Lecture 2

The transient circuits (R-L-C circuits)

Transient circuits

An **active circuit element** is defined as one which is capable of delivering an average power greater than zero to some external device, where the average power is taken over an infinite time interval. Ideal sources are examples of active elements. Transistors, operational amplifiers would be considered active devices.

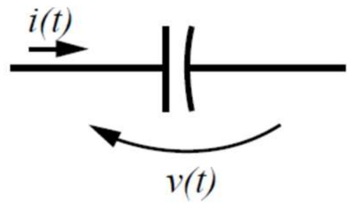
A **passive circuit element** is defined as one which cannot supply an average power which is greater than zero over an infinite time interval. A resistor is an example of a passive element.

So far we have just considered circuits with only constant voltages and currents.

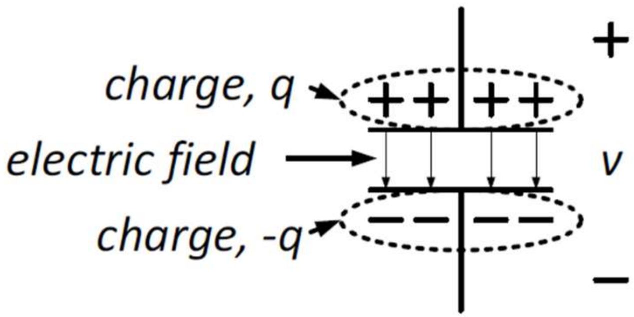
Now we will consider more interesting circuits where the independent sources (forcing functions) and the circuit responses are functions of time. In particular these circuits will contain capacitors and inductors.

Capacitor circuits

The Capacitor is a passive element:



Involves opposite charges
on two separated plates



Energy is stored in the electric field, amount of energy stored depends on the charge, geometry and materials of the capacitor.

The ***Capacitance C*** of the capacitor is determined by geometry and materials of the capacitor.

Capacitance is measured in farads (1F = 1 coulomb/volt). Typical values for components are picofarads to hundreds of millifarads

Capacitor circuits (cont'd)

Current-voltage relationship:

$$i(t) = \frac{dq}{dt} = \frac{d}{dt}(Cv) = C\frac{dv}{dt}\left[+v\frac{dC}{dt}, \textit{but this is usually zero}\right]$$

$$= C\frac{dv}{dt}$$

Its Integral form captures the initial conditions:

$$\int_{t_0}^{t} i(x)dx = C\left(v(t) - v(t_0^-)\right) \Rightarrow v(t) = \frac{1}{C}\int_{t_0}^{t} i(x)dx + v(t_0^-)$$

Energy stored in capacitor:

Instantaneous power: $p(t) = i(t)v(t)$

$$Energy = E = \int_{t_0}^{t} p(x)dx = C\int_{t_0}^{t} v(x)\frac{dv}{dx}dx = C\int_{v(t_0)}^{v(t)} vdv = \frac{1}{2}C\{v(t)^2 - v(t_0)^2\}$$

Capacitor circuits (cont'd)

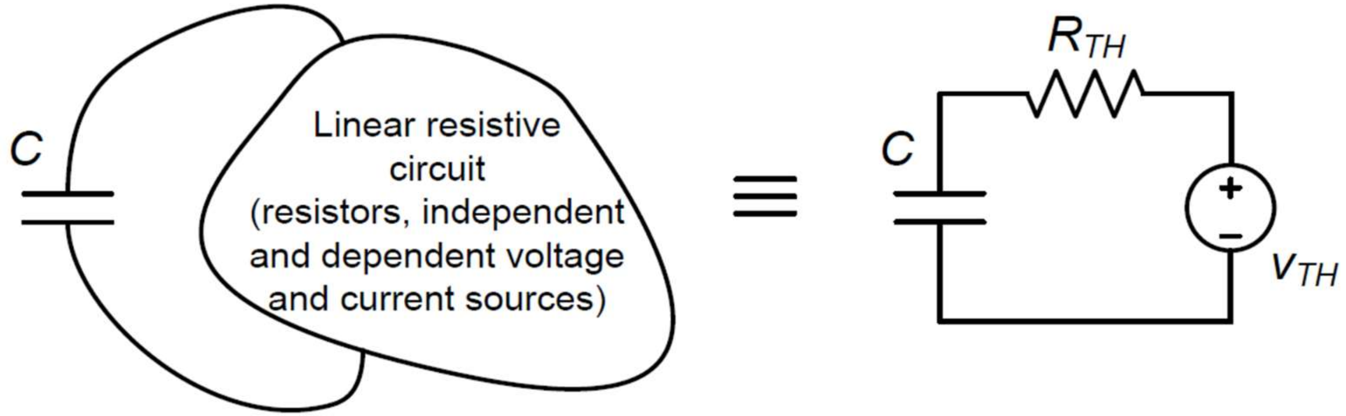
If select $v(t_0) = 0$ then $E = \frac{1}{2}Cv^2$

- Current through a capacitor is zero if voltage across capacitor is constant. A capacitor is therefore an **open circuit** at DC.
- A finite amount of energy can be stored in a capacitor, even if the current through it is zero.
- It is impossible to change the voltage across a capacitor by a finite amount in zero time. Voltage across a capacitor must always be continuous. (Current through the capacitor can change instantaneously though)
- The ideal capacitor never dissipates energy, but only stores it.

First-order RC circuits

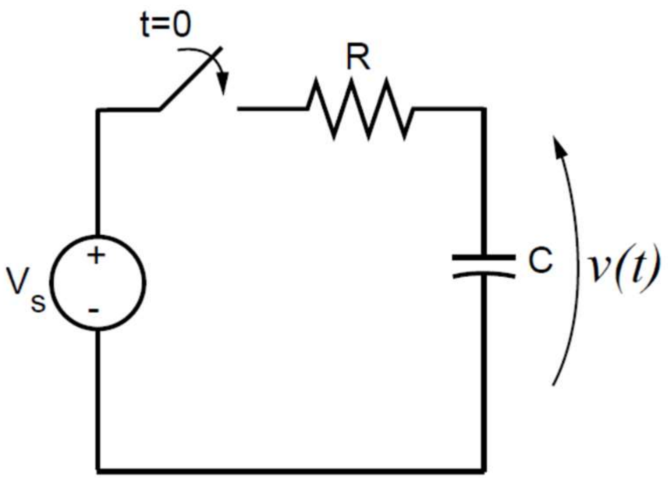
A **first-order RC circuit** consists of a capacitor C (or its equivalent) and a resistor (or its equivalent)

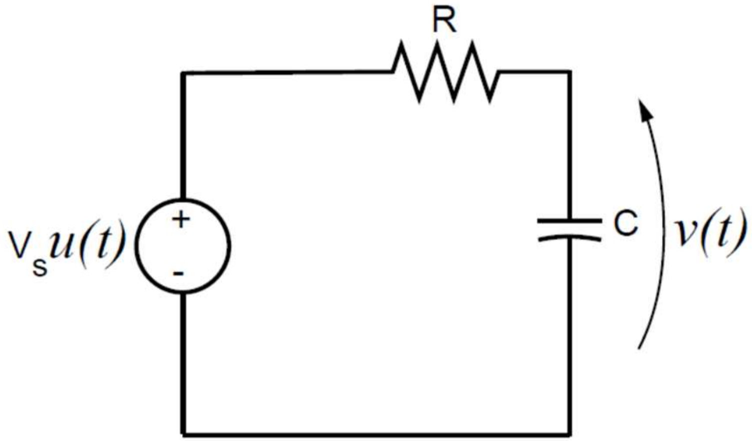
Why is it called first order? What determines the order of a circuit?



Step response of RC circuits

The **step response** of a circuit is its response when the forcing function (either an independent current or voltage source) is a step function *u(t)*.





With forcing function

Assume that the initial condition of the capacitor is: $v(0^-)=V_0$.

Can apply KCL to obtain d.e. :

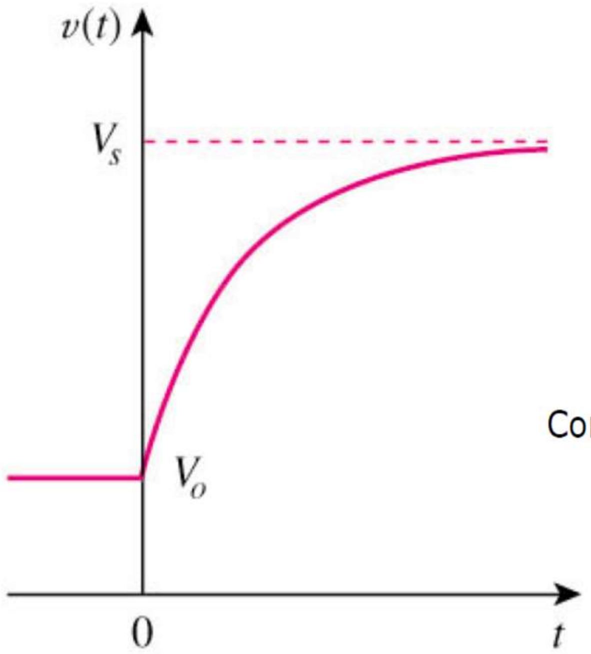
$$C\frac{dv(t)}{dt}+\frac{v(t)-V_s u(t)}{R}=0$$

$$\frac{dv(t)}{dt}=\frac{v(t)-V_s u(t)}{RC}$$

Solution to this differential equation (d.e.) has the form:

$$v(t) = \begin{cases} V_0 & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & t > 0 \end{cases}$$

Where $\tau = RC$ and this value is called the time constant.

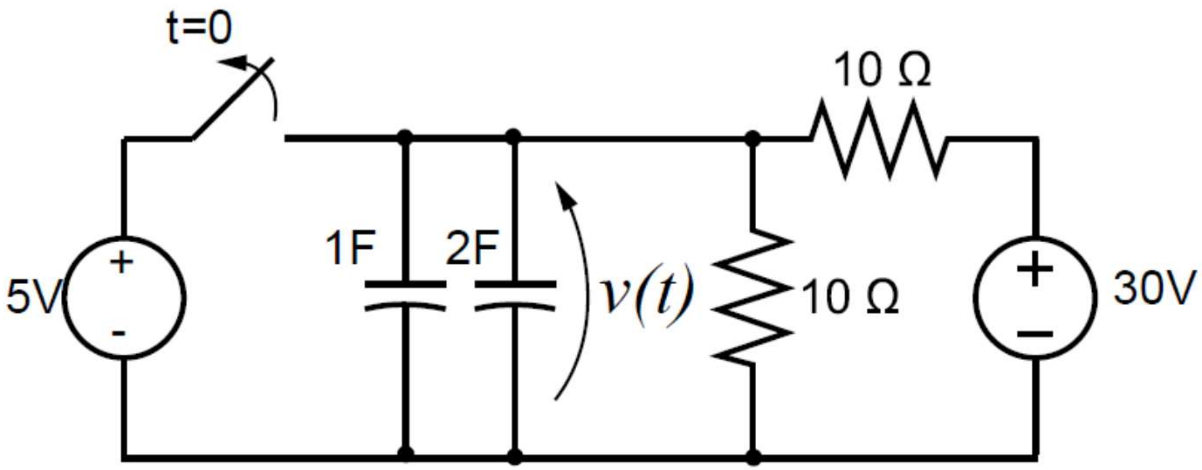


$$v(t) = V_0 e^{-t/\tau} + V_s(1 - e^{-t/\tau})$$

$v(t)$	$V_0 e^{-t/\tau}$	$V_s(1-e^{-t/\tau})$
Complete Response	Natural response (stored energy)	Forced Response (independent source)

RC circuits - Example

Find v(t) for t>0 in the following circuit. Assume the switch has been closed for a long time before being opened at t=0.



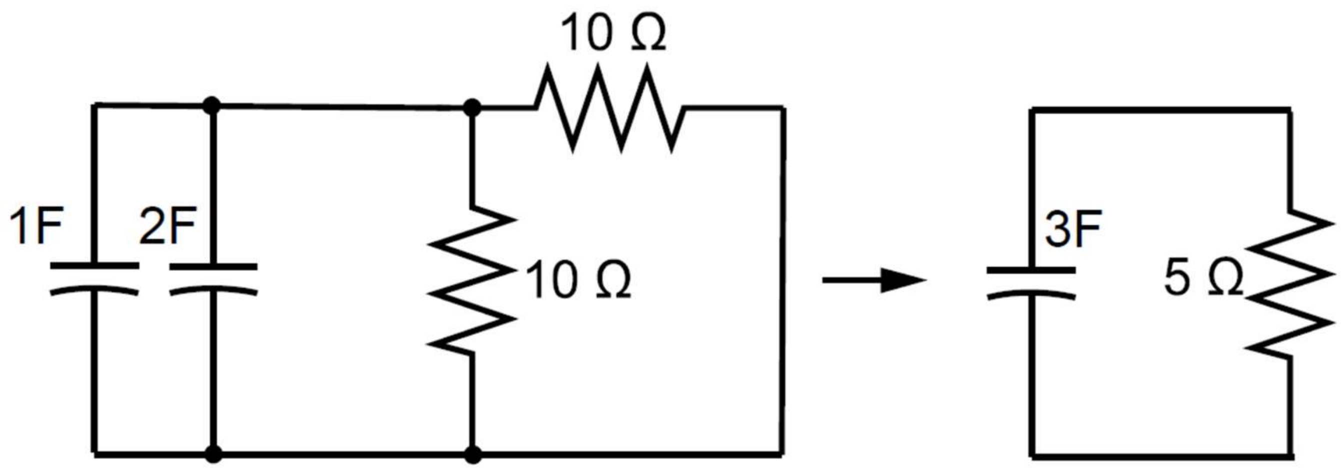
Procedure to find response:

- Remove all independent sources, simplify the circuit to determine R_{eq}, C_{eq} and the time constant $\tau = R_{eq} C_{eq}$.
- With C set as an open circuit, use dc-analysis to find the initial capacitor voltage just prior to the discontinuity. V_0.
- Again with C set as an open circuit, use dc analysis to find the value of capacitor voltage at t= ∞. Call this V_∞
- Solution is:

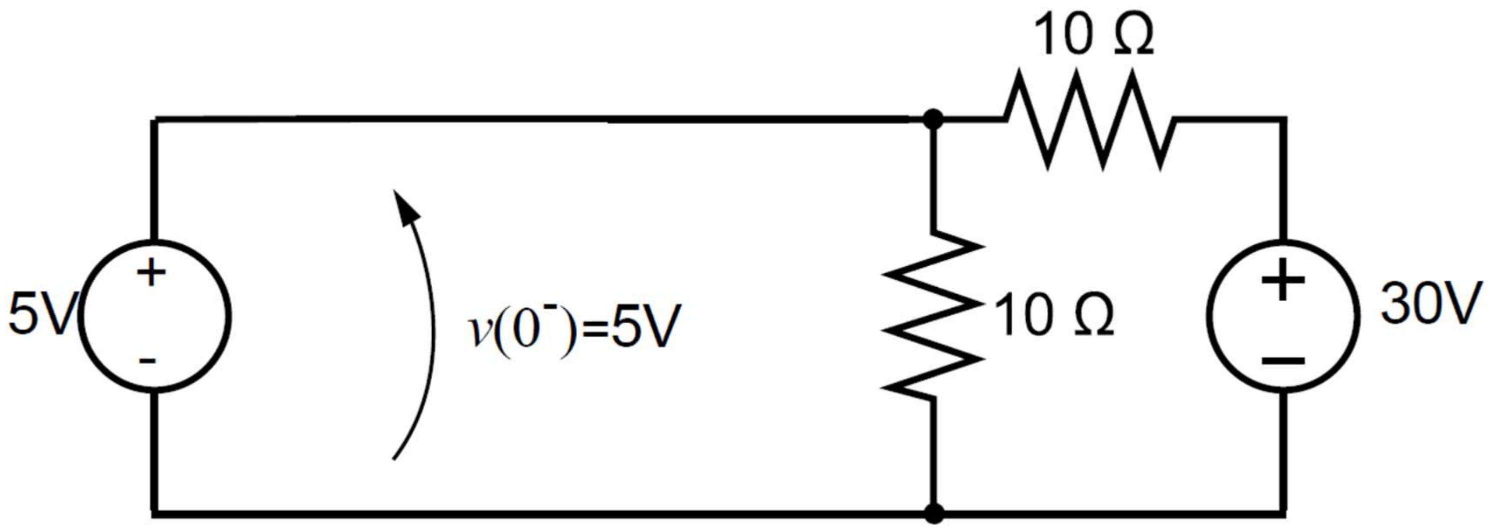
$$\boxed{v(t) = V_\infty + (V_0 - V_\infty)e^{-t/\tau}}$$

Let’s do it:

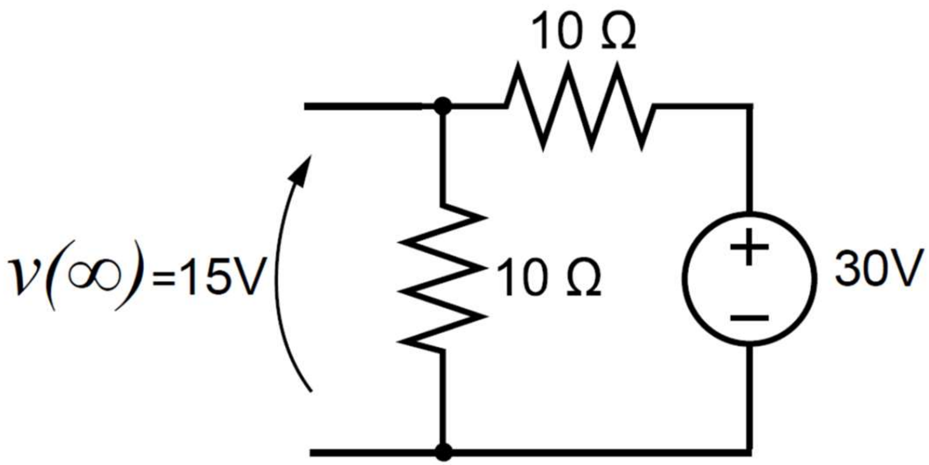
Remove all independent sources, simplify the circuit to determine R_{eq}, C_{eq} and the time constant $\tau = R_{eq} C_{eq}$. Obtain R_{eq}=5 Ohms, C_{eq}=3 F



With C set as an open circuit, use dc-analysis to find the initial capacitor voltage just prior to the discontinuity. V_0=5V



Again with C set as an open circuit, use dc analysis to find the value of capacitor voltage at t= ∞. Call this V_∞ V_∞ =15V

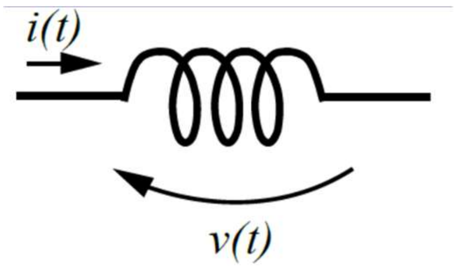


Solution is:

$$\boxed{v(t) = 15 + (5 - 15)e^{-t/15}}$$

Inductor circuits

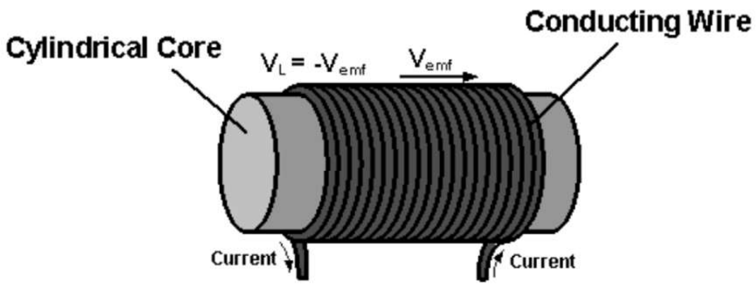
The Inductor is also a passive element:



Energy is stored in the magnetic field created by flowing current.
Amount of energy stored depends on the current, geometry and materials.

The **Inductance *L*** of the inductor is determined by geometry and materials of the inductor.

Typically inductors are made of coiled wires



Inductance is measured in Henrys (1H=1Weber/Ampere).

•Current flow → magnetic field

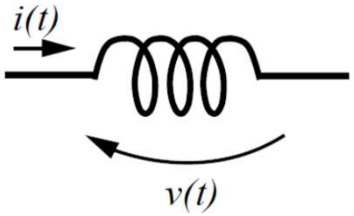
$$\lambda = Li$$

$$magnetic\ flux\ linkage = (inductance) \times (current)$$

•Flux linkage is the total amount of magnetic flux which links the conductor

- Flux linkage depends on geometry and materials
- For a straight wire the flux linkage is small
  → small inductance
- By using coils, the flux lines can link the same conductor more than once
  → large inductance
  → such elements are called inductors
  → for coiled inductors, $L \propto N^2$

Faraday's law gives:



$$v = \frac{d\lambda}{dt} = \frac{d}{dt}(Li) = L\frac{di}{dt}\left[+ i\frac{dL}{dt},\ \textit{but this term is usually zero} \right]$$
$$= L\frac{di}{dt}$$

Current increasing $\Rightarrow$ voltage $\uparrow$

Current decreasing $\Rightarrow$ voltage $\downarrow$

Integral form gives:

$$i(t) = \frac{1}{L}\int_{t'=0}^{t'=t} v(t')dt' + i(t=0)$$

Energy stored in the inductor:

$$\boxed{Instantaneous\ power = p(t) = i(t).v(t)}$$

$$\boxed{Energy = E = \int_{t'=-\infty}^{t} p(t')dt' = \int_{t'=-\infty}^{t} i(t')L\frac{di(t')}{dt'}dt' = L\int_{i(t'=-\infty)}^{i(t)} i(t')di(t') = \frac{1}{2}Li^2}$$

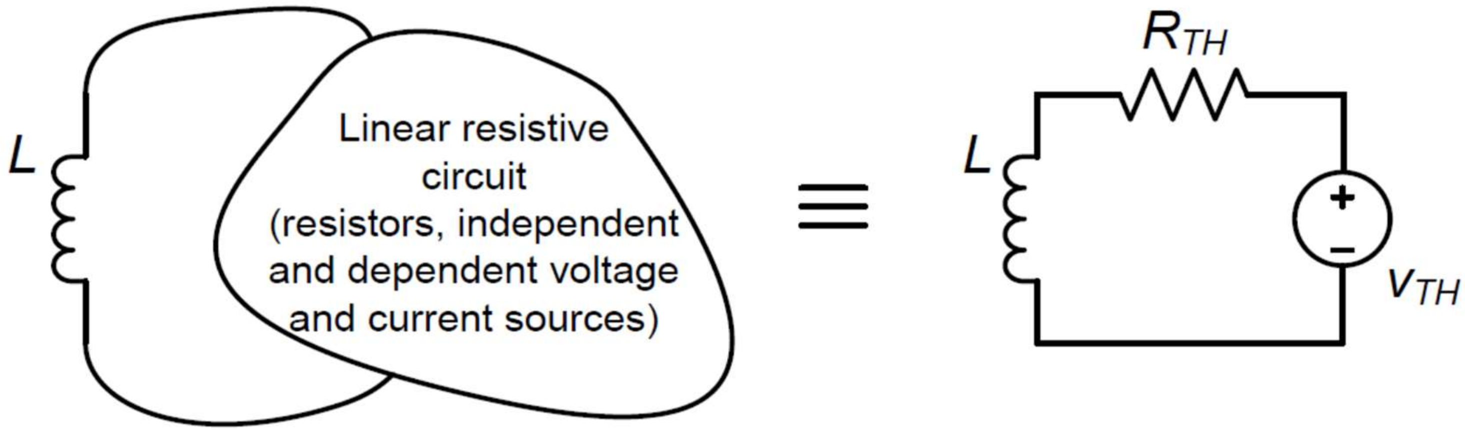
Voltage across an inductor is zero if current through the inductor is constant. An inductor is therefore a short circuit at DC.

It is impossible to change the current through an inductor by a finite amount in zero time. Current through an inductor must always be continuous. (Voltage across the inductor can change instantaneously though)

The ideal inductor never dissipates energy, but only stores it.

RL circuits

A first-order RL circuit consists of an inductor L (or its equivalent) and a resistor (or its equivalent)



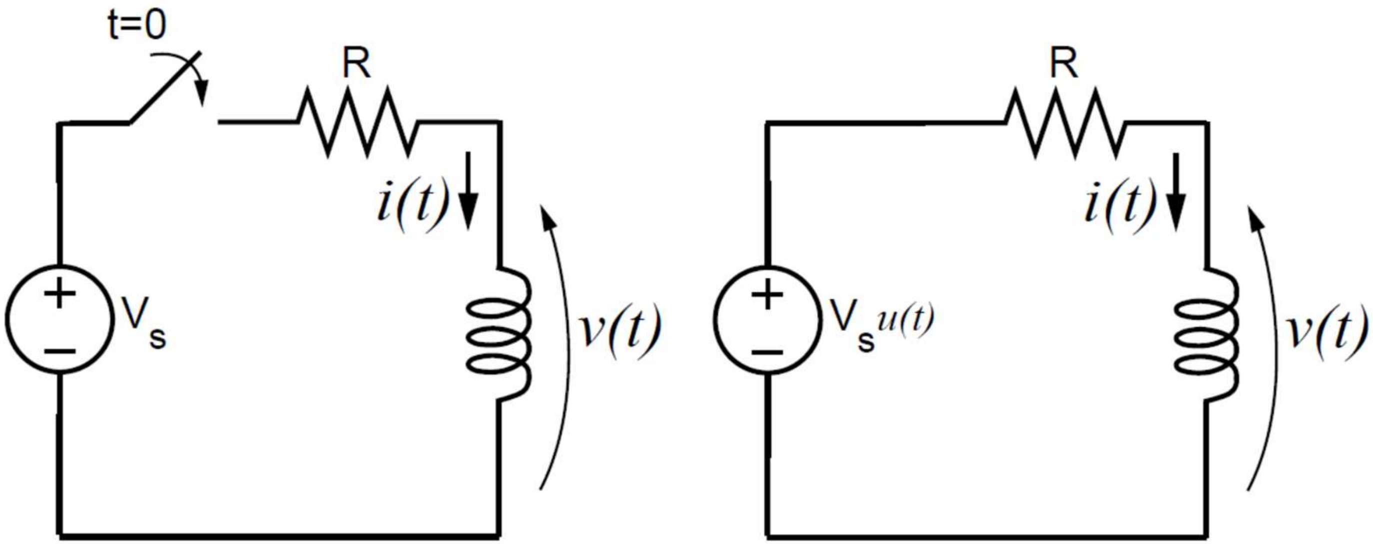
Step response of RL circuits

Assume that the initial condition of the inductor is: $i(0^-)=I_0$.

Can apply KVL to obtain d.e. :

$$L\frac{di(t)}{dt} + Ri(t) - V_s u(t) = 0$$

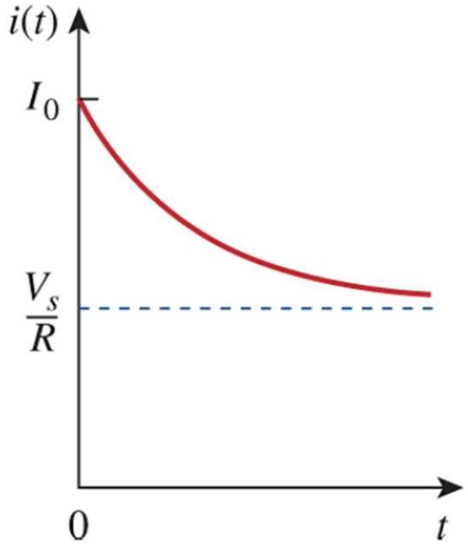
$$\frac{di(t)}{dt} = -\frac{Ri(t) - V_s u(t)}{L}$$



Solution to this d.e. has the form:

$$i(t) = \begin{cases} I_0 & t < 0 \\ \frac{V_s}{R} + (I_0 - \frac{V_s}{R})e^{-t/\tau} & t > 0 \end{cases}$$

Where $\tau = L / R$ and this value is called the time constant.

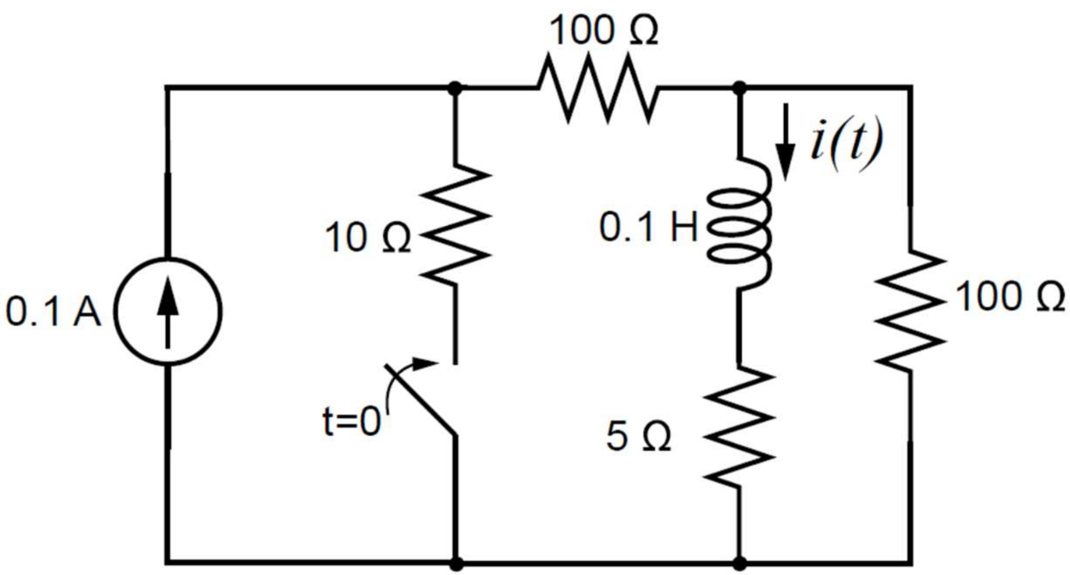


$i(t) = I_0 e^{-t/\tau} + (V_s/R)(1-e^{-t/\tau})$

Complete Response = Natural response (stored energy) + Forced Response (independent source)

Example

Find i(t) for t>0 in the following circuit. Assume the switch has been open for a long time before being closed at t=0.

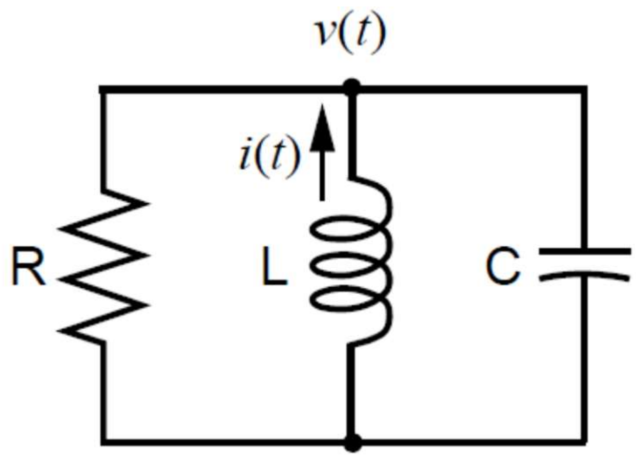


Procedure to find response:

- Remove all independent sources, simplify the circuit to determine R_{eq}, L_{eq} and the time constant $\tau = L_{eq} / R_{eq}$.
- With L set as short circuit, use dc-analysis to find the initial inductor current just prior to the discontinuity. I_0.
- Again with L set as a short circuit, use dc analysis to find the value of inductor current at t= ∞. Call this I_∞
- Solution is: $$i(t) = I_\infty + (I_0 - I_\infty)e^{-t/\tau}$$

RLC circuits – Second order circuits

Consider the following **parallel RLC circuit** without any sources, but with initial conditions for the inductor and capacitor:



Applying KCL at the top node obtain: $\dfrac{v}{R}+\dfrac{1}{L}\int_{t_0}^{t} v\,dt - i(t_0) + C\dfrac{dv}{dt} = 0$

Differentiating this obtain the second order d.e.: $C\dfrac{d^2v}{dt^2}+\dfrac{1}{R}\dfrac{dv}{dt}+\dfrac{1}{L}v=0$
(d.e. = differential equation)

We can try to see if $v(t) = Ae^{st}$ is a possible solution to the d.e. (where A and s may be complex numbers). Substituting $v(t) = Ae^{st}$ into the d.e. we get:

$$Ae^{st}(Cs^2 + \frac{1}{R}s + \frac{1}{L}) = 0$$

Which says our choice of $v(t)$ is a solution to the d.e. providing Ae^{st} is 0 (a trivial solution) or s satisfies:

$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$$

In general there are two solutions for s in the above quadratic equation:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0{}^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0{}^2}$$

Where: $$\alpha = \frac{1}{2RC}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

ω_0 is called the resonant frequency (in radians/s),
α is called the exponential damping coefficient.

Overdamped case --

In general the solution of v(t) is given by the linear combination:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Note this is the natural or homogeneous response of the d.e. as there is no forcing function. We look at solutions with a forcing function later.

Let's do this example:

Lets look at the example where R=10 ohms, L=11.11 H, C=0.01 F.

Say we have initial conditions: $i(0^+) = I_0 = 10\text{A}, \quad v(0^+) = V_0 = 0$

, where I_0 is the initial current in the inductor.

Then obtain α = 5, ω_0 = 3, s_1=-1, s_2= -9.

But we still need to obtain A_1 and A_2. Use initial conditions!

Firstly the initial voltage on the capacitor:

$$v(0^+) = V_0 = 0 = A_1 + A_2$$

Secondly we can look at the derivative of v(t) to get another equation for A_1 and A_2.

$$\frac{dv}{dt} = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$$

We can find dv/dt at t=0, by knowing the current through the capacitor, and also the initial conditions of the current in the inductor:

$$i_c = C\frac{dv}{dt}$$

$$\left.\frac{dv}{dt}\right|_{t=0} = \frac{i_C(0)}{C} = \frac{i(0) - i_R(0)}{C} = \frac{i(0)}{C} = 1000 \text{ V/s} = s_1 A_1 + s_2 A_2$$

Solving both equations obtain: A_1=125, A_2=-125.

Solution: $\boxed{v(t) = 125(e^{-t} - e^{-9t})}$

Three cases of RLC circuits

In general the second order d.e. can have three distinct forms of solution depending on the values of R, L and C:

(a) **Overdamped**:

characterized by $\alpha > \omega_0$
leads to two negative real values for s_1, s_2, and a response expressed as the sum of two negative exponentials.

(b) **Critically damped**:

characterized by $\alpha = \omega_0$
actually leads to a solution of the form: $v(t) = e^{-\alpha t}(A_1 t + A_2)$
which contradicts our original assumptions on the form of $v(t)$

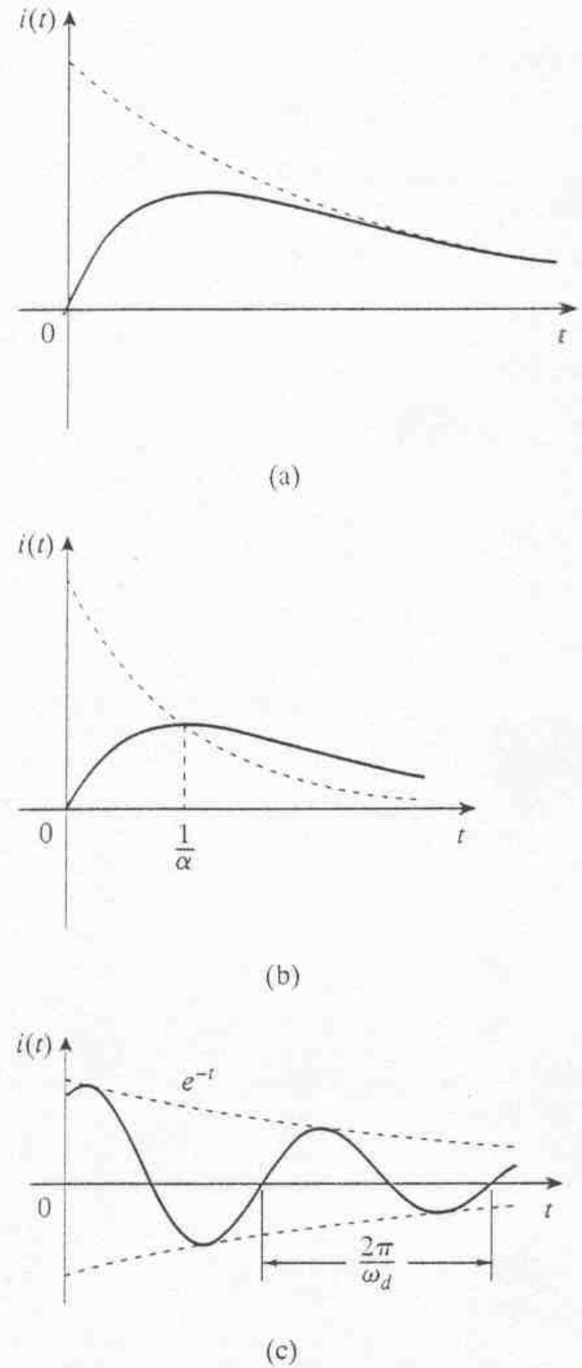
(c) **Underdamped**:

characterized by $\alpha < \omega_0$
leads to two complex values for s_1, s_2,. The response is a sum of two complex exponentials, which can be expressed a a decaying sinusoid.

(a) overdamped response

(b) critically damped response

(c) underdamped response



Note: second order systems described by the d.e. we have been studying occur in many situations and fields, including mechanics and control theory.

Underdamped case --

For the example just done we had $\alpha > \omega_0$ so it was the overdamped case.

Now lets look at what happens when $\alpha < \omega_0$, i.e. the **underdamped** case.

Again assume solution is of the form: $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

However, $s_{1,2}$ will now be complex numbers given by:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^{\,2}} = -\alpha \pm j\sqrt{\omega_0^{\,2} - \alpha^2}$$

Define the *natural resonant frequency* : $\omega_d = \sqrt{\omega_0^{\,2} - \alpha^2}$

And the natural response is then : $v(t) = e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})$

We can re-write the natural response as follows:

$$v(t) = e^{-\alpha t}\left\{(A_1 + A_2)\left[\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right] + j(A_1 - A_2)\left[\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{j2}\right]\right\}$$

Then converting the complex exponential sums to sinusoids:

$$v(t) = e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t]$$

Then finally defining new constants B_1 and B_2:

$$v(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

Which is the final form for the natural response, and note that it only contains real quantities (provided $v(t)$ is real which it must be)

Underdamped case -- Example

Lets do an example where: R=25 ohms, L=11.111 H and C=0.01 F.
And initial conditions are as before: $i(0^+) = I_0 = 10\text{A}$, $v(0^+) = V_0 = 0$

Then obtain α = 2, ω_0 = 3 rad/s, ω_d = $\sqrt{5}$ rad/s

$$v(t) = e^{-2t}(B_1 \cos\sqrt{5}t + B_2 \sin\sqrt{5}t)$$

With the above $v(t)$ and with $v(0)$ = 0 then B_1 must equal 0:

$$v(t) = e^{-2t} B_2 \sin\sqrt{5}t$$

Use the derivative at t=0 to find B_2:

$$\frac{dv}{dt} = \sqrt{5}B_2 e^{-2t}\cos\sqrt{5}t - 2B_2 e^{-2t}\sin\sqrt{5}t$$

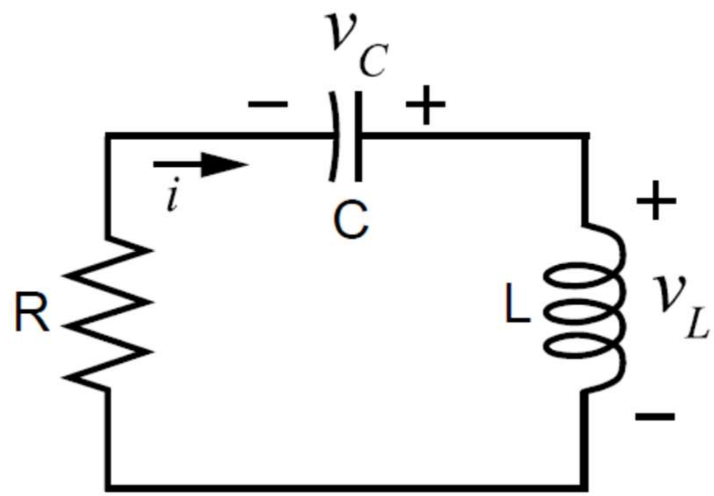
$$\left.\frac{dv}{dt}\right|_{t=0} = \sqrt{5}B_2 = \frac{i(0)}{C} = 1000 \text{ V/s}$$

To get the final expression for the natural response:

$$v(t) = (1000/\sqrt{5})e^{-2t}\sin\sqrt{5}t$$

RLC circuits II – Series RLC

Consider the following **series RLC circuit** without any sources, but with initial conditions for the inductor and capacitor.



This time we can use KVL around the loop to obtain:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{t_0}^{t} i dt - v_C(t_0) = 0$$

(note the direction of current and potential differences as given in the figure)
We can differentiate that equation to obtain:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

We can compare the result to that for the parallel RLC circuit:

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

Clearly we can apply all our results for the parallel circuit to the series circuit, however now in the series circuit we are considering current rather than voltage.

In a manner similar to the parallel circuit we assume a solution of the form:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

And we can find a characteristic equation which has solutions:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}$$

However, for the series circuit: $\boxed{\alpha = \frac{R}{2L}}$

ω_0 is the same as in the parallel circuit: $\omega_0 = \frac{1}{\sqrt{LC}}$

So the form of the ***overdamped*** response is : $i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

The form of the ***critically damped*** response is : $i(t) = e^{-\alpha t}(A_1 t + A_2)$

The form of the ***underdamped*** response is: $i(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$

Where like in the parallel circuit $\omega_d = \sqrt{\omega_0^{\,2} - \alpha^2}$

Note that the constants A_1, A_2 and B_1, B_2 need to be determined with initial conditions on the value of *i*(*t*) and its derivative, in a manner similar to the parallel circuit.

We will look at examples for the series circuit when looking at RLC circuits with sources.

RLC circuits with sources

Thus far we have just been considering the *natural or homogenous response of the RLC circuit*.

Next we consider the case of where *an independent source is included* in the circuit to *produce a forced response* of the circuit. In particular we will look at sources which are multiples of the *unit step function u(t).* So at time t=0 a DC source is effectively switched on.

The complete response of the circuit (arbitrarily assumed to be a voltage response) consists of a ***forced response*** which is constant for DC excitation:

$$v_f(t) = V_f$$

And the ***natural response*** (here the overdamped): $v_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

So the ***complete response is***: $v(t) = v_f(t) + v_n(t) = V_f + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

V_f, ω_0, α and whether we are dealing with an overdamped, critically damped or overdamped system can be obtained from the circuit.

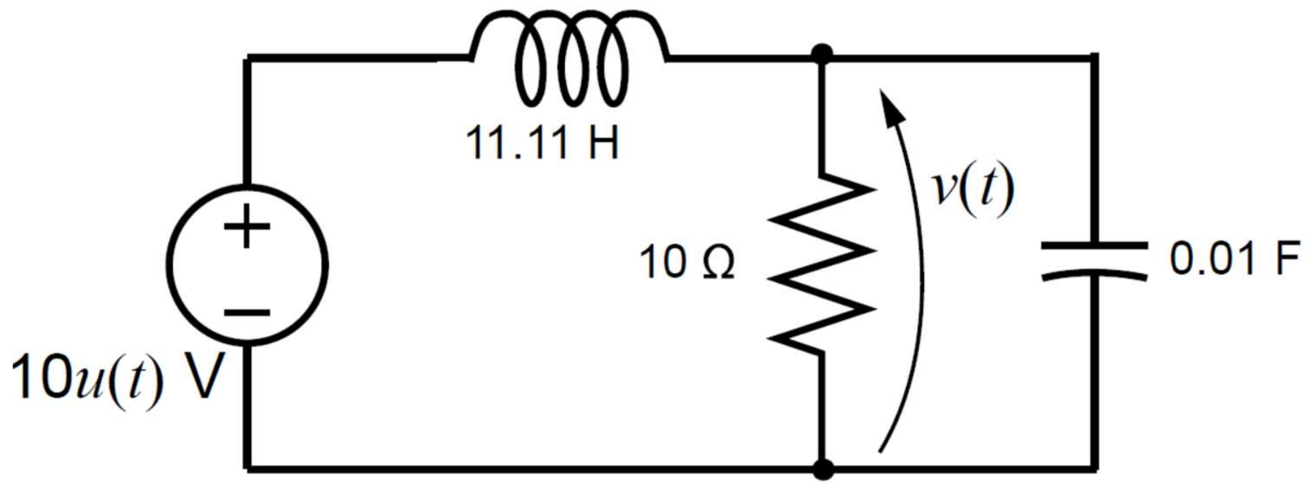
However, we still need to A_1, A_2 for the case of over and critically damped systems and B_1, B_2 for the case of underdamped systems.

These constants need to be found as before by using initial conditions in the circuit carefully to find v(t) and its derivative at $t=0^+$.

With these two values, two equations found which when solved simultaneously will yield the values of A_1, A_2 or B_1, B_2.

Example 1:

Find v(t) in the circuit below



Procedure

- Determine if it is a parallel or series RLC.
  - **For series RLC:** $$\boxed{\alpha = \frac{R}{2L}}$$
  - **For Parallel RLC:** $$\boxed{\alpha = \frac{1}{2RC}}$$
- From component parameters determine V_f, ω_0, α, s_1, s_2
- From ω_0, α determine the form of the solution: over, critically or underdamped
- Use initial conditions to find two equations to solve for to A_1, A_2 or B_1, B_2

Example (cont'd)

Parallel RLC circuit

α = 5, ω_0 = 3, s_1=-1, s_2= -9, V_f=10V

So overdamped natural response

Initial conditions: v(0+)=0, and dv/dt at t=0+ is also 0, due to inductor current being 0 at t=0+.

Two equations to find A_1, A_2 : $v(0^+) = 0 = 10 + A_1 + A_2$

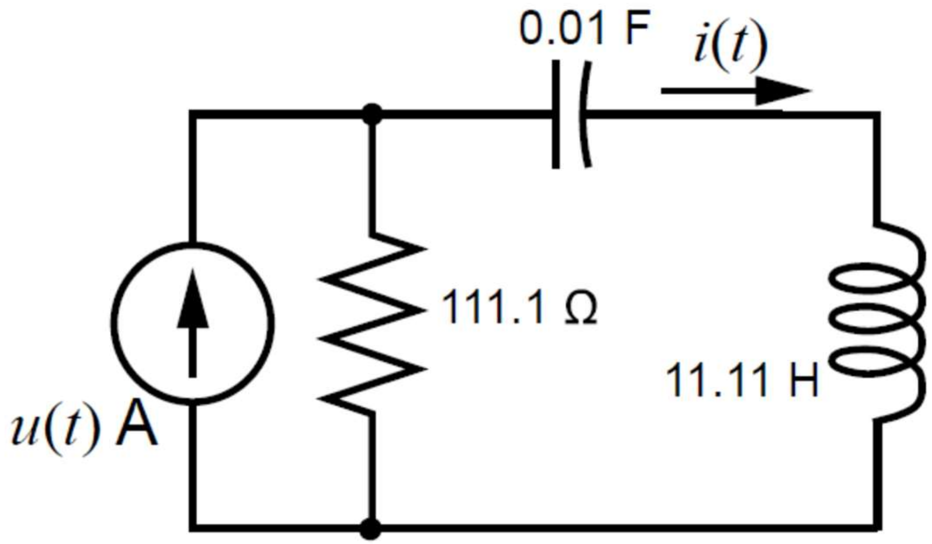
$$\left.\frac{dv}{dt}\right|_{t=0} = 0 = -A_1 - 9A_2$$

So A_1=-11.25, A_2=1.25

And for t>0: $v(t) = 10 - 11.25e^{-t} + 1.25e^{-9t}$

Example 2:

Find i(t) in the circuit below



For the example:

series RLC circuit

$\alpha = 5$, $\omega_0 = 3$, s_1=-1, s_2= -9, I_f=0A

So overdamped natural response

Initial conditions: i(0+)=0, due to capacitor blocking i.

di/dt at t=0+ is 10A/s , due to inductor voltage being Rx1A =111.1 V at t=0+.

Two equations to find A_1, A_2 : $$i(0^+) = 0 = A_1 + A_2$$

$$\left.\frac{di}{dt}\right|_{t=0} = 10 = -A_1 - 9A_2$$

So A_1=1.25, A_2=-1.25

And for t>0: $$i(t) = 1.25e^{-t} - 1.25e^{-9t}$$

RLC circuits Settling Time

Second order systems with a d.e. similar to the RLC we have discussed occur in many varied fields (electrical, control, mechanical engineering, physics) and also in many varied situations in nature.

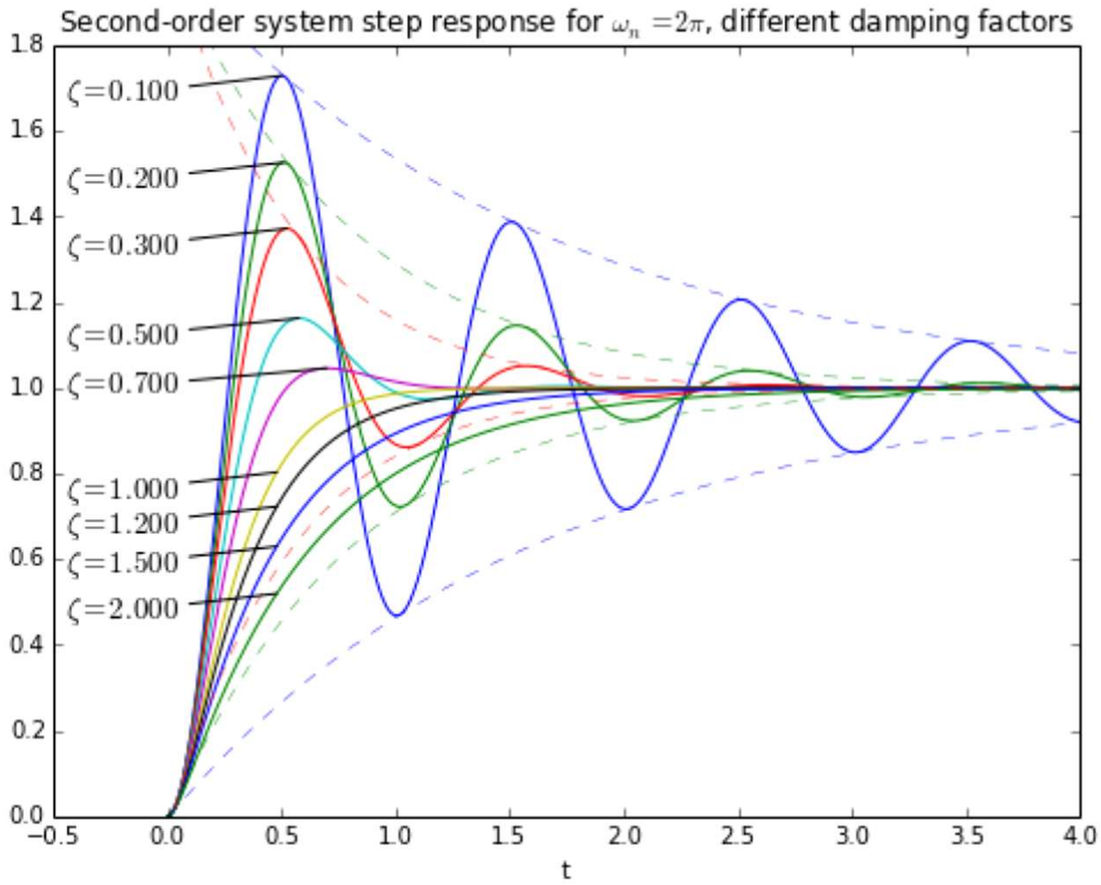
Often it is interesting to know how long the natural response takes to die out and for the forced response to become dominate. For example the response time of the system to a unit step input.

Since the natural response is a decaying exponential it never truly becomes zero. However, we can define a settling time as the time natural response dies away to small percentage of its maximum value, say 1% or 5%.

The fastest settling time without the transient response overshooting the forced value of the system is found when the system is critically damped.

i.e. $\zeta = \alpha / \omega_0 = 1$

The graph below gives you an idea of the response of a second order system to a unit step input at t=0, for varying values of ζ. It shows underdamped, critically damped and overdamped responses.



In the next lecture slides we will look at using a sinusoidal forcing function and develop the *frequency domain* analysis of linear circuits.

Acknowledgments

- Lecture slides are based on lecture materials from various sources, including M. Hill, T. Cantoni, F. Boussaid, and R. Togneri.

- Credit is acknowledged where credit is due. Please refer to the full list of references.