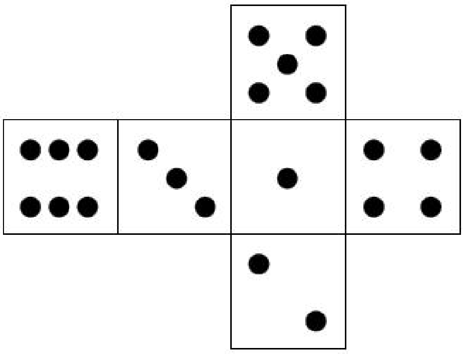
CMPE 548 Monte Carlo Methods
Homework 2

Due on March 16, 2016

1. **(Importance Sampling)**

   Consider the unit p-norm ball in 2D, i.e. the region where $|x|^p + |y|^p \leq 1$. Compute the area of the ball for $p = 0.5$ by using importance sampling with a properly chosen proposal distribution. Report the variance of your estimate for $N = 1000$ samples.

2. **(Markov Chains)**



   Place a 6-sided die (possibly loaded according to a distribution π_0) on a table with the top face having the value 1, i.e. $X_0 = 1$. Successive values X_t for $t = 1, 2, \ldots$ are obtained as follows:

   - If X_{t-1} is on the top face, pick one of the four side faces uniformly at random. We will call this face f_t. For example, if $X_{t-1} = 1$, you would choose f_t from $\{2, 3, 4, 5\}$, with each element having probability $1/4$.
   - Rotate the die f_t times in the direction of f_t and set X_t to the number on the top. For example if $X_{t-1} = 1$ and $f_t = 2$, then $X_t = 6$.

   This procedure defines a *Markov Chain*, $\{X_t\}_{t=0,1,2,\ldots}$

   (a) By carefully examining the standard die layout given above, construct the 6×6 transition matrix A for the chain X_t.

(b) Find the stationary distribution of A (if any) by computing eigenvalues and eigenvectors. Is the stationary distribution the uniform distribution?

(c) Does this process satisfy the *detailed balance condition*?

(d) The chain is said to be approximately reached the stationary distribution when the total variation distance between π and $A^t\pi_0$ (see lecture notes for the definition) is below a small threshold ϵ. Take $\epsilon = 10^{-8}$. We will call the first time the total variation drops below ϵ the mixing time T_{mix}. Find T_{mix}.

(e) Visualize some intermediary powers of A to visually confirm the convergence of the distribution as demonstrated in the lecture (like Fig.7 of the lecture notes). You can use imshow() from matplotlib. Verify the mixing time by plotting a figure (like Fig.8 of the lecture notes).

(f) Run 1000 independent chains of length at least T_{mix}, starting from $X_0 = 1$ and record the whole trajectory of each chain. Compute the histograms of X_t for each $t \leq T_{\text{mix}}$ and generate a figure like Fig. 13 bottom left.

(g) Run a single chain. After discarding the first T_{mix} samples, record 1000 consecutive values for X_t and plot a histogram. Confirm that although these samples are dependent, their histogram agrees with the stationary distribution obtained numerically from the first eigenvector.