Lecture 3

The Frequency Domain

Linearity of differential operators

Before moving forward, let's talk about the linearity of differential operators.

$$\frac{d(v_1(t)+v_2(t))}{dt}=\frac{dv_1(t)}{dt}+\frac{dv_2(t)}{dt}$$

$$\frac{d(\alpha v_1(t))}{dt}=\alpha\frac{dv_1(t)}{dt}$$

Differentiation is a linear operation.

So *inductors and capacitors are considered linear circuit elements*. And methods based on linearity such as *superposition, Thevenin and Norton equivalents* can be applied to circuits with inductors and capacitors.

We will show these concepts are valid when the voltages and currents are not static, but rather time varying functions, e.g. v(t)

In particular we will look at the case when the voltages and currents are sinusoidal functions of time, and more specifically *the forcing functions of a linear circuit are **sinusoidal functions** of time*.

Sinusoids

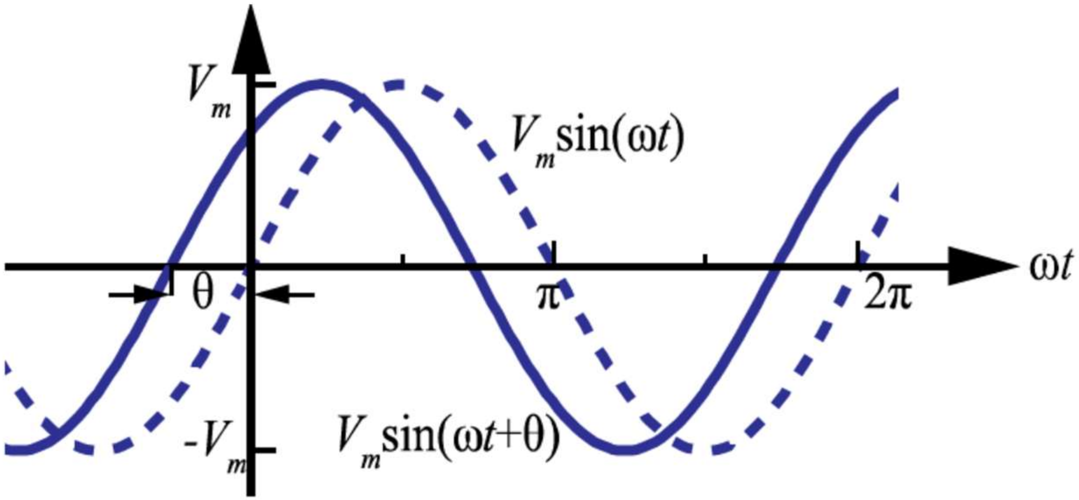
Why choose sinusoids?

- Response of underdamped second order system is a damped sinusoid
- Can represent most periodic function as an infinite sum of sinusoids

  Can use superposition in linear circuit to obtain response
- Derivatives and integrals of sinusoids are sinusoids of the same frequency
- Sinusoids often occur naturally in electric power generation

In the last lecture, we use *step function u(t),* while here we will use *sinusoidal function*.

Sinusoids (Cont'd)



Consider the sinusoidal voltage $v(t) = V_m \sin(\omega t + \theta)$

V_m is the amplitude of the sinewave, ω is the radian or angular frequency, θ is the phase angle. The sinewave has a period T and frequency in Hertz of $f=1/T$. ω=2πf .

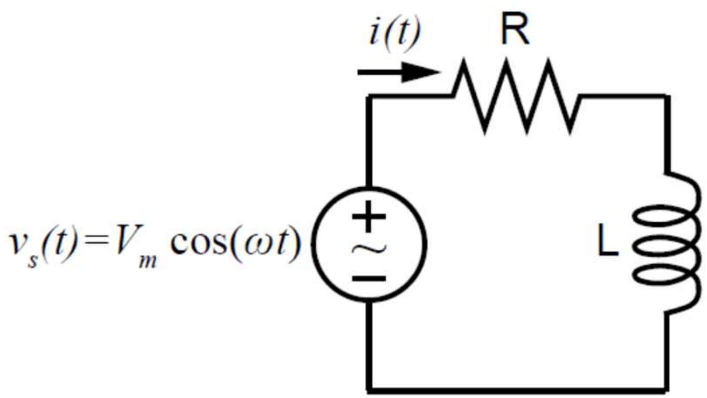
We say the sinusoid $V_m \sin(\omega t + \theta)$ leads the sinusoid $V_m \sin(\omega t)$ by θ radians. The relationship can be seen in the figure above. Alternatively we can say $V_m \sin(\omega t)$ lags $V_m \sin(\omega t + \theta)$ by θ radians.

Forced response to sinusoidal forcing function

We know that the response of a circuit composed of linear elements consists of the *natural (or homogeneous) response* and the *forced response*. We will consider now just the forced response, assuming the natural response has had time to die away.

The *forced response* is synonymous with the steady state response. Note that we are not referring to the DC steady state, but rather a sinusoidal steady state response where voltages are definitely changing with time.

Consider the ***RL circuit below***, with the ***sinusoidal source*** being switched on a long time ago in the past.



The forced response i(t) must satisfy the d.e. :

$$L\frac{di(t)}{dt} + Ri(t) = V_m \cos(\omega t) \qquad \text{(KCL)}$$

Repeated differentiation and integration of the forcing function will only result in two different forms of function: sin(ωt) and cos(ωt).

Hence we can assume the forced response will have the general form:

$$i(t) = I_1 \cos(\omega t) + I_2 \sin(\omega t)$$

, where I_1 and I_2 are real constants which depend on the values of V_m, *R, L and* ω.

We can substitute this form of i(t) into the d.e. and solve for I1 and I2. Furthermore with additional manipulation we can obtain the forced response i(t) as:

$$\boxed{i(t) = I_m \cos(\omega t + \phi) \qquad \text{, where} \quad I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}, \quad \phi = -\tan^{-1}\frac{\omega L}{R}}$$

The result involves a lot of manipulation. However we could have taken a ***simpler approach by choosing a complex forcing function***.

Firstly, note that by changing the instant we call t=0, we can change the *phase of the forcing function*.

Say we have a forcing function $V_m \cos(\omega t + \theta)$ which has the response function $I_m \cos(\omega t + \phi)$

Then shifting the phase of the forcing function by 90º gives

$$V_m \cos(\omega t + \theta - 90^\circ) = V_m \sin(\omega t + \theta)$$

Which has the response function $I_m \cos(\omega t + \phi - 90^\circ) = I_m \sin(\omega t + \phi)$

Let's play some math tricks. If we chose the forcing function:

$$V_m \cos(\omega t + \theta) + jV_m \sin(\omega t + \theta)$$

Since the circuit is linear and we can apply superposition the forced response will be:

$$I_m \cos(\omega t + \phi) + jI_m \sin(\omega t + \phi)$$

From ***Euler's identity*** we know that:

$$V_m e^{j(\omega t+\theta)} = V_m \cos(\omega t + \theta) + jV_m \sin(\omega t + \theta)$$

$$I_m e^{j(\omega t+\phi)} = I_m \cos(\omega t + \phi) + jI_m \sin(\omega t + \phi)$$

We apply a *complex forcing function whose real part is the given real forcing function*. We then obtain a *complex response whose real part is the desired response*. Through this procedure, the integrodifferential equations describing the steady state response will become simple algebraic equations.

Trying the concept with the ***example of the RL circuit***:

Complex forcing function: $V_m e^{j(\omega t)}$ And complex response: $I_m e^{j(\omega t+\phi)}$

These are substituted into the d.e.:

$$L\frac{d}{dt}(I_m e^{j(\omega t+\phi)}) + RI_m e^{j(\omega t+\phi)} = V_m e^{j(\omega t)}$$

Then taking the derivative to obtain a complex algebraic equation:

$$j\omega L I_m e^{j(\omega t+\phi)} + RI_m e^{j(\omega t+\phi)} = V_m e^{j(\omega t)}$$

To obtain I_m and ϕ we divide through by the common factor $e^{j(\omega t)}$

$$I_m e^{j\phi} = \frac{V_m}{R + j\omega L}$$

Expressing the right hand side in polar form to identify I_m and ϕ :

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{j[-\arctan(\omega L/R)]}$$

Giving:

$$I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}, \quad \phi = -\tan^{-1}\frac{\omega L}{R}$$

, which is what we obtained before with real forcing functions.

The Phasor Concept

A sinusoidal current or voltage at a *given frequency ω* is characterized by only two parameters, ***an amplitude and a phase angle***. Similarly for the complex representation.

We can represent the sinusoid by just its amplitude and phase, typically in polar form. For example: $I_m \angle \phi$ represents the real sinusoid:

$$I_m \cos(\omega t + \phi) = \mathrm{Re}(I_m e^{j(\omega t + \phi)})$$

We can represent the current as a complex quantity by dropping the Re.

We can further simplify by suppressing the factor $e^{j\omega t}$

$$\mathbf{I} = I_m \angle \phi = I_m e^{j\phi}$$

The abbreviated complex representation is called the **phasor** representation. Phasors are complex quantities and printed in bold e.g. ***I, V***.

i(t) is a time domain representation, the ***phasor I is a frequency domain*** representation.

Phasor for linear circuit elements

Resistor:

In the time domain: $v(t) = Ri(t)$

Lets apply the complex voltage

and assume a complex current response $I_m e^{j(\omega t+\phi)}$

We obtain $V_m e^{j(\omega t+\theta)} = RI_m e^{j(\omega t+\phi)}$

Dividing both sides by $e^{j\omega t}$

obtain $V_m e^{j\theta} = RI_m e^{j\phi}$

And $V_m \angle\theta = RI_m \angle\phi, \qquad \mathbf{V} = R\mathbf{I}$

Since R is real in this case $\theta = \phi$

Inductor:

In the time domain: $v(t) = L\dfrac{di(t)}{dt}$

Again applying complex voltage, and complex current:

$$V_m e^{j(\omega t+\theta)} = L\frac{d}{dt} I_m e^{j(\omega t+\phi)}$$

Taking the derivative: $V_m e^{j(\omega t+\theta)} = j\omega L I_m e^{j(\omega t+\phi)}$

Dividing both sides by $e^{j\omega t}$: $V_m e^{j\theta} = j\omega L I_m e^{j\phi}$

$$V_m \angle\theta = j\omega L I_m \angle\phi, \qquad \mathbf{V} = j\omega L \mathbf{I}$$

Due to the factor $j\omega L$ $\mathbf{I}$ must lag $\mathbf{V}$ by 90 degrees.

Because $e^{j(-\pi/2)} = -j = 1/j$

Capacitor:

In the time domain: $i(t) = C\dfrac{dv(t)}{dt}$

Again applying complex voltage, and complex current:

$$I_m e^{j(\omega t+\phi)} = C\frac{d}{dt}V_m e^{j(\omega t+\theta)}$$

Taking the derivative: $I_m e^{j(\omega t+\phi)} = j\omega C V_m e^{j(\omega t+\theta)}$

And in a manner similar to the inductor arrive at: $\mathbf{I} = j\omega C\mathbf{V}$

In the capacitor current leads voltage by 90 degrees.

Because $e^{j(\pi/2)} = j$

All the phasor $\mathbf{V}$-$\mathbf{I}$ relationships are algebraic, and also linear.

The equations for inductors and capacitors bear similarities to Ohms law, and we shall use them as we use Ohm's law, but in the *frequency domain*.

We need to show that phasors obey Kirchhoff's two laws.

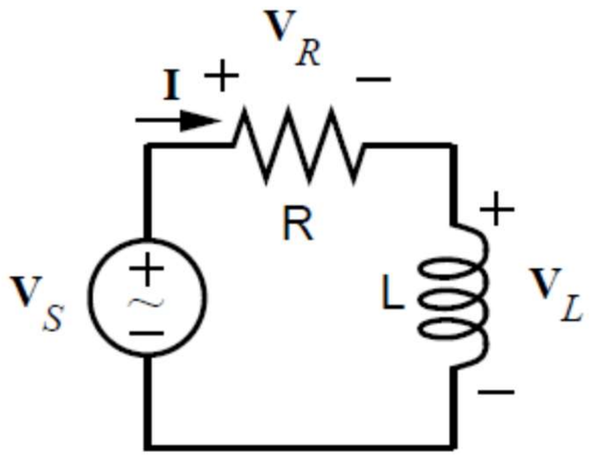
For KVL in the time domain: $v_1(t) + v_2(t) + \cdots + v_N(t) = 0$

We can replace each real voltage (potential difference) by a complex voltage (potential difference) having the same real part, suppress $e^{j\omega t}$ throughout and obtain:

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = 0$$

So KVL holds for phasors, with similar arguments can show KCL holds

Example – RL circuit



Consider the RL circuit from before, but now with phasors.

KVL: $\mathbf{V}_R + \mathbf{V}_L = \mathbf{V}_S$

And using the **V**-**I** relationships: $R\mathbf{I} + j\omega L\mathbf{I} = \mathbf{V}_S$

Solving for **I** $$\mathbf{I} = \frac{\mathbf{V}_S}{R + j\omega L}$$

Which can be written in polar form, and the real part taken to obtain earlier results. A much simpler method, particularly with more complex circuits.

Impedance

Call the ratio of the phasor voltage to the phasor current the **Impedance Z**.

That is:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

For a **resistor** $\mathbf{Z}_R = R$

For a **capacitor** $\mathbf{Z}_C = \dfrac{1}{j\omega C}$

For an **inductor** $\mathbf{Z}_L = j\omega L$

The impedance is a *complex number* having the dimension of ohms.
It is not a phasor and cannot be transformed to the time domain.

Impedance is a part of the frequency domain and NOT a concept which is part of the time domain.

Impedances may be combined in series and parallel by the same rules that were used for resistances. This can be demonstrated by KVL and KCL.

For series impedances: $$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

For parallel impedances: $$1/\mathbf{Z}_{eq} = 1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + \cdots + 1/\mathbf{Z}_N$$

The complex number for **impedance Z** can be represented in polar form, though there are no special symbols, one form could be:

$$\mathbf{Z} = |\mathbf{Z}|\angle\theta$$

In rectangular form, the resistive (real) component is represented by *R (resistance)*, and the reactive (imaginary) component by X (reactance):

$$\mathbf{Z} = R + jX$$

Admittance

Call the ratio of the phasor current to the phasor voltage the **Admittance Y**.

That is:

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}}$$

So:

$$\mathbf{Y} = 1 / \mathbf{Z}$$

The real part of the admittance is the *conductance G*, and the imaginary part is the *susceptance B*:

$$\mathbf{Y} = G + jB$$

Note we will not use the admittance very much in this course.

Circuit Analysis using Phasors

In the first part of the course a number of techniques for circuit analysis were introduced for resistive circuits, i.e. circuits composed of resistors, linear dependent sources, and independent sources.

In particular those methods were: Nodal Analysis, Mesh Analysis, Superposition, Thevenin's theorem, Norton's theorem, source transformations, voltage divider.

All of these techniques can be used to solve for phasor potential differences and currents in a circuit, by treating the impedances in a similar way we treated resistances.

However, we must remember we are only looking at the ***sinusoidal steady state solution***. Furthermore, when we solve for phasors in the circuit we are doing so only at the specific frequency of the source, $\boldsymbol{\omega}$. Additionally all sources need to be of the same frequency $\boldsymbol{\omega}$ when solving for a particular phasor.

If there are sources with different frequencies in the circuit, then we could do the analysis by employing superposition.

Specifically:

1. Do circuit analysis for just one source present in the circuit at time. Replace other voltages sources by a short circuit and other current sources by an open circuit.

2. Convert the result for the one source from the frequency domain back to the time domain. Each result will be a sinusoid with potentially a different frequency.

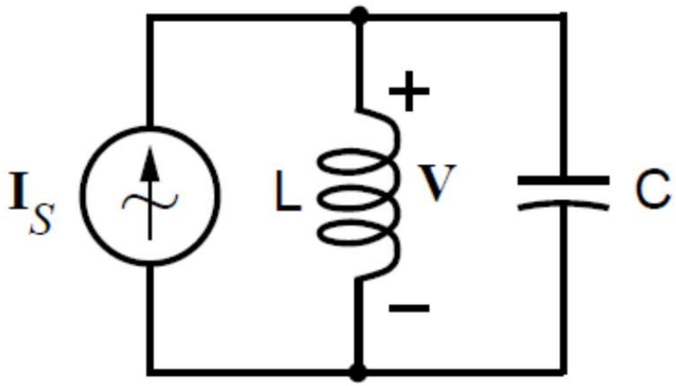
3. Repeat steps 1 and 2 for all the sources. Via superposition we can add up the results of step 2 to find the final answer for the potential difference or current.

Response as a function of frequency

We now briefly look at the response of a circuit with a sinusoidal excitation as a function of the radian frequency ω.

Consider the parallel LC circuit below driven by a sinusoidal current source which has a radian frequency ω.

Using phasors the voltage response of the circuit to the current source, $\boldsymbol{V/I_S}$ can be found to be:



$$\frac{\mathbf{V}}{\mathbf{I}_s} = \mathbf{Z} = -j\frac{1}{C}\frac{\omega}{\omega^2 - 1/LC}$$

Critical frequencies

Letting $\omega_0 = 1/\sqrt{LC}$

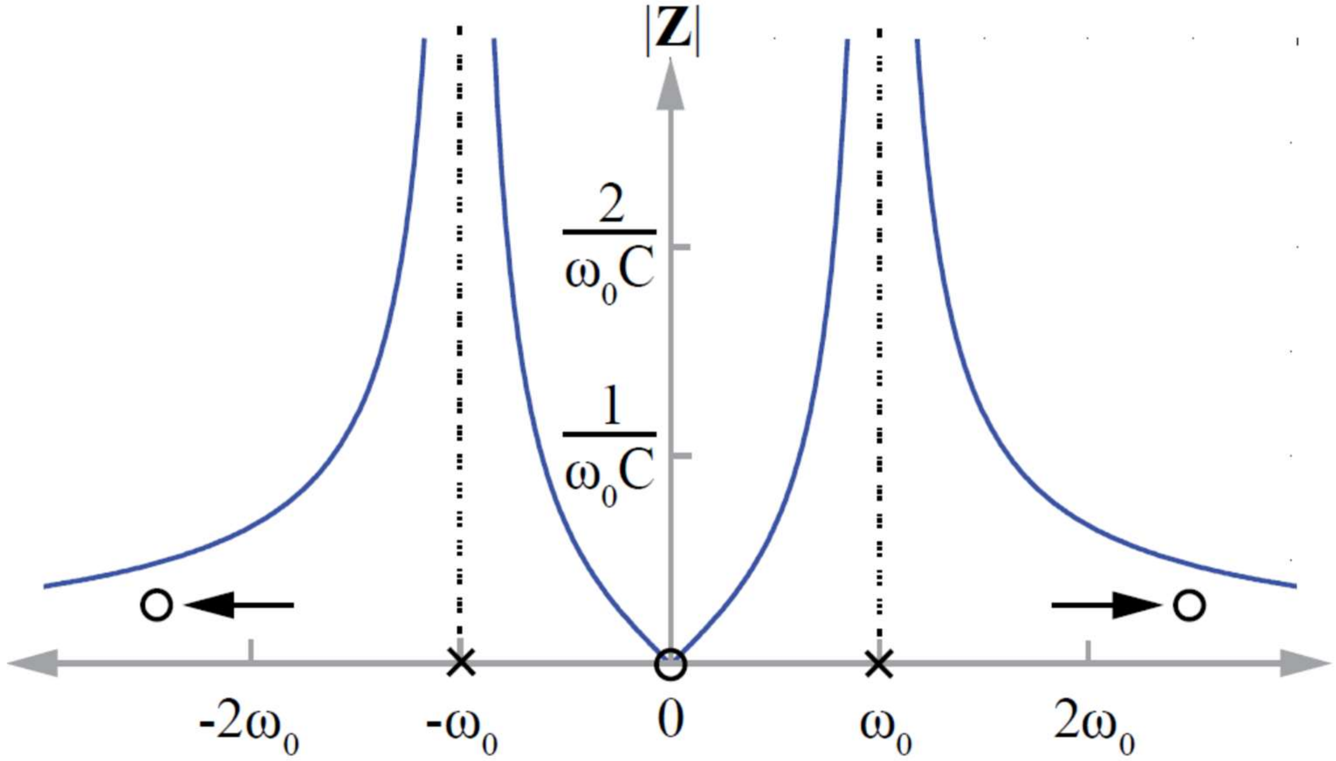
And factoring the expression for $\mathbf{Z}$, the magnitude $\mathbf{Z}$ of may be written to enable easy identification of frequencies where is $\mathbf{Z}$ zero or infinite.

$$|\mathbf{Z}| = \frac{1}{C}\frac{|\omega|}{|(\omega-\omega_0)(\omega+\omega_0)|}$$

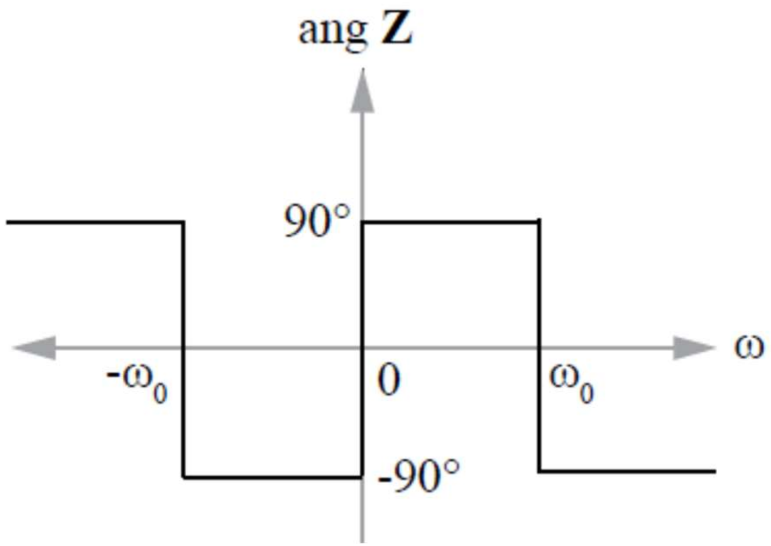
Such frequencies are termed critical frequencies. The response has zero amplitude at ω=0; when this happens we say the response has a ***zero*** at ω=0. The response is of infinite amplitude when $\omega=\omega_0$ and $-\omega_0$; these frequencies are called ***poles***, and the response is said to have a ***pole*** at each of these frequencies. Finally the response has a zero as ω tends to ∞. And so ω= ± ∞ are also zeros.

The critical frequencies are marked on the ω axis by using small circles for zeros and crosses for the poles. Poles or zeros at infinity are indicated by an arrow near the axis.

We call |Z| the ***magnitude response*** of the circuit. In this case it was the ratio of voltage out to current in. Depending on the circuit and the currents and voltages of interest it may be a ratio of voltage out to voltage in etc.



The angle of Z for the parallel LC circuit can be seen from inspection to be either +90° or -90° in certain frequency ranges, with no other values possible. Circuits composed entirely of inductors and capacitors have this property of the angle of the impedance being ± 90°. We call the angle of Z the ***phase response*** of the circuit. Again whether it is ratio of voltage out to current in or some other ratio depends on the circuit and voltages or currents of interest.



We will look in more detail at plotting the response of circuits when we deal with Laplace domain analysis.

Instantaneous Power

Consider a voltage source connected to a resistor. Consider two cases:

A. Where the voltage source and current through the resistor are constant (DC or Direct Current).

B. Where the voltage source and current through the resistor are sinusoidal.

For the DC case the power absorbed is: $P_{DC} = VI = I^2 R = V^2 / R$

For the sinusoidal case we can first define ***instantaneous power***:

$$p(t) = v(t) i(t) = i^2(t) R = v^2(t) / R$$

Further we can define an ***average power*** by averaging the instantaneous power over some time T. For the case of our sinusoidal source, we can make T equal to the period of the sinusoid.

$$P_{AVE} = \frac{1}{T} \int_0^T p(t) dt = \frac{1}{T} \int_0^T i^2(t) R dt = \frac{1}{T} \int_0^T \frac{v^2(t)}{R} dt$$

RMS voltage

By letting $P_{AVE}=P_{DC}$ then we can define the effective current (I_{eff}) or voltage (V_{eff}) for the sinusoidal case that gives the same power dissipation in the resistor as for the DC case:

$$P_{DC} = I_{eff}^2 R = P_{AVE} = \frac{1}{T}\int_0^T i^2(t)R\,dt$$

Finding I_{eff}:

$$I_{eff} = I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

And for V_{eff}:

$$V_{eff} = V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

These values are also called the ***RMS (root mean square)*** values of current and voltage. (And the concept is also applicable to non sinusoidal sources)

For our sinusoidal voltage source case $i(t)$ and $v(t)$ will be of the form:

$$i(t) = I_m \cos(\omega t + \phi), \quad v(t) = V_m \cos(\omega t + \theta)$$

Furthermore:

$$i^2(t) = I_m{}^2 \cos^2(\omega t + \phi) = \frac{I_m{}^2}{2}(1 + \cos(2\omega t + 2\phi))$$

When $i^2(t)$ is averaged over one period of the T of the sinusoid and the square root taken we obtain:

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

Similarly for V_{rms}:
$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

For a sinusoidal voltage source connected to a resistor, the current and voltage will be in phase, i.e. θ=ϕ. Consider now instead of resistor we have a general impedance, in which case θ may not equal ϕ.

Average Power

For this more general situation the instantaneous power will be:

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta)\cos(\omega t + \phi)$$
$$= (1/2)V_m I_m (\cos(\theta - \phi) + \cos(2\omega t + \theta + \phi))$$

Finding now the average power over one period of the sinusoid:

$$\boxed{P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2}V_m I_m \cos(\theta - \phi)}$$

Consider when we have a resistor as the impedance, θ=ϕ and:

$$P = \frac{1}{2}V_m I_m = \frac{1}{2}I_m^2 R = \frac{1}{2}|\mathbf{I}|^2 R$$

Consider when the impedance is purely reactive, so just capacitors or inductors, θ=ϕ ± 90°.

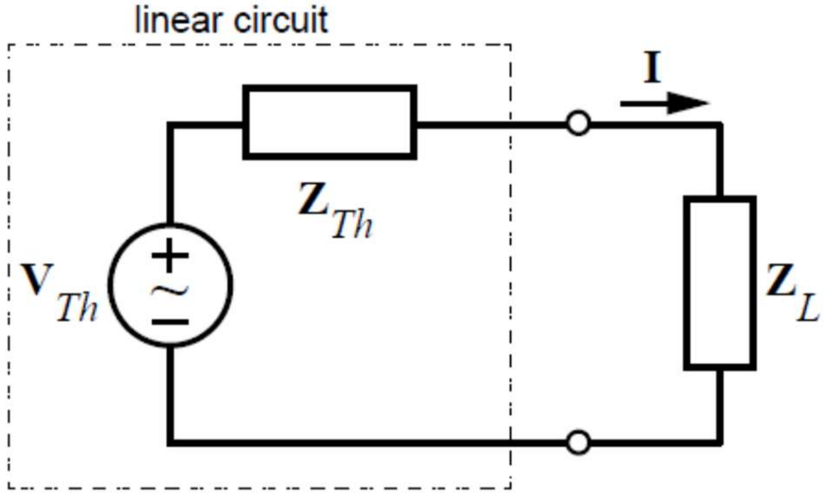
$$P = \frac{1}{2}V_m I_m \cos(90^\circ) = 0$$

Due to *L* and *C* components being lossless and so don't absorb power.

Maximum Power Transfer

So power is only dissipated in the resistive part of a circuit.

Consider the circuit below where a load impedance $\mathbf{Z}_L$ is connected to a linear circuit. The linear circuit can be represented by its Thevenin equivalent $\mathbf{V}_{Th}$ and $\mathbf{Z}_{Th}$.



Where: $\mathbf{Z}_{Th} = R_{Th} + jX_{Th}$, $\mathbf{Z}_L = R_L + jX_L$

We want to determine the power dissipated in the load. Remember only the resistive part of the load dissipates power. From circuit analysis:

$$P_L = \frac{1}{2}|\mathbf{I}|^2 R_L = \frac{1}{2}\left|\frac{\mathbf{V}_{Th}}{\mathbf{Z}_{Th} + \mathbf{Z}_L}\right|^2 R_L = \frac{|\mathbf{V}_{Th}|^2 R_L/2}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

We can find the value of $\mathbf{Z}_L$ which gives a maximum power to delivered to the load by setting first finding the value of X_L where: $\dfrac{\partial P_L}{\partial X_L} = 0$

This value turns out to be X_L =- X_{Th}

Looking now at the value of R_L for which: $\dfrac{\partial P_L}{\partial R_L} = 0$

We can find $R_L = \sqrt{R_{Th}^2 + (X_{Th} + X_L)^2}$

So the maximum power is transferred when: $\mathbf{Z}_L = R_{Th} - jX_{Th} = \mathbf{Z}_{Th}^*$

So maximum power is transferred to the load when the load is ***matched to the source***:

$$\boxed{\mathbf{Z}_L = \mathbf{Z}_{Th}^*}$$

And the maximum power transferred is:

$$\boxed{P_{L\max} = \frac{|\mathbf{V}_{Th}|^2}{8R_{Th}}}$$

Note that if the circuits are purely resistive then maximum power is when the load resistance equals the source resistance. Which for purely resistive networks is true even for non-sinusoidal sources.

Complex Power

Consider the phasors: $\mathbf{V} = V_m \angle \theta, \quad \mathbf{I} = I_m \angle \phi$

Define ***Complex Power S*** as: $S = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2} V_m I_m \angle(\theta - \phi)$

From before real power *P*: $P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2} V_m I_m \cos(\theta - \phi)$

So *P*=Re(*S*), and in general: $S = P + jQ$

Where *P* is the real power, and *Q* is the reactive power.

Acknowledgments

- Lecture slides are based on lecture materials from various sources, including M. Hill, T. Cantoni, F. Boussaid, and R. Togneri.

- Credit is acknowledged where credit is due. Please refer to the full list of references.