\vartheta \\
&= ((n_1,\, h_1)(n_2,\, h_2))\vartheta
\end{aligned}$$

So ϑ is an isomorphism. □

2.2 Group Actions

Some snazzy introduction.

Definition 5. Let G be a group, and Ω be a set, with elements $g \in G$ and $\omega \in \Omega$. Consider a map $\mu : \Omega \times G \to \Omega$, and write ω^g for the image of $(\omega,\, g)$ under μ. So we have:

$$\mu : \Omega \times G \to \Omega \quad \text{defined by} \quad (\omega,\, g) \mapsto \omega^g$$

We say G acts on Ω if for all $g_1,\, g_2 \in G$ and all $\omega \in \Omega$:

(i) $(\omega^{g_1})^{g_2} = \omega^{(g_1 g_2)}$

(ii) $\omega^1 = \omega$

We call μ the group action of G on Ω.

This might remind you of a homomorphism. Indeed we have a result:

Lemma 6. *A group action induces a homomorphism. Specifically, let G be a group which acts on a set Ω, with $g \in G$ and $\omega \in \Omega$, and define:*

$$\rho_g : \Omega \to \Omega \quad \textit{by} \quad \omega \mapsto \omega^g$$

Then:

$$\rho : G \to \operatorname{Sym}\Omega \quad \textit{defined by} \quad g \mapsto \rho_g$$

is a homomorphism.

Proof. Firstly, ρ_g is indeed a permutation of Ω because it is invertible (and therefore a bijection), with:

$$(\rho_g)^{-1} = \rho_{g^{-1}}$$

Consider $g,\, h \in G$ and their corresponding maps, $\rho_g,\, \rho_h \in \operatorname{Sym}\Omega$. Then:

$$\omega(g\rho)(h\rho) = \omega\rho_g\rho_h = (\omega^g)^h = \omega^{(gh)} = \omega\rho_{gh} = \omega(gh)\rho$$

Thus ρ is a homomorphism. □

A group acting on the set its cosets will be very useful:

Definition 6. For a group G with $H \leqslant G$, let $\Omega = \{\, Hg \mid g \in G \,\}$, i.e. the set of cosets of H in G. If $x \in G$, define a group action:

$$\Omega \times G \to \Omega \quad \text{by} \quad (Hg, x) \mapsto Hgx$$

Lemma 7. *The action above is well defined, meaning the action is independent of our choice of representative g.*

Proof. □

Part I
Prime Power Orders

First, we will prove a few useful lemmas:

Lemma 8. *If G is a p-group (i.e. a group of prime power order), then every subgroup of index p is normal.*

Proof. Let H be a subgroup of G, with index p. We know kernels are normal subgroups, so we will show that H is the kernel of some homomorphism. Let Ω be the set of all cosets of H. So by definition, $|\Omega| = p$. By Lemma 6, there is a homomorphism:

$$\rho : G \to S_p$$

Let's investigate the kernel of ρ. If we have $x \in \ker \rho$, then:

$$(H1)x = H1 = H$$

which means $x \in H$. So the kernel of ρ is H. Hence, $H \trianglelefteq G$. □

Lemma 9. *If G is a group of prime power order, the centre of G is non-trivial.*

Proof. Let Z denote the centre of G, and consider the action of G on itself by conjugation. The orbit of an element, $g \in G$ is:

$$g^G = \{\, x^{-1}gx \mid x \in G \,\}$$

which is the conjugacy class of g. So the size of each orbit divides some power of p. In particular, the size of each orbit is divisible by p. So then the sum of the sizes of all of the conjugacy classes is also divisible by p. Looking at the class equation:

$$|G| = |Z| + \sum_{i=1}^{k} |g_i^G|$$

Then reducing mod p gives:

$$|G| \equiv |Z| \mod p$$

Because G is non-trivial, it follows that $|Z| \neq 1$.

□

Lemma 10. *For a group G with centre $\mathrm{Z}(G)$. Then if $G/\mathrm{Z}(G)$ is cyclic, G is abelian.*

Proof. Let $x \in G$ be the element such that $x\,\mathrm{Z}(G)$ generates $G/\mathrm{Z}(G)$. Then $\langle\, x,\, \mathrm{Z}(G)\,\rangle$ contains $\mathrm{Z}(G)$. Because G is the union of cosets of $\mathrm{Z}(G)$, then indeed $\langle\, x,\, \mathrm{Z}(G)\,\rangle = G$. The centraliser of x certainly contains x, and every element of $\mathrm{Z}(G)$ also commutes with x. Hence the centre of G is a subgroup of the centraliser of x. The result follows by concluding:

$$G = \langle\, x,\, \mathrm{Z}(G)\,\rangle = \langle\, \mathrm{Z}(G)\,\rangle = \mathrm{Z}(G)$$

□

Now onto the classification!

3 First Classifications

Let's start with the easiest case: groups of order 1. Any group G must have an identity element, and so that's all our possible elements used up! All groups of order 1 are isomorphic to the trivial group, **1**.

What about groups of prime order? Let G be a group of order p, where p is a prime number. Then Lagrange's Theorem tell us all elements must have order 1 or p. Pick some $x \in G$ with x having order p. Then $\langle x\rangle = G$ so G is cyclic and $G \cong C_p$.

4 Groups of Order p^2

Let G be a group of order p^2. By Lagrange's Theorem, the elements of G have order 1, p or p^2. If $x \in G$ has order p^2, then x generates G so $G \cong C_{p^2}$.

If G does not have an element of order p^2 then all elements, except the identity, have order p. We know that G must have a subgroup of order p, P, and because p is prime, $P \cong C_p$. Pick a generator for P, say x and an element $y \in G$ such that $y \notin P$. Then $y \neq x^i$ for any i.

If $y^j = x^i$ for some i and j, then:

$$\left(y^j\right)^{1-j} = \left(x^i\right)^{1-j} = y^{j-j+1} = y = x^{i(1-j)} = x^k \quad \text{for some } k\text{, a contradiction.}$$

So no power of y is equal to any power of x. Because y has order p, it generates a subgroup of order p, $\bar{P}$, with $P \cap \bar{P} = \mathbf{1}$. Lemma 8 tells us that both P and $\bar{P}$ are normal, and by Lemma 4, $|P\bar{P}| = p^2 = |G|$, so:

$$G = P \times \bar{P} \cong C_p \times C_p$$

If G has no elements of order p or p^2, then it only has elements of order 1, which is the trivial group.

Hence any group of order p^2 is isomorphic to one of:

$$C_{p^2} \quad \text{or} \quad C_p \times C_p$$

5 Groups of Order p^3

Let G be a group of order p^3, where p is a prime number. We will first gain a handle on G by describing its centre, and quotient by it. If G is abelian, we know by the Fundamental Theorem of Finite Abelian Groups that it is isomorphic to one of:

$$C_{p^3}, \quad C_{p^2} \times C_p \quad \text{or} \quad C_p \times C_p \times C_p$$

So from now on, we will focus on the non-abelian groups.

Denote the centre of G by Z and consider its order. Lagrange's Theorem tells us Z must have order dividing p^3. It cannot be p^3 because G is non-abelian, and Lemma 9 tells us that it cannot be 1. If $|Z| = p^2$, then $|G/Z| = p$, so $G/Z \cong C_p$. However Lemma 10 says that then G must be abelian, so then $|Z|$ must be p. Then by our previous classification, G/Z is isomorphic to either C_{p^2} or $C_p \times C_p$. Lemma 10 tells us that it must by the latter.

This gives us a handle to start investigating the structure of G. Another useful tool will be <u>commutators</u>, which we will denote by $[a, b] = a^{-1}b^{-1}ab$. The <u>derived subgroup</u> of G, $G' = \langle\, [x, y] \mid x, y \in G \,\rangle$, is the smallest normal subgroup such that G/G' is abelian. We saw that G/Z is abelian, so $G' \leqslant Z$, but because G' is non-trivial, we must have equality. Now we will prove a useful lemma which holds in G.

Lemma 11. *Suppose G is a group such that $G' \leqslant \mathrm{Z}(G)$. Then for elements a, b, $c \in G$:*

$$[a, bc] = [a, b][a, c]$$

Proof. First we note that:

$$abc = bca[a, bc]$$

Then moving one term at a time:

$$abc = ba[a, b]c = bac[a, b]^c = bca[a, c][a, b]^c$$

Hence:

$$bca[a,\, bc] = bca[a,\, c][a,\, b]^c$$

Now by multiplying on the left by $a^{-1}c^{-1}b^{-1}$ gives:

$$[a,\, bc] = [a,\, c][a,\, b]^c$$

Because $G' \leqslant \mathrm{Z}(G)$, conjugation by c has no effect. Additionally, the two commutators commute, giving:

$$[a,\, bc] = [a,\, b][a,\, c]$$

as required. $\square$

So far, we know $G/Z \cong C_p \times C_p$, and that $G' = Z$, as well as a useful lemma. Now pick two elements, a and b so that aZ and bZ generate G/Z. So then $G = \langle\, Z,\, a,\, b\,\rangle$.

Let $z = [a,\, b]$. If $z = 1$ then that means a and b commute. And by definition, a commutes with Z, so $a \in Z$, which contradicts our choice of a as a generator of G/Z. Hence $z \neq 1$, and in particular, a and b do not commute. Now we know $G' = Z$ which has order p, so $Z \cong C_p$. Moreover, $z \in Z$, and $z \neq 1$ so we can conclude that $\langle\, z\,\rangle = Z$. We can see that although a and b are not in Z, a^p and b^p are, because aZ and bZ have order p in G/Z. Considering the orders of a and b we have 3 cases:

Case 1: Both a and b have order p.

The above descriptions give the presentation:

$$G = \langle\, z,\, a,\, b \mid z^p = a^p = b^p = 1,\ az = za,\ bz = zb,\ [a,\, b] = z\,\rangle$$

We can write an arbitrary $g \in G$ as $a^i b^j z^k$ for integers i, j and k taken mod p. Hence this presentation has order at most p^3.

Now consider the set:

$$\left\{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha,\, \beta,\, \gamma \in \mathbb{F}_p \right\}$$

It can be shown that this is a group under the usual matrix multiplication, and is known as the <u>unitriangular group</u>, denoted $\mathrm{UT}_3(p)$. Taking:

$$z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

we can see that $\mathrm{UT}_3(p)$ satisfies this presentation for $p > 2$. (Indeed, the above presentation is the standard presentation definition for $\mathrm{UT}_3(p)$). Thus there is a single isomorphism class for this case.

The group behaves differently when $p = 2$ because we know that a group whose elements all have order either 1 or 2 is abelian. So the elements cannot have order only 1 or 2. In particular:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

have order 4. We can check that all other non-identity elements have order 2. Thus $\mathrm{UT}_2(2) \cong D_8$.

Part II
Composite Orders

6 Groups of Order pq

Let G be a group of order pq where p, q are prime numbers with $p > q$, and let n_p and n_q denote the number of Sylow p-subgroups and Sylow q-subgroups of G respectively. Then by Sylow's Theorems:

$$n_p \equiv 1 \mod p \quad \text{and} \quad n_p \mid q \implies n_p = 1$$

$$n_q \equiv 1 \mod q \implies n_q = 1,\ q+1,\ 2q+1,\ \ldots \quad \text{and} \quad n_q \mid p$$

So G has a unique Sylow p-subgroup, say $P \trianglelefteq G$, and a Sylow q-subgroup, $Q \leqslant G$. Because p and q are prime numbers, $P \cong C_p$ and $Q \cong C_q$. Pick generators for each, say $\langle x \rangle = P$ and $\langle y \rangle = Q$. We have 2 possibilities for n_q: $p-1$ is a multiple of q or 1.

Case 1: $q \nmid p-1$.

If $p-1$ is not a multiple of q then $n_q = 1$ and $Q \trianglelefteq G$, hence:

$$G = P \times Q \cong C_{pq}$$

Case 2: $q \mid p-1$.

If $p-1$ is a multiple of q then $n_q = p$ and so Q is not normal in G. By Lagrange's Theorem, $P \cap Q = \mathbf{1}$ and by Lemma 4, $|PQ| = pq$. Hence, as well as the direct product, we have $G = P \rtimes Q$, some non-trivial semidirect product.

By Lemma 2, $\operatorname{Aut} C_p \cong (\mathbb{Z}/p\mathbb{Z})^\times \cong C_{p-1}$. So if $\nu \in (\mathbb{Z}/p\mathbb{Z})^\times$, then $x \mapsto x^\nu$ is an automorphism. We know also that C_{p-1} has a unique subgroup of order q, hence G has the presentation:

$$G = \langle\, x, y, \mid x^p = y^q = 1,\ y^{-1}xy = x^a \,\rangle$$

where a is a generator for the subgroup of order q in $(\mathbb{Z}/p\mathbb{Z})^\times$.

Notice that picking different generators are equivalent up to isomorphism because the composition of two isomorphisms is an isomorphism.

So any group of order pq is isomorphic to either:

$$\begin{array}{rl} C_{pq} \quad \text{or} \quad \langle\, x, y \mid x^p = y^q = 1,\ y^{-1}xy = x^a \,\rangle & \text{if } q \mid p-1 \\ C_{pq} & \text{if } q \nmid p-1 \end{array}$$

6.1 Groups of Order $2p$

To illustrate an example of groups of order pq, let's take $q = 2$. Because every prime greater than 2 is odd, $p-1$ is an even number, and so $2 \mid p-1$.

An element $\alpha \in (\mathbb{Z}/p\mathbb{Z})^\times$ of order 2 satisfies $\alpha^2 = 1$, hence $\alpha = 1$ or -1. But 1 has order 1, so α can only be -1. Side-note: from the proof of Lemma 2, this corresponds to the inverse map.

So, in addition to C_{2p}, we have:

$$G \cong \langle\, x, y \mid x^p = y^2 = 1,\ y^{-1}xy = x^{-1} \,\rangle$$

Which is the presentation for the dihedral group of order $2p$, D_{2p}.

Hence a group of order $2p$ is isomorphic to one of:

$$C_{2p} \quad \text{or} \quad D_{2p}$$

7 Some Groups of Order p^2q

Let p and q be distinct prime numbers, and G be a group of order p^2q. To classify G in full generality is beyond this report, so we will focus on the cases when $p = 2$ and when $q = 2$.

7.1 $4q$

Let G be a group of order $4p$, and require $p > 3$. And let n_q denote the number of Sylow q-subgroups. The n_q must divide 4, so could be 1, 2 or 4, and must be congruent to 1 mod q. If $q = 3$, then G could have 4 Sylow q-subgroups, so we will classify groups of order 12 later. If $q = 2$, then we have a group of order p^3, which we have already classified. This is why we took $q > 3$. So G has a normal Sylow q-subgroup, $Q \cong C_q$. Let x generate Q.

Lagrange's Theorem, together with Lemma 4, tell us that a Sylow 2-subgroup, T, intersects trivially with Q, and $|QT| = |G|$. Hence, $G = Q \rtimes T$.

We know by Lemma 2, that $\operatorname{Aut} Q \cong C_{q-1}$. So we have two cases:

Case 1: $T \cong V_4$ i.e. $G \cong C_q \rtimes V_4$.

We saw in our classification of groups of order $2p$, that $(\mathbb{Z}/q\mathbb{Z})^\times$ has a unique element of order 2, corresponding to the inversion map. So Lemma 5 tells us that there is only a single non-trivial homomorphism $\psi : T \to \operatorname{Aut} Q$.

If ψ is trivial, then we obtain the product:

$$G \cong C_q \times V_4 \cong C_{2q} \times C_2$$

If ψ is non-trivial, it maps T to the subgroup generated by the inversion map, isomorphic to C_2. Therefore the kernel is isomorphic to C_2, so pick z such that it generates the kernel. Denote the other generator of T by y, then we obtain the following presentation:

$$G = \langle\, x,\, y,\, z \mid x^q = y^2 = z^2 = 1,\ yz = zy,\ xz = zx,\ y^{-1}xy = x^{-1} \,\rangle$$

Now let $a = xz$, and in a similar calculation to when we classified groups of order 12, we will show that $G \cong D_{4p}$.

Firstly, notice that the order of a is $4q$, and:

$$a^q = x^q z^q = z \quad \text{and} \quad a^{q-1} = x^{q-1}z^{q-1} = x^{q-1}$$

Now consider:

$$y^{-1}ay = y^{-1}xzy = y^{-1}xyz = x^{-1}z = a^{q-1}a^q = a^{2q-1} = a^{-1}$$

Hence:

$$G = \langle\, a,\, y \mid a^{2q} = y^2 = 1,\ y^{-1}ay = a^{-1} \,\rangle$$

which we recognise as D_{4p}.

Case 2: $T \cong C_4$ i.e. $G \cong C_q \rtimes C_4$.

Let t generate T. Assume $4 \nmid q-1$, which means $q \equiv 3 \mod 4$. So then $\operatorname{Aut} Q$ has no subgroup of order 4, and a homomorphism, ψ must map T to either the trivial group, or the group generated by the inverse automorphism.

If $T\psi$ is trivial, then we recover the direct product, $C_q \times C_4 \cong C_{4q}$.

If $T\psi$ is non-trivial, then G has the presentation:

$$G = \langle\, x,\, t \mid x^q = t^4 = 1,\; t^{-1}xt = x^{-1}\,\rangle$$

Let $a = xt^2$. Then:

$$a^q = xt^2 \dots xt^2 = x^q t^{2q} = t^{2q}$$

We know $q \equiv 3 \mod 4$, so for some n, $q = 4n+3$. Thus $2q = 8n+6 = 4(2n+1)+2$. So then:

$$a^q = t^{4(2n+1)+2} = t^2$$

Additionally:

$$t^{-1}at = t^{-1}xt^2t = (t^{-1}xt)t^2 = x^{-1}t^2 = t^2x^{-1} = a^{-1}$$

Hence:

$$G = \langle\, a,\, t \mid a^{2q} = 1,\; a^q = t^2,\; t^{-1}at = a^{-1}\,\rangle$$

which is the dicyclic group of order $4q$, Dic_{4q}.

If $4 \mid q-1$, i.e. $q \equiv 1 \mod 4$, then $\operatorname{Aut} Q$ contains a unique element of order 4, and so has a unique subgroup generated by it. We know by Lemma 2, that $\operatorname{Aut} Q \cong (\mathbb{Z}/q\mathbb{Z})^\times$, so say α is the generator of the subgroup of order 4 in $(\mathbb{Z}/q\mathbb{Z})^\times$. Our homomorphism can map T to this subgroup, and we get a group with the presentation:

$$G = \langle\, x,\, t \mid x^q = t^4 = 1,\; t^{-1}xt = x^\alpha\,\rangle$$

7.2 $2p^2$

Let G be a group of order $2p^2$, with $p > 2$. Denote the number of Sylow p-subgroups by n_p. By Sylow's Theorems, n_p divides 2, and is congruent to 1 mod p, so must be 1. Hence, G has a normal Sylow p-subgroup, P of order p^2.

If T is a Sylow 2-subgroup, then by applying Lagrange's Theorem, and Lemma 4, we can conclude that $G = P \rtimes T$. From our classification of groups of order p^2, we have 2 choices for P:

Case 1: $P \cong C_{p^2}$ i.e. $G \cong C_{p^2} \rtimes C_2$.

From Lemma 2, we know $|\operatorname{Aut} P| = p^2 - p = p(p-1)$. Because p is prime, $2 \nmid p$, but $2 \mid p-1$, so $\operatorname{Aut} P$ has a unique element of order 2. Hence, in addition to the direct product, $G \cong C2p^2$, we have $G \cong C_{p^2} \rtimes C_2$, with C_2 acting by inversion. If x generates P, and y generates T, we have the presentation:

$$G = \langle\, x,\, y \mid x^{p^2} = y^2 = 1,\; y^{-1}xy = x^{-1}\,\rangle$$

which we recognise as D_{2p^2}, the dihedral group of order $2p^2$.

Case 2: $P \cong C_p \times C_p$ i.e. $G \cong C_p \times C_p \rtimes C_2$.

Consider P as the product of the subgroups generated by a and b, i.e. $P = \langle a \rangle \times \langle b \rangle$. Then the action of T on P can either be trivial on both subgroups, invert one, or invert both.

If the action is trivial on both subgroups, then we recover the direct product $G \cong C_p \times C_{2p}$.

If the action is non-trivial on just one of the subgroups, then we can consider only one case. This is because they are equivalent up to an isomorphism of T, and Lemma 5 tells us the resulting semidirect products are isomorphic. So we have:

$$G = \langle a \rangle \times (\langle b \rangle \rtimes T) \cong C_p \times D_{2p}$$

Finally, if we choose to invert both subgroups, then we act on all of P by inversion. So if a and b generate P, then:

$$G = \langle\, a,\, b,\, x \mid a^p = b^p = x^2 = 1,\ ab = ba,\ x^{-1}ax = a^{-1},\ x^{-1}bx = b^{-1} \,\rangle$$

Because C_p has all elements of order p, excluding 1, and they are all $\overline{\text{automorphic}}$ to each other (meaning that some automorphism maps one to the other), $x^{-1}gx = g^{-1}$ for all $g \in P$. Hence:

$$G = \langle\, P,\, x \mid x^2 = 1,\ x^{-1}gx = g^{-1}\ \forall g \in P \,\rangle$$

which is known as the generalised dihedral group for C_p, denoted $\mathrm{Dih}(C_p)$.

Part III
Special Cases

8 Groups of order 12

We have seen that groups of order 12 have slightly different behaviour to groups of order $4q$ in general, and we will need this classification in order to classify groups of order 24.

Let G be a group of order $12 = 2^2 \cdot 3$, and n_3 and n_2 denote the number of Sylow 3-subgroups and Sylow 2-subgroups of G respectively. By Sylow's Theorems:

$$n_2 \equiv 1 \mod 2 \quad \text{and} \quad n_2 \mid 3 \implies n_2 = 1 \text{ or } 3$$

$$n_3 \equiv 1 \mod 3 \quad \text{and} \quad n_3 \mid 4 \implies n_3 = 1 \text{ or } 4$$

Let H be a Sylow 2-subgroup and K be a Sylow 3-subgroup of G.

Lagrange's Theorem tells us H has elements of order 1, 2, and 4, and K has elements of order 1 and 3. Hence $H \cap K = \mathbf{1}$. Lemma 4 tells us:

$$|HK| = |H| \cdot |K| = 12$$

Hence $G = HK$, $H \trianglelefteq G$, and $H \cap K = \mathbf{1}$.

Since an automorphism, φ, must map generators to generators, $\operatorname{Aut} C_4 \cong C_2$ because C_4 has two generators. An automorphism of V_4 corresponds to a permutation of the three non-identity elements, hence $\operatorname{Aut} V_4 \cong S_3$.

If we consider G where $K \trianglelefteq G$, i.e. $G = K \rtimes H$, then we have two cases:

Case 1: $H \cong C_4$ i.e. $G \cong C_3 \rtimes C_4$.

Let $H = \langle y \rangle$.

We know $\operatorname{Aut} C_3 \cong C_2$ so a homomorphism ψ maps H to the trivial group or to $\langle \beta : x \mapsto x^{-1} \rangle$.

If $H\psi = \mathbf{1}$ then $G = K \times H \cong C_4 \times C_3$, which we have already seen.

If $H\psi = \langle \beta \rangle$ then we have:

$$G = \langle\, x, y \mid x^3 = y^4 = 1,\ y^{-1}xy = x^{-1} \,\rangle$$

Now let $a = xy^2$. And remember, $y^{-1}xy = x^{-1}$ means x commutes with y^2. So now:

$$a^3 = xy^2xy^2xy^2 = x^3y^6 = y^2$$

and

$$y^{-1}ay = y^{-1}xy^2y = (y^{-1}xy)y^2 = x^{-1}y^2 = y^2x^{-1} = a^{-1}$$

So:

$$G = \langle\, a,\, y \mid a^6 = 1,\ a^3 = y^2,\ y^{-1}ay = a{-}1 \,\rangle$$

This is known as the binary dihedral or dicyclic group, denoted Dic_{12}. This group is also sometimes denoted by T.

Case 2: $H \cong V_4$ i.e. $G \cong C_3 \rtimes (C_2 \times C_2)$.

If $\psi : H \to \operatorname{Aut} K$ is trivial then we obtain the direct product again. We saw in our classification of groups of order $2p$, that $\operatorname{Aut} K$ only has a single element of order 2, corresponding to the inverse map. So we have 3 choices of elements in H to send to it, but they are all equivalent up to isomorphism, by Lemma 5.

We know that $H/\operatorname{im}\psi \cong \ker\psi$, so $\ker\psi$ must be isomorphic to C_2. Pick z so that it generates the kernel, and so the remaining generator, y is not in the kernel. Then:

$$G = \langle\, x, y, z \mid x^3 = y^2 = z^2 = 1,\ yz = zy,\ xz = zx,\ y^{-1}xy = x \,\rangle$$

Let $a = xz$. The order of $a = \operatorname{lcm}(\mathrm{o}(x), \mathrm{o}(z)) = \operatorname{lcm}(2,3) = 6$ because x and z commute. So:

$$a^3 = x^3z^3 = z$$

and

$$y^{-1}ay = y^{-1}xzy = y^{-1}xyz = x^{-1}z = x^2z = a^2a^3 = a^{-1}$$

Hence:

$$G = \langle\, y, a \mid a^6 = y^2 = 1,\ a^{-1}ay = a^{-1} \,\rangle \cong D_{12}$$

Instead, if G has 4 Sylow 3-subgroups, then there are 8 elements of order 3 in G. So the remaining 4 must form the Sylow 2-subgroup, hence it is normal.

Case 1: $H \cong C_4$ i.e. $G \cong C_4 \rtimes C_3$.

Let $H = \langle y \rangle$.

A homomorphism $\psi : K \to \operatorname{Aut} H \cong C_2$, preserves order and together with Lagrange's Theorem means that the only possibility for ψ is trivial, i.e. $K\psi = \mathbf{1}$.

Hence $G \cong C_4 \times C_3 \cong C_{12}$.

Case 2: $H \cong V_4$ i.e. $G \cong (C_2 \times C_2) \rtimes C_3$.

Let $H = \langle y, z \rangle$.

A trivial homomorphism $K\psi = \mathbf{1}$ yields the direct product. What non-trivial homomorphisms are there? The automorphism group, $\operatorname{Aut} H \cong S_3$ is of order 6, and so has a unique subgroup of order 3, by Sylow's Theorems. We know that a homomorphism $\psi : K \to \operatorname{Aut} H$ is determined by where it sends the generator x, so for ψ to be non-trivial, it must send x to an element of order 3 in $\operatorname{Aut} H$.

There are 2 such elements. Because $\operatorname{Aut} H \cong S_3$, we will think of them as the permutations of order 3 of the set $\{1, 2, 3\}$. Denote them $a = (1\ 2\ 3)$ and $b = (1\ 3\ 2)$. Notice that $b = a^{-1}$, so we have homomorphisms:

$$\psi_1 : x \mapsto a \quad \text{and} \quad \psi_2 : x \mapsto a^{-1}$$

It appears we have 2 choices, but this is not the case. The inverse map, $\beta : x \mapsto x^{-1}$, is an automorphism of K, and so by Lemma 5, the corresponding semidirect products of ψ_1 and ψ_2 are isomorphic. Hence (up to isomorphism) there is one non-trivial homomorphism $\psi : K \to \operatorname{Aut} H$. So $x \in K$ acts by permuting the 3 non-identity elements of H.

We will show that in this case, $G \cong A_4$. First, let's check A_4 has the same subgroup structure as G. There is a subgroup isomorphic to C_3 in A_4, generated by the 3-cycle $(1\ 2\ 3)$:

$$\bar{K} = \langle (1\ 2\ 3) \rangle$$

We can also find a subgroup isomorphic to V_4:

$$\bar{H} = \{\, 1,\ (1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4)(2\ 3) \,\}$$

Indeed, we can check that $\bar{H}$ is normal in A_4. We can see that $\bar{H} \cap \bar{K} = \mathbf{1}$ because $\bar{H}$ contains no 3-cycles, and that $\bar{H}\bar{K} = A_4$. So we can conclude that $A_4 = \bar{H} \rtimes \bar{K}$.

Let's investigate how conjugation behaves. If we let $\alpha = (1\ 2)(3\ 4)$, $\beta = (1\ 4)(2\ 3)$ and $\gamma = (1\ 2\ 3)$, then we can write an element of A_4 as $\alpha^i \beta^j \gamma^k$ for some i, j and k. Define $\phi : A_4 \to G$ by $\phi : \alpha^i \beta^j \gamma^k \mapsto x^i y^j z^k$. Then:

$$\beta\phi = (\gamma^{-1}\alpha\gamma)\phi = c^{-1}ac = b$$

So conjugation acts in the same way. Hence we can conclude that $G \cong A_4$.

So a group G of order 12 is isomorphic to one of:

$$C_{12}, \quad C_2 \times C_6, \quad A_4, \quad D_{12}, \quad \text{or} \quad \mathrm{Dic}_{12}$$

9 Groups of Order 24

Let G be a group of order 24, and let H be a Sylow 3-subgroup of G, so $H \cong C_3$, and let h generate H. Let T by a Sylow 2-subgroup of G, so T has order 8. By Lagrange's Theorem, $H \cap T = \mathbf{1}$ and then applying Lemma 4, $|HT| = 24$. Now let n_3 denote the number of Sylow 3-subgroups, and by Sylow's Theorems:

$$n_3 \equiv 1 \mod 3 \quad \text{and} \quad n_3 \mid 8$$

Hence n_3 is either 1 or 4.

If $n_3 = 1$, then H is normal in G. Thus $G = H \rtimes T$. We'll want a homomorphism $\psi : T \to \operatorname{Aut} H$. We know $\operatorname{Aut} H \cong C_2$, and from our classification of groups of order 8, we have 5 possibilities. An action of T on H will have image isomorphic to C_2, and a kernel isomorphic to a group of order 4. We can classify the possible actions by considering the kernel.

Case 1: $T \cong C_8$ i.e. $G \cong C_3 \rtimes C_8$.

Let t generate T, and so its unique subgroup of order 4 is generated by t^2. Hence $\langle t^2 \rangle$ is the kernel of ψ, so ψ must send t to the identity or inversion map. Hence a non-trivial action of T on H is unique. If the action is trivial, then:

$$G = T \times H \cong C_{24}$$

Otherwise we obtain:

$$G = \langle\, h,\, t \mid h^3 = t^8 = 1,\ h^{-1}th = t^{-1} \,\rangle \cong C_3 \rtimes C_8$$

Case 2: $T \cong (C_4 \times C_2)$ i.e. $G \cong C_3 \rtimes (C_4 \times C_2)$.

In this case, T has subgroups isomorphic to both C_4 and $C_2 \times C_2$, so we have more possibilities for ψ. Firstly, if ψ is trivial, then we obtain the direct product:

$$G \cong C_3 \times C_4 \times C_2$$

Let T be generated by x and y, where $x^4 = y^2 = 1$, and consider non-trivial ψ. Say the kernel of ψ is isomorphic to $C_2 \times C_2$. So it must be generated by the elements of order 2 in T: x^2 and y. Then ψ must map x to the non-identity element in $\operatorname{Aut} H$, inversion. Hence $\langle\, x \,\rangle$ acts by inversion on H, giving:

$$\begin{aligned} G &= (H \rtimes \langle\, x \,\rangle) \times \langle\, y \,\rangle \\ &\cong (C_3 \rtimes C_4) \times C_2 \\ &\cong \mathrm{Dic}_{12} \times C_2 \end{aligned}$$

If instead the kernel is isomorphic to C_4, then it must be generated by an element of order 4 from T. However, all elements of order 4 are automorphic, and so by Lemma 5, we can pick x to generate the kernel, without loss of generality. So then ψ must map y to inversion. Hence $\langle\, x \,\rangle$ acts trivially on H, and $\langle\, y \,\rangle$ acts by inversion. Thus:

$$\begin{aligned} G &= (H \rtimes \langle\, y \,\rangle) \times \langle\, x \,\rangle \\ &\cong (C_3 \rtimes C_2) \times C_4 \\ &\cong S_3 \times C_4 \end{aligned}$$

Case 3: $T \cong (C_2 \times C_2 \times C_2)$ i.e. $G \cong C_3 \rtimes (C_2 \times C_2 \times C_2)$.

Let $\langle\, a,\, b,\, c \,\rangle = T$. All elements in T have order 1 or 2, so cannot have subgroups isomorphic to C_4. However, T does have subgroups isomorphic to $C_2 \times C_2$, which can be generated by 2 of the three generators of T. This gives us 3 subgroups, but permuting the generators a, b and c is an automorphism of T, so Lemma 5 tells us the resulting semidirect products are isomorphic. So choose ψ such that b and c are in the kernel. Then $a\psi$ is either the identity map or the inversion map. If ψ is trivial, then we obtain the direct product:

$$G \cong C_3 \times C_2 \times C_2 \times C_2$$

If $a\psi$ is inversion, then:

$$G = (C_3 \rtimes \langle\, a \,\rangle) \times \langle\, b \,\rangle \times \langle\, c \,\rangle \cong S_3 \times C_2 \times C_2$$

Case 4: $T \cong D_8$ i.e. $G \cong C_3 \rtimes D_8$.

Let r and s generate T with $r^4 = s^2 = 1$. A trivial homomorphism will yield the direct product:

$$G \cong C_3 \times D_8$$

So for a non trivial homomorphism, firstly assume $\ker\psi \cong C_4$. There is a unique subgroup in T isomorphic to C_4, so it's generated by an element of order 4. However the choice of generator is the same up to an isomorphism of T, so Lemma 5 lets us pick r to be the generator, without loss of generality. Hence s cannot be in the kernel, and so $s\psi$ is the inversion map. We obtain the presentation:

$$G = \langle\, x,\, r,\, s \mid x^3 = r^4 = s^2 = 1,\ xr = rx,\ s^{-1}rs = r^{-1},\ s^{-1}xs = x^{-1} \,\rangle$$

Let $a = xr$, and consider:

$$s^{-1}as = s^{-1}xrs = s^{-1}xrs^2s^{-1} = (s^{-1}xs)(srs^{-1}) = x^{-1}r^{-1} = r^{-1}x^{-1} = a^{-1}$$

So we have:

$$G = \langle\, a,\, s \mid a^{12} = s^2 = 1,\ s^{-1}as = a^{-1} \,\rangle$$

Which we recognise as D_{24}, the dihedral group of order 24.

If instead we consider ψ with kernel isomorphic to $C_2 \times C_2$, then the kernel is generated by two elements of order 2. However, T only has two elements of order 2, r^2 and s, so they must generate the kernel. So then ψ must map r to inversion. Hence this action is fully specified. So:

$$G \cong C_3 \rtimes_{V_4} D_8$$

We will use the above notation to mean the unique action with kernel isomorphic to V_4.

Case 5: $T \cong Q_8$ i.e. $G \cong C_3 \rtimes Q_8$.

Let T be generated by i and j, with the product denoted by k. That is:

$$T = \langle\, i,\, j \mid i^4 = j^4 = 1,\ i^2 = j^2,\ j^{-1}ij = i^{-1} \,\rangle$$

There is a single element of order 2 in T, hence T has no subgroup isomorphic to $C_2 \times C_2$. The elements i, j and k each generate a cyclic subgroup in T. So ψ will send one of them to the kernel. We know that permuting these is an automorphism of T, so Lemma 5 tells us the choice results in isomorphic semidirect products.

So take $i \in \ker\psi$. Indeed $\langle\, i \,\rangle = \ker\psi$. Then for a non-trivial homomorphism, we must have $j \notin \ker\psi$. Otherwise:

$$i\psi\ j\psi = (ij)\psi = k\psi \in \ker\psi$$

making ψ trivial.

Thus either ψ is trivial and we obtain:

$$G \cong C_3 \times Q_8$$

or ψ maps j to the inversion map and we obtain the presentation:

$$G = \langle\, x,\, i,\, j \mid x^3 = i^4 = j^4 = 1,\ xi = ix,\ i^2 = j^2,\ j^{-1}xj = x^{-1},\ j^{-1}ij = i^{-1} \,\rangle$$

Now let $a = xi$. So:

$$a^6 = x^6 i^6 = i^2 = j^2$$

And:

$$j^{-1}aj = j^{-1}xij = j^{-1}xji^{-1} = x^{-1}i^{-1} = i^{-1}x^{-1} = a^{-1}$$

Hence:

$$G = \langle\, a,\, j \mid a^{12} = 1,\ a^6 = j^2,\ j^{-1}aj = a^{-1}\,\rangle$$

We recognise this as the dicyclic group of order 24, Dic_{24}.

If $n_3 = 4$ then H is not normal. So then the normaliser of H, $\mathrm{N}_G(H)$ has index 4. Now let G act on the set of the cosets of $\mathrm{N}_G(H)$ by conjugation. Hence we obtain a homomorphism $\rho : G \to S_4$. The kernel is a subgroup of $\mathrm{N}_G(H)$ so must have order dividing 6 by Lagrange's Theorem.

The kernel cannot be of order 3, because G has no normal subgroup of order 3 (because kernels of homomorphism are normal in the domain group). Likewise the kernel cannot be of order 6 because every group of order 6 has a unique Sylow 3-subgroup, which is characteristic. So by Lemma 3, it would be normal in G. Hence the kernel must have order 1 or 2.

If the kernel is of order 1, then ρ is an isomorphism, so $G \cong S_4$.

If the kernel is of order 2, then we know that $G/\ker\rho \cong \operatorname{im}\rho$, so then $\operatorname{im}\rho$ must have order 12. It also cannot have a normal Sylow 3-subgroup, so looking at our classification of groups of order 12, this must be isomorphic to A_4. We know that A_4 has a normal subgroup of order 4, and so by the Correspondence Theorem, G must contain a normal subgroup of order 8, say T. By Lagrange's Theorem and Lemma 4, we can conclude that $G = T \rtimes H$. Again, we have 5 cases, but this time we'll exclude the trivial homomorphism, because that will just give us the direct product which we have already seen:

Case 1: $T \cong C_8$ i.e. $G \cong C_8 \rtimes C_3$.

An automorphism of T, φ, maps generators to generators, so say $\langle x\rangle = T$. Then $x\varphi$ could be x, x^3, x^5 or x^7. Notice that each of these, apart from $\varphi : x \mapsto x$, has order 2. Hence, and Lagrange's Theorem tells us that there are no non-trivial homomorphisms $\psi : H \to \operatorname{Aut} T$. As a bonus: $\operatorname{Aut} C_8 \cong V_4$.

Case 2: $T \cong (C_4 \times C_2)$ i.e. $G \cong (C_4 \times C_2) \rtimes C_3$.

An automorphism of T, ψ preserves element order, so if $\langle\, x,\, y \mid x^4 = y^2 = 1,\ xy = yx\,\rangle = T$, then $x\psi$ must be of order 4, and $y\psi$ must be of order 2. Moreover, $y\psi$ cannot be in $\langle\, x\psi\,\rangle$ because ψ is injective.

So we are reduced to 2 possible choices for $y\psi$, and 4 possible choices for $x\psi$. Because an automorphism is determined by it's effect on generators, this gives us 8 possible automorphisms. Hence $|\operatorname{Aut} T| = 8$, and Lagrange's Theorem tells us that there are no non-trivial homomorphisms $\psi : H \to \operatorname{Aut} T$.

Case 3: $T \cong (C_2 \times C_2 \times C_2)$ i.e. $G \cong (C_2 \times C_2 \times C_2) \rtimes C_3$.

To determine $\operatorname{Aut} T$ it is helpful to think of C_2 as the finite field with two elements. Then T is isomorphic a 3 dimensional vector space over two elements. So an automorphism of that vector space is just any linear map, with non-zero determinant. Thus, $\operatorname{Aut} T \cong \mathrm{GL}_3(2)$.

We can determine that $|\,\mathrm{GL}_3(2)| = 168 = 2^3 \cdot 3 \cdot 7$, so $\operatorname{Aut} T$ has a Sylow 3-subgroup of order 3, isomorphic to C_3.

Sylow's Theorems tells us that all subgroups of order 3 are conjugate, so Lemma 5 tells us there is only one unique action (up to isomorphism) of H on T. As before, pick a homomorphism, ψ, which will let us easily classify the resulting semidirect product.

Write $T = A \times B$ where $A \cong C_2$ and $B \cong C_2 \times C_2$. Then let ψ map H to the subgroup generated by the automorphism which fixes A and permutes the non-identity elements of B in a 3-cycle. This automorphism has order 3 by construction, so we can write:

$$G \cong C_2 \times (V_4 \rtimes C_3)$$

We know already that $V_4 \rtimes C_3 \cong A_4$, so $G \cong C_2 \times A_4$.

Case 4: $T \cong D_8$ i.e. $G \cong D_8 \rtimes C_3$.

Let $\langle\, s, r \mid s^2 = r^4 = 1,\ s^{-1}rs = r^{-1} \,\rangle = T$. An automorphism, ψ, of T preserves element order, so for $r\psi$ we have two choices, r or r^{-1}. We can send $s\psi$ to any element of order 2 which is not in $\langle r\psi \rangle$. This leaves only reflections, of which there are 4: s, rs, r^2s and r^3s. Hence there are 8 possible automorphisms of D_8, so $|\operatorname{Aut} D_8| = 8$. Lagrange's Theorem tells us that there are no non-trivial homomorphisms $\psi : H \to \operatorname{Aut} T$.

Case 5: $T \cong Q_8$ i.e. $G \cong Q_8 \rtimes C_3$.

Firstly, because of the multiplication structure of the quaternions, the image of k is determined by the images of i and j; it is forced. This reduces the possibilities for an automorphism. Additionally, ± 1 are fixed by an automorphism, because they are the only elements of their order. So an automorphism could send i to any of the remaining 6 elements of order 4. The image of j cannot be in the subgroup generated by the image of i, otherwise we wouldn't have an automorphism. Thus there are 4 choices for the image of j, giving us 24 possible automorphisms altogether.

So $\operatorname{Aut} T$ will have a Sylow subgroup of order 3.

Somehow show this is $\mathrm{SL}_2(\mathbb{F}_3)$.

10 Groups of Order 30

Let G be a group of order $30 = 2 \cdot 3 \cdot 5$. So then G has a Sylow 3-subgroup, T, and a Sylow 5-subgroup, F. Let $H = TF$ and by Lagrange's Theorem, $T \cap F = \mathbf{1}$, hence $|H| = 15$ by Lemma 4. We know from our classification of groups of order pq that $H \cong C_{15}$. Because $|H| = 15 = \frac{30}{2}$, the index of H in G is 2, and we know a subgroup of index 2 is normal, so $H \trianglelefteq G$.

Notice that a Sylow 2-subgroup $K \leqslant G$ has order 2, so $K \cong C_2$. Let $\langle k \rangle = K$ and $\langle h \rangle = H$. By the same argument as above, $H \cap K = \mathbf{1}$ and $|HK| = 30$. Hence $G = HK$. Moreover, $G = H \rtimes K$.

By Lemma 2:

$$\operatorname{Aut} C_{15} = (\mathbb{Z}/15\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})^\times \cong (\mathbb{Z}/3\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times \cong C_2 \times C_4$$

A homomorphism, $\psi : C_2 \to C_2 \times C_4$ preserves element order and we know ψ is determined by it's effect on a generator. So then $k\psi$ has four possibilities: either the identity, or one of the three elements of order 2.

Additionally, ψ preserves the Sylow subgroups of H. So write $H = \langle h^3 \rangle \times \langle h^5 \rangle$, the direct product of its Sylow subgroups.

So the action of K on H is either trivial or by inversion on each of the Sylow subgroups of H, giving us 4 possibilities:

Case 1: Trivial action on both Sylow subgroups.

In this case, because the action is trivial on all of H, we recover the direct product, $G = H \times K \cong C_{30}$.

Case 2: Inversion on both Sylow subgroups.

Here, K acts on all of H, so we obtain:

$$G = \langle\, h,\, k \mid h^{15} = k^2 = 1,\ k^{-1}hk = h^{-1} \,\rangle$$

which we recognise as D_{30}.

Case 3: Inversion on $\langle h^5 \rangle$.

We know already, from our classification of groups of order $2p$, that $C_3 \rtimes C_2 \cong D_6$. So then because the action on $\langle h^3 \rangle$ is trivial:

$$G = \langle h^3 \rangle \times (\langle h^5 \rangle \rtimes K) \cong C_5 \times D_6$$

Case 4: Inversion on $\langle h^3 \rangle$.

Similar to above, we obtain:

$$G = \langle h^5 \rangle \times (\langle h^3 \rangle \rtimes K) \cong C_3 \times D_{10}$$

Hence any group of order 30 is isomorphic to one of:

$$C_{30}, \quad D_{15}, \quad C_5 \times D_6, \quad \text{or} \quad C_3 \times D_{10}$$

Part IV
To Do

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