Lecture 4

The S / Laplace Domain

Revisit – frequency domain

In the ***Frequency Domain*** we showed we could represent circuit elements with an impedance which was a ***complex number*** evaluated at the frequency of the sinusoidal source in the circuit.

Using the phasor representation for currents and voltages, and KVL and KCL we could generate algebraic equations describing the circuit, and solve these for any unknown voltage or current phasors.

We could then transform the required voltage or current phasors back to the time domain to find the ***sinusoidal steady state response***.

So by transforming in and out of the frequency domain we can avoid having to directly solve the complex differential equations in the time domain.

Why Laplace domain?

In the ***Laplace Domain*** we can represent circuit elements with an impedance which is a ***complex function*** of the complex variable ***s***.

We can then use KVL and KCL to generate algebraic equations in ***s*** describing the circuit. Independent sources are transformed in to the Laplace domain as function of ***s***. We can solve these equations for any unknown voltage or current and obtain it as a function of ***s***.

We could then transform the required voltage or current back to the time domain to find ***complete*** response of the circuit, which includes both the natural (homogeneous) and forced response.

Again by transforming in and out of the Laplace domain we can avoid having to directly solve the complex differential equations in the time domain.

Laplace transform is a mathematical approach to make complicate circuit calculations in frequency domain to be simple!

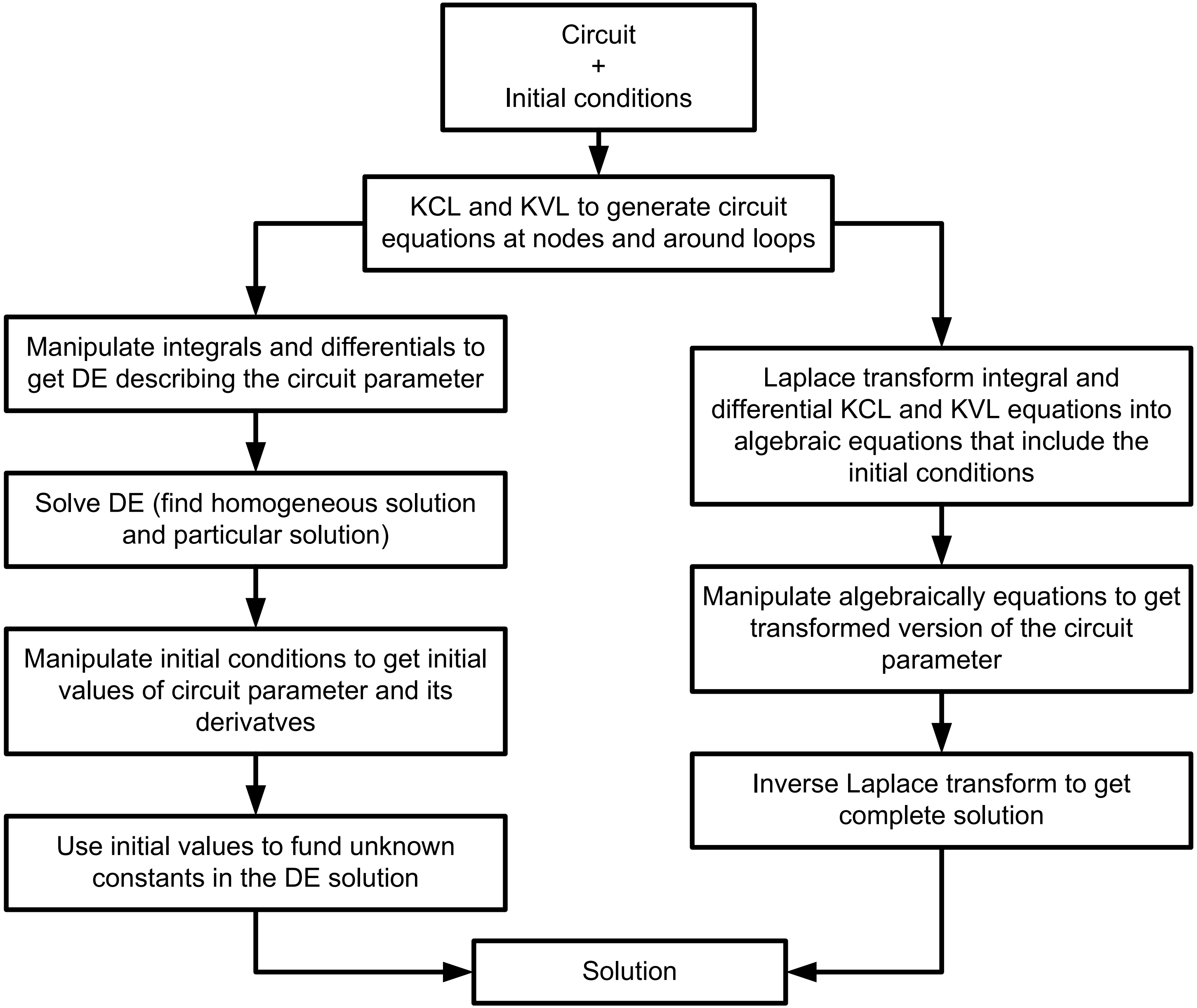
However by using the Laplace transform we have a number of ***advantages*** over the frequency domain:

1. Can include initial conditions

2. Obtain both the natural and forced response

3. Increased range of forcing (sources) functions available.

Laplace transform approach to circuit analysis



Laplace transform

The Laplace transform is an integral transformation of a function *f*(*t*) *from the time domain into the complex frequency domain*, giving *F*(*s*).

$$F(s) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

, where ***s*** is a complex variable given by **s=σ+jω**.

The inverse Laplace transform of F(s) is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{j2\pi}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds$$

However, we generally don’t use the inverse one directly, rather we often use a table lookup method, which be detailed later.

Some important properties of Laplace transform

1) Linearity:

$$\mathcal{L}[af_1(t) + bf_2(t)] = a\mathcal{L}[f_1(t)] + b\mathcal{L}[f_2(t)]$$

2) Time differentiation (one sided transform):

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

This is an important result and can be shown using integration by parts:

$$\int_{0^-}^{\infty} u.\,dv = u.\,v|_{0^-}^{\infty} - \int_{0^-}^{\infty} v.\,du$$

$$u = e^{-st}, \qquad du = -se^{-st}dt, \;\; dv = \frac{df}{dt}dt, \qquad v = f$$

$$\mathcal{L}\left[\frac{df}{dt}\right] = e^{-st}f(t)|_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}dt = sF(s) - f(0^-)$$

This can be extended to higher time derivatives:

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} \left.\frac{df}{dt}\right|_{t=0} - \cdots - \left.\frac{d^{n-1} f}{dt^{n-1}}\right|_{t=0}$$

3) Time integration (one sided transform): $\mathcal{L}\left[\int_{0^-}^{t} f(t')dt'\right] = F(s)/s$

Can be found using integration by parts:

$$\mathcal{L}\left[\int_{0^-}^{t} f(t')dt'\right] = \int_{0^-}^{\infty}\left(\int_{0^-}^{t} f(t')dt'\right)e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\int_{0^-}^{t} f(t')dt'\Big|_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

$$= \frac{F(s)}{s}$$

4) Scaling : $$\mathcal{L}[f(at)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

5) Time shift : $$\mathcal{L}[f(t-\tau)] = F(s)e^{-\tau s}$$

6) Frequency shift : $$\mathcal{L}\left[e^{-at} f(t)\right] = F(s+a)$$

7) Initial value: $$\lim_{s\to\infty} sF(s) = f(0)$$

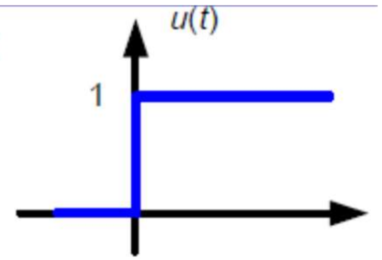
Can be found using time derivative property and is important because allows us to find/check initial value of *f*(*t*) without having to do inverse transform.

8) Final value: $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Also can be found using time derivative property and is important because allows us to find/check final value of *f*(*t*) without having to do inverse transform.

Laplace transform pairs (examples)

1) The Laplace transform of the **unit step function *u*(*t*)** is:



$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_{0^-}^{\infty} = \frac{1}{s}$$

2) The Laplace transform of an exponential:

$$\mathcal{L}[e^{at}] = \int_{0^-}^{\infty} e^{at}e^{-st}\,dt = -\frac{1}{s-a}e^{-(s-a)t}\Big|_{0^-}^{\infty} = \frac{1}{s-a}$$

3) The unit impulse function δ(*t*) :

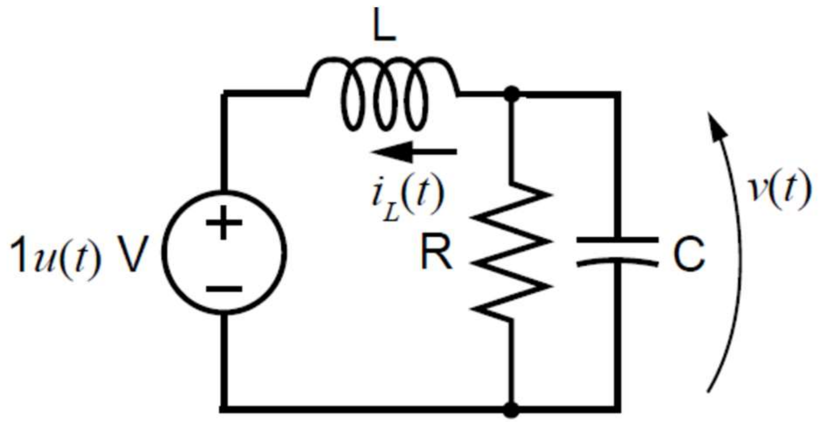
$$L[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = e^{-0} = 1$$

Table of Laplace transform pairs (important!!!)

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$u(t)$	$\frac{1}{s}$	$\left(\frac{1}{\omega}\sin\omega t\right)u(t)$	$\frac{1}{s^2+\omega^2}$
$t.u(t)$	$\frac{1}{s^2}$	$\cos\omega t.u(t)$	$\frac{s}{s^2+\omega^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$(1-\cos\omega t)u(t)$	$\frac{\omega^2}{s(s^2+\omega^2)}$
$e^{at}.u(t)$	$\frac{1}{s-a}$	$\sin(\omega t+\theta).u(t)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$t.e^{at}.u(t)$	$\frac{1}{(s-a)^2}$	$\cos(\omega t+\theta).u(t)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$\frac{t^{n-1}}{(n-1)!}e^{at}u(t)$	$\frac{1}{(s-a)^n}$	$e^{-\alpha t}\sin\omega t.u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$\frac{1}{(a-b)}\left(e^{at}-e^{bt}\right)u(t)$	$\frac{1}{(s-a)(s-b)}$	$e^{-\alpha t}\cos\omega t.u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
$\left[\frac{e^{-at}}{(b-a)(c-a)}+\frac{e^{-bt}}{(a-b)(c-b)}+\frac{e^{-ct}}{(a-c)(b-c)}\right]u(t)$	$\frac{1}{(s+a)(s+b)(s+c)}$	$\sinh\alpha t.u(t)$	$\frac{\alpha}{s^2-\alpha^2}$
$\left(1-e^{at}\right)u(t)$	$\frac{-a}{s(s-a)}$	$\cosh\alpha t.u(t)$	$\frac{s}{s^2-\alpha^2}$

Example to RLC circuits

Consider the following parallel RLC circuit with a source:



We can apply KCL at the top node to find the d.e. describing $v(t)$.

$$\frac{v}{R} + \frac{1}{L}\int_0^t (v(t) - u(t))dt + i_L(0) + C\frac{dv}{dt} = 0$$

Differentiating this obtain the second order d.e.:

$$LC\frac{d^2v(t)}{dt^2} + \frac{L}{R}\frac{dv(t)}{dt} + v(t) = u(t)$$

Procedure with Laplace transform

We could apply Laplace transform techniques to convert this d.e. into an algebraic equation of s, including initial conditions on the capacitor and inductor.

Initial conditions are included in the transform of derivatives.
We could then solve for v(s), and convert the v(s) back to the time domain to obtain v(t).

Note that we have a Laplace transform for the driving function u(t), which in the frequency domain was not available

However, much as we did for phasors, we can define ***impedances*** for the components in the circuit and then use KCL, KVL and other techniques to write down the algebraic equation in s which describes the circuit directly.

Complex Frequency

We can consider the variable ***s*** (s = σ + jω) to represent a complex frequency, where σ and ω are both real.

If we have $v(t) = \mathrm{Re}(V_0 e^{st})$

Then if s is purely real i.e. s= σ, v(t) is an exponential: $v(t) = V_0 e^{\sigma t}$

if s= 0, v(t) is a constant: $v(t) = V_0$

If s is purely imaginary i.e. s= jω , v(t) is a sinusoid:

$$v(t) = \mathrm{Re}(V_0 e^{j\omega t}) = V_0 \cos(\omega t)$$

If s has both real and imaginary components then v(t) is a damped sinusoid:

$$v(t) = \mathrm{Re}(V_0 e^{\sigma} e^{j\omega t}) = V_0 e^{\sigma} \cos(\omega t)$$

We will talk more about the complex frequency later in the frequency response.

Laplace domain models

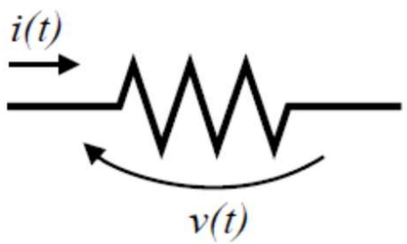
Resistor:

In the time domain:

$$v(t) = Ri(t)$$

$$\frac{v(t)}{i(t)} = R$$

$$\frac{i(t)}{v(t)} = \frac{1}{R} = G$$

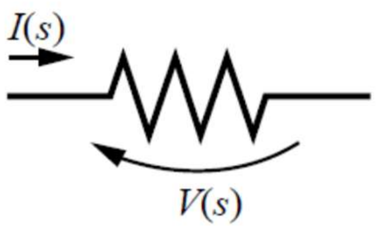


In the Laplace domain:

$$V(s) = RI(s)$$

$$\frac{V(s)}{I(s)} = R$$

$$\frac{I(s)}{V(s)} = \frac{1}{R} = G$$



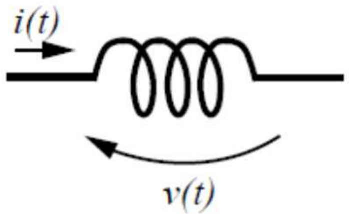
So impedance is: $Z_R(s) = R$ And admittance is: $Y_R(s) = G$

Inductor:

In the time domain:

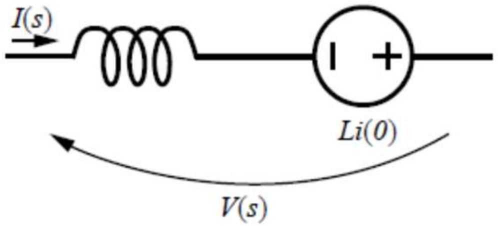
$$v(t) = L\frac{di(t)}{dt}$$

$$\int_0^t v(t)dt = Li(t) - Li(0)$$



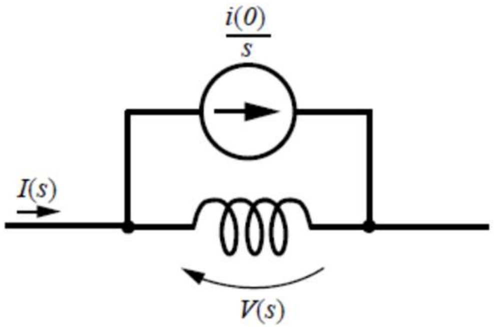
In the Laplace domain,

firstly from v(t) equation:



$$V(s) = sLI(s) - Li(0)$$

Secondly from i(t) equation:

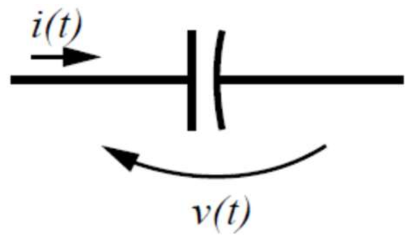


$$I(s) = \frac{1}{sL}V(s) + \frac{i(0)}{s}$$

So impedance is: $Z_L(s) = sL$, admittance is: $Y_L(s) = \frac{1}{sL}$

Capacitor:

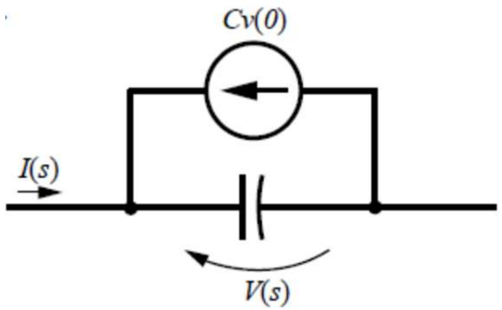
In the time domain: $$i(t) = C\frac{dv(t)}{dt}$$



$$\int_0^t i(t)dt = C\left(v(t) - v(0)\right)$$

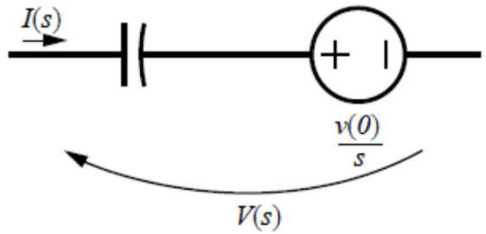
In the Laplace domain,

firstly from i(t) equation:



$$I(s) = sCV(s) - Cv(0)$$

Secondly from v(t) equation:



$$V(s) = \frac{1}{sC}I(s) + \frac{v(0)}{s}$$

So impedance is: $Z_C(s) = \dfrac{1}{sC}$, admittance is: $Y_C(s) = sC$

Note that ***initial conditions*** are incorporated into the Laplace domain model as independent sources.

Furthermore all the *V(s) - I(s)* relationships are algebraic and also linear.

The Laplace domain *impedances for capacitors and inductors are complex* functions. We can use the impedances as we do for resistors and Ohms Law.

It can be shown that KCL and KVL apply for voltages and currents in the s-domain. Furthermore when there are only linear elements in the circuit (or the part of the circuit being considered) then the superposition principle, and Thevenin's and Norton's theorems also are valid.

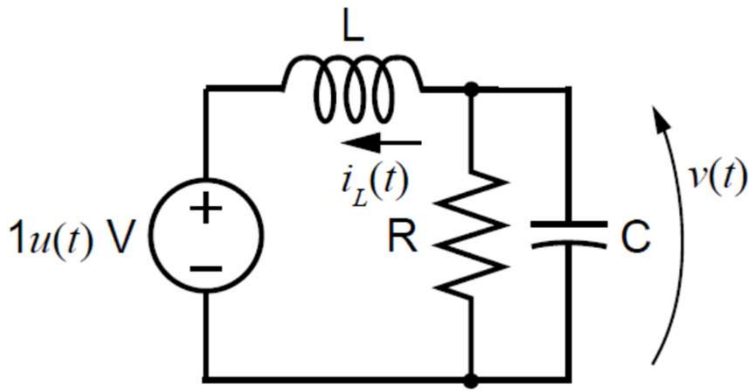
Furthermore, we can combine series and parallel impedances as we did in the frequency domain. However, note that the impedances being combined are the ones in the Laplace domain, so initial conditions have already been taken out as independent sources.

For series impedances: $$\mathbf{Z}_{eq}(s) = \mathbf{Z}_1(s) + \mathbf{Z}_2(s) + \cdots + \mathbf{Z}_N(s)$$

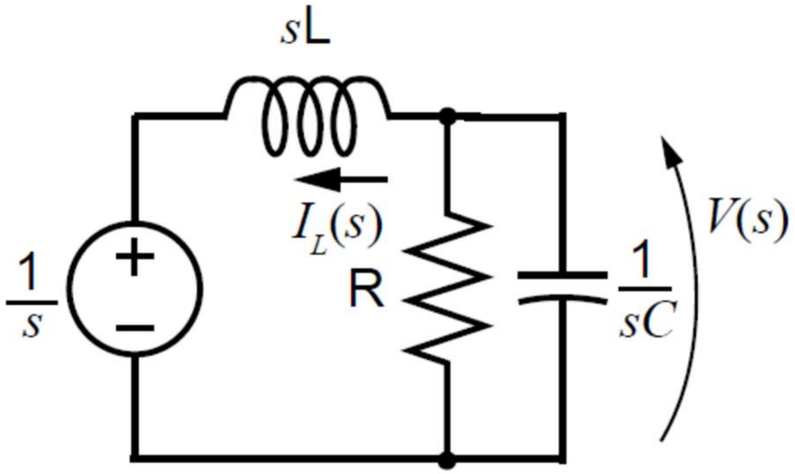
For parallel impedances: $$1/\mathbf{Z}_{eq}(s) = 1/\mathbf{Z}_1(s) + 1/\mathbf{Z}_2(s) + \cdots + 1/\mathbf{Z}_N(s)$$

Example

Lets look at the RLC circuit example. Assume for the moment all initial conditions are zero.



Converting directly to Laplace domain:



Using KCL obtain:

$$sCV(s) + \frac{1}{R}V(s) + \frac{1}{sL}(V(s) - \frac{1}{s}) = 0$$

$$V(s) = \frac{1}{s(LCs^2 + \frac{L}{R}s + 1)}$$

So we could easily obtain V(s).

We can check for final and initial values of v(t) without having to find the inverse transform:

$$v(0) = \lim_{s\to\infty} sV(s) = \frac{1}{(LCs^2 + \frac{L}{R}s + 1)} = 0$$

$$v(\infty) = \lim_{s\to 0} sV(s) = \frac{1}{(LCs^2 + \frac{L}{R}s + 1)} = 1$$

Finding v(t) from V(s) is usually performed via partial fraction expansion and using a table of transforms to find the v(t) for each fraction.

We assume F(s) for which we want to find f(t) has the form:

$$F(s)=\frac{a_0+a_1s+a_2s^2+\cdots+a_ms^m}{b_0+b_1s+b_2s^2+\cdots+b_ns^n}$$

with m<n.

We first need to find the roots of the denominator of F(s), p1,p2,…pn which are the poles of F(s). We can then write F(s) as:

$$F(s)=\frac{a_0+a_1s+a_2s^2+\cdots+a_ms^m}{b_n(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$=\frac{c_1}{s-p_1}+\frac{c_2}{s-p_2}+\cdots+\frac{c_n}{s-p_n}$$

The coefficients c_i can be found for non repeated poles as:

$$c_i = (s - p_i)F(s)\Big|_{s=p_i}$$

Lets look at obtaining v(t) for the example of the parallel RLC circuit we just looked at:

$$V(s) = \frac{1}{s(LCs^2 + \frac{L}{R}s + 1)}$$

$$= \frac{1}{LCs(s^2 + \frac{1}{RC}s + \frac{1}{LC})}$$

Say R=1 Ohm, C=0.2F, L=1/1.2 H

Then: $$V(s) = \frac{6}{s(s^2 + 5s + 6)} = \frac{6}{s(s+2)(s+3)}$$

So have poles at s=0,-2,-3.

For s=0 pole find coefficient c1=1.

For s=-2 pole find coefficient c2=-3.

For s=-3 pole find coefficient c3=2.

$$V(s) = \frac{1}{s} - \frac{3}{(s+2)} + \frac{2}{(s+3)}$$

Using the table of transform pairs can get:

$$v(t) = (1 - 3e^{-2t} + 2e^{-3t})u(t)$$

Another example

If say $$V(s) = \frac{2}{s(s^2+2s+2)} = \frac{2}{s(s+1-j)(s+1+j)}$$

Then we have a pair of complex conjugate poles. We can best try to have the partial fraction expansion with one of the terms having the quadratic denominator.

In this case we also have the pole at s=0, which has a coefficient c_1=1.

Then we can write $$V(s) = \frac{2}{s(s^2+2s+2)} = \frac{1}{s} + \frac{a_1 s + a_2}{s^2+2s+2}$$

By clearing *V(s)* of fractions and equating the coefficients of the powers of *s* we can determine a_1 and a_2:

$$2 = s^2 + 2s + 2 + a_1 s^2 + a_2 s$$

Which gives a_1=-1 and a_2=-2.

$$V(s) = \frac{1}{s} - \frac{s+2}{s^2+2s+2}$$

For the fraction with the quadratic denominator we can use the following transform pair:

$$\frac{As+B}{s^2+2s\alpha+c} \Leftrightarrow re^{-\alpha t}\cos(\beta t+\theta)$$

, where

$$r=\sqrt{\frac{A^2c+B^2-2AB\alpha}{c-\alpha^2}}$$

$$\theta=\tan^{-1}\left(\frac{A\alpha-B}{A\sqrt{c-\alpha^2}}\right)$$

$$\beta=\sqrt{c-\alpha^2}$$

Plugging in the numbers we get: $v(t)=(1-\sqrt{2}e^{-t}\cos(t-45^\circ))u(t)$

Note for the case where F(s) contains a repeated pole then the partial fraction expansion needs to be handled a bit differently. See text books such as: R. A. Gabel and R. A. Roberts, Signals and Linear Systems.

The Frequency Response

In the frequency domain we looked at the ratio of voltage out of a circuit to the driven current input as a function of frequency ω.

In a circuit we are often only interested in the response of the circuit at one point (either a potential difference or current) to the voltage or current at another point which is driven by an independent source. We generally are not interested in solving for every node voltage and every loop current.

The voltages or currents of interest will be at nodes or loops in the circuit which are connected to the ports of circuit elements. Network functions are defined in networks in which at least two ports are of interest. These network functions are called transfer functions since they relate a quantity at one port with a quantity at another port. Transfer functions relate port voltages and port currents and have a number of possible forms:

The **voltage transfer ratio** is the ratio of voltage on one port to the voltage on another port with all *independent sources removed except for one source* at the driven port:

$$G_{km}(s) = \frac{V_m(s)}{V_k(s)}$$

The **current transfer ratio** is the ratio of current on one port to the current on another port with all independent sources removed except for one source at the driven port:

$$\alpha_{km}(s) = \frac{I_m(s)}{I_k(s)}$$

The **transfer impedance** is the ratio of the voltage on one port to the current on another port with all independent sources removed except for one source at the driven port:

$$Z_{km}(s) = \frac{V_m(s)}{I_k(s)}$$

The **transfer admittance** is the ratio of the current on one port to the voltage on another port with all independent sources removed except for one source at the driven port:

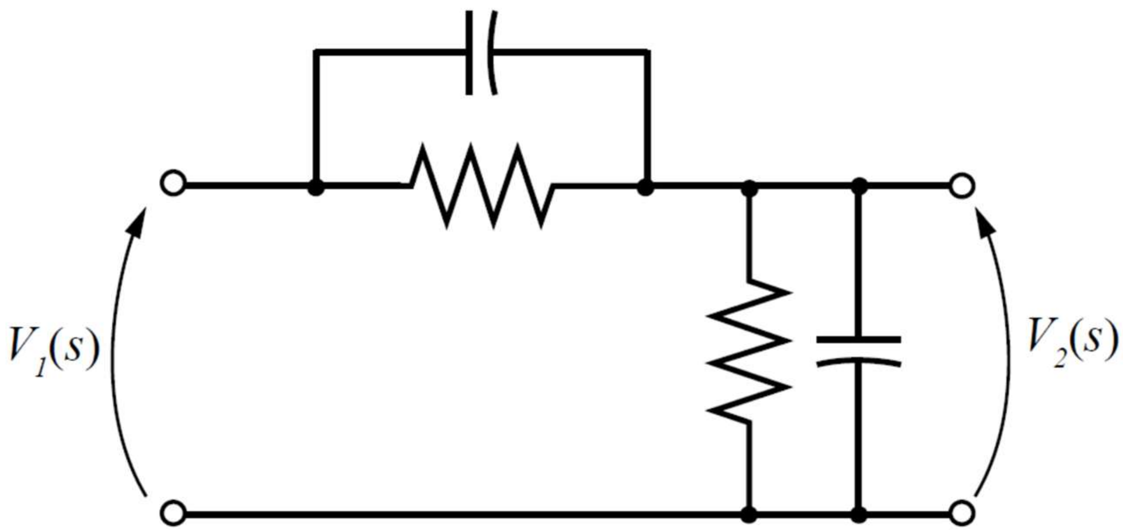
$$Y_{km}(s) = \frac{I_m(s)}{V_k(s)}$$

Note network functions do not capture initial conditions.

Example

As an illustrating example we could consider the example circuit below where there is an input port across which a driving source $V_1(s)$ is connected, and another port, this time across a capacitor where the voltage is measured $V_2(s)$.

So the voltage transfer function is: $$G_{12}(s) = \frac{V_2(s)}{V_1(s)}$$



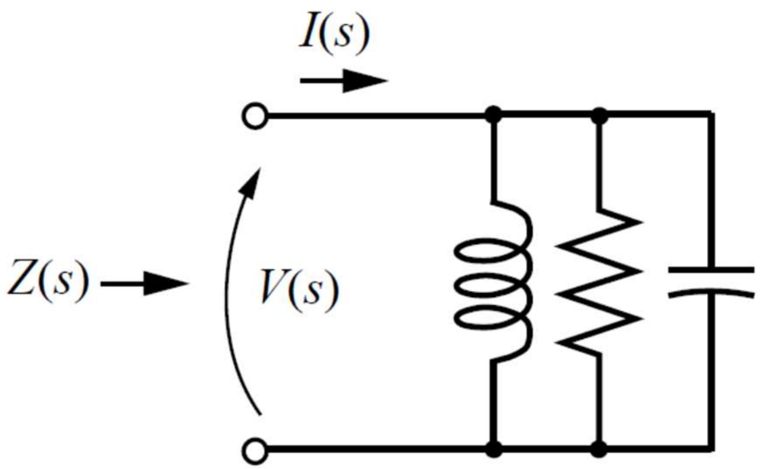
The Frequency Response (cont'd)

There are also other network functions involving just the current and voltage at one port in a circuit.

In particular the driving point impedance:

$$Z_k(s) = \frac{V_k(s)}{I_k(s)}$$

Below is a circuit illustrating the concept of the driving point impedance seen looking into the circuit at a port



The Frequency Response (cont'd)

In the case of linear time invariant (LTI) RLC networks the networks functions (generically labelled here N(s)) have a number of properties.

$$N(s) = \frac{A(s)}{B(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_m s^m}{b_0 + b_1 s + b_2 s^2 + \cdots + b_n s^n}$$

$$= \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{n} (s - z_k)}{\prod_{k=1}^{m} (s - p_k)}$$

, where we say p_k are the poles of the N(s) and z_k the zeros.

The Frequency Response (cont'd)

1. All network functions of RLC networks are quotients of finite polynomials in s

2. The denominator polynomial B(s) has positive real coefficients so all poles network functions of LTI RLC networks are real or occur in complex conjugate pairs.

3. In general the numerator polynomial A(s) may have negative coefficients

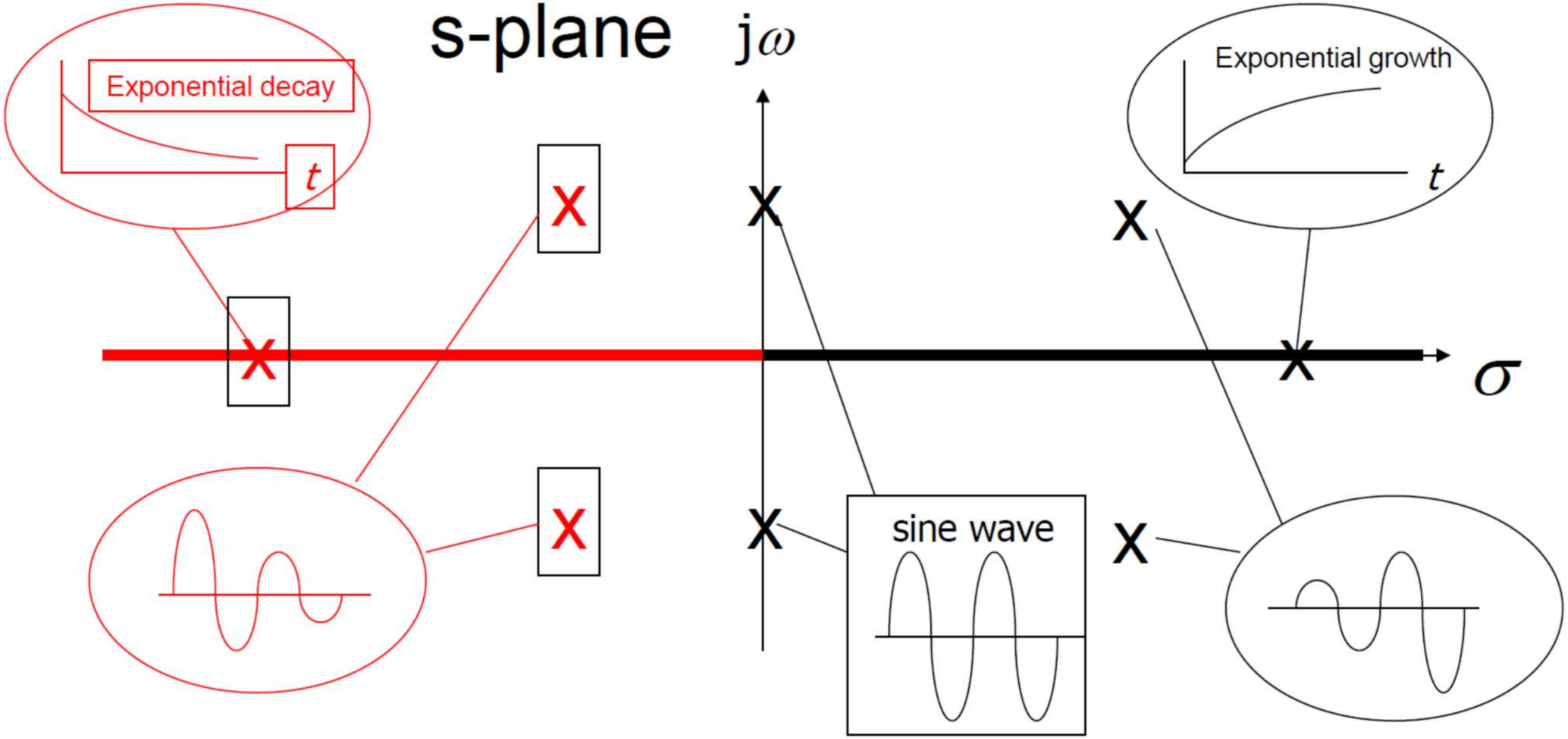
4. The poles of any network function of a given network are the natural frequencies of the network. However, due to possible cancellation between zeros of A(s) and zeros of B(s), not all natural frequencies of a network appear as poles of a network function.

5. All Network functions of a given network have the same poles provided there is no cancellation (as mentioned in 4).

6. For any network function of LTI RLC networks the total number of zeros equals the total number of poles where poles and zeros at zero and infinity are counted in addition to finite poles.

The Frequency Response (cont'd)

We can make plots of the pole and zero locations similar to what we did for the frequency domain. However, now instead of them being on a line ω they sit on a plane with one axis in ω and the other in σ , as s = σ + jω.

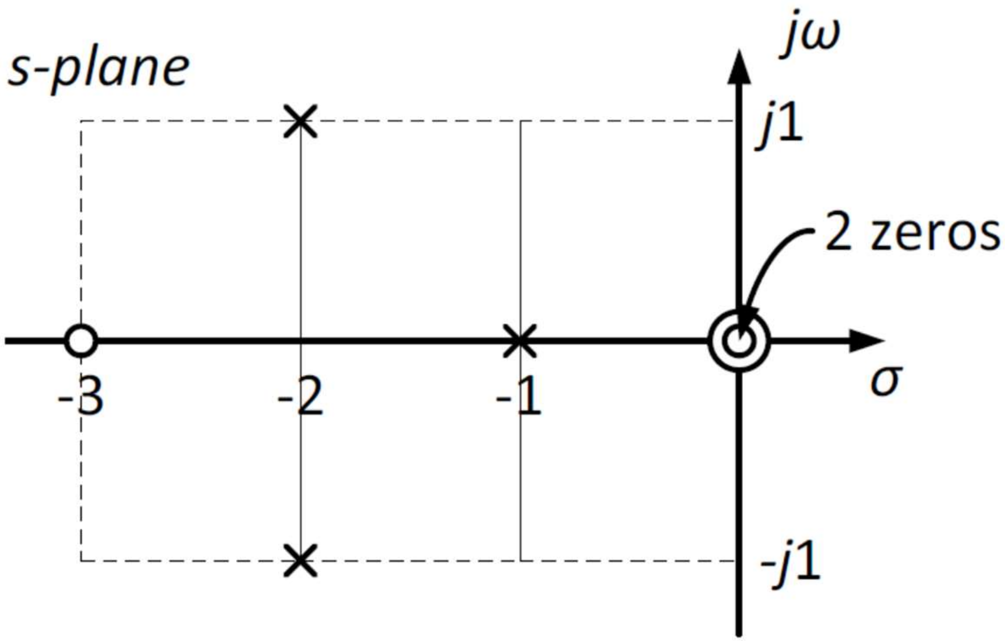


Example

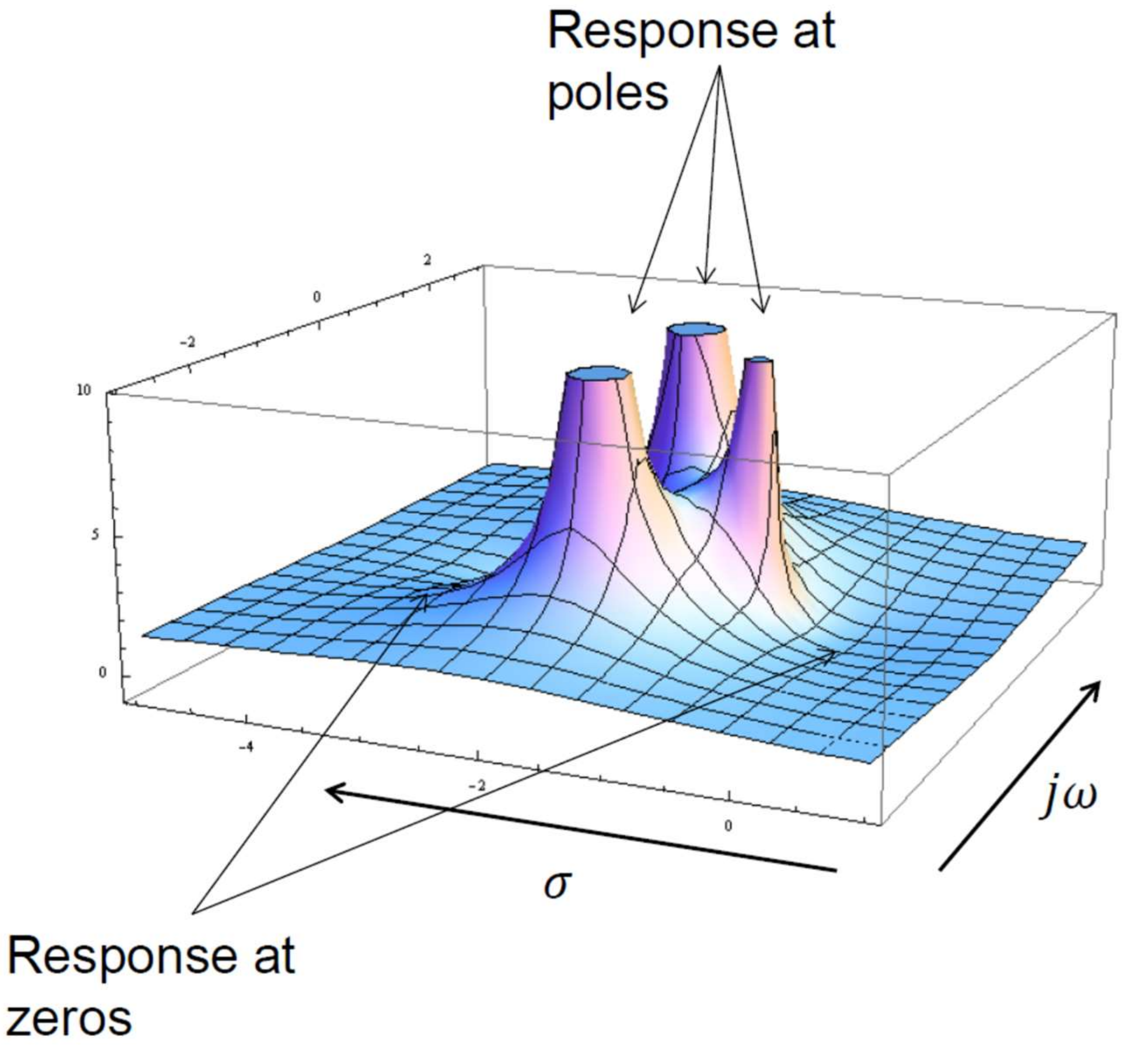
Consider the following network function:

$$N(s) = \frac{A(s)}{B(s)} = \frac{s^2(s+3)}{(s+1)(s+2+j)(s+2-j)}$$

We can plot its poles and zeros a follows:



We can also plot the magnitude of the network function $|N(s)|$



Scaling

Often we may wish to design or analyze a circuit or network using numerical values which are convenient for arithmetic manipulations. However, in practice we are often interested in real circuits where the frequency and impedances are much higher.

We can use magnitude and frequency scaling to convert from a circuit with convenient numerical values to one which operates at the desired frequency and impedance.

Consider two networks: one unscaled network (circuit) and the other scaled network that is obtained from the former by applying the following scaling to each element of the former:

$$R^* = bR$$

$$L^* = \frac{b}{a}L$$

$$C^* = \frac{1}{ab}C$$

Scaling (cont'd)

Where the elements with superscript * are the elements of the scaled network.

The scaling factor ***a*** applies only to inductors and capacitors and is referred to as the frequency scaling.

The scaling factor ***b*** applies to all the elements and is referred to as the magnitude scale factor.

The network functions of the scaled network are related to the network functions of the unscaled network as follows:

Driving point impedances functions are scaled according to:

$$Z^*(s) = bZ(s/a)$$

Voltage transfer network functions are scaled according to:

$$G_{jk}^*(s) = G_{jk}(s/a)$$

Scaling (cont'd)

Both sorts of network function are affected by the frequency scaling with s being replaced by s/a. This means the positions of poles and zeros are simply scaled by a.

The voltage transfer is unaffected by the magnitude scaling.

The driving point impedance scales directly with the magnitude scaling factor b.

Laplace domain dependent sources

Finally, linear dependent sources can also be modelled in the Laplace domain.

For example:

$$i_u(t) = av_x(t) \quad \Leftrightarrow \quad I_u(s) = aV_x(s)$$

$$v_v(t) = ai_x(t) + bv_y(t) \quad \Leftrightarrow \quad V_u(s) = aI_x(s) + bV_y(s)$$

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- Credit is acknowledged where credit is due. Please refer to the full list of references.