Lecture 1

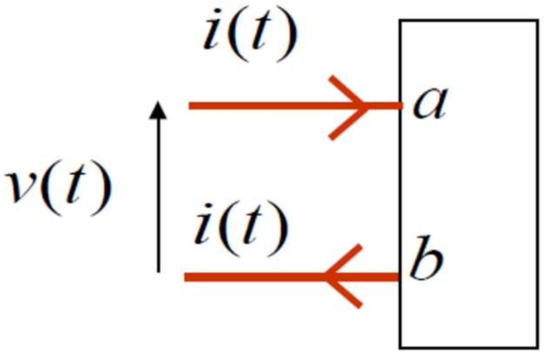
Basic circuit theory

Circuit Theory

- Classical Circuit Theory based on Kirchhoff's voltage law and current law is a self contained abstract mathematical axiomatic theory for dimensionless circuits/networks.

A circuit/network is defined as a system of interconnected circuit elements.

Initially we consider only elements with one port.



One port element

- Each network element has a pair of terminals that constitute the element port .
- Each port has associated with it two directed variables and specific direction conventions.

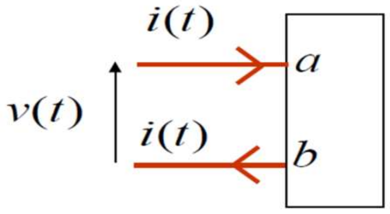
v(t) rise variable, *i(t)* flow variable.

Basic axioms for the circuit theory

The circuit theory is built on the following Axioms:

Axiom 1:

An element is described only by the two (not three-only one i) directed variables.



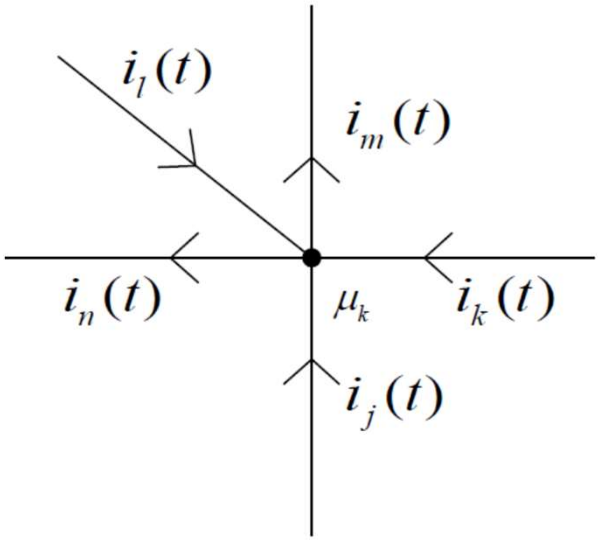
Requires that terminals physically close (in terms of signal frequency), so can use lumped circuit model

Axiom 2:

The relationship between the two variables is assumed to be a characteristic of the element and independent of other elements in the network in which may be connected. The relationship may be linear or nonlinear.

Axiom 3: - Zero Sum Vertex Flow (KCL)

For every node in a network the sum of inward directed flow variables minus the sum of outward directed flow variables is zero.



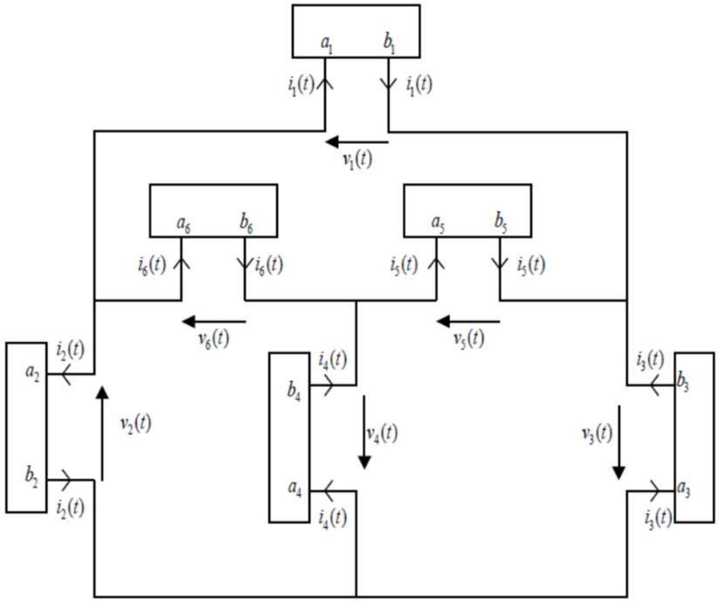
$$i_l(t)+i_k(t)+i_j(t)-i_m(t)-i_n(t)=0$$

A node is a point where two or more elements have a common connection.

Kirchhoff's (1847) current law !

Axiom 4: Zero Sum Loop Rise (KVL):

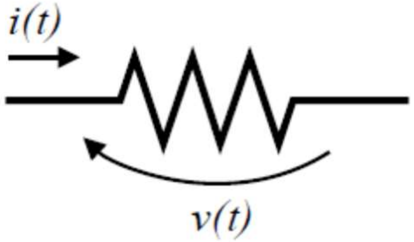
For every closed loop (a circuit) in the network, the sum of clockwise directed rise variables minus the sum of counter clockwise direct rise variables is zero .



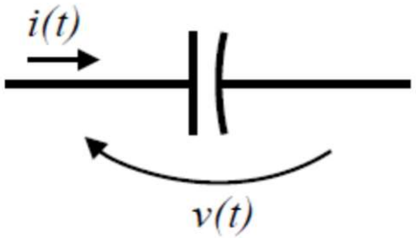
$$v_3(t) - v_4(t) - v_5(t) = 0$$
$$v_2(t) + v_4(t) - v_6(t) = 0$$
$$v_2(t) - v_1(t) + v_3(t) = 0$$
$$v_2(t) + v_3(t) - v_6(t) - v_5(t) = 0$$
$$v_2(t) - v_1(t) + v_4(t) + v_5(t) = 0$$

Kirchhoff's (1847) voltage law !

One port element examples – time domain



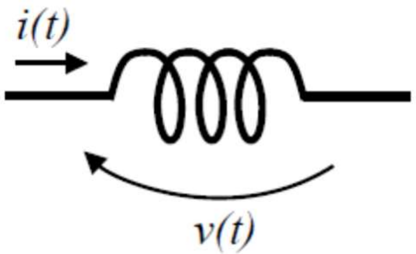
Resistor $v(t) = Ri(t)$



Capacitor

$$i(t) = C\frac{dv}{dt}$$

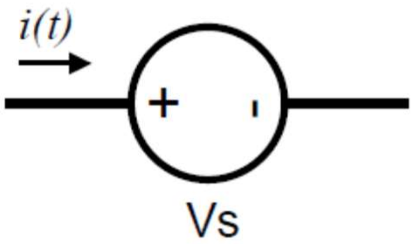
$$\int_{t_0}^{t} i(t)dt = C\left(v(t) - v(t_0^-)\right)$$



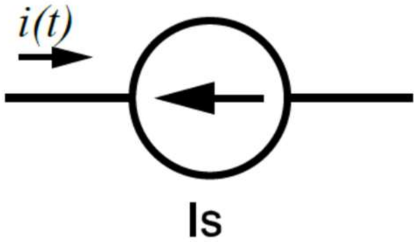
Inductor

$$v(t) = L\frac{di}{dt}$$

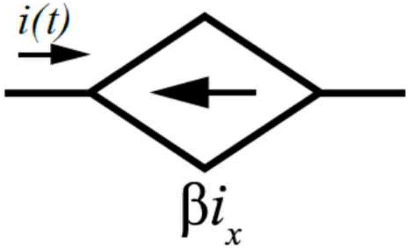
$$\int_{t_0}^{t} v(t)dt = Li(t) - Li\left(t_0^-\right)$$



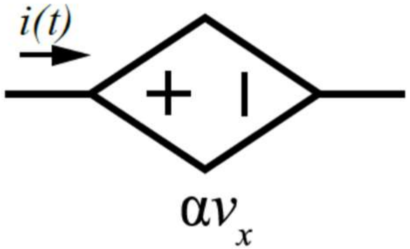
Voltage source $v(t) = Vs$



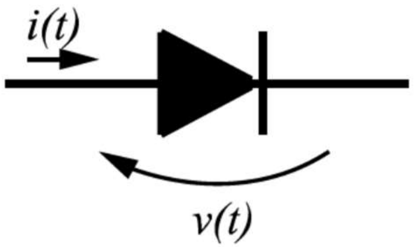
Current source $i(t) = -Is$



Dependent current source $i(t) = -\beta i_x$



Dependent voltage source $v(t) = \alpha v_x$

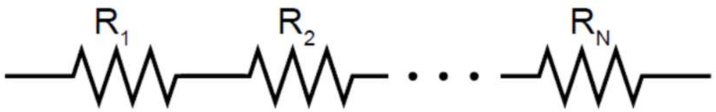


Diode

Series and parallel connected one port elements

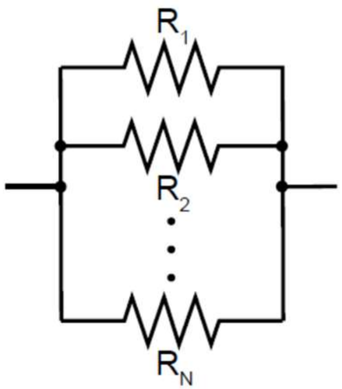
A number of one type of one port element connected in series or parallel can be equivalent to a single element of the same type. This useful result can be derived from the voltage/current relationship of the resistor / capacitor / inductor and network theory.

Resistors in series:



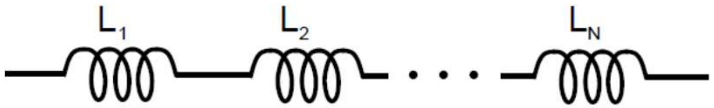
$$R_T = R_1 + R_2 + \cdots + R_N$$

Resistors in parallel:



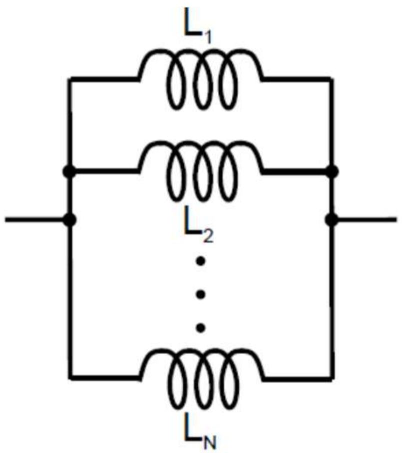
$$1/R_T = 1/R_1 + 1/R_2 + \cdots + 1/R_N$$

Inductors in series:



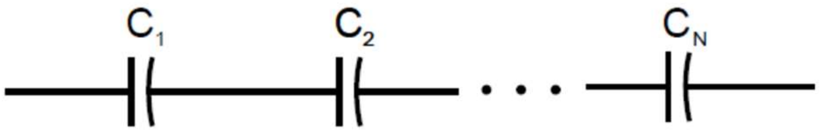
$$I_T = I_1 + I_2 + \cdots + I_N$$

Inductors in parallel:



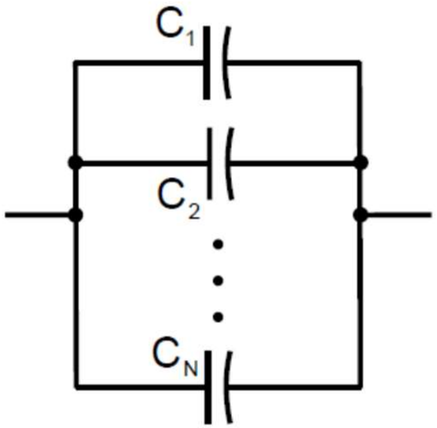
$$1/I_T = 1/I_1 + 1/I_2 + \cdots + 1/I_N$$

Capacitors in series:



$$1/C_T = 1/C_1 + 1/C_2 + \cdots + 1/C_N$$

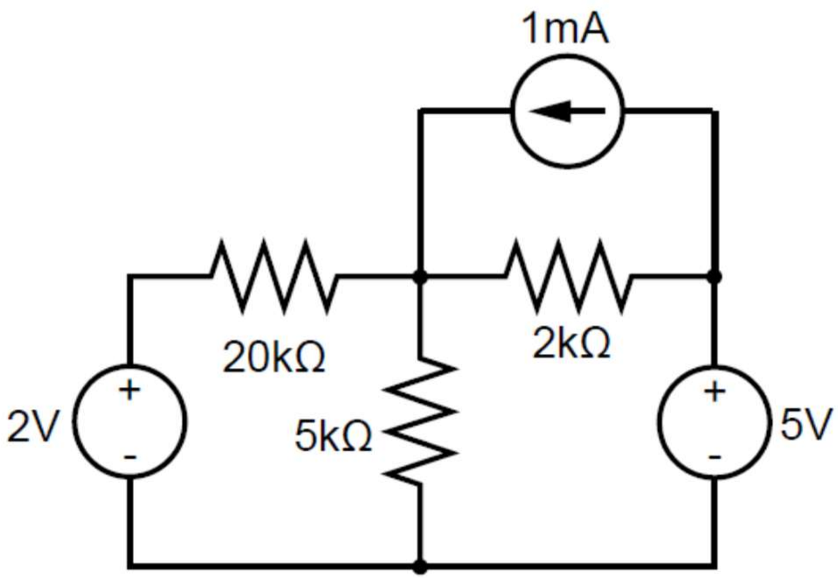
Capacitors in parallel:



$$C_T = C_1 + C_2 + \cdots + C_N$$

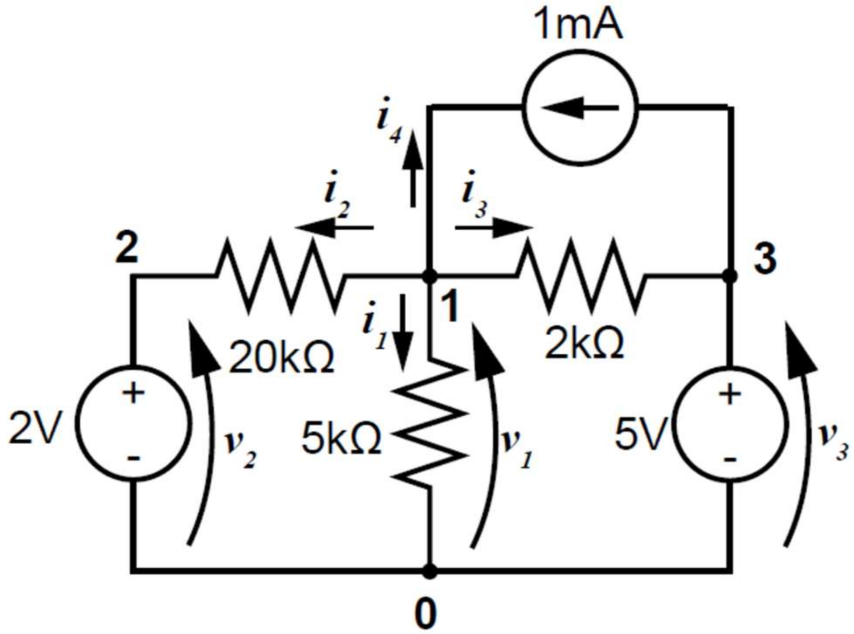
Nodal analysis / KCL example

Often when given a circuit, we need to find unknown currents or voltages. A number of methods can be employed. **One is Nodal analysis.**



Nodal analysis

- Unknowns are the node voltages
- Essentially KCL at every node
  - How PSpice works
- Procedure:
  - Choose a datum (reference) node (generally best to choose the one with the most voltage sources attached)
  - For each node, write KCL using the element relationships between voltage and current
  - For nodes connected by a voltage source, there is no relationship between current and voltage
    - Leave current as an unknown
    - Known difference in voltage between nodes gives the extra equation needed
  - Solve equations



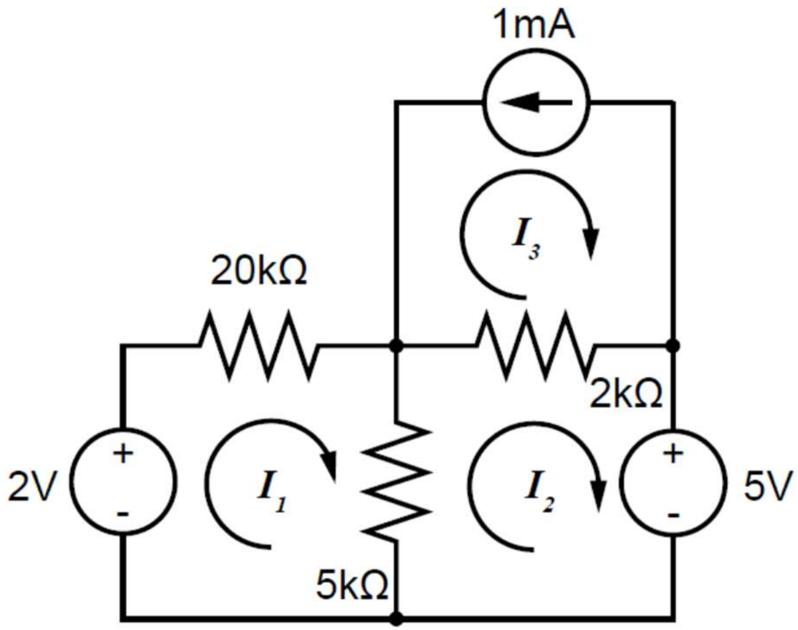
$$i_1 + i_2 + i_3 + i_4 = 0$$
$$v_2 = 2,\ v_3 = 5$$
$$i_1 = v_1 / 5\times10^3$$
$$i_2 = (v_1 - 2)/20\times10^3$$
$$i_3 = (v_1 - 5)/2\times10^3$$
$$i_4 = -1\times10^{-3}$$

Solve to get: $v_1 = 72/15 = 4.8$ *Volts*

Mesh analysis / KVL example

Procedure:

- Unknowns are mesh currents
- Essentially KVL around every mesh
- Not as generally applicable as nodal analysis
- Procedure:
  - Assign a mesh current to every mesh
  - Determine the current through each component in terms of the mesh currents
  - Write KVL for every mesh in terms of the mesh currents
  - For meshes including current sources, there is a problem as the voltage across a current source is independent of the current
    - For meshes including current sources, leave the voltages as unknown
    - The relationships between mesh currents and the currents through the current sources give the extra equations
  - Solve equations



$$20\times10^3\, i_1 + 5\times10^3\,(i_1 - i_2) = 2$$
$$2\times10^3\,(i_2 - i_3) + 5 = 5\times10^3\,(i_1 - i_2)$$
$$i_3 = -1\times10^{-3}$$

Solve to get:

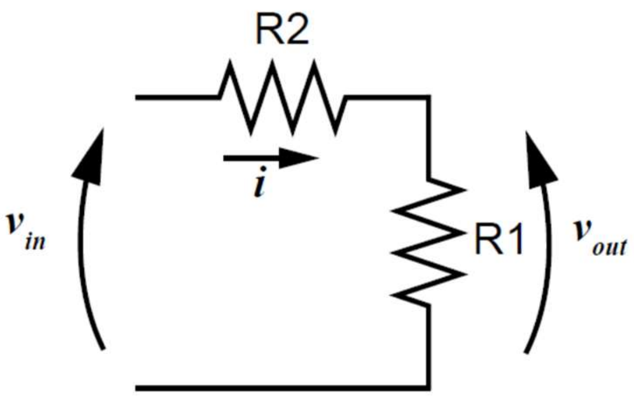
$i_1 = -0.14\times10^{-3}$

$i_2 = -1.1\times10^{-3}$

v_1 = *4.8 Volts (as with the nodal analysis)*

Voltage/Potential divider

One network configuration that appears regularly is the **voltage or potential divider**.



$$v_{out} = \frac{R_1}{R_1 + R_2} v_{in}$$

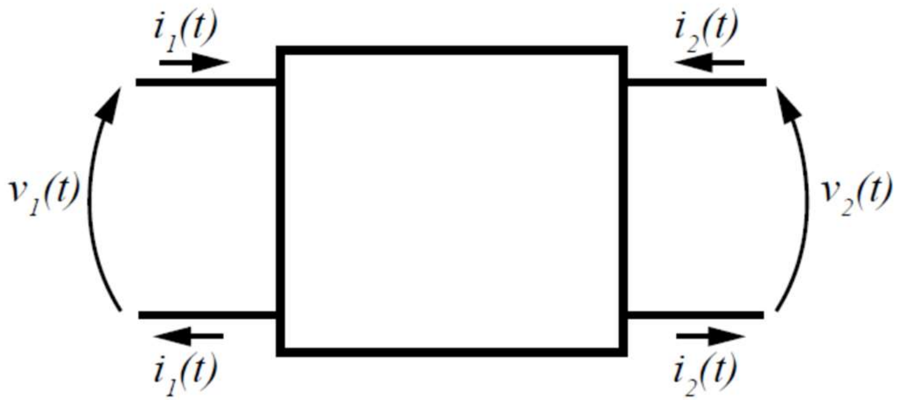
Two Port Network Elements

It is possible to consider a network consisting of elements with more than one port. In general these are called multiport elements.

The most important of these elements is one with two ports, the **two port network element**.

The two port element has four variables associated with it:

$$v_1(t),\ i_1(t),\ v_2(t),\ i_2(t)\ .$$

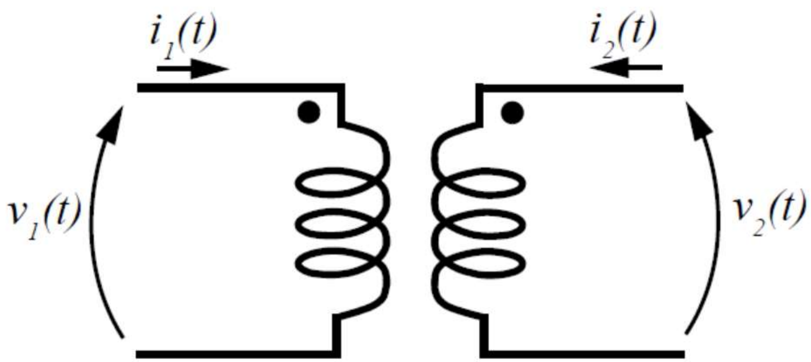


Two port elements are required to allow for dependent sources, coupling between electrical components and subsystems.

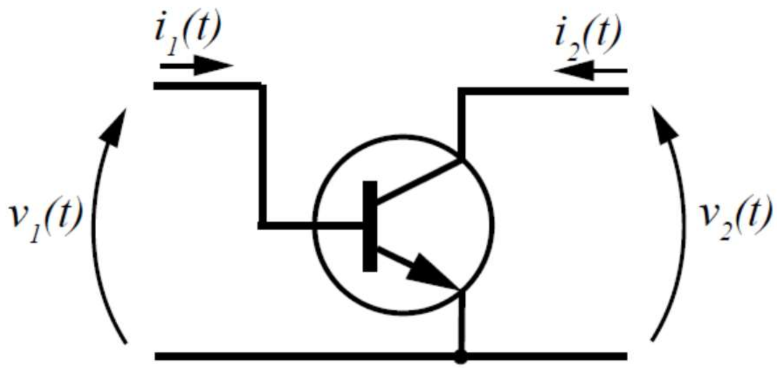
Furthermore to model electronic devices and simple transformers.

Common examples of two port elements are the **transformer**, and the **transistor.**

We will look at two ports in detail later in the course.



transformer



transistor

Linear system - Linearity

A system is said to be ***linear*** if it has the properties of ***Homogeneity*** and ***Additivity.***

Homogeneity: given ***f(u)*** is the output of a system with input ***u***, for the system to have the property of homogeneity, the output of the system for an input **α*u*** where **α** is a constant, must be **α*f(u).*** i.e. the output of the system scales with the input.

Additivity: given ***f(u)*** is the output of a system with input ***u***, and ***f(v)*** is the output of a system with input ***v,*** then for the system to have the property of additivity, the output of the system to input ***u+v*** must be ***f(u)+f(v)***. i.e. the response to a sum of inputs is the sum of the responses to each input applied separately

Linear circuit elements and dependent sources

Linear Elements:

Ideal resistor $v(t)=Ri(t)$

Ideal capacitor $i(t) = C\dfrac{dv}{dt}$ (see later in frequency domain)

Ideal inductor $v(t) = L\dfrac{di}{dt}$ (see later in frequency domain)

Define a ***linear dependent source*** as a dependent current or voltage source whose output current or voltage is proportional only to the first power of some current or voltage variable in the circuit, or to the sum of such quantities.

For example:

$$i_u(t)=a\, v_x(t),$$
$$i_v(t)=b\, i_x(t),$$
$$v_u(t)=c\, v_x(t),$$
$$v_v(t)=d\, i_x(t),$$
$$v_v(t)=a\, i_x(t) + b\, v_y(t)$$

Where a,b,c,d are constants

Linear circuits

Define a **linear circuit** as a circuit composed entirely of independent sources, linear dependent sources, and linear elements.

From the linear circuit definition it is possible to show that multiplication of all independent source voltages and currents in the circuit by a constant K increases all the current and voltage responses by the same factor K.

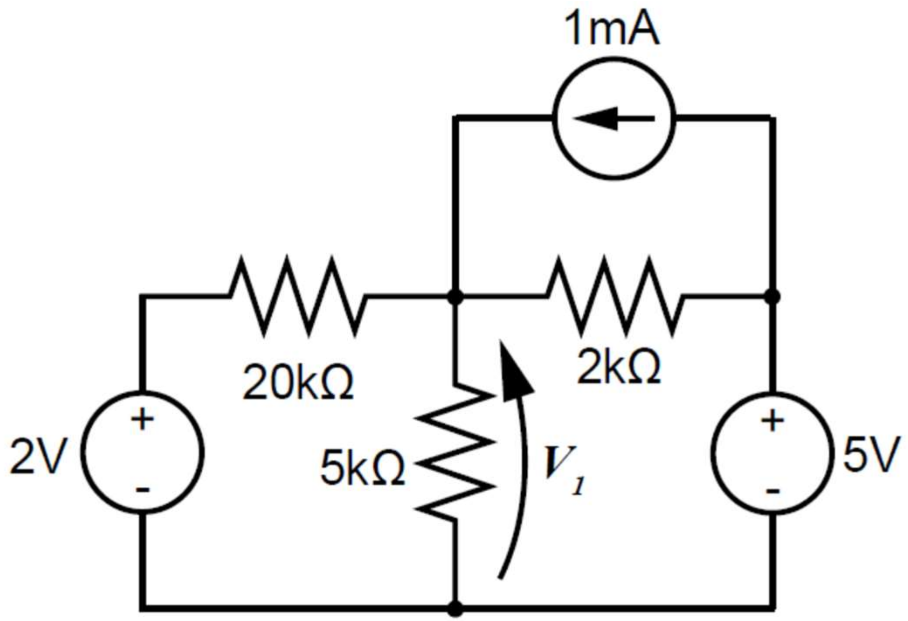
The most important consequence of a linear circuit is that the superposition principle can be applied. The **superposition principle** to linear network:

In any linear resistive network containing several sources, the voltage across or the current through any resistor or source may be calculated by adding algebraically all the individual voltages or currents by each independent source acting alone, with all other independent voltage sources replaced by short circuits and all other independent current sources replaced by open circuits.

Extend to the frequency domain to include networks with capacitors and inductors.

Superposition - Example

Use superposition to find V_1 in the following circuit:



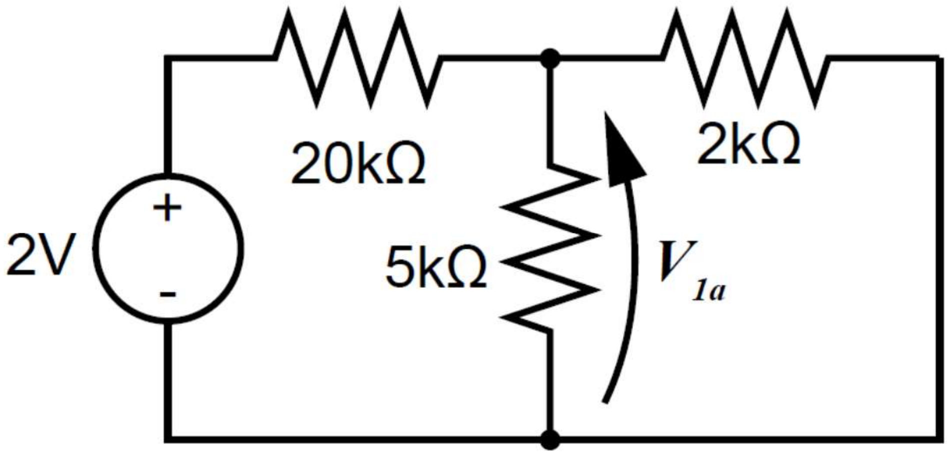
Superposition Steps:

1.Turn off all independent sources except one. To turn off a current source, make it open-circuit, to turn off a voltage source make it short-circuit.

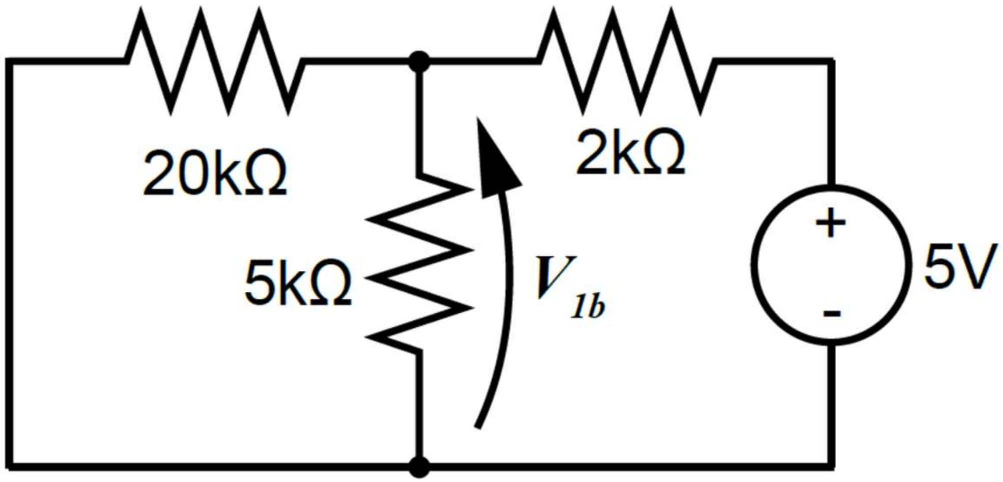
2.Calculate the required output (voltage or current) due to the single active source.

3.Repeats steps 1 and 2 for the remaining independent sources

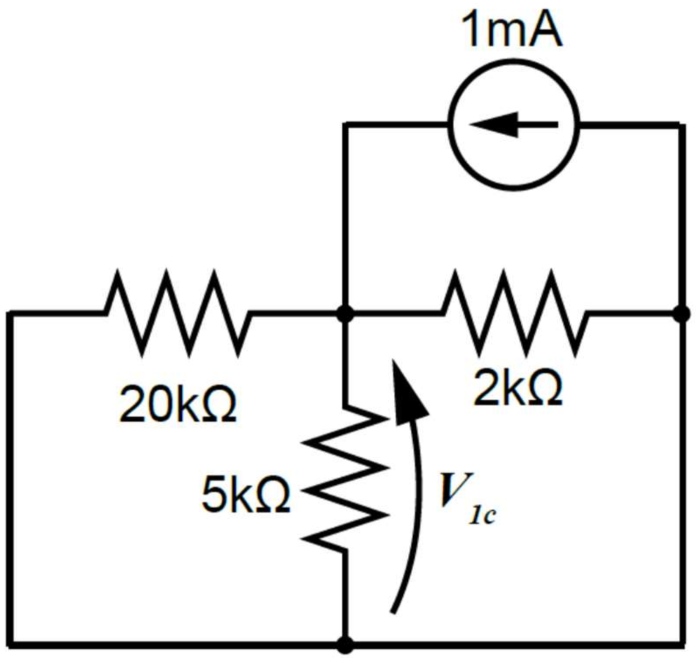
4.Find the true output of the original circuit by adding the individual outputs derives in step 2 for each active source.



$$V_{1a} = 2V \times \frac{2k||5k}{20k+2k||5k} = 0.1333V$$



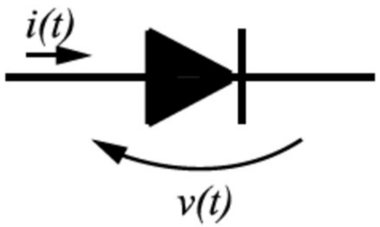
$$V_{1b} = 5V \times \frac{20k||5k}{2k+20k||5k} = 3.333\ V$$



$$V_{1c} = 0.001 \times 20k||5k||2k = 1.333\ V$$

$$V_1 = V_{1a} + V_{1b} + V_{1c} = 4.8V$$

One Port Non Linear Element – Diode



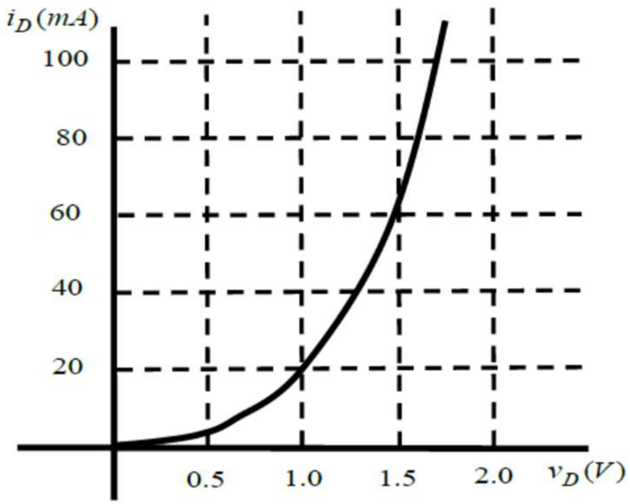
$$i(t) = f(v(t)) \qquad ?$$

$$i(t) = \frac{Q_p}{\tau_p} + \frac{dQ_p}{dt} \qquad Q_p(t) = qAL_p p_{no}\left(e^{\frac{qv(t)}{kT}} - 1\right)$$

Could write the relationship directly in *v*(t) without the Q*p*

For " DC" – "steady state" $\frac{dQ_p}{dt} = 0$

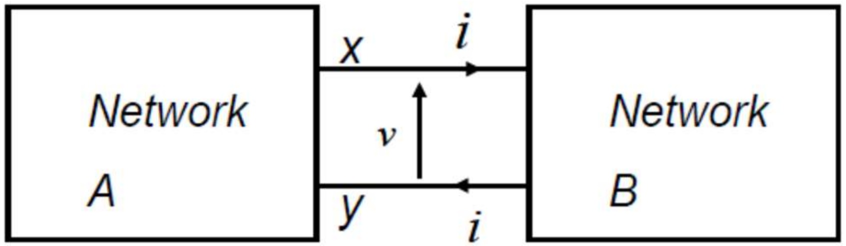
$$I_D = I_{SO}\left(e^{\frac{qV_D}{kT}} - 1\right)$$



Thevenin's Theorem

Thevenin's Theorem states: Any linear electrical network containing only independent and dependent voltage and current sources and resistances can be replaced at terminals X-Y by an equivalent combination of a voltage source V_{th} in a series connection with a resistance R_{th}. (Can be extended to include other linear elements such as capacitors and inductors)

When we are interested in only one part of the network, then the remainder of the network may be replaced by a simple equivalent network by using Thevenin's theorem or its dual Norton's theorem. For example, this approach would be useful in optimizing the load for a power amplifier. You could replace the whole amplifier with its Thevenin or Norton equivalent.



Replace network **A** with its Thevenin equivalent to simplify analysis.

Example (Cont'd)

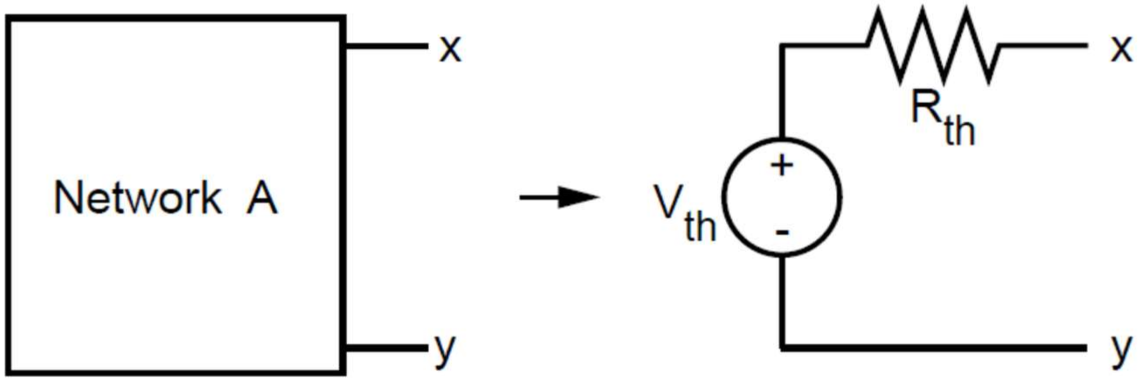
Network A must satisfy the following :

- Consist of Linear elements
- Linear dependent and independent voltage and current sources
- Initial conditions allowed on passive elements
- No coupling to Network B , e.g. no transformers coupling or controlled sources coupling to network B

Network B must satisfy the following :

- Any elements:- Linear, Non-linear elements, Time varying,…
- Any source may be present
- Initial conditions allowed on passive elements
- No coupling to Network A , e.g. no transformers coupling or controlled sources coupling to network A

Example (Cont'd)



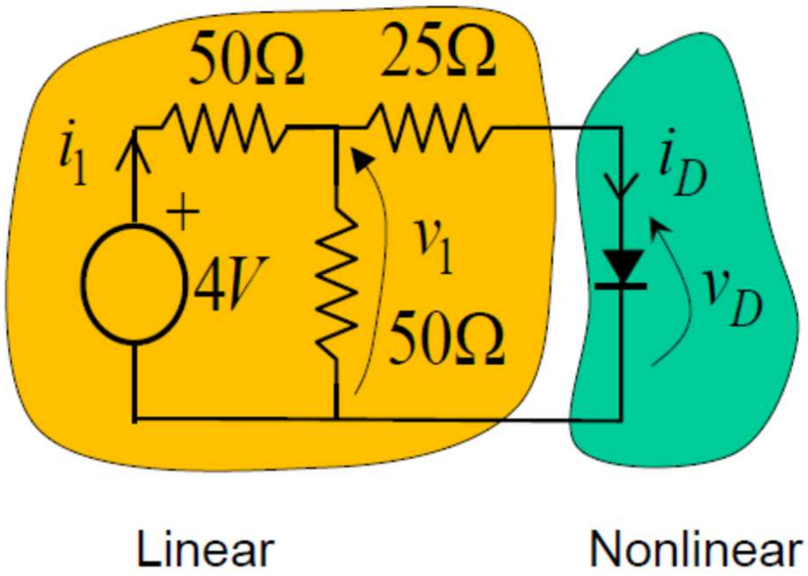
To find V_{th}: Calculate the open circuit voltage at terminals X-Y by whatever method, KVL/KCL, superposition. This open circuit voltage is V_{th}.

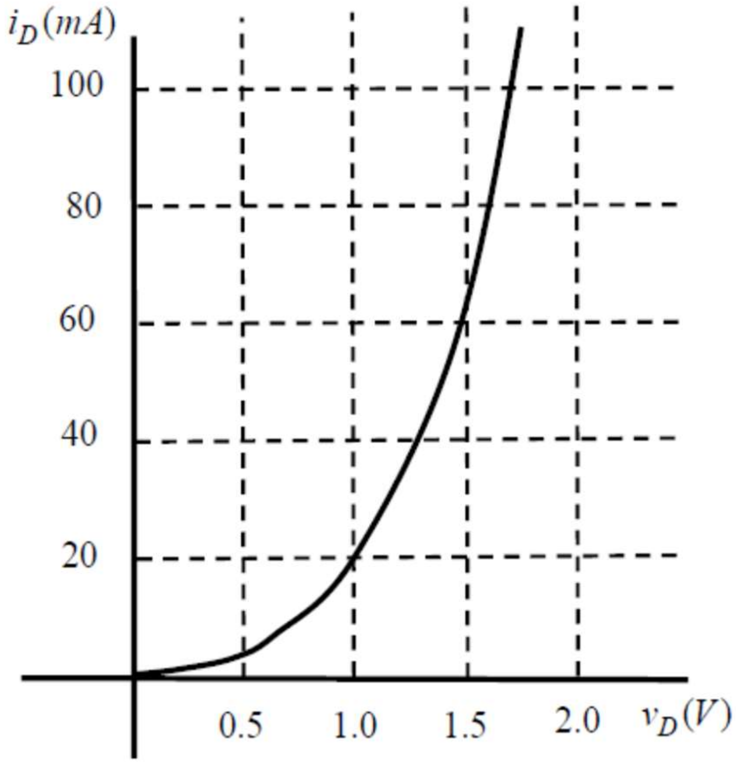
To find R_{th}:

- All independent sources set to zero (voltage sources become short-circuit, current sources become open-circuit) but leave the dependent sources unmodified. All initial conditions set to zero.

- At terminals x-y apply an arbitrary voltage v_0, to facilitate the calculations make v_0 = 1. Then calculate the current from terminals x-y, i_0, keeping in mind how to handle the dependent sources. Then: $R_{th} = v_0 / i_0$

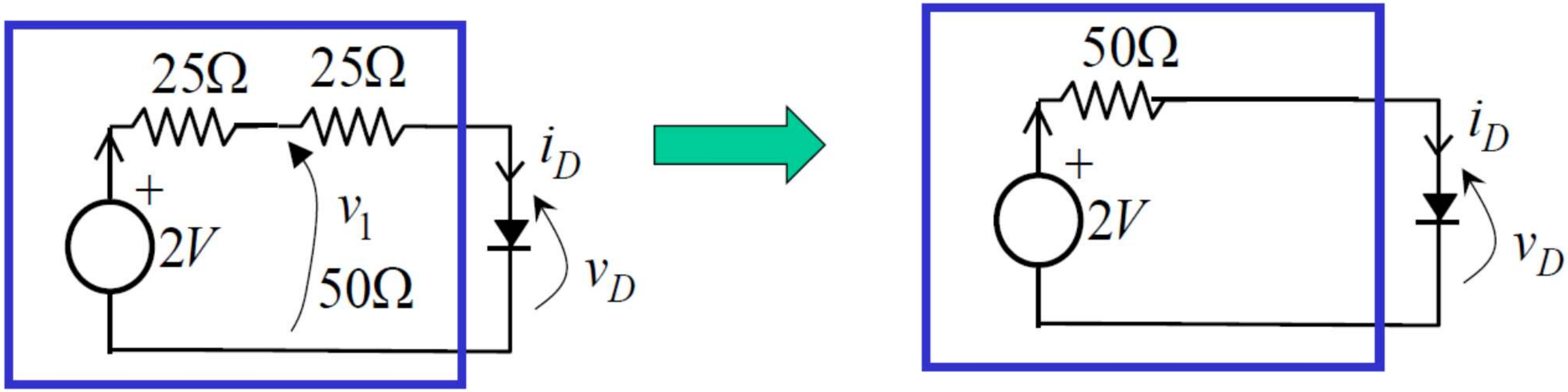
Example (Cont'd)



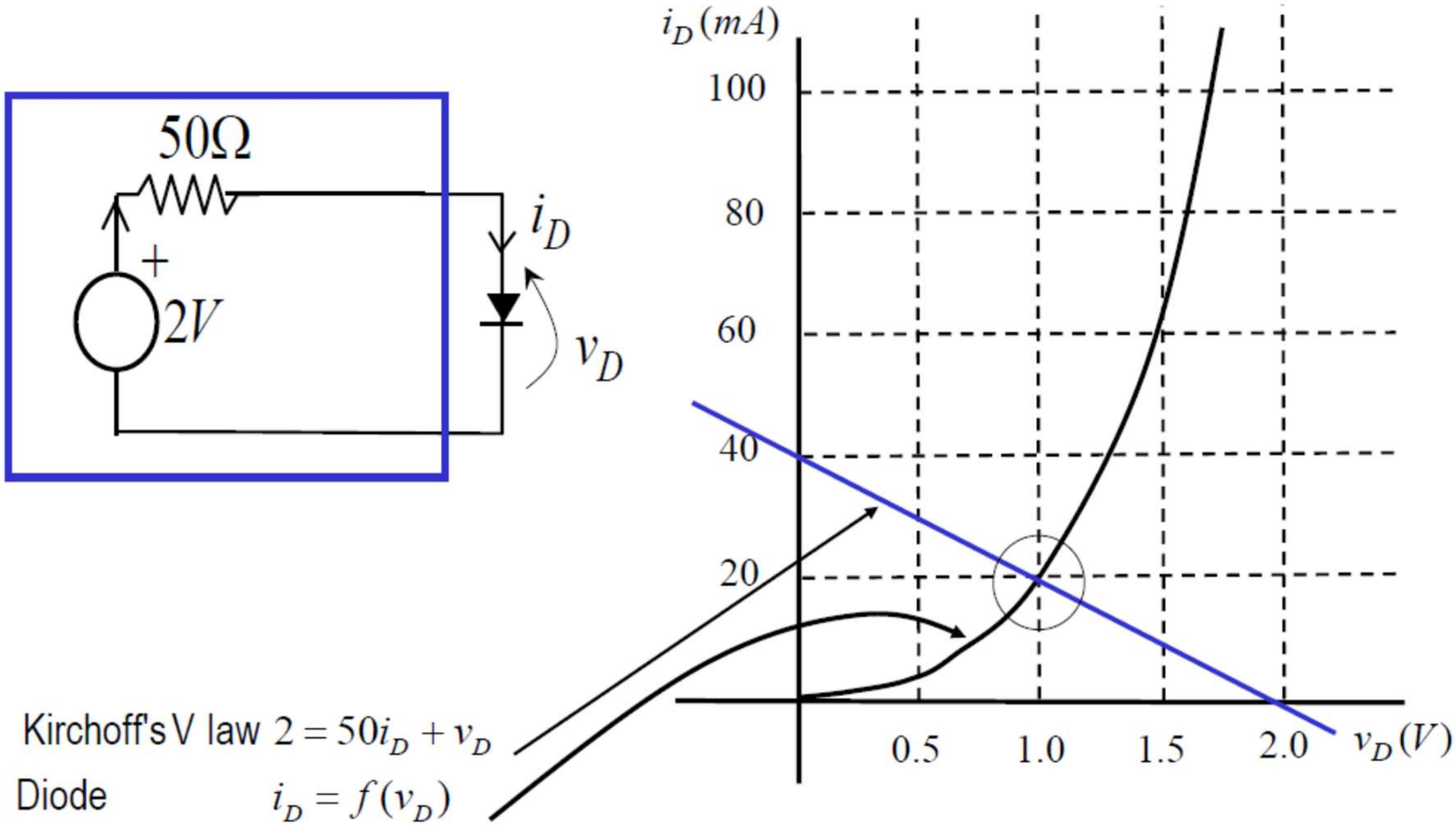


Example (Cont'd)

Thevenin Equivalent Circuit



Example (Cont'd)



Kirchoff's V law $2 = 50 i_D + v_D$

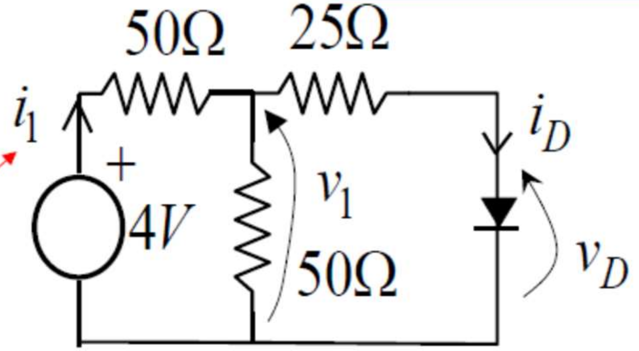
Diode $i_D = f(v_D)$

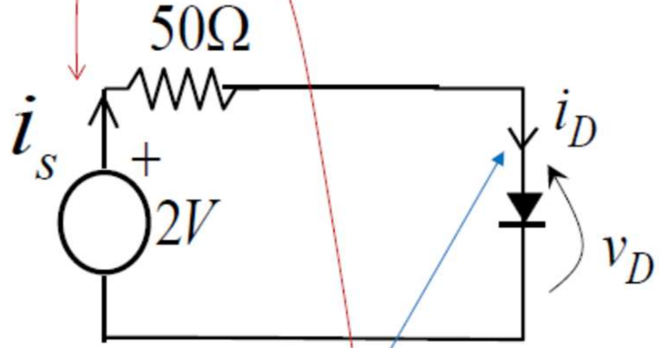
Solve graphically

$i_D = 20mA \quad v_D = 1V$

Example (Cont'd)

What is equivalent ?





$$\boxed{i_s \neq i_1}$$

$$i_1 = \frac{v_1}{50} + i_D = \frac{v_D + 25 i_D}{50} + i_D$$

$$i_1 = 0.02 v_D + 1.5 i_D$$

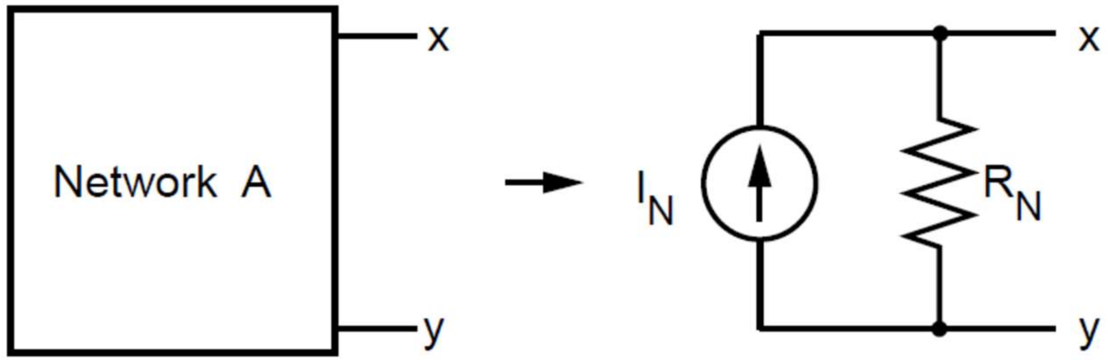
$$i_1 = 0.02 v_D + 1.5 i_s$$

NOTE that Thevenin equivalent circuit tells us nothing about what happens inside network A. It only describes what happens at the terminals x-y.

Norton's Theorem

Norton's theorem states: Any linear electrical network containing only independent and dependent voltage and current sources and resistances can be replaced at terminals x-y by an equivalent combination of a Current source IN in a parallel connection with a resistance Rn. (Can be extended to include other linear elements such as capacitors and inductors)

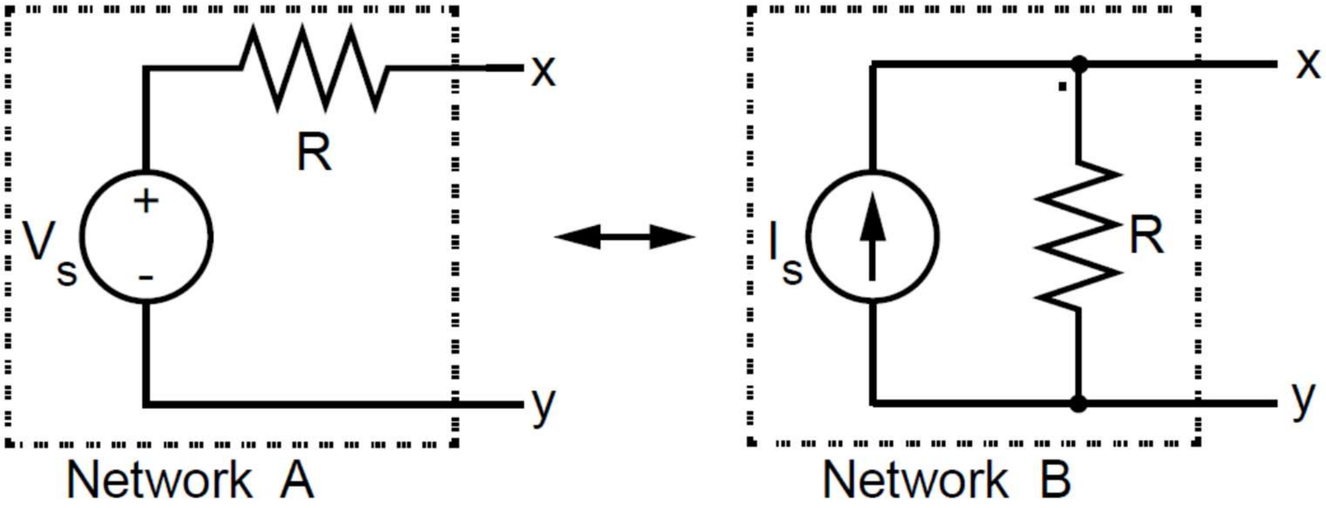
Norton's theorem is the dual of Thevenin's theorem.



To find I_N: Calculate the closed circuit current at terminals X-Y when they are shorted by whatever method, KVL/KCL, superposition. This short circuit current is I_N.

To find R_N: Employ the same method as in finding R_{th}. In fact $R_N = R_{th}$.

Source Transformations



The voltage-current relationship at terminals x-y are identical for Network A and Network B, provided $V_s = R\, I_s$.

Simple case of Norton's and Thevenin's theorems.

Can be extended later to include capacitors and inductors

Allows an ideal voltage source in series with a resistor to be replaced by an ideal current source in parallel with the same resistor, and vice versa.

Acknowledgments

- Lecture slides are based on lecture materials from various sources, including M. Hill, T. Cantoni, F. Boussaid, and R. Togneri.

- Credit is acknowledged where credit is due. Please refer to the full list of references.