Tutorial 2 (Solutions)

1. (a) Show that if $x(t)$ is an even function of t then:

$$X(j\omega) = 2\int_0^\infty x(t)\cos\omega t\,dt$$

And if $x(t)$ as an odd function of t then:

$$X(j\omega) = -2j\int_0^\infty x(t)\sin\omega t\,dt$$

(b) Using the relevant property show that:

$$x(t+T) + x(t-T) \overset{FT}{\Longleftrightarrow} 2X(j\omega)\cos\omega T$$

Answers:

(a)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)\cos\omega t\,dt - j\int_{-\infty}^{\infty} x(t)\sin\omega t\,dt$$

If $x(t)$ is an even function of t then $x(t)\sin\omega t$ is an odd function of t and hence the RHS term goes to zero. Furthermore $x(t)\cos\omega t$ is even and hence the LHS integral evaluates to the same value from $(-\infty, 0]$ as $[0,\infty)$ so that we can say:

$$X(j\omega) = 2\int_0^\infty x(t)\cos\omega t\,dt$$

If $x(t)$ is an odd function of t then $x(t)\cos\omega t$ is an odd function of t and hence the LHS term goes to zero, but $x(t)\sin\omega t$ will be an even function …

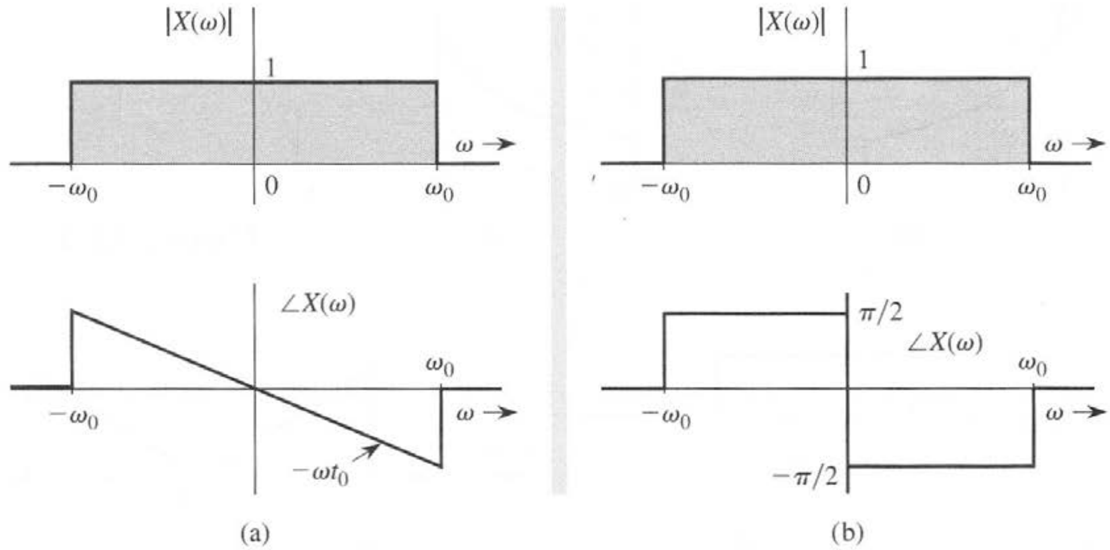
(b) From the time shifting property we have:

$$x(t-t_0) \Leftrightarrow e^{-j\omega t_0}X(j\omega)$$

Hence we can say:

$$x(t+T) + x(t-T) \Leftrightarrow e^{j\omega T}X(j\omega) + e^{-j\omega T}X(j\omega) = 2X(j\omega)\cos\omega T$$

2. Find the inverse Fourier transform of the two different spectra below:



What can you conclude?

Answers:

We note that $X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$

For both (a) and (b), $|X(j\omega)| = \begin{cases} 1 & -\omega_0 < \omega < \omega_0 \\ 0 & else \end{cases}$,

For (a), $\angle X(j\omega) = \begin{cases} -\omega t_0 & -\omega_0 < \omega < \omega_0 \\ 0 & else \end{cases}$ and for (b) $\angle X(j\omega) = \begin{cases} \pi/2 & -\omega_0 < \omega < 0 \\ -\pi/2 & 0 < \omega < \omega_0 \\ 0 & else \end{cases}$

(a) Inverse FT is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1.e^{-j\omega t_0} e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(t-t_0)} d\omega$$
$$= \frac{1}{(2\pi)j(t-t_0)}\left[e^{j\omega(t-t_0)}\right]_{-\omega_0}^{\omega_0} = \frac{\sin\omega_0(t-t_0)}{\pi(t-t_0)}$$

(b) Inverse FT is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\left[\int_{-\omega_0}^{0} 1.e^{j\pi/2} e^{j\omega t} d\omega + \int_{0}^{\omega_0} 1.e^{-j\pi/2} e^{j\omega t} d\omega\right]$$
$$= \frac{1}{2\pi}\left[\int_{-\omega_0}^{0} j e^{j\omega t} d\omega + \int_{0}^{\omega_0} -j e^{j\omega t} d\omega\right] = \frac{j}{2\pi}\left\{\left[\frac{1}{jt} e^{j\omega t}\right]_{-\omega_0}^{0} - \left[\frac{1}{jt} e^{j\omega t}\right]_{0}^{\omega_0}\right\}$$
$$= \frac{1 - \cos\omega_0 t}{\pi t}$$

Even though the magnitude spectra are identical (same spectral content and distribution) these represent entirely different time signals.

3. Find the Fourier transform of the signal:

$$x(t) = \begin{cases} 1 & -T < t < 0 \\ -1 & 0 < t < T \\ 0 & else \end{cases}$$

(a) By direct integration.

(b) Using Fourier transform pair 17 from the lecture notes and relevant properties

(c) Considering the Fourier transform of $dx(t)/dt$ and relevant transform pairs and properties.

Answers: NOTE: $X(\omega) \equiv X(j\omega)$

(a)

$$X(\omega) = \int_{-T}^{0} e^{-j\omega t}\,dt - \int_{0}^{T} e^{-j\omega t}\,dt = -\frac{2}{j\omega}[1-\cos\omega T] = \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

(b)

$$x(t) = \text{rect}\left(\frac{t+T/2}{T}\right) - \text{rect}\left(\frac{t-T/2}{T}\right)$$

$$\text{rect}\left(\frac{t}{T}\right) \iff T\text{sinc}\left(\frac{\omega T}{2}\right)$$

$$\text{rect}\left(\frac{t \pm T/2}{T}\right) \iff T\text{sinc}\left(\frac{\omega T}{2}\right) e^{\pm j\omega T/2}$$

$$\begin{aligned} X(\omega) &= T\text{sinc}\left(\frac{\omega T}{2}\right)[e^{j\omega T/2} - e^{-j\omega T/2}] \\ &= 2jT\text{sinc}\left(\frac{\omega T}{2}\right)\sin\frac{\omega T}{2} \\ &= \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right) \end{aligned}$$

(c)

$$\frac{df}{dt} = \delta(t+T) - 2\delta(t) + \delta(t-T)$$

The Fourier transform of this equation yields

$$j\omega X(\omega) = e^{j\omega T} - 2 + e^{-j\omega T} = -2[1-\cos\omega T] = -4\sin^2\left(\frac{\omega T}{2}\right)$$

Therefore

$$X(\omega) = \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

4. Use tables of transforms and properties to find the FTs of the following signals:

(a) $x(t) = \sin(2\pi t)\, e^{-t}u(t)$

(b) $x(t) = te^{-3|t-1|}$

(c) $x(t) = \left[\frac{2\sin(3\pi t)}{\pi t}\right]\left[\frac{\sin(2\pi t)}{\pi t}\right]$

Answers:

(a) $x(t) = \sin(2\pi t)e^{-t}u(t)$

$$\begin{aligned} x(t) &= \sin(2\pi t)e^{-t}u(t) \\ &= \frac{1}{2j}e^{j2\pi t}e^{-t}u(t) - \frac{1}{2j}e^{-j2\pi t}e^{-t}u(t) \end{aligned}$$

$$\begin{aligned} e^{-t}u(t) &\xleftrightarrow{FT} \frac{1}{1+j\omega} \\ e^{j2\pi t}s(t) &\xleftrightarrow{FT} S(j(\omega-2\pi)) \\ X(j\omega) &= \frac{1}{2j}\left[\frac{1}{1+j(\omega-2\pi)} - \frac{1}{1+j(\omega+2\pi)}\right] \end{aligned}$$

(b) $x(t) = te^{-3|t-1|}$

$$\begin{aligned} e^{-3|t|} &\xleftrightarrow{FT} \frac{6}{9+\omega^2} \\ s(t-1) &\xleftrightarrow{FT} e^{-j\omega}S(j\omega) \\ tw(t) &\xleftrightarrow{FT} j\frac{d}{d\omega}W(j\omega) \\ X(j\omega) &= j\frac{d}{d\omega}\left[e^{-j\omega}\frac{6}{9+\omega^2}\right] \\ &= \frac{6e^{-j\omega}}{9+\omega^2} - \frac{12j\omega^{-j\omega}}{(9+\omega^2)^2} \end{aligned}$$

(c) $x(t) = \left[\frac{2\sin(3\pi t)}{\pi t}\right]\left[\frac{\sin(2\pi t)}{\pi t}\right]$

$$\begin{aligned} \frac{\sin(Wt)}{\pi t} &\xleftrightarrow{FT} \begin{cases} 1 & |\omega| \le W \\ 0, & \text{otherwise} \end{cases} \\ s_1(t)s_2(t) &\xleftrightarrow{FT} \frac{1}{2\pi}S_1(j\omega) * S_2(j\omega) \end{aligned}$$

$$X(j\omega) = \begin{cases} 5 - \frac{|\omega|}{\pi} & \pi < |\omega| \le 5\pi \\ 4 & |\omega| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

5. Use tables of transforms and properties to find the inverse FTs of the following:

(a) $X(j\omega) = \frac{j\omega}{(1+j\omega)^2}$

(b) $X(j\omega) = \frac{4\sin(2\omega-4)}{2\omega-4} - \frac{4\sin(2\omega+4)}{2\omega+4}$

(c) $X(j\omega) = \frac{1}{j\omega(j\omega+2)} - \pi\delta(\omega)$

Answers:

(a) $X(j\omega) = \frac{j\omega}{(1+j\omega)^2}$

$$\frac{1}{(1+j\omega)^2} \xleftrightarrow{FT} te^{-t}u(t)$$
$$j\omega S(j\omega) \xleftrightarrow{FT} \frac{d}{dt}s(t)$$

$$\begin{aligned} x(t) &= \frac{d}{dt}[te^{-t}u(t)] \\ &= (1-t)e^{-t}u(t) \end{aligned}$$

(b) $X(j\omega) = \frac{4\sin(2\omega-4)}{2\omega-4} - \frac{4\sin(2\omega+4)}{2\omega+4}$

$$\frac{2\sin(\omega)}{\omega} \xleftrightarrow{FT} \text{rect}\left(\frac{t}{2}\right) = \begin{cases} 1 & |t| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$
$$S(j2\omega) \xleftrightarrow{FT} \frac{1}{2}s(\frac{t}{2})$$
$$S(j(\omega-2)) \xleftrightarrow{FT} e^{j2t}s(t)$$

$$\begin{aligned} x(t) &= \text{rect}(\frac{t}{2})e^{j2t} - \text{rect}(\frac{t}{2})e^{-j2t} \\ &= 2j\text{rect}(\frac{t}{2})\sin(2t) \end{aligned}$$

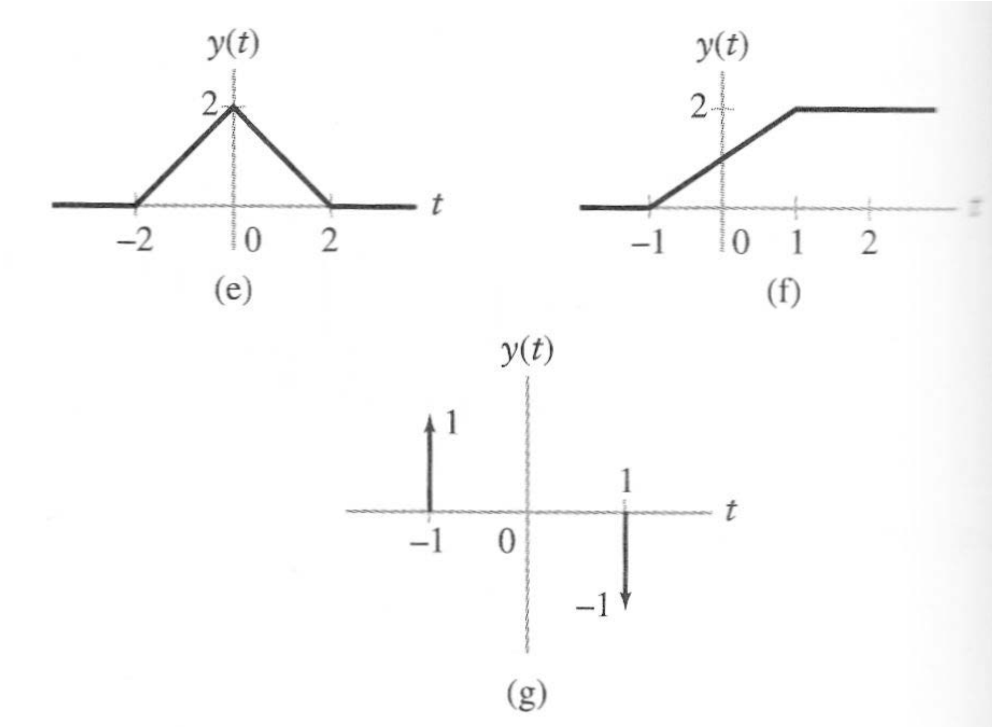
(c) $X(j\omega) = \frac{1}{j\omega(j\omega+2)} - \pi\delta(\omega)$

$$\frac{1}{j\omega} + \pi\delta(j\omega) \xleftrightarrow{FT} u(t)$$
$$\frac{1}{2+j\omega} \xleftrightarrow{FT} e^{-2t}u(t)$$
$$2\pi\delta(\omega) \xleftrightarrow{FT} 1$$
$$X(j\omega) = -0.5\frac{1}{(j\omega+2)} + 0.5\frac{1}{j\omega} + 0.5\pi\delta(\omega) - 1.5\pi\delta(\omega)$$
$$X(j\omega) \xleftrightarrow{FT} x(t) = -0.5e^{-2t}u(t) + 0.5u(t) - \frac{3}{4}$$

6. Use the FT pair:

$$x(t) = \begin{cases} 1 & |t| < 1 \\ 0 & \text{otherwise} \end{cases} \xleftrightarrow{FT} X(j\omega) = \frac{2\sin(\omega)}{\omega}$$

and the FT properties to evaluate the frequency-domain representations of the signals depicted below.



Answers:

(e) $y(t) = x(t) * x(t)$

$$Y(j\omega) = \frac{4\sin^2(\omega)}{\omega^2}$$

(f)) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$$\begin{aligned} Y(j\omega) &= 2\frac{\sin(\omega)}{\omega}\frac{1}{j\omega} + \pi(2)\delta(\omega) \\ &= 2\frac{\sin(\omega)}{j\omega^2} + 2\pi\delta(\omega) \end{aligned}$$

(g) $y(t) = \frac{d}{dt}x(t)$

$$\begin{aligned} Y(j\omega) &= j\omega 2\frac{\sin(\omega)}{\omega} \\ &= j2\sin(\omega) \end{aligned}$$

7. Find the frequency response of the following systems:

(a) $h(t) = \delta(t) - 2e^{-2t}u(t)$, is this a low pass or high pass response?

(b) $x(t) = e^{-t}u(t), \quad y(t) = e^{-2t}u(t) + e^{-3t}u(t)$, what is the impulse response?

Answers:

(a)

$$H(j\omega) = 1 - \frac{2}{2 + j\omega} = \frac{j\omega}{2 + j\omega}$$

$$|H(j\omega)| = \frac{\omega}{\sqrt{4 + \omega^2}}$$

For $\omega \to 0$, then $|H(j\omega)| \to 0$ and for $\omega \to \infty$, then $|H(j\omega)| \to 1$ which is a high pass response.

(b)

$x(t) = e^{-t}u(t), \quad y(t) = e^{-2t}u(t) + e^{-3t}u(t)$

$$\begin{aligned} X(j\omega) &= \frac{1}{1 + j\omega} \\ Y(j\omega) &= \frac{1}{2 + j\omega} + \frac{1}{3 + j\omega} \\ &= \frac{5 + 2j\omega}{(2 + j\omega)(3 + j\omega)} \end{aligned}$$

$$\begin{aligned} H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\ &= \frac{5 + 7j\omega + 2(j\omega)^2}{(2 + j\omega)(3 + j\omega)} \\ &= 2 - \frac{1}{2 + j\omega} - \frac{2}{3 + j\omega} \\ h(t) &= 2\delta(t) - (e^{-2t} + 2e^{-3t})u(t) \end{aligned}$$