1.

The variables of this problem are the decisions made by the company and the government.

a.

I assume costs are paid at the beginning of a period, and revenues are collected at the end of the period.

i)

$$NPV^H = -SC^H - C^H + \frac{R_N^H}{1+r_f} + \pi * \lim_{n\to\infty}\left(\frac{-C^H}{1+r_f} + \frac{R_S^H}{(1+r_f)^2} + \frac{-C^H}{(1+r_f)^2} + \frac{R_S^H}{(1+r_f)^3} + \cdots + \frac{-C^H}{(1+r_f)^{n-1}} + \frac{R_S^H}{(1+r_f)^n}\right) + (1-\pi) * \lim_{n\to\infty}\left(\frac{-C^H}{1+r_f} + \frac{R_N^H}{(1+r_f)^2} + \frac{-C^H}{(1+r_f)^2} + \frac{R_N^H}{(1+r_f)^3} + \cdots + \frac{-C^H}{(1+r_f)^{n-1}} + \frac{R_N^H}{(1+r_f)^n}\right)$$

By using the formula for an infinite geometric series ($a =$first term, $r =$ratio, $|r| < 1$):

$$s_\infty = \frac{a}{1-r}$$

We get that:

$$NPV^H = -SC^H - C^H + \frac{R_N^H}{1+r_f} + \pi * \left(\frac{-C^H}{r_f} + \frac{R_S^H}{(1+r_f)*r_f}\right) + (1-\pi) * \left(\frac{-C^H}{r_f} + \frac{R_N^H}{(1+r_f)*r_f}\right)$$

By plugging the values in, we get:

$$NPV^H = 415788.835$$

ii)

$$NPV^P = -SC^P - C^H + \frac{R_N^H}{1+r_f} + \pi * \lim_{n\to\infty}\left(\frac{-C^P}{1+r_f} + \frac{R_S^P}{(1+r_f)^2} + \frac{-C^P}{(1+r_f)^2} + \frac{R_S^P}{(1+r_f)^3} + \cdots + \frac{-C^P}{(1+r_f)^{n-1}} + \frac{R_S^P}{(1+r_f)^n}\right) + (1-\pi) * \lim_{n\to\infty}\left(\frac{-C^P}{1+r_f} + \frac{R_N^P}{(1+r_f)^2} + \frac{-C^P}{(1+r_f)^2} + \frac{R_N^P}{(1+r_f)^3} + \cdots + \frac{-C^P}{(1+r_f)^{n-1}} + \frac{R_N^P}{(1+r_f)^n}\right)$$

By using the formula for an infinite geometric series ($a =$first term, $r =$ratio, $|r| < 1$):

$$s_\infty = \frac{a}{1-r}$$

We get that:

$$NPV^P = -SC^P - C^H + \frac{R_N^H}{1+r_f} + \pi * (\frac{-C^P}{r_f} + \frac{R_S^P}{(1+r_f)*r_f}) + (1-\pi) * (\frac{-C^P}{r_f} + \frac{R_N^P}{(1+r_f)*r_f})$$

By plugging the values in, we get:

$$NPV^P = 436514.619$$

Since the $NPV^P > NPV^H$, the firm will choose P.

b.

I assume costs are paid at the beginning of a period, and revenues are collected at the end of the period.

NPV adapted to prices:

$$P_1 = \frac{\pi P_S^2 + (1-\pi)P_N^2}{1 + WACC}$$

Arranging to express WACC:

$$WACC = \frac{\pi P_S^2 + (1-\pi)P_N^2}{P_1} - 1$$

By plugging the known values in, we get:

$$WACC = 0.085316$$

Calculating the NPV for H:

$$NPV^H = -SC^H - C^H + \frac{R_N^H}{1+WACC} + \pi * \lim_{n\to\infty}(\frac{-C^H}{1+r_f} + \frac{R_S^H}{(1+r_f)^2} + \frac{-C^H}{(1+r_f)^2} + \frac{R_S^H}{(1+r_f)^3} + \cdots + \frac{-C^H}{(1+r_f)^{n-1}} + \frac{R_S^H}{(1+r_f)^n}) + (1-\pi) * \lim_{n\to\infty}(\frac{-C^H}{1+r_f} + \frac{R_N^H}{(1+r_f)^2} + \frac{-C^H}{(1+r_f)^2} + \frac{R_N^H}{(1+r_f)^3} + \cdots + \frac{-C^H}{(1+r_f)^{n-1}} + \frac{R_N^H}{(1+r_f)^n})$$

By using the formula for an infinite geometric series (a =first term, r =ratio, $|r| < 1$):

$$s_\infty = \frac{a}{1-r}$$

We get that:

$$NPV^H = -SC^H - C^H + \frac{R_N^H}{1+WACC} + \pi * (\frac{-C^H}{r_f} + \frac{R_S^H}{(1+r_f)*r_f}) + (1-\pi) * (\frac{-C^H}{r_f} + \frac{R_N^H}{(1+r_f)*r_f})$$

By plugging the values in, we get:

$$NPV^H = 415342.017$$

Calculating the NPV for P:

$$NPV^P = -SC^P - C^H + \frac{R_N^H}{1+WACC} + \pi * \lim_{n\to\infty}\left(\frac{-C^P}{1+r_f} + \frac{R_S^P}{(1+r_f)^2} + \frac{-C^P}{(1+r_f)^2} + \frac{R_S^P}{(1+r_f)^3} + \cdots + \frac{-C^P}{(1+r_f)^{n-1}} + \frac{R_S^P}{(1+r_f)^n}\right) + (1-\pi) * \lim_{n\to\infty}\left(\frac{-C^P}{1+r_f} + \frac{R_N^P}{(1+r_f)^2} + \frac{-C^P}{(1+r_f)^2} + \frac{R_N^P}{(1+r_f)^3} + \cdots + \frac{-C^P}{(1+r_f)^{n-1}} + \frac{R_N^P}{(1+r_f)^n}\right)$$

By using the formula for an infinite geometric series (a =first term, r =ratio, $|r| < 1$):

$$s_\infty = \frac{a}{1-r}$$

We get that:

$$NPV^P = -SC^P - C^H + \frac{R_N^H}{1+WACC} + \pi * \left(\frac{-C^P}{r_f} + \frac{R_S^P}{(1+r_f)*r_f}\right) + (1-\pi) * \left(\frac{-C^P}{r_f} + \frac{R_N^P}{(1+r_f)*r_f}\right)$$

By plugging the values in, we get:

$$NPV^P = 436067.797$$

Since $NPV^P > NPV^H$, the firm will choose P.

c.

Formula for an infinite geometric series (a =first term, r =ratio, $|r| < 1$):

$$s_\infty = \frac{a}{1-r} \qquad \text{(Q)}$$

Calculating the NPVs in the subsidy case having year two as the reference:

$$NPV_S^H = -SC^H + \lim_{n\to\infty}\left(-C^H + \frac{R_S^H}{1+r_f} + \frac{-C^H}{1+r_f} + \frac{R_S^H}{(1+r_f)^2} + \frac{-C^H}{(1+r_f)^2} + \frac{R_S^H}{(1+r_f)^3} + \cdots + \frac{-C^H}{(1+r_f)^{n-1}} + \frac{R_S^H}{(1+r_f)^n}\right) = -SC^H - \frac{C^H(1+r_f)}{r_f} + \frac{R_S^H}{r_f} \text{ (using Q)}$$

$$NPV_S^P = -SC^P - C^H + \frac{R_S^H}{1+r_f} + \lim_{n\to\infty}\left(\frac{-C^P}{1+r_f} + \frac{R_S^P}{(1+r_f)^2} + \frac{-C^P}{(1+r_f)^2} + \frac{R_S^P}{(1+r_f)^3} + \cdots + \frac{-C^P}{(1+r_f)^{n-1}} + \frac{R_S^P}{(1+r_f)^n}\right) = -SC^P - C^H + \frac{R_S^H}{1+r_f} - \frac{C^P}{r_f} + \frac{R_S^P}{(1+r_f)r_f} \text{ (using Q)}$$

By plugging the values in, we get:

$$NPV_S^H = 438600$$

$$NPV_S^P = 546629.557$$

We notice that $NPV_S^P > NPV_S^H$. So, the company will choose P in the next year if S occurs.

Calculating the NPVs in the no subsidy case having year two as the reference:

$NPV_N^H = -SC^H + \lim_{n\to\infty}(-\ C^H + \frac{R_N^H}{1+r_f} + \frac{-C^H}{1+r_f} + \frac{R_N^H}{(1+r_f)^2} + \frac{-C^H}{(1+r_f)^2} + \frac{R_N^H}{(1+r_f)^3} + \cdots + \frac{-C^H}{(1+r_f)^{n-1}} + \frac{R_N^H}{(1+r_f)^n}) = -SC^H - \frac{C^H(1+r_f)}{r_f} + \frac{R_N^H}{r_f}$ (using Q)

$NPV_N^P = -SC^P - C^H + \frac{R_N^H}{1+r_f} + \lim_{n\to\infty}(\frac{-C^P}{1+r_f} + \frac{R_N^P}{(1+r_f)^2} + \frac{-C^P}{(1+r_f)^2} + \frac{R_N^P}{(1+r_f)^3} + \cdots + \frac{-C^P}{(1+r_f)^{n-1}} + \frac{R_N^P}{(1+r_f)^n}) = -SC^P - C^H + \frac{R_N^H}{1+r_f} - \frac{C^P}{r_f} + \frac{R_N^P}{(1+r_f)r_f}$ (using Q)

By plugging the values in, we get:

$$NPV_N^H = 371933{,}333$$

$$NPV_N^P = 217237.110$$

We notice that $NPV_N^H > NPV_N^P$. So, the company will choose H in the next year if N occurs.

Table of NPVs:

NPV	H	P
N	371933,333	217237.110
S	438600	546629.557

Table 1. Table of NPVs

The expected NPV of the cash flows with the option, discounted to the current year, is:

Expected NPV of cash flows $= \frac{(\pi * NPV_S^P + (1-\pi) * NPV_N^H)}{(1+\text{WACC})} = \frac{(66.77\% * 546629.557 + 33.23\% * 371933{,}333)}{(1+0.085316)} =$ 450171.196

Assuming that the company produces H in the current year before making a decision in the next year, we get that the NPV with the option discounted to the current year (NPV*) is:

$$\text{NPV*} = \text{Expected NPV of cash flows} - C^H + \frac{R_N^H}{1+WACC}$$

If we plug the numbers in:

$$\text{NPV}^* = 455220.930$$

The value of the option is:

$$\text{Option value} = \text{NPV}^* - NPV^P\text{, where } NPV^P \text{ comes from part b.}$$

By plugging the numbers in:

$$\text{Option value} = 19153.133$$

As a result, the company is willing to pay, at most, the option value to get the option.

2.

I assume the principal and agent are risk-neutral.

a.

There are no variables here.

$$\max_e 30e\omega - e^2 - h(\omega) \text{ s.t.}$$
$$\max_e h(\omega) - 0.25e \geq 50$$

Taking the derivate of the manufacturer's problem w.r.t. e and setting it equal to 0, we get:

$$e = 15\omega\text{, where } \omega \text{ is known}$$

The manufacturer wants a contract forcing the manager to put in e $= 15\omega$ and giving him just enough in exchange to make him sign (i.e., h(ω) + T, where T is the smallest positive real number existent used to tip the balance in favour of the manufacturer's contract and beat the outside offer). However, if the manufacturer learns that the manager did not put in the profit-maximising effort, the manager receives nothing.

$$h(\omega) = 50 + \frac{15}{4}\omega + T\text{, if e} = 15\omega$$
$$h(\omega) = 0\text{, if e} \neq 15\omega$$

The contract is:

$$C\big(15\omega, h(\omega)\big)$$

I assume that by knowing the profit function and the contract terms, the manager will deduce the state ω. I also assume that the manufacturer knows that the manager will learn the state ω once presented with the contract.

The manager will decide on the effort to be $e = 15\omega$ since it maximises his benefit and yields him a better offer than 50.

b.

The variables here are: $e \in R^{+}, \sigma \in [0\%, 100\%], \omega \sim U[0,1]$.

$$\max_{\sigma} \int_0^1 \pi(e,\omega) \cdot (1-\sigma)\, d\omega \text{ s.t.}$$

$$\int_0^1 \max_{e}[\pi(e,\omega) \cdot \sigma - 0.25e]\, d\omega \geq \overline{U}$$

Taking the derivative of the function inside of the integral in the manager's problem and equalling the result to 0 and solving for e, we get the manager's optimal effort:

$$e^{*}(\omega,\sigma) = \frac{30\omega\sigma - 0.25}{2\sigma}$$

Plugging this effort into the manager's problem, we get that the manager's expected payoff follows:

$$75\,\sigma + \frac{1}{64\sigma} - \frac{15}{8} \geq \overline{U}$$

Considering the above form and the comparison with $\overline{U}$, we get that:

$$4800\sigma^2 - 3320\sigma + 1 \geq 0$$

Solving the above equation and knowing that σ can take values in [0,1]:

$$\sigma \in [0\%, 0.03\%] \cup [69.136\%, 100\%]$$

Plugging the manager's optimal effort into the manufacturer's problem, we get that the manufacturer's payoff follows:

$$75 - 75\sigma + \frac{1}{64\sigma} - \frac{1}{64\sigma^2}$$

Taking the derivative of the above w.r.t. σ and equalling it to 0, we find:

$$-4800\sigma^3 - \sigma + 2 = 0$$

$$\sigma \cong 7.376\%$$

For this σ, the manufacturer's payoff is 66.8078, and the manager's payoff is 3.8688. As a result, the manufacturer must choose a different σ because the manager's requirement is not met.

Here is the code I wrote to arrive at the result:

```
mod.mod

var THETA;

maximize manufacturer: 75 - 75*THETA + 1/(64*THETA) - 1/(64*THETA**2);

subject to manager: 75*THETA + 1/(64*THETA) - 15/8 >= 50;

subject to percentage: 0<=THETA<=1;
```

Figure 1. Code

```
Console
AMPL
ampl: reset;
ampl: model mod.mod;
ampl: option solver gurobi;
ampl: solve;
Gurobi 11.0.0:            Gurobi 11.0.0: optimal solution; objective 23.13751117
0 simplex iteration(s)
1 branching node(s)
absmipgap=2.21093e-05, relmipgap=9.55562e-07
ampl: display THETA;
THETA = 0.691365

ampl:
```

Figure 2. Result

The optimal σ is (made a very small increase of 0.0001% to address approximations and satisfy the manager's problem):

$$\sigma = 69.1366\%$$

For this optimal σ:

$$\text{Manufacturer's expected payoff} = 23.13746$$

$$\text{Manager's expected payoff} = 50.00005$$

c.

The variables here are: $e \in R^{+}, \sigma \in [0,1], \omega \sim U[0,1]$.

The manager's problem changes because he does not look at his expected utility; he looks at the utility given by that exact state known to him. In other words, the integral is dropped, but the rest of the manager's problem stays the same as in b.. The manager's optimal effort takes the same function as in part b). However, he will put in this effort only if the state known to him (i.e., ω) is more or equally favourable as the state $\widehat{\omega}$ given by the value of σ, where $\widehat{\omega}$ is the state value for which the manager's utility is equal to $\overline{U}$. Otherwise, he will put in zero effort because he will decline the contract. As a result, the manufacturer's problem changes because we need to integrate over the manager's favourable states. In this exercise, ω must be greater than or equal to $\widehat{\omega}$ for the manager to sign the contract. Therefore, the integral in the manufacturer's problem will go from $\widehat{\omega}$ to one (i.e., $\int_{\widehat{\omega}}^{1}$), but the rest of the problem stays the same as in b..

The manager's expected payoff increases. In part b., he wanted to get more than or equal to $\overline{U}$ in expectation. Therefore, in some cases, he would have gotten more than $\overline{U}$ and, in other cases, less than $\overline{U}$, given the uncertainty around ω and that he signed without knowing ω. Now, the manager is getting more than or equal to $\overline{U}$ if he signs. If he doesn't sign, he takes the outside offer and gets $\overline{U}$. Therefore, in expectation, his payoff is greater than $\overline{U}$. As a result, the manager's expected payoff increases.

On the other hand, the manufacturer's expected payoff decreases. If the manager signs the contract, he will put in the same effort in those favourable states as in part b.. So, the profit will be the same in those favourable states, and the manager must be paid more than or equal to $\overline{U}$, exactly as in part b., for the exact same states. However, in the manager's unfavourable states, the manufacturer will now earn zero as the manager will not sign the contract, leading to zero profit. Previously, the manufacturer would have earned a positive value in all those unfavourable states. That is because the manager would have never incurred a loss, so the profit is at least equal to his cost, which is a positive value because effort is positive. Therefore, the manufacturer's expected payoff decreases.

d.

The variables here are: $\pi \sim U[0, \overline{\pi}]$, $\pi^* \in [0, \overline{\pi}]$.

The absolute maximum value the profit can take given all states $= \overline{\pi}$

$\pi \sim U[0, \overline{\pi}]$

$$\max_{\pi*} \int_0^{\pi^*} \pi - C_a \, d\pi + \int_{\pi^*}^{\overline{\pi}} \pi^* \, d\pi$$

$$\text{s. t. } \int_{\pi^*}^{\overline{\pi}} \pi - \pi^* \, d\pi \geq 0$$

The manufacturer's problem looks like this after some calculations:

$$\max_{\pi*} \frac{\pi^* * (2\overline{\pi} - 2C_a - \pi^*)}{2}$$

Taking the derivative of the manufacturer's problem w.r.t π^* and equalling the result to 0 leads to:

$$\pi^* = \bar{\pi} - C_a \qquad \text{(A)}$$

Plugging this value back into the original equation results in:

$$\text{Principal gets} = \frac{(\bar{\pi} - C_a)^2}{2}$$

Solving the integral in the manager's problem leads to:

$$\frac{\bar{\pi}^2}{2} - \bar{\pi}\pi^* + \frac{\pi^{*2}}{2} \geq 0$$

By plugging in the value calculated for π^* at A:

$$\text{Agent gets} = \frac{{C_a}^2}{2}$$

The manager will always put in the profit-maximising effort regardless of the state because there is no downside to doing so. However, if the real profit exceeds the threshold level, there is an upside, plus the amount F for the fine will never be incurred. Thus, the best strategy for the manager is always to choose a profit-maximising effort regardless of the state. This implies that the lowest profit value is 0.