/45

UWA – ENSC3015 Signals and Systems

Please complete your details below:	
Surname: ______________________________	Number: ____________________
Signature: _____________________________	Date: __________________
10:58am, Monday, October 9, 2017 in **Tattersall LT**	
Class Test 3: Fourier Series and Fourier Transform	
Time allowed: 45 minutes Max mark: **45**, Assessment: **5%**[1]	This paper contains: X pages, 5 questions

Candidates should attempt **all** questions and show **all** working with numerical answers to **3** decimal places in the spaces provided after each question, show as much working as possible to gain maximum marks. You can use the blank page on the reverse side for rough working, but these pages will not be marked

FOR THE ATTACHMENTS PLEASE REFER TO THE SEPARATED PAGES

[1] If you do better in the exam this test will not contribute to your unit marks and the 5% will come from the final exam performance. However if you do better in this test compared to the final exam then this test will be included in the unit marks.

Question 1 (5 marks)

Identify the appropriate Fourier representation (FS, FT, DTFT, DTFS) for each of the following signals

(a) $\frac{1}{n} + \sin\left(\frac{\pi n}{5}\right)$

(b) $\sin(2\pi t^2)$

(c) $3 + \left|\cos\left(\frac{\pi n}{3}\right)\right|$

(d) $e^{-2(t-kT)},\ kT < t < (k+1)T,\ k = 0, \pm 1, \pm 2, \ldots$

(e) $\cos(0.01n)$

(a) DTFT, (b) FT, (c) DTFS, (d) FS (e) DTFT

Question 2 (10 marks)

A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period of $T = 8$. The nonzero Fourier series coefficients are specified as:

$$X[1] = X^*[-1] = j, \quad X[-5] = X^*[5] = 2$$

Express $x(t)$ in the form (with the help of Euler):

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$$

We know that:

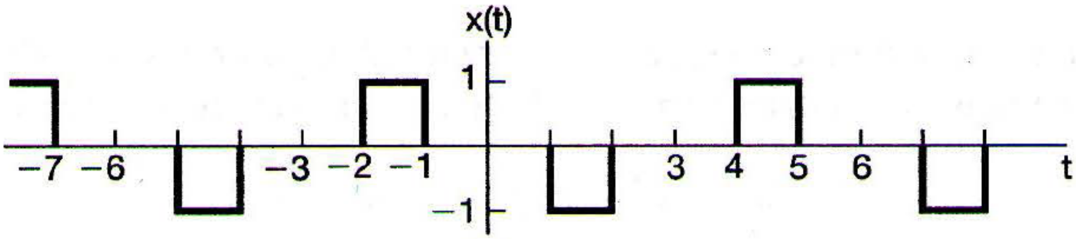
$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \\
&= X[-5]e^{-j5(\pi/4)t} + X[-1]e^{-j(\pi/4)t} + X[1]e^{j(\pi/4)t} + X[5]e^{j5(\pi/4)t} \\
&= 2e^{-j5(\pi/4)t} - je^{-j(\pi/4)t} + je^{j(\pi/4)t} + 2e^{j5(\pi/4)t} \\
&= j\left(e^{j(\pi/4)t} - e^{-j(\pi/4)t}\right) + 2\left(e^{-j5(\pi/4)t} + e^{j5(\pi/4)t}\right) \\
&= j\left(2j\sin\left(\frac{\pi}{4}t\right)\right) + 2\left(2\cos\left(\frac{5\pi}{4}t\right)\right) = -2\sin\left(\frac{\pi}{4}t\right) + 4\cos\left(\frac{5\pi}{4}t\right) \\
&= 2\cos\left(\frac{\pi}{4}t + \frac{\pi}{2}\right) + 4\cos\left(\frac{5\pi}{4}t\right)
\end{aligned}$$

NOTE: If you plot $|X[k]|$ and $\angle X[k]$ you can directly obtain the final answer (just remember to x 2)!

Question 3 (10 marks)

Determine the Fourier series representation (i.e. fundamental period ω_0 and the co-efficients $X[k]$) of the following signal:



Please simplify your expression for $X[k]$ where possible (e.g. use Euler's relation)

We can see that $T_0 = 6$ so that $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}$. To derive the co-efficients we need to do:

$$X[k] = \frac{1}{6}\int_{T_0} x(t) e^{-jk\frac{\pi}{3}t}\, dt$$

Let us choose the interval $-2 \le t < 4$ so that we get:

$$X[k] = \frac{1}{6}\int_{-2}^{4} x(t)e^{-jk\frac{\pi}{3}t}\,dt = \frac{1}{6}\left\{\int_{-2}^{-1}(1).e^{-jk\frac{\pi}{3}t}\,dt + \int_{1}^{2}(-1).e^{-jk\frac{\pi}{3}t}\,dt\right\}$$

$$= \frac{1}{6}\left\{\left[-\frac{3}{jk\pi}e^{-jk\frac{\pi}{3}t}\right]_{t=-2}^{-1} - \left[-\frac{3}{jk\pi}e^{-jk\frac{\pi}{3}t}\right]_{t=1}^{2}\right\} = -\frac{1}{2jk\pi}\left\{\left(e^{jk\frac{\pi}{3}} - e^{jk\frac{2\pi}{3}}\right) - \left(e^{-jk\frac{2\pi}{3}} - e^{-jk\frac{\pi}{3}}\right)\right\}$$

$$= \frac{j}{2k\pi}\left\{\left(e^{jk\frac{\pi}{3}} + e^{-jk\frac{\pi}{3}}\right) - \left(e^{jk\frac{2\pi}{3}} + e^{-jk\frac{2\pi}{3}}\right)\right\} = \frac{j}{k\pi}\left\{\cos\left(k\frac{\pi}{3}\right) - \cos\left(k\frac{2\pi}{3}\right)\right\}$$

Question 4 (10 marks)

(a) Use the defining equation to calculate the Fourier transform, $X(j\omega)$, by direct integration of the following signal:

$$x(t) = e^{-2(t-1)}u(t-1)$$

(b) Repeat but this time use the table of Fourier transform pairs and properties

(a) We do as follows:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-2(t-1)}u(t-1)e^{-j\omega t}dt$$

$$= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt = e^{2}\int_{1}^{\infty} e^{-(2+j\omega)t}dt$$

$$= e^{2}\left[\frac{e^{-(2+j\omega)t}}{-(2+j\omega)}\right]_{t=1}^{\infty} = e^{2}\left[0 + \frac{e^{-(2+j\omega)}}{2+j\omega}\right] = \frac{e^{-j\omega}}{2+j\omega}$$

(b) Let us define:

$$v(t) = e^{-2t}u(t), \text{ so that } x(t) = v(t-1)$$

Then:

$$v(t) = e^{-2t}u(t) \xrightarrow{Pair} V(j\omega) = \frac{1}{2+j\omega}$$

$$x(t) = v(t-1) \xrightarrow{Prop} X(j\omega) = e^{-j\omega}V(j\omega) = \frac{e^{-j\omega}}{2+j\omega}$$

Question 5 (10 marks)

(a) Use the defining equation of the inverse Fourier transform to derive the real-valued signal function, $x(t)$, by direct integration of its Fourier Transform $X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$ where:

$$|X(j\omega)| = \begin{cases} 2 & |\omega| \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\angle X(j\omega) = -\frac{3}{2}\omega + \pi$$

Simplify your expression for $x(t)$ where possible (**Hint:** your Fourier friend is Euler).

(b) Find all values of t such that $x(t) = 0$?

(a) We need to calculate:

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|e^{j\angle X(j\omega)}e^{j\omega t}d\omega \\
&= \frac{1}{\pi}\int_{-3}^{3} e^{j\left(-\frac{3}{2}\omega+\pi\right)}e^{j\omega t}d\omega = \frac{1}{\pi}\int_{-3}^{3} e^{j\left(\omega\left(t-\frac{3}{2}\right)+\pi\right)}d\omega \\
&= \frac{1}{\pi}\left\{\left[\frac{e^{j\left(\omega\left(t-\frac{3}{2}\right)+\pi\right)}}{j\left(t-\frac{3}{2}\right)}\right]_{\omega=-3}^{3}\right\} = \frac{e^{j\pi}}{\pi}\left\{\left[\frac{e^{j\omega\left(t-\frac{3}{2}\right)}}{j\left(t-\frac{3}{2}\right)}\right]_{\omega=-3}^{3}\right\} = -\frac{1}{\pi}\left\{\frac{e^{j3\left(t-\frac{3}{2}\right)}-e^{-j3\left(t-\frac{3}{2}\right)}}{j\left(t-\frac{3}{2}\right)}\right\} \\
&= -\frac{1}{\pi}\frac{2j\sin(3(t-3/2))}{j(t-3/2)} = -\frac{6}{\pi}\frac{\sin(3(t-3/2))}{3(t-3/2)}
\end{aligned}
$$

(b) $x(t) \propto \frac{\sin(x)}{x}$ is zero when $\sin(x) = 0$ ($x \neq 0$) which is true whenever $x = k\pi$ for $k = \pm1, \pm2, \pm3, \ldots$, that is:

$$3\left(t-\frac{3}{2}\right) = k\pi \to t = \frac{k\pi}{3} + \frac{3}{2}$$

PLEASE TEAR THIS PAGE AND KEEP

Time Domain	*Periodic* (t,n)	*Non periodic* (t,n)	
Continuous (t)	Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$ $X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$ $x(t)$ has period T $\omega_o = \frac{2\pi}{T}$	Fourier Transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	*Nonperiodic* (k,ω)
Discrete (n)	Discrete-Time Fourier Series $x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n}$ $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n}$ $x[n]$ and $X[k]$ have period N $\Omega_o = \frac{2\pi}{N}$	Discrete-Time Fourier Transform $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π	*Periodic* (k,Ω)
	Discrete (k)	*Continuous* (ω,Ω)	*Frequency Domain*

Euler's Relation and friends

$$e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)} = 2\cos(\omega t+\phi)$$
$$e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)} = 2j\sin(\omega t+\phi)$$
$$\sin(\omega t+\phi) = \cos(\omega t+\phi-\pi/2)$$

Symmetry

If $x(t) = x(-t)$ is an even signal then $X^*[k] = X[k]$, $X^*(j\omega) = X(j\omega)$. For real signals this implies $X[k]$, $X(j\omega)$ is real (no imaginary component).

If $x(t) = -x(-t)$ is an odd signal then $X^*[k] = -X[k]$, $X^*(j\omega) = -X(j\omega)$. For real signals this implies $X[k]$, $X(j\omega)$ is imaginary (no real component).

If $x(t)$ is a real periodic signal then we have the following conjugate symmetry property:

$$X[-k] = X^*[k], \quad X(-j\omega) = X^*(j\omega)$$

The real component is an even function and the imaginary component is an odd function
The magnitude spectrum, $|X[k]|$, $|X(j\omega)|$ is an even function and the phase spectrum, $\angle X[k]$, $\angle X(j\omega)$ is an odd function

Parseval's Theorem

$$P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x^2(t)dt = \sum_{k=-\infty}^{\infty} |X[k]|^2, \quad E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(j\omega)|^2 df$$

YOU CAN TEAR THIS PAGE AND KEEP

Table of Fourier Transform Pairs and Properties

$x(t) = u(t)$	$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$
$x(t) = e^{-at}u(t), \quad \mathrm{Re}\{a\} > 0$	$X(j\omega) = \frac{1}{a + j\omega}$
$x(t) = te^{-at}u(t), \quad \mathrm{Re}\{a\} > 0$	$X(j\omega) = \frac{1}{(a + j\omega)^2}$
$x(t) = e^{-a\lvert t\rvert}, \quad a > 0$	$X(j\omega) = \frac{2a}{a^2 + \omega^2}$

Linearity	$ax(t) + by(t) \xleftrightarrow{FT} aX(j\omega) + bY(j\omega)$
Time shift	$x(t - t_o) \xleftrightarrow{FT} e^{-j\omega t_o}X(j\omega)$
Frequency shift	$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega - \gamma))$
Scaling	$x(at) \xleftrightarrow{FT} \frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Differentiation in time	$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$
Differentiation in frequency	$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$
Integration/ Summation	$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$
Convolution	$\int_{-\infty}^{\infty} x(\tau)y(t - \tau)\,d\tau \xleftrightarrow{FT} X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t) \xleftrightarrow{FT} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\nu)Y(j(\omega - \nu))\,d\nu$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\,d\omega$
Duality	$X(jt) \xleftrightarrow{FT} 2\pi x(-\omega)$
Symmetry	$x(t)$ real $\xleftrightarrow{FT} X^*(j\omega) = X(-j\omega)$ $x(t)$ imaginary $\xleftrightarrow{FT} X^*(j\omega) = -X(-j\omega)$ $x(t)$ real and even $\xleftrightarrow{FT} \mathrm{Im}\{X(j\omega)\} = 0$ $x(t)$ real and odd $\xleftrightarrow{FT} \mathrm{Re}\{X(j\omega)\} = 0$