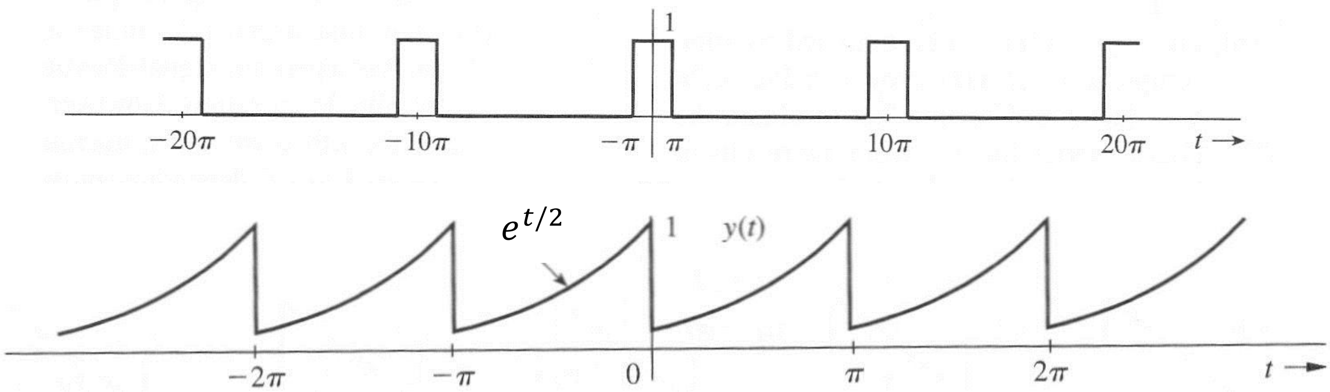
Tutorial 1 (Solutions)

1. For the following periodic signals find the compact trigonometric Fourier series and sketch the amplitude and phase spectra:



Answers:

(a) We have $T_0 = 10\pi \rightarrow \omega_0 = \frac{2\pi}{T_0} = \frac{1}{5}$ and even symmetry implies the sine terms are zero.

$$A[k] = \frac{2}{T_0}\int_{-5\pi}^{5\pi} x(t)\cos(k\omega_0 t)\,dt = \frac{2}{10\pi}\int_{-\pi}^{\pi}\cos\left(\frac{k}{5}t\right)$$

$$= \frac{2}{10\pi}\left[\frac{5}{k}\sin\left(\frac{k}{5}t\right)\right]_{-\pi}^{\pi} = \frac{2}{k\pi}\sin\left(\frac{k\pi}{5}\right)$$

For $A[0] = \frac{1}{T_0}\int_{-5\pi}^{5\pi} x(t)\,dt = \frac{1}{10\pi}(2\pi) = \frac{1}{5}$ and since $B[k] = 0$ then:

$$x(t) = A[0] + \sum_{k=1}^{\infty} A[k]\cos(k\omega_0 t) = \frac{1}{5} + \sum_{k=1}^{\infty}\left[\frac{2}{k\pi}\sin\left(\frac{k\pi}{5}\right)\right]\cos\left(\frac{k}{5}t\right)$$

Amplitude and Phase spectra given by:

$$C[k] = \sqrt{A^2[k] + B^2[k]} = \left|\frac{2}{k\pi}\sin\left(\frac{k}{5}t\right)\right|, \quad C[0] = \frac{1}{5}$$

$$\theta[k] = \tan^{-1}\left(\frac{-B[k]}{A[k]}\right) = 0$$

(b) $T_0 = \pi \rightarrow \omega_0 = 2$ and using the $(-\pi, 0)$ interval over which we have $e^{t/2}$ defined:

$$A[0] = \frac{1}{\pi}\int_{-\pi}^{0} e^{t/2}\,dt = 0.504$$

$$A[k] = \frac{2}{\pi}\int_{-\pi}^{0} e^{t/2}\cos(k\omega_0 t)\,dt = 0.504\left(\frac{2}{1+16k^2}\right)$$

$$B[k] = \frac{2}{\pi}\int_{-\pi}^{0} e^{t/2}\sin(k\omega_0 t)\,dt = -0.504\left(\frac{8k}{1+16k^2}\right)$$

NOTE: You can use integration by parts or from a table of standard integrals:

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$C[k] = 0.504\left(\frac{2}{\sqrt{1+16k^2}}\right), \quad C[0] = 0.504$$
$$\theta[k] = \tan^{-1}(4k)$$

Hence:

$$x(t) = 0.504 + \sum_{k=1}^{\infty} 0.504\left(\frac{2}{\sqrt{1+16k^2}}\right)\cos(k2t + \tan^{-1}(4k))$$

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2. Are the following signals periodic? If so, what is the period and what harmonics are present? If not, why not?

(a) $3\sin t + 2\sin 3t$

(b) $2\sin 3t + 7\cos \pi t$

(c) $7\cos \pi t + 5\sin 2\pi t$

(d) $\sin\frac{5t}{2} + 3\cos\frac{6t}{5} + 3\sin\left(\frac{t}{7} + 30^o\right)$

Answers:

(a) Ratio of frequencies is $\frac{3}{1}$ hence periodic. The GCF is $\omega_0 = 1$ and hence $T_0 = 2\pi$. The 1st and 3rd harmonics are present.

(b) Ratio of frequencies is $\frac{\pi}{3}$ which is not a ratio of integers so NOT periodic.

(c) Ratio of frequencies is $\frac{2\pi}{\pi} = \frac{2}{1}$ hence periodic. The GCF is $\omega_0 = \pi$ and hence $T_0 = 2$. The 1st and 2nd harmonics are present.

(d) Consider all frequency pairs:

$$\frac{6t/5}{5t/2} = \frac{12}{25}, \qquad \frac{t/7}{5t/2} = \frac{2}{35}, \qquad \frac{t/7}{6t/5} = \frac{5}{42}$$

hence periodic. To find the GCF we sort the harmonics and find the common factor:

$$\frac{1}{7}:\frac{6}{5}:\frac{5}{2} \rightarrow \frac{10}{70}:\frac{84}{70}:\frac{175}{70}$$

Hence the GCF is $\omega_0 = \frac{1}{70}$ and $T_0 = 140\pi$ and the 10th, 84th and 175th harmonics are present.

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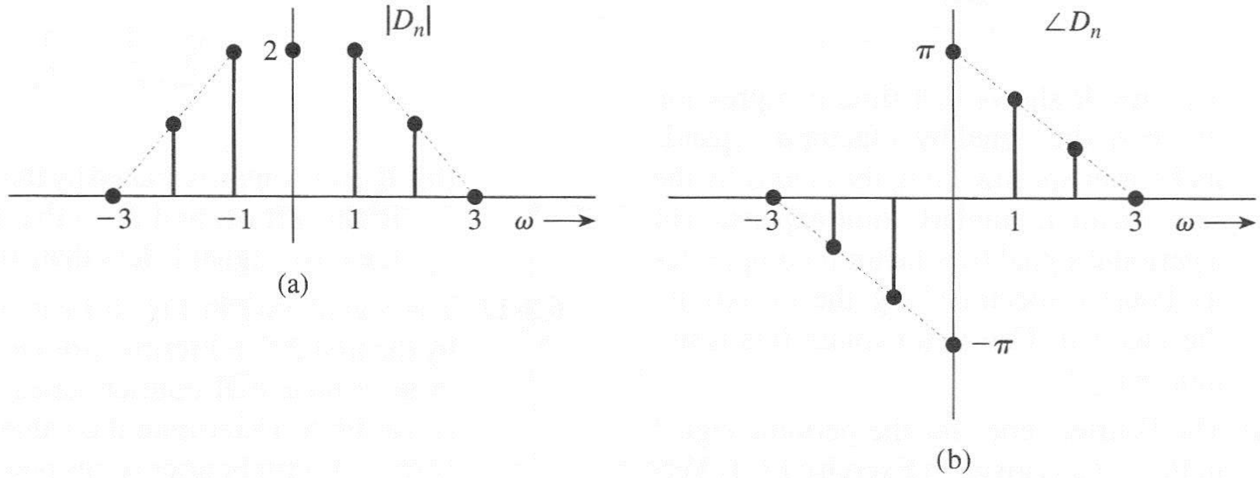
3. Consider the exponential Fourier spectra of the signal $x(t)$ shown below.

(a) Find the exponential Fourier series expression for $x(t)$

(b) Sketch the compact trigonometric Fourier spectra for $x(t)$

(c) Find the trigonometric Fourier series expression for $x(t)$

(d) Show that (a) and (c) are equivalent.



Answers:

(a) Note that $D_n \equiv X[k]$. Since $X[k] = |X[k]|e^{j\angle X[k]}$ then $X[k]e^{jk\omega_0 t} = |X[k]|e^{j(k\omega_0 t + \angle X[k])}$ and thus:

$$x(t) = \sum_{k=-3}^{3} X[k]e^{jk\omega_0 t} = e^{-j\left(2t+\frac{\pi}{3}\right)} + 2e^{-j\left(t+\frac{2\pi}{3}\right)} + 2 + 2e^{j\left(t+\frac{2\pi}{3}\right)} + e^{j\left(2t+\frac{\pi}{3}\right)}$$

(b) We sketch $C[0] = X[0] = 2$, $C[k] = 2|X[k]|$ and $\theta[0] = 0$, $\theta[k] = \angle X[k]$ for $k = 0,1,2,3$

(c) From (a) and (b) we can see that:

$$x(t) = 2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right)$$

(d) Just use the following identity from Euler's formula:

$$\cos(\omega t + \theta) = \frac{1}{2}\left(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right)$$

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4. If a periodic signal $x(t)$ is expressed by the exponential Fourier series.

(a) What happens to the Fourier spectrum when the signal is time-shifted (say $x(t) \rightarrow x(t-T)$)?

(b) What happens to the Fourier spectrum when the signal is compressed/dilated in time (say $x(t) \rightarrow x(at)$)?

Answers:

(a) The exponential FS of $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$$

If we consider $y(t) = x(t-T)$ this becomes:

$$y(t) = \sum_{k=-\infty}^{\infty} Y[k]e^{jk\omega_0 t} = x(t-T) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 (t-T)} = \sum_{k=-\infty}^{\infty} \left(X[k]e^{-jk\omega_0 T}\right)e^{jk\omega_0 t}$$

And thus:

$$Y[k] = X[k]e^{-jk\omega_0 T} \rightarrow |Y[k]| = |X[k]|, \qquad \angle Y[k] = \angle X[k] - k\omega_0 T$$

For a time delay of T the amplitude spectrum is unchanged but the phase spectrum is offset by the factor $-k\omega_0 T$.

(b) If we now consider $y(t) = x(at)$ we have:

$$y(t) = \sum_{k=-\infty}^{\infty} Y[k]e^{jk\omega_0 t} = x(at) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 at} = \sum_{k=-\infty}^{\infty} X[k]e^{jk(a\omega_0)t}$$

Yielding a change in the fundamental frequency $\omega_0 \rightarrow a\omega_0$. If $a > 1$ there is time compression of the signal waveform which would imply a higher fundamental frequency, whereas for $a < 1$ there is time expansion / stretching of the signal waveform which would imply a lower fundamental frequency.

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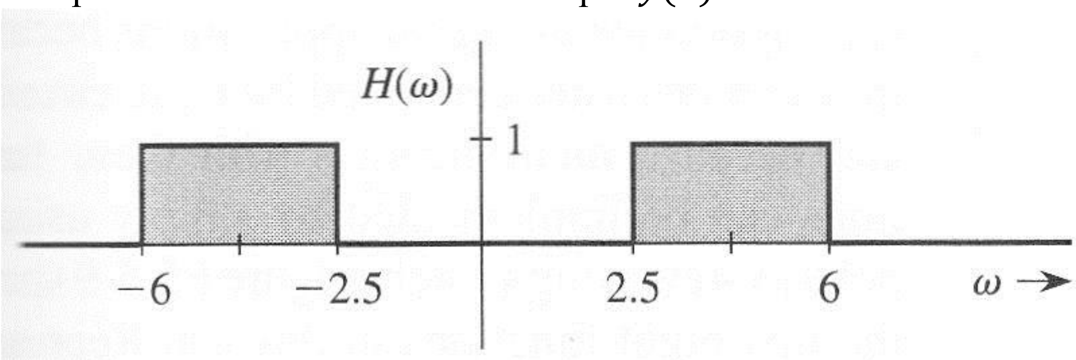
5. (a) Find the exponential Fourier series for a signal $x(t) = \cos 5t \sin 3t$. Use the fact that $2\sin A \cos B = \sin(A + B) + \sin(A - B)$.

(b) Sketch the Fourier spectra.

(c) The signal $x(t)$ is applied to the input of the LTI system with frequency response shown below. Find the output $y(n)$.



Answers:

(a) $x(t) = \cos 5t \sin 3t = \frac{1}{2}(\sin 8t - \sin 2t)$ and using $\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$ we get:

$$x(t) = \frac{1}{4j}\left(e^{j8t} - e^{-j8t}\right) - \frac{1}{4j}\left(e^{j2t} - e^{-j2t}\right)$$
$$= \left(j\frac{1}{4}\right)e^{-j8t} + \left(-j\frac{1}{4}\right)e^{-j2t} + \left(j\frac{1}{4}\right)e^{j2t} + \left(-j\frac{1}{4}\right)e^{j8t}$$

since $1/j = -j$.

(b) Hence we have $|X[-8]| = |X[-2]| = |X[2]| = |X[8]| = 0.25$ and

$$\angle X[-8] = \angle X[2] = \frac{\pi}{2}, \qquad \angle X[-2] = \angle X[8] = -\frac{\pi}{2}$$

Since $\angle j \rightarrow \frac{\pi}{2}$ and $\angle(-j) = -\frac{\pi}{2}$.

(c) We will have $y(t) = 0$ since none of the harmonic components of $x(t)$ at $\omega = 2, 8$ will pass through the passband filter range $\omega \in [2.5 \ldots 6]$.

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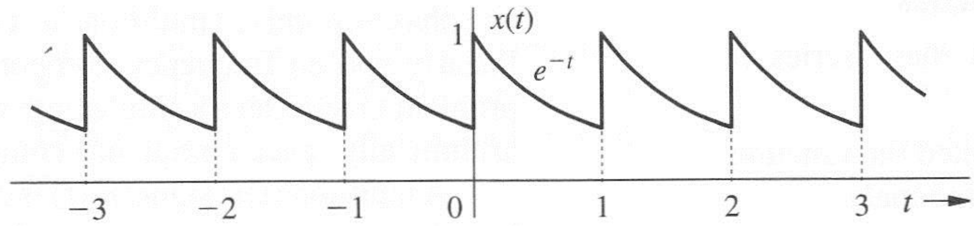
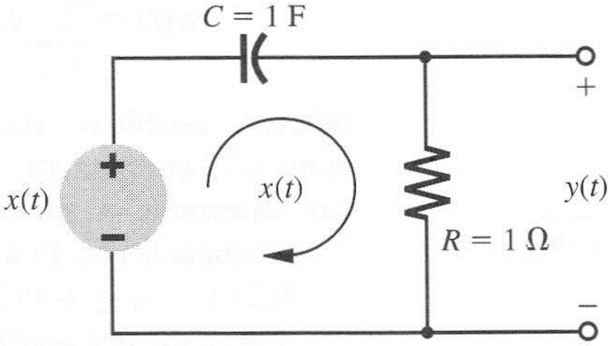
6. Find the exponential Fourier series for the periodic signal $x(t)$ shown below. If $x(t)$ is applied to the RC circuit system shown find the exponential Fourier series expression for $y(t)$.



(a)



(b)

Answers:

We have $T_0 = 1$ and thus $\omega_0 = 2\pi$ so:

$$X[k] = \int_0^1 x(t)e^{-jk\omega_0 t}\,dt = \int_0^1 e^{-t}e^{-j2\pi kt}\,dt = \left[-\frac{e^{-(1+j2\pi k)t}}{(1+j2\pi k)}\right]_0^1 = \frac{(e-1)(1-j2\pi k)}{e(1+4\pi^2k^2)}$$

The RC circuit system response is given by (voltage division using impedances):

$$H(j\omega) = \frac{1}{1+\left(\frac{1}{j\omega}\right)} = \frac{j\omega}{j\omega+1}$$

Thus the output is given by:

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} X[k]H(jk\omega_0)e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} \left(\frac{(e-1)(1-j2\pi k)}{e(1+4\pi^2k^2)}\right)\left(\frac{j2\pi k}{j2\pi k+1}\right)e^{j2\pi kt} \end{aligned}$$