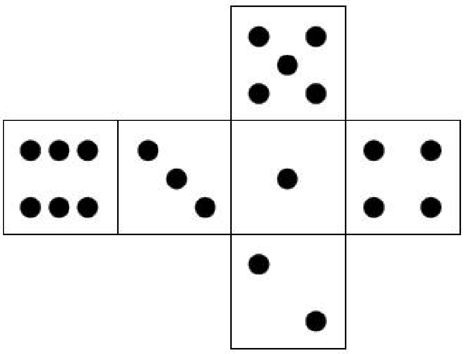
CMPE 548 Monte Carlo Methods
Homework 2

Due on March 16, 2016

1. **(Importance Sampling)**

   Consider the unit p-norm ball in 2D, i.e. the region where $|x|^p + |y|^p \leq 1$. Compute the area of the ball for $p = 0.5$ by using importance sampling with a properly chosen proposal distribution. Report the variance of your estimate for $N = 1000$ samples.

2. **(Markov Chains)**

   Place a 6-sided die (possibly loaded according to a distribution π_0) on a table with the top face having the value 1, i.e. $X_0 = 1$. Successive values X_t for $t = 1, 2, \ldots$ are obtained as follows:

   - If X_{t-1} is on the top face, pick one of the four side faces uniformly at random. We will call this face f_t. For example, if $X_{t-1} = 1$, you would choose f_t from $\{2, 3, 4, 5\}$, with each element having probability 1/4.
   - Rotate the die f_t times in the direction of f_t and set X_t to the number on the top. For example if $X_{t-1} = 1$ and $f_t = 2$, then $X_t = 6$.

   This procedure defines a *Markov Chain*, $\{X_t\}_{t=0,1,2,\ldots}$

   (a) By carefully examining the standard die layout given above, construct the 6×6 transition matrix A for the chain X_t.

(b) Find the stationary distribution of A (if any) by computing eigenvalues and eigenvectors. Is the stationary distribution the uniform distribution?

(c) Does this process satisfy the *detailed balance condition*?

(d) The chain is said to be approximately reached the stationary distribution when the total variation distance between π and $A^t\pi_0$ (see lecture notes for the definition) is below a small threshold ϵ. Take $\epsilon = 10^{-8}$. We will call the first time the total variation drops below ϵ the mixing time T_{mix}. Find T_{mix}.

(e) Visualize some intermediary powers of A to visually confirm the convergence of the distribution as demonstrated in the lecture (like Fig.7 of the lecture notes). You can use imshow() from matplotlib. Verify the mixing time by plotting a figure (like Fig.8 of the lecture notes).

(f) Draw independent samples from the stationary distribution by simulating the chain for as long as it is required. In order to draw truly independent samples, you need to stop the simulation after you reach the stationary distribution at time T_{mix} and record $X_{t=T_{\text{mix}}}$ as a sample. Obtain 1000 independent samples by recording the whole trajectory of each chain and plot the histograms of X_t for each $t \leq T_{\text{mix}}$ and generate a figure like Fig. 13 Bottom left.

(g) Run a single chain. After discarding the first T_{mix} samples, record 1000 consecutive values for X_t and plot a histogram. Confirm that although these samples are dependent, their histogram agrees with the stationary distribution obtained numerically from the first eigenvector.