12:00, Friday, October 18, 202X	
Class Test 2: Fourier Analysis and Sampling	
Time allowed: 45 minutes Max mark: 30, Assessment: **10%**	This paper contains: X pages, 4 questions

IMPORTANT INSTRUCTIONS:

Candidates should attempt **all** questions with numerical answers to **3** decimal places for each question, you need to show **your working to the final answer to gain maximum marks**.

Online Students: Properly space solutions to ensure high quality image scans, **use black/blue pen or 2B pencil on white ruled/plain paper** to ensure sufficient contrast, and ensure **you are in a well-lit area**. You will also need a **scientific calculator** and scratch pad to for draft working.

All Students: Solutions will be marked page by page, so **start questions on new page**.

Question 1 (8 marks)

We define:

$$w(t) = \begin{cases} -1.5 & -1 \leq t < 0 \\ 1.5 & 0 \leq t < 1 \end{cases}$$

Let $x(t)$ be the continuous-time periodic signal defined by:

$$x(t) = w(t \pm 2k), \ \ k = 0,1,2,3, \ldots$$

Calculate the Fourier series co-efficients $X[k]$ of $x(t)$ and simplify the expression.

HINT: Exploit symmetry!

We know that:

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi$$

$$X[k] = \frac{1}{2}\int_{T_0=2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_{-1}^{1} x(t)e^{-jk\pi t}dt$$

$$= \frac{1}{2}\int_{0}^{1} \frac{3}{2}e^{-jk\pi t}dt - \frac{1}{2}\int_{-1}^{0} \frac{3}{2}e^{-jk\pi t}dt = \frac{3}{4}\left\{\left[\frac{1}{-jk\pi}e^{-jk\pi t}\right]_0^1 - \left[\frac{1}{-jk\pi}e^{-jk\pi t}\right]_{-1}^0\right\}$$

$$= \frac{3}{-4jk\pi}\left\{\left(e^{-jk\pi} - 1\right) - \left(1 - e^{jk\pi}\right)\right\} = \frac{3j}{4k\pi}\left\{e^{-jk\pi} + e^{jk\pi} - 2\right\} = \frac{3j}{2k\pi}\{\cos(k\pi) - 1\}$$

Hence:

$$X[k] = j\frac{3}{2k\pi}\{\cos(k\pi) - 1\}$$

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Question 2 (8 marks)

Given that $x(t)$ has the Fourier transform $X(j\omega) = 2/(3 + j\omega)$, express the Fourier transform, $Y(j\omega)$, of:

(a) $y(t) = x(t-1) + x(t+1)$

(b) $y(t) = x(t) + x(-t)$

using the tables of Fourier transform properties. Simplify your expression.

(a) Time-shift Property: $x(t-1) = e^{-j\omega}X(j\omega)$ and $x(t+1) = e^{j\omega}X(j\omega)$

Hence:

$$\begin{aligned} Y(j\omega) &= e^{-j\omega}X(j\omega) + e^{j\omega}X(j\omega) \\ &= \left(e^{-j\omega} + e^{j\omega}\right)X(j\omega) = 2\cos(\omega)\,X(j\omega) \\ &= \frac{4\cos\omega}{3 + j\omega} \end{aligned}$$

(b) Scaling Property: $x(-t) = X(-j\omega)$

Hence:

$$\begin{aligned} Y(j\omega) &= X(j\omega) + X(-j\omega) \\ &= \frac{2}{3 + j\omega} + \frac{2}{3 - j\omega} = \frac{12}{9 + \omega^2} \end{aligned}$$

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Question 3 (6 marks)

What is the minimum sampling rate in Hz (Nyquist rate) to recover each of the following signals and explain your reasoning:

(a) $x(t) = 1 + \cos(1000\pi t) + \sin(3000\pi t)$

(b) $x(t) = \sin(4000\pi t)/(\pi t)$

(c) $x(t) = s(t) + s(t-1)$ where $s(t)$ has a Nyquist rate of $f_S = 5000$ Hz

(d) $x(t) = s(t)\cos(6000\pi t)$ where $s(t)$ has a Nyquist rate of $f_S = 8000$ Hz.

Hint: Use the required Fourier transform pairs and properties and the fact that $x(t) * \delta(t-T) = x(t-T)$

Solution:

(a) The signal is bandlimited to the highest frequency of $\omega_B = 3000\pi$, hence the sampling rate is $\omega_S = 6000\pi$ or $f_S = 3000$ Hz

(b) From the Fourier transform pair with $W = 4000\pi$ the signal is bandlimited to $\omega_B = 4000\pi$, hence the sampling rate is $f_S = 4000$ Hz.

(c) From the Fourier transform properties $X(j\omega) = S(j\omega) + e^{-j\omega}S(j\omega)$ which does not affect the Nyquist rate which remains at $f_S = 5000$ Hz

(d) From the Fourier transform properties:

$$X(j\omega) = \frac{1}{2\pi}S(j\omega) * \{\pi\delta(\omega - 6000\pi) + \pi\delta(\omega + 6000\pi)\}$$
$$= \frac{1}{2}\{S(j(\omega - 6000\pi) + S(j(\omega + 6000\pi)\}$$

We know that $s(t)$ is bandlimited to 4000 Hz (8000π) so for $x(t)$ we will have that $\omega_B = 8000\pi + 6000\pi = 14000\pi$ and $f_S = 14000\ Hz$

Question 4 (8 marks)

A system with impulse response, $h(t) = 2e^{-2t}u(t)$, receives the input signal, $x(t) = 3e^{-t}u(t)$. What is the magnitude spectrum and phase spectrum of the output?

HINT: Use $\theta = \text{atan2}(b, a)$ for complex number $z = a + jb = re^{j\theta}$

Solution:

$$x(t) = 3e^{-t}u(t) \text{ and } h(t) = 2e^{-2t}u(t)$$

$$x(t) = 3e^{-t}u(t) \xleftrightarrow{FT} \frac{3}{1+j\omega} = X(j\omega)$$

$$h(t) = 2e^{-2t}u(t) \xleftrightarrow{FT} \frac{2}{2+j\omega} = H(j\omega)$$

$$y(t) = x(t) * h(t) \xleftrightarrow{FT} X(j\omega)\cdot H(j\omega) = Y(j\omega)$$

Thus:

$$Y(j\omega) = \frac{6}{(1+j\omega)(2+j\omega)} = \frac{6}{(2-\omega^2) + 3j\omega}$$

Magnitude Response:

$$|Y(j\omega)| = \frac{6}{\sqrt{(2-\omega^2)^2 + 9\omega^2}}$$

Phase Response:

$$\angle Y(j\omega) = -\text{atan2}(3\omega, 2-\omega^2)$$

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