Classification of Finite Groups

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Groups of Order pq

Setup

Let *G* be a group of order *pq* where *p* and *q* are distinct prime numbers.

Without loss of generality, take p > q.

Sylow Subgroups

Let n_p and n_q denote the number of Sylow p-subgroups and Sylow q-subgroups of G respectively.

$$n_p \equiv 1 \pmod{p}$$
 and $n_p \mid q$

So *G* has a unique Sylow *p*-subgroup, say $P \subseteq G$, and at least one Sylow *q*-subgroup, $Q \subseteq G$.

$$P\cong C_p, \qquad Q\cong C_q$$

Pick generators: $\langle x \rangle = P$ and $\langle y \rangle = Q$.

Sylow q-subgroup

Because $n_q \equiv 1 \pmod{q}$, we have:

$$n_q = 1$$
, $p + 1$, $2p + 1$, ...

So we have two cases:

$$q \nmid p-1$$
 or $q \mid p-1$

Case 1: $q \nmid p - 1$

$$q \nmid p-1$$

Here,
$$n_q = 1$$
 and so $Q \subseteq G$.

So:

$$G=P\times Q\cong C_{pq}$$

Case 2: q | p - 1

$q \mid p-1$

That was the easy one! Still have $n_q = 1$, but now also the other posibilites!

By Lagrange's Theorem: $P \cap Q = 1$.

By Lemma 4: |PQ| = pq.

So G = PQ. Hence:

$$G=P\rtimes Q$$

This is unique!

Presentation

Describe $P \rtimes Q$ by a presentation.

$$G = \langle x, y | x^p = y^q = 1, y^{-1}xy = x^a \rangle$$

where a is a generator for the subgroup of order q in $(\mathbb{Z}/p\mathbb{Z})^{\times}$.

Examples