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UWA – ENSC3015 Signals and Systems

Please complete your details below:

Surname: ______________________________ Number: ____________________

Signature: ______________________________ Date: __________________

8:55am, Wednesday, October 24, 2018 in **ENCM LT1**

Class Test 4:
Fourier Transform and Sampling

Time allowed: 45 minutes
Max mark: **XX,** Assessment: **5%**[1]

This paper contains:
X pages, x questions

Candidates should attempt **all** questions and show **all** working with numerical answers to **3** decimal places in the spaces provided after each question, show as much working as possible to gain maximum marks. You can use the blank page on the reverse side for rough working, but these pages will not be marked

FOR THE ATTACHMENTS PLEASE REFER TO THE SEPARATED PAGES

[1] If you do better in the exam and you make a fair attempt, this test will not contribute to your unit marks and the 5% will come from the final exam performance. If you do better in this test compared to the final exam then this test will be included in the unit marks.

Question 1 (8 marks)

We prove the conjugate symmetry property $X(-j\omega) = X^*(j\omega)$ for real signals as follows:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \rightarrow X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = X(-j\omega)$$

by noting that for real signals we can state $x^*(t) = x(t)$.

(a) What can we state for purely imaginary signals regarding $x(t)$ and $x^*(t)$?

(b) Use (a) to develop the conjugate symmetry property that applies to purely imaginary signals.

(c) Use (b) and state what we can say about the real component and imaginary component of $X(j\omega)$ for purely imaginary signals. HINT: Let $X(j\omega) = a(\omega) + jb(\omega)$ and use the conjugate symmetry property from (b).

(a) For imaginary signals $x^*(t) = -x(t)$

(b)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \rightarrow X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt = -\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = -X(-j\omega)$$

If $x(t)$ is an imaginary signal then we have the following *conjugate symmetry* property:

$$X^*(j\omega) = -X(-j\omega)$$

(c) Let $X(j\omega) = a(\omega) + jb(\omega)$ then:

$$X^*(j\omega) = a(\omega) - jb(\omega)$$
$$-X(-j\omega) = -a(-\omega) - jb(-\omega)$$

Hence: the real component ($a(-\omega) = -a(\omega)$) is an odd function and the imaginary component ($b(-\omega) = b(\omega)$) is an even function.

Question 2 (10 marks)

Referring to the Fourier transforms below determine the Nyquist sampling rate (in kHz) for the signal:

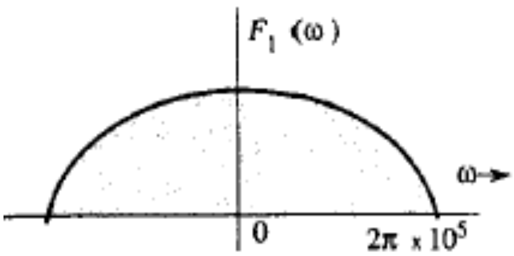
(a) $f_1(t)$

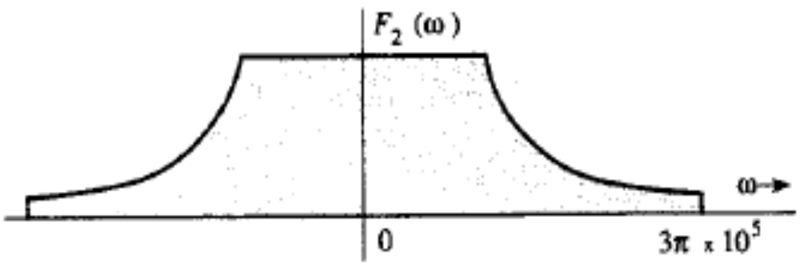
(b) $f_2(t)$

(c) $f_1(t) + f_2(t)$

(d) $f_1(t)f_2(t)$

(e) $f_1(t) * f_2(t)$





HINT: If $F_1(\omega)$ is a signal of bandwidth W_1 and $F_2(\omega)$ is a signal of bandwidth W_2, the bandwidth of $F_1(\omega) * F_2(\omega)$ is $W_1 + W_2$ (Width Property of convolution).

(a) Maximum frequency of $f_1(t)$ is $B = 2\pi(10^5)$ and thus $f_S = 2B = \frac{4\pi(10^5)}{2\pi} = 2(10^5)$ Hz or 200 kHz

(b) Maximum frequency of $f_2(t)$ is $B = 3\pi(10^5)$ and thus $f_S = 2B = \frac{6\pi(10^5)}{2\pi} = 3(10^5)$ Hz or 300 kHz

(c) Maximum frequency of $f_1(t) + f_2(t)$ is $B = 3\pi(10^5)$ and thus $f_S = 2B = \frac{6\pi(10^5)}{2\pi} = 3(10^5)$ Hz or 300 kHz

(d) Since $f_1(t)f_2(t) \leftrightarrow \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$ and the width property of convolution tells us that the bandwidth of the convolution spectrum is the sum of the individual spectra, that is $B = 5\pi(10^5)$ and $f_S = 2B = \frac{10\pi(10^5)}{2\pi} = 5(10^5)$ Hz or 500 kHz

(e) Since $f_1(t) * f_2(t) \leftrightarrow F_1(\omega)F_2(\omega)$ then the bandwidth is $B = 2\pi(10^5)$ and thus $f_S = 2B = \frac{4\pi(10^5)}{2\pi} = 2(10^5)$ Hz or 200 kHz

Question 3 (8 marks)

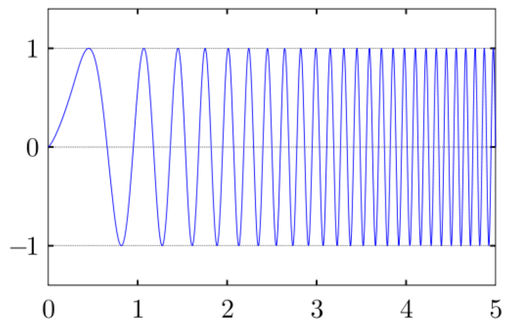
A signal $x_a(t)$ is sampled at a rate of 1600 Hz. Only 300 samples have been collected. You want to apply an FFT to plot the magnitude spectrum. But you want to do this in such a way that the frequency resolution or precision is no more than 2 Hz.

(a) How do you do this if you are allowed to zero-pad the samples?

(b) How do you do this if can go back and acquire more samples?

(c) Why is (b) better than (a)?

(d) Consider $x_a(t)$ as a chirp signal where the frequency increases linearly with time:



Would (b) still be better than (a)?

(a) The minimum number of samples we need is:

$$N_0 = \frac{f_S}{f_0} = \frac{1600}{2} = 800$$

Hence we need to zero-pad by adding an extra 500 samples of zero value.

(b) Just acquire 500 more samples so you have 800 samples in total, and no need to zero-pad.

(c) Method (b) is superior because with more samples of the actual data you have more information to provide a better estimate of the true spectrum. When you only provide a finite sample of data you will always approximate the true spectrum (which needs to be evaluated across all time), zero-padding does not add any new information.

(d) With a signal with variable spectral characteristics it may be better to only acquire a small number of samples to avoid smearing across time (as the frequency varies) so method (a) may actually be better than method (b), it all depends on the signal being measured and how fast it varies with time.

Question 4 (4 marks)

(a) Consider:

$$x[n] = \sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right)$$

Which of the following statements is false and explain your answer:

(i) The Fourier spectrum only exists at specific frequencies

(ii) The signal $x[n]$ is periodic in time hence because of this the Fourier spectrum is periodic in frequency

(iii) If the signal $x[n]$ repeats every N_0 samples then the Fourier spectrum will also repeat every N_0 samples.

(b) Consider:

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

Which of the following statements is false and explain your answer:

(i) The Fourier spectrum exists at all frequencies

(ii) The Fourier spectrum is periodic in frequency because the time-domain signal $x[n]$ is sampled in time.

(iii) Since $x[n]$ is not periodic (and has finite energy) then from Parseval's theorem the Fourier spectrum cannot be periodic (otherwise it would have infinite energy).

(a) <u>(ii) is false</u> because although the Fourier spectrum for the DTFS is periodic this is NOT because $x[n]$ is periodic but because $x[n]$ is discrete sampled in the time domain (Remember! Sampling in one domain implies periodicity in the other domain).

(b) <u>(iii) is false</u>, since we know the Fourier spectrum will be periodic (because the time-domain is sampled). FYI, Parseval's theorem for the DTFT only considers one period of the periodic Fourier spectrum to provide finite energy.

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Time Domain	*Periodic* (t, n)	*Non periodic* (t, n)	
Continuous (t)	Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_o t}$ $X[k] = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_o t}\, dt$ $x(t)$ has period T $\omega_o = \frac{2\pi}{T}$	Fourier Transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$	*Nonperiodic* (k, ω)
Discrete (n)	Discrete-Time Fourier Series $x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n}$ $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}$ $x[n]$ and $X[k]$ have period N $\Omega_o = \frac{2\pi}{N}$	Discrete-Time Fourier Transform $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n}\, d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π	*Periodic* (k, Ω)
	Discrete (k)	*Continuous* (ω, Ω)	*Frequency Domain*

Euler's Relation and friends

$$e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)} = 2\cos(\omega t+\phi)$$
$$e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)} = 2j\sin(\omega t+\phi)$$
$$\sin(\omega t+\phi) = \cos(\omega t+\phi-\pi/2)$$

Symmetry

If $x(t) = x(-t)$ is an <u>even signal</u> then $X^*(j\omega) = X(j\omega)$. For real signals this implies $X(j\omega)$ is real (no imaginary component).

If $x(t) = -x(-t)$ is an <u>odd signal</u> then $X^*(j\omega) = -X(j\omega)$. For real signals this implies $X(j\omega)$ is imaginary (no real component).

If $x(t)$ is a <u>real signal</u> then we have the following *conjugate symmetry* property:

$$X^*(j\omega) = X(-j\omega)$$

The <u>real component is an even function</u> and the <u>imaginary component is an odd function</u>
The <u>magnitude spectrum, $|X(j\omega)|$ is an even function</u> and the <u>phase spectrum, $\angle X(j\omega)$ is an odd function</u>

Nyquist rate

Discrete-time signals which are samples taken from a bandlimited (B Hz) continuous-time signal must be sampled at the **Nyquist rate** of $2B$ samples per second or more to be correctly recovered.

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Table of Fourier Transform Pairs

$x(t) = \begin{cases} 1, & \|t\| \le T_o \\ 0, & \text{otherwise} \end{cases}$	$X(j\omega) = \dfrac{2\sin(\omega T_o)}{\omega}$
$x(t) = \dfrac{1}{\pi t}\sin(Wt)$	$X(j\omega) = \begin{cases} 1, & \|\omega\| \le W \\ 0, & \text{otherwise} \end{cases}$
$x(t) = \delta(t)$	$X(j\omega) = 1$
$x(t) = 1$	$X(j\omega) = 2\pi\delta(\omega)$
$x(t) = u(t)$	$X(j\omega) = \dfrac{1}{j\omega} + \pi\delta(\omega)$
$x(t) = e^{-at}u(t), \quad \text{Re}\{a\} > 0$	$X(j\omega) = \dfrac{1}{a + j\omega}$
$x(t) = te^{-at}u(t), \quad \text{Re}\{a\} > 0$	$X(j\omega) = \dfrac{1}{(a + j\omega)^2}$
$x(t) = e^{-a\|t\|}, \quad a > 0$	$X(j\omega) = \dfrac{2a}{a^2 + \omega^2}$
$x(t) = \dfrac{1}{\sqrt{2\pi}}e^{-t^2/2}$	$X(j\omega) = e^{-\omega^2/2}$
$x(t) = \cos(\omega_o t)$	$X(j\omega) = \pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o)$
$x(t) = \sin(\omega_o t)$	$X(j\omega) = \dfrac{\pi}{j}\delta(\omega - \omega_o) - \dfrac{\pi}{j}\delta(\omega + \omega_o)$
$x(t) = e^{j\omega_o t}$	$X(j\omega) = 2\pi\delta(\omega - \omega_o)$
$x(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT_s)$	$X(j\omega) = \dfrac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\dfrac{2\pi}{T_s}\right)$
$x(t) = \begin{cases} 1, & \|t\| \le T_o \\ 0, & T_o < \|t\| < T/2 \end{cases}$, $x(t + T) = x(t)$	$X(j\omega) = \sum_{k=-\infty}^{\infty}\dfrac{2\sin(k\omega_o T_o)}{k}\delta(\omega - k\omega_o)$

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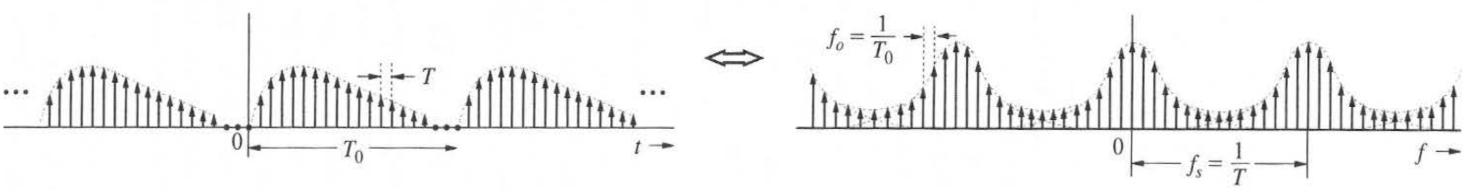
Table of Fourier Transform Properties

Linearity	$ax(t) + by(t) \xleftrightarrow{FT} aX(j\omega) + bY(j\omega)$
Time shift	$x(t - t_o) \xleftrightarrow{FT} e^{-j\omega t_o}X(j\omega)$
Frequency shift	$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega - \gamma))$
Scaling	$x(at) \xleftrightarrow{FT} \dfrac{1}{\lvert a\rvert}X\left(\dfrac{j\omega}{a}\right)$
Differentiation in time	$\dfrac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$
Differentiation in frequency	$-jtx(t) \xleftrightarrow{FT} \dfrac{d}{d\omega}X(j\omega)$
Integration/ Summation	$\displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$
Convolution	$\displaystyle\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,d\tau \xleftrightarrow{FT} X(j\omega)Y(j\omega)$
Multiplication	$\displaystyle x(t)y(t) \xleftrightarrow{FT} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\nu)Y(j(\omega-\nu))\,d\nu$
Parseval's Theorem	$\displaystyle\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\,d\omega$
Duality	$X(jt) \xleftrightarrow{FT} 2\pi x(-\omega)$
Symmetry	$x(t)$ real $\xleftrightarrow{FT} X^*(j\omega) = X(-j\omega)$ $x(t)$ imaginary $\xleftrightarrow{FT} X^*(j\omega) = -X(-j\omega)$ $x(t)$ real and even $\xleftrightarrow{FT} \mathrm{Im}\{X(j\omega)\} = 0$ $x(t)$ real and odd $\xleftrightarrow{FT} \mathrm{Re}\{X(j\omega)\} = 0$

Sampling in time and frequency (digital signal processing)



$$N_0 = \frac{T_0}{T} = \frac{f_S}{f_0},$$ where N_0 is the number of samples over T_0 or f_S