Question 6 (8 marks)

For the following LTI system impulse responses determine whether the system is casual and also whether it is stable. Justify your answers.

(a) $h(t) = e^{-4t}u(t-2)$

(b) $h(t) = e^{-2t}u(t+50)$

(c) $h[n] = (0.5)^n u[-n]$

(d) $h[n] = (-0.5)^n u[n] + (1.01)^n u[n-1]$

(a) Causal because $h(t) = 0$ for $t < 0$; Stable because $\int_{-\infty}^{\infty} |h(t)|dt \to \int_{2}^{\infty} e^{-4t} < \infty$
(b) Not causal because $h(t) \neq 0$ for $-50 < t < 0$; Stable because $\int_{-\infty}^{\infty} |h(t)|dt \to \int_{-50}^{\infty} e^{-2t} < \infty$
(c) Not causal because $h[n] \neq 0$ for $n < 0$; Unstable $\sum_{k=-\infty}^{\infty} |h[k]| \to \sum_{k=-\infty}^{0} (0.5)^k \equiv \sum_{l=0}^{\infty} (2)^l \not< \infty$
(d) Causal because $h[n] = 0$ for $n < 0$; Unstable $\sum_{k=-\infty}^{\infty} |h[k]| \to \sum_{k=1}^{\infty} (1.01)^k + \cdots \not< \infty$

Question 3 (8 marks)

Determine for each of the systems below (where $x(t)$ is the input and $y(t)$ is the output) which and all of the properties that apply: Memoryless, Time invariant, Linear, Causal, Stable.

(a) $y(t) = \begin{cases} 0 & x(t) < 0 \\ x(t) + x(t-2) & x(t) \geq 0 \end{cases}$

(b) $y(t) = x(t/3)$

(c) $y[n] = nx[n]$

(d) $y[n] = x[4n+1]$

(a) Time-invariant, Causal, Stable
(b) Linear, Stable
(c) Memoryless, Linear, Causal
(d) Linear, Stable

Question 3 (6 marks)

What is the minimum sampling rate in Hz (Nyquist rate) to recover each of the following signals and explain your reasoning:

(a) $x(t) = 1 + \cos(1000\pi t) + \sin(3000\pi t)$

(b) $x(t) = \sin(4000\pi t)/(\pi t)$

(c) $x(t) = s(t) + s(t-1)$ where $s(t)$ has a Nyquist rate of $f_S = 5000$ Hz

(d) $x(t) = s(t)\cos(6000\pi t)$ where $s(t)$ has a Nyquist rate of $f_S = 8000$ Hz.

Hint: Use the required Fourier transform pairs and properties and the fact that $x(t) * \delta(t-T) = x(t-T)$

Solution:

(a) The signal is bandlimited to the highest frequency of $\omega_B = 3000\pi$, hence the sampling rate is $\omega_S = 6000\pi$ or $f_S = 3000$ Hz

(b) From the Fourier transform pair with $W = 4000\pi$ the signal is bandlimited to $\omega_B = 4000\pi$, hence the sampling rate is $f_S = 4000$ Hz.

(c) From the Fourier transform properties $X(j\omega) = S(j\omega) + e^{-j\omega}S(j\omega)$ which does not affect the Nyquist rate which remains at $f_S = 5000$ Hz

(d) From the Fourier transform properties:

$$X(j\omega) = \frac{1}{2\pi} S(j\omega) * \{\pi\delta(\omega - 6000\pi) + \pi\delta(\omega + 6000\pi)\}$$
$$= \frac{1}{2}\{S(j(\omega - 6000\pi) + S(j(\omega + 6000\pi)\}$$

We know that $s(t)$ is bandlimited to 4000 Hz (8000π) so for $x(t)$ we will have that $\omega_B = 8000\pi + 6000\pi = 14000\pi$ and $f_S = 14000\ Hz$

Question 5 (10 marks)

(a) Use the defining equation of the inverse Fourier transform to derive the real-valued signal function, $x(t)$, by direct integration of its Fourier Transform $X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$ where:

$$|X(j\omega)| = \begin{cases} 2 & |\omega| \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\angle X(j\omega) = -\frac{3}{2}\omega + \pi$$

Simplify your expression for $x(t)$ where possible (**Hint:** your Fourier friend is Euler).

(b) Find all values of t such that $x(t) = 0$?

(a) We need to calculate:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|e^{j\angle X(j\omega)}e^{j\omega t}d\omega$$

$$= \frac{1}{\pi}\int_{-3}^{3} e^{j\left(-\frac{3}{2}\omega+\pi\right)}e^{j\omega t}d\omega = \frac{1}{\pi}\int_{-3}^{3} e^{j\left(\omega\left(t-\frac{3}{2}\right)+\pi\right)}d\omega$$

$$= \frac{1}{\pi}\left\{\left[\frac{e^{j\left(\omega\left(t-\frac{3}{2}\right)+\pi\right)}}{j\left(t-\frac{3}{2}\right)}\right]_{\omega=-3}^{3}\right\} = \frac{e^{j\pi}}{\pi}\left\{\left[\frac{e^{j\omega\left(t-\frac{3}{2}\right)}}{j\left(t-\frac{3}{2}\right)}\right]_{\omega=-3}^{3}\right\} = -\frac{1}{\pi}\left\{\frac{e^{j3\left(t-\frac{3}{2}\right)} - e^{-j3\left(t-\frac{3}{2}\right)}}{j\left(t-\frac{3}{2}\right)}\right\}$$

$$= -\frac{1}{\pi}\frac{2j\sin(3(t-3/2))}{j(t-3/2)} = -\frac{6}{\pi}\frac{\sin(3(t-3/2))}{3(t-3/2)}$$

(b) $x(t) \propto \frac{\sin(x)}{x}$ is zero when $\sin(x) = 0$ ($x \neq 0$) which is true whenever $x = k\pi$ for $k = \pm 1, \pm 2, \pm 3, \ldots$, that is:

$$3\left(t - \frac{3}{2}\right) = k\pi \rightarrow t = \frac{k\pi}{3} + \frac{3}{2}$$

Question 2 (6 marks)

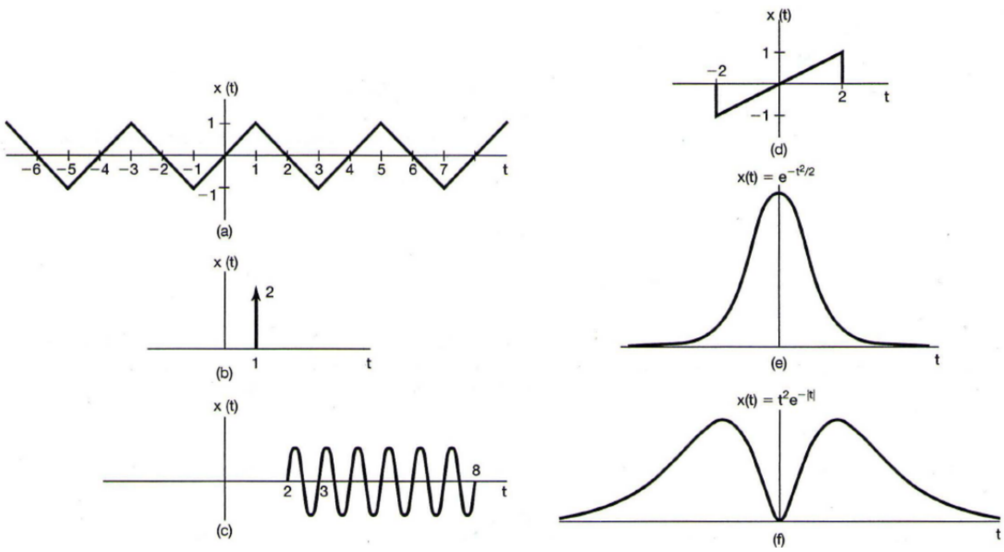
For each condition listed below determine which, if any, of the real signals depicted below have Fourier transforms that satisfy that condition:

(A) $\mathrm{Re}\{X(j\omega)\} = 0$

(B) $\mathrm{Im}\{X(j\omega)\} = 0$

(C) $|X(j\omega)| = 2$, for all ω

(D) $X(j\omega)$ is non-zero only for certain values of ω



Solution:

(A) implies that $x(t)$ must be an odd function about $t = 0$ ($X(j\omega)$ is purely imaginary): Only **(a) and (d)** have this property.
(B) implies that $x(t)$ must be an even function about $t = 0$ ($X(j\omega)$ is purely real): Only **(e) and (f)** have this property.
(C) implies that the spectrum is flat, which can only happen with an impulse function: Only **(b)** has this property.
(D) If $X(j\omega)$ is non-zero only for certain values of ω then this is equivalent to a Fourier series representation and hence $x(t)$ must be periodic. Only **(a)** has this property.

Question 3 (6 marks)

(i) Consider the continuous-time signal:

$$x(t) = 3\cos 6\pi t + \sin 18\pi t + 2\cos 28\pi t$$

What is the expression for the magnitude spectrum, $|X(\omega)|$, and sketch it as a function of ω.
HINT: You have three sinusoids so you will have just three spectral harmonics.

(ii) What is the Nyquist rate, that is, determine the range of possible sampling frequencies (in Hz), f_s, required to be able to reconstruct $x(t)$ from these samples without error?

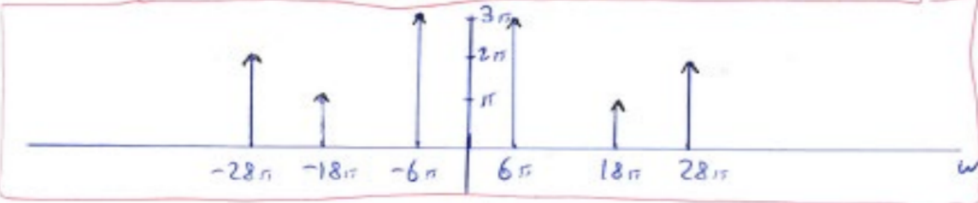
(iii) What is the sampling frequency if you sample $x(t)$ at 25% above the Nyquist rate (i.e. your answer in (ii) x 1.25)? For this sampling frequency carefully sketch the magnitude spectrum of the sampled signal (in Hz) over the range $\pm$ 50 Hz.

$3\pi[\delta(w-6\pi) + \delta(w+6\pi)]$
$+\pi[\delta(w+18\pi) + \delta(w-18\pi)]$
$2\pi[\delta(w-28\pi) + \delta(w+28\pi)]$

Solution:

1 (i) $x(t) = 3\cos(6\pi t) + \sin(18\pi t) + 2\cos(28\pi t)$

$$\therefore \boxed{\begin{aligned} |X(\omega)| = \; & 3\pi[\delta(\omega-6\pi) + \delta(\omega+6\pi)] \\ & + \pi[\delta(\omega+18\pi) + \delta(\omega-18\pi)] \\ & + 2\pi[\delta(\omega-28\pi) + \delta(\omega+28\pi)] \end{aligned}}$$ 1



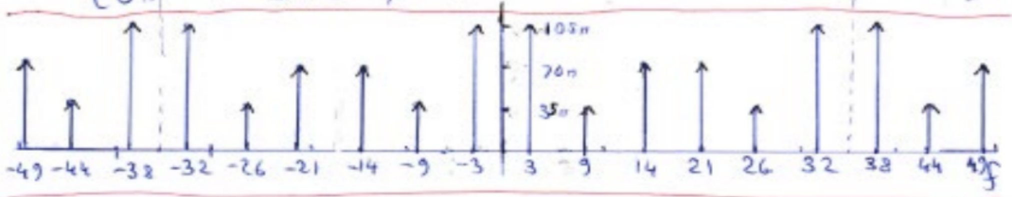
1

(ii) $B = 28\pi$ $\omega_s \geq 2B = 56\pi$

$\therefore \boxed{f_s \geq 28\ Hz}$ 1

(iii) 25% above 28 Hz $\Rightarrow \boxed{f_s = 35\ Hz}$ 1

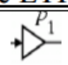
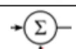
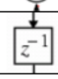
($6\pi \to 3\ Hz$; $18\pi \to 9\ Hz$; $28\pi \to 14\ Hz$)



Gain of 35

$3\pi \to 105\pi$ $2\pi \to 70\pi$ $\pi \to 35\pi$ 2

Computational structures for discrete-time LTI systems require three elements:

gain or multiplier element	
summer or accumulator element	
unit delay element	

Question 5 (10 marks)
Consider a discrete-time LTI system for which the input $x[n]$ and output $y[n]$ are related by the difference equation:

$$y[n-1] - \frac{5}{2}y[n] + y[n+1] = x[n]$$

where it is evident that this system is anti-causal due to the $y[n+1]$.

(a) By using the z-transform determine the transfer function, $H(z)$, expressed as a ratio of two polynomials in z

(b) Sketch the pole-zero pattern of $H(z)$ on the z-plane.

(c) Now by first expressing $H(z)$ as a ratio of two polynomials in z^{-1} carry out a partial fraction expansion (to allow taking the inverse z-transform in the next step).

(d) Find the impulse response sequence $h[n]$ which ensures the system is stable.

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(d) Find the impulse response sequence $h[n]$ which ensures the system is stable.

Question 2 (5 marks)
Consider the following system where $x(t)$ is the input (where $x(t) > 0$, for all t) and $y(t)$ is the output:

$$y(t) = \log\big(x(t+1)\big)$$

Determine whether the system is:

a) memoryless? (yes or no)
b) time-invariant? (yes or no)
c) linear? (yes or no)
d) causal? (yes or no)
e) BIBO stable? (yes or no)

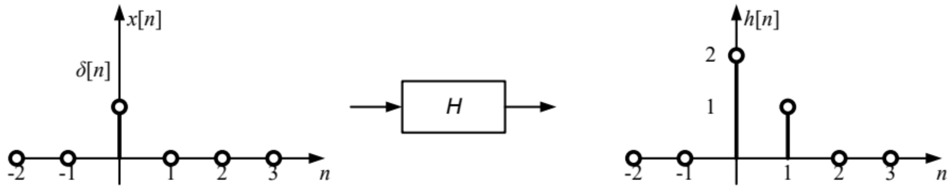
Question 3 (3 marks)
Find the zero-input response for the system described by:

$$3\frac{d}{dt}y(t) + 1 = 4x(t)$$

given initial conditions $y_0(0) = 3$.

Question 2 (6 marks)

A linear, time-invariant system has the impulse response, $h[n]$, shown below (when input $x[n] = \delta[n]$):



Sketch the response of the system to the input when it is:

$$x[n] = \delta[n+1] - \delta[n-2]$$

You must label and scale all axes.

Question 2 (6 marks)

A linear, time-invariant system has the impulse response, $h[n]$, shown in Figure Q2.

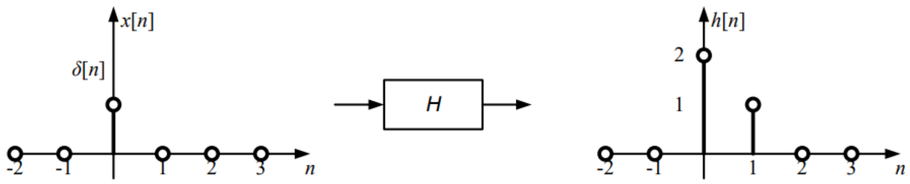


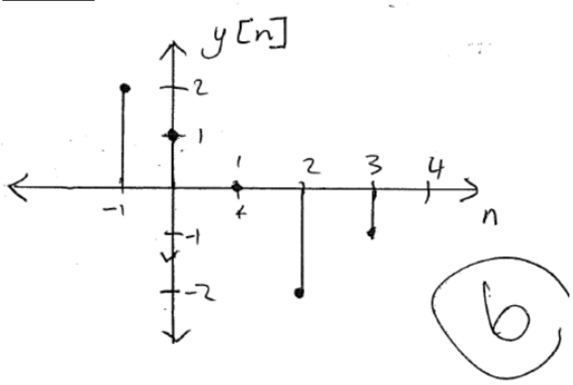
FIGURE Q2

Sketch the response of the system to the input

$$x[n] = \delta[n+1] - \delta[n-2]$$

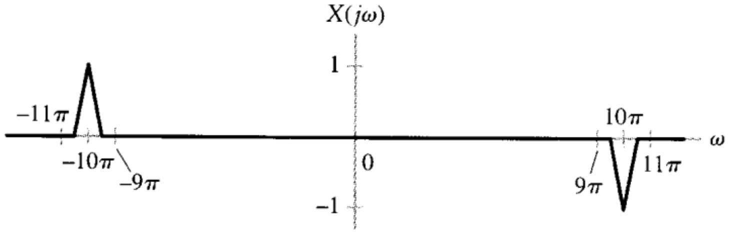
You must label and scale all axes.

<u>Solution:</u>



Question 10 (12 marks)

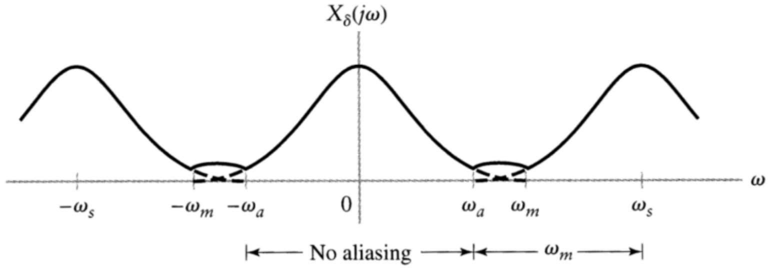
(a) The continuous-time signal $x(t)$ with Fourier Transform as depicted below is sampled:



Sketch the Fourier Transform of the sampled signal for the following sampling periods, T_s, and identify whether aliasing occurs:

(i) $T_s = 1/14$
(ii) $T_s = 1/7$
(iii) $T_s = 1/5$

(b) We sample a continuous-time signal with Fourier spectra $X(j\omega)$ and want to ensure that we can reconstruct $X(j\omega)$ over the interval $-\omega_a < \omega < \omega_a$ given that the signal is band-limited with maximum frequency ω_m but where $\omega_m \geq \omega_a$. This is depicted in the figure below:



What is the maximum sampling period, T_s, we can use?

Solution:

B3 4.26

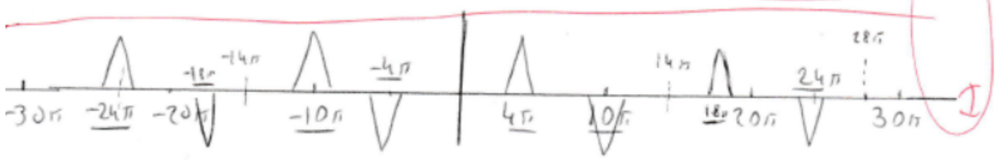
(a) (i) If $T_s = 1/14$ then $\omega_s = \frac{2\pi}{T_s} = 28\pi$ 1

-38π -18π -10π 10π 18π 38π

1

$\boxed{\omega_m = 11\pi}$ 1 $\therefore$ $\boxed{\omega_N = 2\omega_m = 22\pi}$ 1 Since $\boxed{\omega_s > \omega_N \text{ no aliasing}}$

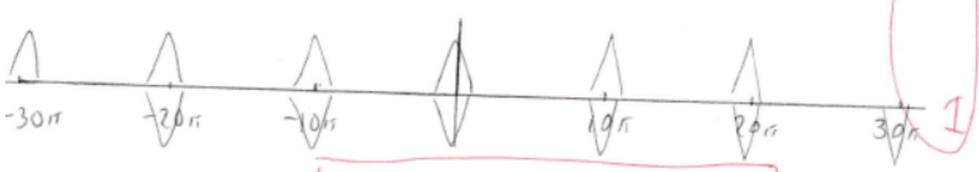
(ii) If $T_s = 1/7$ then $\omega_s = \frac{2\pi}{T_s} = 14\pi$ 1



-24π -10π -4π 4π 10π 24π why 1

$\boxed{\omega_s = 14\pi < \omega_N = 22\pi \quad \text{aliasing!}}$

(iii) If $T_s = 1/5$ then $\omega_s = \frac{2\pi}{T_s} = 10\pi$! 1



$\therefore$ $\boxed{\text{all cancels out} = \text{zero!!}}$

$\boxed{\text{Definitely aliasing is occurring!}}$

(b) From the figure we see that

$$W_a + W_m = W_s = \frac{2\pi}{T_s}$$

And we want $W_s \geq W_a + W_m$

$$\therefore \frac{2\pi}{T_s} \geq W_a + W_n$$

$$\boxed{T_s \leq \frac{2\pi}{W_a + W_m}}$$

Question 2 (10 marks)

Referring to the Fourier transforms below determine the Nyquist sampling rate (in kHz) for the signal:

(a) $f_1(t)$

(b) $f_2(t)$

(c) $f_1(t) + f_2(t)$

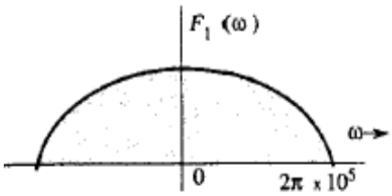
(d) $f_1(t) f_2(t)$

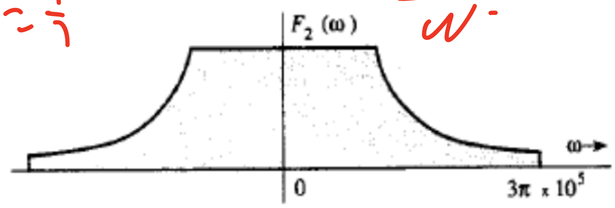
(e) $f_1(t) * f_2(t)$

$T = \frac{2\pi}{W}$

$f_1 = \frac{1}{T}$ $= \frac{2\pi}{W}$ $\frac{4\pi \times 10^5}{2\pi}$

$= 200 kHz$





HINT: If $F_1(\omega)$ is a signal of bandwidth W_1 and $F_2(\omega)$ is a signal of bandwidth W_2, the bandwidth of $F_1(\omega) * F_2(\omega)$ is $W_1 + W_2$ (Width Property of convolution).

(a) Maximum frequency of $f_1(t)$ is $B = 2\pi(10^5)$ and thus $f_S = 2B = \frac{4\pi(10^5)}{2\pi} = 2(10^5)$ Hz or 200 kHz

(b) Maximum frequency of $f_2(t)$ is $B = 3\pi(10^5)$ and thus $f_S = 2B = \frac{6\pi(10^5)}{2\pi} = 3(10^5)$ Hz or 300 kHz

(c) Maximum frequency of $f_1(t) + f_2(t)$ is $B = 3\pi(10^5)$ and thus $f_S = 2B = \frac{6\pi(10^5)}{2\pi} = 3(10^5)$ Hz or 300 kHz

(d) Since $f_1(t) f_2(t) \leftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$ and the width property of convolution tells us that the bandwidth of the convolution spectrum is the sum of the individual spectra, that is $B = 5\pi(10^5)$ and $f_S = 2B = \frac{10\pi(10^5)}{2\pi} = 5(10^5)$ Hz or 500 kHz

(e) Since $f_1(t) * f_2(t) \leftrightarrow F_1(\omega) F_2(\omega)$ then the bandwidth is $B = 2\pi(10^5)$ and thus $f_S = 2B = \frac{4\pi(10^5)}{2\pi} = 2(10^5)$ Hz or 200 kHz

$f_s = 2B = \frac{10\pi \times 10^5}{2\pi}$

$= 5 \times 10^5$

Question 3 (8 marks)

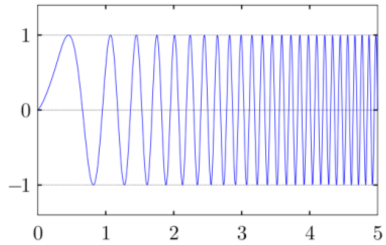
A signal $x_a(t)$ is sampled at a rate of 1600 Hz. Only 300 samples have been collected. You want to apply an FFT to plot the magnitude spectrum. But you want to do this in such a way that the frequency resolution or precision is no more than 2 Hz.

(a) How do you do this if you are allowed to zero-pad the samples?

(b) How do you do this if can go back and acquire more samples?

(c) Why is (b) better than (a)?

(d) Consider $x_a(t)$ as a chirp signal where the frequency increases linearly with time:



Would (b) still be better than (a)?

(a) The minimum number of samples we need is:

$$N_0 = \frac{f_S}{f_0} = \frac{1600}{2} = 800$$

Hence we need to zero-pad by adding an extra 500 samples of zero value.

(b) Just acquire 500 more samples so you have 800 samples in total, and no need to zero-pad.

(c) Method (b) is superior because with more samples of the actual data you have more information to provide a better estimate of the true spectrum. When you only provide a finite sample of data you will always approximate the true spectrum (which needs to be evaluated across all time), zero-padding does not add any new information.

(d) With a signal with variable spectral characteristics it may be better to only acquire a small number of samples to avoid smearing across time (as the frequency varies) so method (a) may actually be better than method (b), it all depends on the signal being measured and how fast it varies with time.

Question 4 (4 marks)

(a) Consider:

$$x[n] = \sin\left(\frac{\pi}{3}n + \frac{\pi}{4}\right)$$

Which of the following statements is false and explain your answer:

(i) The Fourier spectrum only exists at specific frequencies

(ii) The signal $x[n]$ is periodic in time hence because of this the Fourier spectrum is periodic in frequency

(iii) If the signal $x[n]$ repeats every N_0 samples then the Fourier spectrum will also repeat every N_0 samples.

(b) Consider:

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

Which of the following statements is false and explain your answer:

(i) The Fourier spectrum exists at all frequencies

(ii) The Fourier spectrum is periodic in frequency because the time-domain signal $x[n]$ is sampled in time.

(iii) Since $x[n]$ is not periodic (and has finite energy) then from Parseval's theorem the Fourier spectrum cannot be periodic (otherwise it would have infinite energy).

(a) <u>(ii) is false</u> because although the Fourier spectrum for the DTFS is periodic this is NOT because $x[n]$ is periodic but because $x[n]$ is discrete sampled in the time domain (Remember! Sampling in one domain implies periodicity in the other domain).

(b) <u>(iii) is false</u>, since we know the Fourier spectrum will be periodic (because the time-domain is sampled). FYI, Parseval's theorem for the DTFT only considers one period of the periodic Fourier spectrum to provide finite energy.

Question 1 (8 marks)

We prove the conjugate symmetry property $X(-j\omega) = X^*(j\omega)$ for <u>real signals</u> as follows:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \;\rightarrow X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = X(-j\omega)$$

by noting that for real signals we can state $x^*(t) = x(t)$.

(a) What can we state for <u>purely imaginary signals</u> regarding $x(t)$ and $x^*(t)$?

(b) Use (a) to develop the conjugate symmetry property that applies to purely imaginary signals.

(c) Use (b) and state what we can say about the real component and imaginary component of $X(j\omega)$ for <u>purely imaginary signals.</u> HINT: Let $X(j\omega) = a(\omega) + jb(\omega)$ and use the conjugate symmetry property from (b).

(a) For imaginary signals $x^*(t) = -x(t)$

(b)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \rightarrow X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt = -\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = -X(-j\omega)$$

If $x(t)$ is an <u>imaginary signal</u> then we have the following *conjugate symmetry* property:

$$X^*(j\omega) = -\,X(-j\omega)$$

(c) Let $X(j\omega) = a(\omega) + jb(\omega)$ then:

$$X^*(j\omega) = a(\omega) - jb(\omega)$$
$$-X(-j\omega) = -a(-\omega) - jb(-\omega)$$

Hence: the <u>real component ($a(-\omega) = -a(\omega)$) is an odd function</u> and the <u>imaginary component ($b(-\omega) = b(\omega)$) is an even function</u>.