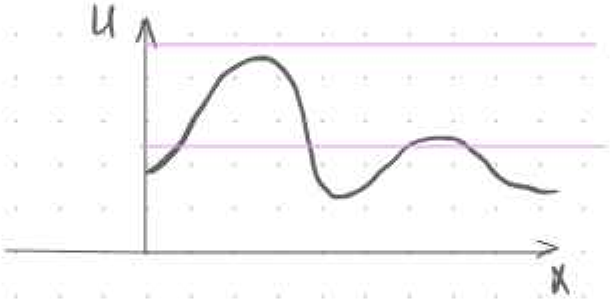
Дарья Васильевна:

книжки: Ландау – Лифшиц : т. 1.
задачник Коткин, Сербо, „Задачи по классической механике"
Коткин, Сербо, Черных – лекции

Одномерное движение в квадратурах



$$E = \frac{m\dot{x}^2}{2}$$

1 ст.св = 2 нач.усл.

$$\frac{m\dot{x}^2}{2} + U(x) = E$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E - U(x))} \Rightarrow \int_{t_0}^{t} dt = \pm\int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}(E - U(x))}}$$

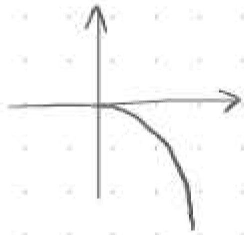
$$t(x) \to x(t)$$

$$E = U(x_0) + \frac{m\dot{x}_0^2}{2}$$

№1.2.

ЭКЗ

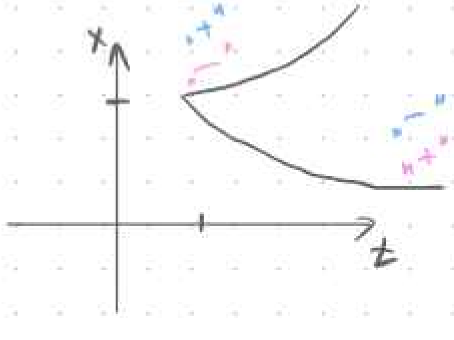
$x(t)$, если $U = -Ax^4$, $E = 0$, x_0 ; $\left.\frac{dx}{dt}\right|_{x_0}$ $\pm$ = ?



$$\int_{t_0}^{t} dt = \pm\int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}Ax^4}} = \pm\int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2A}{m}}\,x^2} = \mp\frac{1}{\sqrt{\frac{2A}{m}}}\frac{1}{x}$$

$$t - t_0^{\to 0} = \mp\frac{\sqrt{m}}{\sqrt{2A}}\left(\frac{1}{x} - \frac{1}{x_0}\right) \Rightarrow \pm\left(\frac{1}{x_0} - \frac{1}{x}\right) = \frac{\sqrt{2A}}{\sqrt{m}}(t - t_0)$$

$$x = \frac{1}{\frac{1}{x_0} \mp \sqrt{\frac{2A}{m}}\,t}$$

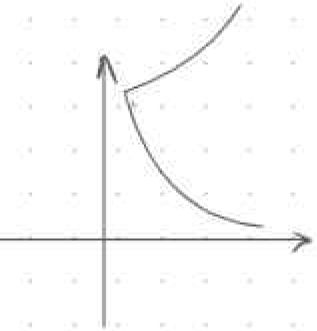


„+" – за конечное время на $\infty \Rightarrow t = \sqrt{\frac{m}{2A}}\frac{1}{x_0}$

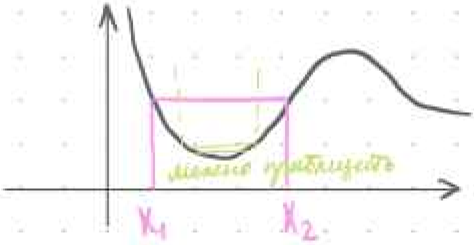
„–" – за бесконечное время к 0

$U=-Ax^2$: $\int\limits_{x_0}^{x}\frac{1}{x}dx = \frac{\sqrt{2A}}{\sqrt{m}}\int\limits_0^t dt \Rightarrow \ln x - \ln x_0 = \sqrt{\frac{2A}{m}}\,t$

$$x = x_0 + e^c, \quad c=\sqrt{\frac{2A}{m}}\,t$$



если:



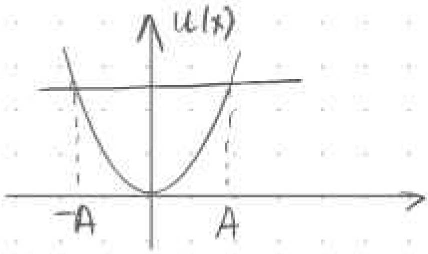
$E=U(x),\ x_1,\ x_2$

$$T(E)=2\int\limits_{x_1}^{x_2}\frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}}$$

$$U(x_0+\Delta x) = U(x_0) + \frac{d}{dx}U(x)\Big|_{x_0}\Delta x + \frac{1}{2}U''(x)\Big|_{x_0}\Delta x^2 + \ldots$$

<u>базовая задача:</u> Дано $T(E) \to U(x)$ −?

№2. найти вероятность $\frac{dw}{dx}$ нахождения осциллятора $U(x)=\frac{m\omega^2x^2}{2}$ в диапазоне x до $x+dx$



$dw \sim dt$

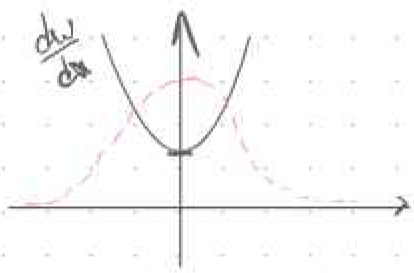
$dw = \frac{2dt}{T}$

$$\frac{m\omega^2A^2}{2} = \frac{m\omega^2x^2}{2} + \frac{m}{2}\left(\frac{dx}{dt}\right)^2 \Rightarrow \frac{dw}{dx} = \frac{2}{Tv} = \frac{2}{T\omega\sqrt{A^2-x^2}} = \frac{1}{\pi\sqrt{A^2-x^2}}$$

$$\int\frac{dx}{\sqrt{A^2-x^2}} = -\arccos\left(\frac{x}{a}\right)+C$$

$$\int dx\,\frac{dw}{dx} = \frac{1}{\pi}\arccos\frac{x}{A}\Big|_{-A}^{A} \Rightarrow =1$$

расх-ти нет: $\frac{1}{\sqrt{A^2-x^2}} = \frac{1}{\sqrt{(A-x)(A+x)}} = \frac{1}{\delta^{1/2}} \Rightarrow$ чаще маятник нах-ся на концах, т.к. он немного зависает на них

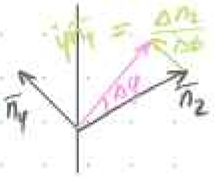


Центральное поле (зависит только от r)

$$\bar F \parallel \bar r \Rightarrow \bar M = 0 \Rightarrow \frac{dp}{dt} = \bar F, \quad M_F = \frac{dM}{dt} = const$$

$$\bar M = m[\bar r \times \bar v] \Rightarrow \bar v \perp \bar M, \quad \bar r \perp \bar M$$

$$\frac{m\bar v^2}{2} = \frac{m\dot r^2}{2} + \frac{m(r\dot\varphi)^2}{2} \qquad \bar r = r(t)\bar n_r, \quad \bar V = \dot{\bar r} = \dot r(t)\bar n_r + r\dot{\bar n}$$



$$E = \frac{m\dot r^2}{2} + \frac{mr^2\dot\varphi^2}{2} + U(r) = \frac{m\dot r^2}{2} + \underbrace{\frac{M}{2mr^2} + U(r)}_{U_{эфф}}$$

$$\bar M = m[\bar r\times\bar v] = \bar e_z m r^2\dot\varphi; \quad M = mr^2\dot\varphi$$

$$\int_{t_0}^{t} dt = \int_{r_0} \frac{dr}{\sqrt{\frac{2}{m}\left(E - \frac{M}{2mr^2} - U(r)\right)}}$$

начальные условия

$$\frac{d\varphi}{dt} = \frac{M}{mr^2(t)} \Rightarrow \varphi-\varphi_0 = \int \frac{M\,dt}{mr^2(t)}$$

$$d\varphi = \frac{\frac{M}{r^2}dr}{\sqrt{2m\left[E-\frac{M^2}{2mr^2} - U(r)\right]}}$$ – траектория

задача Кеплера: $U = -\frac{\alpha}{r}$ $\quad \left(\frac{dr}{r^2} = d\left(\frac1r\right), \ x \sim \left(a+\frac{b}{r}\right)^2\right)$

$$(\varphi-\varphi_0)A = -\arccos\frac{x}{A} \Rightarrow x = \frac{A}{\dots} = \frac{p}{1+e\cos(\varphi-\varphi_0)}$$

$$d\varphi = \frac{M\,dr}{r^2}\Big/\sqrt{2m\left[E-\frac{M}{2mr^2}-U(r)\right]} = \frac{-M\,d\left(\frac1r\right)}{\sqrt{2m\left[E-\frac{M^2}{2mr^2}+\frac{\alpha}{r}\right]}}$$

$$-\frac{M^2}{2mr^2}+\frac{\alpha}{r} = -\left(\frac{M}{2mr^2} - 2\left(\frac{\alpha}{2r}\right)\right) = -\left(\frac{M}{2mr^2} - 2\left(\frac{\alpha}{2r}\right) + \frac{\alpha^2 m}{2M} - \frac{\alpha^2 m}{2m}\right) =$$

$$= -\left(\frac{\sqrt M}{\sqrt{2m}\,r} - \frac{\alpha\sqrt m}{\sqrt2\sqrt M}\right) - \frac{\alpha^2 m}{2M}$$

$$d\varphi = \frac{-M\,d\left(\frac1r\right)}{\sqrt{2m\left[E-\left(\sqrt{\frac{M}{2m}}\frac1r - \alpha\sqrt{\frac{m}{2M}}\right) - \frac{\alpha^2m}{2M}\right]}} = \frac{-M\,dy}{\sqrt{2m\left[E-\left(\sqrt{\frac{M}{2m}}y-\alpha\sqrt{\frac{m}{2M}}\right)-\frac{\alpha^2m}{2M}\right)}}$$

$$2mE - \sqrt{\frac{M4m^2}{2m}}y - \alpha\sqrt{\frac{4m^2m}{2M}}\Big) - \frac{\alpha^2 2m^2}{2M} = 2mE - \sqrt{2mM}\,y - \alpha\sqrt{\frac{2m^3}{M}} - \frac{\alpha^2m^2}{M}$$

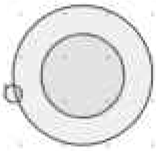
$$d\varphi = \frac{-M\,dy}{\sqrt{\left(-2mE - \sqrt{2mM}\,y - \alpha\sqrt{\frac{2m^3}{M} - \frac{\alpha^2 m^2}{M}}\right)\frac{M^2}{M^2}}} =$$

$$r = \frac{1}{\frac{m\alpha}{M^2} + \sqrt{\frac{\alpha^2 m^2}{M^4} + \frac{2mE}{M^2}}\cos(\varphi-\varphi_0)} = \frac{M^2/m\alpha}{1 + \sqrt{1 + \frac{2EM^2}{\alpha^2 m}}\cos(\varphi-\varphi_0)}$$

$$e = \sqrt{1 + \frac{2EM^2}{\alpha^2 m}}$$

$$p = \frac{M^2}{m\alpha}$$

$$g = \frac{MG}{R}$$



$$\frac{mv^2}{R} = mg$$

$$v^2 = gR, \quad E = \frac{mv^2}{2} - mgh = -\frac{mgR}{2}$$

$$M = mvR$$

$$\frac{\alpha}{r} = \frac{mMGh}{R} \Rightarrow \alpha = \frac{mMGhr}{R}$$

№3. траекторию для $U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}$

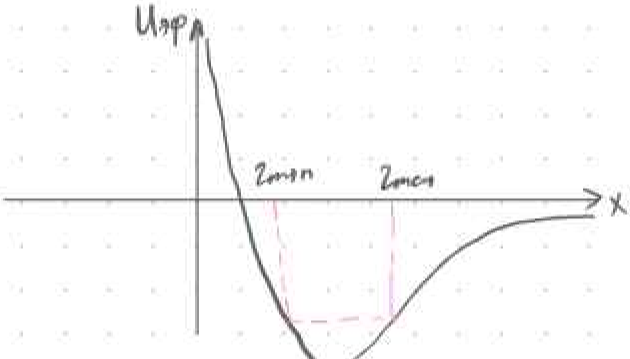
$$d\varphi = \frac{\frac{M}{r^2}dr}{\sqrt{2m\left[E - \frac{M^2}{2mr^2} - \frac{\alpha}{r} - \frac{\beta}{r^2}\right]}}$$

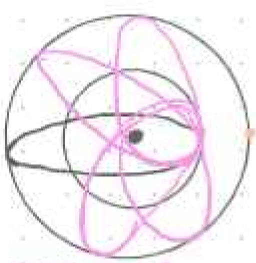
$$\tilde{M}^2 = M^2 + 2\beta m$$

$$\frac{\tilde{M}\,d\varphi}{M} = \frac{\tilde{M}\,dr/r^2}{\sqrt{2m\left[E - \frac{\tilde{M}^2}{2mr^2} - \frac{\alpha}{r}\right]}}$$

$$r(\varphi) = \frac{\tilde{p}}{1 + \tilde{e}\cos\left(\frac{\tilde{M}}{M}(\varphi-\varphi_0)\right)}$$

$$U_{эф} = \frac{M^2}{2mr^2} - \frac{\alpha}{r}$$





прецессия

$$\frac{\tilde{M}}{M}(\varphi-\varphi_0) = \Pi \Rightarrow \tilde{M} > M$$

$$\Delta\varphi = 2(\Pi - \varphi_0) = 2\Pi\frac{\beta m}{M^2}$$

$$\frac{M}{\tilde{M}} = \frac{M}{M\sqrt{1 + \frac{2\beta}{M^2}}} \simeq 1 + \frac{\beta m}{M^2}$$

φ_0 – угол в положении максимума

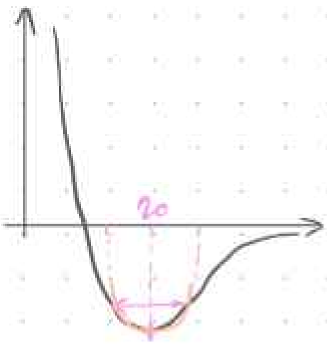
14. 02. 23.

след. вторник: с др. препод.

№1.

$$U(r) = -\frac{\alpha}{r}$$

$$U_{эфф} = \frac{M^2}{2mr^2} - \frac{\alpha}{r}$$



r_0 находим: $\frac{dU}{dr}\Big|_{r_0} = 0 \Rightarrow r_0 = \frac{M^2}{m\alpha} = \left\{ \frac{m^2 v^2 r^2}{m^2 MG} = r_0 \quad \frac{mMG}{r_0^2} = \frac{mv^2}{R} \right\}$

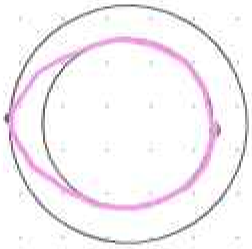
раскладываем $U(r_0 + r) = U(r_0) + \cancel{\frac{dU}{dr}}^{\,0} \Delta r + \frac{1}{2}\frac{d^2U}{dr^2}\Big|_{r_0} \Delta r^2 = U_0 + \frac{k\Delta r^2}{2}$ – приближ. параболой

$$k = \frac{d^2U}{dr^2}\Big|_{r_0} = \frac{m^3\alpha^4}{M^6}$$

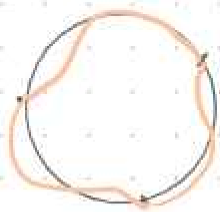
$$\omega^2 = \frac{k}{m} = \frac{m^2\alpha^4}{M^6}$$

$$\dot\varphi \simeq \frac{M}{mr_0^2} = \frac{M}{m}\cdot\frac{m^2\alpha^2}{M^4} = \frac{m\alpha^2}{M^3} = \omega$$

$\frac{\omega_2}{\dot\varphi} = 1$:

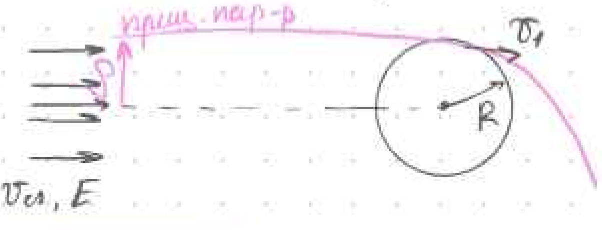


$\frac{\omega_2}{\dot\varphi} = 3$:



g|3: дифф-ное сечение рассеяния

№2. сечение падения



сеч. падения – площадь, с которой пучок попадает в цель

условие падения: $U_{эфф} \le E$

1) $\sigma_{пад} = \pi\rho^2$

2) $\begin{cases} \frac{mv_\infty^2}{2} = \frac{mv_1^2}{2} - \frac{MmG}{r} \; (= mgR_з) & \text{– ЗСЭ} \\ mv_\infty\rho = mv_1R_з & \text{– ЗСМ} \end{cases}$

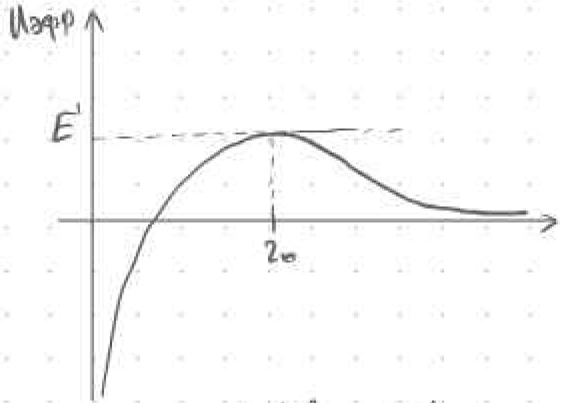
$v_1 = \sqrt{v_\infty^2 + 2gR_з}$

$v_\infty\rho = \sqrt{v_\infty^2 + 2gR_з}\,R_з$

$$\rho = \frac{\sqrt{v_\infty^2 + 2gR_3}\; R_3}{v_\infty} = \sqrt{1 + \frac{2gR_3}{v_\infty^2}}\, R^3 = \sqrt{R_3^2 + \frac{2gR_3^3}{v_\infty^2}}$$

$$\rho^2 = R_3^2\left(1 + \frac{mgR}{E}\right)$$

№3. $U(r) = -\dfrac{\alpha}{r^4}$



$$U_{эф} = \frac{M^2}{2mr^2} - \frac{\alpha}{r^4}$$

усл. падения : $E \geq U_{эф}$

$$U'_{эфф}\big| = 0 : \; -\frac{3M^2}{2mr^3} + \frac{5\alpha}{r^5} = 0 \;\Rightarrow\; -3M^2r^2 + 10\alpha m = 0$$

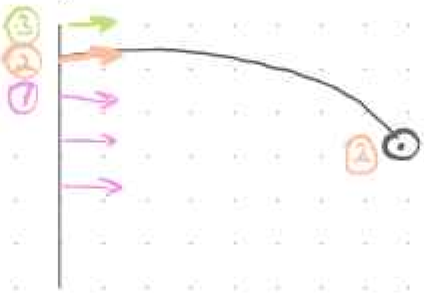
$$r^2 = \frac{10\alpha m}{3M} \;\Rightarrow\; r_0 = \sqrt{\frac{4m\alpha}{M^2}}$$

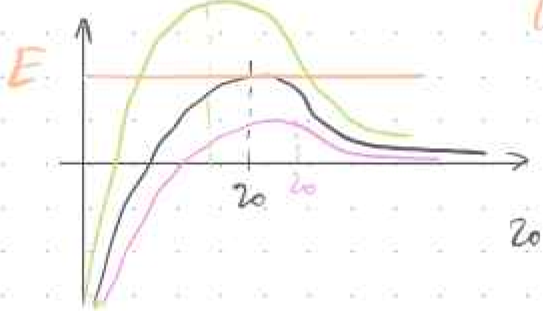
$$\left[\begin{array}{l} M^2 = m^2 v_\infty^2 \rho^2 = 2mE\rho^2 \\ E = \dfrac{m v_\infty^2}{2} \end{array}\right.$$

$$E = U_{эфф}(r_0) = -\frac{M^4}{16m^2\alpha} + \frac{2M^4}{16m^2\alpha} = 2mv_\infty^2\alpha = E^2\rho^4$$

$$\rho^4 = \frac{4\alpha}{E}$$

$$\sigma = \pi\rho^2 = \pi\sqrt{\frac{4\alpha}{E}}$$





(одинаковая у всех)

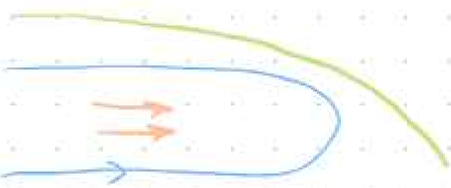
r_0 - точка, где v напр. по кас.

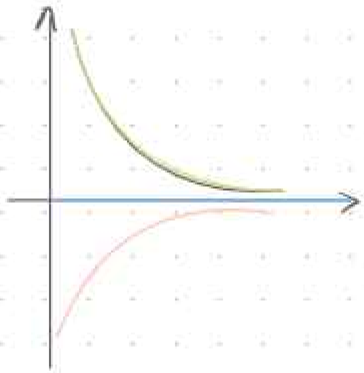
№4. $U(r) = -\dfrac{\alpha}{r^2}$

$$U_{эф} = \frac{M^2}{2mr^2} - \frac{\alpha}{r^2} \;;$$

$M^2 > 2\alpha m$ ~ гип-лы

$M^2 < 2\alpha m$ ~ $1/r^2$





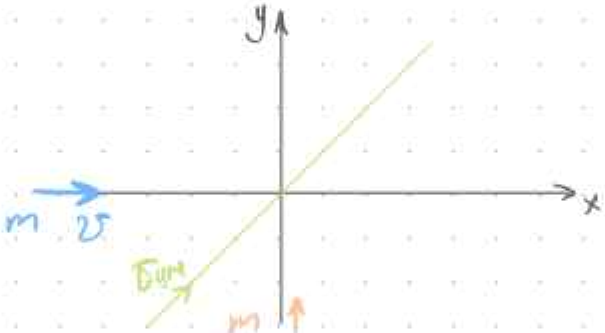
№5. задача двух тел

$m_1, \bar{r}_1$ | $\bar{r} = \bar{r}_1 - \bar{r}_2$ $\Rightarrow \mu$ 2

$m_2, \bar{r}_2$ | $\Rightarrow m_1 + m_2$ $\bar{R} = \dfrac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1 + m_2}$

$U(|\bar{r}_1 - \bar{r}_2|)$

$$E = \frac{m_1\dot{\bar{r}}_1^2}{2} + \frac{m_2\dot{\bar{r}}_2^2}{2} + U(\bar{r}_1 - \bar{r}_2) = \frac{M\dot{R}_{цм}}{2} + \underbrace{\frac{\mu\dot{r}^2}{2} + U(r)}_{\boxed{\frac{\mu\dot{r}^2}{2} + \frac{M}{2\mu r^2} + U(r)}}$$



ошибка: переходит в СО одной из част. (СО неинерциальна)

1) $\bar{r}_1 = (vt, 0, 0)$, $t \to -\infty$
$\bar{r}_2 = (0, v(t-\tau), 0)$

$$\bar{r}_1 - \bar{r}_2 = (vt, v(\tau - t), 0)$$

$$\bar{R}_{ц.м} = \frac{mr_1 + mr_2}{2m} = \frac{m(r_1 + r_2)}{2m} = \frac{r_1 + r_2}{2}$$

$$\dot{R}_{ц.м} = \left(\frac{v}{2}, \frac{v}{2}\right), \quad \mu = \frac{m_1 \cdot m_2}{m_1 + m_2} = \frac{m^2}{2m} = \frac{m}{2}$$

2) $$2\frac{mv^2}{2} = 2m\frac{\dot{R}_{цм}}{2} + \frac{M}{2}\dot{r}^2 + U(r) = \underbrace{\frac{mV^2}{2}}_{E_{цм}} + \underbrace{\frac{m}{2}\cdot\frac{1}{2}\dot{r}^2 + \frac{M^2}{2\mu r^2} + U(r)}_{E_{отн} = \frac{mV_{от}^2}{2}}$$

3) $V_{отн} = (v, -v, 0)$

$$M = \mu(\bar{r} \times \dot{\bar{r}}) = \frac{m}{2}\begin{pmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ vt & v(\tau - t) & 0 \\ v & -v & 0 \end{pmatrix} = \frac{m}{2}\left[\bar{e}_3 \cdot -v^2 t - \bar{e}_3 v^2(\tau - t)\right] =$$

$$= \frac{m}{2}\bar{e}_3 \cdot V^2(-\tau)$$

$$|M| = \frac{mV^2\tau}{2}$$

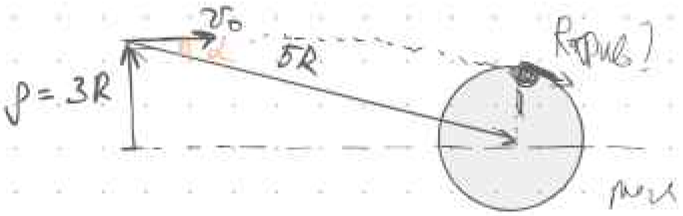
№6.

$U(r) = \dfrac{\alpha}{r}$, на какое r_{min} они сблизятся?

$(\dot{r} = 0 \Rightarrow U_{эфф} = E)$

$U(r) = -\dfrac{\alpha}{r^2}$ при каком τ_{min} они столкнутся?

№1



1) при каких v_0 астероид упадет на землю?

$E_{полн}$ (θmgR)

1) ЗСЭ: $\frac{mv_0^2}{2} - \frac{MmG}{5R} = \frac{mv_1^2}{2} - \frac{MmG}{R}$

ЗСМИ: $m[\bar{r}_1 \times \bar{v}_0] = m[\bar{r}_2 \times \bar{v}_1] = mR_3 v_1 \Rightarrow 3mv_0 R = mR_3 v_1$

$mr_1 v_0 \sin\alpha = m5Rv_0 \sin\alpha =$

$= m5Rv_0 \cdot \frac{3R}{5R} = 3mv_0 R$

$+ v_1 = \frac{3mv_0 R}{mR_3} = 3v_0$

2) $\frac{v_0^2}{2} - \frac{MG}{5R} = \frac{9v_0^2}{2} - \frac{MG}{R}$

$\frac{MG}{R} - \frac{MG}{5R} = \frac{9v_0^2}{2} - \frac{v_0^2}{2} = 4v_0^2$

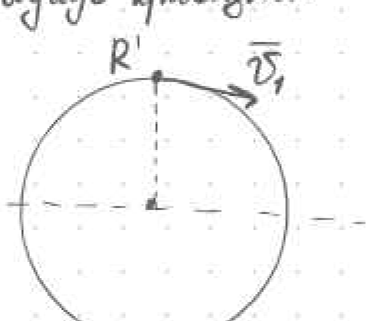
$\frac{4MG}{5R} = 4v_0^2$

$v_0 = \sqrt{\frac{GM}{5R}}$

3) полная энергия:

$E = \frac{mv_0^2}{2} - \frac{mMG}{5R} = \frac{mGM}{10R} - \frac{mGM^2}{5R} = -\frac{mGM}{10R}$

радиус кривизны:



$\bar{a} = \bar{a}_y + \bar{a}_\tau$

$v_{\tau_{R'}} = 0 \Rightarrow \bar{a} = \bar{a}_y = g$

$a_y = \frac{v_1^2}{2} \Rightarrow g2 = v_1^2 = (3v_0)^2 = 9 \frac{GM}{5R_3}$

$2 = \frac{9}{g} \frac{GM}{5R_3} = \frac{9}{5} R_3 \frac{GM}{R_3^2} = g$

$e = ?$

№2

$$U(r) = \alpha r^7$$

$$U_{эфф} = \frac{M^2}{2mr^2} + \alpha r^7$$

1) $\left.\frac{dU_{эфф}}{dr}\right|_{r_0} = 0 \Rightarrow -\frac{2M^2}{2mr^3} + 7\alpha r^6 = 0$

$$-\frac{M^2}{mr^3} + \frac{7\alpha m r^9}{mr^3} = 0 \Rightarrow r_0 = \sqrt[9]{\frac{M^2}{7\alpha m}}$$

2) $\left.\frac{d^2U}{dr}\right|_{r_0} = k \Rightarrow \frac{3M^2}{mr_0^4} + 42\alpha r_0^5 = k$

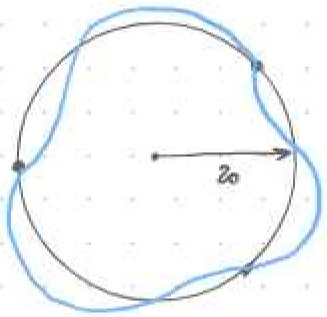
$$\omega^2 = \frac{k}{m} = \left(\frac{3M^2}{mr_0^4} + 42\alpha r_0^5\right)/m = \frac{3M^2}{m^2r_0^4} + \frac{42\alpha r_0^5}{m}$$

3) $\dot{\varphi} \simeq \frac{M}{mr_0^2}$

$$\frac{\omega}{\dot{\varphi}} = \frac{mr_0^2\sqrt{\frac{3M^2 + 42\alpha m r_0^9}{m^2r_0^4}}}{M} = \frac{mr_0^2\sqrt{3M^2 + 42\alpha m r_0^9}}{M \cdot mr_0^2} = \sqrt{\frac{3M^2}{M^2} + \frac{42\alpha m r_0^9}{M^2}} =$$

$$= \sqrt{3 + \frac{42\alpha m}{M^2} \cdot \frac{M^2}{7\alpha m}} = \sqrt{3+6} = 3$$

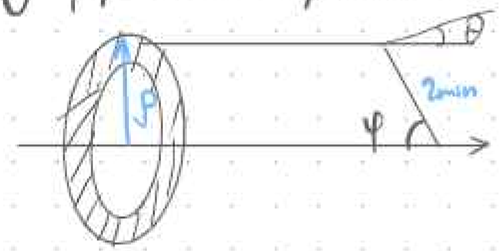
4)



Сечение рассеяния.

21.02.23

дифф. сечение рассеяния:



$$d\sigma = \frac{dN}{j}$$

$$\frac{d\sigma}{d\Omega}(\theta) =$$

$$\frac{d\sigma}{d\Omega} = \frac{2\pi\rho(\theta)d\rho}{2\pi\sin\theta d\theta} \Rightarrow \frac{d\sigma}{d\Omega} = \frac{\rho(\theta)}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| = \left|\frac{d\rho^2(\theta)}{d\theta}\right|\frac{1}{2\sin\theta}$$

Способы нахождения $\rho(\theta)$:

1 в квадратурах

$$d\varphi = \frac{\frac{M}{mr^2}dr}{\sqrt{\frac{2}{m}\left(E - \frac{M^2}{2mr^2} - U(r)\right)}} = \frac{\frac{M}{mr^2}dr}{\sqrt{\underbrace{\tfrac{2}{m}E}_{v_\infty}\left(1 - \frac{\rho^2}{r^2} - \frac{U(r)}{E}\right)}} = \frac{\frac{\rho dr}{r^2}}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{U(r)}{E}}}$$

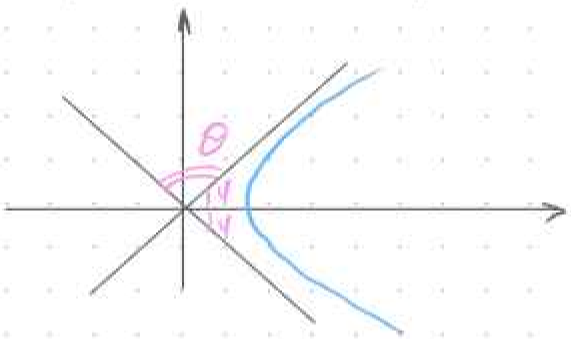
$M = m v_\infty \rho$ | $M^2 = 2E\rho^2 m$

$E = \frac{m v_\infty^2}{2}$

$$\frac{\pi - \theta}{2} = \varphi_0 = \int_{r_{min}}^{\infty} \frac{\rho dr/r^2}{\sqrt{}} = f(\rho, E) \Rightarrow \rho(E, \theta)$$

r_{min} ищем: $\frac{m\dot{r}^2}{2} = 0 = E - \frac{M}{2mr^2} - U(r)$

пример: ф-ла Резерфорда



$$r = \frac{p}{1 - e\cos\varphi}$$

$$\theta + 2\varphi_0 = \pi$$

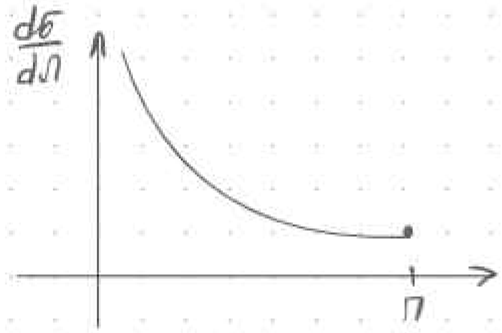
$$r \to \infty: \varphi_0 \to \underset{\sin\frac{\theta}{2}}{\cos\varphi_0} = \frac{1}{e(\rho, E)}$$

$$e = \sqrt{1 + \frac{2EM^2}{\alpha^2 m}}$$

$$\sin\frac{\theta}{2} = \frac{1}{\sqrt{1+\frac{2Em^2\rho^2 v_\infty^2}{m\alpha^2}}} = \frac{1}{\sqrt{1+\frac{4E\rho^2}{\alpha^2}}} \Rightarrow \sin^2\frac{\theta}{2} = \frac{1}{1+\frac{4E^2\rho^2}{\alpha^2}} \Rightarrow$$

$$\rho^2 = \left(\frac{1}{\sin^2\frac{\theta}{2}} - 1\right)\frac{\alpha^2}{4E^2} \Rightarrow \boxed{\rho(\theta) = \frac{\alpha}{2E}\,\mathrm{ctg}\,\frac{\theta}{2}}$$

2) $$\frac{d\sigma}{d\Omega} = \frac{\rho(\theta)}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| = \frac{\alpha}{2E}\mathrm{ctg}\frac{\theta}{2}\cdot\frac{1}{\sin\theta}\cdot\frac{\alpha}{2E}\cdot\frac{1}{2\sin^2\frac{\theta}{2}}\,d\theta = \left(\frac{\alpha}{4E}\right)^2\frac{1}{\sin^4\frac{\theta}{2}}$$ – ф-ла Резерфорда

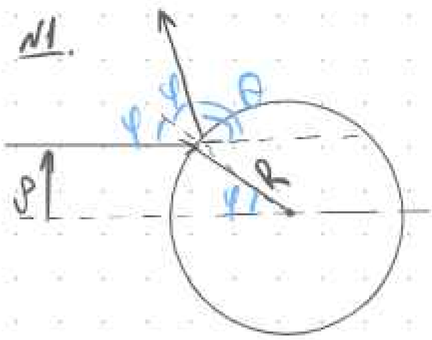
3) 

4) $$\sigma = \int\frac{d\sigma}{d\omega}(\theta)\,\underbrace{\sin\theta\,2\pi d\theta}_{d\omega}$$ – площадь, с которой частицы „чувствуют" друг друга

$$\sigma = \left(\frac{\alpha}{4E}\right)^2\int_0^{\pi}\frac{1}{\sin^4\frac{\theta}{2}}\sin\theta\,2\pi d\theta = -\pi\frac{\alpha^2}{8E^2}\int_0^{\pi}\frac{d\cos\theta}{(1-\cos^2\frac{\theta}{2})^2} = \infty \Rightarrow$$ частица на беск. почувствует заряд

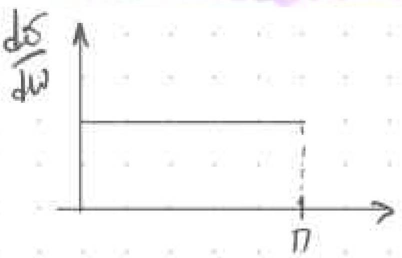
2 геометрический способ (на тв. поверхностях)

№1.



$$\rho = R\sin\psi = R\cos\frac{\theta}{2}$$

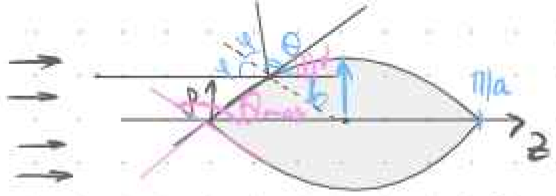
$$\frac{d\sigma}{d\omega} = \frac{\rho(\theta)}{\sin\theta}\left|\frac{d\rho}{d\theta}\right| = \frac{1}{2}R\sin\frac{\theta}{2}R\cos\frac{\theta}{2}\frac{1}{\sin\theta} = \frac{R^2}{4}$$



задача к экзамену

$$\sigma = \int\left|\frac{d\sigma}{d\omega}\right|d\omega = \pi R^2$$

№2 опр-ть на пов-ти вращения рассеяния



$\rho(z) = b\sin\frac{z}{a}$, $0 \le z \le \Pi a$

1) $tg\alpha = \frac{d\rho}{dz} = tg\frac{\theta}{2} = \frac{b}{a}\cos\frac{z}{a}$

2) $\begin{cases} 2\varphi + \theta = \Pi \\ \varphi + (\theta - \alpha) = \frac{\Pi}{2} \end{cases} \Rightarrow 2\left(\frac{\Pi}{2} - (\theta - \alpha)\right) + \theta = \Pi$

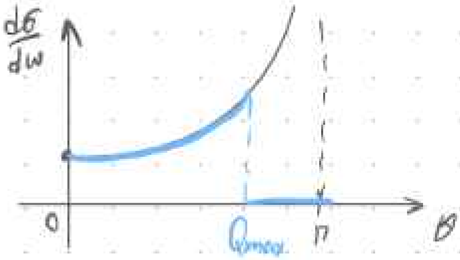
$\Pi - 2\theta + 2\alpha + \theta = \Pi$

$\theta = 2\alpha$

3) $\left(\frac{a}{b}\right)^2 tg^2\frac{\theta}{2} + \left(\frac{\rho}{b}\right)^2 = 1$

$\rho^2 = b^2 - a^2 tg^2\frac{\theta}{2}$

$\frac{d\sigma}{d\omega}(\theta) = \left|\frac{d\rho^2}{d\theta}\right|\left(\frac{1}{2\sin\theta}\right) = a^2 \cdot \frac{1}{2} tg\frac{\theta}{2} \cdot \frac{1}{\cos^2\frac{\theta}{2}} \cdot \frac{1}{2\sin\theta} = \frac{a^2}{4\cos^4\theta/2}$



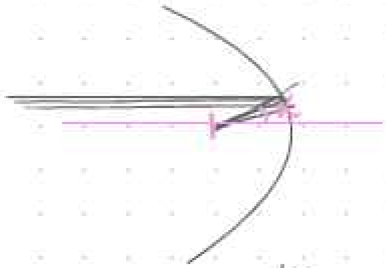
$\frac{d\sigma}{d\omega}(\theta) = \begin{cases} \frac{a^2}{4\cos^4\theta/2}, & \theta < \theta_{max} \\ 0, & \theta > \theta_{max} \end{cases}$

$\sigma = \int_0^{\theta_m} \frac{d\sigma}{d\Omega}(\theta) 2\Pi\sin\theta d\theta = \Pi b^2$

4) $tg\frac{\theta_{max}}{2} = \frac{b}{a}$

$\frac{a^2}{4\cos^4\theta/2} \cdot 2\Pi\sin\theta d\theta = \frac{a^2\Pi}{2\cos^4\theta/2}\sin\theta d\theta = \int_0^{\theta_m} \frac{\Pi a^2 tg\frac{\theta}{2}}{\cos^2\theta/2} d\theta = 2\int_0^{\frac{b}{a}} \Pi a^2 tg\frac{\theta}{2} d tg\frac{\theta}{2} = \Pi a^2 \frac{b^2}{a^2} = \Pi b^2$

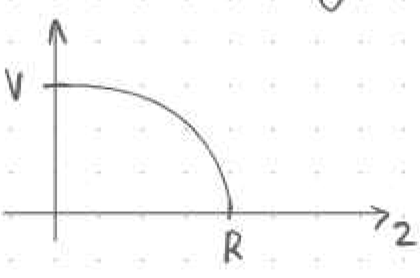
3 метод быстрых частиц ($\theta \ll 1$)

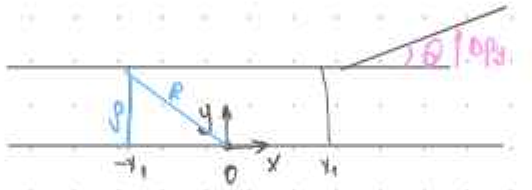


$z = R\alpha = R\sin\alpha$ – малые отклонения

№1. найти $\frac{d\sigma}{d\Omega}(\theta)$ для быстрых частиц в $U(z) = \begin{cases} V\left(1 - \frac{z^2}{R^2}\right), & z < R \\ 0, & z > R \end{cases}$

как связать ρ и θ?





$$-\frac{\partial U}{\partial y} = -\frac{\partial U}{\partial z}\,2y$$

$$\theta = \frac{\Delta p_y}{p_x} = \frac{1}{m v_\infty}\int_{-\infty}^{+\infty} F_y\,dt = \frac{1}{2E}\int_{-x_1}^{x_1} dx\cdot\frac{V}{R^2}\,2y = \frac{\sqrt{R^2-\rho^2}\,V\cdot 2\rho}{R^2 E} \quad \left[x = \frac{\rho}{R}\right]$$

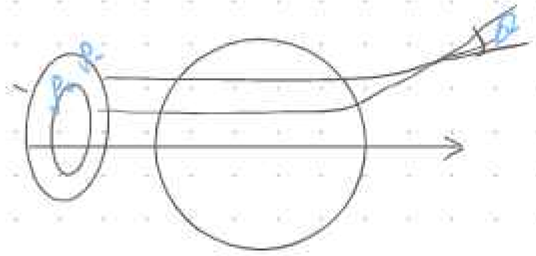
$dt = \frac{dx}{v_\infty}$, $x_1 = \sqrt{R^2-\rho^2}$, $2y \approx \rho$

$$\frac{V}{E} = \theta_0 \ll 1 \Rightarrow \theta = \theta_0\cdot\sqrt{1-x^2}\cdot 2x \Rightarrow \rho^2 = R^2\cdot\frac{1\pm\sqrt{1-\left(\frac{\theta}{\theta_0}\right)^2}}{2}$$

$$\frac{\theta^2}{\theta_0^2} = (1-x^2)4x^2 = 4x^2 - 4x^4 = 0$$

$$\frac{d\sigma}{d\omega}(\theta) = \left(\frac{d\rho^2}{d\theta}\right)\left(\frac{1}{2\sin\theta}\right) = \frac{R^2}{2}\cdot\frac{1}{2}\frac{1}{\sqrt{1-\left(\frac{\theta}{\theta_0}\right)^2}}\cdot 2\theta\,\frac{1}{\theta_0^2}\cdot\frac{1}{2\theta} = R^2\cdot\frac{1}{4\theta_0^2}\frac{1}{\sqrt{1-\left(\frac{\theta}{\theta_0}\right)^2}}$$

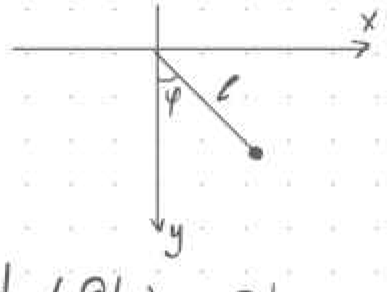
$$\sigma = \int\frac{R^2}{4}\frac{1}{\theta_0^2}\frac{1}{\sqrt{1-\left(\frac{\theta}{\theta_0}\right)^2}}\cdot 2\pi\theta\,d\theta = -\frac{R^2}{4}\pi\int_0^{\theta_0}\frac{1}{\sqrt{1-\left(\frac{\theta}{\theta_0}\right)^2}}\,d\left(1-\left(\frac{\theta}{\theta_0}\right)^2\right) = -\frac{\pi R^2}{2}\sqrt{1-\left(\frac{\theta}{\theta_0}\right)^2}\,\Bigg|_0^{\theta_0} = \frac{\pi R^2}{2}$$



28.02.23

Функция Лагранжа

1) $\delta S = 0$ (система описывается экстремумом некоторого функционала)
2) $S = \int L(\dot q, q, t)\,dt$ [c]
3) $\{q\}$ – обобщ. коорд-ты



способы задания: 1) угол 2) координата x

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot q_i}}_{p_i}\Big) - \underbrace{\frac{\partial L}{\partial q_i}}_{F_i} = 0$$

S – степ. свободы → S обычных дифф. ур-ний 2-го порядка

2S начальных условий

$p_i = \frac{\partial L}{\partial \dot q_i}$ – обобщ. импульс

$F_i = \frac{\partial L}{\partial q_i}$ – обобщ. сила

$\dot q_i = f_i(q_i)$

1) потенц. поле : $U(x) \Rightarrow L = \frac{m\dot x^2}{2} - U(x)$

$p_x = \frac{\partial L}{\partial \dot x} = m\dot x$

$m\ddot x = -\frac{\partial U}{\partial x}$ (2 закон Ньютона)

2) $L(x, \dot x, t) = \frac{e^{\alpha t}}{2}(\dot x^2 - \omega^2 x^2)m$

$\frac{\partial L}{\partial \dot x} = e^{\alpha t} m \cdot \dot x \Rightarrow \frac{d}{dt}(\ ) = \ddot x m e^{\alpha t} + \dot x \alpha e^{\alpha t} m = e^{\alpha t} m(\ddot x + \alpha \dot x)$

$\frac{\partial L}{\partial x} = -\frac{e^{\alpha t} m}{2}\omega^2 \cdot 2x = -e^{\alpha t} m \omega^2 x$

$\ddot x + \alpha \dot x + \omega^2 x = 0$ – затухание

(*): если есть $L(\dot q, q, t)$

- если $\frac{\partial L}{\partial q_i} = 0 \Rightarrow p_i = const \Rightarrow$ интеграл движения
- если $\frac{\partial L}{\partial t} = 0 \Rightarrow \sum^n p_i \dot q_i - L = E_n$ (сохраняется)

(*)

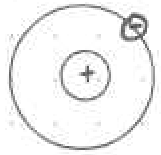
№1.

$U(z)$, $L = m\left(\frac{\dot z^2 + z^2\dot\varphi^2}{2}\right) - U(z)$

$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow p_\varphi = \frac{\partial L}{\partial \dot\varphi} = mz^2\dot\varphi$

$p_\varphi \cdot \varphi = [Дж \cdot c]$
$p_z \cdot z = [Дж \cdot c]$

$\Rightarrow E = p_\varphi \dot\varphi + p_z \dot z - L = \frac{mz^2\dot\varphi}{2} + m\dot z^2 - L =$
$= \frac{m(z^2\dot\varphi + \dot z^2)}{2} + U(z)$



$m v z = \hbar \cdot z$
атом Бора

№2. функция Лагранжа св. нерелят. частицы

$L = -mc^2\sqrt{1 - \left(\frac{\dot x}{c}\right)^2}$

$p_x = \frac{\partial L}{\partial \dot x} = \frac{-mc^2}{2\sqrt{1-(\frac{\dot x}{c})^2}} \cdot -\frac{2\dot x}{c^2} = \frac{\dot x m}{\sqrt{1-(\frac{\dot x}{c})^2}}$

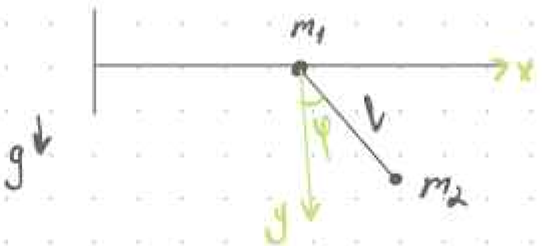
$$E = p_x\dot{x} - L = \frac{m\dot{x}^2}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} + mc^2\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2} = \frac{m\dot{x}^2 + mc\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} = \frac{mc^2}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}}$$

$$\frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}}\right) = \frac{m\ddot{x}}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} - \frac{1}{2}\frac{m\dot{x}\ddot{x}}{\left(1-\left(\frac{\dot{x}}{c}\right)^2\right)^{3/2}}\left(-\frac{2\dot{x}}{c^2}\right) = 0$$

$$p_x = \frac{m\dot{x}^2}{\sqrt{1-\left(\frac{\dot{x}}{c}\right)^2}} = p_0 \Rightarrow \frac{m^2\dot{x}^2}{p_0^2} = 1 - \frac{\dot{x}^2}{c^2} \Rightarrow \dot{x} = const$$

д/з: Л.Л. §5 - 4 задачи

№1.

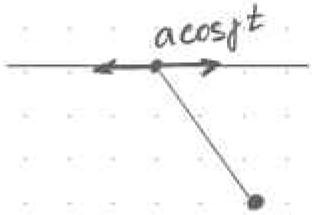


$$x_2 = x + l\sin\varphi$$
$$y_2 = l\cos\varphi$$
$$\dot{x}_2 = \dot{x} + l\dot{\varphi}\cos\varphi$$
$$\dot{y}_2 = -l\sin\varphi\,\dot{\varphi}$$
$$\dot{x}_2^2 + \dot{y}_2^2 = \dot{x}^2 + l^2\dot{\varphi}^2\cos^2\varphi + l^2\sin^2\varphi\,\dot{\varphi}^2$$

$$U = -mgy$$

$$L = \frac{m_1\dot{x}^2}{2} + \frac{m_2}{2}\Big[\underbrace{\dot{x}^2}_{\text{пост. дв.}} + \underbrace{l^2\dot{\varphi}^2}_{\text{конст.}} + \underbrace{2l\dot{x}\dot{\varphi}\cos\varphi}_{\text{зависимость движения}}\Big] + mgl\cos\varphi$$

№2.



$$L = \frac{m}{2}\left(l^2\dot{\varphi}^2 + 2l\dot{\varphi}\sin\gamma t\cos\varphi\cdot a\gamma + a^2\gamma^2\sin^2\gamma t\right) + mgl\cos\varphi$$

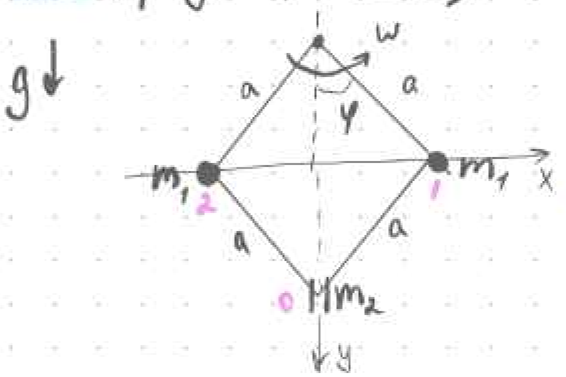
$$\frac{\partial L}{\partial \varphi} \neq \frac{\partial L}{\partial \varphi}(t)\ ;\quad \frac{\partial L}{\partial \dot{\varphi}} \neq \frac{\partial L}{\partial \dot{\varphi}}(t)$$

можем выкинуть зав-ть от t, т.к. ур-е Лагранжа от t не зависит

$$U = -mgy$$

$$x_2 = a\cos(\gamma t) + l\sin\varphi \qquad \dot{x}_2 = l\dot{\varphi}\cos\varphi - a\gamma\sin\gamma t$$
$$y_2 = l\cos\varphi \qquad \dot{y}_2 = -l\dot{\varphi}\sin\varphi$$

№3 (регулятор Уатта)



1) написать функцию Лагранжа
2) найти E
3) найти ω^2

если задано ур-ние дв-ния: -1 ст. св.

для этой сист. 1 ст. св. $= \varphi$

$$\omega = \dot{\theta}$$

1) $U = -4mga\cos\varphi = -2(m_1+m_2)ga\cos\varphi$
2) $y_0 = 2a\cos\varphi$
3) $V_1^2 = w^2(a\sin\varphi)^2 + (a\cos\varphi)^2$, $V_2^2 = w^2(a\sin\varphi)^2 + \dot\varphi^2(-a\cos\varphi)^2$, $V_3^2 = (2a\sin\varphi w)^2$

$$K_0 = \frac{m(2a\sin\varphi \cdot w)^2}{2}$$

$$K_\Sigma = \frac{m_1(w^2a^2\sin^2\varphi + a^2\cos^2\varphi + w^2a^2\sin^2\varphi + a^2\cos^2\varphi)}{2} + 2m_2a^2\sin^2\varphi \cdot w^2 =$$

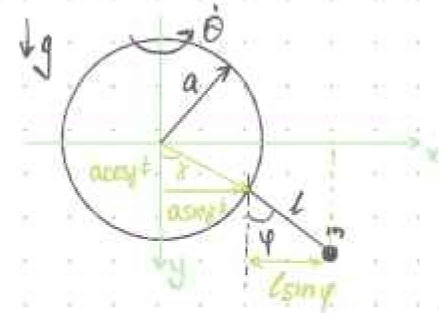
$$L = 4mga\cos\varphi + \frac{m}{2}\left(4a^2\sin\alpha \cdot \dot\alpha + 2(a\sin\alpha \cdot w)^2 + 2l^2\dot\alpha^2\right)$$

$$L' = 4mga\cos\varphi + \frac{m}{2}\left(4a^2\sin^2\alpha \cdot \dot\alpha^2 + 2\dot\alpha l^2 + 2(a\sin\alpha)^2\dot\varphi\right)^2$$

подзадача: $L = A\dot x^2 + B\dot x + C \Rightarrow E = \frac{\partial L}{\partial \dot x}\dot x - L = (2A\dot x + B)\dot x - A\dot x^2 - B\dot x - C =$
$= A\dot x^2 - C$

$$p_x = 2A\dot x + B \Rightarrow \dot p_x = 2\dot A\dot x + 2A\ddot x + \dot B = \frac{\partial A}{\partial x}\dot x + \frac{\partial B}{\partial x}\dot x$$

N3.

a) 

$\dot\theta = \gamma = const$

1 ст. свободы, т.к. задано ур-е движения $\Rightarrow q = \varphi$

$x = l\sin\varphi + a\cos\gamma t$
$y = l\cos\varphi - a\sin\gamma t$

$U = -mgl\cos\varphi$

$$\upsilon_x^2 = (l\cos\varphi \cdot \dot\varphi - a\gamma\sin\gamma t)^2 = l^2\cos^2\varphi \cdot \dot\varphi^2 - 2la\cos\varphi\sin\gamma t \cdot \dot\varphi\gamma + a^2\gamma^2\sin^2\gamma t)$$

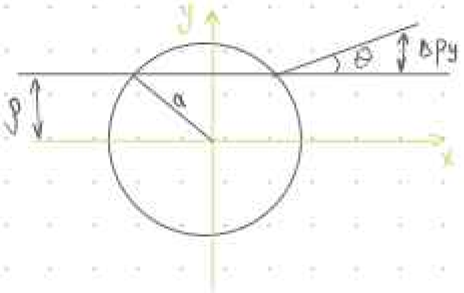
$$\upsilon_y^2 = (-l\sin\varphi \cdot \dot\varphi - a\gamma\cos\gamma t)^2 = l^2\sin^2\varphi \cdot \dot\varphi^2 + 2la\gamma\dot\varphi\sin\varphi \cdot \cos\gamma t + a^2\gamma^2\cos^2\gamma t$$

$$K = \frac{m}{2} \cdot l^2\dot\varphi^2(\cos^2\varphi + \sin\varphi^2) + \frac{m}{2} \cdot 2la\gamma(-\cos\varphi\sin\gamma t + \sin\varphi\cos\gamma t) + \frac{m}{2} \cdot a^2\gamma^2 =$$

$$= \frac{ml^2\dot\varphi^2}{2} + mla\gamma(\sin\varphi - \gamma t) + \frac{ma^2\gamma^2}{2}$$

№4.

$$U(r) = \begin{cases} V\ln\frac{r}{a}, & r<a \\ 0, & r>a \end{cases}$$



$E \gg V$

$\sin\theta \sim \theta$

$$\frac{\partial U}{\partial r} = V\cdot\frac{1}{\frac{r}{a}}\cdot\frac{1}{a}\cdot 2y\cdot\frac{1}{2}$$

1) $$\theta \sim \sin\theta = \frac{\Delta p_y}{p_x} = \frac{1}{mv_\infty}\cdot\int_{-\infty}^{+\infty} F_y\,dt = \frac{1}{mv_\infty}\int_{-x'}^{x'} -\frac{\partial U}{\partial r}\,\frac{dx}{v_\infty} = -\frac{V}{2E}\int_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} \frac{\rho\,dx}{\sqrt{x^2+\rho^2}} =$$

$$= -\frac{V}{2E}\,\rho\cdot\frac{1}{\rho}\,\mathrm{arctg}\,\frac{x}{\rho}\Big|_{-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} = -\underbrace{\frac{V}{E}}_{\theta_0}\,\mathrm{arctg}\,\frac{\sqrt{a^2-\rho^2}}{\rho}$$

$$\frac{\theta}{\theta_0} = -\mathrm{arctg}\sqrt{\frac{a^2}{\rho^2}-1}$$

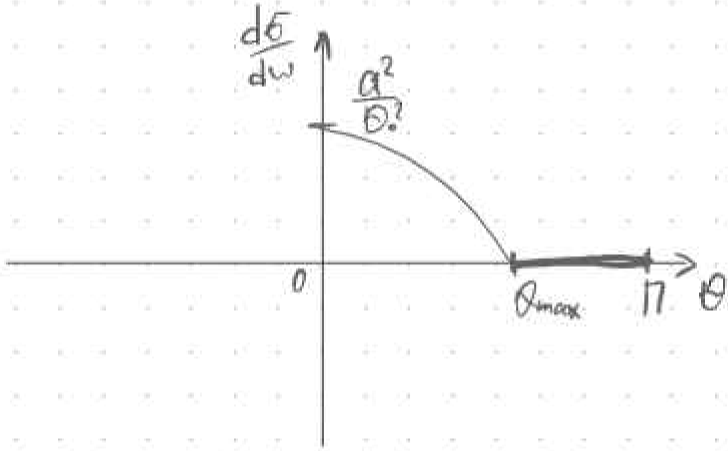
$$\mathrm{tg}\,\frac{\theta}{\theta_0} = -\sqrt{\frac{a^2}{\rho^2}-1} \Rightarrow \mathrm{tg}^2\frac{\theta}{\theta_0} = \frac{a^2}{\rho^2}-1 \Rightarrow \rho^2(\theta) = \frac{a^2}{\mathrm{tg}^2\frac{\theta}{\theta_0}+1} = a^2\cos^2\frac{\theta}{\theta_0}$$

2) $$\frac{d\sigma}{d\omega} = \left|\frac{d\rho^2}{d\theta}\right|\frac{1}{2\sin\theta} = \frac{a^2\cdot 2\cos\frac{\theta}{\theta_0}\cdot\sin\frac{\theta}{\theta_0}\cdot\frac{1}{\theta_0}}{2\theta} = \frac{a^2\sin\left(2\frac{\theta}{\theta_0}\right)}{2\theta\cdot\theta_0}$$

3) $$\cos^2\frac{\theta}{\theta_0} = 0 \Rightarrow \cos\frac{\theta}{\theta_0} = 0 \Rightarrow \frac{\theta}{\theta_0} = \frac{\pi}{2} \Rightarrow \theta_{max} = \frac{\pi\theta_0}{2}$$

4) $$\sigma = \int_0^{\theta_{max}} \frac{a^2\sin(2\theta/\theta_0)}{2\theta\cdot\theta_0}\cdot 2\pi\theta\,d\theta = \int_0^{\theta_{max}} \frac{a^2\sin(2\theta/\theta_0)\cdot\pi\,d\theta}{\theta_0} = \frac{a^2\pi}{\theta_0}\cdot\frac{\theta_0}{2}\int_{\theta_0}^{\theta_{max}} \sin\frac{2\theta}{\theta_0}\,d\frac{2\theta}{\theta_0} =$$

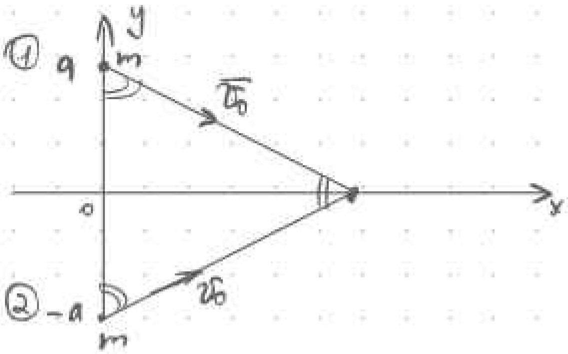
$$= \frac{a^2\pi}{2}\cos\frac{2\theta}{\theta_0}\Big|_{\theta_{max}}^{0} = \frac{\pi a^2}{2}\left(\cos\frac{2\cdot\pi\theta_0}{2\theta_0} - \cos\frac{2\cdot 0}{\theta_0}\right) = \pi a^2$$



$$\frac{a^2\sin\frac{2\theta}{\theta_0}}{\frac{2\theta}{\theta_0}\cdot\theta_0\cdot\theta_0} = \frac{a^2}{\theta_0^2}\,\mathrm{sinc}\,\frac{2\theta}{\theta_0}$$

№3.

$$U(r) = -\frac{b}{r^2}$$



1) $\bar{r}_1 = (v_0 \cos 30^\circ \tau, -v_0 \sin 30 \cdot \tau, 0)$ | $\dot{\bar{r}}_1 = (v_0 \cos 30, -v_0 \sin 30, 0)$

$\bar{r}_2 = (v_0 \cos 30^\circ \tau, v_0 \sin 30 \cdot \tau, 0)$ | $\dot{\bar{r}}_2 = (v_0 \cos 30, v_0 \sin 30, 0)$

$$\bar{r} = \bar{r}_1 - \bar{r}_2 = (0, -2v_0 \sin 30, 0) = (0, -v_0, 0)$$

2) $$E = \frac{M\dot{\bar{R}}_{ц.м}}{2} + \frac{\mu \dot{r}^2}{2} + \frac{M}{2\mu r^2} + U(r)$$

$$M = \mu[\bar{r} \times \dot{\bar{r}}] = \frac{m}{2}\begin{pmatrix} \bar{e}_x & \bar{e}_y & \bar{e}_z \\ 0 & -v_0\tau & 0 \\ 0 & -v_0 & 0 \end{pmatrix} = 0$$

$$E = \frac{\mu \dot{r}^2}{2} - \frac{b}{r^2} \Rightarrow \frac{\mu \dot{r}^2}{2} = \pm\sqrt{\frac{2}{\mu}\left(E + \frac{b}{r^2}\right)}$$

$$\int_{2a}^{0} \frac{dr}{\sqrt{\frac{4}{m}\left(E + \frac{b}{r^2}\right)}} = \int_0^{\tau} dt$$

$$\int_{2a}^{0} \frac{r\,dr}{\sqrt{\frac{4}{m}(Er^2 + b}} = \sqrt{\frac{m}{4}}\frac{2}{2}\int_0^{2a}\frac{dr^2}{\sqrt{Er^2+b}} = \frac{\sqrt{m}}{2E}\sqrt{Er^2+b}\,\Big|_{2a}^{0} = \frac{\sqrt{m}}{2E}\left(\sqrt{4a^2\left(\frac{mv_0^2}{4} - \frac{b}{4a^2}\right) + b}\right.$$

$$\left.-\sqrt{b}\right) = \frac{\sqrt{m}}{E}\left(\sqrt{a^2 m v_0^2} - \sqrt{b}\right) = \frac{maV_0 - \sqrt{mb}}{\frac{2mv_0^2}{4}\left[1 + \frac{b}{mv_0^2 a^2}\right]} = \frac{2av_0 m\left(1 - \frac{\sqrt{mb}}{mav_0}\right)}{mv_0^2\left(1 + \frac{b}{mv_0^2 a^2}\right)} =$$

$$= \frac{2a}{v_0}\cdot\frac{1}{1 + \sqrt{\frac{b}{ma^2v_0^2}}}$$

3) $$\Delta l = \frac{a\sqrt{3}}{2} - v_0\tau\frac{\sqrt{3}}{2} = \sqrt{3}a\left(1 - \frac{1}{\sqrt{1 + \frac{b}{mv_0^2a^2}}}\right)$$

Функция Лагранжа

7.03.23

$$L = 4mag\cos\alpha + \frac{m}{2}\left(4a^2\sin^2\alpha\,\dot\alpha^2 + 2\dot\alpha^2 l^2 + 2(a\sin\alpha)^2\dot\varphi^2\right)$$

$$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow p_\varphi = \frac{\partial L}{\partial \dot\varphi} = 2a^2\sin^2\alpha\,\dot\varphi$$

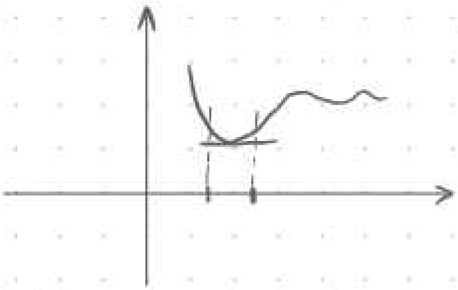
$$L = 4mga\cos\alpha + \dot\alpha^2\{\ \} + \frac{p_\varphi^2}{\dots}$$

$$E_n = \dot\varphi p_\varphi + \dot\alpha p_\alpha - L = \left[\dot\varphi = \frac{p_\varphi}{2\sin\alpha}\right] \Rightarrow$$ задача с 1 ст. свободы $\sim \dot\alpha^2 f(\alpha) + U + \frac{p_\varphi^2}{a^2\sin^2\alpha}$

возвращаясь к нач. задаче:

$$E_n = p_\alpha\dot\alpha - L = -4ma\cos\alpha\cdot g + \frac{m}{2}\left[4a^2\sin\dot\alpha + 2l^2\dot\alpha\right] - ma\sin\alpha\,\omega^2$$

$$E = \frac{f(\alpha)\cdot\dot\alpha^2}{2} - U(x)$$



раскладываем: $U = U_0 + \left.\frac{dU}{dx}\right|_{x=x_0}\Delta x + \frac{1}{2}\left.\frac{d^2U}{\partial x^2}\right|_{x=x_0}\Delta x^2$

$\Rightarrow$ можем получить ур-я колебаний

! в системе, где ЗСЭ не выполняется (кто-то действует на систему с пост силой), в качестве сохр энергии можно взять интеграл движения

№2

$$L = -mc^2\sqrt{1 - \frac{\dot x^2}{c^2}}$$

$$L = \frac{m}{2}\left(\dot z^2 + z^2\dot\varphi^2\right) - U(z) \Longrightarrow$$

$\frac{\partial L}{\partial t} = 0 \Rightarrow E$ – сохр-ся

$\frac{\partial L}{\partial \varphi} = 0 \Rightarrow M$ – сохр-ся (не зависит от угла поворота)

замена: $q\,\mathrm{ch}\,\lambda + \frac{\tau\,\mathrm{sh}\,\lambda}{c} = x$

$\tau\,\mathrm{ch}\,\lambda + \frac{q}{c}\mathrm{sh}\,\lambda = t$

БУСТ – поворот в пр-ве Минковского

$$\left(\frac{dx}{d\tau}\right)^2 = \mathrm{ch}\,\lambda\frac{dq}{d\tau} + c\,\mathrm{sh}\,\lambda\frac{d\tau}{d\tau} = \mathrm{ch}^2\lambda\left(\frac{dq}{d\tau}\right)^2 + c^2\mathrm{sh}^2\lambda$$

$$\left(\frac{dt}{d\tau}\right)^2 = \mathrm{ch}\,\lambda\frac{d\tau}{d\tau} + \frac{\mathrm{sh}\,\lambda}{c}\frac{dq}{d\tau} = \mathrm{ch}^2\lambda - \frac{1}{c^2}\mathrm{sh}^2\lambda\left(\frac{dq}{d\tau}\right)^2$$

$$S = \int L(\dot{x}, t)\,dt \to extr$$
$$= \int L'(\dot{q}, q, \tau)\,d\tau \to extr$$
$$\Rightarrow L'(q,\tau) = L(x,t)\cdot\frac{dt}{d\tau} \ominus$$

$$\ominus -mc^2\sqrt{1-\frac{1}{c^2}\left(\frac{dx}{dt}\right)^2}\left(\frac{dt}{d\tau}\right) = -mc^2\sqrt{\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2\cdot\left(\frac{dt}{d\tau}\right)^2} = -mc^2\sqrt{1-\frac{1}{c^2}\left(\frac{dq}{d\tau}\right)^2}$$

№3. движение в поле диполя

$$\bar{A} = \frac{[\bar{\mu}\times\bar{r}]}{r^3}$$

$$L = \frac{m\bar{v}^2}{2} - e\varphi(\bar{r}) + \frac{e}{c}\bar{v}\cdot\bar{A}$$

пусть $=0$

$$\mu \parallel Oz \Rightarrow \bar{v}\underbrace{[\bar{\mu}\times\bar{r}]}_{\parallel e_\varphi} = (v_z e_z + v_r e_r + r\dot{\varphi} e_\varphi)[\bar{\mu}\cdot\bar{r}] = r^2\mu\dot{\varphi}$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) + \frac{e}{c}\frac{\mu r^2\dot{\varphi}}{(r^2+z^2)^{3/2}}$$

1) $$\frac{\partial L}{\partial z} = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = m\ddot{z} = \frac{3}{2}\frac{e\mu\, r^2\dot{\varphi}\cdot 2z}{c(r^2+z^2)^{5/2}}$$

пусть: $\dot{z}(0) = 0$
$z(0) = 0$

в $t=0$, $\dot{z}=0 \Rightarrow \dot{z}(t+dt) = 0$

частный случай:

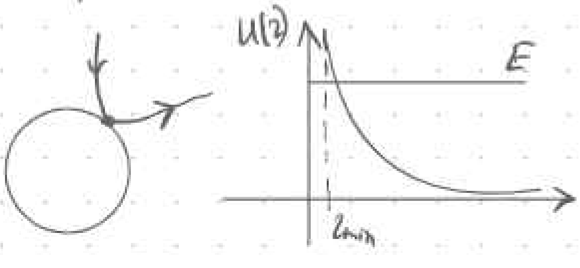
$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{e\mu\dot{\varphi}}{cr}$$

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = mr^2\dot{\varphi} + \frac{e\mu}{cr} = const \Rightarrow \dot{\varphi} = \frac{p_\varphi - \frac{e\mu}{cr}}{mr^2}$$

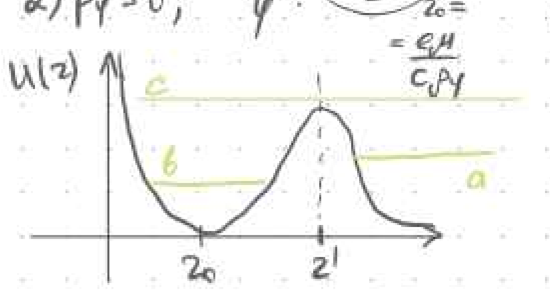
$$E = const = \frac{m\dot{r}^2}{2} + \frac{1}{2mr^2}\left(p_\varphi - \frac{e\mu}{cr}\right)^2$$

рисуем:

1) $p_\varphi < 0$; $\dot{\varphi} < 0$;



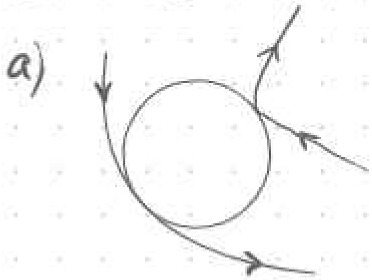
2) $p_\varphi > 0$; $\dot{\varphi}$: $r_0 = \frac{e\mu}{cp_\varphi}$;

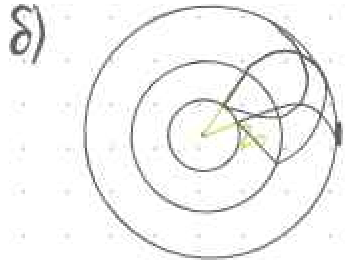


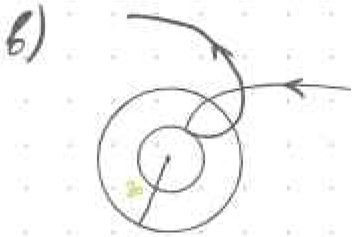
$$U' = \left(\frac{1}{2mz^2}\left(p_y - \frac{eM}{cz}\right)^2\right)' = \frac{-2}{2mz^3}\left(p_y - \frac{eM}{cz}\right)^2 + \frac{1}{2mz^2}\cdot 2\left(p_y - \frac{eM}{cz}\right)\cdot\frac{eM}{cz^2} =$$

$$= \left(p_y - \frac{eM}{cz}\right)\frac{1}{mz^2}\left(\frac{1}{z}\left(p_y - \frac{eM}{cz}\right) + \frac{eM}{cz^2}\right) = 0$$

$z' = 2z_0$

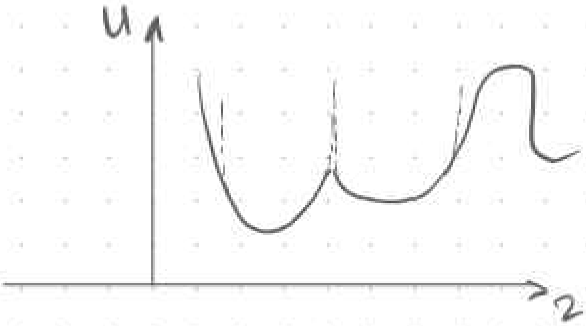




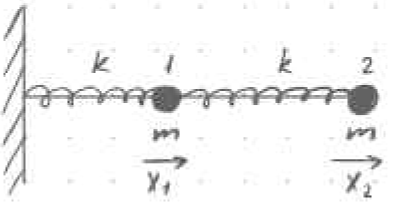


Колебания

14.03.23



№1.



1) $L = K - U = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2}(x_1^2 + (x_2 - x_1)^2)$

2) $m\ddot{x}_1 + k(2x_1 - x_2) = 0$
$m\ddot{x}_2 + k(x_2 - x_1) = 0$

$\omega_0^2 = \frac{k}{m}$

$x_i = a_i \cos(\omega t + \varphi)$: $(-\omega^2 + 2\omega_0^2)a_1 - \omega_0^2 a_2 = 0$
$-\omega_0^2 a_1 + (-\omega^2 + \omega_0^2)a_2 = 0$

$$\frac{a_2}{a_1} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} = \frac{\omega_0^2}{\omega_0^2 - \omega^2}$$

$$\det\begin{vmatrix} -\omega^2 + 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 \end{vmatrix} = 0$$

$$L = \frac{m}{2}\left[\dot{x}_1^2 + \dot{x}_2^2\right] - \frac{k}{2}\left[2x_1^2 - 2x_1x_2 + x_2^2\right]$$

$$L = \frac{\hat{m}_{ij}\dot{x}_i\dot{x}_j}{2} - \frac{\hat{k}_{ij}x_ix_j}{2}$$

$$(-\hat{m}\omega^2 + \hat{k}) = 0$$

3) $$\det\begin{vmatrix} -\omega^2 + 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 \end{vmatrix} = 0$$

$$(-\omega^2 + 2\omega_0^2)(-\omega^2 + \omega_0^2) - \omega_0^4 = 0$$

$$\omega^4 - 2\omega_0^2\omega^2 - \omega_0^2\omega^2 + 2\omega_0^4 - \omega_0^4 = 0$$
$$\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$

$$\omega^2 = t: \quad t^2 - 3t\omega_0^2 + \omega_0^4 = 0$$

$$D = 9\omega_0^4 - 4\omega_0^4 = 5\omega_0^4$$
$$t_1 = \frac{3\omega_0^2 + \sqrt{5}\omega_0^2}{2}$$
$$t_1 = \frac{3\omega_0^2 - \sqrt{5}\omega_0^2}{2}$$

$$\omega^2 = \omega_0^2 \cdot \left(\frac{3 \pm \sqrt{5}}{2}\right) \qquad \left(\begin{matrix} 1 \Rightarrow \text{,,}+\text{''} \\ 2 \Rightarrow \text{,,}-\text{''} \end{matrix}\right)$$

4) 1) $$\frac{a_2}{a_1} = \frac{1-\sqrt{5}}{2}$$

2) $$\frac{a_2}{a_1} = \frac{1+\sqrt{5}}{2}$$

$$\bar{z} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1 \underbrace{\begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}}_{z_1} \cos(\omega_1 t + \varphi_1) + A_2 \underbrace{\begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}}_{z_2} \cos(\omega_2 t + \varphi_2)$$

z_1 - вектора собств. колебаний - z_2

(*) собств. число за ω_0

5) $$z_1|M|z_2 = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} = 1 + \frac{1-5}{4} = 0$$

$$z_1\begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}z_2 = \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}^T = 0$$

Теорема: собств. век-ра ортогональны в пр-ве масс или жесткостей

№2.
$$x_1 = q_1 + q_2$$
$$x_2 = \frac{1-\sqrt{5}}{2}q_1 + \frac{1+\sqrt{5}}{2}q_2$$

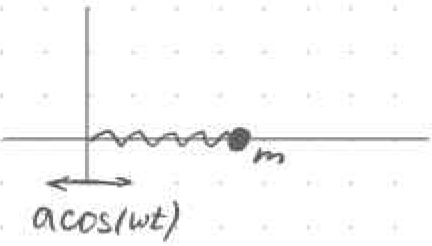
$$L = \frac{m}{2}\left(\dot x_1^2 + \dot x_2^2\right) - \overbrace{\frac{k}{2}\left(x_1^2 + (x_2 - x_1)^2\right)}^{①}$$

$$①\begin{cases} q_1^2\left(2 + \left(\frac{1-\sqrt5}{2}\right)\cdot 2 + (1+\sqrt5)\right) \\ q_1 q_2\left(4 + 2\left(\frac{1-\sqrt5}{2}\right)\left(\frac{1+\sqrt5}{2}\right) - 2\right) = 0 \\ q_2^2\left(2 + \left(\frac{1+\sqrt5}{2}\right)\cdot 2 + (1-\sqrt5)\right) \end{cases} \Rightarrow q_1^2(2 + 1 - \sqrt5 + 1 + \sqrt5) + q_2(2 + 1 + \sqrt5 + 1 - \sqrt5) = \\ = 4q_1^2 + 4q_2^2$$

$$L = \frac{m\dot q_1^2}{2}\cdot\frac{(5-\sqrt5)}{2} - \frac{\overset{m\omega_0^2}{k}\, q_1^2}{2}\cdot\frac{(5+\sqrt5)^2(5-\sqrt5)}{2\,(5-\sqrt5)(5+\sqrt5)}$$

$$L = \frac{m\cdot(5-\sqrt5)}{4}\left[\dot q_2^2 - q_2\omega_2^2\right] + \frac{1}{2}\cdot\frac{m}{2}\cdot(5+\sqrt5)\left(\dot q_2 - \omega_2^2 q_2\right)$$ – моды колебаний

№3.

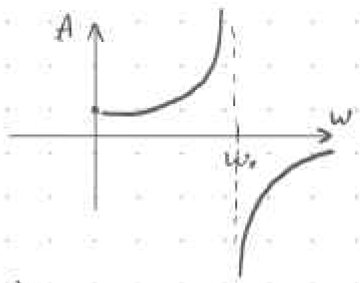


1) $m\ddot x = -k(x - a\cos\omega t)$

ищем в виде: $x = A\cos(\omega t) + B\sin(\omega_0 t + \varphi_0)$

$$-mA\omega^2 + kA = ka$$

$$A = a\,\frac{\omega_0^2}{\omega_0^2 - \omega^2}$$



$$L = \frac{m}{2}\left(\dot x_1^2 + \dot x_2^2\right) - \frac{k}{2}\left((x_1 - a(t))^2 + (x_2 - x_1)^2\right)$$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot x}$$

$$\frac{\partial L}{\partial x_1} = -\frac{k}{2}\left[2(x_1 - a(t)) - 2(x_2 - x_1)\right] = -k(x_1 - a(t)) + k(x_2 - x_1)$$

$$\frac{\partial L}{\partial x_2} = -k(x_2 - x_1)$$

$$\left[\begin{aligned} m\ddot x_1 &= -k(2x_1 - x_2 - a\cos(\omega t)) \\ m\ddot x_2 &= -k(x_2 - x_1) \end{aligned}\right. \quad \left|\quad \begin{aligned} \ddot x_1 &= -\omega_0^2(2x_1 - x_2 - a\cos(\omega t)) \\ \ddot x_2 &= -\omega_0^2(x_2 - x_1) \end{aligned}\right.$$

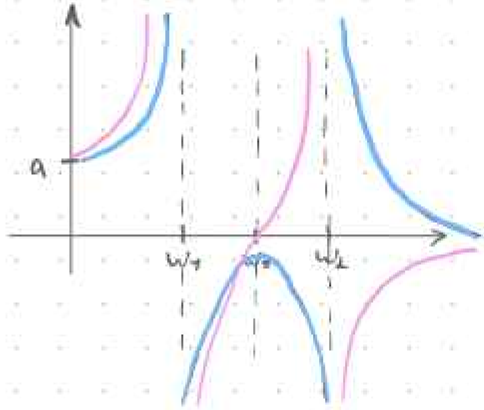
$x_1 = A_1\cos(\omega t)$: $\quad -\omega^2 A_1\cos\omega t = -\omega_0^2(2A_1\cos\omega t - A_2\cos\omega t - a\cos\omega t)$

$x_2 = A_2\cos(\omega t)$: $\quad -\omega^2 A_2\cos\omega t = -\omega_0^2(A_2\cos\omega t - A_1\cos\omega t)$

$$\begin{aligned} -\omega^2 A_1 &= -2A_1\omega_0^2 + \omega_0^2 A_2 + \omega_0^2 a \\ -\omega^2 A_2 &= -A_2\omega_0^2 + A_1\omega_0^2 \end{aligned} \Rightarrow -\omega^2(A_1 + A_2) = -\omega_0^2 A_1 + \omega_0^2 a$$

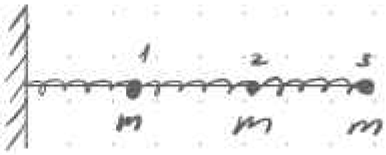
$$A_1 = \frac{a(w_0^2 - w^2)w_0^2}{-3w_0^2 w^2 + w_0^4 + w^4}$$

$$A_2 = \frac{A_1 w_0^2}{w^2 - w_0} = \frac{a w_0^4}{w^2 - w_0^2}$$



$$(w^2 - 2w_0^2)(w_0^2 - w^2) = w_0^2 w^2 - w^4 - 2w_0^4 + 2w_0^2 w^2$$

если:



собств. частоты: $\tilde{w}_1 = 3{,}24 w_0^2 \ (\tilde{w}_1 = 1{,}8 w_0)$

$\tilde{w}_2 = 1{,}98 w_0^2 \ (\tilde{w}_2 = 1{,}4 w_0)$

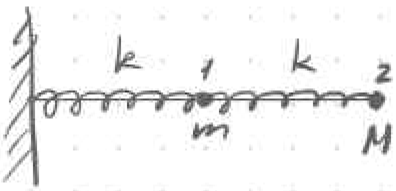
$\tilde{w}_3 = 1{,}55 w_0^2 \ (\tilde{w}_3 = 1{,}25 w_0)$

$$\tilde{A}_1 = \frac{(w^2 - w_1^2)(w^2 - w_2^2) w_0^2 a}{(\ )(\ )(\ )}$$

$$\tilde{A}_2 = \frac{a(w^2 - w_0^2) w_0^4}{(\tilde{w}_1^2 - w_0^2)(\tilde{w}_2^2 - w_0^2)}$$

$$\tilde{A}_3 = \frac{w_0^6 a}{(\tilde{w}_1^2 - w_0^2)(\tilde{w}_2^2 - w_0^2)(\tilde{w}_3^2 - w_0^2)}$$

N4.



$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$\begin{cases} m\ddot{x}_1 = -\frac{k}{2}\left(x_1^2 - (x_2 - x_1)^2\right) = -\frac{k}{2}(x_1^2 - x_2^2 + 2x_1x_2 - x_1^2) = -\frac{k}{2}(x_2^2 + 2x_1x_2) \\ M\ddot{x}_2 = -\frac{k}{2}(x_2 - x_1)^2 \end{cases}$$

$$w_1^2 = \frac{k(m + 2M) + k\sqrt{m^2 + 4M^2}}{2mM} = \frac{mk\left(1 + \frac{2M}{m}\right) + km\sqrt{1 + \frac{4M^2}{m^2}}}{2mM}$$

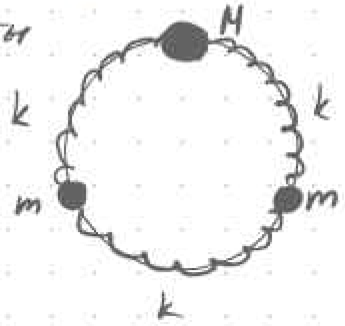
$$w_2^2 = \frac{k(m + 2M) - k\sqrt{m^2 + 4M^2}}{2mM}$$

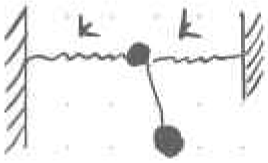
$M >> m$: $$w_1^2 = \frac{\cancel{M}k\left(\frac{m}{M} + 2\right) + k\cancel{M}\left(\sqrt{\frac{m^2}{M^2} + 4}\right)}{2m\cancel{M}} = \frac{2k}{m}$$

$$w_2^2 = \frac{\cancel{M}k\left(\frac{m}{M} + 2\right) - k\cancel{M}\left(\sqrt{\frac{m^2}{M^2} + 4}\right)}{2m\cancel{M}} \to 0$$

$M << m$: $\dfrac{\cancel{m}k\left(1+\frac{2M}{m}\right)+k\cancel{m}\sqrt{1+\frac{4M^2}{m^2}}}{2\cancel{m}M} = \dfrac{k}{2M}$

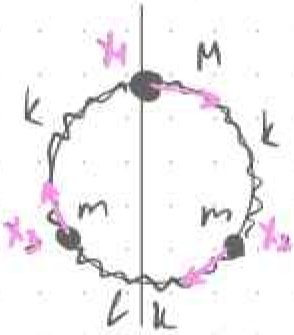
Д/з: найти частоты



№7. (правка) 

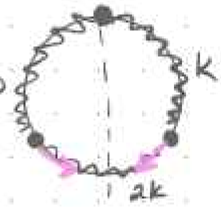
21.03.23

№1.



1) симм-я отн-но поворота на π вокруг l $\Rightarrow$ собств. в-ра $\hat S_l$

2) $\hat S_l \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_3 \\ -x_2 \end{pmatrix} \Rightarrow \left[\begin{array}{l} \hat S z_s = z_s \text{ - сим-ни} \\ S z_a = -z_a \text{ - антисимм-ни} \end{array} \right.$

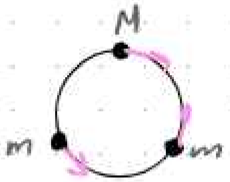
3) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_3 \\ -x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = z_1 \Rightarrow$ 

$\omega_1^2 = \frac{3k}{m}$

4) $\begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_3 \\ -x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  $\omega_0^2 = 0$

сим-м ортог-ть в метрике масс : $\hat z_2 M z_0 = \begin{pmatrix} \alpha \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} M & & \\ & m & \\ & & m \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$



$\alpha M + m + m = 0 \Rightarrow \alpha = -\frac{2m}{M}$

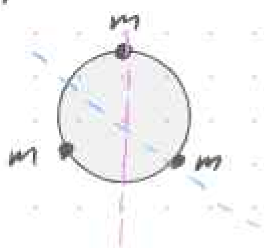
(сог-ся ЗСИ)

$\omega_2^2 = \frac{2k}{\mu} = \frac{k(m+M)}{m \cdot M}$

5) $z(t) = (C_1 t + C_2) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + A \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos\left(\sqrt{\frac{3k}{m}} t + \varphi_A \right) + B \begin{pmatrix} -\frac{2m}{M} \\ 1 \\ 1 \end{pmatrix} \cos\left(\sqrt{\frac{k(2m+M)}{mM}} t + \varphi_B \right)$

если дано 6 нач. усл: $\bar z_0 (t=0)$, $\bar v_0 (t=0)$

6) если $M = m$:



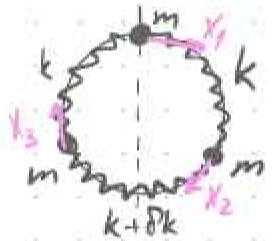
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \underset{\tilde z_1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}, \underset{\tilde z_2}{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$

собств. числа одинаковые $\Rightarrow$ вырождение

$\tilde z_1 = \frac{1}{2}(z_1 - z_2)$, $\tilde z_2 = -\frac{1}{2}(z_1 + z_2)$

№2.



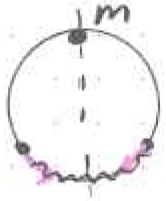
$\dot{\bar{z}}\big|_{t=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\bar{z}\big|_{t=0} = \begin{pmatrix} -a \\ 0 \\ a \end{pmatrix}$

если $z|_{t=0} \neq 0 \Rightarrow \cos$
если $v|_{t=0} \neq 0 \Rightarrow \sin$

$z(t) = ?$

1) ищем собств-ые вектора: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_3 \\ -x_2 \end{pmatrix}$

2) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_3 \\ -x_2 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 1 \\ x_3 = -1 \end{matrix} \Rightarrow z_1 :$



$\omega_1^2 = \dfrac{3k + 2\delta k}{m}$

$$z(t) = (Ct + C_0)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + A\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\cos\Bigg(\underbrace{\sqrt{\frac{3k+2\delta k}{m}}}\,t + \varphi_A\Bigg) + B\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\cos\Bigg(\underbrace{\sqrt{\frac{3k}{m}}}_{\omega_0}\,t + \varphi_B\Bigg)$$

$\omega_1 = \frac{3k}{m}\left(1 + \frac{2}{3}\frac{\delta k}{k}\right) = \omega_0\left(1 + \frac{\delta k}{k}\right)$

3) $C, \varphi_A, \varphi_B = 0$

4) $\begin{cases} -a = C_0 + A \cdot 0 - 2B \\ 0 = C_0 + A + B \\ a = C_0 - A + B \end{cases} \Rightarrow \begin{matrix} C_0 = 0 \\ B = a/2 \\ A = -a/2 \end{matrix}$

$$z(t) = \frac{a}{2}\left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\cos(\omega_1 t) + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\cos(\omega_0 t)\right]$$

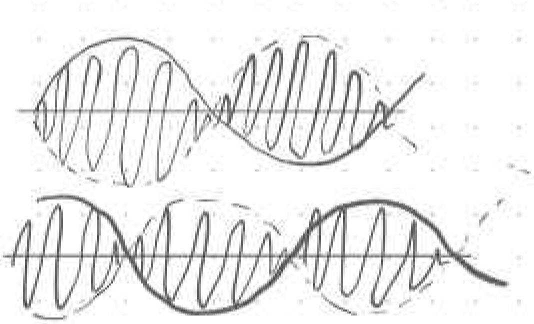
$x_1 = -a\cos(\omega_0 t)$, $x_2 = \frac{a}{2}\left(\cos(\omega_0 t) - \cos(\omega_1 t)\right)$, $x_3 = \frac{a}{2}\left(\cos(\omega_0 t) + \cos(\omega_1 t)\right)$

__вывод:__ $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

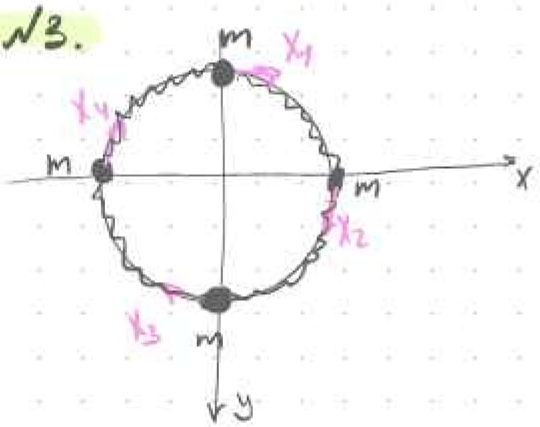
$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$x_2 \simeq a\sin(\omega_0 t)\sin\left(\frac{\delta k \cdot \omega_0 t}{6k}\right)$

$x_3 \simeq a\cos(\omega_0 t)\cos\left(\frac{\delta k\, \omega_0 t}{6k}\right)$



№3.



$\hat{S}_y$, $\bar{S}_x$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

$\hat{S}_x \bar{z} = \begin{pmatrix} -x_3 \\ -x_2 \\ -x_1 \\ -x_4 \end{pmatrix}$; $\hat{S}_y \bar{z} = \begin{pmatrix} -x_1 \\ -x_4 \\ -x_3 \\ -x_2 \end{pmatrix}$

$Sim\,S_x\,Asim\,S_y$

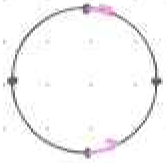
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_2 \\ -x_1 \\ -x_4 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_4 \\ -x_3 \\ -x_2 \end{pmatrix}$$

нет омеги

$Asim\,S_y\,sim\,S_x$

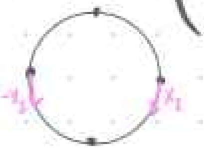
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_2 \\ -x_1 \\ -x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\omega_2^2 = \frac{2k}{m}$$



$Asim\,S_x\,sim\,S_y$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_4 \\ -x_3 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$



$$\omega_1 = \frac{2k}{m}$$

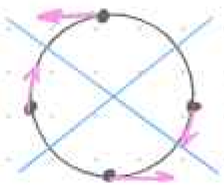
$Asim\,S_x\,Asim\,S_y$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 1 \\ \alpha \\ 1 \end{pmatrix}$$

$\alpha = 1: \ \omega_0 = 0$

$$\begin{pmatrix} \alpha \\ 1 \\ \alpha \\ 1 \end{pmatrix} \begin{pmatrix} m & & & \\ & m & & \\ & & m & \\ & & & m \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$\alpha m + m + \alpha m + m = 0$$
$$2\alpha m + 2m = 0 \Rightarrow \alpha = -1$$

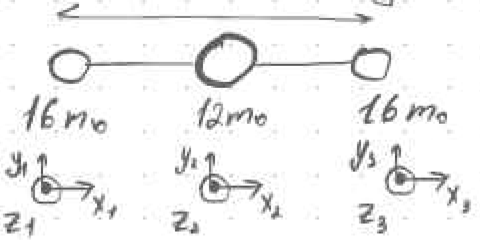


$$\omega_3^2 = \frac{4k}{m}$$

$$x(t) = (Ct + C_0)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + A\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\cos\left(\sqrt{\frac{2k}{m}}\,t + \varphi_A\right) + B\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\cos\left(\sqrt{\frac{2k}{m}}\,t + \varphi_B\right) +$$
$$+ D\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}\cos\left(\sqrt{\frac{4k}{m}}\,t + \varphi_D\right)$$

N 4.

O – C – O



$$\hat{S}_L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_2 \\ -x_3 \end{pmatrix}$$

$S_x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \sim \omega_1^2$

$\begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ → трансляция по x

$\begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ -8/3 \\ 1 \end{pmatrix} - \omega_2^2$

$$\hat{S}_L\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_3 \\ y_2 \\ y_1 \end{pmatrix}$$

$S_y = \begin{pmatrix} 1 \\ 9 \\ 1 \end{pmatrix}$

$aS_y = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ – вращение вокруг Oz

$$\hat{S}_L\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -z_3 \\ -z_2 \\ -z_1 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ – вращение вокруг Oy

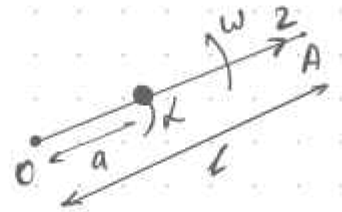
$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ – трансляция по z

$\begin{pmatrix} -1 \\ 8/3 \\ 1 \end{pmatrix}$

если $E = E_0 \cos(\omega t - kx)$, $\lambda \gg d$

Подготовка к контрольной:

№1:



1) $\omega = const \Rightarrow$ есть внешняя сила $\Rightarrow$ ЗСЭ не сохр-ся

2) $\bar{v} = \dot{z} + \bar{z}\dot{\varphi}$

3) $U = 0$

$K = \dfrac{m(\dot{z}^2 + z\dot{\varphi}^2)}{2}$

4) $L = \dfrac{m(\dot{z}^2 + (z\omega)^2)}{2}$

$\dfrac{\partial L}{\partial t} = 0 \Rightarrow E_\Lambda = p_z \dot{z} - L = const \Rightarrow \dfrac{m\dot{z}^2}{2} - \dfrac{z^2\omega^2 m}{2} = E_\Lambda$

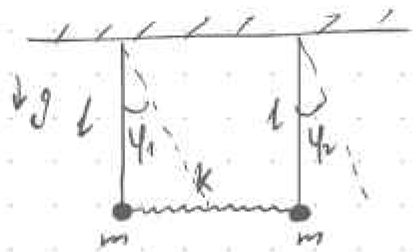
5) $\dot{z} = \dfrac{2}{m}\sqrt{E_\Lambda + \dfrac{z^2\omega^2 m}{2}}$, $E_\Lambda = -\dfrac{m\omega^2 a^2}{2}$

$\dot{z} = \omega\sqrt{z^2 - a^2}$

$\dfrac{m\dot{z}^2}{2} - \dfrac{ml^2\omega^2}{2} = -\dfrac{ma^2\omega^2}{2}$

$\dot{z}^2 = (l^2 - a^2)\omega^2 \Rightarrow v^2 = \dot{z}^2 + \omega^2 l^2 = (2l^2 - a^2)\omega^2$

№4.



$L = \dfrac{m}{2}(\dot{\varphi}_1^2 + \dot{\varphi}_2^2)\cdot l^2 - mg(2 - \cos\varphi_1 - \cos\varphi_2)$

$- \dfrac{k(l\sin\varphi_2 - l\sin\varphi_1)^2}{2} = \left[\cos\varphi = 1 - \dfrac{\varphi^2}{2}\right]$

$= \dfrac{ml^2}{2}(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) - mgl\left(\dfrac{\varphi_1^2 + \varphi_2^2}{2}\right) - \dfrac{k}{2} l^2(\varphi_2 - \varphi_1)^2$

$M = \begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}$, $K = \begin{pmatrix} mgl + kl^2 & -kl^2 \\ -kl^2 & mgl + kl^2 \end{pmatrix}$

$\omega_1^2 = \dfrac{g}{l}$, $\omega_2^2 = \dfrac{2k}{m} + \dfrac{g}{l}$

№5

$$U(x) = V\cos(\alpha x) + F\cdot x\ , \quad |F| \le |V\cdot\alpha|$$

$$\omega^2 = \frac{k}{m}$$

$$U(x)\big|_{x=x_0} = U(x_0) + U'(x_0)\cdot x + \frac{\overbrace{U''(x_0)}^{=k}(x-x_0)^2}{2} + o((x-x_0)^2)$$

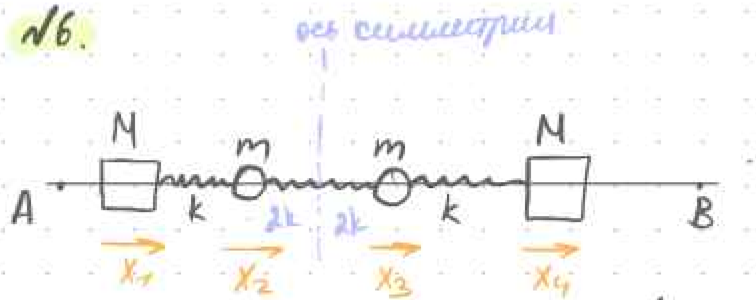
$$U'(x) = -V\sin\alpha x\cdot\alpha + F = 0 \Rightarrow \sin\alpha x = \frac{F}{\alpha V}$$

$$U''(x) = -\alpha^2 V\cos\alpha x = -\alpha^2 V\sqrt{1-\sin^2\alpha x} = -\alpha^2 V\sqrt{1-\frac{F^2}{(\alpha V)^2}} \approx -\alpha^2 V\left(1-\frac{F^2}{(2\alpha V)^2}\right) =$$

$$= -\alpha^2 V + \frac{F^2\alpha^2 V}{2\alpha^2 V^2} = -\alpha^2 V + \frac{F^2}{2V}$$

$$\omega^2 = \frac{\frac{F^2}{2V} - \alpha^2 V}{m} = \frac{F^2 - 2\alpha^2V^2}{2Vm} \Rightarrow \omega = \sqrt{\frac{F^2-2\alpha^2V^2}{2Vm}}$$

№6.



1) симметричное преобразование:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_4 \\ -x_3 \\ -x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -\alpha \\ \alpha \\ 1 \end{pmatrix}$$

$$L = \frac{M\dot x_1^2}{2} + \frac{m\dot x_2^2}{2} - \frac{k(x_1-x_2)^2}{2} - \frac{2kx_2^2}{2}$$

$$\left[\begin{aligned} &M\ddot x_1 + \frac{k}{2}\cdot 2(x_1-x_2) = 0 \\ &m\ddot x_2 + 2kx_2 - \frac{k}{2}\,2(x_1-x_2) = 0 \end{aligned}\right. \Rightarrow \left[\begin{aligned} &M\ddot x_1 + kx_1 - kx_2 = 0 \\ &m\ddot x_2 + 2kx_2 - kx_1 + kx_2 = m\ddot x_2 - kx_1 + 3kx_2 = 0 \end{aligned}\right.$$

$$k = \begin{pmatrix} k & -k \\ -k & 3k \end{pmatrix},\ M = \begin{pmatrix} M & 0 \\ 0 & m \end{pmatrix} \Rightarrow \begin{vmatrix} k-\omega^2 M & -k \\ -k & 3k-\omega^2 m \end{vmatrix} = 0$$

$$(k-\omega^2M)(3k-\omega^2m) - k^2 = 3k^2 - 3k\omega^2M - \omega^2mk + \omega^4mM - k^2 = \omega^4mM - \omega^2k(m+3M) + 2k^2$$

$$D = k^2(m+3M)^2 - 8k^2mM = k^2m^2 + 9M^2k^2 - 2k^2mM$$

$$\omega^2_{1,2} = \frac{k(m+3M) \pm \sqrt{k^2m^2 + 9M^2k^2 - 2k^2mM}}{2mM}$$

$$\left(\begin{array}{cc|c} k-\omega^2M & -k & 0 \\ -k & 3k-\omega^2m & 0 \end{array}\right) \Rightarrow \left[\begin{aligned} &x_2 = \frac{k-M\omega^2}{k}x_1 \\ &x_1 \in R \end{aligned}\right.$$

$$U_1 = \left(-1 \quad \frac{k - Mw_1^2}{k} \quad \frac{Mw_1^2 - k}{k} \quad 1\right)^T, \quad w_1^2 = \frac{k(m+3M) + \sqrt{k^2m^2 + 9M^2k^2 - 2k^2mM}}{2mM}$$

$$M \gg m: \quad w_1^2 = \frac{k\left(\frac{m}{M} + 3\right) + \sqrt{k^2\frac{m^2}{M^2} + 9k^2 - 2k^2\frac{mM}{M^2}}}{2m} \xrightarrow{M\to\infty} \frac{3k+3k}{2m} = \frac{6k}{2m} = \frac{3k}{m}$$

$$M \gg m: \quad w_2^2 = \frac{k\left(\frac{m}{M} + 3\right) - \sqrt{k^2\frac{m^2}{M^2} + 9k^2 - 2k^2\frac{mM}{M^2}}}{2m} = \frac{k\left(\frac{m}{M}+3\right)}{2m} - \frac{3k\sqrt{1 - \frac{2}{9}\frac{m}{M}}}{2m} =$$

$$= \frac{k}{2M} + \frac{3k}{2m} - \frac{3k}{2m} + \frac{3k}{2m}\cdot\frac{1}{2}\cdot\frac{2}{9}\frac{m}{M} = \frac{k}{2M} + \frac{1k}{6M} = \frac{3k+k}{6M} = \frac{4k}{6M} = \frac{2k}{3M}$$

$$U_2 = \left(-1 \quad \frac{k - Mw_2^2}{k} \quad \frac{Mw_2^2 - k}{k} \quad 1\right)^T, \quad w_2^2 = \frac{k(m+3M) - \sqrt{k^2m^2 + 9M^2k^2 - 2k^2mM}}{2mM}$$

$$m \gg M: \quad w_1^2 = \frac{k\left(1 + \frac{3M}{m}\right) + \sqrt{k^2 + 9k^2\frac{M^2}{m^2} - 2k^2\frac{Mm}{m^2}}}{2M} \to \frac{2k}{2M} = \frac{k}{M}$$

$$m \gg M: \quad w_2^2 = \frac{k\left(1 + \frac{3M}{m}\right) - \sqrt{k^2 + 9k^2\frac{M^2}{m^2} - 2k^2\frac{M}{m}}}{2M} = \frac{k\left(1+\frac{3M}{m}\right)}{2M} - \frac{k\sqrt{1 - 2\frac{M}{m}}}{2M} =$$

$$= \frac{k}{2M} + \frac{3k}{m} - \frac{k}{2M}\left(1 + \frac{1}{2}\cdot 2\frac{M}{m}\right) = \frac{k}{2M} + \frac{3k}{m} - \frac{k}{2M} - \frac{k}{m} = \frac{2k}{m}$$

3) антисимметричное преобразование:

$$\begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \\ -x_4 \end{pmatrix} = \begin{pmatrix} -x_4 \\ -x_3 \\ -x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \beta \\ \beta \\ 1 \end{pmatrix}; \qquad \text{при } \beta = 1: \quad U_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad w_3^2 = 0$$

$$(1\ \beta\ \beta\ 1)\begin{pmatrix} M & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & M \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow \beta = -\frac{M}{m} \Rightarrow U_4 = \begin{pmatrix} 1 \\ -M/m \\ -M/m \\ 1 \end{pmatrix}, \quad w_4^2 = \frac{k}{\mu} = \frac{k(m+M)}{mM}$$

4) если $M = m = m$

$$w_{1,2}^2 = \frac{k(m+3M) \pm \sqrt{k^2m^2 + 9M^2k^2 - 2k^2mM}}{2mM}$$

$$w_{1,2}^2 = \frac{k\cdot 4m \pm \sqrt{k^2m^2 + 9m^2k^2 - 2k^2m^2}}{2m^2} = \frac{4mk \pm 2\sqrt{2}mk}{2m^2} = \frac{k}{m}(2 \pm \sqrt{2})$$

$w_3 = 0$

$w_4 = \frac{2k}{m}$

$$\varphi = \frac{\pi S}{N} \Rightarrow \varphi_0 = 0,\ \varphi_1 = \frac{\pi}{4},\ \varphi_2 = \frac{\pi}{2};\ \varphi_3 = \frac{3\pi}{4}$$

$$\varphi_0 = 0 \Rightarrow \omega_1 = 2\tilde{\omega}\sin 0 = 0 \Rightarrow U_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\varphi_1 = \frac{\pi}{4} \Rightarrow \omega_2 = 2\tilde{\omega}\sin\frac{\pi}{8} = 2\tilde{\omega}\,\frac{\sqrt{2-\sqrt{2}}}{2} = \tilde{\omega}\sqrt{2-\sqrt{2}}$$

$$\varphi_2 = \frac{\pi}{2} \Rightarrow \omega_3 = 2\tilde{\omega}\sin\frac{\pi}{4} = 2\tilde{\omega}\cdot\frac{\sqrt{2}}{2} = \sqrt{2}\,\tilde{\omega}$$

$$\varphi_3 = \frac{3\pi}{8} \Rightarrow \omega_4 = 2\tilde{\omega}\,\frac{\sqrt{2+\sqrt{2}}}{2} = \tilde{\omega}\sqrt{2-\sqrt{2}}$$

Нелинейные колебания: 28.03.22

№8.1.

$$U = \frac{m\omega^2 x^2}{2} + \delta U$$

$$\delta U = \frac{m x^4 \beta}{4}$$

$$L = \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2} - \frac{m x^4 \beta}{4}$$

$$\ddot{x} + \omega_0^2 x = -x^3\beta\ ,\quad a^2\beta \ll \omega_0^2$$

$$x_0 = a\cos(\omega t + \varphi_0) = \frac{Ae^{i\omega t} + A^* e^{-i\omega t}}{2}\ ,\ \text{где}\quad A = \frac{a}{2}e^{i\varphi_0}$$

$$x(t) = x_0 + \delta x$$

$$\ddot{x}_0 + \ddot{\delta x} + \omega_0^2 x + \omega_0^2\delta x = -\beta x_0^3$$

$$(Ae^{i\omega t} + A^* e^{-i\omega t})^3 = A^3 e^{3i\omega t} + \overbrace{3A^2A^* e^{i\omega t} + 3A^{*2}Ae^{-i\omega t}}^{3A^2(Ae^{i\omega t} + Ae^{-i\omega t})} + (A^*)^3 e^{-3i\omega t} = \frac{a^3}{4}\cos(3\omega t + \varphi_0) + 3|A|^2 a\cos(\omega t + \varphi_0)$$

$$= -\beta\left(\frac{a^3}{4}\cos 3(\omega t+\varphi_0)\right) + \frac{3a^2}{4}a\cos(\omega t+\varphi_0))$$

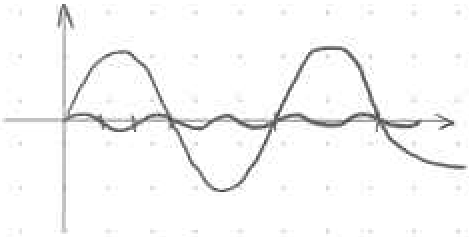
$$= -a\omega^2 + \omega_0^2 a = -\frac{\beta a^3 \cdot 3}{4} \Rightarrow \omega^2 = \omega_0^2 + \frac{3\beta a^2}{4}$$ – подкручивание частоты

$$\omega^2 = \omega_0^2 + 3\beta|A|^2$$

$$\delta x = b\cos(3\omega t + 3\varphi_0) \Rightarrow b - 9\omega^2 b + \omega_0 b = -\frac{a^3\beta}{4}$$

$$b = \frac{\beta a^3}{32\omega_0^2}$$

$$x(t) = a\cos(\omega t+\varphi) + \frac{\beta a^3}{32\omega_0^2}\cos(3(\omega t+\varphi_0))$$



№2.

$$\delta U = \frac{m\alpha x^3}{3}$$

$$L = \frac{m\dot{x}^2}{2} - \frac{m\omega_0^2 x^2}{2} - \frac{m\alpha x^3}{3}$$

$$\ddot{x} + \omega_0 x = -\alpha x^2$$

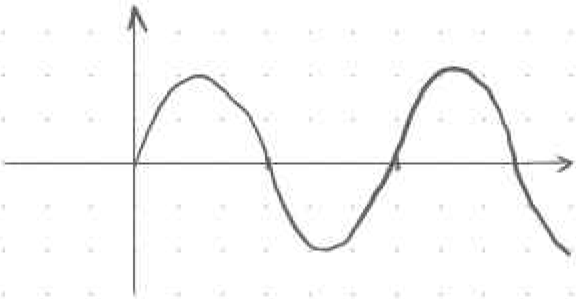
$$x(t) = x_0 + \delta x \Rightarrow \ddot{x}_0 + \delta\ddot{x} + \omega_0^2 x_0 + \delta x_0 \omega_0^2 = -\alpha x^2 = -\alpha\left(\frac{1+\cos 2x}{2}\right)$$

$$x_0 = a\cos(\omega t+\varphi)$$

$$\delta x = b\cdot\cos 2(\omega t+\varphi_0) + C$$

$$b = -\frac{a^2\alpha}{3\omega_0^2}, \quad C = -\frac{\alpha a^2}{2\omega_0^2}$$

$$x(t) = a\cos(\omega_0 t+\varphi_0) + \frac{\alpha a^2}{6\omega_0^2}\cos 2(\omega_0 t+\varphi_0) - \frac{\alpha a^2}{2\omega_0^2}$$



Д/З: для этого потенциала второй член

Подготовка к контрольной

№1. Траектория: $-1, 2$ – эллипс

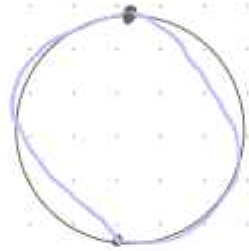
$$U(r) = \frac{\tilde{k} r^2}{2}, \quad \frac{\omega_r}{\langle \dot\varphi \rangle} = ?$$

$$U_{эфф} = \frac{M^2}{2mr^2} + \frac{kr^2}{2}$$

$$\left.\frac{dU_{эф}}{dr}\right|_{r_0} = 0 = \frac{-\cancel{2}M^2}{\cancel{2}mr^3} + \frac{\cancel{2}\tilde{k}r}{\cancel{2}} = 0 \Rightarrow -2M^2 + 2krmr^3 = 0$$

$$r^4 km = M^2 \Rightarrow r_0 = \sqrt[4]{\frac{M^2}{\tilde{k}m}} =$$

$$\left.\frac{d^2U_{эф}}{dr^2}\right|_{r_0} = k = \frac{3M^2}{mr_0^4} + \tilde{k}$$



$$\omega^2 = \frac{k}{m} = \frac{3M^2 + \tilde{k}mr_0^4}{m^2 r_0^4}$$

$$\dot\varphi = \frac{M}{mr_0^2}$$

$$\frac{\omega}{\dot\varphi} = \sqrt{\frac{\frac{3M^2 + \tilde{k}mr_0^4}{m^2r_0^4} \cdot m^2 r_0^4}{M^2}} = \sqrt{\frac{3M^2 + \tilde{k}mr_0^4}{M^2}} = \sqrt{3 + \frac{\tilde{k}mr_0^4}{M^2}} = \sqrt{3 + \frac{r_0^4}{r_0^4}} = 2$$

№2.

$$U(r) = -\frac{\beta}{r^2} \qquad \xrightarrow{E}$$

условие падения: $E \le U_{эфф}(R)$

$$U_{эфф} = \frac{M^2}{2mr^2} - \frac{\beta}{r^2}$$

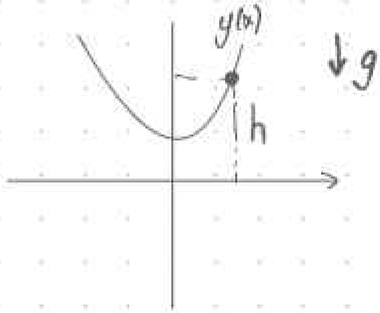
$$M^2 = m^2 v_\infty^2 \rho^2 = 2mE\rho^2$$

$$\frac{2mE\rho^2}{2mR^2} - \frac{\beta}{r^2} = E \Rightarrow E\rho^2 - \beta = ER^2$$

$$\rho^2 = \frac{ER^2 + \beta}{E}$$

$$\sigma = \pi\rho^2 = \pi\left(R^2 + \frac{\beta}{E}\right)$$

№3.



$y = a\cdot ch\left(\frac{x}{b}\right)$

$K = \frac{m\dot{x}^2}{2}$, $U = mgy = mgach\left(\frac{x}{b}\right)$

$U'(x) = \frac{mgash\left(\frac{x}{b}\right)}{b}$

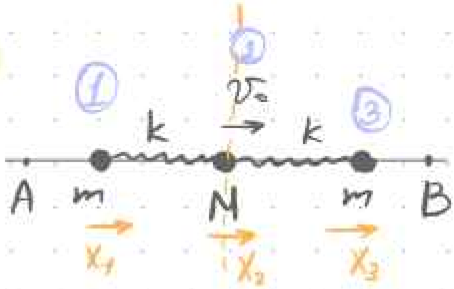
$U''(x) = \frac{mga\cdot ch\left(\frac{x}{b}\right)}{b\cdot b}$

$U''(0) = \frac{mga}{b^2}\cdot ch(0) = \frac{mga}{b^2}$

$\omega^2 = \frac{k}{m} = \frac{mga}{b^2} = \frac{ag}{b^2}$

$L(x,\dot{x}) = \frac{m\dot{x}^2}{2} + mgach\left(\frac{x}{b}\right)$

№4.



$\bar{\dot{z}}\big|_{t=0} = \begin{pmatrix} 0 \\ v_0 \\ 0 \end{pmatrix}$, $\bar{z}\big|_{t=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

1) симметричное преобр-е: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ -\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\omega^2 = \frac{k}{m}$

антисимм-е преобр-е: $\begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \alpha \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\omega^2 = 0$

$\begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$, $\omega^2 = \frac{k(2m+M)}{mM}$

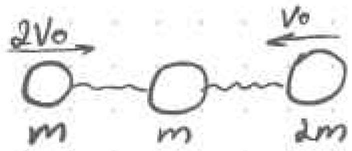
$\begin{pmatrix} 1 & \beta & 1 \end{pmatrix} \begin{pmatrix} m & & \\ & M & \\ & & m \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (0)$

$m + \beta M + m = 0 \Rightarrow 2m + \beta M = 0 \Rightarrow \beta = -\frac{2m}{M}$

$$x(t) = (C_1 t + C_2)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + A\sin\left(\sqrt{\frac{k}{m}}\,t + \varphi_1\right)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + B\sin\left(\sqrt{\frac{(2m+M)k}{mM}}\,t + \varphi_2\right)\begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$

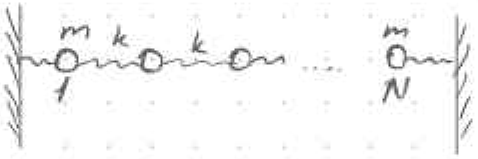
$$x(t) = \frac{v_0}{\left(1+\frac{2m}{M}\right)}\,t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{v_0}{\sqrt{\frac{(2m+M)k}{mM}}\left(\frac{2m}{M}+1\right)}\sin\left(\sqrt{\frac{(2m+M)k}{mM}}\,t\right)\begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$

1) повт-ть $\frac{d\sigma}{d\omega}$



Цепочки

4.04.23.



$$L = \sum \frac{m\dot{x}_i^2}{2} - \left\{ \frac{kx_1^2}{2} + \sum_{i=1}^{N-1} \frac{k(x_{i+1}-x_i)^2}{2} + \frac{kx_N^2}{2} \right.$$

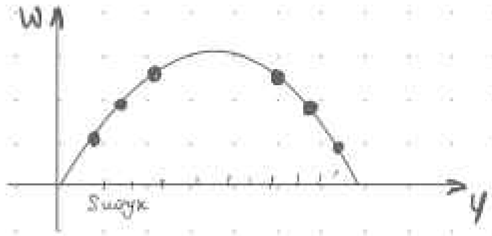
$$\begin{cases} m\ddot{x}_1 = -k(2x_1 - x_2) \\ \dots \\ m\ddot{x}_i = -k(2x_i - x_{i-1} - x_{i+1}), \quad i = 2, N-1 \\ \dots \\ m\ddot{x}_N = -k(2x_N - x_{N-1}) \end{cases} \Rightarrow \begin{cases} m\ddot{x}_i = -k(2x_i - x_{i-1} - x_{i+1}), \quad i = 1, \dots, N \quad (*) \\ x_0 = 0 \\ x_{N+1} = 0 \end{cases}$$

$$\omega_0^2 = \frac{k}{m}$$

ищем в виде: $x_n = A^+ e^{i(\omega t + n\psi)} + A^- e^{i(\omega t - n\psi)}$

$(*) x_n m \left[\omega^2 = \omega_0^2 (2 - (e^{i\psi} + e^{-i\psi})) \right] \Rightarrow \omega^2 = \omega_0^2 (1 - \cos\psi) \cdot 2 = \omega_0^2 \cdot 4 \sin^2\frac{\psi}{2}$

• если $x_0 = 0$: $A^+ + A^- = 0 \Rightarrow x_n = Ae^{i\omega t}(e^{in\psi} - e^{-in\psi}) = 2iAe^{i\omega t}\sin n\psi$



$$\sum_s \begin{pmatrix} \sin\psi_s \\ \sin 2\psi_s \\ \vdots \\ \sin N\psi_s \end{pmatrix} \cdot a_s \cos(\omega_s t + \varphi_s)$$

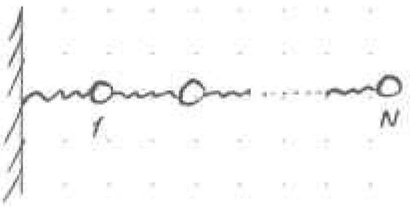
, S - номер собств. колебания

• если $x_{N+1} = 0$: $\sin(N+1)\psi = 0 \Rightarrow \psi = \frac{\pi s}{N+1}$, $S = 1, \dots, N$

• $\omega_s = 2\omega_0 \sin\frac{\psi_s}{2}$

N2.

гран. усл: $x_{N+1} = x_N$ (нет деформации, не принимает участия в движении)



$x_n = 2iAe^{i\omega t}\sin n\psi$

если $x_{N+1} = x_N$: $\sin(N+1)\psi = \sin N\psi \Rightarrow \sin(N+1)\psi - \sin N\psi = 2\cos\frac{2N+1}{2}\sin\frac{\psi}{2} = 0$

$\cos\left(\frac{2N+1}{2}\right)\psi = 0 \Rightarrow \psi\frac{2N+1}{2} = \frac{\pi}{2} + \pi s \Rightarrow \psi_s = \frac{\pi(2s+1)}{2N+1}$, $S = 0, \dots, N-1$

$\sin\frac{\psi}{2} = 0 \Rightarrow \psi = 0$ (вырожд. в-р)

если $N = 2$: $\omega_1 = 2\omega_0 \sin\frac{\pi(2+1)}{2(2\cdot 2+1)} = 2\omega_0 \sin\frac{3\pi}{25} \approx 1{,}6\omega_0$; $\omega_2 = 2\omega_0 \sin\frac{\pi}{10} \approx 0{,}6\omega_0$

$$\omega_1^2 = \omega_0^2\left(\frac{3+\sqrt{5}}{2}\right) \approx 1,61\,\omega_0;\quad \omega_2^2 = \omega_0^2\left(\frac{3-\sqrt{5}}{2}\right) \approx 0,61\,\omega_0$$

№3.

$$\begin{cases} x_0 = x_1 \\ x_{N+1} = x_N \end{cases}$$

$$x_n = A^+ e^{i(\omega t + n\psi)} + A^- e^{i(\omega t - n\psi)}$$

$$x_0 = x_1:\quad A^+ \cancel{e^{i\omega t}} + A^- \cancel{e^{i\omega t}} = A^+ \cancel{e^{i\omega t}} e^{i\psi} + A^- \cancel{e^{i\omega t}} e^{-i\psi}$$

$$A^+(1 - e^{i\psi}) + A^-(1 - e^{-i\psi}) = 0 \Rightarrow A^+(1 - e^{i\psi}) = A^-(-1 + e^{-i\psi})$$

$$x_{N+1} = x_N:\quad A^+ e^{i(\omega t + (N+1)\psi)} + A^- e^{i(\omega t - (N+1)\psi)} = A^+ e^{i(\omega t + N\psi)} + A^- e^{i(\omega t - N\psi)}$$

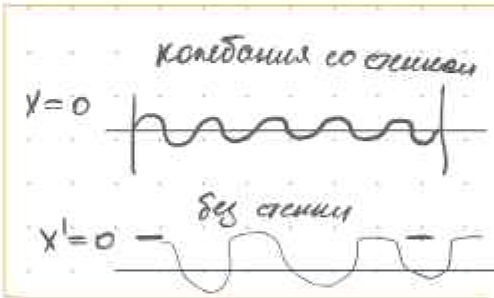
$$A^+ e^{i(N+1)\psi} + A^- e^{-i(N+1)\psi} = A^+ e^{iN\psi} + A^- e^{-iN\psi}$$

$$A^+ e^{iN\psi} e^{i\psi} + A^- e^{-iN\psi} e^{-i\psi} = A^+ e^{iN\psi} + A^- e^{-iN\psi}$$

$$A^+ e^{iN\psi}(e^{i\psi} - 1) + A^- e^{-iN\psi}(e^{-i\psi} - 1) = 0$$

$$A^+ e^{iN\psi}(e^{i\psi} - 1) + e^{-iN\psi} A^+ (1 - e^{i\psi}) = 0 \Rightarrow A^+ = \ldots\ldots \text{ (надо посчитать)}$$

2 способ: сдвинем n на $\frac{1}{2}$



$$x_n = A^+ e^{i(\omega t + (n - \frac{1}{2})\psi)} + A^- e^{i(\omega t - (n - \frac{1}{2})\psi)}$$

$$x_0 = x_1:\quad A^+ e^{-\frac{i\psi}{2}} + A^- e^{\frac{i\psi}{2}} = A^+ e^{\frac{i\psi}{2}} + A^- e^{-\frac{i\psi}{2}} \Rightarrow A^+ = A^-$$

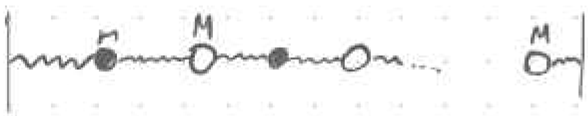
$$x_n = A e^{i\omega t}\cos\left((n - \tfrac{1}{2})\psi\right)$$

$$x_{N+1} = x_N:\quad \cos(N - \tfrac{1}{2})\psi = \cos(N + \tfrac{1}{2})\psi \Rightarrow \cos\left((N - \tfrac{1}{2})\psi\right) - \cos\left((N + \tfrac{1}{2})\psi\right) = 0 \Rightarrow$$

$$\Rightarrow \sin N\psi = 0 \Rightarrow \psi = \frac{\pi s}{N},\quad s = 0, \ldots, N-1$$

$$\Rightarrow \begin{pmatrix} \cos(-\frac{\psi}{2}) \\ \cos(\psi/2) \\ \cos(3\psi/2) \\ \vdots \\ \cos((n + \frac{1}{2})\psi) \end{pmatrix}$$

№4.



$2N$ штук

$$\begin{cases} x_0 = 0 \\ x_{2N+1} = 0 \end{cases}$$

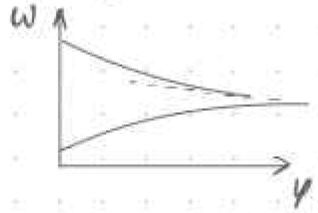
$$\begin{cases} m\ddot{x}_{2n-1} + k[2x_{2n-1} - 2x_{2n-2} - x_{2n}] = 0 \\ M\ddot{x}_{2n} + k[2x_{2n} - x_{2n-1} - x_{2n+1}] = 0 \end{cases}$$

$$\begin{cases} X_{2n-1} = A^{\pm} e^{i(\omega t \pm (2n-1)\varphi)} \\ X_{2n} = B^{\pm} e^{i(\omega t \pm 2n\varphi)} \end{cases}$$

$$-\omega^2 m A + k[2A - [Be^{-i\varphi} + Be^{i\varphi}]] = 0$$
$$-\omega^2 M B + k[2B - A \cdot 2\cos\varphi] = 0$$

$$\begin{vmatrix} -\omega^2 m + 2k & -2\cos\varphi\, k \\ -2\cos\varphi \cdot k & -\omega^2 M + 2k \end{vmatrix} = 0 \Rightarrow \omega^2 Mm + 2\omega^2 k(m+M) + 4k^2\sin^2\varphi = 0$$

$$\omega^2 = k\left[\frac{(M+m) \pm \sqrt{(M+m)^2 - 4mM\sin^2\varphi}}{Mm}\right.$$ $\Rightarrow$ график с двумя ветвями



$$\omega^2 = \frac{k}{\mu}\left[1 \pm \sqrt{1 - \frac{4\sin^2\varphi \cdot \mu}{M+m}}\right]$$

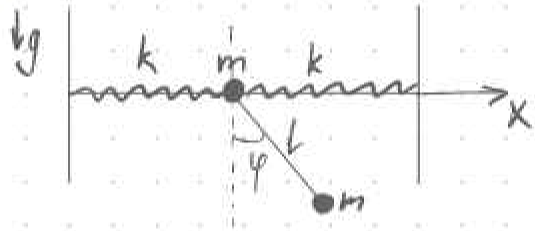
нач. условие

$$x_0 = B^+ + B^- = 0 \Rightarrow X_{2n} = b_s \sin 2n\varphi \cdot \cos(\omega_s t + \varphi_s)$$

$$X_{2n+1} = 0 \Rightarrow \sin(2N+1)\varphi = 0 \Rightarrow \varphi_s = \frac{\pi s}{2N+1}$$

Д/з: если m и M разные, то как две ветви сойдутся
сколько частот
ф-я Гамильтона, ур-е Г, скобки Пуассона

№7.



1) 

2) 

$\dot{y} = ((x + l\sin\varphi)')^2 + (l\cos\varphi)'^2 = (\dot{x} + l\dot{\varphi}\cos\varphi)^2 + (l\dot{\varphi}\sin\varphi)^2$

1) функция Лагранжа: $L = \frac{m\dot{x}^2}{2} + \frac{m}{2}((\dot{x} + l\dot{\varphi}\cos\varphi)^2 + (l\dot{\varphi}\sin\varphi)^2) + mgl\cos\varphi - \frac{kx^2}{2} - \frac{kx^2}{2} =$

$$= \frac{m}{2}(\dot{x}^2 + \dot{x}^2 + 2\dot{x}\dot{\varphi}l\cos\varphi + l^2\dot{\varphi}^2\cos^2\varphi + l^2\dot{\varphi}\sin^2\varphi) + mgl\cos\varphi - kx^2 =$$

$$= \frac{m}{2}(2\dot{x}^2 + 2\dot{x}\dot{\varphi}l\cos\varphi + l^2\dot{\varphi}^2) + mgl\cos\varphi - kx^2$$

2) составляем систему ур-ний: $\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \end{cases}$

$$\begin{cases} 2m\ddot{x} + ml\ddot{\varphi} + 2kx = 0 \\ m\ddot{x}l + ml^2\ddot{\varphi} + mgl \cdot \varphi = 0 \end{cases} \quad \Bigg| \quad M = \begin{pmatrix} 2m & ml \\ ml & ml^2 \end{pmatrix}, \; K = \begin{pmatrix} 2k & 0 \\ 0 & mgl \end{pmatrix}$$

$$M = \begin{pmatrix} 2m & ml \\ ml & ml^2 \end{pmatrix}, \quad K = \begin{pmatrix} 2k & 0 \\ 0 & mgl \end{pmatrix}$$

$$|K - w^2 M| = 0: \quad \begin{vmatrix} 2k - w^2 2m & -w^2 ml \\ -w^2 ml & mgl - w^2 ml^2 \end{vmatrix} = 0$$

$$(mgl - w^2 ml^2)(2k - 2m w^2) - w^4 m^2 l^2 = 0$$

$$2kmgl - 2k w^2 m l^2 - 2m^2 w^2 g l + 2 w^4 m^2 l^2 - w^4 m^2 l^2 = 0$$

$$w^4 m^2 l^2 - w^2 \cdot 2(kml^2 + m^2 g l) + 2kmgl = 0 \qquad g/l << \frac{k}{m}$$

$$D = 4(kml^2 + m^2 g l)^2 - 4m^2 l^2 \cdot 2kmgl$$

$$w_{1,2}^2 = \frac{\cancel{2}(kml^2 + m^2 g l) \pm \cancel{2}\sqrt{(kml^2 + m^2 g l)^2 - 2m^3 l^3 k g}}{\cancel{2} m^2 l^2} =$$

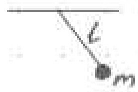
$$= \left(\frac{k}{m} + \frac{g}{l}\right) \pm \frac{\sqrt{k^2 m^2 l^4 + \cancel{2m^3 k g l^3} + m^4 g^2 l^2 - \cancel{2m^3 l^3 k g}}}{m^2 l^2} =$$

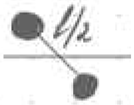
$$= \left(\frac{k}{m} + \frac{g}{l}\right) \pm \sqrt{\frac{k^2}{m^2} + \frac{g^2}{l^2}}$$

1) $\frac{g}{l} << \frac{k}{m}$: $\quad w_1^2 = \frac{k}{m} + \frac{k}{m} = \frac{2k}{m}$ 

$$w_2^2 = \frac{k}{m} - \frac{k}{m}\sqrt{1 + \frac{g^2}{l^2} \cdot \frac{m^2}{k^2}} = \frac{g}{l} + \frac{k}{m} - \frac{k}{m}\left(1 + \frac{1}{2}\left(\frac{gm}{lk}\right)^2\right) =$$

$$= \frac{g}{l} + \frac{k}{m} - \frac{k}{m} - \frac{1}{2}\frac{g}{l} \cdot \frac{m}{k} = \frac{g}{l}\left(1 - \frac{1}{2}\frac{m \cdot g}{k \cdot l}\right) = \frac{g}{l}$$



2) $\frac{k}{m} << \frac{g}{l}$: $\quad w_1^2 = \frac{g}{l} + \frac{g}{l} = \frac{2g}{l}$ 

$$w_2^2 = \frac{g}{l} + \frac{k}{m} - \frac{g}{l}\sqrt{1 + \frac{l^2}{g^2} \cdot \frac{k^2}{m^2}} = \frac{g}{l} + \frac{k}{m} - \frac{g}{l}\left(1 + \frac{1}{2}\frac{l^2 k^2}{g^2 m^2}\right) =$$

$$= \frac{g}{l} + \frac{k}{m} - \frac{g}{l} - \frac{1}{2}\frac{l}{g} \cdot \frac{k^2}{m^2} = \frac{k}{m}\left(1 - \frac{1}{2}\frac{k}{m} \cdot \frac{l}{g}\right) = \frac{k}{m}$$

Функция Гамильтона

11.04.23

$\delta S = 0$

$L(q_s, \dot q_s, t) \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot q_s} - \frac{\partial L}{\partial q_s} = 0$ 2S ОДУ 2 порядка

$p_s = \frac{\partial L}{\partial \dot q_s}$, $q_s(t)$ – решение

ф-м Гамильтона: $p_s = \frac{\partial L}{\partial \dot q_s}$ – независимые величины

из S ур-ний классических → 2S ур-ний с q_i и p_i , $\frac{\partial p_i}{\partial q_i} = 0$

№1. вывод

$L(q, \dot q, t) \rightarrow$

$p = \frac{\partial L}{\partial \dot q}$, $\dot q(p, q) \Rightarrow$ $\sum p \cdot \dot q - L = H(p,q)$ | разные пр-ва
$\sum p \cdot \dot q - L = L(q, \dot q)$

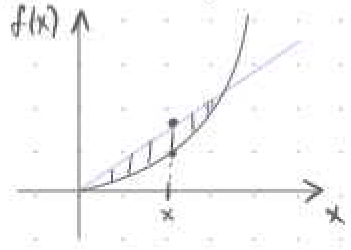
$\dot p = -\frac{\partial H}{\partial q}$, $\dot q = \frac{\partial H}{\partial p}$

вывод минуса: $\frac{m\dot x^2}{2} + U(x)$, $\dot x = \frac{p}{m} \Rightarrow H = \frac{p^2}{2m} + U(x) \Rightarrow \dot p = -\frac{\partial H}{\partial x} \rightarrow \frac{\partial p}{\partial t} = F - -\frac{\partial H}{\partial x}$

$\dot x = \frac{\partial H}{\partial p} = \frac{p}{m}$

преобр-е Лежандра – вспомог. прием, обеспечивающих переход к кас-му пр-ву

рецепт: пусть $f''(x) > 0$, тогда: $\{x\} \rightarrow \{p\}$



строим $y = p \cdot x$, $\forall p$ соответствует такое x: $px - f(x)$ имеет max

$p = \frac{\partial f}{\partial x} = p(x) \rightarrow x(p)$ (арнольд)

$$g(p) = F(p, x(p)) = p \cdot x(p) - f(x(p))$$

пример: пусть $f(x) = \frac{mx^2}{2} \Rightarrow p = \frac{\partial f}{\partial x} = mx \rightarrow x = \frac{p}{m}$

$g(p) = \frac{p \cdot p}{m} - \frac{p^2}{2m} = \frac{p^2}{2m}$

обр. преобр: $g(p) = \frac{p^2}{2m} \Rightarrow \frac{\partial g}{\partial p} = \frac{p}{m} = x \Rightarrow \hat f(x) = x \cdot p - \frac{p^2}{2m} = \frac{mx^2}{2}$

№2.

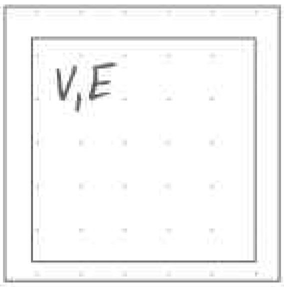
$$f(x) = \frac{x^\alpha}{\alpha}$$

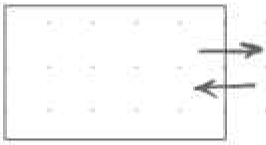
$$p = \frac{\partial f}{\partial x} = x^{\alpha-1} \Rightarrow x = p^{\frac{1}{\alpha-1}}$$

$$g(p) = p \cdot x(p) - f(x(p)) = \frac{\alpha-1}{\alpha} p^{\frac{\alpha}{\alpha-1}} = \left(1-\frac{1}{\alpha}\right) p^{\frac{1}{1-1/\alpha}} = \frac{p^\beta}{\beta}$$

$$\frac{1}{\beta} + \frac{1}{\alpha} = 1$$

№3.





$$TdS = pdV + dE$$

$$dE(S,V) = TdS - pdV$$

$$E + mgh = E + pV = H$$

$$dH = dE + pdV + Vdp = TdS + Vdp$$

$$dF = d(E - ST) = -SdT - pdV$$

$$\frac{\partial(p,V)}{\partial(S,T)} = 1 \Rightarrow p,V \text{ и } S,T \text{ — кас-нне пр-ва}$$

№4.

> ! Гамильтонова система — система, где нет диссип-ых сил

1) $L = \frac{m(\dot{z}^2 - z^2\dot{\varphi}^2)}{2} - U(z)$ — потенц. поле

2) $L = e^{\alpha t}\left(\frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}\right)$

3) $L = -mc\sqrt{1 - \left(\frac{\dot{x}}{c}\right)^2}$

4) $H = \frac{p^2}{2m} - pa$

5) $L = \frac{m\bar{V}^2}{2} - \frac{e}{c}(\bar{V} \cdot \bar{A})$

① $p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \dot{z} = \frac{p_z}{m}$

$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mz^2\dot{\varphi} \Rightarrow \dot{\varphi} = \frac{p_\varphi}{mz^2}$

$$H(p,z,y) = p_z\dot z + p_y\cdot\dot y - L = \frac{p_z^2}{m} + \frac{p_y^2}{mz^2} - \frac{p_z^2}{2m} - \frac{p_y^2}{2mz^2} + U(z) = \frac{1}{2m}\left[p_z^2 + \frac{p_y^2}{z^2}\right] + U(z)$$

② $$p_x = \frac{\partial L}{\partial \dot x} = e^{\alpha t} m\dot x \Rightarrow \dot x = \frac{p_x}{e^{\alpha t} m}$$

$$\frac{p_x \cdot p_x}{e^{\alpha t}\cdot m} - e^{\alpha t}\left(\frac{m}{2}\frac{p_x^2}{e^{2\alpha t} m^2} - \frac{m\omega^2 x}{2}\right) = \frac{p_x^2}{e^{\alpha t} m} - \frac{p_x^2}{2e^{\alpha t} m} + \frac{m\omega^2 x^2}{2} e^{\alpha t} = \frac{p_x^2}{2e^{\alpha t} m} + \frac{m\omega^2 x^2 e^{\alpha t}}{2}$$

$$\dot p = -\frac{\partial H}{\partial x} = -m\omega^2 x e^{\alpha t}$$

$$\dot x = \frac{\partial H}{\partial p} = \frac{p_x}{e^{\alpha t} m}$$

③ $$p_x = \frac{\partial L}{\partial \dot x} = \frac{-mc^2\cdot 2\dot x / c^2}{\sqrt{}} = \frac{m\dot x}{\sqrt{1-\left(\frac{\dot x}{c}\right)^2}} \Rightarrow$$

$$p_x^2 = \frac{m^2\dot x^2}{1-\left(\frac{\dot x}{c}\right)^2} \Rightarrow p_x^2\left(1-\left(\frac{\dot x}{c}\right)^2\right) = m^2\dot x^2 \Rightarrow 1 - \frac{\dot x^2}{c^2} = \frac{m^2}{p_x^2}\dot x^2$$

$$\dot x^2 = \frac{1}{\frac{m^2}{p_x^2} + \frac{1}{c^2}} \Rightarrow \dot x = \frac{1}{\sqrt{\frac{m^2}{p_x^2} + \frac{1}{c^2}}}$$

$$H(p,x) = p_x\cdot\frac{1}{\sqrt{\frac{m^2}{p_x^2} + \frac{1}{c^2}}} + mc^2\sqrt{1 - \frac{1}{\left(\frac{m^2}{p_x^2} + \frac{1}{c^2}\right)}\frac{1}{c^2}} = \frac{p_x + m^2 c}{\sqrt{m^2 + \frac{p_x}{c^2}}} = \frac{c\left(\frac{p_x}{c^2} + m^2\right)}{\sqrt{m^2 + \frac{p_x}{c^2}}} =$$

$$= c\sqrt{p_x^2 + m^2c^2}$$

$$S = (ct, \bar x),\quad S^2 = ct^2 - x^2,\quad cp = (E, c\bar p) \Rightarrow E^2 - p^2c^2 = (mc^2)^2$$

4 $$\dot x = \frac{\partial H}{\partial p} = \frac{p}{m} - a$$

$$p = m(\dot x + a)$$

$$L = p\cdot\dot x - H = \dot x m(\dot x + a) - \frac{m(\dot x + a)^2}{2} + ma(\dot x + a) = \frac{m(\dot x + a)^2}{2}$$

5 $$L = \frac{m(\dot x^2 + \dot y^2 + \dot z^2)}{2} - \frac{e}{c}\left[\dot x A_x + \dot y A_y + \dot z A_z\right]$$

$$p_x = \frac{\partial L}{\partial \dot x} = m\dot x - \frac{eA_x}{c} \Rightarrow \dot x = \frac{p_x}{m} - \frac{eA_x}{cm}$$

$$H = p_x\cdot\dot x - L = \frac{p_x^2}{m} - \frac{p_x A_x e}{cm} - \frac{m}{2}\left(\frac{p_x}{m} - \frac{eA_x}{cm}\right)^2 - \frac{e}{c}\left[\frac{p_x}{m} - \frac{eA_x}{cm}\right]A_x =$$

$$\frac{p_x^2}{m} - \frac{p_x A_x e}{cm} - \frac{m}{2}\left(\frac{p_x}{m} - \frac{eA_x}{cm}\right)^2 - \frac{e}{c}\left[\frac{p_x}{m} - \frac{eA_x}{cm}\right]A_x = \frac{p_x^2}{m} - \cancel{\frac{p_x A_x e}{cm}} - \frac{mp_x^2}{2m^2} - \cancel{\frac{p_x e A_x m}{cm^2}} -$$

$$+ \frac{e^2 A_x^2}{c^2 m^2}\cdot\frac{m}{2} - \frac{e p_x A_x}{cm} + \frac{e^2 A_x^2}{c^2 m} = \left(\frac{p_x}{2m} + \frac{eA_x}{cm}\right)^2 = \frac{1}{2m}\left(p_x + \frac{e}{c}A_x\right)^2$$

$$H = \frac{1}{2m}\left(\bar{p} + \frac{e}{c}\bar{A}\right)^2$$

Д/З: скобки Пуассона №10.4, 10.5, 10.6

тензоры: $T_{\alpha,\beta}$ – тензор : $\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$

$\partial_\mu T_{\alpha,\beta}$ – не факт, что тензор

$D_\mu T_{\alpha,\beta} = \partial_\mu T_{\alpha,\beta} + \Gamma^\mu_{\alpha,\mu} X_\mu$

$p, x \to \hat{x} = \cdot x \Rightarrow \hat{x}\Psi = x\cdot\Psi,\ \hat{p}\Psi = -i\hbar\frac{\partial}{\partial x}\Psi$

$\Psi(x)\cdot\Psi^*(x)$ – плотность вероят-ти для x, p

– если $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$

применим $\Psi(x) \to \Psi(x)e^{i\varphi(x)} \Rightarrow \hat{H}\Psi = E\Psi \Rightarrow \hat{H} = (\hbar^2\frac{\partial^2}{\partial x^2} + x^2)\Psi$

$\partial(\Psi(x)e^{i\varphi(x)}) = e^{i\varphi(x)}\partial\Psi(x) \pm i\dot{\varphi}(x)e^{i\varphi(x)}\Psi(x)$

$\partial_x \to D_x = \partial_x + A_x$ → отсюда возникает необходимость калибровки и введение вектор-потенциала

№8.

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{4}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 + \frac{1}{2}q_1 + 2q_1\dot{q}_2$$

$$p_{q_1} = \frac{\partial L}{\partial \dot{q}_1} = \frac{1}{4}\cdot 2\cdot\dot{q}_1 = \frac{\dot{q}_1}{2} \Rightarrow \dot{q}_1 = 2p_{q_1}$$

$$p_{q_2} = \frac{\partial L}{\partial \dot{q}_2} = \frac{1}{2}\cdot 2\dot{q}_2 + 2q_1 = \dot{q}_2 + 2q_1 \Rightarrow \dot{q}_2 = p_{q_2} - 2q_1$$

$$H(p_{q_1}, p_{q_2}, q_1, q_2) = p_{q_1}\cdot\dot{q}_1 + p_{q_2}\cdot\dot{q}_2 - L = p_1\cdot 2p_1 + p_2^2 - 2q_1 p_2 - \frac{1}{4}\cdot 4p_1^2 - \frac{1}{2}p_2^2 + 2p_2 q_1 - 2q_1^2 +$$
$$- \frac{1}{2}q_1 - 2q_1 p_2 + 4q_1^2 =$$

$$= p_1^2 + \frac{1}{2}p_2^2 - 2p_2 q_1 - \frac{1}{2}q_1 + 2q_1^2$$

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = 2p_1\ ;\quad \dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2 - 2q_1$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = 2p_2 + \frac{1}{2} - 4q_1\ ;\quad \dot{p}_2 = -\frac{\partial H}{\partial q_2} = 0$$

Функция Гамильтона. Скобки Пуассона. 18.04.23

1. частица в м.п.

$\bar{B}\uparrow$ $$H = \frac{(\bar{p} - \frac{e}{c}\bar{A}(\bar{z}))^2}{2m}$$

$\bar{B} = (0, 0, B_z)$, $\bar{B} = \mathrm{rot}\bar{A}$, $B_z = \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x$

$\bar{A} = (0, Bx, 0)$

$$H = \frac{p_x^2 + p_z^2 + (p_y - \frac{e}{c}Bx)^2}{2m}$$

$\dot{x} = \frac{\partial H}{p_x} = \frac{p_x}{m}$ $\qquad \dot{p}_x = -\frac{\partial H}{\partial x} = \frac{e}{cm}B_x\left(p_y - \frac{e}{c}B_x\right)$

$\dot{z} = p_z/m$ $\qquad \dot{p}_z = -\partial H/\partial z = 0$ $\qquad$ положим $\omega = \frac{eB}{cm}$

$\dot{y} = \frac{2}{2m}(p_y - \frac{e}{c}Bx)$ $\qquad \dot{p}_y = -\frac{\partial H}{\partial y} = 0$

если $\bar{p}_0, \bar{z}_0$ – начальные усл:

1 $p_z = p_{0z} = \mathrm{const}$, $z = z_0 + \frac{p_{0z}}{m}t$

$p_y = p_{0y} = \mathrm{const}$,

2 $\ddot{p}_x = \omega(\dot{p}_y - \frac{e}{c}B\dot{x}) = -m\omega^2 p_x$ или $\ddot{x} = \frac{\dot{p}_x}{m} = \frac{\omega}{m}\left(p_{0y} - \frac{eB}{c}x\right)$

$$\ddot{x} = \frac{\omega}{m}p_{0y} - \omega^2 x$$

$$x(t) = A\cos(\omega t + \varphi_0) + \frac{p_{0y}}{m\omega}$$

$$p_x(t) = -A\omega m\sin(\omega t + \varphi_0)$$

3 $\dot{y}(t) = -\omega A\cos(\omega t + \varphi)$
$y(t) = -A\sin(\omega t + \varphi) + y_0$

$\Rightarrow$ получим прямол. дв-е по z и колебания по x и y $\Rightarrow$ движ. по окр-ти

№2. $H(x,p) = V\cos(ap) - xF$

$$\dot{x} = \frac{\partial H}{\partial p} = -aV\sin(ap) = -aV\sin(a(Ft + p_0)) = -aV\sin(aFt + ap_0)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = F \Rightarrow p = Ft + p_0$$

$$x = \frac{aV\cos(aFt + ap_0)}{aF} - \frac{V}{F}\cos(ap_0) + x_0 = \frac{V}{F}\left[\cos(ap_0 + Ft) - \cos(ap_0)\right] + x_0$$

• если $H(q_1, p_1)$ не зависит от времени, то: $V\cos ap_0 - x_0 F = V_0\cos(p(t)a) - x(t)F$

№3. $H(x, y, p_x, p_y) = c\sqrt{p_x^2 + p_y^2} - xF$

$$\dot{x} = \frac{\partial H}{\partial p_x} = c\,\frac{1}{2}(p_x^2 + p_y^2)^{-1/2}\cdot 2p_x = \frac{cp_x}{\sqrt{p_x^2 + p_y^2}}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{cp_y}{\sqrt{p_x^2 + p_y^2}}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = F \Rightarrow p_x = Ft + p_{0x}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = 0 \Rightarrow p_y = p_{0y}$$

$$c\sqrt{p_x^2(t) + p_y^2(t)} - x(t)F = c\sqrt{p_{0x}^2 + p_{0y}^2} - x_0 F \Rightarrow x(t) = \frac{cp_{0y}}{F}\left(\sqrt{1 + \frac{Ft + p_{0x}}{p_y}} - \sqrt{1 + \frac{p_{0x}}{p_{0y}}}\right) + x_0$$

$$\dot{y} = \frac{cF}{\sqrt{1 + \left(\frac{p_x}{p_{0y}}\right)^2}}\,dp_x =$$

скобки Пуассона:

$H(p, q)$; p, q – координ., если $f = f(p, q)$:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum\left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) = \frac{\partial f}{\partial t} + \sum\left(\frac{\partial f}{\partial q_i}\cdot\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\cdot\frac{\partial H}{\partial q_i}\right)$$

$$\{f, g\} = \sum\left(\frac{\partial f}{\partial p_i}\cdot\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\cdot\frac{\partial g}{\partial p_i}\right)$$

тогда: $\dfrac{df}{dt} = \dfrac{\partial f}{\partial t} + \{H, f\}$

св-ва скобок Пуассона:

1) $\{f, g\} = -\{g, f\}$
2) $\{f, const\} = 0$
3) $\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$
4) $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$
5) тожд. Якоби: $f\{g, h\} + h\{f, g\} + g\{h, f\} = 0$,

б) $\{f, g\} = const$, если f — интегр. движения, то и g — инт. дв. (т. Пуассона)

$\Sigma(H, f) = 0$, если $f(p, q)$ — сохр-ся во времени

$$\left.\begin{aligned} \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{p_i, q_j\} &= \delta_{ij} \end{aligned}\right] \Rightarrow \text{если} \left[\begin{aligned} \{Q_i, Q_j\}_{q,p} &= 0 \\ \{P_i, P_j\}_{q,p} &= 0 \\ \{P_i, Q_j\}_{q,p} &= 0 \end{aligned}\right.$$

! критерий на интегралы
критерий на преобразования координат
понадобится в квантах

№1.

$$\{f, q_k\} = \frac{\partial f}{\partial p_k}$$

$$\{f, p_k\} = -\frac{\partial f}{\partial q_k}$$

№2.

$$\frac{\partial}{\partial p_x}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial p_x} +$$

$$\frac{\partial}{\partial p_y}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial p_y} +$$

$$\frac{\partial}{\partial p_z}\frac{\partial}{z} - \frac{\partial}{\partial z}\frac{\partial}{\partial p_z} +$$

1) $M_x = yp_z - zp_y$
$M_y = zp_x - xp_z$
$M_z = xp_y - yp_x$
$\Rightarrow$
$\{M_x, p_y\} = -p_z$
$\{M_x, y\} = -z$
$\{M_x, M_y\} = M_z$

$$\{M_x, M^2\} = \{M_x, M_x^2 + M_y^2 + M_z^2\}$$

$$\{M_x, M_x^2\} = 2M_x\{M_x, M_x\} = 0$$

$$\{M_x, M_y^2\} = 2M_y\{M_x, M_y\} = -2M_yM_z$$

$$\{M_x, M_z^2\} = 2M_z\{M_x, M_z\} = 2M_z\{yp_z - zp_y, xp_y - yp_x\} = 2M_zM_y$$

$$\{M_x, M^2\} = -2M_yM_z + 2M_yM_z = 0$$

№3. $H = \frac{(p + \frac{e}{c}\bar{A}(z))^2}{2m}$

$\upsilon_x = \dot{x} = \frac{\partial H}{\partial p} = \frac{p_x}{m} + \frac{e}{cm} A_x(x,y,z)$, $\upsilon_y = \frac{p_y}{m} + \frac{e}{cm} A_y(x,y,z)$

$\{\upsilon_x, \upsilon_y\} = \frac{e}{cm^2}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \frac{e}{cm^2} \mathrm{rot}_z A_z = \frac{e}{cm^2} B_z$

$M_x = yp_z - zp_y$
$M_y = zp_x - xp_z$
$M_z = xp_y - yp_x$

№9.

a) • $\{p_x^2, M_z\} = 2p_x\{p_x, M_z\} = 2p_x\left(\frac{\partial p_x}{\partial p_x}\cdot\frac{\partial M_z}{\partial x} - \frac{\partial p_x}{\partial x}\cdot\frac{\partial M_z}{\partial p_x}\right) = 2p_xp_y$ $\quad \frac{\partial}{\partial p_x}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial p_x} +$

• $\{p_y^2, M_z\} = 2p_y\{p_y, M_z\} = 2p_y\left(\frac{\partial p_y}{\partial p_y}\cdot\frac{\partial M_z}{\partial y} - \frac{\partial M_z}{\partial p_y}\cdot\frac{\partial p_y}{\partial x}\right) = -2p_xp_y$ $\quad \frac{\partial}{\partial p_y}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial p_y} +$

• $\{p_z^2, M_z\} = 2p_z\{p_z, M_z\} = 2p_z\left(\frac{\partial p_z}{\partial p_z}\cdot\frac{\partial M_z}{\partial z} - \frac{\partial M_z}{\partial p_z}\cdot\frac{\partial p_z}{\partial z}\right) = 0$ $\quad \frac{\partial}{\partial p_z}\frac{\partial}{z} - \frac{\partial}{\partial z}\frac{\partial}{\partial p_z} +$

• $\{\bar{p}^2, M_z\} = \{p_x^2, M_z\} + \{p_y^2, M_z\} + \{p_z^2, M_z\} = 2p_x\{p_x, M_z\} + 2p_y\{p_y, M_z\} + 2p_z\{p_z, M_z\} = 0$

• $\{\bar{p}, (\bar{a}, \bar{z})\} = \{\bar{p}, (a_xx + a_yy + a_zz)\} = \{\bar{p}, a_xx\} + \{\bar{p}, a_yy\} + \{\bar{p}, a_zz\} = a_x\bar{e}_x + a_y\bar{e}_y + a_z\bar{e}_z$

• $\{M_y, x^3p_z^5\} = \frac{\partial M_y}{\partial p_x}\cdot\frac{\partial(x^3p_z^5)}{\partial x} - \frac{\partial M_y}{\partial z}\cdot\frac{\partial(x^3p_z^5)}{\partial p_z} = 3x^2p_z^5\cdot z - p_x5x^3p_z^4 =$

$= x^2p_z^4(3zp_z - 5p_xx)$

• $\{M_x, p_y^2p_z^4\} = \frac{\partial M_x}{\partial p_y}\cdot\frac{\partial(p_y^2p_z^4)}{\partial y} - \frac{\partial M_x}{\partial y}\cdot\frac{\partial(p_y^2p_z^4)}{\partial p_y} - \frac{\partial M_x}{\partial z}\cdot\frac{\partial(p_y^2p_z^4)}{\partial p_z} = -p_z\cdot 2p_yp_z^4 + p_y4p_y^2p_z^3$

$= -2p_yp_z^5 + 4p_z^3p_y^3 = -p_z^3p_y(2p_z^2 - 4p_y^2)$

б) • $\{p_3q_1, p_1^2 - q_2q_3\} = \{p_3q_1, p_1^2\} - \{p_3q_1, q_2q_3\} = -2p_1p_3 - q_1q_3$

• $\{q_3p_2 - q_1^2, p_1q_2\} = \{q_3p_2, p_1q_2\} - \{q_1^2, p_1q_2\} = q_3p_1 + 2q_1q_2$

• $\{q_1q_2 - p_3^2, p_2q_3\} = \{q_1q_2, p_2q_3\} - \{p_3^2, p_2q_3\} = -2p_3p_2 - q_1q_3$

Канонические преобразования. 25.04.23

$$\left\{\frac{\partial}{\partial p}\frac{\partial}{\partial q} - \frac{\partial}{\partial q}\frac{\partial}{\partial p}\right\} \quad и \quad \frac{\partial q_i}{\partial p_i} = 0$$

$$\delta S = \int L dt = \int p dq - H dt$$

$$H dt = p dq - L dt$$

новые координаты: $\begin{cases} Q(p,q), \\ P(q,p) \end{cases}$

если $L = L(q_i, \dot{q}_i, t) \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q} = 0$

$$\underbrace{L(Q, \dot{Q}, \tau)}_{=\frac{m(\dot{\tau}^2 + \tau^2\dot{\varphi}^2)}{2}} d\tau = \underbrace{L(q, \dot{q}, t)}_{=\frac{m(\dot{x}^2+\dot{y}^2)}{2}} dt$$

$Q = f(q)$, $\quad \begin{matrix} x = \tau\cos\varphi \\ y = \tau\sin\varphi \end{matrix}$

$$p dq - H dt = P dQ - H' dt + dF(Q, q, t)$$

$$dF(Q, q, t) = p dq - P dQ + (H' - H) dt$$

$$\begin{cases} p_i(Q, q, t) = \frac{\partial F}{\partial q_i} \\ P_i(Q, q, t) = -\frac{\partial F}{\partial Q_i} \\ H' = H + \frac{\partial F}{\partial t} \end{cases} \Rightarrow \begin{matrix} Q(q,p) \\ P(q,P) \end{matrix} \Rightarrow \begin{matrix} \dot{Q} = \frac{\partial H'(P,Q)}{\partial P} \\ \dot{P} = -\frac{\partial H'(P,Q)}{dQ} \end{matrix}$$

весь этот подход эквив-н следующему:

$$\begin{cases} \{P_i, Q_j\}_{pq} = \delta_{ij} \\ \{Q_i, Q_j\}_{pq} = 0 \\ \{P_i, P_j\}_{pq} = 0 \end{cases}$$ – критерий каноничности

$$\Phi(q, P, t) = F + PQ$$

$$d\Phi = (H' - H) dt + p dq + Q dP$$

$$p = \frac{\partial \Phi}{\partial q}, \quad Q = \frac{\partial \Phi}{\partial P}$$

№1.

$F(q,Q)=qQ$

$$\begin{cases} p_i(Q,q,t)=\frac{\partial F}{\partial q_i}=Q \\ P=-\frac{\partial F}{\partial Q}=-q \\ H'=H+\frac{\partial F}{\partial t}=H \end{cases}$$

$$\begin{cases} \frac{\partial H}{\partial p}=\dot q \Rightarrow \frac{\partial H'}{\partial Q}=-\dot P \\ \frac{\partial H}{\partial q}=-\dot p \Rightarrow \frac{\partial H'}{\partial P}=\dot Q \end{cases}$$

$$\{P,Q\}=\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}=1$$

№2.

$\Phi(q,P)=qP$

$$p=\frac{\partial\Phi}{\partial q}=P$$

$$q=\frac{\partial\Phi}{\partial P}=q$$

$$F(q,Q)=\Phi(q,P)-PQ=qP-PQ=0$$

№3

а) $\Phi(\bar r,\bar P)=\bar r\bar P+\overline{\delta\alpha}\,\bar P$, $\overline{\delta\alpha}=const$ – трансляция

$\bar R=\bar r+\overline{\delta\alpha}$

$\bar p=\bar P$

$\bar P[\overline{\delta\varphi}\times r]=\bar r[P\times\overline{\delta\varphi}]$

б) $\Phi(\bar r,\bar P)=\bar r\bar P+\overline{\delta\varphi}[\bar r\times\bar P]$ – поворот

$$\vec R=\frac{\partial\Phi}{\partial P}=\vec r+[\overline{\delta\varphi}\times\bar r] \qquad \bar R=\bar r+[\overline{d\varphi}\times\bar r]$$

$$\bar p=\frac{d\Phi}{dr}=\bar P+[\bar P\times\overline{\delta\varphi}] \xrightarrow{|\delta\varphi|\ll 1} \bar P=\bar p-[\bar P\times\overline{\delta\varphi}]=\bar p-[\bar p\times\delta\varphi]=p+[\overline{\delta\varphi}\times\bar p]$$

г) $\Phi(q,P)=qP+\delta\alpha(q^2+P^2)$

$$Q=\frac{\partial\Phi}{\partial P}=q+\delta\alpha\cdot 2P \xrightarrow{|\delta\alpha|\ll 1} q+\delta\alpha\cdot 2p$$

$\begin{pmatrix} 1 & 2\delta\alpha \\ -2\delta\alpha & 1\end{pmatrix}$ – поворот в фазовом пр-ве

$p=P+\delta\alpha\cdot 2q \Rightarrow P=p-2q\delta\alpha$

в) $\Phi(q,P)=qP+\delta\tau H(q,P)$

$$Q=\frac{\partial\Phi}{\partial P}=q+\delta\tau\cdot\frac{\partial H}{\partial P}=q+\delta\tau\cdot\dot q=q(t+\delta\tau)$$

$$p=P+\delta\tau\frac{\partial H}{\partial q} \Rightarrow P=p+\delta\tau\cdot\dot P=p+\delta\tau\cdot\dot p=p(t+\delta\tau)$$

№4.

$x = z\cos\psi$
$y = z\sin\psi$

$p_x = m\dot z\cos\psi - z\dot\psi\sin\psi = p_z\cos\psi - \frac{p_\psi\sin\psi}{z}$
$p_y = m\dot z\sin\psi + z\dot\psi\cos\psi = p_z\sin\psi + \frac{p_\psi\cos\psi}{z}$

$$\frac{\partial}{\partial p_z}\frac{\partial}{\partial z} - \frac{\partial}{\partial z}\frac{\partial}{\partial p_z} + \frac{\partial}{\partial p_\psi}\frac{\partial}{\partial \psi} - \frac{\partial}{\partial \psi}\cdot\frac{\partial}{\partial p_\psi}$$

$p_z = m\dot z$
$p_\psi = m z^2\dot\psi$

$\{x, y\}_{z,\psi} = 0$:

$\{p_x, p_y\} = 0$

$\{p_x, y\} = 0$

$\{p_y, x\} = 0$

$\{p_x, x\} = 1$

$\{p_y, y\} = 1$:

$$\{p_y, y\} = \frac{\partial\left(p_z\sin\psi + \frac{p_\psi\cos\psi}{z}\right)}{\partial p_z}\,\frac{\partial z\sin\psi}{\partial z} - \frac{\partial\left(p_z\sin\psi + \frac{p_\psi\cos\psi}{z}\right)}{\partial z}\cdot\cancelto{0}{\frac{\partial z\sin\psi}{\partial p_z}} +$$
$$+ \frac{\partial\left(p_z\sin\psi + \frac{p_\psi\cos\psi}{z}\right)}{\partial p_\psi}\cdot\frac{\partial z\sin\psi}{\partial\psi} - \frac{\partial\left(p_z\sin + \frac{p_\psi\cos\psi}{z}\right)}{\partial\psi}\cdot\cancelto{0}{\frac{\partial z\sin\psi}{\partial p_\psi}} = \sin\psi\cdot\sin\psi + \frac{\cos\psi}{z}\cdot\cos\psi\cdot z = 1$$

$$\{p_x, p_y\} = \frac{\partial}{\partial p_z}\left(p_z\cos\psi - \frac{p_\psi\sin\psi}{z}\right)\frac{\partial}{\partial z}\left(p_z\sin\psi - \frac{p_\psi\cos\psi}{z}\right) - \frac{\partial}{\partial z}\left(p_z\cos\psi - \frac{p_\psi\sin\psi}{z}\right)\frac{\partial}{\partial p_z}\left(p_z\sin\psi - \frac{p_\psi\cos\psi}{z}\right) +$$
$$+ \frac{\partial}{\partial p_\psi}\left(\left(p_z\cos\psi - \frac{p_\psi\sin\psi}{z}\right)\right)\frac{\partial}{\partial\psi}\left(p_z\sin\psi - \frac{p_\psi\cos\psi}{z}\right) - \frac{\partial}{\partial\psi}\left(p_z\sin\psi - \frac{p_\psi\cos\psi}{z}\right)\frac{\partial}{\partial p_\psi}\left(\left(p_z\cos\psi - \frac{p_\psi\sin\psi}{z}\right)\right) =$$
$$= 0$$

$$\Phi = P_x z\cos\psi + P_y z\sin\psi$$

$x = \frac{\partial\Phi}{\partial P_x} = z\cos\psi$
$y = \frac{\partial\Phi}{\partial P_y} = z\sin\psi$

$p_z = \frac{\partial\Phi}{z} = P_x\cos\psi + P_y\sin\psi$
$p_\psi = zP_x(-\sin\psi) + P_y z\cos\psi$

№5.

$$H_0=\frac{p^2}{2m}+\frac{m\omega^2x^2}{2}+\alpha x^3 m \qquad \text{надо: } H=\frac{P^2}{2m}+\frac{Q^2\omega^2 m}{2}$$

$\alpha a \ll m\omega^2$

$$Q=A\cos(\omega t+\psi)$$
$$P=-A\omega\sin(\omega t+\psi)$$

$$\Phi(q,P)=qP+aq^2P+bP^3$$

$$Q=\frac{\partial\Phi}{\partial P}=q+aq^2+3bP^2$$

$$p=\frac{\partial\Phi}{\partial q}=P+2aqP\approx P+2aQP$$

$$q=Q-aQ^2+3bP^2$$

$$H=\frac{1}{2m}\left(P^2+4QP^2a\right)+\frac{m\omega^2}{2}\left(Q^2-2aQ^3-6bQP^2\right)+\alpha Q^3 m=H_0+QP\left(\frac{2a}{m}-3m\omega^2 b\right)+$$
$$+Q^3(\alpha-am\omega^2)$$

$a=\alpha/\omega^2$; $b=2a/3m^2\omega^2=2\alpha/3m^2\omega^4$

$$q=A\cos(\omega t+\psi)-\frac{\alpha}{\omega^2}A^2\cos^2(\omega t+\psi)-\frac{2\alpha}{m^2\omega^4}A^2m^2\omega^2\sin^2(\omega t+\psi)$$

ответ из задачи про нелинейные кол-ния: $x=a\cos(\ )-\frac{\alpha a^2}{2\omega^2}+\frac{\alpha a^2\cos 2(\ )}{6\omega^2}$

$$q=A\cos(\omega t+\psi)-\frac{\alpha A^2}{\omega^2}\frac{1+\cos2(\omega t+\psi)}{2}-\frac{2\alpha A^2}{\omega^2}\frac{1-\sin2(\omega t+\psi)}{2}=$$

$$=A\cos(\omega t+\psi)-\frac{\alpha A^2}{2\omega^2}\cos2(\omega t+\psi)+\frac{\alpha A^2}{\omega^2}\sin2(\omega t+\psi)$$

№6.

$$H=\frac{p^2}{2m}+\frac{m\omega^2x^2}{2}=a\bar{a}\cdot\omega=\frac{P}{i}\cdot Q\omega=-iPQ\omega$$

$$a(x,p,t)=\frac{m\omega x+ip}{\sqrt{2m\omega}}e^{i\omega t}$$

$$\bar{a}(x,p,t)=\frac{m\omega x-ip}{\sqrt{2m\omega}}e^{-i\omega t}$$

$$a\cdot\bar{a}=\frac{H}{\omega}$$

$$\{\bar{a},a\}=\frac{-ie^{-i\omega t}}{\sqrt{2m\omega}}\cdot\frac{m\omega e^{i\omega t}}{\sqrt{2m\omega}}-\frac{m\omega e^{-i\omega t}}{\sqrt{2m\omega}}\cdot\frac{ie^{i\omega t}}{\sqrt{2m\omega}}=-i$$

$P=i\bar{a}$
$Q=a$ $\Rightarrow\{P,Q\}=1$

$$H' = H + \frac{\partial F}{\partial t}$$

11.5 ($\beta = 0$)
11.17
11.26 (а, б)

$$F(x, Q, t) = ?$$

$$Q = \frac{(m\omega x + ip)e^{i\omega t}}{\sqrt{2m\omega}}$$

$$P = \frac{(im\omega x + p)e^{-i\omega t}}{\sqrt{2m\omega}} = \frac{(im\omega x - iQ\sqrt{2m\omega}\,e^{-i\omega t} + mix\omega)e^{-i\omega t}}{\sqrt{2m\omega}}$$

$$\frac{\partial F}{\partial x} = p(x, Q, t) = -iQ\sqrt{2m\omega}\,e^{-i\omega t} + m\omega x i$$

$$\frac{\partial F}{\partial Q} = -P(x, Q, t) = iQe^{-2i\omega t} - i\sqrt{2m\omega}\,xe^{-i\omega t}$$

$$F(x, Q, t) = -iQx\sqrt{2m\omega}\,e^{-i\omega t} + \frac{im\omega x^2}{2} + \frac{iQ^2e^{-2i\omega t}}{2} + A(t)$$

$$\frac{\partial F}{\partial t} = -\omega Qx\sqrt{2m\omega}\,e^{-i\omega t} - \frac{2\omega Q^2e^{-2i\omega t}}{2} = Q\left(-x\omega\sqrt{2m\omega}\,e^{-i\omega t} + Q\omega e^{-2i\omega t}\right) =$$

$$= Q\omega iP$$

$$\Rightarrow H' = 0$$

№10.

$$\Phi(q, P, t) = qP + (aq - bP)t$$

$$p = \frac{\partial \Phi}{\partial q} = P + at$$

$$Q = \frac{\partial \Phi}{\partial P} = q - bt$$

$$\frac{\partial \Phi}{\partial t} = aq - bP = a(Q + bt) - bP$$

проверка каноничности:

$$\{P, Q\}_{pq} = \delta_{ij} : \{p - at, q + bt\} = 1 \cdot 1 - 0 = 1$$

$$\{Q, Q\}_{pq} = \{q - bt, q - bt\} = 0 - 0 = 0$$

$$\{P, P\}_{pq} = \{p - at, p - at\} = 0 - 0 = 0$$

для свободной частицы: $L = \frac{m\dot{x}^2}{2}$, $p = \frac{\partial L}{\partial \dot{q}} = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$

$$H = \sum \dot{q}_i p_i - L(q, p, t) = \frac{p}{m} \cdot p - \frac{m\frac{p^2}{m^2}}{2} = \frac{p^2}{2m}$$

$$H' = H + \frac{\partial \mathcal{P}}{\partial t} = \frac{P^2}{2m} + a(Q+bt) - bP = \frac{(P+at)^2}{2m} + a(Q+bt) - bP$$

$$\begin{cases} \dot{Q} = \frac{\partial H'}{\partial P} = \frac{(P+at)}{m} - b \Rightarrow \dot{Q} = \frac{-at + C_1 + at}{m} - b = \frac{C_1}{m} - b \Rightarrow Q = \frac{C_1}{m} t - bt + C_2 \\ \dot{P} = -\frac{\partial H'}{\partial Q} = -a \Rightarrow P = -at + C_1 \end{cases}$$

№11.

$$U(\bar{z}) = -\frac{(\bar{a}\cdot\bar{z})^2}{z^4}, \text{ дано: } \bar{z}_0, \bar{v}_0$$

$$L = \frac{m\dot{z}^2}{2} + \frac{(\bar{a}\cdot\bar{z})^2}{z^4}$$

$$L = \frac{m}{2}\left[\dot{z}^2 + (z\sin\theta\dot{\varphi})^2 + z^2\dot{\theta}^2\right] + \frac{(\bar{a}\cdot\bar{z})^2}{z^4}$$

$p_\varphi = mz^2\sin\theta\dot{\varphi}$, $p_\theta = z^2\dot{\theta}m$, $p_z = m\dot{z}$

$x = z\cos\varphi\sin\theta$, $y = z\sin\varphi\sin\theta$, $z = z\cos\theta$

$$H = \sum p_i \dot{q}_i - L = \frac{p_z^2}{m} + \frac{p_\varphi^2}{mz^2\sin\theta} + \frac{p_\theta^2}{mz^2} - \frac{m}{2}\left[\frac{p_z^2}{m^2} + z^2\sin^2\theta\frac{p_\varphi^2}{m^2z^4\sin^2\theta} + z^2\frac{p_\theta^2}{m^2z^4}\right] + \frac{(\bar{a}\cdot\bar{z})^2}{z^4}$$

$$= \frac{p_z^2}{2m} + \frac{p_\varphi^2}{2mz^2\sin\theta} + \frac{p_\theta^2}{2mz^2} + \frac{a^2\cos^2\theta}{z^2}$$

ур-е Г-Я: $\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 + \frac{1}{2mz^2\sin\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 + \frac{1}{2mz^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{a^2\cos^2\theta}{z^2} = 0$

пусть $S = -Et + \alpha_\theta\theta + \alpha_\varphi\varphi + S_z(z)$

$$-E + \frac{1}{2m}\left(\frac{dS}{dz}\right)^2 + \frac{\alpha_\varphi^2}{2mz^2\sin\theta} + \frac{\alpha_\theta^2}{2mz^2} + \frac{a^2\cos^2\theta}{z^2} = 0$$

$$-E + \frac{1}{2m}\left(\frac{dS}{dz}\right)^2 + \frac{1}{z^2}\left(\frac{\alpha_\varphi^2}{2m\sin\theta} + \frac{\alpha_\theta^2}{2m} + a^2\cos^2\theta\right) = 0$$

$$S_z = \int\sqrt{\left(E - \frac{1}{z^2}\left(\frac{\alpha_\varphi^2}{2m\sin\theta} + \frac{\alpha_\theta^2}{2m} + a^2\cos^2\theta\right)\right)\cdot 2m}\, dz$$

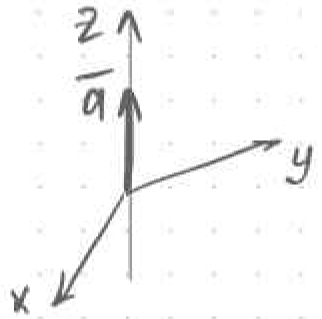
$$S = -Et + \alpha_\theta\theta + \alpha_\varphi\varphi + \int dS_z$$

$$\frac{\partial S}{\partial E} = -t - \frac{1}{2}\int\frac{2m\,dz}{\sqrt{E - \frac{1}{z^2}\left(\frac{\alpha_\varphi^2}{2m\sin\theta} + \frac{\alpha_\theta^2}{2m} + a^2\cos^2\theta\right)2m}} =$$

$$= -t - \frac{\sqrt{2m}}{2}\cdot\sqrt{Ez^2 - \underbrace{\frac{\alpha_\varphi^2}{2m\sin\theta} + \frac{\alpha_\theta^2}{2m} + a^2\cos^2\theta}_{C(z_0,v_0)}}\cdot\frac{2}{Ez} = \beta_0$$

$$\beta_0 + t = f(z)$$

$$-t - \frac{\sqrt{2m}}{2}\cdot\sqrt{Ez^2 - \frac{\alpha_\varphi^2}{2m\sin\theta} + \frac{\alpha_\theta^2}{2m} + a^2\cos^2\theta}\cdot\frac{2}{Ez} = \beta_0$$



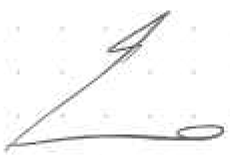
$$t + \beta = -\sqrt{\frac{m}{2}}\cdot\frac{1}{E}\sqrt{Ez^2 + const}$$

$$H = \frac{p_r^2}{2m} + \frac{1}{r^2}\{\quad\} = E(\bar{r}_0, \bar{v}_0)$$

$$A(\bar{r}_0, \bar{v}_0)$$

1. $\{A, H\} = 0 \Rightarrow \dot{r} = \sqrt{\frac{2}{m}\left(E - \frac{1}{r^2}A\right)}$

2. $\left\{\frac{p_\theta^2}{2m} + \frac{p_\varphi^2}{\sin^2\theta\, 2m} + \cos\theta\right\}$



Метод Гамильтона-Якоби

I. $L(q,\dot{q},t)$: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$ (N-степеней св. N штук ОДУ II пор. $2N$ штук нач. усл. α_i, β_i)

$q(t;\text{ну})$

II. $H(q,p,t)$: $\frac{\partial H}{\partial p_i} = \dot{q}_i$; $\frac{\partial H}{\partial q_i} = -\dot{p}_i$ ($2N$ ОДУ I порядка)

$p(t)$, $q_i(t)$

III. получали: $H' = H + \frac{\partial F}{\partial t}$

$H'(P,Q) = 0 \Rightarrow \dot{P} = -\frac{\partial H'}{\partial Q_i} = 0$

$2N$: $P = const = P(q,p,t)$

$Q = const = Q(q,p,t)$

итак: $dS = L dt = p dq - H dt = p(q,t)dq - H\left(\frac{\partial S}{\partial q}, q, t\right)dt$

$\Rightarrow$: $S = S(q,t,\alpha_i) \Rightarrow \boxed{\frac{\partial S}{\partial t} = -H\left(\frac{\partial S}{\partial q_i}, q_i, t\right)}$ - 1 ур-ние в частных производных

решение

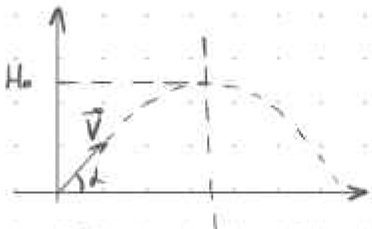
$\frac{\partial S}{\partial q_i} = p$

$S(q,\overset{=P_i}{\alpha_i},t) = \Phi(q,P,t)$

$H' = H + \frac{\partial S}{\partial t} = 0 \Rightarrow p_i = \frac{\partial S}{\partial q_i} \Rightarrow p_i(q,t)$

$Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i} = \beta_i(q,\alpha_i,t) \Rightarrow q(\alpha,\beta,t)$

№1.



$L = \frac{m(\dot{x}^2+\dot{y}^2)}{2} - mgy$

$H = \frac{p_x^2+p_y^2}{2m} + mgy$

ур-е Гамильтона-Якоби: $\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy = 0$ (*)

$S = \alpha_0 t + \alpha_x x + S_y(y) = -Et + \alpha_x x + S_y(y)$

(*): $-E + \frac{1}{2m}\left[\alpha_x^2 + \left(\frac{dS_y}{dy}\right)^2\right] + mgy = 0 \Rightarrow \frac{dS_y(y)}{dy} = \sqrt{(E-mgy)2m - \alpha_x^2}$

$S_y = \int\sqrt{(E-mgy)2m - \alpha_x^2}\,dy = \frac{((E-mgy)2m - \alpha_x^2)^{3/2}}{3m^2 g}$

$$S = -Et + \alpha_x x - \frac{\sqrt{2m(E-mgy) - \alpha_x^2}}{3m^2 g}$$

проверим: $2mE - \alpha_x^2 = m^2 v_0^2 - m^2 V_x^2 = m^2 v_y^2$
$= m[2mg H_0] = 2m^2 g H_0$

алгебра
$$\begin{cases} \beta_x = \dfrac{\partial S}{\partial \alpha_x} = x + \dfrac{\partial S_y}{\partial \alpha_x} \Rightarrow x - \beta_x = -\dfrac{\partial S_y}{\partial \alpha_x} = f(y) \text{ — траектория } y(x) \\ \beta_0 = \dfrac{\partial S}{\partial t} = -t + \dfrac{\partial S_y}{\partial E} \Rightarrow \beta_0 + t = \dfrac{\partial S_y}{\partial E} \text{ — зависимость } y(t) \end{cases}$$

$$\frac{\partial S_y}{\partial \alpha_x} = \frac{1}{3m^2 g} \cdot \frac{3}{2} \cdot \sqrt{(E-mgy)2m - \alpha_x^2} \cdot -2\alpha_x = -\frac{\alpha_x \sqrt{(E-mgy)2m - \alpha_x^2}}{m^2 g} = x - \beta_x = x'$$

$$\frac{\alpha_x^2}{m^4 g^2}\left[(E-mgy)2m - \alpha_x^2\right] = x' \Rightarrow \frac{x' m^4 g^2}{\alpha_x^2} = (E-mgy)2m - \alpha_x^2 \Rightarrow \underbrace{2mE - \alpha_x^2}_{2m^2 g H_0} - mgy2m$$

$$\frac{x' m^4 g^2}{\alpha_x^2} = 2m^2 g H_0 - mgy2m \Rightarrow y = H_0 - \frac{x' m^2 g}{2\alpha_x^2}$$

$$\frac{\partial S_y}{\partial t} = y(t+\beta_0) = H - \frac{g}{2}(t+\beta_0)^2$$

№2. (движение заряда в поле диполя)

$$L = \frac{m}{2}\left[\dot{z}^2 + (z\sin\theta\dot{\varphi})^2 + z^2\dot{\theta}^2\right] - U(\bar{z})$$
$p_\varphi = m z^2 \sin\theta \dot{\varphi}$
$p_\theta = z^2 \dot{\theta} m$
$p_z = m\dot{z}$

$$U = \frac{e(\bar{a}\cdot\bar{z})}{z^3}, \quad \bar{a} \parallel \bar{z}$$

$$H = \frac{1}{2m}\left\{p_z^2 + \frac{p_\theta^2}{z^2} + \frac{p_\varphi^2}{z^2 \sin^2\varphi}\right\} + \frac{ea\cos\theta}{z^2}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial z}\right)^2 + \frac{1}{z^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \varphi}\right)^2 + 2ema\cos\theta\right]\right] = 0$$

1) ищем в виде: $S = -Et + \alpha_\varphi \varphi + S_\theta(\theta) + S_z(z)$

2) $\alpha_\theta = [\quad] = \left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{\alpha_\varphi^2}{\sin^2\theta} + 2ema\cos\theta \Rightarrow S_\theta = \int\left(\alpha_\theta - 2ema\cos\theta - \frac{\alpha_\varphi^2}{\sin^2\theta}\right)^{1/2} d\theta$

3) $\frac{\partial S}{\partial t} = -E + \frac{1}{2m}\left[\left(\frac{dS_z}{dz}\right)^2 + \frac{\alpha_\theta}{z^2}\right] = 0 \Rightarrow S_z = \int\sqrt{2m\left(E - \frac{\alpha_\theta}{z^2}\right)}\, dz$

4) решение: $S = -Et + \alpha_\varphi \varphi + \int\left(\alpha_\theta - 2ema\cos\theta - \frac{\alpha_\varphi^2}{\sin^2\theta}\right)^{1/2} d\theta + \int\sqrt{2m\left(E - \frac{\alpha_\theta}{z^2}\right)}\, dz$

5) $\beta_0 = \frac{\partial S}{\partial E} = -t + \int\frac{m\,dz}{\sqrt{2mE - \alpha_\theta/z^2}} \Rightarrow z(t)$

6) $\beta_\theta = \frac{\partial S}{\partial \alpha_\theta} = \int\frac{d\theta}{2\left[\alpha_\theta - 2ema\cos\theta - \frac{\alpha_\varphi^2}{\sin^2\theta}\right]^{1/2}} + \int \frac{dz}{\quad} \Rightarrow z(\theta)$

7) $p_\psi = \frac{\partial S}{\partial \psi} = \psi + \int d\theta \qquad \Rightarrow \psi(\theta)$

№3.

$$H = \frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{m\omega^2x^2}{2} = 0 \quad (*):$$

ищем в виде: $S = -Et + S_x(x)$

$$(*): \; -E + \frac{1}{2m}\left(\frac{dS}{dx}\right)^2 + \frac{m\omega^2x^2}{2} = 0 \Rightarrow \left(\frac{dS}{dx}\right)^2 = \left(E - \frac{m\omega^2x^2}{2}\right)2m \Rightarrow$$

$$dS = \sqrt{\left(E - \frac{m\omega^2x^2}{2}\right)2m}\,dx = \sqrt{2mE - m^2\omega^2x^2}\,dx = \sqrt{2mE\left(1 - \frac{m^2\omega^2x^2}{2mE}\right)}\,dx =$$

$$= \int\sqrt{1 - \frac{m^2\omega^2x^2}{2mE}}\cdot\frac{2E}{\omega}\cdot d\left(\frac{\omega\sqrt{m}\,x}{\sqrt{2E}}\right) = \left[\begin{array}{ll}\frac{\sqrt{m}\omega x}{\sqrt{2E}} = t, & t = \sin u \\ & dt = \cos u\, du \\ & \sqrt{1-t^2} = \cos u\end{array}\right] =$$

$$= \frac{2E}{\omega}\int \cos^2 u\,du = \frac{2E}{\omega}\left(\frac{u}{2} + \frac{\sin 2u}{4}\right) = \frac{\arcsin t}{2} + \frac{t}{2}\sqrt{1-t^2}$$

$$S_x = \frac{E}{\omega}\arcsin\sqrt{\frac{m}{2E}}\,\omega x + \sqrt{\frac{mE}{2}}\,\frac{x}{2}\sqrt{1 - \frac{m\omega^2x^2}{2E}} = \frac{x}{2}\sqrt{\frac{mE}{2} - \frac{m^2\omega^2x^2}{4}} + \frac{E}{\omega}\arcsin\sqrt{\frac{m}{2E}}\,\omega x$$

$$S = -Et + \frac{x}{2}\sqrt{\frac{mE}{2} - \frac{m^2\omega^2x^2}{4}} + \frac{E}{\omega}\arcsin\sqrt{\frac{m}{2E}}\,\omega x$$

$$\frac{\partial S}{\partial E} = -t + \frac{x}{2}\cdot\frac{m/2}{\sqrt{\frac{mE}{2} - \frac{m^2\omega^2x^2}{4}}} + \frac{1}{\omega}\arcsin\sqrt{\frac{m}{2E}}\,\omega x + \frac{E}{\omega}\cdot\frac{1}{\sqrt{\left(1 - \frac{m\omega^2x^2}{2E}\right)}}\cdot x\omega/2m =$$

$$= -t + \frac{1}{\omega}\arcsin\left(\sqrt{\frac{m}{2E}}\,\omega x\right) = \beta_0$$

$$\beta_0 + t = \frac{1}{\omega}\arcsin$$

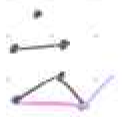
$$\sin(\omega(\beta_0 + t)) = \sqrt{\frac{m}{2E}}\,\omega x \Rightarrow x = \sqrt{\frac{2E}{m}}\frac{1}{\omega}\sin(\omega(\beta_0 + t))$$

Тензор инерции

12.05.23

$$T = \sum_n \frac{m_n V_n^2}{2} \ominus$$

тв. тело – модель, где расст-е между частицами фиксированно



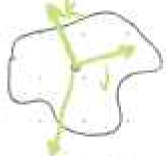
ст.св – 3 · кол-во част – кол-во связей

$$\bar V_n = \underbrace{\bar V_0}_{\text{центр.м}} + [\bar\Omega \times \bar z_n]$$

$$\ominus \sum_n \frac{m_n}{2}\left[V_0^2 + 2V_0[\Omega \times z_n] + [\Omega \times z_n]^2\right]$$

$$= \underbrace{\frac{MV_0^2}{2}}_{\text{эн. ц.м}} + \underbrace{\sum m_n z_n[\bar V \times \bar\Omega]}_{0,\ \text{т.к. в ц.м}} + \underbrace{\sum \frac{m_n}{2}\left[(\Omega^2 z_n)^2 - (\bar\Omega \bar z_n)^2\right]}_{\Rightarrow}$$

введем i, j, k – оси, связанные с телом в сист. ц.м.

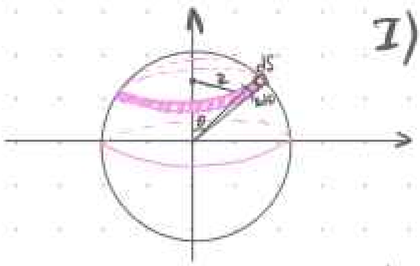


$$\Rightarrow \sum \frac{m_n}{2}\left[(\Omega_i \Omega_k \delta_{ik} \cdot z_n^2) - (\Omega_i z_{in})(\Omega_k z_{kn})\right] = \frac{\Omega_i \Omega_j}{2} \underbrace{\sum(\delta_{ik} z^2 - z_i z_k) m_i}_{I_{ik} - \text{тензор инерции}}$$

в дек-вой СК: $I_{ik} = \begin{vmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{vmatrix}$

если $I = \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix}$ – x, y, z — главные оси энерции

момент инерции сферы:

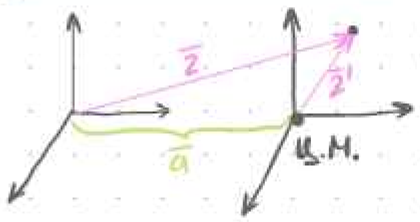


I) $dI = dm z^2 = \frac{M}{4\pi R^2} dS z^2 = \frac{M z^2}{4\pi R^2} \cdot R d\theta \cdot 2\pi z = \frac{M}{2R}\int d\theta R^3 \sin^3\theta =$

$$= \frac{MR^2}{2} \cdot 2\int_0^{\pi/2} \sin^3\theta\, d\theta = \frac{2}{3} MR^2$$

II) оси равноправны $\Rightarrow I_{xx} + I_{yy} + I_{zz} = 3I_{xx} = 2MR^2$

теорема Штейнера:

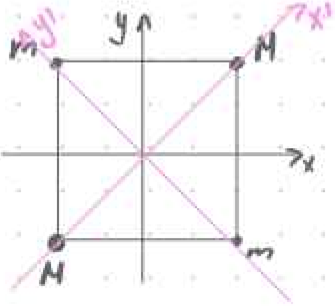


$$\bar z = \bar z' + \bar a$$

$$\left.\begin{aligned} I'_{ik} &= \sum m(\delta_{ik} z'^2 - z'_i z'_k) \\ I_{ik} &= \sum m(\delta_{ik} z^2 - z_i z_k) \end{aligned}\right| \Rightarrow$$

$$\sum m\left[\delta_{ik}(z'^2 + 2z'a + a^2) - (z'_i + a_i)(z'_k + a_k)\right] \Rightarrow I_{ik} = I'_{ik} + M[\delta_{ik}\bar a^2 - a_i a_k]$$

№1.



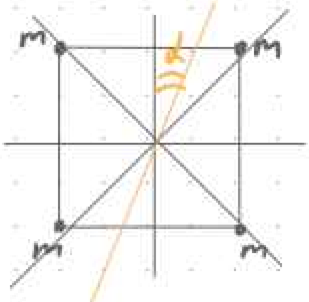
1. $M(a,a,0)$
   $m(a,-a,0)$
   $M(-a,-a,0)$
   $m(-a,a,0)$

2. $M(\sqrt{2}a,0,0)$
   $m(0,-\sqrt{2}a,0)$
   $M(-\sqrt{2}a,0,0)$
   $m(0,\sqrt{2}a,0)$

1) $$I_{ik} = \begin{vmatrix} 2a^2(M+m) & 2a^2(m-M) & 0 \\ 2a^2(m-M) & 2a^2(M+m) & 0 \\ 0 & 0 & 4a^2(M+m) \end{vmatrix}$$

2) $$I'_{ik} = \begin{vmatrix} 4a^2m & & \\ & 4a^2M & \\ & & 4a^2(M+m) \end{vmatrix}$$

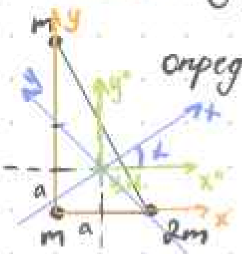
! если у тела есть ось симметрии, то она является главной

3) 

лин. комб главных осей тоже главная ось

! если $I_{xx} = I_{yy} = \rightarrow$ симметричный волчок

№2. (задача к экзамену!) (№9.2)

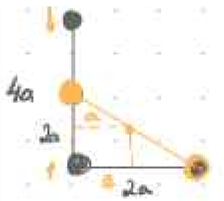


определить главные оси

1) гл. оси кривые, но они находятся в Ц.М. ⇒ находим Ц.М
2) I'' в $\forall$ осях x'', y'' через Ц.М.
3) поворачиваем тензор

1) $$x_{цм} = \frac{2m\cdot 2a}{4m} = a$$

$$y_{цм} = \frac{m\cdot 4a}{4m} = a$$



2) $$I''_{ik} = 4ma^2 \begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

3) хотим пов-ть оси на α, чтобы избавиться от диаг. членов

$$\bar{z} = \underbrace{U}_{\text{матр. поворота}} \bar{z}'' \Rightarrow \begin{cases} x = x''\cos\alpha + y''\sin\alpha \\ y = -x''\sin\alpha + y''\cos\alpha \end{cases} \Rightarrow U = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1 $I_{xx} = \sum my^2 = \sum m(x''^2 \sin^2\alpha + y''^2\cos^2\alpha - 2x''y''\sin\alpha\cos\alpha) = I''_{xx}\cos^2\alpha + I''_{yy}\sin^2\alpha + I''_{xy}\sin 2\alpha$

2 $I_{xx} = \sum_{i,j} U_{x_i} U_{x_j} I''_{ij} =$
$U_{xx}U_{xx}I''_{xx} = \cos^2 I''_{xx}$
$U_{xy}U_{xy}I''_{yy} = \sin^2 I''_{yy}$
$U_{xy}U_{xx}I''_{yx} = \frac{\sin 2\alpha}{2} I''_{yx}$
$U_{xx}U_{xy}I''_{xy} = \frac{\sin 2\alpha}{2} I''_{xy}$

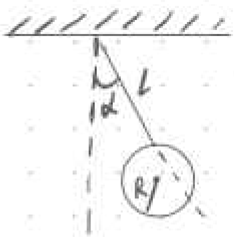
$I_{yy} = \sum mx^2 = \sum m(x''^2\cos^2\alpha + y''^2\sin^2\alpha + 2x''y''\cos\alpha\sin\alpha) = I''_{xx}\cos^2\alpha + I''_{yy}\sin^2\alpha - I''_{xy}\sin 2\alpha$

$I_{yy} = \sum_{i,j} U_{y_i} U_{y_j} I''_{ij} =$
$U_{yy}U_{yy}I''_{yy} =$
$U_{yx}U_{yy}I''_{xy} =$
$U_{yy}U_{yx}I''_{yx} =$
$U_{yx}U_{yx}I''_{xx} =$

$I_{xy} = \sum_{i,j} U_{x_i} U_{y_j} I''_{ij} =$
$U_{xx}\ U_{yx}\ I''_{xx}$
$U_{xx}\ U_{yy}\ I''_{xy}$
$U_{xy}\ U_{yx}\ I''_{yx}$
$U_{xy}\ U_{yy}\ I''_{yy}$
$= \frac{\sin 2\alpha (I''_{yy} - I''_{xx})}{2} + \cos 2\alpha\, I''_{yx}$

3 $\operatorname{tg} 2\alpha = 1 \Rightarrow \alpha = \frac{\pi}{8}$

<u>меcн!</u>

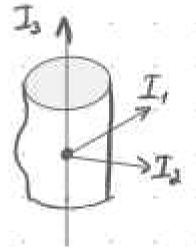


2 ст. свободн

$y = l\cos\alpha + R\cos\beta \Rightarrow$ по малым колебаниям разложить

задача о симметричном волчке

$T_{вр} = \frac{\Omega_i \Omega_j}{2} I_{ij}$

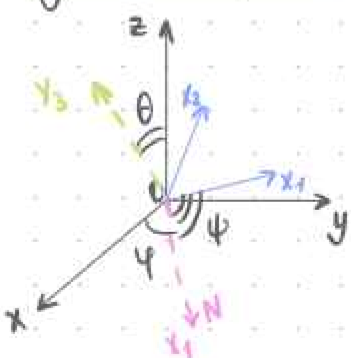


1, 2, 3 – связаны с телом
x, y, z – связаны с внешним миром

$I_{xx} = I_{yy} = I_1 = I_2$

если 1, 2, 3 – оси симметрии : $T = \frac{\Omega_1^2 I_1}{2} + \frac{\Omega_2^2 I_2}{2} + \frac{\Omega_3^2 I_3}{2}$

углы Эйлера:



1. поворот вокруг Oz на ψ ($\psi \in (0; 2\pi)$)
2. ON – линия узлов, отн-но нее отклоним на θ $(0, \pi)$
3. Ox_3 повернули на φ $(0, 2\pi)$

1) $\Omega_1 = \dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi$

$\Omega_2 = -\dot\theta\sin\psi + \dot\varphi\sin\theta\cos\psi$

$\Omega_3 = \dot\psi + \dot\varphi\cos\theta$

$\dot\psi$ - вращение вокруг X_3

$\dot\theta$ - вращение вокруг ON

2) $T = \frac{I_1(\Omega_1^2+\Omega_2^2)}{2} + \frac{I_3\Omega_3^2}{2}$

$M \parallel Oz$: $M_3 = M\cos\theta = I_3\Omega_3 = I_3(\dot\psi + \dot\varphi\cos\theta) \Rightarrow \dot\psi = \dot\varphi\cos\theta\frac{(-I_1+I_3)}{I_1}$

$M_2 = M\sin\theta\cdot\cos\psi = I_2\Omega_2 = I_2\dot\varphi\sin\theta \Rightarrow \dot\varphi = \frac{M}{I_1}$

$M_1 = M\sin\theta\cdot\sin\psi = 0 \Rightarrow \dot\theta = 0$

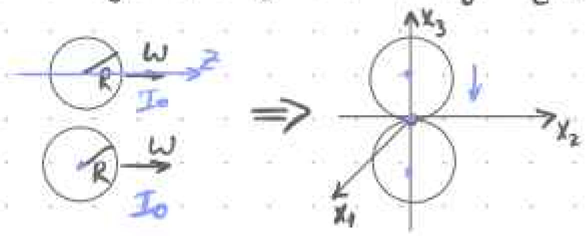
выбираем $\psi = 0$, но $\dot\psi \neq 0 \Rightarrow$ в каждый момент времени X_1 совпадает с ON

$I_1 = I_2$

3) $T_{вр} = \frac{I_1\Omega_2^2}{2} + \frac{I_3\Omega_3^2}{2} = \frac{I_1(\dot\varphi^2\sin^2\theta) + I_3(\dot\psi+\dot\varphi\cos\theta)^2}{2}$

задача о двух спутниках (№ 9.9)

$I_{ш} = \int dI_{сф} = \int\frac{2}{3}dNz^2 = \int\frac{2}{3}\frac{MdV}{\frac{4}{3}\pi R^3}z^2 = \int_0^R\frac{M4\pi z^2dz}{\frac{4}{3}\pi R^3} = \frac{2}{5}Mz^2 = I_0$



1) $I_3 = 2I_0$

$I_1 = 2(I_0 + MR^2) = 7I_0 = I_2$

$M = 2I_0\omega$

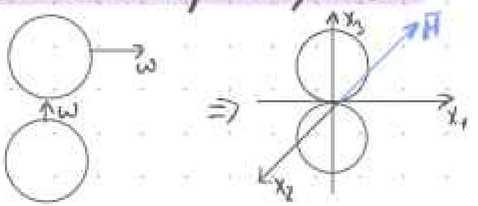
$\theta = \frac{\pi}{2}$, $\dot\varphi = \frac{2}{7}\omega$, $\dot\psi = 0$

2) было: $E_0 = 2\frac{I_0\omega^2}{2} = I_0\omega^2$

$T_{вр} = \frac{4}{49}\frac{7I_0\omega^2}{2} = \frac{2}{7}E_0$

$\frac{Q}{E_0} = \frac{5}{7} = 71\%$

поменяли траекторию:



$M = \sqrt{2}I_0\omega$

$\begin{cases} I_3 = 2I_0 \\ I_1 = 2(I_0+MR^2) = 7I_0 = I_2 \end{cases}$ - св-ва тела, постоянны

$|M| = \sqrt{2} I_0$

$\theta = \frac{\pi}{4}$, $\dot{\varphi} = \frac{\sqrt{2} I_0 \omega}{7 I_0} = \frac{\sqrt{2}}{7}\omega$

$\dot{\psi} = \frac{\sqrt{2}}{7}\omega \cdot \frac{\sqrt{2}}{2}\left(\frac{2I_0 - 7I_0}{2I_0}\right) = \frac{5}{14}\omega$

$T_{вр} = \frac{I_1(\dot{\varphi}^2 \sin^2\theta) + I_3(\dot{\psi} + \dot{\varphi}\cos\theta)^2}{2} = \frac{2I_0\left(\frac{2}{49}\omega^2 \cdot \frac{2}{4}\right) + 7I_0\left(\frac{5}{14}\omega + \frac{\sqrt{2}}{7} \cdot \frac{\sqrt{2}}{2}\omega\right)^2}{2} =$

$= \frac{9}{28} I_0 \omega^2$

$\frac{Q}{E_0} = \frac{19}{28} \approx 67\%$

№12.

$$H(p,q) = \frac{p^2}{2m} + \frac{m\omega^2x^2}{2} + \lambda\left(\frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right)^4$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2x - 4\lambda\left(\frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right)^3 \cdot m\omega^2x$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} + 4\lambda\left(\frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right)^3 \cdot \frac{p}{2m} \Rightarrow p = \frac{m\dot{x}}{1+4\lambda\left(\frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right)^3}$$

$$\frac{\partial H}{\partial t} = 0 \Rightarrow E_\lambda - \text{сохраняется} \Rightarrow \frac{p^2}{2m} + \frac{m\omega^2x^2}{2} = const = E$$

$$p = \frac{m\dot{x}}{1+4\lambda\left(\frac{p^2}{2m} + \frac{m\omega^2x^2}{2}\right)^3} = \frac{m\dot{x}}{1+4\lambda E^3}$$

$$\dot{p} = \frac{m\ddot{x}}{(1+4\lambda E^3)} = -m\omega^2x - 4\lambda E^3 m\omega^2 x$$

$$\ddot{x} = (1+4\lambda E^3)(-m\omega^2x - 4\lambda E^3\omega^2x) = (1+4\lambda E^3)(-\omega^2 - 4\lambda E^3\omega^2)x$$

$$\ddot{x} + (1+4\lambda E^3)^2\omega_0^2 x = 0$$

$$\omega = (1+4\lambda E^3)\omega_0$$

$$x(t) = A\cos\left((1+4\lambda E^3)\omega t\right) + B\sin\left((1+4\lambda E^3)\omega t\right)$$

$$x(t) = A\cos\left((1+4\lambda E^3)\omega t + \varphi\right)$$

$$p = -A\omega m\sin\left(\omega t(1+4\lambda E^3) + \varphi\right)$$

$$E = \frac{A^2\omega^2m^2\sin^2(\;)}{2m} + \frac{m\omega^2A^2\cos^2(\;)}{2} = \frac{m\omega^2A^2}{2}$$

$$x = A\cos\left(\omega t(1+2\lambda m\omega^2A^2) + \varphi\right)$$

$$p = -A\omega m\sin\left(\omega t(1+2\lambda m\omega^2A^2) + \varphi\right)$$

№13.

$$H = \frac{1}{2m}\left(p_x^2 + (m\omega_x x)^2 + p_y^2 + (m\omega_y y)^2\right) - m\alpha x y^2$$

$$\Phi(x, y, P_x, P_y) = xP_x + yP_y + axyP_y + bP_xP_y^2 + cy^2P_x$$

1) $X = \frac{\partial\Phi}{\partial P_x} = x + bP_y^2 + cy^2 \Rightarrow x = X - bP_y^2 - cy^2 \approx X - bP_y^2 - cY^2$

$Y = \frac{\partial\Phi}{\partial P_y} = y + axy + 2bP_xP_y \Rightarrow y = Y - axy - 2aP_xP_y \approx Y - aXY - 2bP_xP_y$

$p_x = \frac{\partial\Phi}{\partial x} = P_x + ayP_y \Rightarrow p_x = P_x + aYP_y$

$p_y = \frac{\partial\Phi}{\partial y} = P_y + axP_y + 2cyP_x \Rightarrow p_y = P_y + aXP_y + 2cYP_x$

2) $H' = \frac{1}{2m}\Big(P_x^2 + 2\alpha YP_xP_y + \alpha^2Y^2P_y^2 + m\omega_x(X^2 - \beta X^2P_y^2 - cXY^2 - \beta P_y^2X + \beta^2P_y^4 + cY^4\beta P_y^2 -$

$- cY^2X + cY^2\beta P_y^2 + c^2Y^4) + P_y^2 + \alpha XP_y^2 + 2cYP_xP_y + \alpha XP_y^2 + \alpha^2X^2P_y^2 + 2cYP_x\alpha XP_y$

$+ 2cYP_xP_y + \alpha XP_y2cYP_x + 4c^2Y^2P_x^2 + m^2\omega_y^2(Y^2 - \alpha XY^2 - 2\beta P_xP_yY - \alpha XY^2 + \alpha^2X^2Y^2 +$

$2\beta P_xP_y\alpha XY + 2\beta P_xP_yY + \alpha XY2\beta P_xP_y + 4\beta^2P_x^2P_y^2)\Big) - m\alpha(X - \beta P_y^2 - cY^2)(Y - \alpha XY - 2\beta P_xP_y)^2 =$

$= \frac{1}{2m}\left(P_x^2 + P_y^2 + m^2\omega_x^2X^2 + m^2\omega_y^2Y^2\right) + \frac{1}{2m}\Big(X^2P_y^2(-2bm^2\omega_x^2 - 2a) +$

$+ X\cdot Y^2\left(-2cm^2\omega_x^2 - m^2\omega_y^2 2a - \frac{2ma}{m}\right) + YP_yP_x(2a + 4c - m^2\omega_y^2 4b)$

$$\begin{cases} -2bm^2\omega_x^2 - 2a = 0 \\ -2cm^2\omega_x^2 - m^2\omega_y^2 2a - 2a = 0 \\ 2a + 4c - m^2\omega_y^2 4b = 0 \end{cases} \quad \begin{aligned} &\Rightarrow a = bm^2\omega_x^2 \\ &\Rightarrow b(m^2\omega_x^2 - 2m^2\omega_y^2) = -2c \\ &\Rightarrow -bm^2\omega_x^2\omega_y^2 + \omega_x^2 b(m^2\omega_x^2 - 2m^2\omega_y^2) - \alpha = 0 \end{aligned}$$

$b\left(-m^2\omega_x^2\omega_y^2 + \frac{1}{2}m^2\omega_x^4\right) = \alpha \Rightarrow b = \frac{2\alpha}{m^2\omega_x^2(\omega_x^2 - 4\omega_y^2)}$

$a = \frac{2\alpha}{\omega_x^2 - 4\omega_y^2}$; $c = \frac{\alpha}{m^2\omega_x^2(4\omega_y^2 - \omega_x^2)}$

3) $H' = \frac{1}{2m}\left(P_x^2 + P_y^2\right) + \frac{m}{2}\left[\omega_x^2X^2 + \omega_y^2Y^2\right]$

4) $X = A\cos(\omega_x t + \varphi_x)$ $\quad p_x \approx P_x = -m\omega_x A\sin(\omega_x t + \varphi_x)$

$Y = B\cos(\omega_y t + \varphi_y)$ $\quad p_y \approx P_y = -m\omega_y B\sin(\omega_y t + \varphi_y)$

$$x(t) = X - bP_y^2 - cY^2 = A\cos(\omega_x t + \varphi_x) - \frac{2\alpha}{m^2\omega_x^2(\omega_x^2 - 4\omega_y^2)} \cdot m^2\omega_y^2 B^2 \sin^2(\omega_y t + \varphi_y) -$$

$$- \frac{\alpha B^2 \cos^2(\omega_y t + \varphi_y)}{m^2\omega_x^2(4\omega_y^2 - \omega_x^2)}$$

$$y(t) = Y - aXY - 2bP_x P_y = B\cos(\omega_y t + \varphi_y) - \frac{\alpha AB\cos(\omega_x t + \varphi_x)\cos(\omega_y t + \varphi_y)}{(\omega_x^2 - 4\omega_y^2)} -$$

$$- \frac{4\alpha m^2\omega_x\omega_y^2 AB\sin(\omega_x t + \varphi_x)\sin(\omega_y t + \varphi_y)}{m^2\omega_x^2(\omega_x^2 - 4\omega_y^2)}$$

№14.

$$L = \frac{ml^2\dot{y}^2}{2} - \frac{mgly^2}{2} \qquad p_y = \frac{\partial L}{\partial \dot{y}} = ml^2\dot{y} \Rightarrow \dot{y} = \frac{p_y}{ml^2}$$

1) $$H(p_y, y) = p_y \cdot \dot{y} - L = ml^2\dot{y}^2 - \frac{ml^2\dot{y}^2}{2} + \frac{mgly^2}{2} = \frac{ml^2\dot{y}^2}{2} + \frac{mgly^2}{2} =$$

$$= \frac{p_y^2}{2ml^2} + \frac{mgly^2}{2}$$

2) ур-е Гамильтона-Якоби: $\frac{\partial S}{\partial t} + \frac{1}{2ml^2}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{mgly^2}{2} = 0$

$$S = -Et - S_y(y) \Rightarrow -E + \frac{1}{2ml^2}\left(\frac{dS_y}{dy}\right)^2 + \frac{mgly^2}{2} = 0$$

$$\frac{dS_y}{dy} = \sqrt{\left(E - \frac{mgly^2}{2}\right)2ml^2} \Rightarrow S_y = \int\sqrt{\left(E - \frac{mgly^2}{2}\right)2ml^2}\,dy$$

3) $$S = -Et - \int\sqrt{\left(E - \frac{mgly^2}{2}\right)2ml^2}\,dy$$

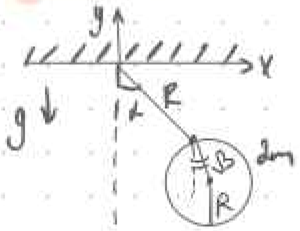
$$\frac{\partial S}{\partial E} = -t + \frac{1}{2}\int\frac{dy}{\sqrt{\left(E - \frac{mgly^2}{2}\right)2ml^2}} = -t + \sqrt{\frac{ml^2}{2}}\int\frac{dy}{\sqrt{E - \frac{mgly^2}{2}}} = \beta_0$$

$$\beta_0 + t = \sqrt{\frac{ml^2}{2E}}\int\frac{dy}{\sqrt{1 - \frac{mgly^2}{2E}}} = \sqrt{\frac{ml^2}{2E}} \cdot \sqrt{\frac{2E}{mgl}}\int\frac{dy}{\sqrt{1 - \frac{mgly^2}{2E}}}$$

$$\beta_0 + t = \sqrt{\frac{l}{g}}\arcsin\left(\sqrt{\frac{mgl}{2E}}\,y\right)$$

$$\sin\left(\frac{g}{l}(\beta_0 + t)\right) = \sqrt{\frac{mgl}{2E}}\,y \Rightarrow y = \sqrt{\frac{2E}{mgl}}\sin\left(\sqrt{\frac{g}{l}}(\beta_0 + t)\right)$$

№15.



$$\begin{cases} x = R\sin\alpha + R\sin\beta \\ y = -R\cos\alpha - R\cos\beta \end{cases}$$

$$\begin{cases} \dot x = R\cos\alpha\cdot\dot\alpha + R\cos\beta\cdot\dot\beta \\ \dot y = R\sin\alpha\cdot\dot\alpha + R\sin\beta\cdot\dot\beta \end{cases}$$

1) $L = 2mgR(\cos\alpha + \cos\beta) + \frac{2m}{2}\left((R\cos\alpha\cdot\dot\alpha + R\cos\beta\cdot\dot\beta)^2 + (R\sin\alpha\cdot\dot\alpha + R\sin\beta\cdot\dot\beta)^2\right) + \frac{I\omega^2}{2} =$

$= 2mgR(\cos\alpha + \cos\beta) + mR^2(\dot\alpha^2\cos^2\alpha + 2\dot\alpha\dot\beta\cos\alpha\cos\beta + \cos^2\beta\cdot\dot\beta^2 + \sin^2\alpha\dot\alpha^2 + 2\dot\alpha\dot\beta\sin\alpha\sin\beta + \sin^2\beta\dot\beta^2)$
$+ \frac{2m\cdot R^2\cdot\dot\beta^2}{2\cdot 2} =$

$= 2mgR(\cos\alpha + \cos\beta) + mR^2\left(\dot\alpha^2 + \frac{3}{2}\dot\beta^2 + 2\dot\alpha\dot\beta\cos(\alpha-\beta)\right)$

2) $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\alpha}\right) - \frac{\partial L}{\partial\alpha} = 0 \Rightarrow$

$\frac{d}{dt}\left(2mR^2\dot\alpha + mR^2 2\dot\beta\cos(\alpha-\beta)\right) + 2mgR\sin\alpha - 2mR^2\dot\alpha\dot\beta\sin(\alpha-\beta) = 0$

$\cancel{2m}R^2\ddot\alpha + \cancel{m}R^2\cancel{2}\ddot\beta\,\overset{1}{\cancel{\cos(\alpha-\beta)}} + \cancel{2m}gR\,\overset{\alpha}{\cancel{\sin\alpha}} = 0$

$\ddot\alpha R + \ddot\beta R + g\alpha = 0$

3) $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\beta}\right) - \frac{\partial L}{\partial\beta} = 0 \Rightarrow$

$\frac{d}{dt}\left(mR^2\frac{3}{2}\cdot 2\dot\beta + mR^2 2\dot\alpha\cos(\alpha-\beta)\right) + 2mgR\sin\beta = 0$

$3\cancel{m}R^{\cancel{2}}\ddot\beta + 2\cancel{m}R^{\cancel{2}}\ddot\alpha + 2\cancel{m}g\cancel{R}\beta = 0 \;/ :2$

$\frac{3}{2}R\ddot\beta + R\ddot\alpha + g\beta = 0$

4) $\hat M = \begin{pmatrix} R & R \\ R & \frac{3}{2}R \end{pmatrix}, \quad \hat k = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$

$|k - \omega^2 M| = \begin{vmatrix} g - R\omega^2 & -\omega^2 R \\ -\omega^2 R & g - \frac{3}{2}R\omega^2 \end{vmatrix} = g^2 - R\omega^2 g - \frac{3}{2}R\omega^2 g + \frac{3}{2}R^2\omega^4 - \omega^4 R^2 = 0$

$\frac{1}{2}\omega^4 R^2 - \frac{5}{2}\omega^2 gR + g^2 = 0 \;/\cdot 2$

$\omega^4 R^2 - 5\omega^2 gR + 2g^2 = 0$

$D = 25g^2R^2 - 8g^2R^2 = 17g^2R^2$

$$\omega_{1,2} = \frac{5Rg \pm \sqrt{17}Rg}{2R^2} = \frac{g(5 \pm \sqrt{17})}{2R}$$

$$\begin{vmatrix} g - R\omega^2 & -\omega^2 R \\ -\omega^2 R & g - \frac{3}{2}R\omega^2 \end{vmatrix}$$

4) $H_1: \begin{pmatrix} g - \frac{\cancel{R}\cdot g^2(5+\sqrt{17})^2}{4R^2} & -\frac{g^2\cancel{R}(5+\sqrt{17})^2}{4R^2} \\ -\frac{g^2\cancel{R}(5+\sqrt{17})^2}{4R^2} & g - \frac{3}{2}\frac{\cancel{R}g^2(5+\sqrt{17})}{4R^2} \end{pmatrix}$

$$\left(g - \frac{(5+\sqrt{17})^2 g^2}{4R}\right)x_1 = \frac{g^2(5+\sqrt{17})^2}{4R}x_2$$

пусть $x_1 = 1$: $\quad x_2 = \frac{4R}{g(5+\sqrt{17})^2} - 1$

5) $H_2: \begin{pmatrix} g - \frac{g^2(5-\sqrt{17})^2}{4R} & -\frac{g^2(5-\sqrt{17})^2}{4R} \\ -\frac{g^2(5-\sqrt{17})^2}{4R} & g - \frac{3}{2}\frac{g^2(5-\sqrt{17})}{4R} \end{pmatrix}$

пусть $x_1 = 1$: $\quad x_2 = \frac{4R}{g(5-\sqrt{17})^2} - 1$

Адиабатический инвариант

23.05.23

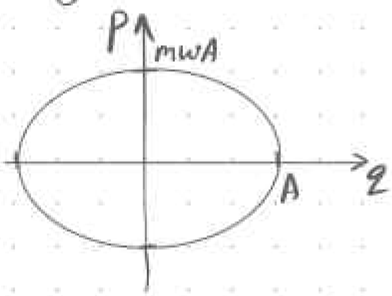
теорема Лиувилля: в гамильтоновых системах сохр-ся фазовый объем

канонические преобразования сохраняют объем: $\tau H(p,Q)+pQ$

$I=\frac{1}{2\pi}\oint p\,dq$ – для псевдофинитного движения

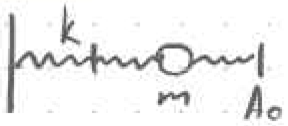
$T\cdot\frac{d\lambda}{dt}\ll\lambda$

адиабатические инварианты:



$I=\frac{S}{2\pi}=\pi\frac{m\omega A^2}{2\pi}=\frac{E}{\omega}$ – для осциллятора

N1.



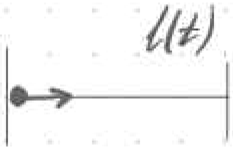
$A=A_0 g(t)$, где $g(t)$ медленная

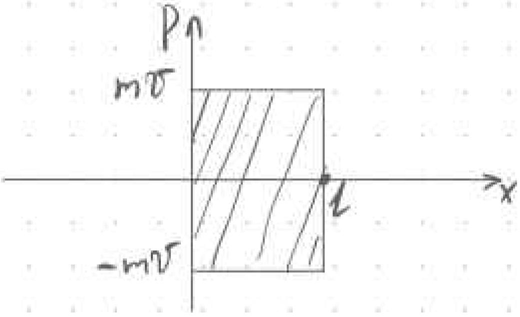
$k(t)=?$

$\frac{E}{\omega}=\frac{kA^2}{2}\cdot\sqrt{\frac{m}{k}}=const$

$k_0A_0^4=k(t)A^4(t)$

N2.





$mvl=const$

N3.

$X(t)=?$

$|\dot x|\gg|\dot X|$

$E=\frac{m\dot x^2}{2}+\frac{M\dot X^2}{2}=\frac{M\dot X^2}{2}+\frac{C^2}{2mX}$

$C = m|\dot{x}|X$

$$\frac{dX}{dt} = \sqrt{\frac{2}{M}\left(E - \frac{C^2}{2mX^2}\right)} \Rightarrow \int_{t_0}^{t} dt = \int_{X_0}^{X} \frac{dX}{\sqrt{\frac{2}{M}\left(E - \frac{C^2}{2mX^2}\right)}}$$

$$X(t) = \frac{dX}{\sqrt{\frac{2}{M}}\sqrt{\frac{2mEX^2 - C^2}{2mX^2}}} = \frac{mdX^2}{\sqrt{\frac{2}{m}}\sqrt{2mEX^2 - C^2}} = \sqrt{\frac{M}{2}} \cdot \sqrt{\frac{2}{2E}}\sqrt{2mEX^2 - C^2}$$

$$X(t) = \sqrt{\left(\left((t - t_0)\frac{2E}{M}\right)^2 + \frac{C^2}{Mm}\right)\frac{M}{2E}}$$

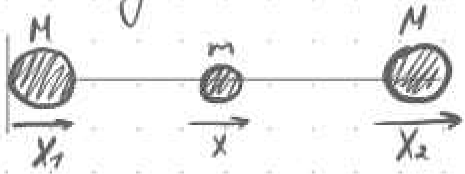
$E = E(\dot{x}_0, \dot{X}_0)$

$C = C(\dot{x}_0, X_0)$

$t_0 = t_0(E, C, X_0)$

№4.

модель M_2:



m прит-ся к M с $f = const$

$U_1 = f(X_1 - x)$, $U_2 = f(x - X_2)$

свести задачу к одномерной $X = X_2 - X_1$

в поле $U(x)$ $\min U(x)$

$$E = \frac{M\dot{X}_1^2}{2} + \frac{M\dot{X}_2^2}{2} + \frac{m\dot{x}^2}{2} + U = \frac{M\dot{X}_1^2}{2} + \frac{M\dot{X}_2^2}{2} + \frac{m\dot{x}^2}{2} + fX$$

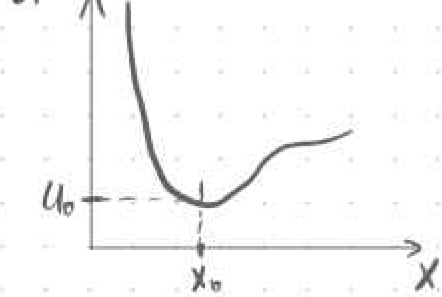
$C = |\dot{x}||X|$

$$E = \underbrace{\frac{M\dot{X}_1^2}{2} + \frac{M\dot{X}_2^2}{2}}_{T} + \underbrace{fX + \frac{mC^2}{2X^2}}_{U(X)}$$

$|\dot{X}_1| = |\dot{X}_2| \Rightarrow |\dot{X}| = |\dot{X}_1| + |\dot{X}_2|$

$$E = \frac{M\dot{X}^2}{4} + fX + \frac{mC^2}{2X^2}$$

$$U(X) = fX + \frac{mC^2}{2X^2}$$



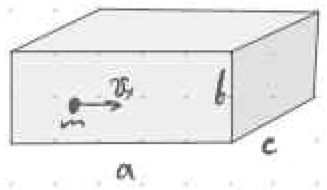
$$U'(X) = f - \frac{2mC^2}{2X^3} = 2X^3 f - 2mC^2 = 0 \Rightarrow X_0^3 = \frac{mC^2}{f}$$

$$U''(X) = \frac{3mc^2}{X^4}\Big|_{X_0} = \frac{3mc^2}{\left(\frac{mc^2}{f}\right)^{4/3}} = 3f\left(\frac{f}{mc^2}\right)^{1/3}$$

$$E \approx \frac{M\dot{X}^2}{4} + \frac{k}{2}(X-X_0)^2 + U_0$$

$$\omega^2 = \frac{2k}{M} = \frac{6f}{M}\left(\frac{f}{mc^2}\right)^{1/3}$$

№5. ур-ние состояния для газа в кубе $a(t)$! задача к экзамену



полное давление: $p = N\underline{p_1}$ давление одной частицы

$$pV = \nu RT$$

$$\mathcal{E} = \frac{3}{2}kT = \frac{mV_T^2}{2} \Rightarrow \frac{3}{2}RT = \frac{\mu V_T^2}{2} \Rightarrow V_T = \sqrt{\frac{3RT}{\mu}} = \sqrt{\frac{3\cdot 25\cdot 300}{3\cdot 30\cdot 10^{-3}}} = 500\,м/с$$

$$p_1 = \frac{F}{S} = \frac{\Delta p}{\Delta t\cdot S} = \frac{v_x}{2a}\cdot\frac{2mv_x}{b\cdot c} = \frac{mv_x^2}{V}$$

(чтобы $u' = u + 2v$)

$$p = N\bar{p}_1 = \frac{m\overline{v_1^2}}{V} = \frac{m\overline{v^2}N}{3V} = \frac{2E_k}{3V}\cdot N$$

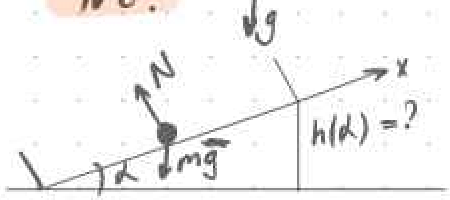
$$k = 1{,}38\cdot 10^{-23}\ \frac{Дж}{K}$$
$$N_A = 6\cdot 10^{23}\ \frac{шт}{моль}$$

$$pV = \frac{2}{3}NE_k$$

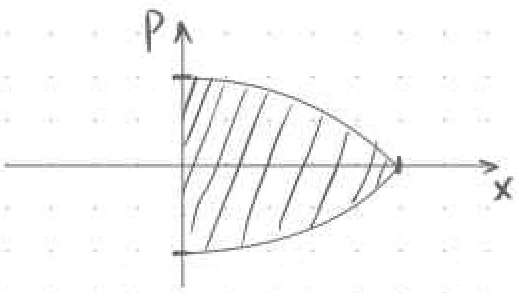
$$p \sim \frac{\Delta p}{\Delta t\cdot S} \sim \frac{mV_x^2}{a^3}$$

$$p \sim \frac{V_x^2}{a^3} \sim \frac{1}{a^5} \Rightarrow pa^5 = const$$
$$pV^{5/3} = const$$

№6.



$h \sim \cos^{-4}\alpha$



$$\begin{cases} v = -gt \\ x^{\prime} = -\frac{gt^2}{2} \end{cases}$$

$$\dot{x} = v_0 - g\sin\alpha t \Rightarrow t_0 = \frac{v_0}{g\sin\alpha}$$

$$x = v_0 t - g\sin\alpha \frac{t^2}{2}$$

$$I = \frac{1}{2\pi}\int m\dot{x}dx = \frac{1}{2\pi}\int mx\,dv_x = -\frac{1}{2\pi}\int m\left(v_0 t - g\sin\alpha\frac{t^2}{2}\right)dt\cdot g\sin\alpha =$$

$$=$$

$$I \sim \int p\,dq \sim \int_0^{\frac{h}{\sin\alpha}} \sqrt{2g\sin\alpha\, x}\,dx \sim \sqrt{2g\sin\alpha}\; x^{3/2}\Big|_0^{\frac{h}{\sin\alpha}} = \sqrt{2g\sin\alpha}\frac{h^{3/2}}{(\sin\alpha)^{3/2}}$$

$$\Rightarrow \frac{h^{2/3}}{\sin\alpha} \sim I \Rightarrow h \sim (\sin\alpha)^{3/2}$$

Подготовка к к/р.

2018.

№2.

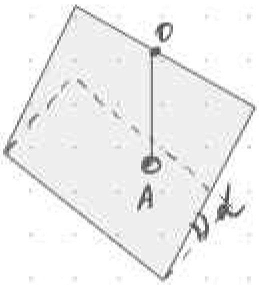
$$H(x,p) = V\cos(ap) - xF, \quad V, F = const$$

$$\dot{p} = -\frac{\partial H}{\partial x} = F \Rightarrow p(t) = Ft + p_0$$

$$\dot{x} = \frac{\partial H}{\partial p} = -Va\sin(ap) = -Va\sin(aFt + ap_0)$$

$$x = \int_0^{t'} -Va\sin(aFt + ap_0) = \frac{Va}{Fa}\cos(aFt + ap_0)\Big|_0^{t'} = \frac{V}{F}\left(\cos(aFt' + ap_0) - \cos(ap_0)\right) + x_0$$

№3.

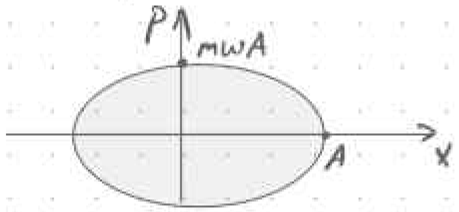


$\alpha : [\alpha_1 = 60°;\ \alpha_2 = 90°]$ медленно

$\frac{\max \varphi_2}{\max \varphi_1} = ?$

$$E = \frac{ml^2\dot{\varphi}^2}{2}$$

для осциллятора:



$$E = \frac{p^2}{2m} = \frac{m^2\omega^2A^2}{2m} \Rightarrow A = \sqrt{\frac{2E}{m\omega^2}}$$

$$p_{max} = m\omega A = \sqrt{2Em}$$

$$\Rightarrow I = \frac{S}{2\pi} = \frac{\pi A\cdot p_0}{2\pi} = \frac{2\pi E}{\omega 2\pi} = \frac{E}{\omega} = const$$

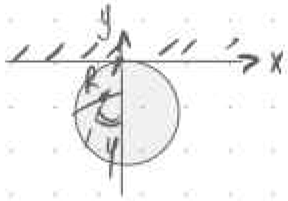
для φ_1: $x_1 = \sqrt{\frac{2E}{m\omega_1^2}} = \sqrt{\underbrace{\frac{2}{m}\cdot\frac{E}{\omega_1}}_{const=\frac{2E}{m\omega}}}\cdot\frac{1}{\sqrt{\omega_1}}$

для φ_2: $x_2 = \sqrt{\frac{2E}{m\omega_2^2}} = \sqrt{\underbrace{\frac{2}{m}\cdot\frac{E}{\omega_2}}_{const=\frac{2E}{m\omega}}}\cdot\frac{1}{\sqrt{\omega_2}}$

$\Rightarrow \frac{x_2}{x_1} = \sqrt{\frac{\omega_1}{\omega_2}}$

$$\frac{\max\varphi_2}{\max\varphi_1} = \sqrt{\frac{\omega_1}{\omega_2}} = \sqrt{\frac{\sqrt{\frac{g}{l}\sin\alpha_1}}{\sqrt{\frac{g}{l}\sin\alpha_2}}} = \sqrt[4]{\frac{\sin\alpha_1}{\sin\alpha_2}} = \sqrt[4]{\frac{\frac{\sqrt{3}}{2}}{1}} = \sqrt[4]{\frac{\sqrt{3}}{2}}$$

№4.



момент инерции тонкой сферы:

$I = \frac{2}{3}mr^2$

$$T = \frac{mv^2}{2} = \frac{m(\omega^2 R)^2}{2} = \frac{m\dot{\varphi}^2R^2}{2} + \frac{I\dot{\varphi}^2}{2} = \frac{m\dot{\varphi}^2R^2}{2} + \frac{mR^2\dot{\varphi}^2}{3}$$

$$U = mgR(1-\cos\varphi) \approx \frac{mgR\varphi^2}{2}$$

! привести Лагранжиан к виду $A\dot{q}^2 + Bq^2 \Rightarrow \omega = \sqrt{\frac{B}{A}}$

$$L = T - U = \frac{5}{6}mR^2\dot{\varphi}^2 - \frac{mgR\varphi^2}{2}$$

$$\omega = \sqrt{\frac{mgR}{\frac{5}{3}mR^2}} = \sqrt{\frac{3g}{3R}}$$

моменты инерции тел:

шар: $\frac{2}{5}mR^2$

диск: $\frac{1}{2}mR^2$

сплошной цилинд: $\frac{1}{2}mR^2$

сфера тонкая: $\frac{2}{3}mR^2$

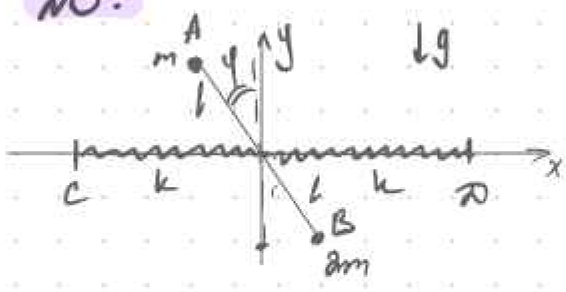
диск: $\frac{1}{4}mR^2$

толстост. цилиндр: $\frac{1}{2}m(r_1^2+r_2^2)$

однор. стержень: $\frac{1}{12}ml^2$

тонк. цилиндр: mR^2

№5.



$$\begin{cases} x_A = x + l\sin\varphi \\ x_B = x - l\sin\varphi \end{cases} \quad \begin{aligned} \dot{x}_A &= \dot{x} + \dot{\varphi}l\cos\varphi \\ \dot{x}_B &= \dot{x} - \dot{\varphi}l\cos\varphi \end{aligned}$$

$$\begin{cases} y_A = l\cos\varphi \\ y_B = -l\cos\varphi \end{cases} \quad \begin{aligned} \dot{y}_A &= -l\sin\varphi\cdot\dot{\varphi} \\ \dot{y}_B &= l\sin\varphi\cdot\dot{\varphi} \end{aligned}$$

$$U = -mgl(1-\cos\varphi) + 2mgl(1-\cos\varphi) =$$

$$H = \frac{p_z^2}{2m} + \frac{p_\theta^2}{2mz^2} + \frac{p_\psi^2}{2mz^2\sin\theta} - \frac{a^2\cos^2\theta}{z^2}$$

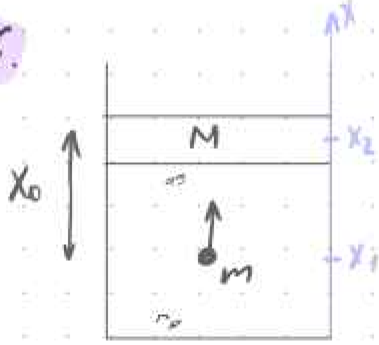
$$f = \frac{p_\psi^2}{2m\sin^2\theta} + \frac{p_\theta^2}{2m} - a^2\cos^2\theta$$

$$\{H, f\} = \frac{\partial H}{\partial p_z}\frac{\partial f}{\partial z} - \frac{\partial H}{\partial z}\frac{\partial f}{\partial p_z} + \frac{\partial H}{\partial p_\psi}\cdot\frac{\partial f}{\partial \psi} - \frac{\partial H}{\partial \psi}\frac{\partial f}{\partial p_\psi} + \frac{\partial H}{\partial p_\theta}\cdot\frac{\partial f}{\partial\theta} - \frac{\partial H}{\partial\theta}\cdot\frac{\partial f}{\partial p_\theta} = 0$$

$$\frac{\partial H}{\partial p_\theta}\cdot\frac{\partial f}{\partial\theta} = \frac{p_\theta}{mz^2}\cdot\left[-\frac{p_\psi^2}{2m}\cdot\frac{1}{\sin^{\cancel{3}\,2}\theta}\cdot 2\cancel{\sin\theta}\cos\theta + a^2 2\cos\theta\sin\theta\right]$$

$$\frac{\partial H}{\partial\theta}\frac{\partial f}{\partial p_\theta} = \left(-\frac{p_\psi^2\cos\theta}{2mz^2\sin^2\theta} + \frac{a^2 2\cos\theta\sin\theta}{z^2}\right)\cdot\frac{p_\theta}{m}$$

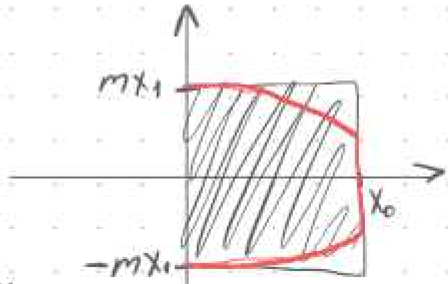
№16.



$\vartheta_м \Rightarrow \vartheta_{порш}$

инвариант: $m\vartheta l = const$

$l(t_0) = X_0$



$$I = \frac{S}{2\pi} = \frac{2m\dot{x}_1 x_2}{2\pi} \Rightarrow \dot{x}_1 = \frac{\pi I}{m x_2}$$

$$E = \frac{m\dot{x}_1^2}{2} + \cancel{mgx_1} + Mgx_2 + \frac{M\dot{x}_2^2}{2} \approx \frac{M\dot{x}_2^2}{2} + \underbrace{\frac{\pi^2 I^2}{2m x_2^2}}_{U_{эфф}} + Mgx_2$$

$$U_{эфф} = \frac{\pi^2 I^2}{2x^2 m} + Mgx \Rightarrow U'_{эфф} = -\frac{\pi^2 I^2}{x_0^3 m} + Mg = 0 \Rightarrow x_0 = \sqrt[3]{\frac{\pi^2 I^2}{Mmg}}$$

$$U''_{эфф} = \frac{3\pi^2 I^2}{x_0^4 m} = \frac{3\pi^2 I^2}{m}\cdot\left(\frac{Mmg}{\pi^2 I^2}\right)^{4/3} = \frac{3Mg}{x_0}$$

$$L = \frac{M\dot{x}}{2} - \frac{3Mg}{2x_0}(x - x_0)^2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow$$

$$M\ddot{x} + \frac{3Mg}{x_0}x = 0 \Rightarrow \ddot{x} + \frac{3g}{x_0}x = 0 \Rightarrow \omega = \frac{3g}{x_0}$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{M}(E - U_{эф})} \Rightarrow dt = \frac{dx}{\sqrt{\frac{2}{M}\left(E - \frac{\pi^2 I^2}{2x^2 m} - Mgx\right)}}$$