12:00, Monday, October 18, 2021	
Class Test 2: Fourier Analysis and Sampling	
Time allowed: 45 minutes Max mark: 30, Assessment: **10%**	This paper contains: X pages, 4 questions

IMPORTANT INSTRUCTIONS:

Candidates should attempt **all** questions with numerical answers to **3** decimal places for each question, you need to show **your working to the final answer to gain maximum marks**.

Properly space solutions to ensure high quality image scans, **use black/blue pen or 2B pencil on white ruled/plain paper** to ensure sufficient contrast, and ensure **you are in a well-lit area**. You will also need a **scientific calculator** and scratch pad to for draft working.

Solutions will be marked page by page, so **start questions on new page**.

Question 1 (10 marks)

Determine the Fourier series representation , $X[k]$, of:

$$x(t) = 2\sin(2\pi t - 3) + \sin(6\pi t)$$

HINT: Expand using *Euler's relation* and match with the *inverse Fourier series* expression.

Solution:

Period $T = 1$ sec; Fundamental Frequency, $\omega_0 = 2\pi$ radians

$$\begin{aligned} x(t) &= \tfrac{1}{2j}\left\{2e^{j2\pi t}\cdot e^{-j3} - 2e^{-j2\pi t}\cdot e^{j3} + e^{j6\pi t} - e^{-j6\pi t}\right\} \\ &= \tfrac{j}{2}e^{j(-3)2\pi t} + je^{j3}\cdot e^{j(-1)2\pi t} - je^{-j3}\cdot e^{j(1)2\pi t} - \tfrac{j}{2}e^{j(3)2\pi t} \\ &= X[-3]e^{j(-3)2\pi t} + X[-1]e^{j(-1)2\pi t} + X[1]e^{j(1)2\pi t} + X[3]e^{j(3)2\pi t} \\ &= \sum_{k=-\infty}^{\infty} X[k]e^{jk2\pi t} \end{aligned}$$

Note that $X[k] = 0$ for k values other than $-3, -1, 1, 3$.

$$X[k] = \begin{cases} j/2 & k = -3 \\ je^{j3} & k = -1 \\ -je^{-j3} & k = 1 \\ -j/2 & k = 3 \\ 0 & otherwise \end{cases}$$

Question 2 (6 marks)

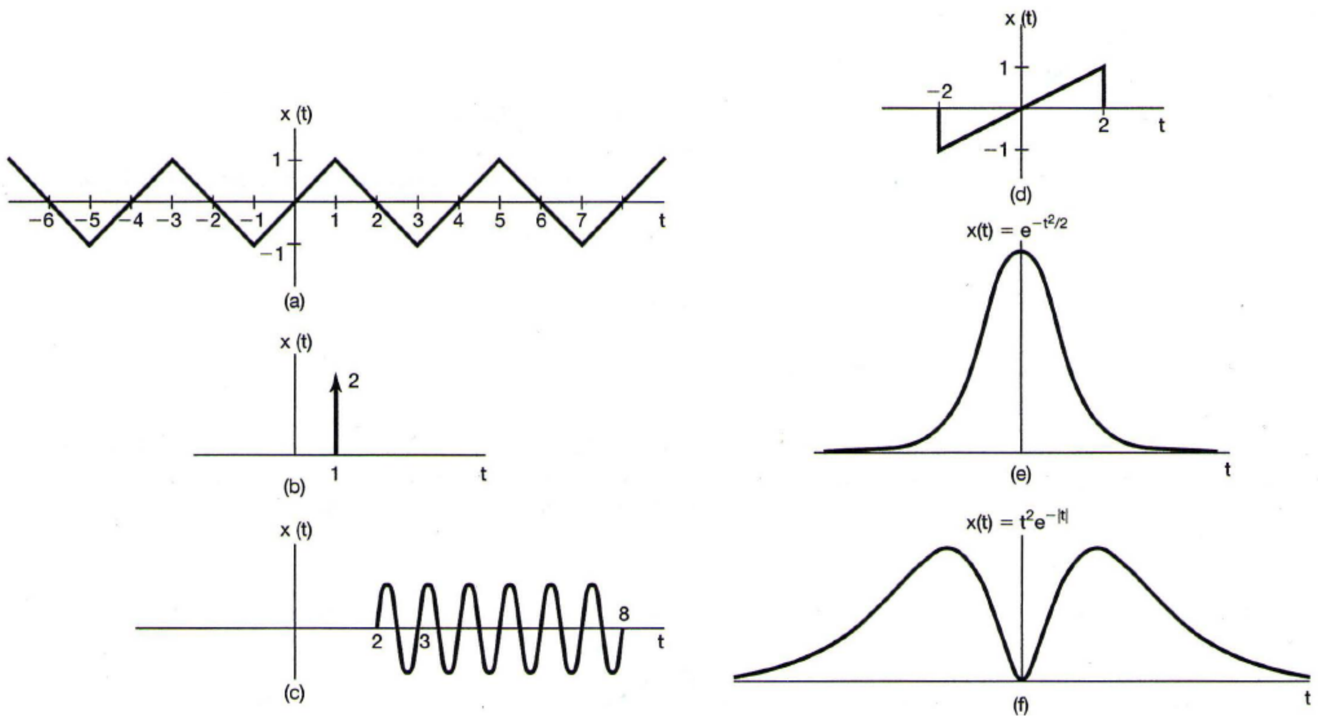
For each condition listed below determine which, if any, of the real signals depicted below have Fourier transforms that satisfy that condition:

(A) $\mathrm{Re}\{X(j\omega)\} = 0$

(B) $\mathrm{Im}\{X(j\omega)\} = 0$

(C) $|X(j\omega)| = 2$, for all ω

(D) $X(j\omega)$ is non-zero only for certain values of ω



Solution:

(A) implies that $x(t)$ must be an odd function about $t = 0$ ($X(j\omega)$ is purely imaginary): Only **(a) and (d)** have this property.
(B) implies that $x(t)$ must be an even function about $t = 0$ ($X(j\omega)$ is purely real): Only **(e) and (f)** have this property.
(C) implies that the spectrum is flat, which can only happen with an impulse function: Only **(b)** has this property.
(D) If $X(j\omega)$ is non-zero only for certain values of ω then this is equivalent to a Fourier series representation and hence $x(t)$ must be periodic. Only **(a)** has this property.

Question 3 (6 marks)

(i) Consider the continuous-time signal:

$$x(t) = 3\cos 6\pi t + \sin 18\pi t + 2\cos 28\pi t$$

What is the expression for the magnitude spectrum, $|X(\omega)|$, and sketch it as a function of ω.
HINT: You have three sinusoids so you will have just three spectral harmonics.

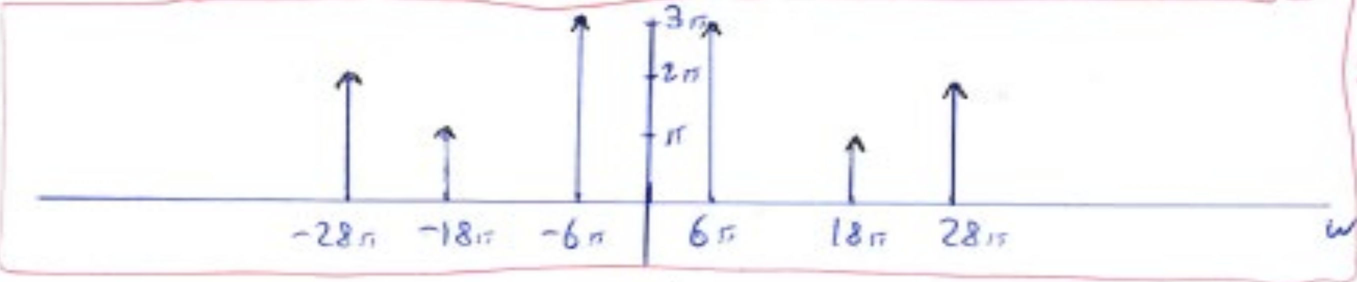
(ii) What is the Nyquist rate, that is, determine the range of possible sampling frequencies (in Hz), f_s, required to be able to reconstruct $x(t)$ from these samples without error?

(iii) What is the sampling frequency if you sample $x(t)$ at 25% above the Nyquist rate (i.e. your answer in (ii) x 1.25)? For this sampling frequency carefully sketch the magnitude spectrum of the sampled signal (in Hz) over the range $\pm$ 50 Hz.

Solution:

1 (i) $x(t) = 3\cos(6\pi t) + \sin(18\pi t) + 2\cos(28\pi t)$ *

$$\therefore \boxed{\begin{aligned} |X(\omega)| = {} & 3\pi\left[\delta(\omega - 6\pi) + \delta(\omega + 6\pi)\right] \\ & + \pi\left[\delta(\omega + 18\pi) + \delta(\omega - 18\pi)\right] \\ & + 2\pi\left[\delta(\omega - 28\pi) + \delta(\omega + 28\pi)\right] \end{aligned}}$$ 1



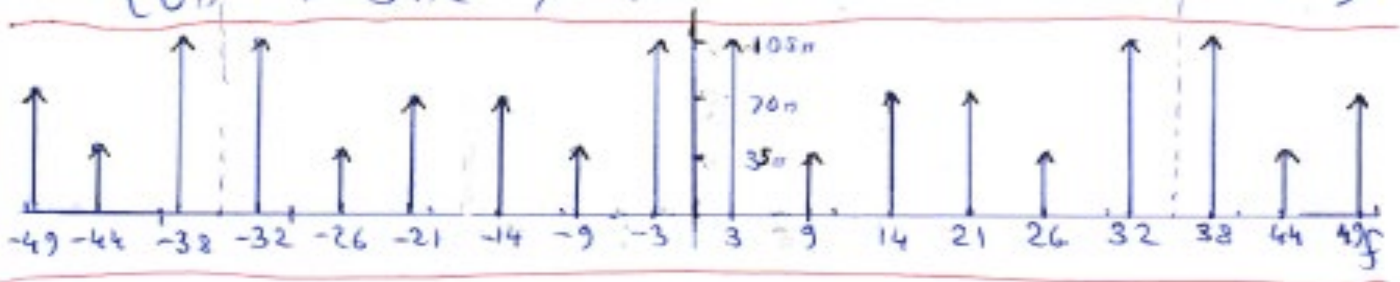
1

(ii) $B = 28\pi$ $\omega_s \geq 2B = 56\pi$

$\therefore \boxed{f_s \geq 28 \text{ Hz}}$ 1

(iii) 25% abov 28 Hz $\Rightarrow \boxed{f_s = 35 \text{ Hz}}$ 1

($6\pi \to 3$ Hz ; $18\pi \to 9$ Hz ; $28\pi \to 14$ Hz)



Gain of 35

$3\pi \to 105\pi$ $2\pi \to 70\pi$ $\pi \to 35\pi$ 2

Question 4 (8 marks)

Consider an LTI system frequency response:

$$H(j\omega) = \frac{3(3 + j\omega)}{8 - \omega^2 + 6j\omega}$$

(a) What is the magnitude response, $|H(j\omega)|$?

(b) What is the phase response, $\angle H(j\omega)$? HINT: Use the atan2(y,x) function for the phase of $x + iy$

(c) If the input is $x(t) = 3\cos(2t)$, what is the output $y(t)$?

(d) By evaluating $|H(j\omega)|$ at $\omega = 0$, $\omega^2 = 8$ and $\omega \to \infty$ indicate whether this LTI system represents a low-pass, high-pass or band-pass response?

Solution:

$$H(j\omega) = \frac{9 + j3\omega}{(8 - \omega^2) + j6\omega}$$

(a)

$$|H(j\omega)| = \frac{3\sqrt{9 + \omega^2}}{\sqrt{(8 - \omega^2)^2 + 36\omega^2}}$$

(b)

$$\angle H(j\omega) = \text{atan2}(\omega, 3) - \text{atan2}(6\omega, 8 - \omega^2)$$

(NOTE: $\text{atan2}(3\omega, 9)$ also acceptable, but common factors should be removed)

(c)

For $\omega = 2$ then:

$$|H(j2)| = \frac{10.817}{12.649} = 0.8552$$

$$\angle H(j2) = \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{12}{4}\right) = 0.5880 - 1.2490 = -0.6610$$

Hence:

$$y(t) = 2.5656\cos(2t - 0.6610)$$

(d) From (a) we see that:

$$|H(j0)| = 1.1250, \quad \left|H\left(j\sqrt{8}\right)\right| = 0.7289, \quad |H(j\infty)| = 0$$

This is a **low-pass filter** response