Question 1 (8 marks)

Determine the total energy of the signal $x(t) = 1 - \frac{|t|}{2}$ for $|t| < 2$. **HINT:** Split the integral to handle the $|t|$.

Solution:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt.$$

$$= \int_{-2}^{2} \left|1 - \frac{|t|}{2}\right|^2 dt.$$

$$= \int_{-2}^{2} 1 - |t| + \frac{t^2}{4} \, dt$$

$$= \int_{-2}^{0} 1 + t + \frac{t^2}{4} dt + \int_{0}^{2} 1 - t + \frac{t^2}{4} dt$$

$$= \left[t + \frac{t^2}{2} + \frac{t^3}{12}\right]_{-2}^{0} + \left[t - \frac{t^2}{2} + \frac{t^3}{12}\right]_{0}^{2}$$

$$= 0 - \left(-2 + 2 - \frac{8}{12}\right) + \left[2 - 2 + \frac{8}{12} - 0\right]$$

$$= \frac{8}{12} + \frac{8}{12}$$

$$= \frac{16}{12}$$

$$E = \frac{4}{3} \text{ J.}$$

6

Question 2 (6 marks)

A linear, time-invariant system has the impulse response, $h[n]$, shown in Figure Q2.

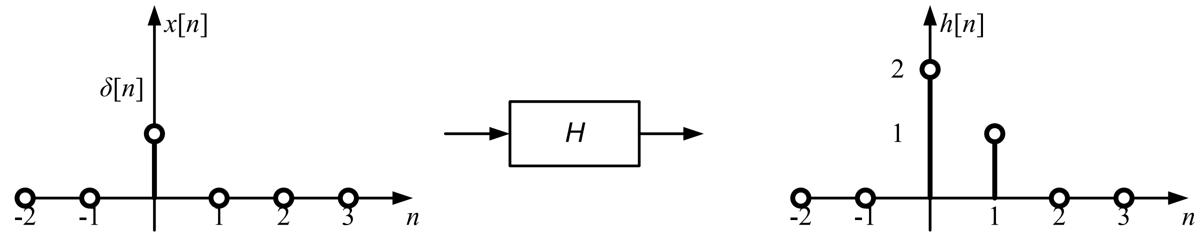


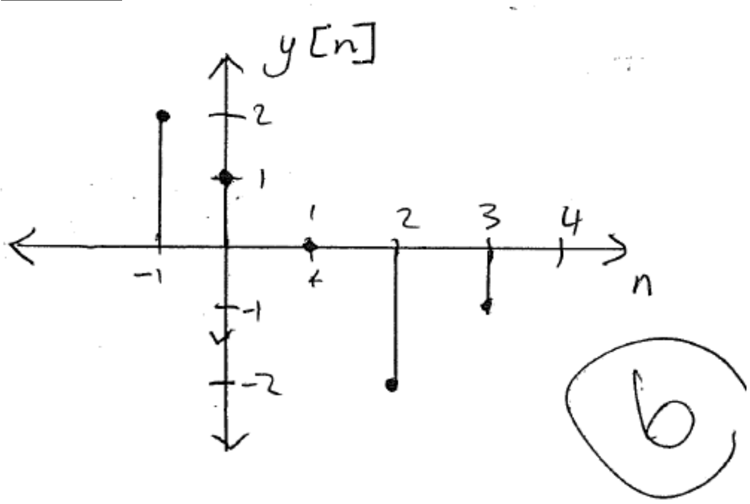
FIGURE Q2

Sketch the response of the system to the input

$$x[n] = \delta[n+1] - \delta[n-2]$$

You must label and scale all axes.

Solution:



Question 3 (8 marks)

Determine the response, $y(t)$, given an LTI system described by:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

with input $x(t) = 0$ and initial conditions $y(0^-) = 1$ and $\dot{y}(0^-) = 2$. Do NOT use the Laplace transform for this.

<u>Solution</u>

As the input is zero then we need the zero-input response. The characteristic equation is:

$$\lambda^2 + 6\lambda + 8 = 0$$

which yields roots of $\lambda_1 = -2$ and $\lambda_2 = -4$

Thus:

$$y(t) = c_1e^{-2t} + c_2e^{-4t}$$
$$y'(t) = -2c_1e^{-2t} - 4c_2e^{-4t}$$

Using the initial conditions:

$$y(0) = c_1 + c_2 = y(0^-) = 1$$
$$y'(0) = -2c_1 - 4c_2 = \dot{y}(0^-) = 2$$

This gives $c_1 = 3$ and $c_2 = -2$ and hence:

$$y(t) = (3e^{-2t} - 2e^{-4t})u(t)$$

Question 4 (6 marks)

Determine the Laplace transform of $x(t) = tu(t-1)$ using the table of Laplace transforms pairs and properties. **HINT**: Use the fact that $(t-1)u(t-1) = tu(t-1) - u(t-1)$, the time-shift property and the pair (for $k = 0, 1$ and 2):

$$\frac{1}{k!}t^k u(t) \leftrightarrow \frac{1}{s^{k+1}}$$

<u>Solution</u>

From the pair $u(t) \leftrightarrow \frac{1}{s}$ and $\frac{1}{k!}t^k u(t) \leftrightarrow \frac{1}{s^{k+1}}$ for $k = 1$ we have that:

$$tu(t) \leftrightarrow \frac{1}{s^2}$$

So using the time-shift property:

$$u(t-1) \leftrightarrow e^{-s}U(s) = \frac{e^{-s}}{s}$$
$$(t-1)u(t-1) \leftrightarrow \frac{e^{-s}}{s^2}$$

Since:

$$x(t) = tu(t-1) = (t-1)u(t-1) + u(t-1)$$

Then:

$$X(s) = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}$$

Question 5 (6 marks)

The transfer function of a causal continuous-time LTI system is given below:

$$H(s) = \frac{s+5}{s^2+4s+3}$$

(i) State with reason if the system is stable and minimum phase.

(ii) Describe the differential equation relating the input $x(t)$ and the output $y(t)$.

Solution:

(i)

$$H(s) = \frac{s+5}{s^2+4s+3} = \frac{s+5}{(s+3)(s+1)}$$

The poles are $p_1 = -3$ and $p_2 = -1$ and since they are both in the LHP the system is stable. The zero is $z_1 = -5$ which is in the LHP, hence the system is also minimum phase.

(ii)

$$\frac{Y(s)}{X(s)} = H(s) = \frac{s+5}{s^2+4s+3}$$

Hence:

$$s^2Y(s) + 4sY(s) + 3Y(s) = sX(s) + 5X(s)$$

Which gives:

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + 5x(t)$$

Question 6 (4 marks)

Consider the discrete-time LTI system with difference equation:

$$y[n] = x[n] - \frac{1}{2}x[n-1] - \frac{3}{4}y[n-1]$$

What is the difference equation of the inverse system?

Solution

Take the z-transform:

$$Y(z) = X(z) - \frac{1}{2}z^{-1}X(z) - \frac{3}{4}z^{-1}Y(z)$$

And from the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}}$$

And the inverse system is given by:

$$H^{inv}(z) = \frac{1}{H(z)} = \frac{1+\frac{3}{4}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{Y(z)}{X(z)}$$

From which the difference equation is:

$$y[n] = x[n] + \frac{3}{4}x[n-1] + \frac{1}{2}y[n-1]$$

Question 7 (10 marks)
A discrete system is described by the following transfer function:

$$H(z) = \frac{(1 - 0.5z^{-1})}{(1 + 0.5z^{-1})(1 - z^{-1})}$$

(i) Find the system response to input $x[n] = 2^{-n}u[n]$ if all initial conditions are zero.
(ii) Write the difference equation relating the output $y[n]$ to input $x[n]$ for this system.

Solution:

(i) With $x[n] = 2^{-n}u[n] = (0.5)^n u[n]$ we know that:

$$X(z) = \frac{1}{1 - 0.5z^{-1}}$$

Thus:

$$Y(z) = H(z)X(z) = \frac{(1 - 0.5z^{-1})}{(1 + 0.5z^{-1})(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{1}{(1 + 0.5z^{-1})(1 - z^{-1})}$$

We use a partial fraction expansion:

$$Y(z) = \frac{1}{(1 + 0.5z^{-1})(1 - z^{-1})} = \frac{A_1}{(1 + 0.5z^{-1})} + \frac{A_2}{(1 - z^{-1})}$$

$$A_1 = (1 + 0.5z^{-1})Y(z)\big|_{z=-0.5} = \left.\frac{1}{(1 - z^{-1})}\right|_{z=-0.5} = \frac{1}{3}$$

$$A_2 = (1 - z^{-1})Y(z)\big|_{z=1} = \left.\frac{1}{(1 + 0.5z^{-1})}\right|_{z=1} = \frac{2}{3}$$

Thus:

$$Y(z) = \frac{1/3}{(1 + 0.5z^{-1})} + \frac{2/3}{(1 - z^{-1})} \quad \leftrightarrow \quad y[n] = \frac{1}{3}(-0.5)^n u[n] + \frac{2}{3}u[n]$$

(ii)

$$\frac{Y(z)}{X(z)} = H(z) = \frac{(1 - 0.5z^{-1})}{(1 + 0.5z^{-1})(1 - z^{-1})} = \frac{(1 - 0.5z^{-1})}{1 - 0.5z^{-1} - 0.5z^{-2}}$$

Hence:

$$Y(z) - 0.5z^{-1}Y(z) - 0.5z^{-2}Y(z) = X(z) - 0.5z^{-1}X(z)$$

yielding:

$$y[n] - 0.5y[n-1] - 0.5y[n-2] = x[n] - 0.5x[n-1]$$

Question 8 (8 marks)

Use Euler's relation and the fact that $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$ to derive the Fourier series, $X[k] = |X[k]|e^{j\angle x[k]}$, for the following periodic signal:

$$x(t) = \cos t + 0.5\cos(4t + \pi/3)$$

Solution:

We can obviously see that $\omega_0 = 1$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt} + \frac{1}{4}e^{j\pi/3}e^{j4t} + \frac{1}{4}e^{-j\pi/3}e^{-j4t}$$

$$= \frac{1}{4}e^{-j\pi/3}e^{-j4t} + \frac{1}{2}e^{-jt} + \frac{1}{2}e^{jt} + \frac{1}{4}e^{j\pi/3}e^{j4t}$$

$$= X[-4]e^{-j4t} + X[-1]e^{-jt} + X[1]e^{jt} + X[4]e^{j4t}$$

Hence:

$$X[k] = \begin{cases} 0.25e^{-j\pi/3} & k = -4 \\ 0.5 & k = -1 \\ 0.5 & k = 1 \\ 0.25e^{j\pi/3} & k = 4 \\ 0 & otherwise \end{cases}$$

Question 9 (8 marks)

Calculate the Fourier Transform by direct integration using the defining equations of the following time-domain signal (where $u(t)$ is the unit step function):

$$x(t) = e^{1+t}u(-t+2)$$

Solution:

B2

(b)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad 1$$

$$= \boxed{\int_{-\infty}^{+\infty} \left(e^{1+t} u(-t+2)\, e^{-j\omega t}\, dt\right.}$$

$$= \boxed{\int_{-\infty}^{2} e^{1+t}\, e^{-j\omega t}\, dt} \qquad 2$$

$$= \int_{-\infty}^{2} e^{(1+t)-j\omega t}\, dt$$

$$= \boxed{e \int_{-\infty}^{2} e^{t(1-j\omega)}\, dt} \qquad 2$$

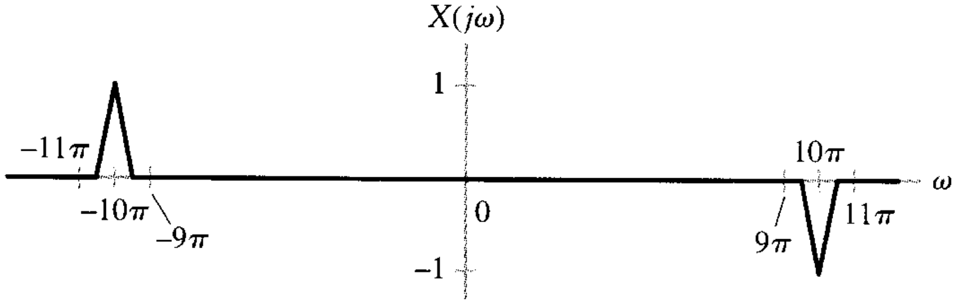
$$= \boxed{e \left[\frac{e^{t(1-j\omega)}}{(1-j\omega)}\right]_{-\infty}^{2}}$$

$$= e\left[\left(\frac{e^{2(1-j\omega)}}{1-j\omega}\right) - (0)\right]$$

$$\boxed{X(j\omega) = \frac{e^{3-j2\omega}}{1-j\omega}} \qquad 1$$

Question 10 (12 marks)

(a) The continuous-time signal $x(t)$ with FT as depicted below is sampled:



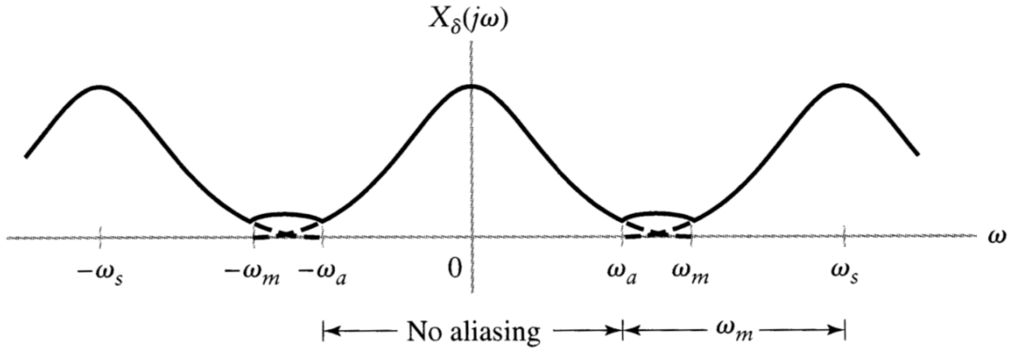
Sketch the FT of the sampled signal for the following sampling intervals and identify whether aliasing occurs:

(i) $T_s = 1/14$

(ii) $T_s = 1/7$

(iii) $T_s = 1/5$

(b) We sample a continuous-time signal with Fourier spectra $X(j\omega)$ and want to ensure that we can reconstruct $X(j\omega)$ over the interval $-\omega_a < \omega < \omega_a$ given that the signal is band-limited with maximum frequency ω_m but where $\omega_m \geq \omega_a$. This is depicted in the figure below:

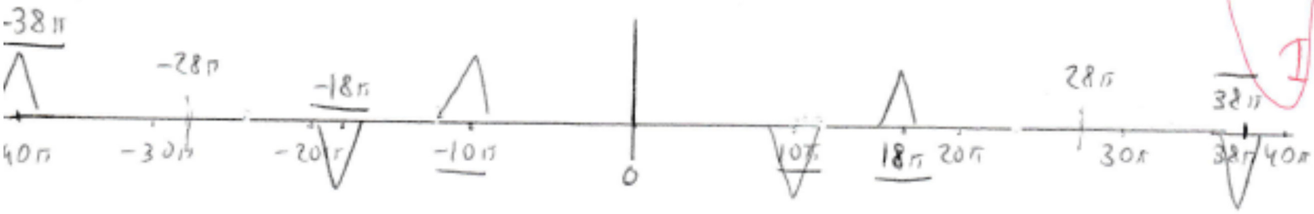


What is the maximum sampling interval, T_s, we can use?

Solution:

B5 4.26

(a) (i) If $T_s = 1/14$ then $\omega_s = \frac{2\pi}{T_s} = 28\pi$ 1

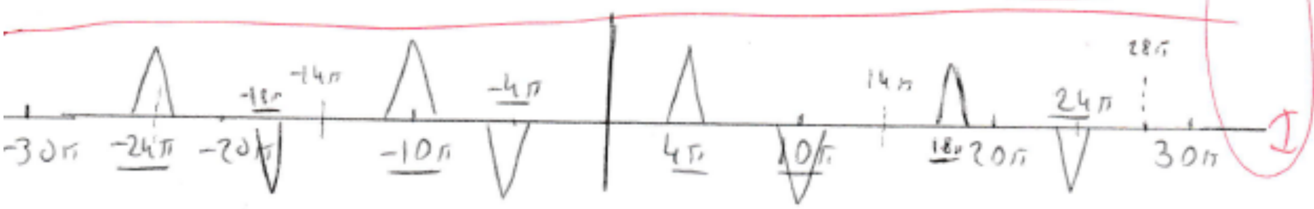


$-38\pi \quad -18\pi \quad -10\pi \quad 10\pi \quad 18\pi \quad 38\pi$

1

$\boxed{\omega_m = 11\pi}$ 1 ∴ $\boxed{\omega_N = 2\omega_m = 22\pi}$ 1 Since $\boxed{\omega_s > \omega_N \text{ no aliasing}}$

(ii) If $T_s = 1/7$ then $\omega_s = \frac{2\pi}{T_s} = 14\pi$ 1

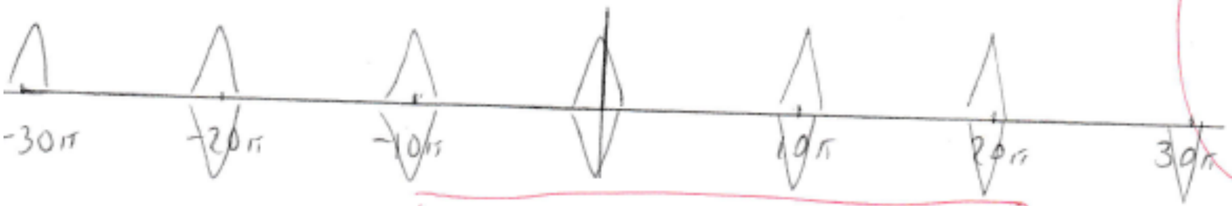


$-24\pi \quad -10\pi \quad -4\pi \quad 4\pi \quad 10\pi \quad 24\pi$

why 1

$\boxed{\omega_s = 14\pi < \omega_N = 22\pi \quad \text{aliasing!}}$

(iii) If $T_s = 1/5$ then $\omega_s = \frac{2\pi}{T_s} = 10\pi$! 1



∴ $\boxed{\text{all cancels out} = \text{zero!!}}$

$\boxed{\text{Definitely aliasing is occurring!}}$

(b) From the figure we see that

$$w_a + w_m = w_s = \frac{2\pi}{T_s}$$

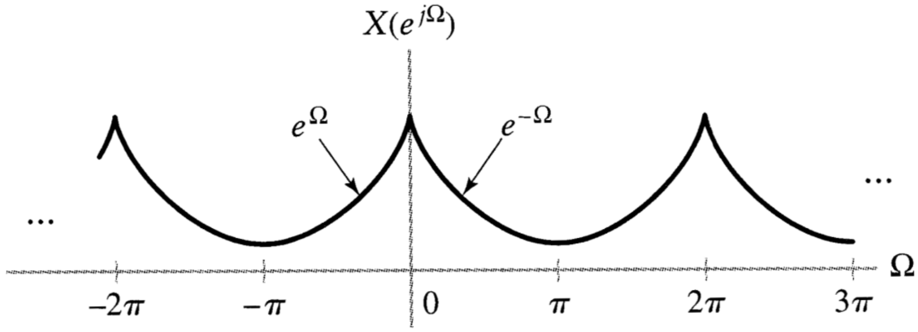
And we want $w_s \geq w_a + w_m$

$$\therefore \frac{2\pi}{T_s} \geq w_a + w_m$$

$$\boxed{T_s \leq \frac{2\pi}{w_a + w_m}}$$

Question 11 (8 marks)

Use the equation describing the DTFT representation to determine the time-domain signal corresponding to the following DTFT:



Solution:

B4

(b)

$$\boxed{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega}$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} (e^{\Omega}) e^{j\Omega n}\, d\Omega + \int_{0}^{\pi} (e^{-\Omega}) e^{j\Omega n}\, d\Omega\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} e^{\Omega(1+jn)}\, d\Omega + \int_{0}^{\pi} e^{\Omega(jn-1)}\, d\Omega\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{\Omega(jn+1)}}{jn+1}\right]_{-\pi}^{0} + \left[\frac{e^{\Omega(jn-1)}}{jn-1}\right]_{0}^{\pi}\right\}$$

$$= \boxed{\frac{1}{2\pi}\left\{\frac{1-e^{-\pi(jn+1)}}{jn+1} + \frac{e^{\pi(jn-1)}-1}{jn-1}\right\}}$$

Can be simplified to show $x[n]$ is a real-valued function of n