The application of ordinary differentiation

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$$\frac{dN}{dt} = -\lambda N$$

$$N(t_0) = N_0$$

problem $dN/dt = -\lambda N$, $N(t_0) = N_0$ is

$$N(t) = N_0 \exp\left(-\lambda \int_{t_0}^{t} ds\right) = N_0 e^{-\lambda(t-t_0)}$$

or $N/N_0 = \exp(-\lambda(t-t_0))$. Taking logarithms of both sides we obtain that

$$-\lambda(t-t_0) = \ln \frac{N}{N_0}. \tag{2}$$

Now, if $N/N_0 = \frac{1}{2}$ then $-\lambda(t-t_0) = \ln \frac{1}{2}$ so that

$$(t-t_0) = \frac{\ln 2}{\lambda} = \frac{0.6931}{\lambda}. \tag{3}$$

$$\frac{dy}{dt} = -\lambda y + r(t), \qquad y(t_0) = y_0.$$

$$\frac{d}{dt} e^{\lambda t} y = r e^{\lambda t}.$$

$$e^{\lambda t} y(t) - e^{\lambda t_0} y_0 = \frac{r}{\lambda}(e^{\lambda t} - e^{\lambda t_0})$$

$$y(t) = \frac{r}{\lambda}(1 - e^{-\lambda(t - t_0)}) + y_0 e^{-\lambda(t - t_0)}.$$

$$\frac{dp}{dt} = ap - bp^2, \qquad p(t_0) = p_0.$$

This is a separable differential equation, and from Equation (10), Section 1.4,

$$\int_{p_0}^{p} \frac{dr}{ar - br^2} = \int_{t_0}^{t} ds = t - t_0.$$

To integrate the function $1/(ar - br^2)$ we resort to partial fractions. Let

$$\frac{1}{ar - br^2} \equiv \frac{1}{r(a - br)} = \frac{A}{r} + \frac{B}{a - br}.$$

To find A and B, observe that

$$\frac{A}{r} + \frac{B}{a - br} = \frac{A(a - br) + Br}{r(a - br)} = \frac{Aa + (B - bA)r}{r(a - br)}.$$

Therefore, $Aa + (B - bA)r = 1$. Since this equation is true for all values of r, we see that $Aa = 1$ and $B - bA = 0$. Consequently, $A = 1/a$, $B = b/a$, and

$$\int_{p_0}^{p}\frac{dr}{r(a-br)}=\frac{1}{a}\int_{p_0}^{p}\left(\frac{1}{r}+\frac{b}{a-br}\right)dr$$

$$=\frac{1}{a}\left[\ln\frac{p}{p_0}+\ln\left|\frac{a-bp_0}{a-bp}\right|\right]=\frac{1}{a}\ln\frac{p}{p_0}\left|\frac{a-bp_0}{a-bp}\right|.$$

Thus,

$$a(t-t_0)=\ln\frac{p}{p_0}\left|\frac{a-bp_0}{a-bp}\right|. \tag{2}$$

Now, it is a simple matter to show (see Exercise 1) that

$$\frac{a-bp_0}{a-bp(t)}$$

is always positive. Hence,

$$a(t-t_0)=\ln\frac{p}{p_0}\frac{a-bp_0}{a-bp}.$$

Taking exponentials of both sides of this equation gives

$$e^{a(t-t_0)} = \frac{p}{p_0}\,\frac{a-bp_0}{a-bp},$$

or

$$p_0(a-bp)e^{a(t-t_0)} = (a-bp_0)\,p.$$

Bringing all terms involving p to the left-hand side of this equation, we see that

$$\left[a-bp_0+bp_0e^{a(t-t_0)}\right]p(t) = ap_0e^{a(t-t_0)}.$$

Consequently,

$$p(t) = \frac{ap_0e^{a(t-t_0)}}{a-bp_0+bp_0e^{a(t-t_0)}} = \frac{ap_0}{bp_0+(a-bp_0)e^{-a(t-t_0)}}. \tag{3}$$

Let us now examine Equation (3) to see what kind of population it predicts. Observe that as $t\to\infty$,

$$p(t)\to\frac{ap_0}{bp_0} = \frac{a}{b}.$$

Thus, *regardless of its initial value, the population always approaches the limiting value* a/b. Next, observe that $p(t)$ is a monotonically increasing function of time if $0<p_0<a/b$. Moreover, since

$$\frac{d^2p}{dt^2} = a\frac{dp}{dt} - 2bp\frac{dp}{dt} = (a-2bp)p(a-bp),$$

we see that dp/dt is increasing if $p(t)<a/2b$, and that dp/dt is decreasing if $p(t)>a/2b$. Hence, if $p_0<a/2b$, the graph of $p(t)$ must have the form given in Figure 1. Such a curve is called a logistic, or S-shaped curve. From its shape we conclude that the time period before the population reaches half its limiting value is a period of accelerated growth. After this point, the rate of growth decreases and in time reaches zero. This is a period of diminishing growth.

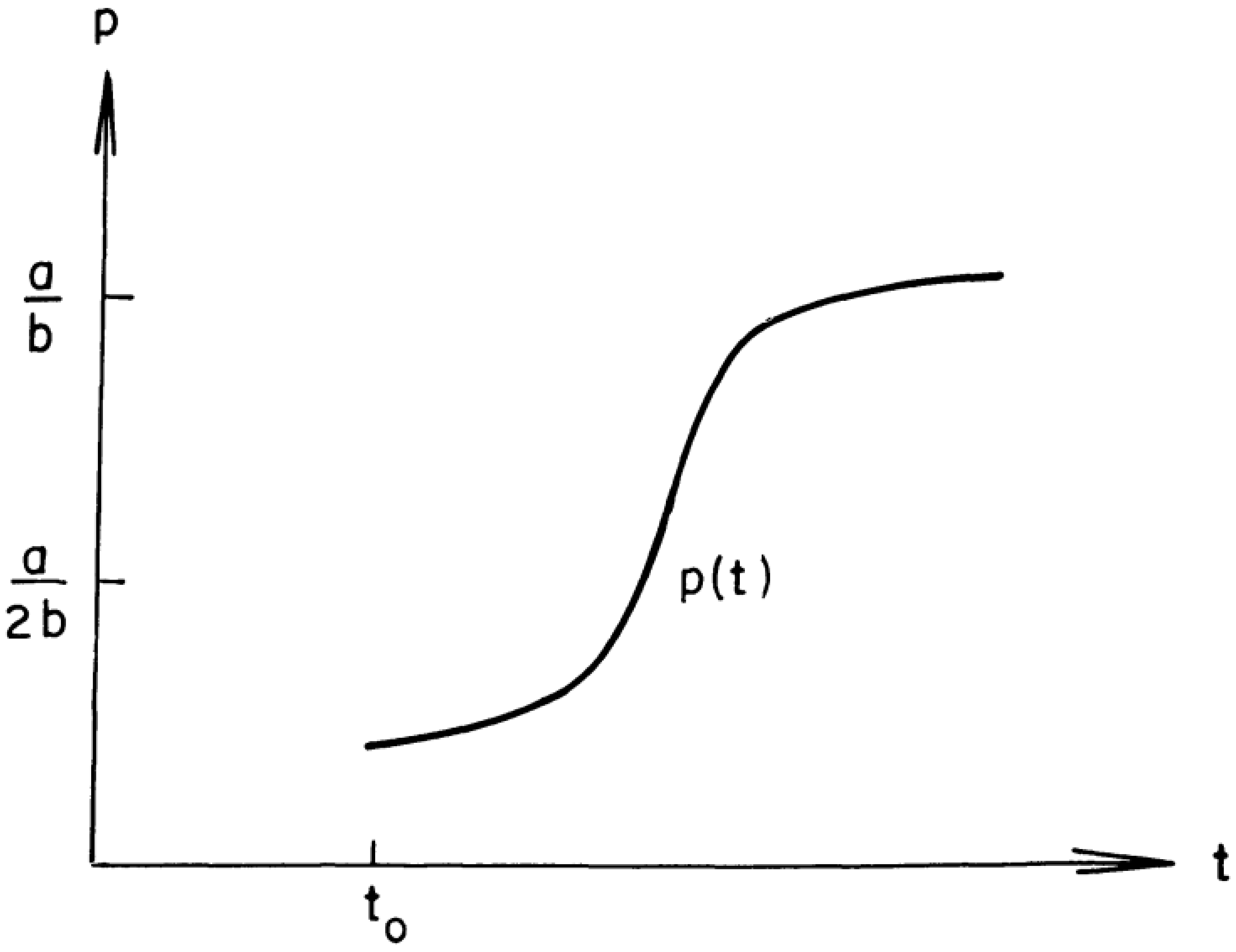


Figure 1. Graph of $p(t)$

$b = 2.309/375$. Consequently, the logistic law predicts that

$$p(t) = \frac{(2.309)5}{\frac{(2.309)5}{375} + \left(2.309 - \frac{(2.309)5}{375}\right)e^{-2.309t}}$$

$$= \frac{375}{1 + 74e^{-2.309t}}. \tag{4}$$

(We have taken the initial time t_0 to be 0.) Figure 2 compares the graph of $p(t)$ predicted by Equation (4) with the actual measurements, which are denoted by o. As can be seen, the agreement is remarkably good.

In order to apply our results to predict the future human population of the earth, we must estimate the vital coefficients a and b in the logistic equation governing its growth. Some ecologists have estimated that the natural value of a is 0.029. We also know that the human population was increasing at the rate of 2% per year when the population was $(3.34)10^9$. Since $(1/p)(dp/dt) = a - bp$, we see that

$$0.02 = a - b\,(3.34)10^9.$$

Consequently, $b = 2.695 \times 10^{-12}$. Thus, according to the logistic law of population growth, the human population of the earth will tend to the limiting value of

$$\frac{a}{b} = \frac{0.029}{2.695 \times 10^{-12}} = 10.76 \text{ billion people}$$

Note that according to this prediction, we were still on the accelerated

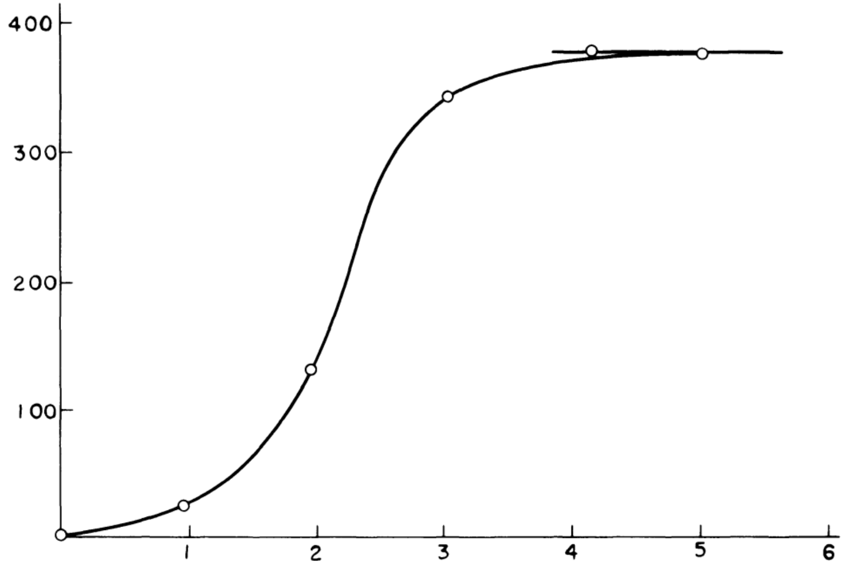


Figure 2. The growth of paramecium

$$p(t) = \frac{197{,}273{,}000}{1 + e^{-0.03134(t - 1913.25)}}$$

Let us now use Equation (3) to predict the population of the earth in the year 2000. Setting $a = .029$, $b = 2.695 \times 10^{-12}$, $p_0 = 3.34 \times 10^9$, $t_0 = 1965$, and $t = 2{,}000$ gives

$$\begin{aligned} p(2000) &= \frac{(.029)(3.34)10^9}{.009 + (.02)e^{-(.029)35}} \\ &= \frac{29(3.34)}{9 + 20e^{-1.015}} 10^9 \\ &= 5.96 \text{ billion people!} \end{aligned}$$

This is another spectacular application of the logistic equation.

yet unaware. Hence, $\Delta p = cp(N-p)\Delta t$ or $\Delta p/\Delta t = cp(N-p)$ for some positive constant c. Letting $\Delta t \to 0$, we obtain the differential equation

$$\frac{dp}{dt} = cp(N-p). \tag{1}$$

This is the logistic equation of the previous section if we set $a = cN$, $b = c$. Assuming that $p(0) = 1$; i.e., one farmer has adopted the innovation at time $t = 0$, we see that $p(t)$ satisfies the initial-value problem

$$\frac{dp}{dt} = cp(N-p), \qquad p(0) = 1. \tag{2}$$

The solution of (2) is

$$p(t) = \frac{Ne^{cNt}}{N-1+e^{cNt}} \tag{3}$$

proportional to the number of farmers who do not yet know, i.e.,

$$\Delta p = c'(N-p)\Delta t$$

for some positive constant c'. Letting $\Delta t \to 0$, we see that $c'(N-p)$ farmers, per unit time, learn of the innovation through the mass communication media. Thus, if $p(0)=0$, then $p(t)$ satisfies the initial-value problem

$$\frac{dp}{dt} = cp(N-p) + c'(N-p), \qquad p(0)=0. \tag{4}$$

The solution of (4) is

$$p(t) = \frac{Nc'\left[e^{(c'+cN)t}-1\right]}{cN + c'e^{(c'+cN)t}}, \tag{5}$$

and in Exercises 2 and 3 we indicate how to determine the shape of the curve (5).

We would now like to show that the differential equation

$$dp/dt = cp(N-p)$$

also governs the rate at which firms in such diverse industries as bituminous coal, iron and steel, brewing, and railroads adopted several major innovations in the first part of this century. This is rather surprising, since we would expect that the number of firms adopting an innovation in one of these industries certainly depends on the profitability of the innovation and the investment required to implement it, and we haven't mentioned these factors in deriving Equation (1). However, as we shall see shortly, these two factors are incorporated in the constant c.

Let n be the total number of firms in a particular industry who have adopted an innovation at time t. It is clear that the number of firms Δp who adopt the innovation in a short time interval Δt is proportional to the number of firms $n-p$ who have not yet adopted; i.e., $\Delta p = \lambda(n-p)\Delta t$. Letting $\Delta t \to 0$, we see that

$$\frac{dp}{dt} = \lambda(n-p).$$

The proportionality factor λ depends on the profitability π of installing this innovation relative to that of alternative investments, the investment s required to install this innovation as a percentage of the total assets of the firm, and the percentage of firms who have already adopted. Thus,

$$\lambda = f(\pi, s, p/n).$$

Expanding f in a Taylor series, and dropping terms of degree more than two, gives

$$\lambda = a_1 + a_2\pi + a_3 s + a_4\frac{p}{n} + a_5\pi^2 + a_6 s^2 + a_7\pi s + a_8\pi\left(\frac{p}{n}\right) + a_9 s\left(\frac{p}{n}\right) + a_{10}\left(\frac{p}{n}\right)^2.$$

In the late 1950's, Edwin Mansfield of Carnegie Mellon University investigated the spread of twelve innovations in four major industries. From his exhaustive studies, Mansfield concluded that $a_{10}=0$ and

$$a_1 + a_2\pi + a_3 s + a_5\pi^2 + a_6 s^2 + a_7\pi s = 0.$$

Thus, setting

$$k = a_4 + a_8\pi + a_9 s, \tag{6}$$

we see that

$$\frac{dp}{dt} = k\frac{p}{n}(n-p).$$

(This is the equation obtained previously for the spread of innovations among farmers, if we set $k/n = c$.) We assume that the innovation is first adopted by one firm in the year t_0. Then, $p(t)$ satisfies the initial-value problem

$$\frac{dp}{dt} = \frac{k}{n}p(n-p), \qquad p(t_0) = 1 \tag{7}$$

and this implies that

$$p(t) = \frac{n}{1+(n-1)e^{-k(t-t_0)}}.$$

$$D = 0.08\,V\,\frac{(\mathrm{lb})(\mathrm{s})}{\mathrm{ft}}.$$

Now, set $y=0$ at sea level, and let the direction of increasing y be downwards. Then, W is a positive force, and B and D are negative forces. Consequently, from (1),

$$\frac{d^2y}{dt^2} = \frac{1}{m}(W - B - cV) = \frac{g}{W}(W - B - cV).$$

We can rewrite this equation as a first-order linear differential equation for $V = dy/dt$; i.e.,

$$\frac{dV}{dt} + \frac{cg}{W}V = \frac{g}{W}(W - B). \tag{2}$$

Initially, when the drum is released in the ocean, its velocity is zero. Thus, $V(t)$, the velocity of the drum, satisfies the initial-value problem

$$\frac{dV}{dt} + \frac{cg}{W}V = \frac{g}{W}(W - B), \qquad V(0) = 0, \tag{3}$$

and this implies that

$$V(t) = \frac{W - B}{c}\left[1 - e^{(-cg/W)t}\right]. \tag{4}$$

Equation (4) expresses the velocity of the drum as a function of time. In order to determine the impact velocity of the drum, we must compute the time t at which the drum hits the ocean floor. Unfortunately, though, it is impossible to find t as an explicit function of y (see Exercise 2). Therefore, we cannot use Equation (4) to find the velocity of the drum when it hits the ocean floor. However, the A.E.C. can use this equation to try and prove that the drums do not crack on impact. To wit, observe from (4) that $V(t)$ is a monotonic increasing function of time which approaches the limiting value

$$V_T = \frac{W-B}{c}$$

as t approaches infinity. The quantity V_T is called the terminal velocity of the drum. Clearly, $V(t) \leqslant V_T$, so that the velocity of the drum when it hits the ocean floor is certainly less than $(W-B)/c$. Now, if this terminal velocity is less than 40 ft/s, then the drums could not possibly break on impact. However,

$$\frac{W-B}{c} = \frac{527.436-470.327}{0.08} = 713.86 \text{ ft/s},$$

and this is way too large.

It should be clear now that the only way we can resolve the dispute between the A.E.C. and the engineers is to find $v(y)$, the velocity of the drum as a function of position. The function $v(y)$ is very different from the function $V(t)$, which is the velocity of the drum as a function of time. However, these two functions are related through the equation

$$V(t)=v(y(t))$$

if we express y as a function of t. By the chain rule of differentiation, $dV/dt=(dv/dy)(dy/dt)$. Hence

$$\frac{W}{g}\frac{dv}{dy}\frac{dy}{dt}=W-B-cV.$$

But $dy/dt=V(t)=v(y(t))$. Thus, suppressing the dependence of y on t, we see that $v(y)$ satisfies the first-order differential equation

$$\frac{W}{g}v\frac{dv}{dy}=W-B-cv, \quad \text{or} \quad \frac{v}{W-B-cv}\frac{dv}{dy}=\frac{g}{W}.$$

Moreover,

$$v(0)=v(y(0))=V(0)=0.$$

Hence,

$$\int_0^v \frac{r\,dr}{W-B-cr}=\int_0^y \frac{g}{W}\,ds=\frac{gy}{W}.$$

Now,

$$\int_0^v \frac{r\,dr}{W-B-cr}=\int_0^v \frac{r-(W-B)/c}{W-B-cr}\,dr+\frac{W-B}{c}\int_0^v \frac{dr}{W-B-cr}$$

$$=-\frac{1}{c}\int_0^v dr+\frac{W-B}{c}\int_0^v \frac{dr}{W-B-cr}$$

$$=-\frac{v}{c}-\frac{(W-B)}{c^2}\ln\frac{|W-B-cv|}{W-B}.$$

We know already that $v<(W-B)/c$. Consequently, $W-B-cv$ is always positive, and

$$\frac{gy}{W}=-\frac{v}{c}-\frac{(W-B)}{c^2}\ln\frac{W-B-cv}{W-B}. \tag{5}$$

At this point, we are ready to scream in despair since we cannot find v as an explicit function of y from (5). This is not an insurmountable difficulty, though. As we show in Section 1.11, it is quite simple, with the aid of a digital computer, to find $v(300)$ from (5). We need only supply the computer with a good approximation of $v(300)$ and this is obtained in the following manner. The velocity $v(y)$ of the drum satisfies the initial-value problem

$$\frac{W}{g} v \frac{dv}{dy} = W - B - cv, \qquad v(0) = 0. \tag{6}$$

Let us, for the moment, set $c = 0$ in (6) to obtain the new initial-value problem

$$\frac{W}{g} u \frac{du}{dy} = W - B, \qquad u(0) = 0. \tag{6'}$$

(We have replaced v by u to avoid confusion later.) We can integrate (6′) immediately to obtain that

$$\frac{W}{g}\frac{u^2}{2} = (W - B)y, \quad \text{or} \quad u(y) = \left[\frac{2g}{W}(W - B)y\right]^{1/2}.$$

In particular,

$$u(300)=\left[\frac{2g}{W}(W-B)300\right]^{1/2}=\left[\frac{2(32.2)(57.109)(300)}{527.436}\right]^{1/2}$$

$$\cong\sqrt{2092}\cong 45.7\ \mathrm{ft/s}.$$

We claim, now, that $u(300)$ is a very good approximation of $v(300)$. The proof of this is as follows. First, observe that the velocity of the drum is always greater if there is no drag force opposing the motion. Hence,

$$v(300)<u(300).$$

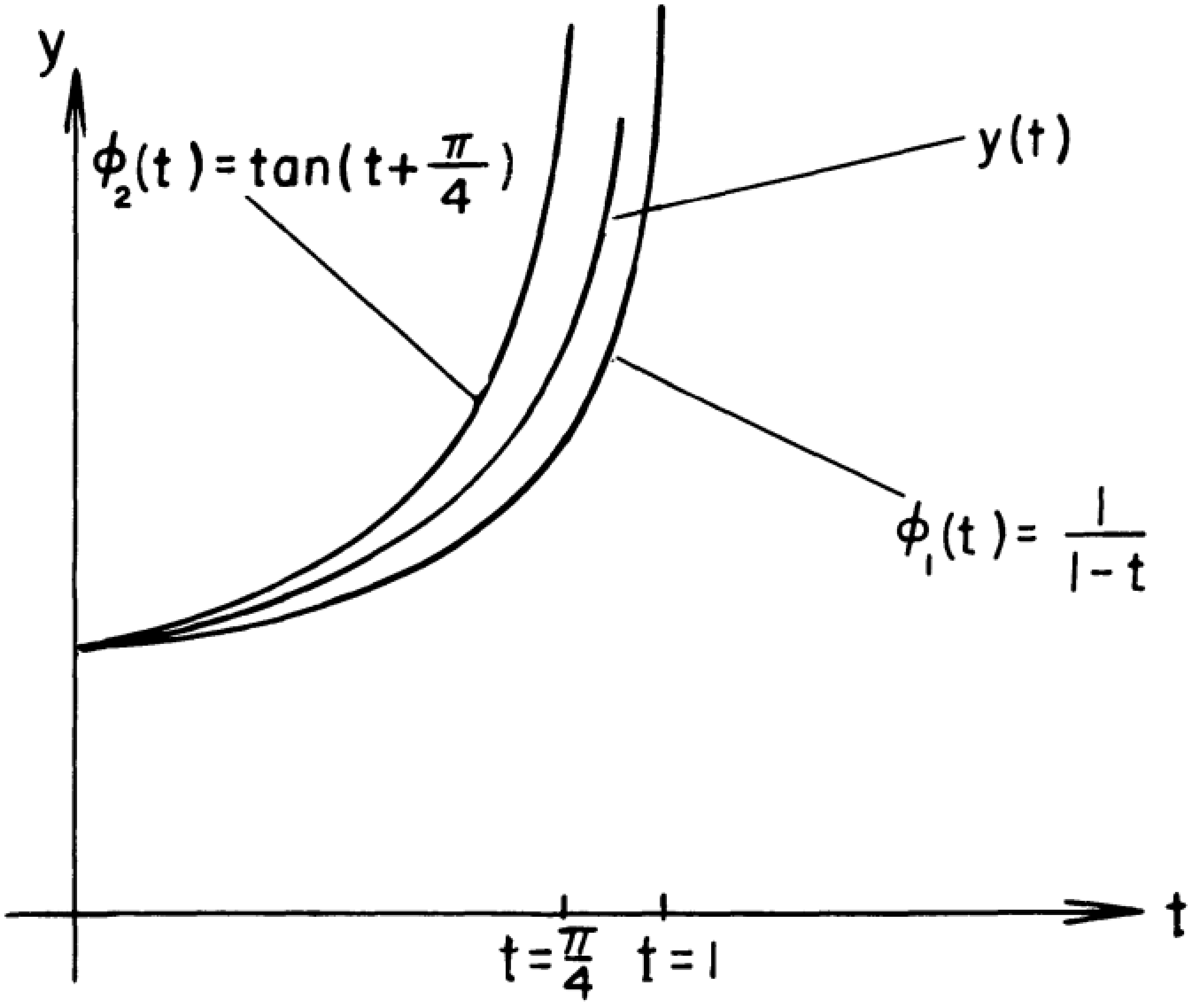


Figure 1

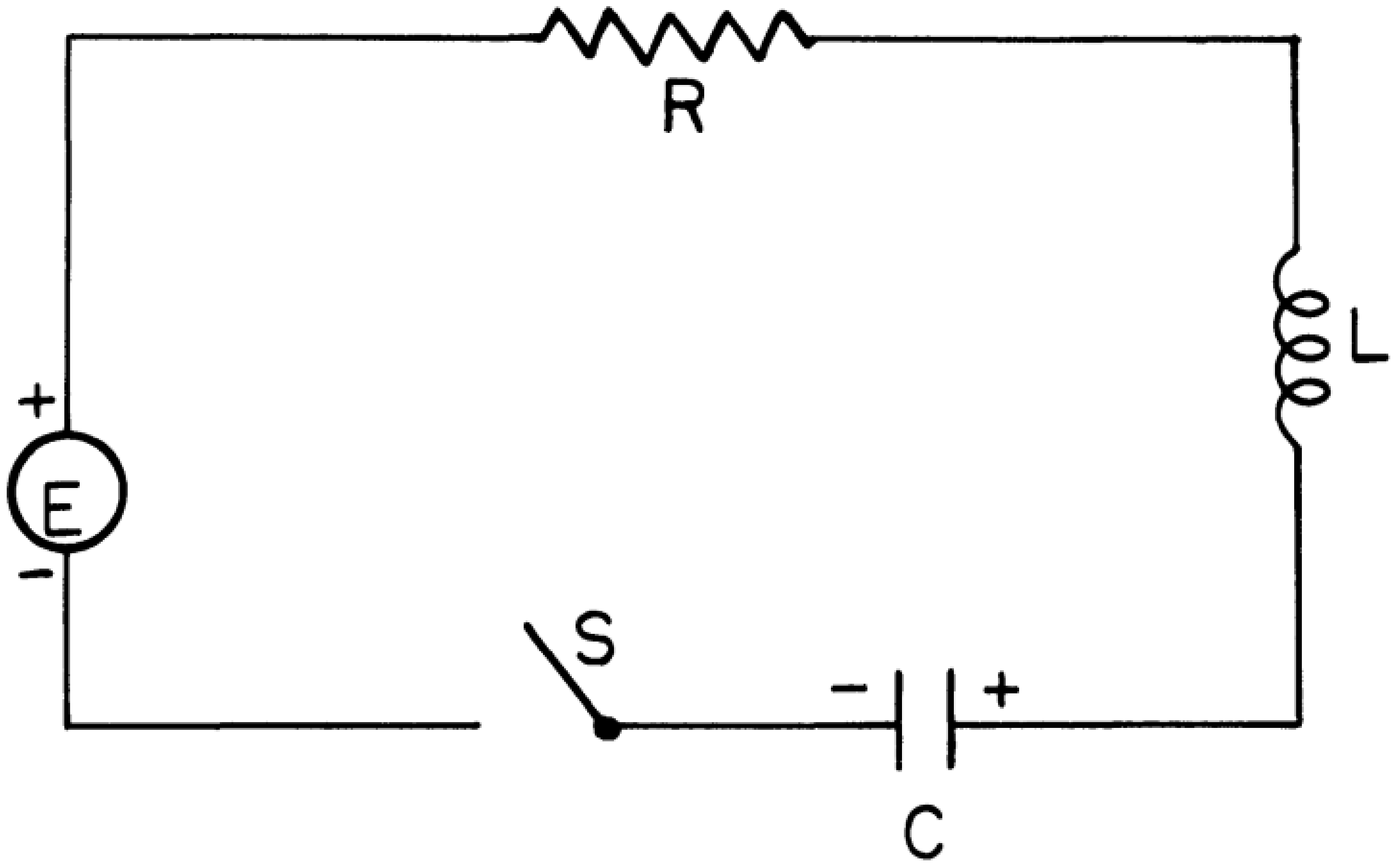


Figure 1. A simple series circuit

Kirchoff's second law: In a closed circuit, the impressed voltage equals the sum of the voltage drops in the rest of the circuit.

Now,

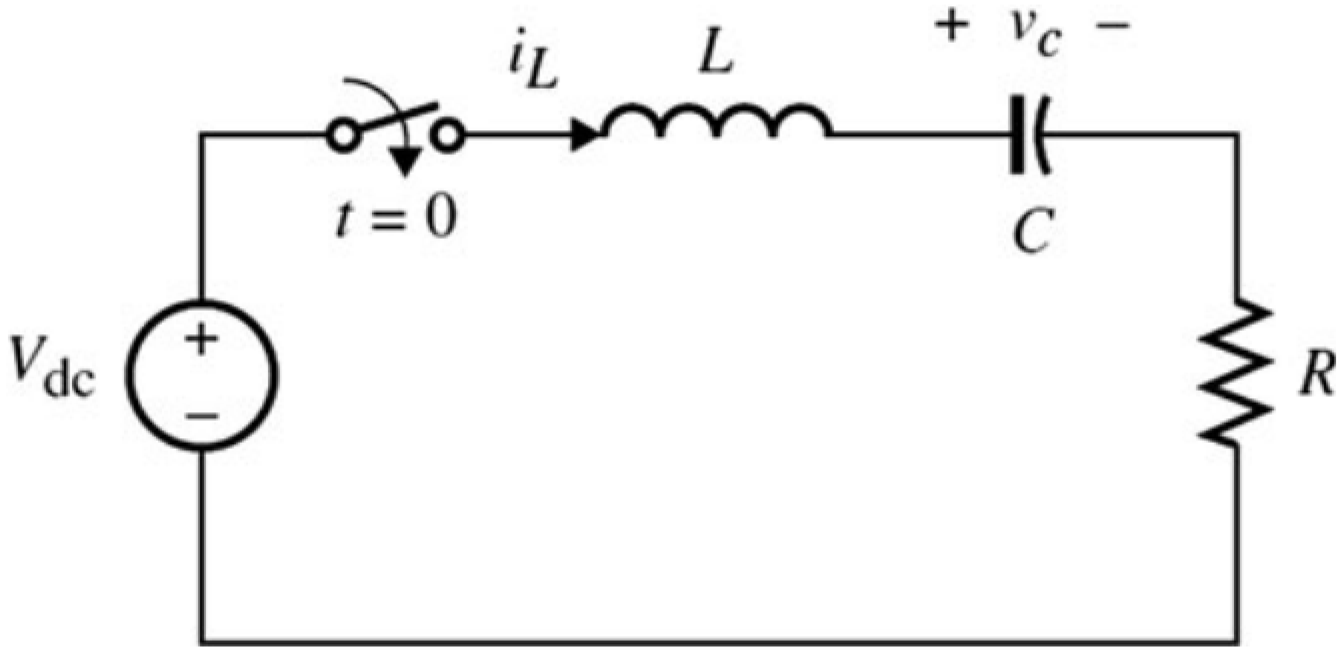
(i) The voltage drop across a resistance of R ohms equals RI (Ohm's law).
(ii) The voltage drop across an inductance of L henrys equals $L(dI/dt)$.
(iii) The voltage drop across a capacitance of C farads equals Q/C.

Hence,

$$E(t) = L\frac{dI}{dt} + RI + \frac{Q}{C},$$

and since $I(t) = dQ(t)/dt$, we see that

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t). \tag{1}$$


i_L
L
$+ \ v_c \ -$
$t = 0$
C
V_{dc}
+
−
R

$$L\frac{di_L}{dt} + Ri_L + \frac{1}{C}\int_0^T i_L dt = V_{dc}$$

$$\frac{d^2 i_L}{dt^2} + \frac{R}{L}\frac{di_L}{dt} + \frac{1}{LC} i_L(t) = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

$$s_{1,2} = -\frac{R}{2L}$$

$$i_L(t) = (A_1 + A_2 t)e^{-(R/2L)t}$$

$$\begin{aligned} v_{\mathrm{c}}(t) &= V_{dc} - L\frac{di_{\mathrm{L}}}{dt} - Ri_{\mathrm{L}} \\ &= V_{dc} + \left[A_2\left(1 - \frac{R}{2L}t\right) - A_1\frac{R}{2L}\right]e^{-(R/2L)t} \end{aligned}$$

$$\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$$

$$i_L(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v_C(t) = V_{dc} - (Ls_1 - R)A_1 e^{s_1 t} - (Ls_2 - R)A_2 e^{s_2 t}$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$$

$$i_L(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

$$\begin{aligned} v_C(t) = V_{dc} - Le^{-\alpha t}[(A_2\omega_d - A_1\alpha)\cos(\omega_d t) - (A_2\alpha + A_1\omega_d)\sin(\omega_d t)] \\ - Re^{-\alpha t}[A_2 \sin(\omega_d t) + A_1 \cos(\omega_d t)] \end{aligned}$$

$$\boldsymbol{s_1 = -\alpha + j\omega_d} \qquad \boldsymbol{\omega_d = \sqrt{\omega_0^2 - \alpha^2}}$$

$$\boldsymbol{s_2 = -\alpha - j\omega_d} \qquad \alpha = \frac{R}{2L}$$

$$\delta \equiv \frac{\alpha}{\omega_0} = \frac{R}{2\sqrt{L/C}} \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

Example 1. A tank contains S_0 lb of salt dissolved in 200 gallons of water. Starting at time $t=0$, water containing $\frac{1}{2}$ lb of salt per gallon enters the tank at the rate of 4 gal/min, and the well stirred solution leaves the tank at the same rate. Find the concentration of salt in the tank at any time $t>0$.

Solution. Let $S(t)$ denote the amount of salt in the tank at time t. Then, $S'(t)$, which is the rate of change of salt in the tank at time t, must equal the rate at which salt enters the tank minus the rate at which it leaves the tank. Obviously, the rate at which salt enters the tank is

$$\tfrac{1}{2}\ \text{lb/gal times 4 gal/min} = 2\ \text{lb/min}.$$

After a moment's reflection, it is also obvious that the rate at which salt leaves the tank is

$$4\ \text{gal/min times}\ \frac{S(t)}{200}.$$

Thus

$$S'(t)=2-\frac{S(t)}{50}, \qquad S(0)=S_0,$$

and this implies that

$$S(t)=S_0e^{-0.02t}+100(1-e^{-0.02t}). \tag{5}$$

Hence, the concentration $c(t)$ of salt in the tank is given by

$$c(t)=\frac{S(t)}{200}=\frac{S_0}{200}e^{-0.02t}+\tfrac{1}{2}(1-e^{-0.02t}). \tag{6}$$