# Computability, Decidability, and Complexity

**Lecture Notes #2: Decidability**

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# Decidability

1. Introduction to decidability

2. Undecidability

3. Decidability

# 1. Introduction to decidability

When a formalism is developed, the following questions are crucial:

- expressive power – What can I say?
- decidable questions – What can I prove?
- complexity questions – How hard is to prove it?
- axiomatics – How should I prove it?

# 1. Introduction to decidability

What is an algorithmic problem? An algorithmic problem is a function $f : \mathcal{I} \to \mathcal{F}$, where $\mathcal{I}$ and $\mathcal{F}$ are two sets at most countable.

As we will only consider algorithmic problems, we will simply call them problems.

$\mathcal{I}$ is called the set of initial data or instances of $f$, and $\mathcal{F}$ is the set of final data.

When $|\mathcal{F}| = 2$ ($\mathcal{F} = \{0, 1\}$ or $\mathcal{F} = \{\top, \bot\}$ or $\mathcal{F} = \{yes, no\}$ etc.), $f$ is called a decision problem; otherwise, it is called a computational problem.

**Example 1**

- $f : \mathbf{N}^2 \rightarrow \mathbf{N}$ given by $f(x,y) = x + y$, is a computational problem (the "addition problem"). Each pair $(x,y) \in \mathbf{N}^2$ is an instance of this problem;
- $f : \mathbf{N} \rightarrow \{0,1\}$ given by $f(x) = 1$ if and only if $x$ is a prime, is a decision problem. Each $x \in \mathbf{N}$ is an instance of this problem.

Let $f : \mathcal{I} \rightarrow \mathcal{F}$ be a problem. As $\mathcal{I}$ and $\mathcal{F}$ are at most countable, they can be encoded as words over a given alphabet $\Sigma$. Therefore, we may assume that $\mathcal{I}, \mathcal{F} \subseteq \Sigma^*$.

**Example 2** Examples of encodings:

- $A \subseteq \mathbf{N} \quad \rightsquigarrow \quad \{a^x | x \in A\}$, over $\Sigma = \{a\}$;
- $A \subseteq \mathbf{N}^2 \quad \rightsquigarrow \quad \{a^x \# b^y | (x, y) \in A\}$, over $\Sigma = \{a, b, \#\}$;

# 1. Introduction to decidability

Turing machines are a good model for the study of algorithms, since we can conceive of

- computations with arbitrarily large inputs on their tapes, using an
- arbitrarily large amount of intermediate storage during a computation, and taking an
- arbitrarily large amount of time.

Moreover, Turing machines are universal, in the sense that every known algorithm can be executed by some Turing machine.

Consider the following algorithm:

<u>`Algorithm` $\mathcal{A}$</u>
`input`: $x \in \mathbf{N}$;
`output`: "yes", if $x < 5$, and "no", if $x = 5$;
`begin`
  $i := x$;
  `while` $i > 5$ `do` $i := i + 1$;
  `if` $i < 5$ `then` "yes" `else if` $i = 5$ `then` "'no";
`end`.

- $accept(\mathcal{A}) = \{0, 1, 2, 3, 4\}$
- $reject(\mathcal{A}) = \{5\}$
- $loop(\mathcal{A}) = \{x \in \mathbf{N} | x > 5\}$

# 1. Introduction to decidability

Let $f : \mathcal{I} \rightarrow \{0, 1\}$ be a decision problem, where $\mathcal{I} \subseteq \Sigma^*$. The language associated to $f$ is the set $L_f = \{w \in \mathcal{I} | f(w) = 1\}$.

$f$ is called decidable if its language is recursive.

> $f$ decidable $\Leftrightarrow$ there exists an algorithm (Turing machine) that decides $f$ ($L_f$)

$f$ is called semi-decidable if its language is recursively enumerable.

> $f$ semi-decidable $\Leftrightarrow$ there exists an algorithm (Turing machine) that semi-decides $f$ ($L_f$)

$f$ is called undecidable if it is not decidable.

A decision problem $f : \mathcal{I} \rightarrow \{0,1\}$ is reducible to a decision problem $g : \mathcal{I}' \rightarrow \{0,1\}$, abbreviated $f \prec g$, if there exists an algorithm (Turing machine) $M$ such that:

- $(\forall x \in \mathcal{I})(M(x) \in \mathcal{I}')$;
- $(\forall x \in \mathcal{I})(f(x) = 1 \;\Leftrightarrow\; g(M(x)) = 1)$.

**Proposition 1** Let $f$ and $g$ be decision problems.

- If $f \prec g$ and $g$ is decidable, then $f$ is decidable.
- If $f \prec g$ and $f$ is undecidable, then $g$ is undecidable.

# 2. Undecidability

2.1. The halting problem

2.2. Rice's theorem revised

2.3. Post correspondence problem

2.4. Domino problems

2.5. Hilbert's 10th problem and consequences

2.6. The word problem for finitely presented monoids

2.7. Valid and invalid computations

2.8. Greibach's theorem and applications

# 2.1. The Halting Problem

2.1.1. The halting problem and its undecidability

2.1.2. Stack machines. Counter machines

2.1.3. Applications to Petri nets

2.1.4. Applications to security protocols

# 2.1.1. The Halting Problem and its Undecidability

The halting problem for a given algorithmic formalism is the problem of whether or not a given procedure of the formalism when executed with a given input eventually terminates.

The Halting Problem

Instance: algorithm $\mathcal{A}$ (Turing machine $M$) and input $x$;

Question: does $\mathcal{A}$ $(M)$ halt on $x$?

**Theorem 1** The halting problem for Turing machines is undecidable.

**Proof** Assume that there exists an algorithm $\mathcal{A}$ that decides the halting problem. Denote by $\langle \mathcal{B} \rangle$ an arbitrary but fixed encoding of an algorithm $\mathcal{B}$. Let $\mathcal{D}$ be the following algorithm:

```
Algorithm D
  input:   algorithm B;
  output:  0 if B(⟨B⟩)↑;
  begin
    y := A(B, ⟨B⟩);
    if y = 0 then 0 else loop forever
  end.
```

It is easy to see that $\mathcal{D}(\langle \mathcal{D} \rangle)\uparrow \;\Leftrightarrow\; \mathcal{D}(\langle \mathcal{D} \rangle)\downarrow$, which is a contradiction. $\square$

There are several variants on the halting problem.

The Empty-input Halting Problem

Instance: algorithm $\mathcal{A}$ (Turing machine $M$);

Question: does $\mathcal{A}$ ($M$) halt on the empty-input?

**Corollary 1** The empty-input halting problem for Turing machines is undecidable.

**Proof** We exhibit a reduction from the halting problem:

$$(\mathcal{A}, x) \quad \rightsquigarrow \quad \mathcal{A}'$$

where $\mathcal{A}'$, on the empty-input, generates $x$ and then simulates $\mathcal{A}$ on $x$.

Given an algorithm $\mathcal{A}$ and an input $x$ for it, define the algorithm $\mathcal{A}'$ as follows:

<u>**Algorithm** $\mathcal{A}'$</u>

**input**: none;
**output**: $z = \mathcal{A}(x)$;
**begin**
  $z := \mathcal{A}(x)$;
**end**.

Clearly, $\mathcal{A}(x)\downarrow$ iff $\mathcal{A}'\downarrow$. $\square$

The Uniform Halting Problem

Instance: algorithm $\mathcal{A}$ (Turing machine $M$);

Question: does $\mathcal{A}$ ($M$) halt on all inputs?

**Corollary 2** The uniform halting problem for Turing machines is undecidable.

**Proof** We exhibit a reduction from the halting problem:

$$(\mathcal{A}, x) \quad \rightsquigarrow \quad \mathcal{A}'$$

where $\mathcal{A}'$, on an arbitrary input $y$, erases $y$, generates $x$, and then simulates $\mathcal{A}$ on $x$.

# 2.1.1. The Halting Problem and its Undecidability

Given an algorithm $\mathcal{A}$ and an input $x$ for it, define the algorithm $\mathcal{A}'$ as follows:

<u>`Algorithm` $\mathcal{A}'$</u>

  `input`: $\quad y$;

  `output`: $\quad z = \mathcal{A}(x)$;

  `begin`

    $z := \mathcal{A}(x)$;

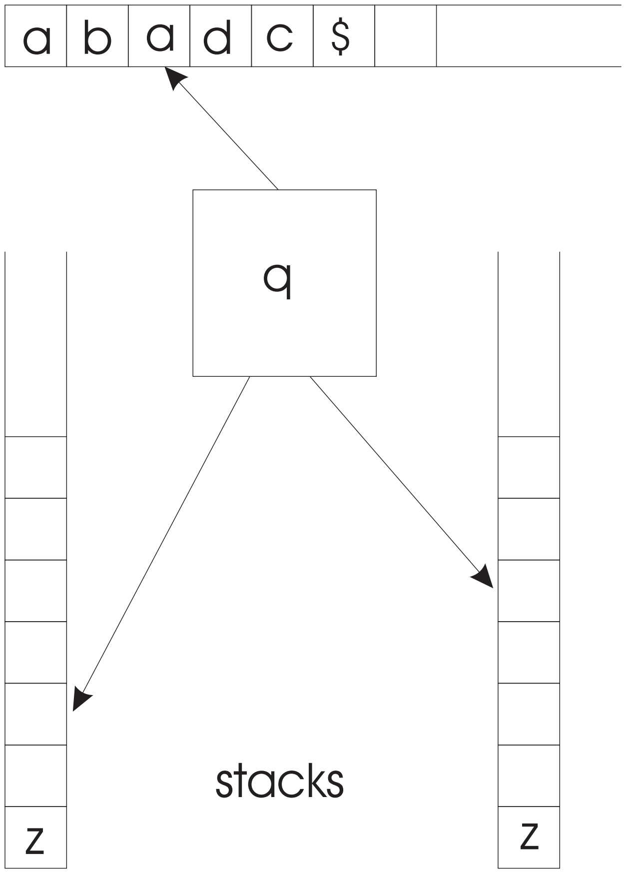
  `end`.

Clearly, $\mathcal{A}(x)\downarrow$ iff $(\forall y)(\mathcal{A}'(y)\downarrow$. $\square$

A $k$-stack machine, abbreviated k-SM, is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, Z, F)$, where:

- $Q$ is a non-empty finite set of states
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the final set of states
- $\Sigma$ is the input alphabet
- $\Gamma$ is the stack alphabet
- $Z \in \Gamma - \Sigma$ is the bottom-of-stack marker
- $\delta : Q \times (\Sigma \cup \{\lambda\}) \times (\Gamma)^k \rightsquigarrow Q \times (\Gamma^*)^k$ is the transition function satisfying the property that the bottom-of-stack marker $Z$ "cannot be erased" and it "cannot appear elsewhere on the stacks".

# 2.1.2. Multistack Machines. Counter Machines




input tape - read-only

$ - endmarker

Z - bottom-of-stack marker

Stack machine

Computation relation:

$$(q, u|av, Zu_1X_1, \ldots, Zu_kX_k) \vdash (q', ua|v, \gamma_1, \ldots, \gamma_k)$$

iff

$$\delta(q, a, X_1, \ldots, X_k) = (q', \gamma_1, \ldots, \gamma_k)$$

where $u, v \in \Sigma^*$, $a \in \Sigma \cup \{\lambda\}$, $u_1X_1, \ldots, u_kX_k \in \Gamma^*$.

**Theorem 2** A language is accepted by a Turing machine iff it is accepted by a 2-stack machine.

**Corollary 3** The halting problem for 2-stack machines is undecidable.

# 2.1.2. Multistack Machines. Counter Machines

A k-counter machine, abbreviated k-CM, is a k-SM $M = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$ such that $|\Gamma| = 2$.

**Theorem 3** A language is accepted by a Turing machine iff it is accepted by a 2-CM.

**Corollary 4** The halting problem for 2-counter machines is undecidable.

A "simplified' version of counter machines:

$$M = (Q, q_0, q_f, C, x_0, I),$$

where:

- $Q$ is a non-empty finite set of states;
- $q_0 \in Q$ is the initial state, and $q_f \in Q$ is the final state;
- $C$ is a finite set of counters, each of which being able to hold a natural number;
- $x_0 : C \to \mathbf{N}$ is the initial content of counters;
- $I$ is a finite set of instructions. For each state there is at most an instruction that can be executed at that state; for $q_f$ there is no instruction. Each instruction is of the one of the following forms:

– increment instruction $I(q, c, q')$

```
q :  begin
       c := c + 1;
       go to q'
     end
```

– test instruction $I(q, c, q', q'')$

```
q :  begin
       if c = 0 then go to q'
        else begin
                   c := c - 1;
                   go to q''
                 end
     end
```

A configuration is a pair $(q, x)$, where $q \in Q$ and $x : C \rightarrow N$. A configuration $(q, x)$ is called initial if $q = q_0$ and $x = x_0$. A configuration $(q, x)$ is called final if $q = q_f$.

Computation:

$$(q, x) \vdash (q', x')$$

iff one of the following holds:

- there exists $I(q, c, q')$ such that $x'(c) = x(c)+1$ and $x'(c') = x(c')$, for all $c' \in C - \{c\}$;
- there exists $I(q, c, q_1, q_2)$ such that
  - if $x(c) = 0$, then $q' = q_1$ and $x' = x$;
  - if $x(c) \neq 0$, then $q' = q_2$, $x'(c) = x(c) - 1$, and $x'(c') = x(c)$, for all $c' \in C - \{c\}$.

# 2.1.3. Applications to Petri Nets

Petri nets, abbreviated PN, have been introduced by Carl Adam Petri in 1962 as models of distributed systems, where concurrency and communication play an important role.

A PN is a system $\Sigma = (S, T, F, W)$, where:

- $S$ is a finite non-empty set of places;
- $T$ is a finite non-empty set of transitions;
- $S \cap T = \emptyset$;
- $F \subseteq S \times T \cup T \times S$ is the flow relation;
- $W : S \times T \cup T \times S \rightarrow \mathbf{N}$ is the weight function satisfying $W(x, y) = 0$ iff $(x, y) \notin F$.

# 2.1.3. Applications to Petri Nets

Configurations in Petri net theory are called markings, and they are defined as functions $M : S \rightarrow \mathbf{N}$.

Because $S$ is a finite set, markings are usually represented as $S$-dimensional vectors.

Computation (firing) rule:

- A transition $t$ is enabled at $M$, denoted $M[t\rangle$, if

$$W(s,t) \geq M(s),$$

  for all $s \in S$;

- If $t$ is enabled at $M$ then $t$ may fire yielding a new marking $M'$ given by

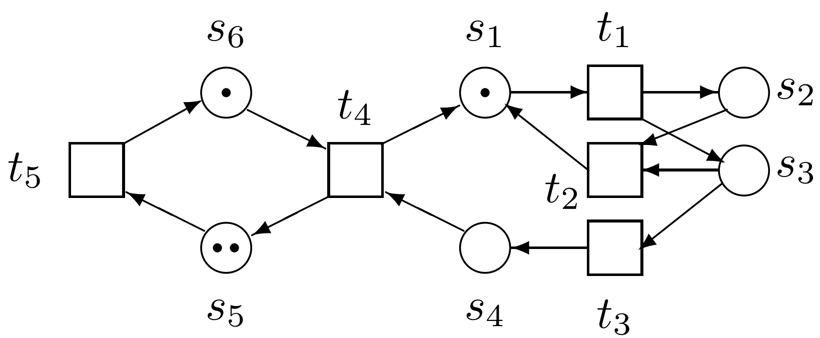
$$M'(s) = M(s) - W(s,t) + W(t,s),$$

  for all $s \in S$. We denote this by $M[t\rangle M'$.

# 2.1.3. Applications to Petri Nets

Graphical representation:

**Example 3** Vending machine:




| | |
|---|---|
| $s_1 = \mathit{ready}$ | $t_1 = \mathit{insert}$ |
| $s_2 = \mathit{counter}$ | $t_2 = \mathit{reject}$ |
| $s_3 = \mathit{inserted}$ | $t_3 = \mathit{accept}$ |
| $s_4 = \mathit{accepted}$ | $t_4 = \mathit{dispense}$ |
| $s_5 = \mathit{warm}$ | $t_5 = \mathit{brew}$ |
| $s_6 = \mathit{cold}$ | |

$$(1,0,0,0,2,1)[t_1\rangle(0,1,1,0,2,1)[t_3\rangle(0,1,0,1,2,1)[t_4\rangle(1,1,0,0,3,0)$$

# 2.1.3. Applications to Petri Nets

A pair $\gamma = (\Sigma, M_0)$, where $\Sigma$ is a Petri net and $M_0$ is a marking of $\Sigma$ is called a marked Petri net.

A marking $M$ is reachable in $\gamma$ if there exists a sequence of transitions $w \in T^*$ such that $M_0[w\rangle M$.

A marking $M$ is coverable in $\gamma$ if there exists a reachable marking $M'$ in $\gamma$ such that $M' \geq M$ (the inequality on vectors is componentwise understood).

$\gamma$ is bounded if there exists $n \in \mathbf{N}$ such that $M(s) \leq n$, for any $s \in S$ and reachable marking $M$.

A transition $t$ of $\gamma$ is live if for any reachable marking $M$ there exists $M'$ reachable from $M$ such that $M'[t\rangle$. If all transitions are live, the $\gamma$ is called live.

## 2.1.3. Applications to Petri Nets

Basic decision problems in Petri net theory: reachability, coverability, boundedness, and liveness.

All these problems are decidable for Petri nets (details will be provided in a separate section). However, they are undecidable for almost all Petri net extensions. For instance, we will prove that they are undecidable for inhibitor Petri nets (see `InhibitorPetriNets.pdf`).

# 2.1.4. Applications to Security Protocols

see `SecurityProtocols.pdf`

# 2.3. Post's Correspondence Problem

# 2.3.1. Post's Correspondence Problem

- Emil Post. *A Variant of a Recursively Unsolvable Problem*, Bulletin of the AMS 52, 1946, 264–268 (see `Post1946.pdf`).

<u>Post's Correspondence Problem (PCP)</u>

| | |
|---|---|
| Instance: | list of pairs of words $L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$ |
| Question: | Is there any list of numbers $i_1, \ldots, i_k$ s.t. $u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$? |

Any list of numbers $i_1, \ldots, i_k$ such that

$$u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$$

is called a solution of $L$.

**Example 4** The list 1,2,1,3 is a solution to the PCP instance

$$L = \{(a^2, a^2b), (b^2, ba), (ab^2, b)\}$$

**Example 5** The PCP instance

$$L = \{(a^2b, a^2), (a, ba^2)\}$$

has no solution.

**Example 6** The list 1,3,2,3 is a solution to the PCP instance

$$L = \{(1, 101), (10, 00), (011, 11)\}$$

**Proposition 2** The PCP instance

$$L = \{(a^{k_1}, a^{l_1}), \ldots, (a^{k_n}, a^{l_n})$$

has solutions if and only if

1. there exists $i$ such that $k_i = l_i$, or
2. there exist $i$ and $j$ such that $k_i > l_i$ and $k_j < l_j$.

**Corollary 5** PCP over one-letter alphabets is decidable.

# 2.3.1. Post's Correspondence Problem

**Proposition 3** Any PCP instance over an alphabet $\Sigma$ with $|\Sigma| \geq 2$ is equivalent to a PCP instance over an alphabet $\Delta$ with $|\Delta| = 2$.

**Proof** Assume $\Sigma = \{a_1, \ldots, a_n\}$ and $n > 2$. Let $\Delta = \{a, b\}$, where $a \neq b$.

Encode $a_i$ by $ba^ib$, for any $i$. □

Define:

- $PCP_1$ – PCP instances over one-letter alphabets
- $PCP_2$ – PCP instances over two-letter alphabets

**Theorem 4** PCP is undecidable.

**Proof** Show that the halting problem for Post machines, which are equivalent to Turing machines, can be reduced to PCP. □

**Corollary 6** $PCP_2$ is undecidable.

Summary:

- $PCP_1$ is decidable
- $PCP_2$ is undecidable

# 2.3.1. Post's Correspondence Problem

Other variations:

- $PCP(n)$ – PCP instances of length $n$

$$(L = \{(u_1, v_1), \ldots, (u_n, v_n)\})$$

- $PCP_1(n)$
- $PCP_2(n)$

$PCP_1(n)$ is decidable, for all $n$.

# 2.3.1. Post's Correspondence Problem

**Theorem 5** (Ehrenfeucht, Karhumaki, Rozenberg, 1982)
PCP(2) is decidable.

For a simpler proof than the original one see HaHH2000.pdf.

**Theorem 6** Matiyasevich, Senizergues, 1996)
PCP(7) is undecidable.

**Proof** See MaSe1996.pdf. □

Open problems: PCP(3),...,PCP(6)

# 2.3.2. Applications of PCP to First-order Logic

Validity problem for first-order logic (VPFOL)

Instance: First-order formula $\phi$

Question: Is $\phi$ valid?

Satisfiability problem for first-order logic (SPFOL)

Instance: First-order formula $\phi$

Question: Is $\phi$ satisfiable?

These two decision problems are equivalent because

$$\phi \text{ is valid} \quad \Leftrightarrow \quad \neg\phi \text{ is not satisfiable}$$

**Theorem 7** VPFOL is undecidable.

**Proof** Reduce $PCP_2$ to SPFOL. Given a $PCP_2$ instance

$$L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$$

over $\Sigma = \{0, 1\}$, define a formula $\phi$ such that

$$L \text{ has solutions} \quad \Leftrightarrow \quad \phi \text{ is satisfiable}$$

$\phi$ is defined as follows:

- let $a$ be a constant. It will be interpreted by $\lambda$ in some interpretation $\mathcal{I}$;
- let $f_0$ and $f_1$ be function symbols. They will be interpreted by $\mathcal{I}(f_0)(x) = x0$ and $\mathcal{I}(f_1)(x) = x1$. We will simply write $f_{b_1 \cdots b_k}(x)$ instead of $f_{b_k}(\cdots f_{b_1}(x) \cdots)$;

# 2.3.2. Applications of PCP to First-order Logic

- let $P$ be a predicate symbol. It will be interpreted by

$$\mathcal{I}(P)(x,y) \;\Leftrightarrow\; x,y \in \Sigma^* \;\wedge\; x = u_{i_1}\cdots u_{i_k} \;\wedge\; y = v_{i_1}\cdots v_{i_k},$$

$$\text{for some } i_1,\ldots,i_k$$

- let $\phi_1$ be the formula $\phi_1 = \wedge_{i=1}^{n} P(f_{u_i}(a), f_{v_i}(a))$
- let $\phi_2$ be the formula $\phi_2 = (\forall u,v)(P(u,v) \Rightarrow \wedge_{i=1}^{n} P(f_{u_i}(u), f_{v_i}(v))$
- let $\phi_3$ be the formula $\phi_3 = (\exists x)(P(x,x))$
- let $\phi$ be the formula $\phi = (\phi_1 \;\wedge\; \phi_2 \Rightarrow \phi_3)$

Then,

$$L \text{ has solutions} \;\Leftrightarrow\; \phi \text{ is satisfiable}$$

which concludes the proof. $\square$

Intersection problem for CFL (IPCFL)

Instance: context-free grammars $G_1$ and $G_2$

Question: Is $L(G_1) \cap L(G_2) \neq \emptyset$?

**Theorem 8** IPCFL is undecidable.

**Proof** Reduce PCP to IPCFL. Given a PCP instance

$$L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$$

over $\Sigma$, define two CF-grammars $G_1$ and $G_2$ such that

$$L \text{ has solutions } \Leftrightarrow L(G_1) \cap L(G_2) \neq \emptyset$$

The grammars are:

- $G_1$ : $S \rightarrow iSu_i | iu_i$, for all $i$
- $G_2$ : $S \rightarrow iSv_i | iv_i$, for all $i$

□

Equivalence problem for CFG (EPCFG)

Instance: context-free grammars $G_1$ and $G_2$

Question: Is $L(G_1) = L(G_2)$?

**Theorem 9** EPCFG is undecidable.

**Proof** Reduce ¬PCP to EPCFG. Given a PCP instance

$$L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$$

over $\Sigma$, define two CF-grammars $G_1$ and $G_2$ such that

$$L \text{ has no solution} \Leftrightarrow L(G_1) = L(G_2)$$

Define two grammars $G_1$ and $G_2$ such that

- $G_1$ generates $L(G_1) = \{1, \ldots, n\}^* \# \Sigma^*$, where # is a new symbol;

- $G_2$ generates $L(G_2) = (L(G_1) - A) \cup (L(G_1) - B)$, where
  - $A = \{i_1 \cdots i_k \# u_{i_k} \cdots u_{i_1} | i_1, \ldots, i_k \in \{1, \ldots, n\}\}$
  - $B = \{i_1 \cdots i_k \# v_{i_k} \cdots v_{i_1} | i_1, \ldots, i_k \in \{1, \ldots, n\}\}$

It is easy to see that two context-free grammars $G_1$ and $G_2$ as above exist, and $L$ has no solution iff $L(G_1) = L(G_2)$. $\square$

<u>Ambiguity problem for CFG (APCFG)</u>

Instance: context-free grammar $G$

Question: Is $G$ ambiguous?

**Theorem 10** APCFG is undecidable.

**Proof** Reduce PCP to APCFG. Given a PCP instance

$$L = \{(u_1, v_1), \ldots, (u_n, v_n)\}$$

over $\Sigma$, define a CFG $G$ such that

$$L \text{ has solutions } \Leftrightarrow G \text{ is ambiguous}$$

Define $G$ by

- $S \rightarrow S_1 | S_2, \quad S_1 \rightarrow u_i S_1 i | u_i i, \quad S_2 \rightarrow v_i S_2 i | v_i i,$

for all $i$. □

# 2.6. The word problem for finitely presented monoids

A semi-group $(S, \cdot)$ is called finitely presented if there exists a finite set $A$ of generators for $S$ and a finite set $E$ of equations over $A$ (i.e., pairs of words over $A$) such that any valid equation in $S$ can be obtained by derivation from $E$. That is, if $t = t'$ is valid in $S$, then $t \stackrel{*}{\Rightarrow}_E t'$.

**Example 7** Let $S$ be a semi-group generated by $A = \{a_1, a_2, a_3\}$ under the equations

- $a_2 a_1 = a_1 a_2$
- $a_3 a_2 = a_2 a_2 a_3$
- $a_3 a_1 = a_1$.

Then, $a_1 a_2 a_2 = a_1 a_2$ is valid in $S$.

Word Problem for Semi-groups (WPSG)

| | |
|---|---|
| Instance: | finite semi-group presentation $(A, E)$ and equation $t = t'$ |
| Question: | Does $t = t'$ hold true in the semi-group presented by $(A, E)$? |

This problem was shown to be undecidable in:

- Emil Post. *Recursive Unsolvability of a Problem of Thue*, Journal of Symbolic Logic 12, 1947, 1–11.

The problem can be reduced to the reachability problem for Thue systems.

- Axel Thue. *Probleme über Veranderungen von Zeichenreihen nach gegeben regeln*, Skr. Vid. Kristiania, I Mat. Naturv. Klasse 10, 1914.

A Thue system over an alphabet $\Sigma$ is any set of unordered pairs of words over $\Sigma$. Each pair $\{t, t'\}$ is usually written as $t = t'$.

A semi-Thue system over an alphabet $\Sigma$ is any set of ordered pairs of words over $\Sigma$. Each pair $(t, t')$ is usually written as $t \rightarrow t'$.

Reachability Problem for Semi-Thue Systems (RPSTS)

| | |
|---|---|
| Instance: | semi-Thue system $R$ and words $t$ and $t'$ |
| Question: | Does $t \stackrel{*}{\Rightarrow}_R t'$? |

## 2.6. The word problem for finitely presented monoids

**Theorem 11** The reachability problem for (semi-)Thue systems is undecidable.

**Proof** Reduce the halting problem for Turing machines to this problem. □

**Corollary 7** The word problem for finitely presented semigroups (monoids) is undecidable.

## 2.6. The word problem for finitely presented monoids

Term rewriting systems and related problems:

`TermRewritingSystems.pdf`

Techniques for proving decidability:

- reducibility: if a problem $A$ is reducible to a problem $B$ and $B$ is decidable, then $A$ is decidable;
- ad hoc techniques.

## Coverability tree based techniques

General remarks:

- a coverability tree reduces the analysis of an infinite state space to the analysis of a finite state space;
- cut off infinite branches and add extra information to the leaf nodes;
- some properties of the original state space (reachability tree) may be lost.

We illustrate the technique on vector addition systems.

A vector addition system (VAS) is a couple $\mathcal{W} = (W, v_0)$, where:

- $W$ is a finite set of $n$-dimensional vectors with integer components ($W = \{v_1, \ldots, v_k\} \subseteq \mathbf{Z}^n$);
- $v_0$ is an $n$-dimensional vector with positive integer components ($v_0 \in \mathbf{N}^n$).

**Example 8** $\mathcal{W} = (W, v_0)$, where

$$W = \{(-1,1,0,1), (0,-1,0,0), (1,0,0,-1), (0,0,-1,1)\}$$

and $v_0 = (1,0,0,1)$, is a vector addition system.

# 3. Decidability

Let $\mathcal{W} = (W, v_0)$ be a VAS, $x \in \mathbf{Z}^n$, and $v \in W$.

- $v$ is enabled at $x$, denoted $x[v\rangle$ or $x \xrightarrow{v}$, if $x + v \geq 0$. $W(x)$ stands for the set $\{v \in W | x[v\rangle\}$;
- if $v$ is enabled at $x$ then $v$ may be applied yielding a new vector $x'$ given by $x' = x + v$. We denote this by $x[v\rangle x'$ or $x \xrightarrow{v} x'$;
- $\Rightarrow = \bigcup_{v \in W} \xrightarrow{v}$;
- $\overset{+}{\Rightarrow}$ is the reflexive and transitive closure of $\Rightarrow$;
- $x$ is reachable in $\mathcal{W}$ if $d \overset{+}{\Rightarrow} x$;
- $[v_0\rangle$ is the set of all reachable vectors in $\mathcal{W}$, called the reachability set of $\mathcal{W}$;
- $x$ is coverable in $\mathcal{W}$ if $v_0 \overset{+}{\Rightarrow} x'$ and $x' \geq x$, for some $x'$;
- $v$ is dead in $\mathcal{W}$ if $\neg(x[v\rangle)$, for any $x$ reachable in $\mathcal{W}$.

Let $\mathcal{W} = (W, v_0)$ be a VAS. A labeled tree $\mathcal{R} = (V, E, l_1, l_2)$ is a reachability tree of $\mathcal{W}$ if:

1. its root $x_0$ is labeled by $v_0$, i.e., $l_1(x_0) = v_0$;
2. $\forall x \in V$, $|x^+| = |W(l_1(x))|$;
3. $\forall x \in V$ with $|x^+| > 0$ and $\forall v \in W(l_1(x))$ there exists $x' \in x^+$ such that:
   (a) $l_1(x') = l_1(x) + v$;
   (b) $l_2(x, x') = v$.

Any two reachability trees of $\mathcal{W}$ are isomorphic. Therefore, we may talk about the reachability tree of $\mathcal{W}$, denoted $\mathcal{R}(\mathcal{W})$.

**Proposition 4** Let $\mathcal{W} = (W, v_0)$ be a VAS. Then,

1. $\mathcal{R}(\mathcal{W})$ is finitely branched;
2. $x$ in $\mathcal{R}(\mathcal{W})$ is a leaf node iff no vector in $W$ is enabled at $l_1(x)$;
3. $[v_0\rangle = \{l_1(x)|x \in V\}$.

$\mathcal{R}(\mathcal{W})$ may be infinite even if $[v_0\rangle$ is finite !

We will derive a finite structure from $\mathcal{R}(\mathcal{W})$.

Let $\omega \notin \mathbf{Z}$ and $\mathbf{Z}_\omega = \mathbf{Z} \cup \{\omega\}$. Extend $+$ and $<$ to $\mathbf{Z}_\omega$ by:

- $n + \omega = \omega + n = \omega$, for any $n \in \mathbf{Z}$;
- $n < \omega$, for any $n \in \mathbf{Z}$.

The notation $x[v\rangle$ etc. is usually extended to vectors over $\mathbf{Z}_\omega$.

Let $\mathcal{W} = (W, v_0)$ be a VAS. A labelled tree $\mathcal{T} = (V, E, l_1, l_2)$ is a coverability tree of $\mathcal{W}$ if:

1. its root $x_0$ is labelled by $v_0$, i.e., $l_1(x_0) = v_0$;
2. $\forall x \in V$,

$$|x^+| = \begin{cases} 0, & W(l_1(x)) = \emptyset \textit{ or} \\ & (\exists x' \in d_{\mathcal{T}}(x_0, x))(x \neq x' \ \wedge \ l_1(x) = l_1(x')) \\ |W(l_1(x))|, & \text{otherwise} \end{cases}$$

3. $\forall x \in V$ with $|x^+| > 0$ and $\forall v \in W(l_1(x))$ there exists $x' \in x^+$ such that:
   (a) $l_1(x')(i) = \omega$ if $(\exists x'' \in d_{\mathcal{T}}(x_0, x))(l_1(x'') \leq l_1(x) + v \ \wedge \ l_1(x'')(i) < (l_1(x)+v)(i))$, and $l_1(x')(i) = (l_1(x)+v)(i)$, otherwise (for any $i$);
   (b) $l_2(x, x') = v$.

Any two coverability trees of $\mathcal{W}$ are isomorphic. Therefore, we may talk about the coverability tree of $\mathcal{W}$, denoted $\mathcal{T}(\mathcal{W})$.

**Proposition 5** Let $\mathcal{W} = (W, v_0)$ be a VAS and $\mathcal{T}(\mathcal{W}) = (V, E, l_1, l_2)$ its coverability tree. Then:

1. $\mathcal{T}(\mathcal{W})$ is finitely branched;
2. $x$ in $\mathcal{T}(\mathcal{W})$ is a leaf node iff $W(l_1(x)) = \emptyset$ or there exists $x' \in d_{\mathcal{T}}(x_0, x)$ such that $x \neq x'$ and $l_1(x) = l_1(x')$;
3. let $x_{i_0}, x_{i_1}, \ldots, x_{i_m}$ be pairwise distinct nodes such that $x_{i_j} \in d_{\mathcal{T}(\gamma)}(x_0, x_{i_{j+1}})$, for any $0 \leq j \leq m-1$.
   (a) if $l_1(x_{i_0}) = l_1(x_{i_1}) = \cdots = l_1(x_{i_m})$, then $m \leq 1$;
   (b) if $l_1(x_{i_0}) < l_1(x_{i_1}) < \cdots < l_1(x_{i_m})$, then $m \leq n$;
4. $\mathcal{T}(\gamma)$ is finite.

**Theorem 12** Let $\mathcal{W} = (W, v_0)$ be a VAS and $\mathcal{T}(\mathcal{W}) = (V, E, l_1, l_2)$ its coverability tree. Then, a vector $x$ is coverable in $\mathcal{W}$ iff it is coverable in $\mathcal{T}(\mathcal{W})$.

**Corollary 8** Coverability, deadness and finiteness problems are decidable for VASs.