Question 1 (10 marks)

Determine the Fourier series representation , $X[k]$, of:

$$x(t) = 2\sin(2\pi t - 3) + \sin(6\pi t)$$

HINT: Expand using *Euler's relation* and match with the *inverse Fourier series* expression.

Solution:

Period $T = 1$ sec; Fundamental Frequency, $\omega_0 = 2\pi$ radians

$$\begin{aligned}
x(t) &= \tfrac{1}{2j}\left\{2e^{j2\pi t}\cdot e^{-j3} - 2e^{-j2\pi t}\cdot e^{j3} + e^{j6\pi t} - e^{-j6\pi t}\right\} \\
&= \tfrac{j}{2}e^{j(-3)2\pi t} + je^{j3}\cdot e^{j(-1)2\pi t} - je^{-j3}\cdot e^{j(1)2\pi t} - \tfrac{j}{2}e^{j(3)2\pi t} \\
&= X[-3]e^{j(-3)2\pi t} + X[-1]e^{j(-1)2\pi t} + X[1]e^{j(1)2\pi t} + X[3]e^{j(3)2\pi t} \\
&= \sum_{k=-\infty}^{\infty} X[k]e^{jk2\pi t}
\end{aligned}$$

Note that $X[k] = 0$ for k values other than $-3, -1, 1, 3$.

$$X[k] = \begin{cases} j/2 & k = -3 \\ je^{j3} & k = -1 \\ -je^{-j3} & k = 1 \\ -j/2 & k = 3 \\ 0 & otherwise \end{cases}$$

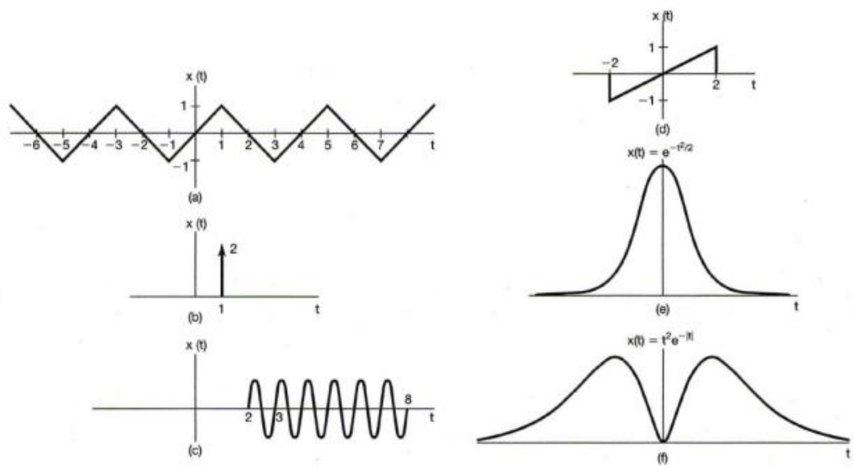
For each condition listed below determine which, if any, of the real signals depicted below have Fourier transforms that satisfy that condition:

(A) $\text{Re}\{X(j\omega)\} = 0$

(B) $\text{Im}\{X(j\omega)\} = 0$

(C) $|X(j\omega)| = 2$, for all ω

(D) $X(j\omega)$ is non-zero only for certain values of ω



Solution:

(A) implies that $x(t)$ must be an odd function about $t = 0$ ($X(j\omega)$ is purely imaginary): Only **(a) and (d)** have this property.

(B) implies that $x(t)$ must be an even function about $t = 0$ ($X(j\omega)$ is purely real): Only **(e) and (f)** have this property.

(C) implies that the spectrum is flat, which can only happen with an impulse function: Only **(b)** has this property.

(D) If $X(j\omega)$ is non-zero only for certain values of ω then this is equivalent to a Fourier series representation and hence $x(t)$ must be periodic. Only **(a)** has this property.

Consider an LTI system frequency response:

$$H(j\omega) = \frac{3(3 + j\omega)}{8 - \omega^2 + 6j\omega}$$

(a) What is the <u>magnitude response</u>, $|H(j\omega)|$?
(b) What is the <u>phase response</u>, $\angle H(j\omega)$? <u>HINT:</u> Use the atan2(*y*,*x*) function for the phase of $x + iy$
(c) If the input is $x(t) = 3\cos(2t)$, what is the output $y(t)$?
(d) By evaluating $|H(j\omega)|$ at $\omega = 0$, $\omega^2 = 8$ and $\omega \to \infty$ indicate whether this LTI system represents a low-pass, high-pass or band-pass response?

<u>Solution:</u>

$$H(j\omega) = \frac{9 + j3\omega}{(8 - \omega^2) + j6\omega}$$

(a)

$$|H(j\omega)| = \frac{3\sqrt{9 + \omega^2}}{\sqrt{(8 - \omega^2)^2 + 36\omega^2}}$$

(b)

$$\angle H(j\omega) = \mathrm{atan2}(\omega, 3) - \mathrm{atan2}(6\omega, 8 - \omega^2)$$

(NOTE: $\mathrm{atan2}(3\omega, 9)$ also acceptable, but common factors should be removed)

(c)
For $\omega = 2$ then:

$$|H(j2)| = \frac{10.817}{12.649} = 0.8552$$

$$\angle H(j2) = \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{12}{4}\right) = 0.5880 - 1.2490 = -0.6610$$

Hence:

$$y(t) = 2.5656\cos(2t - 0.6610)$$

(d) From (a) we see that:

$$|H(j0)| = 1.1250, \quad \left|H\left(j\sqrt{8}\right)\right| = 0.7289, \quad |H(j\infty)| = 0$$

This is a **low-pass filter** response

1. Identify the appropriate Fourier representation (FS, FT, DTFT, DTFS) for each of the following signals

(a) $\frac{1}{n} + \sin\left(\frac{\pi n}{5}\right)$
(b) $\sin(2\pi t^2)$
(c) $3 + \left|\cos\left(\frac{\pi n}{3}\right)\right|$
(d) $e^{-2(t-kT)},\ kT < t < (k+1)T,\ k = 0, \pm 1, \pm 2, \ldots$
(e) $\cos(0.01n)$

Answers:

(a) DTFT, (discrete-time since a function of sample number, *n*, and NOT periodic)

(b) FT, (continuous-time since a function of time, *t*, and NOT periodic due to t^2)

(c) DTFS, (discrete-time since a function of sample number, *n*, and periodic)

(d) FS (continuous-time since a function of time, *t*, and periodic)

(e) DTFT (discrete-time since a function of sample number, *n*, and NOT periodic as $F = \frac{\Omega}{2\pi} = \frac{0.01}{2\pi} = \frac{1}{200\pi}$ is NOT a rational number)

(i) Consider the continuous-time signal:

$$x(t) = 3\cos 6\pi t + \sin 18\pi t + 2\cos 28\pi t$$

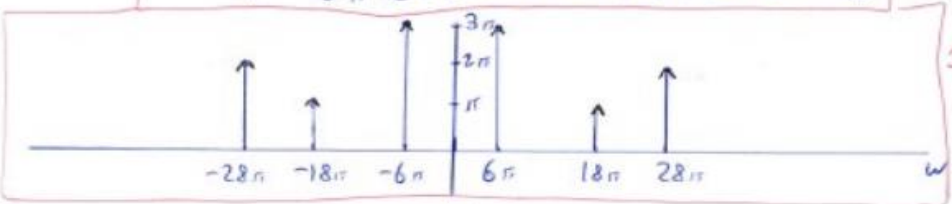
What is the expression for the magnitude spectrum, $|X(\omega)|$, and sketch it as a function of ω.
HINT: You have three sinusoids so you will have just three spectral harmonics.

(ii) What is the Nyquist rate, that is, determine the range of possible sampling frequencies (in Hz), f_s, required to be able to reconstruct $x(t)$ from these samples without error?

(iii) What is the sampling frequency if you sample $x(t)$ at 25% above the Nyquist rate (i.e. your answer in (ii) x 1.25)? For this sampling frequency carefully sketch the magnitude spectrum of the sampled signal (in Hz) over the range $\pm$ 50 Hz.

1 (i) $x(t) = 3\cos(6\pi t) + \sin(18\pi t) + 2\cos(28\pi t)$

$$\therefore \boxed{\begin{aligned}|X(\omega)| = {} & 3\pi\,[\delta(\omega-6\pi)+\delta(\omega+6\pi)] \\ & + \pi\,[\delta(\omega+18\pi)+\delta(\omega-18\pi)] \\ & + 2\pi\,[\delta(\omega-28\pi)+\delta(\omega+28\pi)]\end{aligned}}$$ 1



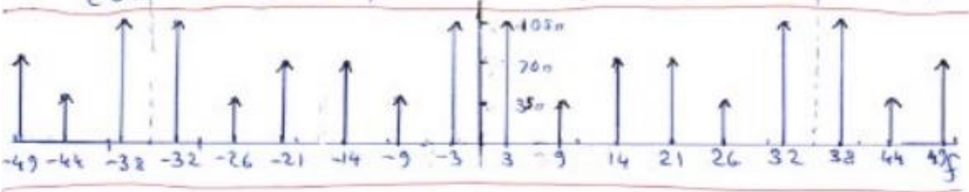
1

(ii) $B = 28\pi$ $\qquad \omega_s \geq 2B = 56\pi$

$\therefore \boxed{f_s \geq 28\ \text{Hz}}$ 1

(iii) 25% above 28 Hz $\Rightarrow \boxed{f_s = 35\ \text{Hz}}$ 1

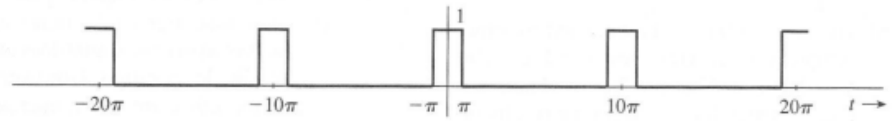
($6\pi \to 3$ Hz ; $18\pi \to 9$ Hz ; $28\pi \to 14$ Hz)



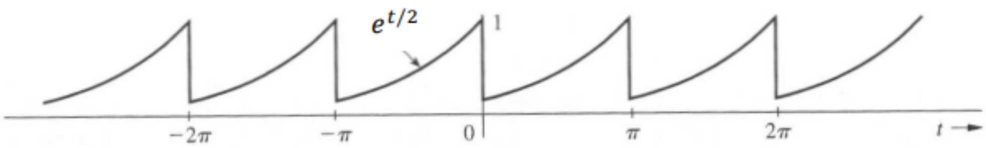
Gain of 35

$3\pi \to 105\pi$ $\quad 2\pi \to 70\pi$ $\quad \pi \to 35\pi$ 2

1. For the following periodic signals find the compact trigonometric Fourier series anc sketch the amplitude and phase spectra:



(a)



(b)

Answers:

(a) We have $T_0 = 10\pi \rightarrow \omega_0 = \frac{2\pi}{T_0} = \frac{1}{5}$ and even symmetry implies the sine terms are zero.

$$A[k] = \frac{2}{T_0}\int_{-5\pi}^{5\pi} x(t)\cos(k\omega_0 t)\,dt = \frac{2}{10\pi}\int_{-\pi}^{\pi}\cos\left(\frac{k}{5}t\right)dt$$

$$= \frac{2}{10\pi}\left[\frac{5}{k}\sin\left(\frac{k}{5}t\right)\right]_{-\pi}^{\pi} = \frac{2}{k\pi}\sin\left(\frac{k\pi}{5}\right)$$

For $A[0] = \frac{1}{T_0}\int_{-5\pi}^{5\pi} x(t)\,dt = \frac{1}{10\pi}(2\pi) = \frac{1}{5}$ and since $B[k] = 0$ then:

$$x(t) = A[0] + \sum_{k=1}^{\infty} A[k]\cos(k\omega_0 t) = \frac{1}{5} + \sum_{k=1}^{\infty}\left[\frac{2}{k\pi}\sin\left(\frac{k\pi}{5}\right)\right]\cos\left(\frac{k}{5}t\right)$$

Amplitude (where we allow negative values) and Phase spectra given by:

$$C[k] = A[k] = \frac{2}{k\pi}\sin\left(\frac{k\pi}{5}\right),\quad C[0] = \frac{1}{5} \qquad \theta[k] = \tan^{-1}\left(\frac{-B[k]}{A[k]}\right) = 0$$

Or if we insist on $C[k]$ being positive then:

$$C[k] = \sqrt{A^2[k] + B^2[k]} = \left|\frac{2}{k\pi}\sin\left(\frac{k\pi}{5}\right)\right|,\quad C[0] = \frac{1}{5} \qquad \theta[k] = \begin{cases} 0 & A[k] > 0 \\ \pi & A[k] < 0 \end{cases}$$

2. Are the following signals periodic? If so, what is the period and what harmonics are present? If not, why not?

(a) $3\sin t + 2\sin 3t$

(b) $2\sin 3t + 7\cos \pi t$

(c) $7\cos \pi t + 5\sin 2\pi t$

(d) $\sin\frac{5t}{2} + 3\cos\frac{6t}{5} + 3\sin\left(\frac{t}{7} + 30^o\right)$

Answers:

(a) Ratio of frequencies is $\frac{3}{1}$ hence periodic. The GCF is $\omega_0 = 1$ and hence $T_0 = 2\pi$. The 1st and 3rd harmonics are present.

(b) Ratio of frequencies is $\frac{\pi}{3}$ which is not a ratio of integers so NOT periodic.

(c) Ratio of frequencies is $\frac{2\pi}{\pi} = \frac{2}{1}$ hence periodic. The GCF is $\omega_0 = \pi$ and hence $T_0 = 2$. The 1st and 2nd harmonics are present.

(d) Consider all frequency pairs:

$$\frac{6/5}{5/2} = \frac{12}{25},\qquad \frac{1/7}{5/2} = \frac{2}{35},\qquad \frac{1/7}{6/5} = \frac{5}{42}$$

hence periodic. To find the GCF we sort the harmonics and find the common factor:

$$\frac{1}{7}:\frac{6}{5}:\frac{5}{2} \rightarrow \frac{10}{70}:\frac{84}{70}:\frac{175}{70}$$

Hence the GCF is $\omega_0 = \frac{1}{70}$ and $T_0 = 140\pi$ and the 10th, 84th and 175th harmonics are present.

4. If a periodic signal $x(t)$ is expressed by the exponential Fourier series.

(a) What happens to the Fourier spectrum when the signal is time-shifted (say $x(t) \to x(t-T)$?

(b) What happens to the Fourier spectrum when the signal is compressed/dilated in time (say $x(t) \to x(at)$)?

Answers:

(a) The exponential FS representation of $x(t)$ is:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$$

If we let $y(t) = x(t-T)$ then:

$$y(t) = \sum_{k=-\infty}^{\infty} Y[k]e^{jk\omega_0 t} \equiv x(t-T) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 (t-T)} = \sum_{k=-\infty}^{\infty} \left(X[k]e^{-jk\omega_0 T}\right)e^{jk\omega_0 t}$$

And thus:

$$Y[k] = X[k]e^{-jk\omega_0 T} \to \quad |Y[k]| = |X[k]|, \quad \angle Y[k] = \angle X[k] - k\omega_0 T$$

For a time delay of T the amplitude spectrum is unchanged but the phase spectrum is offset by the factor $-k\omega_0 T$.

(b) If we now consider $y(t) = x(at)$ we have:

$$y(t) = \sum_{k=-\infty}^{\infty} Y[k]e^{jk\omega_0 t} \equiv x(at) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 at} = \sum_{k=-\infty}^{\infty} X[k]e^{jk(a\omega_0)t}$$

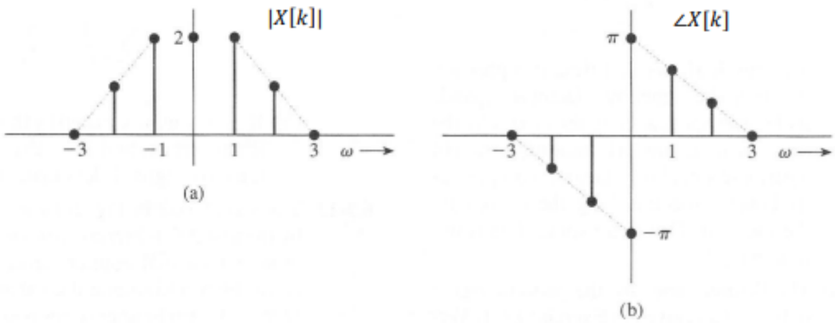
Yielding a change in the fundamental frequency $\omega_0 \to a\omega_0$. If $a > 1$ there is time compression of the signal waveform which would imply a higher fundamental frequency, whereas for $a < 1$ there is time expansion / stretching of the signal waveform which would imply a lower fundamental frequency.

3. Consider the exponential Fourier spectra below for the signal $x(t)$ where $\omega_0 = 1$.

(a) Find the exponential Fourier series expression for $x(t)$

(b) Find the compact trigonometric Fourier expression for $x(t)$

(c) Show that (a) and (b) are equivalent.



Answers:

Since $|X[\pm 3]| = 0$ in the following we will ignore $k = \pm 3$.

(a) Since $X[k] = |X[k]|e^{j\angle X[k]}$ then $X[k]e^{jk\omega_0 t} = |X[k]|e^{j(k\omega_0 t + \angle X[k])}$ and thus:

$$x(t) = \sum_{k=-2}^{2} X[k]e^{jk\omega_0 t} = X[-2]e^{-j2t} + X[-1]e^{-jt} + X[0] + X[1]e^{jt} + X[2]e^{j2t}$$

$$= e^{-j\left(2t+\frac{\pi}{3}\right)} + 2e^{-j\left(t+\frac{2\pi}{3}\right)} - 2 + 2e^{j\left(t+\frac{2\pi}{3}\right)} + e^{j\left(2t+\frac{\pi}{3}\right)}$$

where:

$$X[-2] = 1e^{-j\frac{\pi}{3}}, \quad X[-1] = 2e^{-j\frac{2\pi}{3}}, \quad X[0] = 2e^{-j\pi} = 2.e^{j\pi} = -2$$
$$X[1] = 2e^{j\frac{2\pi}{3}}, \quad X[2] = 1e^{j\frac{\pi}{3}}$$

(b) We have that $C[0] = X[0] = -2$, $C[k] = 2|X[k]|$ and $\theta[k] = \angle X[k]$ for $k = 1,2$ and the compact trigonometric Fourier series expression ($\omega_0 = 1$) becomes:

$$x(t) = -2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right)$$

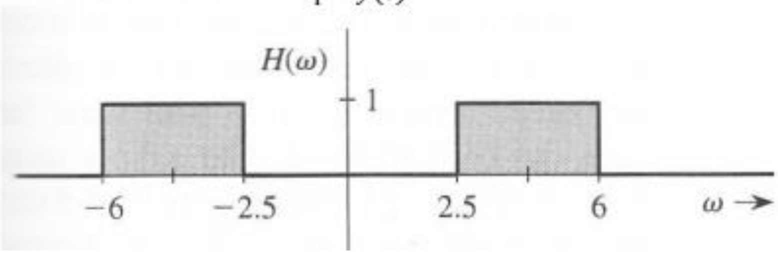
(c) Just use the following identity from Euler's formula in (b) to get (a)

$$2\cos(\omega t + \theta) = \left(e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}\right)$$

5. (a) Find the exponential Fourier series for the signal $x(t) = \cos 5t \sin 3t$. Use the fact that $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$.

(b) Sketch the Fourier spectra.

(c) The signal $x(t)$ is applied to the input of the LTI system with frequency response shown below. Find the output $y(t)$.



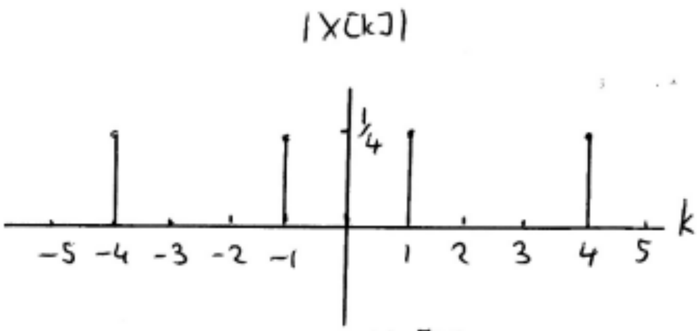
(a) $x(t) = \cos 5t \sin 3t = \frac{1}{2}(\sin 8t - \sin 2t)$ giving $\omega_0 = 2$ and using Euler's relation $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$ we see that (using the fact that $1/j = -j$):

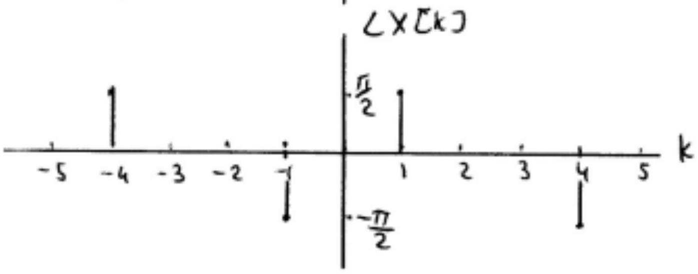
$$\begin{aligned}
x(t) &= \frac{1}{4j}\left(e^{j8t} - e^{-j8t}\right) - \frac{1}{4j}\left(e^{j2t} - e^{-j2t}\right) \\
&= \left(j\frac{1}{4}\right)e^{-j8t} + \left(-j\frac{1}{4}\right)e^{-j2t} + \left(j\frac{1}{4}\right)e^{j2t} + \left(-j\frac{1}{4}\right)e^{j8t} \\
&\equiv \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} X[k]e^{jk2t} \\
&= X[-4]e^{j(-4)2t} + X[-1]e^{j(-1)2t} + X[1]e^{j(1)2t} + X[4]e^{j(4)2t}
\end{aligned}$$

From which: $X[-4] = \left(j\frac{1}{4}\right), X[-1] = \left(-j\frac{1}{4}\right); X[1] = \left(j\frac{1}{4}\right); X[4] = \left(-j\frac{1}{4}\right)$ where $\angle j \rightarrow \frac{\pi}{2}$ and $\angle(-j) = -\frac{\pi}{2}$ and we note $e^{jk\omega_0 t} \equiv e^{jk2t}$ so that only the 1st and 4th $(k = 1, 4)$ harmonics exist (corresponding to $\omega = k\omega_0 = 2, 8$).

(b) We have $|X[-4]| = |X[-1]| = |X[1]| = |X[4]| = 0.25$ and

$$\angle X[-4] = \angle X[1] = \frac{\pi}{2}, \quad \angle X[-1] = \angle X[4] = -\frac{\pi}{2}$$

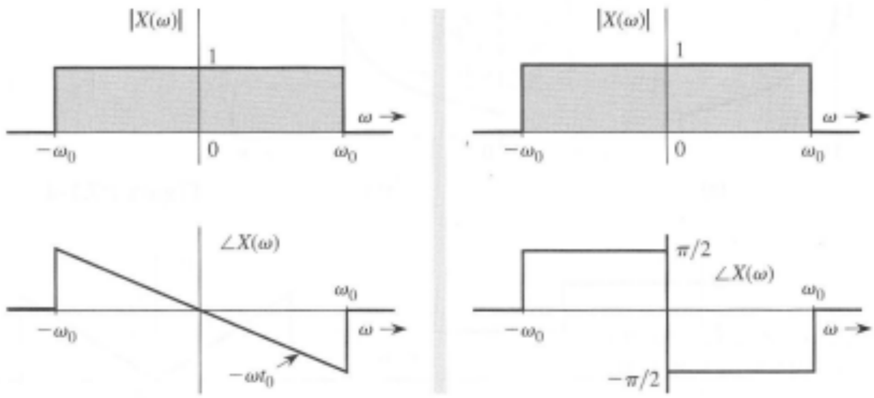




(c) We will have $y(t) = 0$ since none of the harmonic components of $x(t)$ at $\omega = k\omega_0 = 2, 8$ will pass through the passband filter range $\omega \in [2.5, \ldots, 6]$.

Find the fundamental frequency w and then rewrite the equation in terms of kw. This will then give you the harmonics and by inspection we can find X[k] for different values of k.

2. Find the inverse Fourier transform of the two different spectra below:



(a) (b)

What can you conclude?

Answers:

We note that $X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$

For both (a) and (b), $|X(j\omega)| = \begin{cases} 1 & -\omega_0 < \omega < \omega_0 \\ 0 & else \end{cases}$,

For (a), $\angle X(j\omega) = \begin{cases} -\omega t_0 & -\omega_0 < \omega < \omega_0 \\ 0 & else \end{cases}$ and for (b) $\angle X(j\omega) = \begin{cases} \pi/2 & -\omega_0 < \omega < 0 \\ -\pi/2 & 0 < \omega < \omega_0 \\ 0 & else \end{cases}$

(a) Inverse FT is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1.e^{-j\omega t_0}e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(t-t_0)}d\omega$$
$$= \frac{1}{(2\pi)j(t-t_0)}\left[e^{j\omega(t-t_0)}\right]_{-\omega_0}^{\omega_0} = \frac{\sin\omega_0(t-t_0)}{\pi(t-t_0)}$$

(b) Inverse FT is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\left[\int_{-\omega_0}^{0} 1.e^{j\pi/2}e^{j\omega t}d\omega + \int_{0}^{\omega_0} 1.e^{-j\pi/2}e^{j\omega t}d\omega\right]$$
$$= \frac{1}{2\pi}\left[\int_{-\omega_0}^{0} je^{j\omega t}d\omega + \int_{0}^{\omega_0} -je^{j\omega t}d\omega\right] = \frac{j}{2\pi}\left\{\left[\frac{1}{jt}e^{j\omega t}\right]_{-\omega_0}^{0} - \left[\frac{1}{jt}e^{j\omega t}\right]_{0}^{\omega_0}\right\}$$
$$= \frac{1-\cos\omega_0 t}{\pi t}$$

Even though the magnitude spectra are identical (same spectral content and distribution) these represent entirely different time signals.

(c) $x(t) = \left[\frac{2\sin(3\pi t)}{\pi t}\right]\left[\frac{\sin(2\pi t)}{\pi t}\right]$

$$\frac{\sin(Wt)}{\pi t} \xleftrightarrow{FT} \begin{cases} 1 & |\omega| \le W \\ 0, & \text{otherwise} \end{cases}$$

$$s_1(t)s_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}S_1(j\omega) * S_2(j\omega)$$

$$X(j\omega) = \begin{cases} 5 - \frac{|\omega|}{\pi} & \pi < |\omega| \le 5\pi \\ 4 & |\omega| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

8. For the signal:

$$x(t) = \frac{2a}{t^2 + a^2}$$

Determine the "essential" bandwidth B (in Hz) of $x(t)$ such that the energy containe in the spectral components of $x(t)$ for $f \le B$ is 99% of the total signal energy E_x.

Answers:

From the FT pair:

$$x(t) = e^{-a|t|} \leftrightarrow \frac{2a}{a^2 + \omega^2} = X(j\omega)$$

we use the FT duality property:

$$X(jt) = \frac{2a}{t^2 + a^2} \leftrightarrow 2\pi e^{-\alpha|\omega|} = 2\pi x(-\omega)$$

The signal energy is given by:

$$E_x = \frac{1}{\pi}\int_0^\infty |X(j\omega)|^2 d\omega = \frac{1}{\pi}\int_0^\infty \left|2\pi e^{-\alpha|\omega|}\right|^2 d\omega = 4\pi \int_0^\infty e^{-2\alpha\omega} d\omega = \frac{2\pi}{\alpha}$$

The energy contained with the band $0 \le f \le W$, where $W = 2\pi B$ is:

$$E_B = \frac{1}{\pi}\int_0^W |X(j\omega)|^2 d\omega = 4\pi \int_0^W e^{-2\alpha\omega} d\omega = \frac{2\pi}{\alpha}(1 - e^{-2\alpha W})$$

If $E_B = 0.99E_x$ then:

$$\frac{E_B}{E_x} = (1 - e^{-2\alpha W}) = 0.99 \Longrightarrow W = \frac{2.3025}{\alpha} \text{ rad/s} \Longrightarrow B = \frac{0.366}{\alpha} \text{ Hz}$$

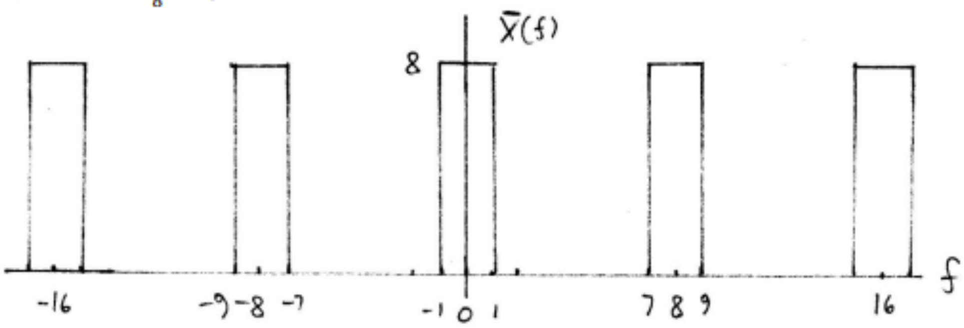
(a) First determine the FT spectrum of $x(t) = \frac{1}{\pi t}\sin(2\pi t)$. From the FT table we have:

$$\frac{\sin(Wt)}{\pi t} \overset{FT}{\Longleftrightarrow} \begin{cases} 1 & |\omega| < W \\ 0 & \text{otherwise} \end{cases}$$

With $W = 2\pi$ the FT spectrum exists over the range: $-2\pi < \omega < 2\pi$ or $-1 < f < 1$.

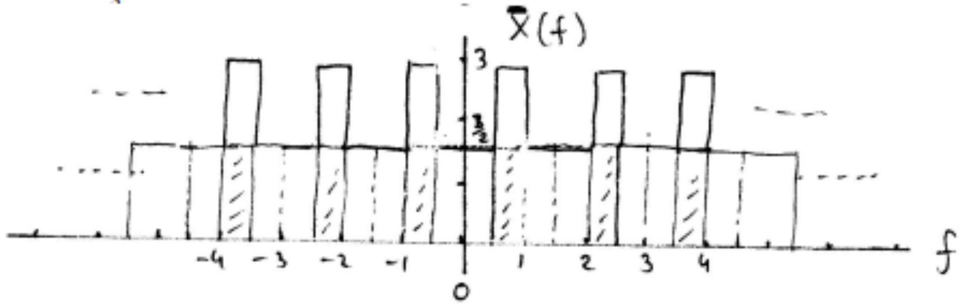
For each case below the FT spectrum will be given by: $f_s \sum_{k=-\infty}^{\infty} X(f - kf_s)$

(a) With $T = \frac{1}{8} \rightarrow f_s = 8$ hence:



as there is no interference from the replicas there is no aliasing.

(d) With $T = \frac{2}{3} \rightarrow f_s = 1.5$ hence we have aliasing as shown:



as there is interference from the replicas to the original spectrum ($|f| < 1$) there is aliasing.

2. For each of the following signals, sampled with sampling interval T_s, determine the bounds on T_s, which guarantee that there will be no aliasing.

(a) $x(t) = \frac{1}{t}\sin(3\pi t) + \cos(2\pi t)$

(b) $x(t) = \cos(12\pi t)\frac{\sin(\pi t)}{2t}$

(c) $x(t) = e^{-6t}u(t) * \frac{\sin(Wt)}{\pi t}$

Answers:

(a) $x(t) = \frac{1}{t}\sin(3\pi t) + \cos(2\pi t)$

$$\frac{\sin(3\pi t)}{t} = \pi\frac{\sin(3\pi t)}{\pi t} \overset{FT}{\Longleftrightarrow} \begin{cases} \pi & |\omega| < 3\pi \\ 0 & \text{otherwise} \end{cases}$$

$$\cos(2\pi t) \overset{FT}{\Longleftrightarrow} \pi[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$$

$$\omega_m = 3\pi \rightarrow f_m = 1.5$$

$$f_s > 2f_m \rightarrow f_s > 3 \rightarrow T_s < \frac{1}{3}$$

b

$$\cos(12\pi t) \overset{FT}{\Longleftrightarrow} \pi[\delta(\omega - 12\pi) + \delta(\omega + 12\pi)]$$

$$\frac{\sin(\pi t)}{2t} \overset{FT}{\Longleftrightarrow} \begin{cases} \pi/2 & |\omega| < \pi \\ 0 & \text{otherwise} \end{cases}$$

$$x_1(t)x_2(t) \overset{FT}{\Longleftrightarrow} \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

$$X(j\omega) = \begin{cases} \pi/4 & |\omega - 12\pi| < \pi \\ 0 & \text{otherwise} \end{cases} + \begin{cases} \pi/4 & |\omega + 12\pi| < \pi \\ 0 & \text{otherwise} \end{cases}$$

This implies the positive sided spectrum will span $11\pi < \omega < 13\pi$ and hence:

$$\omega_m = 13\pi \rightarrow f_m = 6.5$$

$$f_s > 2f_m \rightarrow f_s > 13 \rightarrow T_s < \frac{1}{13}$$

(c) $x(t) = e^{-6t}u(t) * \frac{\sin(Wt)}{\pi t}$

$$e^{-6t}u(t) \overset{FT}{\Longleftrightarrow} \frac{1}{6 + j\omega}$$

$$\frac{\sin(Wt)}{\pi t} \overset{FT}{\Longleftrightarrow} \begin{cases} 1 & |\omega| < W \\ 0 & \text{otherwise} \end{cases}$$

$$x_1(t) * x_2(t) \overset{FT}{\Leftrightarrow} X_1(j\omega)X_2(j\omega)$$

$$X(j\omega) = \begin{cases} \dfrac{1}{6 + j\omega} & |\omega| < W \\ 0 & \text{otherwise} \end{cases}$$

Hence:

$$\omega_m = W \rightarrow f_m = W/2\pi$$

$$f_s > 2f_m \rightarrow f_s > \frac{W}{\pi} \rightarrow T_s < \frac{\pi}{W}$$

7. For a signal $x(t)$ that is timelimited to 10 ms and has an essential bandwidth of 10 kHz, determine N_0 the number of signal samples necessary to compute the FFT with a frequency resolution f_0 of at most 50 Hz. Explain whether any zero padding is necessary.

Answers:

We have that

$$f_0 = 50 \text{ Hz}$$

and:

$$B = 10 \text{ kHz}, \text{ hence } f_s \geq 2B = 20000$$
$$T = \frac{1}{f_s} = \frac{1}{20000} = 50\ \mu s$$

Thus:

$$N_0 = \frac{f_s}{f_0} = \frac{20000}{50} = 400$$

We also have:

$$T_0 = \frac{1}{f_0} = \frac{1}{50} = 20 \text{ ms}$$
$$T = \frac{1}{f_s} = \frac{1}{20000} = 50\ \mu s$$

Since $T = 50\ \mu s$ and $T_0 = 20$ ms, then $f_0 = 1/T_0 = 50.0$ Hz as required. Since $x(t)$ is of 10 ms duration, we need *zero padding* over 10 ms (to ensure $T_0 = 20$ ms).

Property: Any periodic signal $x(t)$ with period T_0 can be expressed as a sum of the sinusoid of fundamental frequency $f_0 = 1/T_0$ or $\omega_0 = 2\pi/T_0$, and all of its harmonics (kf_0 or $k\omega_0$ for the k^{th} harmonic). Conversely the sum of a sinusoid with fundamental frequency f_0, and all of its harmonics will yield a periodic signal $x(t)$ with period $T_0 = 1/f_0 = 2\pi/\omega_0$.
What is the catch? For perfect synthesis you will usually need $k \to \infty$, in practice the synthesis is lossy since you will always have finite frequency bandwidth, B (i.e. $k\omega_0 < B$)

5.1.1 Compact Trigonometric

When $x(t)$ is real the sinusoidal sum expression which is most useful for spectral analysis is:

$$x(t) = C[0] + \sum_{k=1}^{\infty} C[k] \cos(k\omega_0 t + \theta[k])$$

where $C[k]$ represents the magnitude or amplitude of the k^{th} harmonic and $\theta[k]$ represents the phase or time offset of the k^{th} harmonic. Plotting $C[k]$ as a function of k yields the amplitude or **magnitude spectrum**, and plotting $\theta[k]$ as a function of k yields the **phase spectrum**.

To calculate $C[k]$ and $\theta[k]$ from $x(t)$ it is easiest to consider the trigonometric FS first.

5.1.2 Trigonometric FS

A periodic signal $x(t)$ can be expressed as the following sum:

$$x(t) = A[0] + \sum_{k=1}^{\infty} A[k] \cos(k\omega_0 t) + \sum_{k=1}^{\infty} B[k] \sin(k\omega_0 t)$$

And it can be shown ([1: 594-598]) that:

$$A[0] = \frac{1}{T_0}\int_{T_0} x(t)dt\,, \quad A[k] = \frac{2}{T_0}\int_{T_0} x(t)\cos(k\omega_0 t)\,dt$$
$$B[k] = \frac{2}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,dt$$

where $\int_{T_0}$ is integration over any convenient interval of length T_0

Comparing to the compact trigonometric expression we can immediately state:

$$C[0] = A[0]$$

And we can show that since $A[k] = C[k]\cos(\theta[k])$ and $B[k] = -C[k]\sin(\theta[k])$ for $k > 0$ then:

$$C[k] = \sqrt{A^2[k] + B^2[k]}$$
$$\theta[k] = \operatorname{atan2}(-B[k], A[k])$$

PROOF: Using the identity:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

then:

$$C[k]\cos(k\omega_0 t + \theta[k]) = C[k]\cos(\theta[k])\cos(k\omega_0 t) - C[k]\sin(\theta[k])\sin(k\omega_0 t)$$

The Fourier Series (FS) (aka exponential FS)

For a periodic signal, $x(t)$, its Fourier Series representation, $X[k]$, forms the *paired* relationship denoted by:

$$x(t) = \mathrm{FS}^{-1}\{X[k]\} = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$$

$$X[k] = \mathrm{FS}\{x(t)\} = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

where FS^{-1} is the Inverse FS (Fourier domain back to time domain) which is the Fourier series representation of $x(t)$. The $\int_{T_0}$ denotes any periodic interval for convenient integration (especially if even or odd symmetry can be exploited).

Conversion between the exponential and trigonometric Fourier Series

For $k = 0$ (DC component) we have $A[0] = C[0] = X[0]$ and $B[0] = \theta[0] = 0$

For $k > 0$:

$$A[k] = X[k] + X[-k]$$
$$B[k] = j(X[k] - X[-k])$$

and hence:

$$C[k] = 2|X[k]|$$
$$\theta[k] = \angle X[k]$$

NOTE: Since $X[k] = X_R[k] + jX_I[k]$ is a complex quantity then:

$$|X[k]| = \sqrt{X_R^2[k] + X_I^2[k]} \quad , \quad \angle X[k] = \operatorname{atan2}(X_I[k], X_R[k])$$

$$X_R[k] = |X[k]|\cos\angle X[k] \ , \ X_I[k] = |X[k]|\sin\angle X[k] \ \rightarrow \ X[k] = |X[k]|e^{j\angle X[k]}$$

For the FS representation the theorem states that the power of the signal can be expressed directly from the FS co-efficients. That is:

$$P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x^2(t)dt = C^2[0] + \frac{1}{2}\sum_{k=1}^{\infty} C^2[k] = \sum_{k=-\infty}^{\infty} |X[k]|^2$$

The $|X[k]|^2$ defines the **power spectrum** since its sum gives the total power of the signal.

For infinite-duration signals note that the signal energy is zero.

If for an arbitrary sum of sinusoids the ratio of all pairs of frequencies is of the form m/n where m and n are integers then the frequencies are harmonically related, the resultant sum is a periodic signal, and the fundamental frequency is given by greatest common factor (GCF) of all frequencies in the sum.

Even and Odd signals and the trigonometric FS

If $x(t)$ is an even signal (i.e $x(t) = x(-t)$) then the FS consists only of cosine terms, that is, $B[k] = 0$

If $x(t)$ is an odd signal (i.e $x(t) = -x(-t)$) then the FS consists only of sine terms, that is, $A[k] = 0$

5.3 LTI System Response to Periodic Inputs

[1: 637-640]

For an LTI system we know that the response of the system to an input $x(t) = e^{j\omega t}$ is:

$$y(t) = H(j\omega)e^{j\omega t}$$

Thus if the LTI system is excited by a periodic signal, $x(t)$, with FS representation:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$$

Then from the linearity property and noting $\omega = k\omega_0$ the output of the system is given by:

$$y(t) = \sum_{k=-\infty}^{\infty} X[k]H(jk\omega_0)e^{jk\omega_0 t} \equiv \sum_{k=-\infty}^{\infty} Y[k]e^{jk\omega_0 t}$$

where the output is also a periodic signal of the same period as the input. We also have that:

$$Y[k] = H(jk\omega_0)X[k]$$

NOTE: $H(j\omega)$ is the *Fourier Transform* (FT) of the system impulse response function.

6.2.2 Parseval's Theorem and $|X(j\omega)|^2$

For finite-duration signals we define the signal energy (note that the signal power is zero):

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

For the FT representation the theorem states that the energy of the signal can be expressed directly from the FT. That is:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

If we make use of $\omega = 2\pi f$ we can express:

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Which allows us to interpret $|X(f)|^2$ as the **energy spectral density** (energy per df → unit bandwidth in hertz).

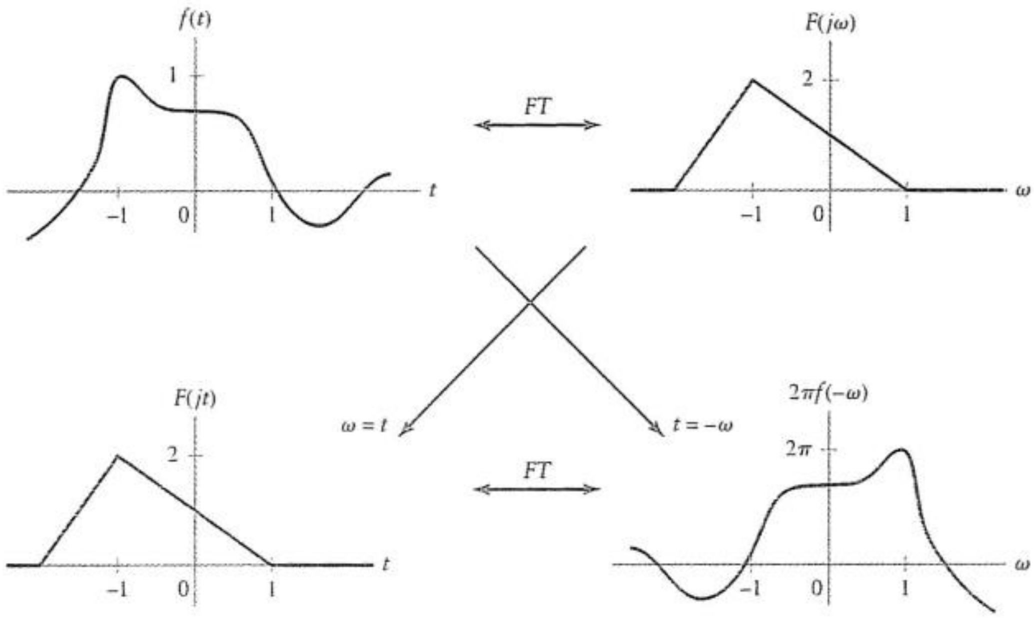
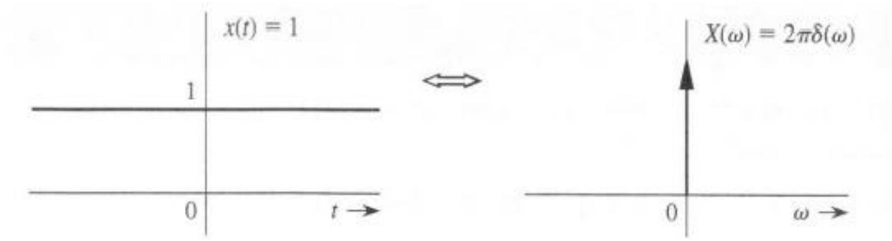


FIGURE 3.74 The FT duality property.



Consider the Fourier spectrum of the rectangular pulse with parameter τ in Figure 6-2. It has a time duration of:

$$T_d = \tau$$

Also most of the energy in the Fourier spectrum is concentrated over the interval $(-\,2\pi/\tau\,, 2\pi/\tau)$ and hence we can state that the double-sided bandwidth is:

$$B_{ds} = \frac{4\pi}{\tau}$$

If we form the time-bandwidth product this is a constant:

$$T_d B_{ds} = 4\pi$$

Definition of Bandwidth, *B*

The measurement of bandwidth is important in communications systems and filtering. It is loosely defined as the major spectral extent of a signal and can be manifested as a baseband bandwidth (main spectral content from $-W$ to 0 to W, so $B = W$ as in the rectangular pulse of Figure 6-5 where $W = \pi/2$) or a passband bandwidth (main spectral content centred around ω_c, as in the modulated rectangular pulse of Figure 6-5 where $B = 2W = \pi$). We can express B in either Hz (human friendly) or rad/s (maths friendly). In electrical engineering systems when dealing with low-pass and band-pass systems we usually define the bandwidth as the frequency at which the response is at $1/\sqrt{2}$ or $20\log_{10}(1/\sqrt{2}) \cong -3$ dB of its peak value.

6.7 LTI System Response

[1: 717-725][2: 260-267]

Frequency Response, $H(j\omega)$

Let $h(t)$ be the impulse response of an LTI system. If the input to the system is $x(t)$ then we know that the output is given by the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

From the convolution property for the Fourier Transform this gives:

$$Y(j\omega) = H(j\omega)X(j\omega)$$

where:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \mathcal{F}[h(t)]$$

is known as the frequency response of the LTI system, $|H(j\omega)|$ is known as the magnitude response and $\angle H(j\omega)$ is known as the phase response

NOTE: $|Y(j\omega)| = |H(j\omega)||X(j\omega)|$ and $\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$

Consider an LTI system with a magnitude response, $|H(j\omega)|$ which can be described as being unity ($|H(j\omega)| = 1$) over a range a frequencies and close to zero ($|H(j\omega)| \approx 0$) over the remaining range of frequencies (or vice versa).

- With $|H(j\omega)| = 1$ then $Y(j\omega) = X(j\omega)$ and the system passes through the signal unaltered over this range of frequencies called the passband.
- With $|H(j\omega)| \approx 0$ then $Y(j\omega) \approx 0$ and the system "kills off" the signal over this range of frequencies called the stopband.

LTI systems which modify the frequency spectrum of the input in a particular way are called filters.

When Magnitude H(jw) starts at 0 and tends to 1 as w tends to infinity it is a high pass filter

When Magnitude H(jw) starts at 1 and tends to 0 as w tends to infinity it is a low pass filter

(Nyquist) Sampling Theorem

A real, continuous-time signal whose spectrum is bandlimited to B Hz (i.e. $X(\omega) = 0$ for $|\omega| > 2\pi B$) can be reconstructed exactly (without any error) from its samples taken uniformly at a rate $f_s > 2B$ samples per second. In other words the minimum *sampling frequency* is $f_s = 2B$ Hz. Once a continuous-time signal, $x(t)$, is sampled it becomes a discrete-time signal, $x[n] \equiv x(nT)$ where $T = 1/f_s$ is the *sampling period*

w = 2πf

Ω = m / F(naught)

7.1.4 Undersampling and Oversampling

From Example 8.1 of [1]:

Consider a signal whose spectrum is bandlimited to 5 Hz / 10π rad/s (Figure 7-5(a)(b)) (spectral width of 10 Hz / 20π rad/s). This implies a Nyquist rate of $f_N = 10$ Hz.

Case 1 (Undersampling with $f_s = 5$): Since $f_s < f_N$ then the spectrum replicas of spectral width 10 Hz will be repeated every $k5$ Hz leading to obvious overlap/interference (see Figure 7-5(c)(d))

Case 2 (Nyquist rate with $f_s = 10$): Since $f_s = f_N$ then the spectrum replicas of spectral width 10 Hz will be repeated every $k10$ Hz and the replicas just touch but do not overlap (see Figure 7-5(e)(f))

Case 3 (Oversampling with $f_s = 20$): Since $f_s > f_N$ then the spectrum replicas of spectral width 10 Hz will be repeated every $k20$ Hz and the there is a gap between successive replicas (see Figure 7-5(g)(h))

Case 1 is not acceptable and has to be avoided. Cases 2 and 3 are both acceptable. Case 2 is more efficient (minimum number of samples to reconstruct signal) but Case 3 is needed to handle the sampling and reconstruction of practical signals.

So what happens if a signal is not bandlimited to $f_s/2$? We get **aliasing** and this is shown by Figure 7-9 (a)-(c). There are two visible effects of aliasing:

a) The *folded tail* distorts frequencies $f < f_s/2$
b) There is a *loss of spectral content* $f > f_s/2$

where the tail area is any spectral content of the original signal $> f_s/2$.

Effect b) is unavoidable but represents loss rather than distortion of information at higher, usually less significant, frequencies. Effect a) can and must be avoided as it represents distortion of information at lower, more important, frequencies. We can address effect a) by using an **analogue anti-aliasing filter prior to sampling the signal.**

With aliasing the actual analog signal we get upon reconstruction is that which falls within the ***fundamental band*** $[-f_s/2, f_s/2]$. This will include any aliased components $f > f_s/2$ which get reflected back to the fundamental band from either the positive or negative frequencies.

$$e^{ix} = \cos x + i \sin x$$