Tutorial 3 (Solutions)

1. Consider sampling the signal $x(t) = \frac{1}{\pi t}\sin(2\pi t)$. Sketch the FT of the sampled signal for the following sampling intervals:

(a) $T = \frac{1}{8}$; (b) $T = \frac{1}{3}$; (c) $T = \frac{1}{2}$; (d) $T = \frac{2}{3}$

Answers:

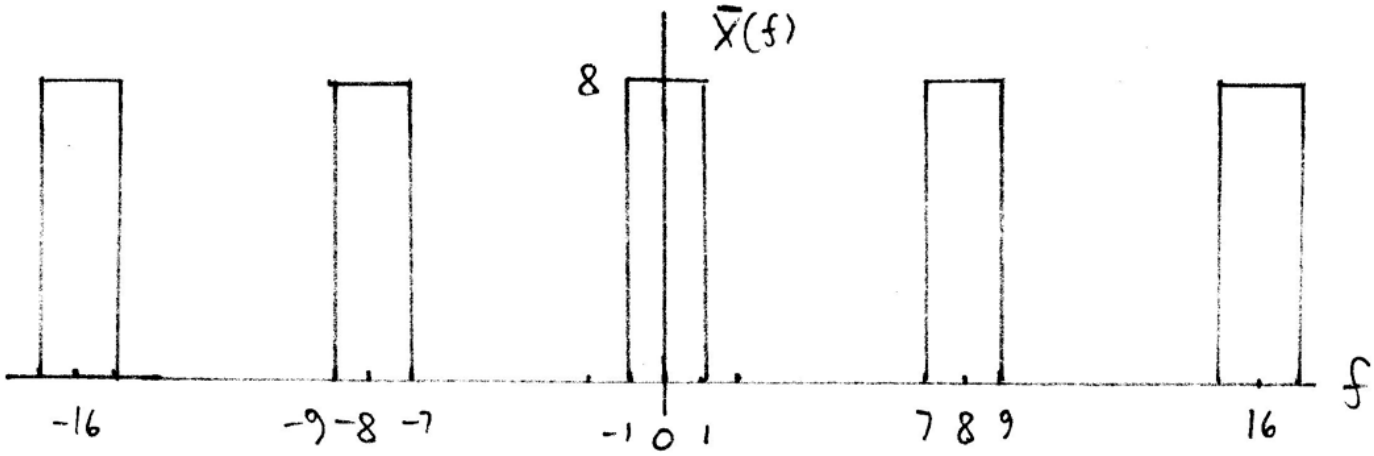
(a) First determine the FT spectrum of $x(t) = \frac{1}{\pi t}\sin(2\pi t)$. From the FT table we have:

$$\frac{W}{\pi}\frac{\sin(Wt)}{Wt} = \frac{\sin(Wt)}{\pi t} \overset{FT}{\Longleftrightarrow} \begin{cases} 1 & |\omega| < W \\ 0 & \text{otherwise} \end{cases}$$

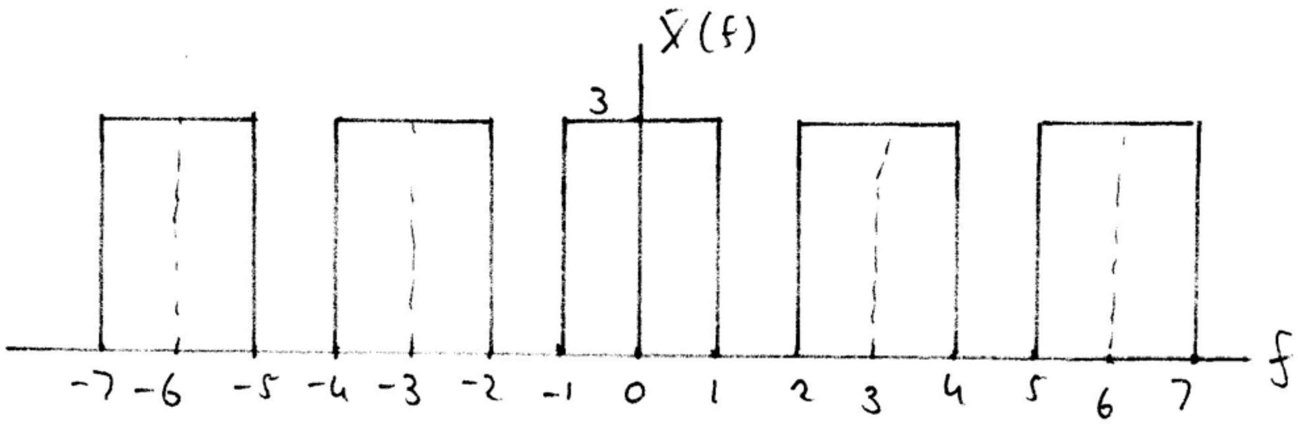
With $W = 2\pi$ the FT spectrum exists over the range: $-2\pi < \omega < 2\pi$ or $-1 < f < 1$.

For each case below the FT spectrum will be given by: $f_s \sum_{k=-\infty}^{\infty} X(f - kf_s)$

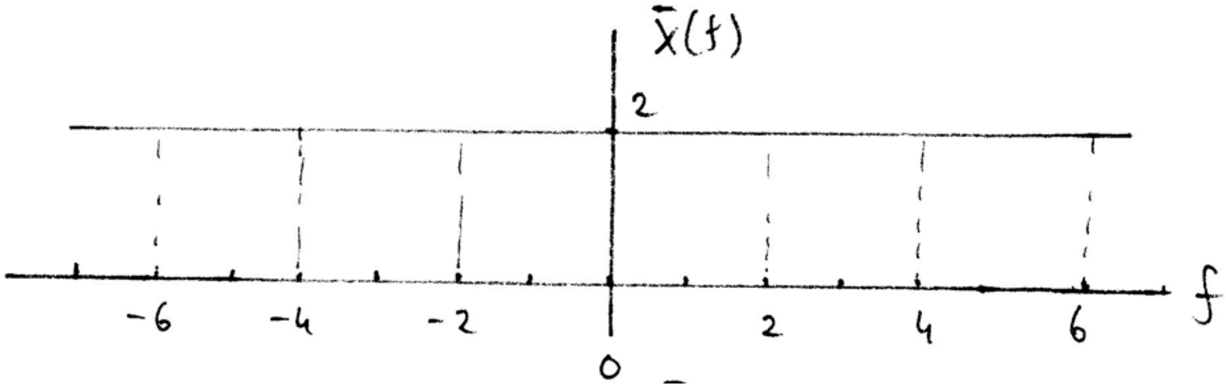
(a) With $T = \frac{1}{8} \to f_s = 8$ hence:



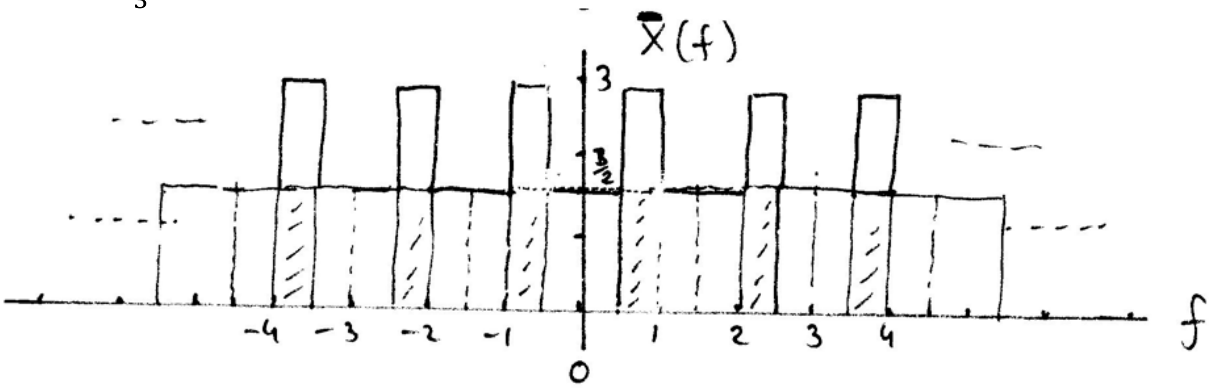
(b) With $T = \frac{1}{3} \to f_s = 3$ hence:



(c) With $T = \frac{1}{2} \rightarrow f_s = 2$ hence:



(d) With $T = \frac{2}{3} \rightarrow f_s = 1.5$ hence we have aliasing as shown:



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2. For each of the following signals, sampled with sampling interval T_s, determine the bounds on T_s, which guarantee that there will be no aliasing.

(a) $x(t) = \frac{1}{t}\sin(3\pi t) + \cos(2\pi t)$

(b) $x(t) = \cos(12\pi t)\frac{\sin(\pi t)}{2t}$

(c) $x(t) = e^{-6t}u(t) * \frac{\sin(Wt)}{\pi t}$

Answers:

(a) $x(t) = \frac{1}{t}\sin(3\pi t) + \cos(2\pi t)$

$$\frac{\sin(3\pi t)}{t} \overset{FT}{\Longleftrightarrow} \begin{cases} \pi & |\omega| < 3\pi \\ 0 & \text{otherwise} \end{cases}$$

$$\cos(2\pi t) \overset{FT}{\Longleftrightarrow} \pi[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$$

$$\omega_m = 3\pi \rightarrow f_m = 1.5$$

$$f_s > 2f_m \rightarrow f_s > 3 \rightarrow T_s < \frac{1}{3}$$

(b) $x(t) = \cos(12\pi t)\frac{\sin(\pi t)}{2t}$

$$\cos(12\pi t) \overset{FT}{\Longleftrightarrow} \pi[\delta(\omega - 12\pi) + \delta(\omega + 12\pi)]$$

$$\frac{\sin(\pi t)}{2t} \overset{FT}{\Longleftrightarrow} \begin{cases} \pi/2 & |\omega| < \pi \\ 0 & \text{otherwise} \end{cases}$$

$$x_1(t)x_2(t) \overset{FT}{\Longleftrightarrow} \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

$$X(j\omega) = \begin{cases} \pi/4 & |\omega - 12\pi| < \pi \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \pi/4 & |\omega + 12\pi| < \pi \\ 0 & \text{otherwise} \end{cases}$$

This implies the positive sided spectrum will span $11\pi < \omega < 13\pi$ amd hence:

$$\omega_m = 13\pi \rightarrow f_m = 6.5$$

$$f_s > 2f_m \rightarrow f_s > 13 \rightarrow T_s < \frac{1}{13}$$

(c) $x(t) = e^{-6t}u(t) * \frac{\sin(Wt)}{\pi t}$

$$e^{-6t}u(t) \overset{FT}{\Longleftrightarrow} \frac{1}{6 + j\omega}$$

$$\frac{\sin(Wt)}{\pi t} \overset{FT}{\Longleftrightarrow} \begin{cases} 1 & |\omega| < W \\ 0 & \text{otherwise} \end{cases}$$

$$x_1(t) * x_2(t) \overset{FT}{\Leftrightarrow} X_1(j\omega)X_2(j\omega)$$

$$X(j\omega) = \begin{cases} \frac{1}{6 + j\omega} & |\omega| < W \\ 0 & \text{otherwise} \end{cases}$$

Hence:

$$\omega_m = W \rightarrow f_m = W/2\pi$$

$$f_s > 2f_m \rightarrow f_s > \frac{W}{\pi} \rightarrow T_s < \frac{\pi}{W}$$

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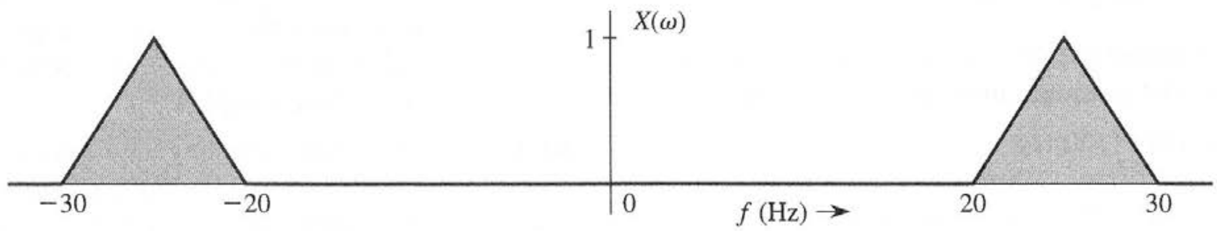
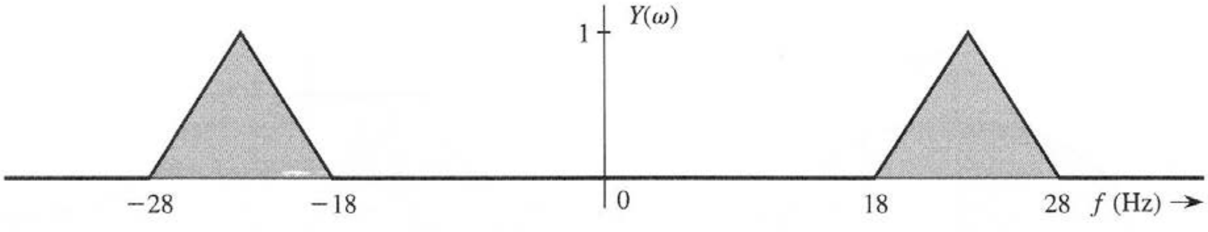
3. Consider the signal $x(t)$ with spectrum as shown below:



(a) Since the highest frequency is 30 Hz the minimum sampling frequency is 60 Hz. Show how you can reconstruct $x(t)$ from these samples.

(b) A certain student looks at $X(\omega)$ and concludes the bandwidth is really 10 Hz and decides that the sampling rate 20 Hz is adequate. Sketch the sampled spectrum and determine whether if there is any way to reconstruct $x(t)$.

(c) The same student now looks at $Y(\omega)$ below and using the same reasoning also concludes a sampling rate of 20 Hz is sufficient to sample $y(t)$. Sketch the sampled spectrum and determine whether if there is any way to reconstruct $y(t)$.



Answers:

Figure S8.1-8a shows $X(\omega)$, which is a bandpass spectrum with bandwidth 10 Hz and band centred at 25 Hz. The highest frequency is 30 Hz. If we use 60 Hz sampling frequency, the spectrum will shift by $\pm 60n (n = 1, 2, 3, \cdots)$ as shown in Figure S8.1-8b. The negative frequency spectrum is labeled P_0 and the positive frequency spectrum is labeled Q_0. Both these segments repeat with period 60 Hz. Let us label P_k as the segment P_0 shifted by 60k Hz and P_{-k} is P_0 shifted by -60k Hz. Similarly Q_k and Q_{-k} represent Q_0 shifted by $\pm$60k Hz. The sampled signal spectrum in Figure S8.1-8b shows the labels of various repeated segments. It is clear from Figure S8.1-8b that we can reconstruct the signal $x(t)$ from this spectrum by passing it through an ideal bandpass filter (shown dotted) centered at 25 Hz and having bandwidth 10Hz. The filter gain is $T = 1/60$.

Figure S8.1-8c shows the sampled signal spectrum with f_s =20 Hz (spectrum repeating at every 20 Hz). It is clear from this figure that we can reconstruct $x(t)$ from this spectrum also because the original segments P_0 and Q_0 are still intact without being overlapped by any other repeating segments.

Figure S8.1-8d shows the spectrum of the signal $y(t)$ sampled at a rate 20 Hz (spectrum repeating at every 20 Hz). The figure shows that P_0 overlaps with Q_{-2} and Q_0 overlaps with P_2. Hence it is impossible to reconstruct $y(t)$ from this spectrum.

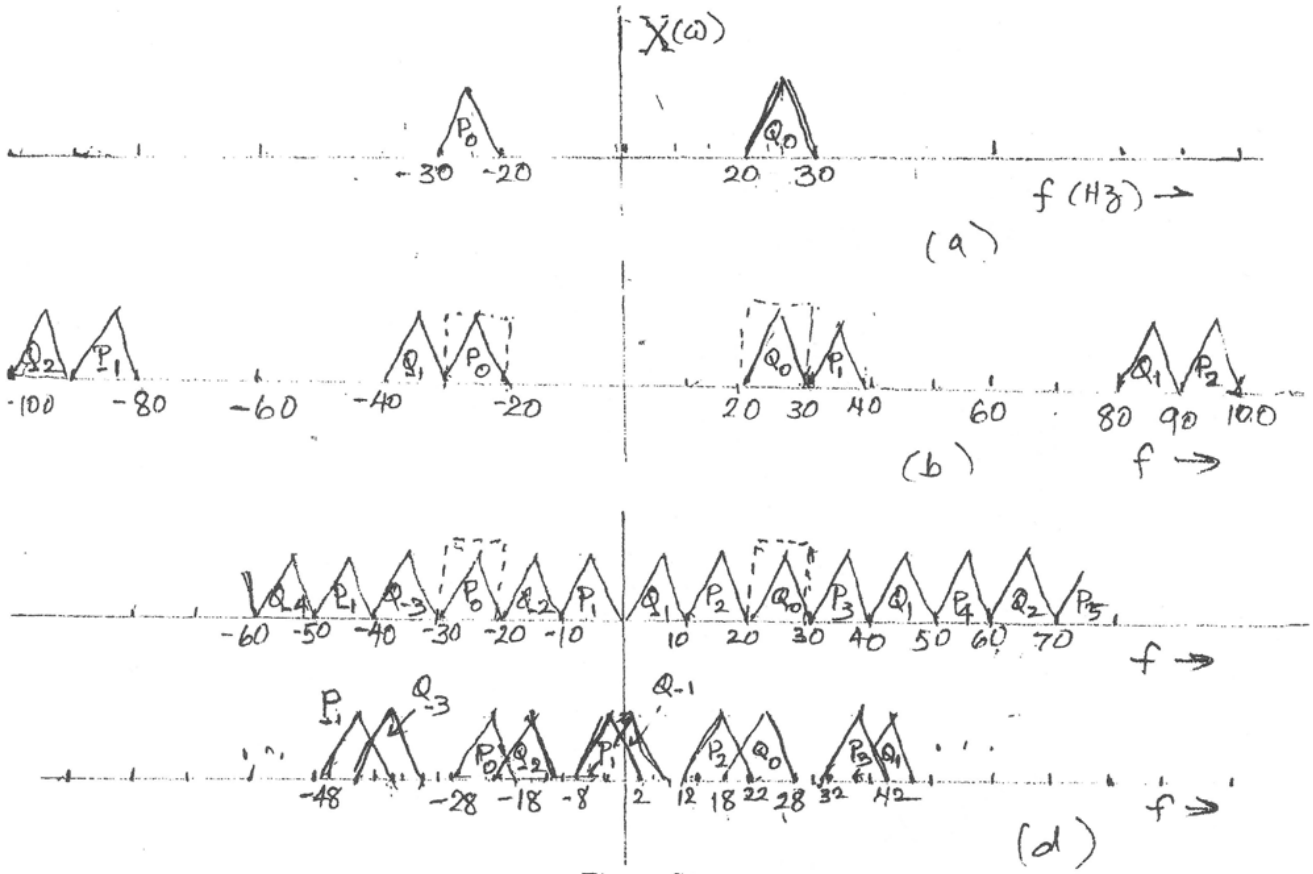


Figure S8.1-8

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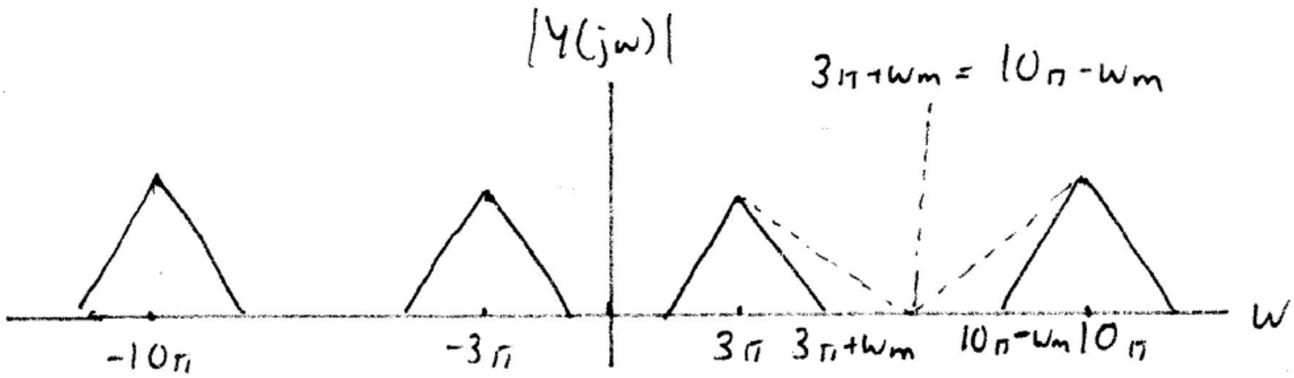
4. Let $|X(j\omega)| = 0$ for $|\omega| > \omega_m$. Form the signal $y(t) = x(t)[\cos(3\pi t) + \sin(10\pi t)]$. Determine the maximum value of ω_m for which $x(t)$ can be reconstructed from $y(t)$, and specify a system that will perform the reconstruction.

Answers:

We can easily see that:

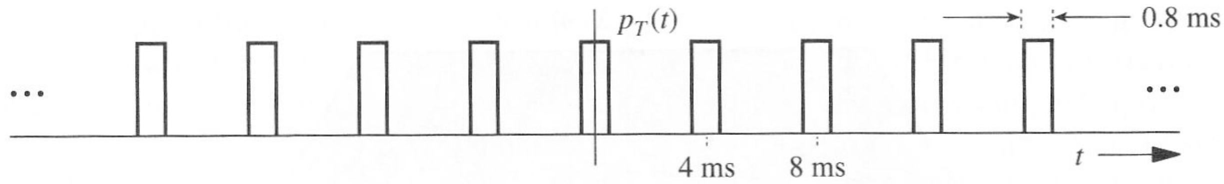
$$Y(j\omega) = \frac{1}{2}[X(j(\omega - 3\pi)) + X(j(\omega + 3\pi)) - jX(j(\omega - 10\pi)) + jX(j(\omega + 10\pi))]$$

And can sketch the spectrum as:



We require $10\pi - \omega_m > 3\pi + \omega_m$ to avoid overlap and permit reconstruction, that is $\omega_m < 3.5\pi$. Since this implies there may be overlap between $X(j(\omega - 3\pi))$ and $X(j(\omega + 3\pi))$ reconstruction proceeds by bandpass filtering the $jX(j(\omega - 10\pi))$ component, using a filter with bandpass $6.5\pi \le \omega \le 13.5\pi$ and then bringing to the baseband with multiplication with $2\sin(10\pi t)$.

5. A signal $x(t) = \text{sinc}(200\pi t) = \frac{\sin(200\pi t)}{200\pi t}$ is sampled (multiplied) by a periodic pulse train $p_T(t)$ shown below. Find and sketch the spectrum of the sampled signal. Will you be able to reconstruct the signal? Find the filter output if the sampled signal is passed through an ideal lowpass filter of bandwidth 100 Hz and unit gain. What happens for other lowpass filter bandwidths > 100 Hz?



Answers:

. The signal $x(t) = \text{sinc}(200\pi t)$ is sampled by a rectangular pulse sequence $p_T(t)$ whose period is 4 ms so that the fundamental frequency (which is also the sampling frequency) is 250 Hz. Hence, $\omega_s = 500\pi$. The Fourier series for $p_T(t)$ is given by

$$p_T(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\omega_s t$$

Use of the FS eqns yields $C_0 = \frac{1}{5}$, $C_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{5}\right)$, that is,

$$C_0 = 0.2, \quad C_1 = 0.374, \quad C_2 = 0.303, \quad C_3 = 0.202, \quad C_4 = 0.093, \quad C_5 = 0, \cdots$$

Consequently

$$\overline{x}(t) = x(t)p_T(t) = 0.2\,x(t) + 0.374x(t)\cos 500\pi t + 0.303\,x(t)\cos 1000\pi t + 0.202\,x(t)\cos 1500\pi t + \cdot$$

and

$$\begin{aligned}\overline{X}(\omega) &= 0.2\,X(\omega) + 0.187\,[X(\omega - 500\pi) + X(\omega + 500\pi)] \\ &\quad + 0.151\,[X(\omega - 1000\pi) + X(\omega + 1000\pi)] \\ &\quad + 0.101\,[X(\omega - 1500\pi) + X(\omega + 1500\pi)] + \cdots\end{aligned}$$

In the present case $X(\omega) = 0.005\,\text{rect}(\frac{\omega}{400\pi})$. The spectrum $\overline{X}(\omega)$ is shown in Figure S8.2-1. Observe that the spectrum consists of $X(\omega)$ repeating periodically at the interval of 500π rad/s (250 Hz). Hence, there is no overlap between cycles, and $X(\omega)$ can be recovered by using an ideal lowpass filter of bandwidth 100 Hz. An ideal lowpass filter of unit gain (and bandwidth 100 Hz) will allow the first term on the he right-side of the above equation to pass fully and suppress all the other terms. Hence the output $y(t)$ is

$$y(t) = 0.2\,x(t)$$

Because the spectrum $\overline{X}(\omega)$ has a zero value in the band from 100 to 150 Hz, we can use an ideal lowpass filter of bandwidth B Hz where $100 < B < 150$. But if $B > 150$ Hz, the filter will pick up the unwanted spectral components from the next cycle, and the output will be distorted.

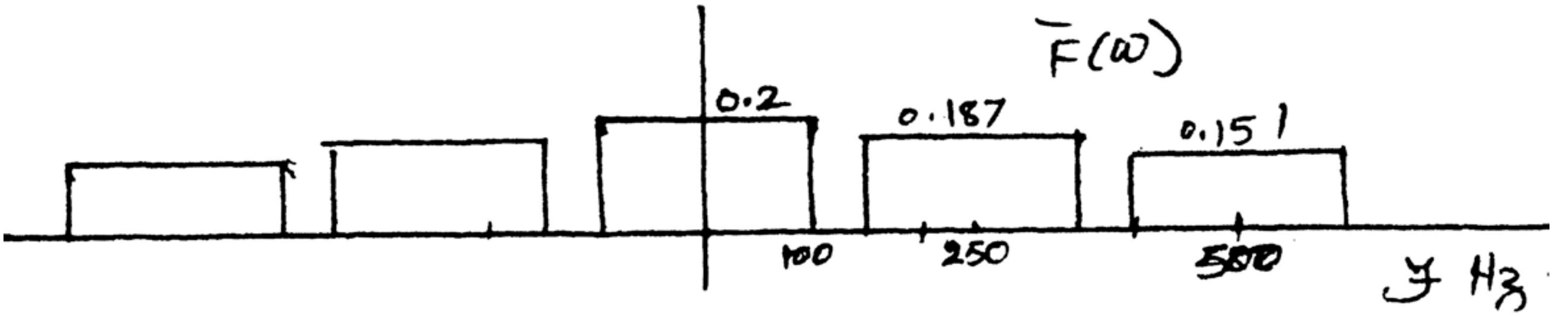


Figure S8.2-1

For the even pulse function the FS eqns are ($C_n \equiv A[k], n \equiv k$):

$$x(t) = A[0] + \sum_{k=1}^{\infty} A[k]\cos(k\omega_s t)$$

$$A[0] = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$$A[k] = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_s t)\,dt$$

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6. A TV signal (video and audio) has a bandwidth of 4.5 MHz. This signal is sampled, quantized and binary coded.

(a) Determine the sampling rate of the signal is to be sampled at a rate 20% above the Nyquist rate?

(b) If the samples are quantized into 1024 levels, what number of binary pulses is required to encode each sample?

(c) Determine the binary pulse rate (bps) of the binary-coded signal.

Answers:

(a) The Nyquist rate is $2 \times 4.5 \times 10^6 = 9$ MHz. The actual sampling rate is $1.2 \times 9 = 10.8$ MHz.

(b) $1024 = 2^{10}$, so that 10 bits or binary pulses are needed to encode each sample.

(c) $10.8 \times 10^6 \times 10 = 108 \times 10^6$ or 108 Mbps.

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7. For a signal $x(t)$ that is timelimited to 10 ms and has an essential bandwidth of 10 kHz, determine N_0 the number of signal samples necessary to compute a power-of-2 FFT with a frequency resolution f_0 of at least 50 Hz. Explain whether any zero padding is necessary.

Answers:

$$T_0 = \frac{1}{f_0} = \frac{1}{50} = 20 \text{ ms}$$

$$B = 10 \text{ kHz}, \ \text{hence } f_s \geq 2B = 20000$$

$$T = \frac{1}{f_s} = \frac{1}{20000} = 50\,\mu s$$

Thus:

$$N_0 = \frac{T_0}{T} = \frac{20 \times 10^{-3}}{50 \times 10^{-6}} = 400$$

Since N_0 must be a power of 2 we choose $N_0 = 512$. Since $T = 50\,\mu s$ and $T_0 = N_0 T = 512 \times 50\mu s = 25.6$ ms, then $f_0 = 1/T_0 = 39.0625$ Hz which is better than 50 Hz. Since $x(t)$ is of 10ms duration, we need zero padding over 15.6 ms (to ensure $T_0 = 25.6$ ms)